Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered and operator stable processes

Serge Cohen * Jan Rosiński †

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Abstract

Problem of simulation of multivariate Lévy processes is investigated. The method based on shot noise series expansions of such processes combined with Gaussian approximation of the remainder is established in full generality. Formulas that can be used for simulation of tempered stable, operator stable and other multivariate processes are obtained.

Key-words: Lévy processes, Gaussian approximation, shot noise series expansions, simulation, tempered stable processes, operator stable processes.

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1 Introduction

In this work we consider a general problem of simulation of multivariate Lévy processes with applications to stable-like and other processes. Simulation of stochastic processes is widely used in science, engineering and economy to model complex phenomena. There is a vast literature on this subject, just to mention classical monographs [9] on numerical solutions of

*Université Paul Sabatier, Laboratoire de Statistique et de Probabilités, 118, Route de Narbonne, 31062 Toulouse, France.

†Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA. Research supported, in part, by a grant from the National Science Foundation. The work was begun during his visit at the Université Paul Sabatier.

SDE's and [6] on simulation of one-dimensional stable processes. Applications of Lévy processes to stochastic finance and physics created the need to have efficient simulation schemes for such processes. In this regard we refer the reader to the recent monograph [3] and references therein. Beside applications, computer simulation can be useful to understand the structure of some multivariate Lévy processes for which we have now many interesting theoretical results but for which our empirical understanding of the sample paths is limited. For instance, we think that simulation of multivariate tempered stable and operator stable processes will improve their understanding and the way that they can be used as model.

Contrary to the one-dimensional case, close formulas for simulation of increments of multidimensional Lévy processes are rarely available. Thus one needs to use approximate methods. At the first level of approximation of a Lévy process one can use an appropriate compound Poisson process. However, when the Lévy process has paths of infinite variation, the error (the remainder process) of such approximation can be significant. In the one–dimensional case it was shown in [1] that the remainder process can often be approximated by a Brownian motion with small variance. Adding such small Brownian motion to the compound Poisson process will account for the variability between the epochs of the latter process, improving the approximation in general (see [1]).

There are two main issues related to the extension of this method to the multidimensional setting. The first one is the construction of a family of successive compound Poisson approximations of a multivariate Lévy process. We use generalized shot noise series expansions of Lévy processes (cf. [15]) for this purpose. Such expansions can also be related to Lévy copulas (see [3], Sect. 6.6) but we do not consider them here. The second issue is the availability of a Gaussian approximation for the remainder process in the multivariate setting. We want to approximate the remainder by a Brownian motion with small covariance matrix.

Computational problems are the first features which distinguish the multidimensional and one-dimensional cases. Verification that a given process admits Gaussian approximation can be involved and may constitute a result by itself. Please see the operator stable case in Section 3.4. Once a Gaussian approximation of the remainder is establish, we seek to replace the original normalizing matrices by their asymptotics, wherever possible, to give simpler formulas for simulation (cf. the tempered stable case, Section 3.3). Another difference is that the remainder process in the one-dimensional case is simply a small-jump part of the Lévy process (cf. [1]). In the multidimensional case such approach is too restrictive and may lead to substantial technical difficulties (see Section 3.3 and Remark 3.5). Generally, a choice of the remainder depends on the geometry of the Lévy measure of the process under consideration.

The paper is organized as follows. The general problem of Gaussian approximation is considered in Section 2. In Section 3 we give applications to multivariate stable, tempered stable and operator stable processes. Simulation of stochastic integral processes driven by an infinitely divisible random measure is discussed in Section 4.

In this article theoretical issues have been adressed. In an upcoming article practical questions as well as efficiency of the simulation will be considered.

2 Gaussian approximation

In this section we give necessary and sufficient conditions for normal approximation of a multivariate infinitely divisible distributions.

Let $\{X_\epsilon\}_{\epsilon\in(0,1]}$ be a family of infinitely divisible random vectors in $\mathbb{R}^d$ with characteristic functions of the form

$$\mathbf{E}e^{i\langle y,X_\epsilon\rangle} = \exp\left\{\int_{\mathbb{R}^d}[e^{i\langle y,x\rangle} - 1 - i\langle y,x\rangle]\nu_\epsilon(dx)\right\} \tag{2.1}$$

where

$$\int_{\mathbb{R}^d}\|x\|^2\,\nu_\epsilon(dx) < \infty. \tag{2.2}$$

X_ϵ has zero mean and covariance matrix Σ_ϵ which can also be written as

$$\Sigma_\epsilon = \int_{\mathbb{R}^d} xx^\top\,\nu_\epsilon(dx). \tag{2.3}$$

In the sequel we will assume that Σ_ϵ is nonsingular. It is helpful to relate this assumption to the properties of ν_ϵ and the distribution $\mathcal{L}(X_\epsilon)$ of X_ϵ. Let us first state a general lemma.

Lemma 2.1 *Let ν be a measure such that $\int_{\mathbb{R}^d}\|x\|^2\,\nu(dx) < \infty$ and $\Sigma = \int_{\mathbb{R}^d} xx^\top\,\nu(dx)$. The following conditions are equivalent*

(i) Σ is non singular;

(ii) the smallest linear space supporting ν equals $\mathbb{R}^d$.

Proof: Suppose Σ is singular, that is $\Sigma y = 0$ for some y such that $\|y\| = 1$. Consider

$$V = \{x \in \mathbb{R}^d \ : \langle y, x\rangle = 0\}.$$

For any $z \in \mathbb{R}^d$ we have

$$\begin{aligned} 0 = \langle \Sigma y, z\rangle &= \int_{\mathbb{R}^d} \langle y, x\rangle\langle z, x\rangle\, \nu(dx) \\ &= \int_{V^c} \langle y, x\rangle\langle z, x\rangle\, \nu(dx) \end{aligned}$$

where $V^c = \mathbb{R}^d \setminus V$. In particular

$$\int_{V^c} \langle y, x\rangle^2 \nu(dx) = 0.$$

Since $|\langle y, x\rangle| > 0$ when $x \in V^c$, $\nu(V^c) = 0$ which means that ν is concentrated on a proper subspace V of $\mathbb{R}^d$.

Conversely, if ν is concentrated on a proper subspace V of $\mathbb{R}^d$ and y is any unit vector perpendicular to V, then the above computation shows that $\langle \Sigma y, z\rangle = 0$ for any $z \in \mathbb{R}^d$. Therefore Σ is singular. $\square$

If we apply this equivalence to Σ_ϵ, then not only the smallest linear space supporting ν_ϵ equals $\mathbb{R}^d$, but also $\mathcal{L}(X_\epsilon)$ is not concentrated on any proper hyperplane of $\mathbb{R}^d$, which follows from Proposition 24.17 (ii-3) in [17].

In the sequel I_d will denote the identity matrix of rank d, $\mathcal{N}(0, I_d)$ will stand for a standard normal vector in $\mathbb{R}^d$, and "$\xrightarrow{(d)}$" will mean the convergence in distribution. Our first theorem is a starting point of a method that will be further developed in more specific situations.

Theorem 2.2 *Suppose that Σ_ϵ is nonsingular for every $\epsilon \in (0, 1]$. Then, as $\epsilon \to 0$,*

$$\Sigma_\epsilon^{-1/2} X_\epsilon \xrightarrow{(d)} \mathcal{N}(0, I_d) \tag{2.4}$$

if and only if for every $\kappa > 0$

$$\int_{\langle \Sigma_\epsilon^{-1} x, x\rangle > \kappa} \langle \Sigma_\epsilon^{-1} x, x\rangle\, \nu_\epsilon(dx) \to 0. \tag{2.5}$$

Proof: We have

$$\begin{aligned} \mathbf{E} e^{i\langle y, \Sigma_\epsilon^{-1/2} X_\epsilon\rangle} &= \exp\left\{ \int_{\mathbb{R}^d} [e^{i\langle y, \Sigma_\epsilon^{-1/2} x\rangle} - 1 - i\langle y, \Sigma_\epsilon^{-1/2} x\rangle]\, \nu_\epsilon(dx) \right\} \\ &= \exp\left\{ i\langle y, b_\epsilon\rangle + \int_{\mathbb{R}^d} [e^{i\langle y, x\rangle} - 1 - i\langle y, x\rangle \mathbf{1}(\|x\| \le 1)]\, \tilde{\nu}_\epsilon(dx) \right\} \end{aligned}$$

where $\tilde{\nu}_\epsilon = \nu_\epsilon \circ \Sigma_\epsilon^{1/2}$ is the push forward of ν_ϵ by the map $x \to \Sigma_\epsilon^{-1/2} x$ and

$$b_\epsilon = -\int_{\|x\|>1} x \tilde{\nu}_\epsilon(dx) = -\int_{\|\Sigma_\epsilon^{-1/2} x\|>1} \Sigma_\epsilon^{-1/2} x \, \nu_\epsilon(dx).$$

To prove that (2.5) implies (2.4) we verify conditions from Theorem 15 in [8]. We need to show thus as $\epsilon \to 0$

$$b_\epsilon \to 0 \tag{2.6}$$

$$\int_{\|x\|\le 1} x x^\top \, \tilde{\nu}_\epsilon(dx) \to I_d \tag{2.7}$$

$$\tilde{\nu}_\epsilon(\|x\| \ge \kappa) \to 0, \quad \forall \kappa > 0. \tag{2.8}$$

(2.6) holds because

$$\begin{aligned} \|b_\epsilon\| &\le \int_{\|\Sigma_\epsilon^{-1/2} x\|>1} \|\Sigma_\epsilon^{-1/2} x\| \, \nu_\epsilon(dx) \\ &\le \int_{\|\Sigma_\epsilon^{-1/2} x\|>1} \|\Sigma_\epsilon^{-1/2} x\|^2 \, \nu_\epsilon(dx) \\ &= \int_{\langle \Sigma_\epsilon^{-1} x, x\rangle>1} \langle \Sigma_\epsilon^{-1} x, x\rangle \, \nu_\epsilon(dx) \to 0. \end{aligned}$$

by the assumption (2.5). Notice that

$$\begin{aligned} \int_{\mathbb{R}^d} x x^\top \, \tilde{\nu}_\epsilon(dx) &= \int_{\mathbb{R}^d} (\Sigma_\epsilon^{-1/2} x)(\Sigma_\epsilon^{-1/2} x)^\top \, \nu_\epsilon(dx) \\ &= \Sigma_\epsilon^{-1/2} \int_{\mathbb{R}^d} x x^\top \, \nu_\epsilon(dx) \, \Sigma_\epsilon^{-1/2} = I_d. \end{aligned} \tag{2.9}$$

Denoting also by $\|\cdot\|$ the operator norm we get

$$\begin{aligned} \|I_d - \int_{\|x\|\le 1} x x^\top \, \tilde{\nu}_\epsilon(dx)\| &= \| \int_{\|\Sigma_\epsilon^{-1/2} x\|>1} (\Sigma_\epsilon^{-1/2} x)(\Sigma_\epsilon^{-1/2} x)^\top \, \nu_\epsilon(dx)\| \\ &\le \int_{\|\Sigma_\epsilon^{-1/2} x\|>1} \|\Sigma_\epsilon^{-1/2} x\|^2 \, \nu_\epsilon(dx) \to 0 \end{aligned}$$

as above, which proves (2.7). To get (2.8) we observe that

$$\begin{aligned} \tilde{\nu}_\epsilon(\|x\| > \kappa) &\le \kappa^{-2} \int_{\|x\|>\kappa} \|x\|^2 \, \tilde{\nu}_\epsilon(dx) \\ &= \kappa^{-2} \int_{\|\Sigma_\epsilon^{-1/2} x\|>\kappa} \|\Sigma_\epsilon^{-1/2} x\|^2 \, \nu_\epsilon(dx) \to 0. \end{aligned}$$

Therefore, we proved that (2.5) implies (2.4).

To obtain the converse, note that from Theorem 15.14 [8], we must have $\forall \kappa > 0$

$$\int_{\|x\|\le\kappa} xx^\top \, \tilde{\nu}_\epsilon(dx) \to I_d$$

as $\epsilon \to 0$. By (2.9) we have

$$\int_{\|x\|>\kappa} xx^\top \, \tilde{\nu}_\epsilon(dx) \to 0.$$

We infer that

$$\begin{aligned}\int_{\langle\Sigma_\epsilon^{-1}x,x\rangle>\kappa} \langle\Sigma_\epsilon^{-1}x, x\rangle \, \nu_\epsilon(dx) &= \int_{\|\Sigma_\epsilon^{-1/2}x\|>\kappa^{1/2}} \|\Sigma_\epsilon^{-1/2}x\|^2 \, \nu_\epsilon(dx) \\ &= \int_{\|x\|>\kappa^{1/2}} \|x\|^2 \, \tilde{\nu}_\epsilon(dx) = \sum_{i=1}^d \int_{\|x\|>\kappa^{1/2}} \langle e_i, x\rangle^2 \, \tilde{\nu}_\epsilon(dx) \\ &= \sum_{i=1}^d \langle e_i, \int_{\|x\|>\kappa^{1/2}} xx^\top \, \tilde{\nu}_\epsilon(dx) \, e_i\rangle \to 0\end{aligned}$$

where $\{e_i\}_{i=1}^d$ is an orthonormal basis in $\mathbb{R}^d$. The proof of Theorem 2.2 is complete. $\square$

Note that in order to verify (2.5) it is enough to show it for some lower bound for Σ_ϵ. We state this simple observation for a convenient reference.

Lemma 2.3 *Suppose that for every $\kappa > 0$ there exist $\epsilon(\kappa) > 0$ and a family of positive definite matrices $\{\tilde{\Sigma}_\epsilon : \epsilon \in (0, \epsilon(\kappa))\}$ such that $\forall \epsilon \in (0, \epsilon(\kappa))$*

$$\Sigma_\epsilon \ge \tilde{\Sigma}_\epsilon \tag{2.10}$$

and

$$\lim_{\epsilon\to 0} \int_{\langle\tilde{\Sigma}_\epsilon^{-1}x,x\rangle>\kappa} \langle\tilde{\Sigma}_\epsilon^{-1}x, x\rangle \, \nu_\epsilon(dx) = 0. \tag{2.11}$$

Then (2.5) *holds.*

Proof: From (2.10) we have $\Sigma_\epsilon^{-1} \le \tilde{\Sigma}_\epsilon^{-1}$. Hence, for $0 < \epsilon < \epsilon(\kappa)$,

$$\int_{\langle\Sigma_\epsilon^{-1}x,x\rangle>\kappa} \langle\Sigma_\epsilon^{-1}x, x\rangle \, \nu_\epsilon(dx) \le \int_{\langle\tilde{\Sigma}_\epsilon^{-1}x,x\rangle>\kappa} \langle\tilde{\Sigma}_\epsilon^{-1}x, x\rangle \, \nu_\epsilon(dx) \to 0$$

as $\epsilon \to 0$. $\square$

Corollary 2.4 *Let* $d = 1$ *and* $\nu_\epsilon(dx) = \mathbf{1}_{(-\epsilon,\epsilon)}(x)\nu(dx)$*, where* ν *is a Lévy measure on* $\mathbb{R}$*. Put*

$$\sigma^2(\epsilon) = \int_{(-\epsilon,\epsilon)} x^2 \, \nu(dx).$$

By Theorem 2.2,

$$\sigma^{-1}(\epsilon) X_\epsilon \xrightarrow{(d)} \mathcal{N}(0,1) \tag{2.12}$$

if and only if $\forall \ \kappa > 0$

$$\sigma^{-2}(\epsilon) \int_{\{|x|>\kappa^{1/2}\sigma(\epsilon),\, |x|<\epsilon\}} x^2 \, \nu(dx) \to 0 \tag{2.13}$$

as $\epsilon \to 0$*.*

It is easy to see that (2.13) is equivalent to the condition of [1]: $\forall \ \kappa > 0$

$$\frac{\sigma(\epsilon \wedge \kappa\sigma(\epsilon))}{\sigma(\epsilon)} \to 1. \tag{2.14}$$

A simpler than the above and sufficient condition for (2.12) is

$$\lim_{\epsilon \to 0} \frac{\sigma(\epsilon)}{\epsilon} = \infty.$$

This is also a necessary condition when ν does not have atoms in a neighborhood of zero, see [1].

We can extend (2.14) (equivalently (2.13)) to a more general setting as follows. Suppose ν_ϵ is given in polar coordinates as

$$\nu_\epsilon(dr, du) = \mu_\epsilon(dr\,|u)\lambda(du) \quad r > 0, \ u \in S^{d-1} \tag{2.15}$$

where λ is a finite measure on the unit sphere S^{d-1} and $\{\mu_\epsilon(\cdot\,|u) : u \in S^{d-1}\}$ is a measurable family of Lévy measures on $(0,\infty)$. Define

$$\sigma_\epsilon^2(u) = \int_0^\infty r^2 \, \mu_\epsilon(dr\,|u). \tag{2.16}$$

Theorem 2.5 *Let* ν_ϵ *be Lévy measures on* $\mathbb{R}^d$ *given by* (2.15) *such that the support of* λ *is not contained in any proper subspace of* $\mathbb{R}^d$*. Suppose there exists a function* $b : (0,1] \mapsto (0,\infty)$ *such that*

$$\liminf_{\epsilon \to 0} \frac{\sigma_\epsilon(u)}{b(\epsilon)} > 0 \quad \lambda - \text{a.e.} \tag{2.17}$$

and for every $\kappa > 0$

$$\lim_{\epsilon\to 0} b(\epsilon)^{-2} \int_{\|x\|>\kappa b(\epsilon)} \|x\|^2 \, \nu_\epsilon(dx) = 0. \tag{2.18}$$

Then Σ_ϵ is nonsingular for sufficiently small ϵ and

$$\Sigma_\epsilon^{-1/2} X_\epsilon \xrightarrow{(d)} \mathcal{N}(0, I_d) \quad \textit{as } \epsilon \to 0.$$

Proof: Consider

$$\Lambda = \int_{S^{d-1}} uu^\top \, \lambda(du).$$

Λ is nonsingular by Lemma 2.1. Hence $\inf_{v\in S^{d-1}} \langle \Lambda v, v\rangle =: 2a > 0$. For any Borel subset B of S^{d-1} put

$$\Lambda_B = \int_B uu^\top \, \lambda(du). \tag{2.19}$$

There exists a $\delta > 0$ such that $\|\Lambda - \Lambda_B\| < a$ whenever $\lambda(S^{d-1} \setminus B) < \delta$. For such a set B and any $v \in S^{d-1}$

$$\langle \Lambda_B v, v\rangle \ge \langle \Lambda v, v\rangle - \|\Lambda - \Lambda_B\| > a.$$

Hence

$$\Lambda_B \ge a I_d \tag{2.20}$$

whenever $\lambda(S^{d-1} \setminus B) < \delta$.

From (2.17) we can find $\epsilon_0, \epsilon_1 \in (0, 1]$ such that the set

$$B := \{u \in S^{d-1} : \inf_{0<\epsilon<\epsilon_0} b(\epsilon)^{-2} \sigma_\epsilon^2(u) > \epsilon_1\}$$

satisfies $\lambda(S^{d-1} \setminus B) < \delta$. Using (2.20) we get

$$\begin{aligned}\Sigma_\epsilon &= \int_{S^{d-1}} \int_0^\infty r^2 \, \mu_\epsilon(dr|u) uu^\top \, \lambda(du) \ge \int_B \sigma_u^2(\epsilon) uu^\top \, \lambda(du) \\ &\ge a\epsilon_1 b(\epsilon)^2 I_d =: \tilde\Sigma_\epsilon.\end{aligned}$$

Thus for any $\kappa > 0$ we have

$$\int_{\langle \tilde\Sigma_\epsilon^{-1} x, x\rangle > \kappa} \langle \tilde\Sigma_\epsilon^{-1} x, x\rangle \, \nu_\epsilon(dx) \le a^{-1}\epsilon_1^{-1} b(\epsilon)^{-2} \int_{\|x\|>(a\epsilon_1\kappa)^{1/2} b(\epsilon)} \|x\|^2 \, \nu_\epsilon(dx).$$

Since the last expression converges to 0 by (2.18), Lemma 2.3 concludes the proof. □

Typically ν_ϵ is of the form

$$\nu_\epsilon = \mathbf{1}_{D_\epsilon}\,\nu \tag{2.21}$$

and

$$\Sigma_\epsilon = \int_{D_\epsilon} xx^\top\,\nu(dx). \tag{2.22}$$

where ν is a Lévy measure and D_ϵ is a bounded neighborhood of the origin. This is a natural extension of the case studied in [1] for $d = 1$. However, contrary to the one–dimensional case, a centered ball is not always the best choice for D_ϵ. We illustrate this point on examples in the next section. On the other hand, the case when D_ϵ is a ball of radius ϵ is typical and important

Suppose a Lévy measure ν has a representation in polar coordinates

$$\nu(dr, du) = \mu(dr\,|u)\lambda(du) \quad r > 0,\ u \in S^{d-1}. \tag{2.23}$$

Here $\{\mu(\cdot\,|u) : u \in S^{d-1}\}$ is a measurable family of Lévy measures on $(0, \infty)$ and λ is a finite measure on the unit sphere S^{d-1} such that the support of λ is not contained in any proper subspace of $\mathbb{R}^d$.

Theorem 2.6 *Let ν be a Lévy measure on $\mathbb{R}^d$ given by (2.23). Consider $\nu_\epsilon(dx) = \mathbf{1}_{D_\epsilon}(x)\,\nu(dx)$ where*

$$D_\epsilon = \{x \in \mathbb{R}^d : \|x\| < \epsilon\}.$$

If

$$\lim_{\epsilon\to 0} \epsilon^{-2} \int_{(0,\epsilon)} r^2\,\mu(dr\,|u) = \infty \quad \lambda-\text{a.e.} \tag{2.24}$$

then Σ_ϵ is nonsingular and

$$\Sigma_\epsilon^{-1/2} X_\epsilon \xrightarrow{(d)} \mathcal{N}(0, I_d) \quad \text{as } \epsilon \to 0,$$

where Σ_ϵ is given by (2.22).

Proof: We have

$$\sigma_\epsilon^2(u) = \int_{(0,\epsilon)} r^2\,\mu_\epsilon(dr\,|u).$$

Therefore, condition (2.24) says that

$$\lim_{\epsilon\to 0}\frac{\sigma_\epsilon(u)}{\epsilon}=\infty \quad \lambda-\text{a.e.} \tag{2.25}$$

We can choose a sequence $\epsilon_k \searrow 0$ such that the sets

$$B_k:=\left\{u\in S^{d-1}: \inf\left\{\frac{\sigma_\epsilon(u)}{\epsilon}: 0<\epsilon\le\epsilon_k\right\}>k^2\right\}$$

satisfy $\lambda(B_k)>\lambda(S^{d-1})(1-2^{-k})$, $k=1,2,\ldots$. Then $S_0=\bigcup_{n=1}^\infty\bigcap_{k=n}^\infty B_k$ has full λ-measure. Put $b(\epsilon)=\epsilon k$ for $\epsilon_{k+1}<\epsilon\le\epsilon_k$. It is clear that

$$\lim_{\epsilon\to 0}\frac{b(\epsilon)}{\epsilon}=\infty. \tag{2.26}$$

and for each $u\in S_0$ there is an n such that $u\in B_k$ for $k\ge n$. Hence for $\epsilon_{k+1}<\epsilon\le\epsilon_k\le\epsilon_n$

$$\frac{\sigma_\epsilon(u)}{b(\epsilon)}=\frac{\sigma_\epsilon(u)}{k\epsilon}>k,$$

which proves that

$$\lim_{\epsilon\to 0}\frac{\sigma_\epsilon(u)}{b(\epsilon)}=\infty \quad \lambda-\text{a.e.}$$

Thus (2.17) of Theorem 2.5 holds. Using (2.26) and the form of ν_ϵ we infer that for each $\kappa>0$ and sufficiently small $\epsilon>0$,

$$\int_{\|x\|\ge\kappa b(\epsilon)}\|x\|^2\,\nu_\epsilon(dx)=\int_{\{\|x\|\ge\kappa b(\epsilon),\|x\|<\epsilon\}}\|x\|^2\,\nu(dx)=0.$$

This verifies condition (2.18) and Theorem 2.5 concludes the proof. □

3 Application to simulation of multivariate stable–like Lévy processes

3.1 General Lévy processes setting

Consider a Lévy process $\mathbf{X}=\{X(t): t\ge 0\}$ in $\mathbb{R}^d$ determined by its characteristic function in the Lévy–Khintchine form

$$\begin{aligned}&\mathbf{E}\exp i\langle y,X(t)\rangle\\ &\quad=\exp\left\{t\left[i\langle a,y\rangle+\int_{\mathbb{R}^d}(e^{i\langle y,x\rangle}-1-i\langle y,x\rangle\mathbf{1}(\|x\|\le 1))\,\nu(dx)\right]\right\}.\end{aligned} \tag{3.1}$$

We will say that ν is the Lévy measure of $\mathbf{X}$. Let $\nu_\epsilon \leq \nu$ be a measure satisfying (2.2), that is

$$\int_{\mathbb{R}^d} \|x\|^2 \, \nu_\epsilon(dx) < \infty,$$

and such that $\nu^\epsilon := \nu - \nu_\epsilon$ is a finite measure for every $\epsilon \in (0, 1]$. Decompose $\mathbf{X}$ into a sum of two independent Lévy processes

$$\mathbf{X} = \mathbf{X}^\epsilon + \mathbf{X}_\epsilon \tag{3.2}$$

where $\mathbf{X}_\epsilon$ is determined by

$$\mathbf{E} \exp i\langle y, X_\epsilon(t)\rangle = \exp\left\{ t \left[\int_{\mathbb{R}^d} (e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle) \, \nu_\epsilon(dx) \right] \right\}.$$

Process $\mathbf{X}^\epsilon$ can be represented as

$$\mathbf{X}^\epsilon = \mathbf{a}_\epsilon + \mathbf{N}^\epsilon \tag{3.3}$$

where $\mathbf{a}_\epsilon = \{ta_\epsilon : t \geq 0\}$,

$$a_\epsilon = \int_{\|x\|>1} x \, \nu_\epsilon(dx) - \int_{\|x\|\leq 1} x \, \nu^\epsilon(dx), \tag{3.4}$$

and $\mathbf{N}^\epsilon = \{N^\epsilon(t) : t \geq 0\}$ is a compound Poisson process with Lévy measure ν^ϵ. There are various methods to simulate process $\mathbf{N}^\epsilon$. If $\Sigma_\epsilon^{-1/2} X_\epsilon(1) \xrightarrow{(d)} \mathcal{N}(0, I_d)$, then we may approximate $\mathbf{X}_\epsilon$ by $\mathbf{W}_\epsilon$, a centered Brownian motion with covariance matrix Σ_ϵ, where

$$\Sigma_\epsilon = \int_{\mathbb{R}^d} xx^\top \, \nu_\epsilon(dx).$$

Sometimes, Σ_ϵ may be too complicated for practical use. In this case we may try to replace it by a simpler asymptotic. The method of approximation of $\mathbf{X}$ is precisely stated in the following proposition.

Proposition 3.1 *Let* $\mathbf{X} = \{X(t) : t \geq 0\}$ *be a Lévy process in* $\mathbb{R}^d$ *determined by* (3.1). *Suppose that*

$$\Sigma_\epsilon^{-1/2} X_\epsilon(1) \xrightarrow{(d)} \mathcal{N}(0, I_d) \quad \text{as } \epsilon \to 0$$

and for some family of nonsingular matrices $\{A_\epsilon\}_{\epsilon \in (0,1]}$ *of rank* d *we have*

$$A_\epsilon^{-1} \Sigma_\epsilon (A_\epsilon^{-1})^\top \to I_d \quad \text{as } \epsilon \to 0. \tag{3.5}$$

Let $\mathbf{a}_\epsilon$, $\mathbf{N}^\epsilon$ be as in (3.3), and let $\mathbf{W} = \{W(t) : t \geq 0\}$ be a standard Brownian motion in $\mathbb{R}^d$ independent of $\mathbf{N}^\epsilon$.

Then for every $\epsilon \in (0,1]$ there exists a cadlag process $\mathbf{Y}_\epsilon = \{Y_\epsilon(t) : t \geq 0\}$ such that

$$\mathbf{X} \stackrel{(d)}{=} \mathbf{a}_\epsilon + A_\epsilon \mathbf{W} + \mathbf{N}^\epsilon + \mathbf{Y}_\epsilon \tag{3.6}$$

in the sense of equality of finite dimensional distributions and such that for each $T > 0$

$$\sup_{t\in[0,T]} \|A_\epsilon^{-1} Y_\epsilon(t)\| \xrightarrow{(\mathbb{P})} 0 \quad \textit{as } \epsilon \to 0. \tag{3.7}$$

Proof: Consider the polar decomposition

$$A_\epsilon^{-1}\Sigma_\epsilon^{1/2} = C_\epsilon U_\epsilon$$

where C_ϵ positive definite and U_ϵ is on orthogonal matrix, see, e.g., [4], Theorem 12.2.22. Then

$$C_\epsilon^2 = C_\epsilon C_\epsilon^\top = A_\epsilon^{-1}\Sigma_\epsilon (A_\epsilon^{-1})^\top \to I_d.$$

Let $\epsilon_n \to 0$. There exists a subsequence $\{n_k\}$ such that $U_{\epsilon_{n_k}} \to U$, for some orthogonal matrix matrix U. Hence

$$A_{\epsilon_{n_k}}^{-1}\Sigma_{\epsilon_{n_k}}^{1/2} = C_{\epsilon_{n_k}} U_{\epsilon_{n_k}} \to U.$$

Consequently,

$$A_{\epsilon_{n_k}}^{-1} X_{\epsilon_{n_k}}(1) = \left(A_{\epsilon_{n_k}}^{-1}\Sigma_{\epsilon_{n_k}}^{1/2}\right)\Sigma_{\epsilon_{n_k}}^{-1/2} X_{\epsilon_{n_k}}(1) \xrightarrow{(d)} \mathcal{N}(0, UU^\top) = N(0, I_d).$$

Thus $A_\epsilon^{-1} X_\epsilon(1) \xrightarrow{(d)} \mathcal{N}(0, I_d)$. By a theorem of Skorohod (cf. [8], Theorem 15.17) there exist Lévy processes $\mathbf{Z}_\epsilon = \{Z_\epsilon(t) : t \geq 0\}$ such that $\mathbf{Z}_\epsilon \stackrel{(d)}{=} A_\epsilon^{-1}\mathbf{X}_\epsilon$ and

$$\sup_{t\in[0,T]} \|Z_\epsilon(t) - W(t)\| \xrightarrow{(\mathbb{P})} 0 \quad \text{as } \epsilon \to 0 \tag{3.8}$$

for each $T > 0$. Making $\mathbf{W}$ and $\mathbf{N}^\epsilon$ depend on different coordinates of a large enough probability space, we may also assume that $\mathbf{Z}_\epsilon$ and $\mathbf{N}^\epsilon$ are independent. Put

$$\mathbf{Y}_\epsilon = A_\epsilon(\mathbf{Z}_\epsilon - \mathbf{W}).$$

Then

$$\begin{aligned}\mathbf{X} &\stackrel{(d)}{=} \mathbf{a}_\epsilon + \mathbf{X}_\epsilon + \mathbf{N}^\epsilon \stackrel{(d)}{=} \mathbf{a}_\epsilon + A_\epsilon \mathbf{Z}_\epsilon + \mathbf{N}^\epsilon \\ &= \mathbf{a}_\epsilon + A_\epsilon \mathbf{W}_\epsilon + \mathbf{N}^\epsilon + \mathbf{Y}_\epsilon.\end{aligned}$$

This proves (3.6), (3.7) follows from (3.8). □

Conditions (3.6)–(3.7) make precise the statement

$$\mathbf{X} \approx \mathbf{a}_\epsilon + A_\epsilon \mathbf{W} + \mathbf{N}^\epsilon. \tag{3.9}$$

There are two practical issues related to implementation of (3.9). The first one is how to define ν_ϵ so that Σ_ϵ (or its asymptotic) is computable. The second one is how to generate $\mathbf{N}^\epsilon$ efficiently. We will illustrate these issues in the following examples.

3.2 Stable processes

The Lévy measure ν of an α–stable process $\mathbf{X}$ in $\mathbb{R}^d$ has the form in polar coordinates

$$\nu(dr, du) = \alpha r^{-\alpha-1}\, dr\, \lambda(du) \tag{3.10}$$

where $\alpha \in (0, 2)$ and λ is a finite measure on S^{d-1}. Denote $\|\lambda\| = \lambda(S^{d-1})$. Assume that $X(1)$ is not concentrated on a proper hyperplane of $\mathbb{R}^d$, which by Lemma 2.1 means that λ is not concentrated on a proper subspace of $\mathbb{R}^d$.

The compound Poisson process $\mathbf{N}^\epsilon$ of (3.3) can be generated from a shot noise expansion of a stable process. Namely, let $\{e'_j\}$ be an iid sequence of exponential random variables with parameter 1 and $\gamma_j = e'_1 + \cdots + e'_j$. $\{\gamma_j\}$ forms a Poisson point process on $(0, \infty)$ with the Lebesgue intensity measure. Let $\{\tau_j\}$ be an iid sequence of uniform on $[0, T]$ random variables. Finally, let $\{v_j\}$ be an iid sequence of random vectors taking values in S^{d-1} with the common distribution $\lambda/\|\lambda\|$. Assume that $\{v_j\}$, $\{\tau_j\}$, and $\{e'_j\}$ are independent. For $\epsilon \in (0, 1]$ and $t \in [0, T]$ define

$$N^\epsilon(t) = (T\|\lambda\|)^{1/\alpha} \sum_{\gamma_j \le \epsilon^{-1}} I_{(0,t]}(\tau_j)\gamma_j^{-1/\alpha} v_j. \tag{3.11}$$

It is elementary to check that $\mathbf{N}^\epsilon$ is a compound Poisson process with characteristic function

$$\mathbf{E} \exp i\langle y, N^\epsilon(t)\rangle = \exp\left\{ t \int_{\|x\| \ge \epsilon_0^{1/\alpha}} (e^{i\langle y,x\rangle} - 1)\, \nu(dx) \right\}$$

where

$$\epsilon_0 := \|\lambda\| T \epsilon. \tag{3.12}$$

Therefore, we define $\nu_\epsilon(dx) = \mathbf{1}_{D_\epsilon}(x)\nu(dx)$, where $D_\epsilon = \{x : \|x\| < \epsilon_0^{1/\alpha}\}$. ν_ϵ is of the form (2.21). Then, by (2.22),

$$\Sigma_\epsilon = \frac{\alpha}{2-\alpha}\epsilon_0^{2/\alpha-1}\Lambda \tag{3.13}$$

where

$$\Lambda = \int_{S^{d-1}} uu^\top \lambda(du). \tag{3.14}$$

Since ν is of the form (2.23) with $\mu(dr|u) = \alpha r^{-1-\alpha}dr$ and

$$\epsilon^{-2}\int_{(0,\epsilon)} r^2 \mu(dr|u) = \frac{\alpha}{2-\alpha}\epsilon^{-\alpha} \to \infty,$$

we infer from Theorem 2.6 that

$$\Sigma_\epsilon^{-1/2}X_\epsilon(1) \xrightarrow{(d)} \mathcal{N}(0, I_d).$$

Therefore, the approximation (3.9) applies. By Proposition 3.1 we have

Proposition 3.2 *Let* $\mathbf{X}$ *be an* α*–stable Lévy process with Lévy measure given by* (3.10)*. Suppose that the support of* λ *is not contained in a proper subspace of* $\mathbb{R}^d$*. Let* T *be fixed and recall that* $\epsilon_0 := \|\lambda\|T\epsilon$*. Let* $\mathbf{N}^\epsilon$ *be given by* (3.11)*,* $\mathbf{W}$ *be a standard Brownian motion in* $\mathbb{R}^d$ *independent of* $\mathbf{N}^\epsilon$*, and* $\mathbf{a}_\epsilon$ *be a shift determined by* (3.4)*. Then, for every* $\epsilon \in (0, 1]$ *there exists a cadlag process* $\mathbf{Y}_\epsilon$ *such that on* $[0, T]$

$$\mathbf{X} \stackrel{(d)}{=} \mathbf{a}_\epsilon + \epsilon_0^{1/\alpha-1/2}(\frac{\alpha}{2-\alpha})^{1/2}\Lambda^{1/2}\mathbf{W} + \mathbf{N}^\epsilon + \mathbf{Y}_\epsilon \tag{3.15}$$

in the sense of equality of finite dimensional distributions and such that

$$\epsilon_0^{1/2-1/\alpha} \sup_{t\in[0,T]} \|Y_\epsilon(t)\| \xrightarrow{(\mathbb{P})} 0 \quad \text{as } \epsilon \to 0. \tag{3.16}$$

Proof: We apply Proposition 3.1 to $A_\epsilon = \Sigma_\epsilon^{1/2}$, where Σ_ϵ is given by (3.13). We obtain (3.15) and

$$\|Y_\epsilon(t)\| \le \|A_\epsilon\|\|A_\epsilon^{-1}Y_\epsilon(t)\| \le \epsilon_0^{1/\alpha-1/2}(\frac{\alpha}{2-\alpha})^{1/2}\|\Lambda^{1/2}\|\|A_\epsilon^{-1}Y_\epsilon(t)\|.$$

Since Λ is nonsingular, (3.7) yields (3.16). □

3.3 Tempered stable processes

Recall that the Lévy measure of a tempered α–stable process $\mathbf{X}$ in $\mathbb{R}^d$ is of the form

$$\nu(dr, du) = \alpha r^{-\alpha-1} q(r, u)\, dr\, \lambda(du), \tag{3.17}$$

in polar coordinates , where $\alpha \in (0, 2)$, λ is a finite measure on S^{d-1}, and $q : (0, \infty) \times S^{d-1} \mapsto (0, \infty)$ is a Borel function such that, for each $u \in S^{d-1}$, $q(\cdot, u)$ is completely monotone with $q(0+, u) = 1$ and $q(\infty, u) = 0$. For this and further facts on tempered stable distributions and processes see [16]. As in the previous section, assume that λ is not concentrated on a proper subspace of $\mathbb{R}^d$ and put $\|\lambda\| = \lambda(S^{d-1})$. Define a finite measure Q on $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$ by

$$Q(A) := \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(ru)\, Q(dr|u) \lambda(du)$$

where $\{Q(\cdot|u)\}_{u \in S^{d-1}}$ is a measurable family of probability measures on $\mathbb{R}_+$ determined by $q(r, u) = \int_0^\infty e^{-rs}\, Q(ds|u)$. Note that $Q(\mathbb{R}^d_0) = \|\lambda\|$.

Let $\{v_j\}$ be an idd sequence in $\mathbb{R}^d_0$ with the common distribution $Q/\|\lambda\|$. Let $\{u_j\}$ and $\{e_j\}$ be iid sequences of uniform on $(0, 1)$ and exponential with parameter 1 random variables, respectively. Let $\{\gamma_j\}$ and $\{\tau_j\}$ be as in the previous section. Assume that $\{v_j\}$, $\{u_j\}$, $\{e_j\}$, $\{\gamma_j\}$ and $\{\tau_j\}$ are independent of each other.

Again, it is natural to take as the process $\mathbf{N}^\epsilon$ the partial sum of a shot noise representation of a tempered stable process. Such representation is given in [16], Theorem 5.4. Therefore, we have for $\epsilon \in (0, 1]$ and $t \in [0, T]$

$$N^\epsilon(t) = \sum_{\gamma_j \le \epsilon^{-1}} \mathbf{1}_{(0,t]}(\tau_j) \left(\left(\frac{\gamma_j}{T\|\lambda\|} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|}. \tag{3.18}$$

$\mathbf{N}^\epsilon$ is a compound Poisson process with characteristic function

$$\mathbf{E} \exp i\langle y, N^\epsilon(t) \rangle = \exp \left\{ t \int_{\mathbb{R}^d_0} (e^{i\langle y, x\rangle} - 1)\, \nu^\epsilon(dx) \right\}.$$

From the proof of Theorem 5.1 [16] or otherwise, we can verify that

$$\nu^\epsilon(dr, du) = k^\epsilon(r, u)\, dr \lambda(du),$$

in polar coordinates, where

$$k^\epsilon(r, u) = \begin{cases} \epsilon_0^{-1} \alpha \left[r^{-1} q(r, u) - r^{\alpha-1} \int_r^\infty \alpha s^{-\alpha-1} q(s, u)\, ds \right] & , 0 < r < \epsilon_0^{1/\alpha}, \\ \alpha r^{-\alpha-1} q(r, u) & , r \ge \epsilon_0^{1/\alpha} \end{cases}$$

and $\epsilon_0 = T\|\lambda\|\epsilon$, as in (3.12). Notice that the process $\mathbf{N}^\epsilon$ has both large and small jumps, which is different from the stable case treated in the previous section. Moreover, since $k^\epsilon(r,u) \le \epsilon_0^{-1}\alpha r^{-1}q(r,u) \le \alpha r^{-\alpha-1}q(r,u)$, if $0 < r < \epsilon_0^{1/\alpha}$, we have $\nu^\epsilon \le \nu$. Hence $\nu_\epsilon = \nu - \nu^\epsilon$ has polar representation $\nu_\epsilon(dr,du) = k_\epsilon(r,u)\,dr\lambda(du)$ where

$$k_\epsilon(r,u) = \alpha(r^{-\alpha-1} - \epsilon_0^{-1}r^{-1})q(r,u) + \epsilon_0^{-1}\alpha r^{\alpha-1}\int_r^\infty \alpha s^{-\alpha-1}q(s,u)\,ds$$

if $0 < r < \epsilon_0^{1/\alpha}$ and $k_\epsilon(r,u) = 0$ if $r \ge \epsilon_0^{1/\alpha}$. ν_ϵ is of the form (2.15) (but not as in (2.21)). We will use Theorem 2.5 to show the normal approximation for $X_\epsilon(1)$. We begin with an estimate for $\sigma_\epsilon^2(u)$.

$$\begin{aligned}\sigma_\epsilon^2(u) &= \int_0^\infty r^2 k_\epsilon(r,u)\,dr \ge \alpha q(\epsilon_0^{1/\alpha},u)\int_0^{\epsilon_0^{1/\alpha}} (r^{-\alpha+1} - \epsilon_0^{-1}r)\,dr \\ &= \frac{\alpha^2}{2(2-\alpha)}\epsilon_0^{2/\alpha-1}q(\epsilon_0^{1/\alpha},u).\end{aligned}$$

Therefore, condition (2.17) holds with $b(\epsilon) = \epsilon_0^{1/\alpha-1/2}$. (2.18) trivially holds because ν_ϵ is concentrated on a ball of radius $\epsilon_0^{1/\alpha}$ and we have

$$\int_{\|x\|>\kappa\epsilon_0^{1/\alpha-1/2}} \|x\|^2\,\nu_\epsilon(dx) = \int_{\{\|x\|>\kappa\epsilon_0^{1/\alpha-1/2},\|x\|<\epsilon_0^{1/\alpha}\}} \|x\|^2\,\nu(dx) = 0$$

for sufficiently small ϵ. Consequently, $\Sigma_\epsilon^{-1/2}X_\epsilon(1) \xrightarrow{(d)} \mathcal{N}(0, I_d)$. As it is, Σ_ϵ can be very complicated. However, we will prove that

$$\epsilon_0^{1-2/\alpha}\Sigma_\epsilon \to \frac{\alpha}{2-\alpha}\Lambda \quad \text{as } \epsilon \to 0 \tag{3.19}$$

which shows (3.5) for

$$A_\epsilon = \epsilon_0^{1/\alpha-1/2}(\frac{\alpha}{2-\alpha})^{1/2}\Lambda^{1/2}. \tag{3.20}$$

To establish (3.19) we will need bounds for $\sigma_\epsilon^2(u)$. First we notice that

$$\sigma_\epsilon^2(u) \le \alpha\int_0^{\epsilon_0^{1/\alpha}} r^{-\alpha+1}q(r,u)\,dr \le \frac{\alpha}{2-\alpha}\epsilon_0^{2/\alpha-1}$$

which yields

$$\epsilon_0^{1-2/\alpha}\Sigma_\epsilon = \epsilon_0^{1-2/\alpha}\int_{S^{d-1}} \sigma_\epsilon^2(u)uu^\top\,\lambda(du) \le \frac{\alpha}{2-\alpha}\Lambda. \tag{3.21}$$

To get a lower bound we write

$$\begin{aligned}\sigma_\epsilon^2(u) &= \alpha \int_0^{\epsilon_0^{1/\alpha}} r^2 (r^{-\alpha-1} q(r,u) - \epsilon_0^{-1} \ell_\epsilon(r,u))\, dr \\ &\geq q(\epsilon_0^{1/\alpha}, u) \frac{\alpha}{2-\alpha} \epsilon_0^{2/\alpha - 1} - \alpha \epsilon_0^{-1} \int_0^{\epsilon_0^{1/\alpha}} r^2 \ell_\epsilon(r,u)\, dr \end{aligned} \tag{3.22}$$

where

$$\ell_\epsilon(r,u) = r^{-1} q(r,u) - r^{\alpha-1} \int_r^\infty \alpha s^{-\alpha-1} q(s,u)\, ds.$$

Then

$$\begin{aligned}\alpha \epsilon_0^{-1} \int_0^{\epsilon_0^{1/\alpha}} r^2 \ell_\epsilon(r,u)\, dr &\leq \alpha \epsilon_0^{2/\alpha - 1} \int_0^{\epsilon_0^{1/\alpha}} \ell_\epsilon(r,u)\, dr \\ &= \epsilon_0^{2/\alpha-1} \int_0^{\epsilon_0^{1/\alpha}} \frac{\partial}{\partial r} \Big(- r^\alpha \int_r^\infty \alpha s^{-\alpha-1} q(s,u)\, ds \Big)\, dr \\ &= \epsilon_0^{2/\alpha-1} \Big[1 - \epsilon_0 \int_{\epsilon_0^{1/\alpha}}^\infty \alpha s^{-\alpha-1} q(s,u)\, ds \Big] =: \epsilon_0^{2/\alpha-1} k(\epsilon_0, u).\end{aligned}$$

Notice that $0 \leq k(\epsilon_0, u) \leq 1$ and $\lim_{\epsilon \to 0} k(\epsilon_0, u) = 0$. Combining the above estimate with (3.22) we get

$$\sigma_\epsilon^2(u) \geq \epsilon_0^{2/\alpha-1} \Big[q(\epsilon_0^{1/\alpha}, u) \frac{\alpha}{2-\alpha} - k(\epsilon_0, u) \Big].$$

which together with (3.21) yields

$$\begin{aligned}\frac{\alpha}{2-\alpha} \Lambda \geq \epsilon_0^{1-2/\alpha} \Sigma_\epsilon &\geq \frac{\alpha}{2-\alpha} \Lambda \\ &- \int_{S^{d-1}} \Big[(1 - q(\epsilon_0^{1/\alpha}, u)) \frac{\alpha}{2-\alpha} - k(\epsilon_0, u) \Big] u u^\top \lambda(du).\end{aligned}$$

Applying now the Dominated Convergence Theorem we establish (3.19) and so (3.20). In conclusion, using Proposition 3.1 we obtain

Theorem 3.3 *Let $\mathbf{X}$ be a tempered α–stable Lévy process with Lévy measure given by* (3.17)*. Suppose that the support of λ is not contained in a proper subspace of $\mathbb{R}^d$. Let T be fixed and ϵ_0 be as in* (3.12)*. Let $\mathbf{N}^\epsilon$ be given by* (3.18)*, $\mathbf{W}$ be a standard Brownian motion in $\mathbb{R}^d$ independent of $\mathbf{N}^\epsilon$, and $\mathbf{a}_\epsilon$ be a shift determined by* (3.4)*.*

Then, for every $\epsilon \in (0,1]$ there exists a cadlag process $\mathbf{Y}_\epsilon$ such that on $[0,T]$

$$\mathbf{X} \stackrel{(d)}{=} \mathbf{a}_\epsilon + \epsilon_0^{1/\alpha-1/2}(\frac{\alpha}{2-\alpha})^{1/2}\Lambda^{1/2}\mathbf{W} + \mathbf{N}^\epsilon + \mathbf{Y}_\epsilon \tag{3.23}$$

in the sense of equality of finite dimensional distributions and such that

$$\epsilon_0^{1/2-1/\alpha} \sup_{t\in[0,T]} \|Y_\epsilon(t)\| \stackrel{(\mathbb{P})}{\longrightarrow} 0 \quad \textit{as } \epsilon \to 0. \tag{3.24}$$

The basic difference between (3.23) and (3.15) is in the form of $\mathbf{N}^\epsilon$ (and implicitly, in $\mathbf{a}_\epsilon$ and $\mathbf{Y}_\epsilon$).

3.4 Operator stable processes

A Lévy process $\mathbf{X}$ as in (3.1) is said to be operator stable with exponent $B \in \mathrm{GL}(\mathbb{R}^d)$ if for every $t > 0$ there exists a vector $b(t) \in \mathbb{R}^d$ such that

$$X(t) \stackrel{(d)}{=} t^B X(1) + b(t) \tag{3.25}$$

where $t^B = \exp(B \log t)$. A comprehensive introduction to operator stable laws can be found in the monographs [7] and [12]. Since $\mathbf{X}$ has no Gaussian part, the necessary and sufficient condition for B to be an exponent is that all the roots of the minimal polynomial of B have real parts greater than 1/2 (cf. [7], Theorem 4.6.12). For a description of the Lévy measure of $\mathbf{X}$ it is convenient to use a norm $\|\cdot\|_B$ on $\mathbb{R}^d$ that depends on B as follows

$$\|x\|_B = \int_0^1 \|s^B x\| s^{-1}\, ds.$$

Let S_B denote the unit sphere in $\mathbb{R}^d$ with respect to $\|\cdot\|_B$. Then there exists a finite Borel measure λ on S_B such that the Lévy measure ν of $\mathbf{X}$ is of the form

$$\nu(A) = \int_{S_B}\int_0^\infty \mathbf{1}_A(s^B u)s^{-2}\, ds\lambda(du) \tag{3.26}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$ (cf. [7], Proposition 4.3.4).

To establish the approximation (3.9) we start with the compound Poisson process $\mathbf{N}$. It is natural to look at a shot noise representation of $\mathbf{X}$. As suggested by a remark following Corollary 4.4 [14], we may take

$$N^\epsilon(t) = \sum_{\gamma_j \le \epsilon^{-1}} I_{(0,t]}(\tau_j)\left(\frac{\gamma_j}{T\|\lambda\|}\right)^{-B} v_j. \tag{3.27}$$

This is a complete analogy of (3.11) with the same notation, except $1/\alpha$ is replaced by B and S^{d-1} by S_B. It is elementary to check that $\mathbf{N}^\epsilon$ is a compound Poisson process with characteristic function

$$\mathbf{E}\exp i\langle y, N^\epsilon(t)\rangle = \exp\left\{t\int_{S_B}\int_{\epsilon_0}^{\infty}(e^{i\langle y, s^B u\rangle}-1)s^{-2}\,ds\lambda(du)\right\}$$

where $\epsilon_0 = \|\lambda\|T\epsilon$, as in (3.12). Combining this with (3.26) we have

$$\nu_\epsilon(A) = \int_{S_B}\int_0^{\epsilon_0}\mathbf{1}_A(s^B u)s^{-2}\,ds\lambda(du).$$

From (2.3) we get

$$\begin{aligned}\Sigma_\epsilon &= \int_{S_B}\int_0^{\epsilon_0}(s^B u)(s^B u)^\top\, s^{-2}\,ds\lambda(du)\\ &= \int_0^{\epsilon_0} s^B\Lambda(s^B)^\top\, s^{-2}ds \qquad (3.28)\end{aligned}$$

where Λ is given by (3.14) (with S^{d-1} replaced by S_B). We also observe that

$$\Sigma_\epsilon = \epsilon_0^{-1}\int_0^1(\epsilon_0 r)^B\Lambda((\epsilon_0 r)^B)^\top\, r^{-2}dr = \epsilon_0^{-1}\epsilon_0^B\Sigma_1(\epsilon_0^B)^\top. \qquad (3.29)$$

Let $\text{lin}_B(\text{supp}\lambda)$ denote the smallest B–invariant subspace of $\mathbb{R}^d$ containing the support of λ. If ν is as in (3.26), then the support of ν is not contained in a proper subspace of $\mathbb{R}^d$ if and only if

$$\text{lin}_B(\text{supp }\lambda) = \mathbb{R}^d \qquad (3.30)$$

cf. [7], Corollary 4.3.5. If λ does not lie in a proper subspace of $\mathbb{R}^d$, then (3.30) clearly holds.

Theorem 3.4 *Let $\mathbf{X}$ be an operator stable Lévy process with exponent B and Lévy measure given by* (3.26) *such that* (3.30) *holds. Let T be fixed and ϵ_0 be as in* (3.12). *Let $\mathbf{N}^\epsilon$ be as in* (3.27), *$\mathbf{W}$ be a standard Brownian motion in $\mathbb{R}^d$ independent of $\mathbf{N}^\epsilon$, and $\mathbf{a}_\epsilon$ be a shift determined by* (3.4).

Then, for every $\epsilon \in (0,1]$ there exists a cadlag process $\mathbf{Y}_\epsilon$ such that on $[0,T]$

$$\mathbf{X} \overset{(d)}{=} \mathbf{a}_\epsilon + \epsilon_0^{-1/2}\epsilon_0^B\Sigma_1^{1/2}\mathbf{W} + \mathbf{N}^\epsilon + \mathbf{Y}_\epsilon \qquad (3.31)$$

in the sense of equality of finite dimensional distributions and such that for every $\delta > 0$

$$\epsilon_0^{1/2-\min\{b_1,\dots,b_d\}+\delta} \sup_{t\in[0,T]} \|Y_\epsilon(t)\| \xrightarrow{(\mathbb{P})} 0 \quad \textit{as } \epsilon \to 0 \tag{3.32}$$

where $b_1, \dots, b_d$ are the real parts of the eigenvalues of B.

Proof. First we will prove that Σ_1 is nonsingular. By Lemma 2.1 we need to show that $\operatorname{supp} \nu_1 = \mathbb{R}^d$. Following Corollary 4.3.5 [7] we have

$$\operatorname{supp} \nu_1 = \{x : x = s^B u,\ 0 \le s \le 1,\ u \in \operatorname{supp} \lambda\}.$$

Let

$$L = \operatorname{lin}(\operatorname{supp} \nu_1)$$

be the linear space spanned by $\operatorname{supp} \nu_1$. We will show that L is B–invariant. To this end it is enough to show that if $x = s^B u \in \operatorname{supp} \nu_1$, $0 < s \le 1$ and $u \in \operatorname{supp} \lambda$, then $Bs^B u \in L$. Take $\theta < 1$ such that $\theta s < s \le 1$. Then $(\theta s)^B u \in \operatorname{supp} \nu_1$ and hence

$$Bs^B u = \lim_{\theta \nearrow 1} \frac{(\theta s)^B u - s^B u}{\log \theta} \in L.$$

Since L is B–invariant and contains support of λ, $L = \mathbb{R}^d$ by (3.30). Thus Σ_1 is nonsingular. We infer that

$$\Sigma_1 \ge c_1 I_d$$

where $c_1 = \min_{\|x\|=1} \langle \Sigma_1 x, x\rangle > 0$. By (3.29) we get

$$\Sigma_\epsilon \ge c_1 \epsilon_0^{-1} \epsilon_0^B (\epsilon_0^B)^\top. \tag{3.33}$$

We will use Lemma 2.3 in the proof of normal approximation. To this end we need to find a lower bound $\tilde{\Sigma}_\epsilon$ of Σ_ϵ which satisfies condition (2.11).

The Jordan decomposition of the exponent B states that

$$B = D + N \tag{3.34}$$

where D is semi-simple and N is a nilpotent matrix such that $DN = ND$. See e.g., Theorem 2.1.18 [12]. Then, for any $s > 0$,

$$s^B (s^B)^\top = s^D s^N (s^N)^\top (s^D)^\top.$$

Choose $\delta > 0$ to be determined later. Since $-N^\top$ is also nilpotent, the real part of all eigenvalues of $-N^\top$ are zeros. Therefore, there exists $c_2 > 0$ such that for all $s \in (0, 1]$

$$\|s^{-N^\top}\| \leq c_2 s^{-\delta} \tag{3.35}$$

(see [11], for instance). Hence

$$\|x\| \leq \|s^{-N^\top}\| \|s^{N^\top} x\| \leq c_2 s^{-\delta} \|s^{N^\top} x\|$$

for $s \in (0, 1]$. This yields

$$\|x\|^2 \leq c_2^2 s^{-2\delta} \langle s^N (s^N)^\top x, x\rangle$$

or

$$s^N (s^N)^\top \geq c_3 s^{2\delta} I_d$$

for $s \in (0, 1]$; $c_3 = c_2^{-2}$. Consequently, for $s \in (0, 1]$

$$s^B (s^B)^\top \geq c_3 s^{2\delta} s^D (s^D)^\top. \tag{3.36}$$

Since D is semi-simple, $D = UEU^{-1}$ where U is a unitary matrix and

$$E = \operatorname{diag}(e_1, \ldots, e_d),$$

is a diagonal matrix with the diagonal $e_1, \ldots, e_d$, where $e_k = b_k + ib'_k$ are the eigenvalues of B. Thus

$$\begin{aligned} s^D (s^D)^\top &= (U s^E U^{-1})(U s^{E^*} U^{-1}) \\ &= U \operatorname{diag}(s^{2b_1}, \ldots, s^{2b_d})\, U^{-1} \end{aligned}$$

where E^* is the complex conjugate transposed of E. Combining this with (3.36) we have for $s \in (0, 1]$

$$s^B (s^B)^\top \geq c_3\, U \operatorname{diag}(s^{2b_1+2\delta}, \ldots, s^{2b_d+2\delta})\, U^{-1}.$$

Combining this with (3.33) we get for $0 < \epsilon_0 \leq 1$

$$\Sigma_\epsilon \geq c_4 \epsilon_0^{2\delta-1}\, U \operatorname{diag}(\epsilon_0^{2b_1}, \ldots, \epsilon_0^{2b_d})\, U^{-1} =: \tilde{\Sigma}_\epsilon.$$

where $c_4 = c_1 c_3$.

To verify condition (2.11) we first estimate $\langle \tilde{\Sigma}_\epsilon^{-1} x, x\rangle$ for $x = s^B u$, where $0 < s \leq \epsilon_0 \leq 1$ and $u \in S_B$. We get

$$\begin{aligned} \langle \tilde{\Sigma}_\epsilon^{-1} s^B u, s^B u\rangle &= \langle (s^D)^\top \tilde{\Sigma}_\epsilon^{-1} s^D s^N u, s^N u\rangle \\ &\leq c_5 s^{-2\delta} \|(s^D)^\top \tilde{\Sigma}_\epsilon^{-1} s^D\| \\ &= c_5 c_4^{-1} \epsilon_0^{1-2\delta} s^{-2\delta} \Big\|U \operatorname{diag}\Big((\frac{s}{\epsilon_0})^{2b_1}, \ldots, (\frac{s}{\epsilon_0})^{2b_d}\Big)\, U^{-1}\Big\| \\ &= c_5 c_4^{-1} \epsilon_0^{1-4\delta} \Big\|\operatorname{diag}\Big((\frac{s}{\epsilon_0})^{2b_1-2\delta}, \ldots, (\frac{s}{\epsilon_0})^{2b_d-2\delta}\Big)\Big\|. \end{aligned}$$

In the first inequality we used the fact that the bound (3.35) holds for any nilpotent matrix (with possibly different constant) and the fact that S_B is bounded. If $\delta = 1/8$ and $c_6 = c_5 c_4^{-1}$, then the above bound shows that

$$\langle \tilde{\Sigma}_\epsilon^{-1} s^B u, s^B u\rangle \le c_6 \epsilon_0^{1/2}, \tag{3.37}$$

whenever $0 < s \le \epsilon_0 \le 1$ and $u \in S_B$.

Then, for every $\kappa > 0$ and $\epsilon_0 \le 1$ we have

$$\begin{aligned}\int_{\langle \tilde{\Sigma}_\epsilon^{-1} x, x\rangle > \kappa} &\langle \tilde{\Sigma}_\epsilon^{-1} x, x\rangle \,\nu_\epsilon(dx) \\ &= \iint_{\{(s,u)\in(0,\epsilon_0]\times S_B:\ \langle \tilde{\Sigma}_\epsilon^{-1} s^B u, s^B u\rangle > \kappa\}} \langle \tilde{\Sigma}_\epsilon^{-1} s^B u, s^B u\rangle\, s^{-2}\, ds \lambda(du) \\ &= 0\end{aligned}$$

when $\epsilon_0 < c_6^{-2}\kappa^2$ (or when $\epsilon < c_6^{-2}\kappa^2\|\lambda\|^{-1}T^{-1}$ by (3.12)). Indeed, in view of (3.37) the region of integration is empty for $\epsilon_0 < c_6^{-2}\kappa^2$. Therefore, (2.11) trivially holds and $\Sigma_\epsilon^{-1/2} X_\epsilon(1) \xrightarrow{(d)} \mathcal{N}(0, I_d)$.

Applying Proposition 3.1 for

$$A_\epsilon = \epsilon_0^{-1/2} \epsilon_0^B \Sigma_1^{1/2}$$

we get (3.31) and that

$$\sup_{t\in[0,T]} \|A_\epsilon^{-1} Y_\epsilon(t)\| \xrightarrow{(\mathbb{P})} 0 \quad \text{as } \epsilon \to 0. \tag{3.38}$$

Proceeding as above we can find positive constants c_1' and c_2' such that for every $\epsilon_0 \le 1$ and $\delta > 0$

$$\begin{aligned} A_\epsilon A_\epsilon^\top &= \Sigma_\epsilon \le c_1' \epsilon_0^{-1} \epsilon_0^B (\epsilon_0^B)^\top \\ &\le c_2' \epsilon_0^{-1-2\delta} \epsilon_0^D (\epsilon_0^D)^\top \\ &= c_2' \epsilon_0^{-1-2\delta} U \operatorname{diag}(\epsilon_0^{2b_1}, \ldots, \epsilon_0^{2b_d}) U^{-1} \\ &\le c_2' \epsilon_0^{-1-2\delta+2\min\{b_1,\ldots,b_d\}} I_d. \end{aligned}$$

This yields

$$\|A_\epsilon\| \le c \epsilon_0^{-1/2-\delta+\min\{b_1,\ldots,b_d\}}.$$

where $c = \sqrt{c_2'}$. Therefore,

$$\|Y_\epsilon(t)\| \le \|A_\epsilon\| \|A_\epsilon^{-1} Y_\epsilon(t)\| \le c \epsilon_0^{-1/2-\delta+\min\{b_1,\ldots,b_d\}} \|A_\epsilon^{-1} Y_\epsilon(t)\|,$$

which together with (3.38) yields (3.32). The proof is complete. □

Remark 3.5 *One may also consider*

$$\widetilde{N}^\epsilon(t) = \sum_j I_{(0,t]}(\tau_j) I\Big(\Big\|\Big(\frac{\gamma_j}{T\|\lambda\|}\Big)^{-B} v_j\Big\| \geq \epsilon\Big)\Big(\frac{\gamma_j}{T\|\lambda\|}\Big)^{-B} v_j.$$

That is, the remainder process $\tilde{\mathbf{X}}_\epsilon$ has jumps of magnitude less than ϵ. From both theoretical and computational point of views the compound Poisson approximation $\mathbf{N}^\epsilon$ of (3.27) is more tractable than the above. As a matter of fact, we were unable to establish Gaussian approximation of $\tilde{\mathbf{X}}_\epsilon$ in full generality, for any operator stable process.

4 Other applications

4.1 Independent marginals

The normal approximation for independent marginals can be deduced from the one–dimensional result of [1]. The aim of this example is to show how the approximation follows from our general scheme. Suppose that the coordinates of the multivariate Lévy process $X(t) = (X_1(t), \ldots, X_d(t))$ are independent. Let ν_i be the Lévy measure of X_i. Then

$$\nu(dx_1, \ldots, dx_d) = \sum_{i=1}^{d} \delta(dx_1) \otimes \cdots \otimes \nu_i(dx_i) \otimes \cdots \otimes \delta(dx_d)$$

where δ are Dirac masses at 0. Let $D_\epsilon = \{x \in \mathbb{R}^d : \|x\| < \epsilon\}$ and $\nu_\epsilon = \nu_{|D_\epsilon}$. Define $\sigma_i^2(\epsilon) = \int_{|x_i|<\epsilon} x_i^2 \nu_i(dx)$ for $1 \leq i \leq d$. Then

$$\Sigma_\epsilon = \int_{D_\epsilon} xx^\top \nu(dx) = \operatorname{diag}\left(\sigma_1^2(\epsilon), \ldots, \sigma_d^2(\epsilon)\right)$$

where $\operatorname{diag}(a_i)$ is a diagonal matrix with the diagonal $a_1, \ldots, a_d$. In view of Corollary 2.4, (2.5) holds if and only if for every $i = 1, \ldots, d$ and $\kappa > 0$

$$\frac{\sigma_i(\kappa\sigma_i(\epsilon) \wedge \epsilon)}{\sigma_i(\epsilon)} \to 1 \quad \text{as } \epsilon \to 0. \tag{4.1}$$

If (4.1) holds then by Proposition 3.1

$$\operatorname{diag}\left(\sigma_1^{-1}(\epsilon), \ldots, \sigma_d^{-1}(\epsilon)\right) \mathbf{X}_\epsilon \xrightarrow{(d)} \mathbf{W}.$$

4.2 Normal asymptotic for stochastic integral processes

Let $\mathbf{X} = \{X(t) : t \in \mathbb{T}\}$ be a stochastic process represented as a stochastic integral

$$X(t) = \int_S f(t,s)\, M(ds)$$

where f is a deterministic function and M is an infinitely divisible random measure with no Gaussian part. For more information on such integrals, see [13]. To simulate $\mathbf{X}$ it is useful to have the normal asymptotic of the part corresponding to "small jumps" of M. Then one can write an approximation of the type (3.9) with $\mathbf{W}$ being a Gaussian process. Such approximation was used in [10, 2] to simulate locally self-similar processes defined by a fractional integral of a Lévy random measure.

To illustrate how this method fits into our general framework, we consider the following simple situation. Let N be a Poisson random measure on $S \times \mathbb{R}$, where S is a Borel set. Let $m \otimes Q$ be the intensity measure of N, where $\int x^2\, Q(dx) < \infty$ and $Q(\{0\}) = 0$. Consider a random measure

$$M(A) = \int_A \int_{\mathbb{R}} x\, [N(ds,dx) - m(ds)Q(dx)],$$

where $A \in \mathcal{B}(S),\ m(A) < \infty$. Clearly, the process

$$X(t) = \int_S f(t,s)\, M(ds) = \int_S \int_{\mathbb{R}} x f(t,s)\, [N(ds,dx) - m(ds)Q(dx)]$$

is well defined when $f(t,\cdot) : S \mapsto \mathbb{R}$ is a Borel function such that

$$\int_S \int_{\mathbb{R}} x^2 |f(t,s)|^2\, m(ds)Q(dx) < \infty$$

for every $t \in \mathbb{T}$. Put

$$M_\epsilon(A) = \int_A \int_{|x|<\epsilon} x\, [N(ds,dx) - m(ds)Q(dx)],$$

and

$$X_\epsilon(t) = \int_S f(t,s)\, M_\epsilon(ds).$$

We will investigate normal asymptotic of the process $\mathbf{X}_\epsilon = \{X_\epsilon(t) : t \in \mathbb{T}\}$. To this end, define

$$\sigma^2(\epsilon) = \int_{|x|<\epsilon} x^2\, Q(dx)$$

and

$$K(t,u) = \mathrm{Cov}(X(t), X(u)) = \int_S f(t,s) f(u,s)\, m(ds). \tag{4.2}$$

Proposition 4.1 *Suppose that for each $\kappa > 0$*

$$\frac{\sigma(\kappa\sigma_\epsilon \wedge \epsilon)}{\sigma(\epsilon)} \to 1 \quad \textit{as } \epsilon \to 0.$$

Then

$$\sigma^{-1}(\epsilon)\mathbf{X}_\epsilon \xrightarrow{(d)} \mathbf{W} \quad \textit{as} \quad \epsilon \to 0 \tag{4.3}$$

in the sense of convergence of finite dimensional distributions, where $\mathbf{W} = \{W(t) : t \in \mathbb{T}\}$ is a centered Gaussian process with the covariance function K given by (4.2).

Proof: Let $t_1, \ldots, t_n \in \mathbb{T}$, $a_1, \ldots, a_n \in \mathbb{R}$ and $n \geq 1$. Then

$$\sum_{j=1}^{n} a_j X_\epsilon(t_j) = \int_S h(s)\, M_\epsilon(ds)$$

where

$$h(s) = \sum_{j=1}^{n} a_j f(t_j, s).$$

We need to show that

$$\sigma^{-1}(\epsilon) \sum_{j=1}^{n} a_j X_\epsilon(t_j) \xrightarrow{(d)} \mathcal{N}(0, \sigma_h^2) \tag{4.4}$$

where

$$\sigma_h^2 = \sum_{i,j=1}^{n} a_i a_j K(t_i, t_j) = \int_S |h(s)|^2\, m(ds).$$

For every $u \in \mathbb{R}$ we have

$$\mathbf{E} \exp\left\{ iu \sum_{j=1}^{n} a_j X_\epsilon(t_j) \right\} = \exp\left\{ \int_{\mathbb{R}} [e^{iuv} - 1 - iuv]\, \nu_\epsilon(dv) \right\}$$

where

$$\nu_\epsilon(B) = m \otimes Q((s,x) : |x| < \epsilon,\ xh(s) \in B \setminus \{0\})$$

for any $B \in \mathcal{B}(\mathbb{R})$. Notice that ν_ϵ is not obtained by a truncation of Lévy measure and may have unbounded support. Nevertheless, we can use Theorem 2.2 in this (one–dimensional!) case. We compute

$$\Sigma_\epsilon = \sigma^2(\epsilon) \int_S |h(s)|^2 \, m(ds) = \sigma^2(\epsilon) \|h\|_2^2.$$

If $\|h\|_2 = 0$, then $\sum_{j=1}^n a_j X_\epsilon(t_j) = 0$ a.s. and (4.4) trivially holds. Otherwise, we verify condition (2.5)

$$\begin{aligned} \sigma^{-2}(\epsilon)\|h\|_2^{-2} & \int_{|x|>\kappa^{1/2}\|h\|_2\sigma(\epsilon), |x|<\epsilon} x^2 \, \nu_\epsilon(dx) \\ &= \sigma^{-2}(\epsilon) \int_{|x|>\kappa^{1/2}\|h\|_2\sigma(\epsilon), |x|<\epsilon} x^2 \, Q(dx) \\ &\le 1 - \sigma^{-2}(\epsilon)\sigma^2(\epsilon \wedge \kappa^{1/2}\|h\|_2\sigma(\epsilon)) \to 0. \end{aligned}$$

Theorem 2.2 gives (4.4). □

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