Optimal investment and consumption in a Black-Scholes market with stochastic coefficients driven by a non-Gaussian Ornstein-Uhlenbeck process

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Abstract

In this paper we investigate an optimal investment and consumption problem for an investor who trades in a Black-Scholes financial market with stochastic coefficients driven by a non-Gaussian Ornstein-Uhlenbeck process. We assume that an agent makes consumption and investment decisions based on a HARA utility function. By applying the usual separation method in the variables, we are faced with the problem of solving a non-linear (semilinear) first-order partial integro-differential equation. A candidate solution is derived via the Feynman-Kac representation. By using the properties of an operator, defined in a suitable functional space, we prove uniqueness and smoothness of the solution. Optimality is verified by applying a classical verification theorem.

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1 Introduction

A fundamental problem in financial mathematics is the allocation of funds between assets in order to provide sufficiently large payments during the duration of an investment contract, as well as to arrive at a high return at maturity. This optimization problem has its origin in a seminal paper by Merton (1971), where it is formulated as a utility maximization problem and an optimal strategy is derived via the Bellman equation. Since then, there has been a growing interest in consumption and investment problems and the classical Merton problem has been extended in many directions. One of the generalizations considers financial coefficients (risk-free return, drift and volatility) affected by an external stochastic factor.

In this paper we extend the results of Benth, Karlsen & Reikvam (2003) and Lindberg (2006). We investigate a Black-Scholes-type financial model with coefficients depending on a background driving process. The dependence is described through general functions which satisfy linear growth conditions. An external stochastic factor is chosen as Ornstein-Uhlenbeck process driven by a subordinator. The Barndorff-Nielsen & Shephard model, considered in Benth *et al.* (2003) and Lindberg (2006), arises as a special case. As an additional possibility, the investor is allowed to withdraw (consume) funds during the term of the contract. This leads to an optimal consumption and investment problem which is more complex than a pure investment problem. From an analytical point of view, the difference is that, after applying the usual separation of variables, we arrive at a non-linear partial integro-differential equation, whereas Benth *et al.* (2003) and Lindberg (2006) have to deal with a linear one.

The first goal of this paper is to show that a candidate value function is the classical solution of a corresponding Hamilton-Jacobi-Bellman equation. This requires proving the existence of a classical solution to a non-linear (semilinear) first-order partial integro-differential equation. It is well known, see Pham (1999) and Chapter 12.2 in Cont & Tankov (2005), that the regularity of solutions to equations with an integral term is uncertain, especially in the degenerate case. There exists some results concerning the smoothness of a solution to a linear partial integro-differential equation, see e.g. Chapters 3.3 and 3.8 in Bensoussan & Lions (1984), Pham (1999) and Chapter 12.2 in Cont & Tankov (2005), but they all deal only with the non-degenerate second-order case. The degenerate case can be handled by applying a viscosity approach (e.g. Chapter 12.2 in Cont & Tankov (2005)), which we want to avoid following instead Benth *et al.* (2003), where the existence of a classical solution to a linear first-order partial integro-differential equation is established. We believe that our proof (in Sections 4 and 5) of the existence of a unique classical solution to a non-linear first-order partial integro-differential equation contributes to the present state of the literature.

Our second goal is to provide an explicit formula for the optimal consumption. In the case of HARA utility functions, it is intuitively easy to foresee a formula for the optimal investment, by simply replacing deterministic coefficients by functions, which relate coefficients to an external factor, and thus make the strategy adapted to an underlying filtration. This is no longer obvious as far as the consumption strategy is concerned. To the best of our knowledge the formula for the optimal consumption in the model investigated in this paper is new (see Theorem 6.2).

Portfolio optimization in stochastic factor models has recently gained much attention in the financial literature. The majority of papers applies a HARA utility function and considers a Black-Scholes financial market with an external stochastic factor of diffusion type. In this setting, it is well known that we have to solve a non-degenerate non-linear second-order partial differential equation. Several methods have been proposed to deal with this problem. In Zariphopoulou (2001), in the case of a pure investment problem, a power transformation was introduced, which makes the non-linear term disappear. In Kraft & Steffensen (2006) a similar transformation has been applied, but because of the possible consumption, a linear partial differential equation appears only in the case of perfectly (positively) correlated Brownian motions or for logarithmic utility. More effective methods have been proposed in Fleming & Hernández-Hernández (2003) and Castañeda-Leyva & Hernández-Hernández (2005). In the first paper a change of measure transformation is applied, and the resulting optimization control problem is investigated, whose value function depends only on time and a factor variable. In the second paper the dual problem is considered, whose control process belongs to a set of equivalent local martingale measures. Again, the value function of the dual problem depends only on time and a factor variable. This method has also been successfully applied in a robust utility maximization model in Hernández-Hernández & Schied (2006) recently. In all three above-mentioned papers the existence of a classical solution to the Hamilton-Jacobi-Bellman equation is proved in three steps: first, by constraining the values of the control process to a compact set, second, by applying results from the theory of non-degenerate linear partial differential equations (see Chapter VI.6 and Appendix E in Fleming & Rishel (1975)) and, thus, showing that the constrained problem has a unique classical solution, and third, by studying the asymptotic limit. It seems that this method cannot be successfully applied for our problem.

In this paper a candidate solution is first derived heuristically via the Feynman-Kac representation. This leads to a fixed point equation. The existence of a solution is established by the Banach fixed point theorem and its differentiability is proved by using the properties of a suitable operator. Finally, we show that the candidate solution verifies our integro-differential equation and that this solution is unique. The idea of finding a

solution to a control problem through a fixed point theorem is not new. The fixed point method is, for example, mentioned in Castañeda-Leyva & Hernández-Hernández (2005). In Becherer & Schweizer (2005), the existence of a solution to a non-degenerate non-linear (semilinear) partial differential equation is proved by the Banach fixed point theorem. The smoothness follows from the Hölder estimates for a solution of a non-degenerate linear partial differential equation. We would like to point out that, in particular, in Becherer & Schweizer (2005) an exponent in the Feynman-Kac formula is assumed to be bounded, which leads to a bounded solution, while we are dealing with a solution which satisfies only an exponential growth condition. We also would like to mention that in the context of optimal control, the results from Becherer & Schweizer (2005) are directly applied in Delong (2007), where an investment and consumption problem is investigated in the presence of default, triggered by a one-jump counting process with a stochastic intensity of diffusion type.

Throughout this paper we assume that the external factor is observable (as in all above mentioned publications). An alternative would be a partially observed control problem, whose optimal strategy would then be based on an estimate of the underlying factor. We refer to Pham & Quenez (2001) or Bäuerle & Rieder (2005), where a portfolio problem is solved in a diffusion setting with an unobserved volatility process of diffusion type and of Markov-switching type, respectively.

Our paper is structured as follows. In Section 2 we introduce the financial market. The optimization problem is formulated in Section 3. The uniqueness of a solution is proved in Section 4, whereas the differentiability is established in Section 5. The verification theorem and the optimality of the derived strategy is proved in Section 6.

2 The financial market

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F})_{0\leq t\leq T}$, where T denotes a finite time horizon. The filtration is assumed to satisfy the usual conditions of completeness and right continuity. The measure $\mathbb{P}$ is the real-world, objective probability measure. All expectations are taken with respect to $\mathbb{P}$, unless it is stated otherwise.

We consider a Black-Scholes market with coefficients driven by an external stochastic factor. Let $Y := (Y(t), 0 \leq t \leq T)$ denote this economic factor and let us assume that its dynamics is given by a stochastic differential equation of the Ornstein-Uhlenbeck type

$$dY(t) = -\lambda Y(t-)dt + dL(\lambda t), \quad Y(0) = y > 0, \tag{2.1}$$

where $\lambda > 0$ denotes the reversion rate and $L := (L(t), 0 \leq t \leq T)$ is an $\mathbb{F}$-adapted subordinator with càdlàg sample paths. Recall that a subordinator is a Lévy process with

a.s. nondecreasing sample paths. For definitions and more background on Lévy processes; we refer to Applebaum (2004), Bertoin (1996) or Sato (1999).

Our financial market consists of two instruments. The price of a (locally) risk-free asset $B := (B(t), 0 \leq t \leq T)$ is described by the differential equation

$$\frac{dB(t)}{B(t)} = r(Y(t-))dt, \quad B(0) = 1, \tag{2.2}$$

whereas the dynamics of the price of a risky asset, $S := (S(t), 0 \leq t \leq T)$, is given by the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu(Y(t-))dt + \sigma(Y(t-))dW(t), \quad S(0) = s > 0, \tag{2.3}$$

where $W := (W(t), 0 \leq t \leq T)$ denotes an $\mathbb{F}$-adapted Brownian motion, independent of the subordinator L. We make the following assumptions concerning the functions r, μ and σ:

(A1) the functions $r : (0, \infty) \to [0, \infty), \mu : (0, \infty) \to [0, \infty)$ and $\sigma : (0, \infty) \to (0, \infty)$ are continuous and satisfy the linear growth conditions:

$$r(y) \leq A_r + B_r y, \quad \mu(y) \leq A_\mu + B_\mu y, \quad \sigma^2(y) \leq A_\sigma + B_\sigma y, \quad y > 0$$

with nonnegative constants,

(A2) the derivatives $\frac{dr}{dy} : (0, \infty) \to \mathbb{R}, \frac{d\mu}{dy} : (0, \infty) \to \mathbb{R}$ and $\frac{d\sigma^2}{dy} : (0, \infty) \to \mathbb{R}$ are continuous and satisfy analogous linear growth conditions as r, μ, σ^2,

(A3) $\inf_{y \in \mathcal{D}_2} \sigma(y) > 0$, where the set $\mathcal{D}_2$ will be specified in (3.7).

Notice that the assumptions (A1)-(A3) are more general than in Fleming & Hernández-Hernández (2003), Castanañeda-Leyva & Hernández-Hernández (2005) and Hernández-Hernández & Schied (2006), where uniform boundedness of the functions r, μ, σ^2 and their first derivatives is required. Our conditions are similar to those of Zariphopoulou (1999), where Lipschitz continuity of the coefficients is assumed, together with a linear growth condition.

A prominent example of the above financial model is a Barndorff-Nielsen & Shephard model, introduced in Barndorff-Nielsen & Shephard (2001), which can be described by the following set of equations

$$\frac{dB(t)}{B(t)} = rdt,$$

$$\frac{dS(t)}{S(t)} = (\mu + \beta Y(t-))dt + \sqrt{Y(t-)}dW(t). \tag{2.4}$$

Besides the above paper we refer also to Chapter 15 in Cont & Tankov (2005), Benth *et al.* (2003), Lindberg (2006) and references therein for more information about the properties of non-Gaussian stochastic volatility models in the context relevant for our paper.

We shall need some further results and notation for Y and its background driving Lévy process L. The subordinator L has the representation (see e.g. Chapter 1.3.2 in Applebaum (2004))

$$L(t) = \int_{(0,t]} \int_{z>0} zN(ds, dz), \tag{2.5}$$

where $N((0,t] \times A) = \#\{0 < s \leq t : (L(s) - L(s-)) \in A\}$ denotes a Poisson random measure with a deterministic, time-homogeneous intensity measure $\nu(dz)ds$ satisfying $\int_{0<z<1} z\nu(dz) < \infty$. The representation (2.5) is used in the proof of the verification theorem 6.1 in Section 6. The fundamental result in the theory of infinitely divisible random variables is the Lévy-Kintchine formula, which presents the moment generating function of a subordinator as follows

$$\mathbb{E}[e^{wL(t)}] = e^{t\psi(w)} = \exp\{t \int_{z>0} \big(e^{wz} - 1\big)\nu(dz)\}, \quad w \leq \bar{w}, \tag{2.6}$$

for some $\bar{w} \in [0, \infty]$. The function $\psi(w)$ is called the Laplace exponent of L. Notice that $\psi(w)$ exists at least for all $w \leq 0$ and $\psi(w) > 0$ for $w > 0$, provided it exists.

Let us now investigate the stochastic differential equation (2.1). Its unique solution is given by (cf. Chapter 6.3 in Applebaum (2004))

$$Y(s) = ye^{-\lambda(s-t)} + \int_t^s e^{-\lambda(u-t)} dL(\lambda u), \quad Y(t) = y. \tag{2.7}$$

We abbreviate the process (2.7) by $Y^{t,y} := (Y^{t,y}(s), t \leq s \leq T)$ and would like to point out that it has a.s. càdlàg sample paths of finite variation, and that the mapping $y \mapsto Y^{t,y}$ is continuous $\mathbb{P}$-a.s.. Moreover

$$\frac{\partial}{\partial y} Y^{t,y}(s) = e^{-\lambda(s-t)}, \quad \mathbb{P} - a.s.. \tag{2.8}$$

Finally, it is straightforward to establish the following relations for all $0 \leq t \leq s \leq T$ and $y > 0$:

$$\begin{aligned} Y^{t,y}(s) &\leq y + L(\lambda s) - L(\lambda t), & (2.9)\\ \lambda \int_t^s Y^{t,y}(u)du &= y + L(\lambda s) - L(\lambda t) - Y^{t,y}(s), & (2.10)\\ &\leq y + L(\lambda s) - L(\lambda t) \;=\; y + L(\lambda(s-t)). & (2.11) \end{aligned}$$

The above relations hold $\mathbb{P}$-a.s., except for the last equality, which holds in distribution.

3 Formulation of the optimization problem

We consider an investor who takes decisions concerning consumption and investment of a portfolio based on a power utility function of the form x^γ for $\gamma \in (0,1)$.

Consider the wealth process $X^{c,\pi} := (X^{c,\pi}(t), 0 \leq t \leq T)$ of an agent. Its dynamics is given by the stochastic differential equation

$$\begin{aligned} dX^{c,\pi}(t) &= \pi(t)X^{c,\pi}(t)\big(\mu(Y(t-))dt + \sigma(Y(t-))dW(t)\big) \\ &\quad + \big(1-\pi(t)\big)X^{c,\pi}(t)r(Y(t-)dt - c(t)dt, \end{aligned} \tag{3.1}$$

where $\pi(t)$ denotes a fraction of the wealth invested in the risky asset and $c(t)$ denotes the rate of consumption at time t. We are dealing with the following optimization problem

$$\sup_{c,\pi} \mathbb{E}[\int_0^T \big(c(s)\big)^\gamma ds + \big(X^{c,\pi}(T)\big)^\gamma \mid X(0) = x, Y(0) = y]. \tag{3.2}$$

The corresponding optimal value function is defined as

$$V(t,x,y) = \sup_{(c,\pi)\in\mathcal{A}} \mathbb{E}[\int_t^T \big(c(s)\big)^\gamma ds + \big(X^{c,\pi}(T)\big)^\gamma \mid X(t) = x, Y(t) = y]. \tag{3.3}$$

Let us introduce the set $\mathcal{A}$ of admissible strategies.

Definition 3.1. *A strategy $(c,\pi) := (c(t), \pi(t), 0 \leq t \leq T)$ is admissible, and we write $(c,\pi) \in \mathcal{A}$, if it satisfies the following conditions:*

(1) $(c,\pi) : (0,T] \times \Omega \to [0,\infty) \times [0,1]$ *is a progressively measurable mapping with respect to the filtration $\mathbb{F}$,*

(2) $\int_0^T c(s)ds < \infty$ $\mathbb{P}$*-a.s.,*

(3) the stochastic differential equation (3.1) *has a unique, positive solution $X^{c,\pi}$ on $[0,T]$.*

We would like to mention that for every $(c,\pi) \in \mathcal{A}$ the wealth process $X^{c,\pi}$, which satisfies (3.1), is an Itô diffusion; i.e. in particular a semimartingale with $\mathbb{P}$-a.s continuous sample paths.

Notice that we exclude the possibility of borrowing from the bank account and short-selling the asset, as in Benth *et al.* (2003) and Lindberg (2006). Technically, there is no problem in solving the unconstrained optimization problem. In particular, if $\frac{\mu(y)-r(y)}{\sigma^2(y)}$ is positive and uniformly bounded, all our results remain the same.

One can associate a Hamilton-Jacobi-Bellman equation with the optimization problem (3.3). One arrives at the following partial integro-differential equation

$$\begin{aligned}\sup_{(c,\pi)\in[0,\infty)\times[0,1]}\{c^\gamma &+ \frac{\partial v}{\partial t}(t,x,y) + \frac{\partial v}{\partial x}(t,x,y)\left(\pi x(\mu(y)-r(y)) + xr(y) - c\right)\\ &+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x,y)\pi^2x^2\sigma^2(y) - \frac{\partial v}{\partial y}(t,x,y)\lambda y\\ &+\lambda\int_{z>0}\big(v(t,x,y+z)-v(t,x,y)\big)\nu(dz)\} = 0, \quad v(T,x,y) = x^\gamma.\end{aligned} \tag{3.4}$$

As we use a power utility function it is natural to try to find a solution of the form $v(t,x,y) = x^\gamma f(t,y)$ for some function f. With this choice of the value function, the optimal strategy $(\hat{c},\hat{\pi})$, which maximizes the left hand side of (3.4), is given by

$$\hat{c} = xf(t,y)^{-\frac{1}{1-\gamma}}, \tag{3.5}$$

$$\hat{\pi} = \arg\max_{\pi\in[0,1]}\{\pi(\mu(y)-r(y)) - \frac{1}{2}\pi^2(1-\gamma)\sigma^2(y)\}. \tag{3.6}$$

Let us investigate the formula for the investment strategy more closely. Define three sets

$$\begin{aligned}\mathcal{D}_1 &= \{y>0, \mu(y)-r(y)<0\},\\ \mathcal{D}_2 &= \{y>0, \mu(y)-r(y)>0, (1-\gamma)\sigma^2(y) > \mu(y)-r(y)\},\\ \mathcal{D}_3 &= \{y>0, \mu(y)-r(y)>0, (1-\gamma)\sigma^2(y) < \mu(y)-r(y)\}.\end{aligned} \tag{3.7}$$

The strategy $\hat{\pi}$ is given by

$$\hat{\pi} = \begin{cases} 0 & y\in\mathcal{D}_1,\\ \dfrac{\mu(y)-r(y)}{(1-\gamma)\sigma^2(y)} & y\in\mathcal{D}_2,\\ 1 & y\in\mathcal{D}_3.\end{cases} \tag{3.8}$$

The following lemma is a counterpart of Lemma 5.1 of Benth *et al.* (2003).

Lemma 3.2. *Define the function*

$$\begin{aligned}Q(y) &= \max_{\pi\in[0,1]}\{\pi(\mu(y)-r(y)) - \frac{1}{2}\pi^2(1-\gamma)\sigma^2(y)\} + r(y) \qquad (3.9)\\ &= \begin{cases} r(y) & y\in\mathcal{D}_1,\\ \dfrac{(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} + r(y) & y\in\mathcal{D}_2,\\ \mu(y) - \frac{1}{2}(1-\gamma)\sigma^2(y) & y\in\mathcal{D}_3.\end{cases}\end{aligned}$$

The function Q is nonnegative, continuous and satisfies the linear growth condition

$$0\le r(y)\le Q(y)\le A+By, \quad y>0, \tag{3.10}$$

for nonnegative A and B. The derivative $\frac{dQ}{dy}$ is continuous and satisfies also a linear growth condition: for nonnegative C and D we have

$$|\frac{dQ}{dy}(y)| \leq C + Dy, \quad y > 0.$$

Proof. First, notice that the sets $\mathcal{D}_1$ and $\mathcal{D}_2$ have common boundary

$$\partial\mathcal{D}_{12} = \{y > 0, \mu(y) = r(y)\}, \tag{3.11}$$

and $\mathcal{D}_2$ and $\mathcal{D}_3$ have common boundary

$$\partial\mathcal{D}_{23} = \{y > 0, (1-\gamma)\sigma^2(y) = \mu(y) - r(y)\}. \tag{3.12}$$

The sets $\mathcal{D}_1$ and $\mathcal{D}_3$ do not have a common boundary.

It is straightforward to show that Q is continuous in $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, as well as over the boundaries $\partial\mathcal{D}_{12}$ and $\partial\mathcal{D}_{23}$. The linear growth condition clearly holds in the sets $\mathcal{D}_1$ and $\mathcal{D}_3$. Notice that in $\mathcal{D}_2$ the following inequality holds

$$\frac{(\mu(y) - r(y))^2}{2(1-\gamma)\sigma^2(y)} + r(y) \leq \frac{1}{2}(\mu(y) - r(y)) + r(y) = \frac{1}{2}(\mu(y) + r(y)), \tag{3.13}$$

from which the linear growth condition of the function Q in the set $\mathcal{D}_2$ follows from (A1).

We differentiate the function Q and obtain

$$\frac{dQ}{dy}(y) = \begin{cases} \frac{dr}{dy}(y) & y \in \mathcal{D}_1, \\ \frac{(\mu(y) - r(y))(\frac{d\mu}{dy}(y) - \frac{dr}{dy}(y))}{(1-\gamma)\sigma^2(y)} - \frac{(\mu(y) - r(y))^2 \frac{d\sigma}{dy}(y)}{(1-\gamma)\sigma^3(y)} + \frac{dr}{dy}(y) & y \in \mathcal{D}_2, \\ \frac{d\mu}{dy}(y) - \frac{1}{2}(1-\gamma)\frac{d\sigma^2}{dy}(y) & y \in \mathcal{D}_3. \end{cases}$$

Again, it is easy to show that this derivative is continuous in $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ and over the boundaries $\partial\mathcal{D}_{12}$ and $\partial\mathcal{D}_{23}$, and that a linear growth condition holds in $\mathcal{D}_1$ and $\mathcal{D}_3$. To prove the linear growth condition in the set $\mathcal{D}_2$, notice that

$$\big|\frac{dQ}{dy}(y)\big| \leq \big|\frac{d\mu}{dy}(y)\big| + \frac{1}{2}(1-\gamma)\big|\frac{d\sigma^2}{dy}(y)\big|,$$

holds for $y \in \mathcal{D}_2$. □

Remark 3.3. Condition (A3) is required in order to guarantee that the strategy and the function Q are well-defined over the set $\mathcal{D}_2$. When investigating the unconstrained

optimization problem, $\pi \in \mathbb{R}$, the set $\mathcal{D}_2$ must coincide with the whole positive real line, and one has to assume a uniform lower bound of the function σ; i.e., $\inf_{y>0} \sigma(y) > 0$. In the Barndorff-Nielsen & Shephard model this condition does not hold, unless we introduce reversion to a strictly positive constant (i.e. a linear drift term with positive mean reverting level). However, by considering a constrained strategy, one can overcome the global lower uniform boundedness and work with uniform boundedness only over some subset, see Benth *et al.* (2003) and Lindberg (2006) for the structure of the set $\mathcal{D}_2$. $\square$

We would like to point out that the linear growth condition (3.10) and the relations (2.8)-(2.11) will be applied frequently when proving our results.

By substituting (3.5) and (3.6) into (3.4) we arrive at the non-linear first-order partial integro-differential equation for the function f:

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t}(t,y) - \lambda \frac{\partial f}{\partial y}(t,y) y + \lambda \int_{z>0} \big(f(t,y+z) - f(t,y)\big)\nu(dz) \\ &\quad + \gamma f(t,y) Q(y) + (1-\gamma) f(t,y)^{-\frac{\gamma}{1-\gamma}}, \quad f(T,y) = 1. \end{aligned} \tag{3.14}$$

We will show that there exists a unique classical solution to this equation.

4 Existence of the solution

We introduce an operator $\mathcal{L}$ acting on functions f as follows:

$$(\mathcal{L}f)(t,y) = \mathbb{E}\big[(1-\gamma)\int_t^T e^{\gamma \int_t^s Q(Y^{t,y}(u))du} f(s, Y^{t,y}(s))^{-\frac{\gamma}{1-\gamma}} ds + e^{\gamma \int_t^T Q(Y^{t,y}(s))ds}\big], \tag{4.1}$$

for Q as in Lemma 3.2. By applying (heuristically) the Feynman-Kac formula to (3.14), we arrive at the following fixed point equation

$$(\mathcal{L}f)(t,y) = f(t,y), \quad (t,y) \in [0,T] \times (0,\infty). \tag{4.2}$$

In this section we prove that equation (4.2) has a unique solution $\hat{f}$. In Section 5 we shall show that this solution satisfies the partial integro-differential equation (3.14) in the classical sense.

We start with some observations. Notice that it is easy to derive a lower bound for the optimal value function V:

$$V(t,x,y) \geq x^\gamma \mathbb{E}[e^{\gamma \int_t^T r(Y(s))ds}], \quad (t,x,y) \in [0,T] \times (0,\infty) \times (0,\infty), \tag{4.3}$$

where the left hand side of (4.3) is the pay-off, when the agent does not consume and invests everything in the bank account. We conclude that the solution to (4.2) should satisfy the inequality

$$f(t,y) \;\geq\; \mathbb{E}[e^{\gamma\int_t^T r(Y(s))ds}] \geq 1, \quad (t,y)\in[0,T]\times(0,\infty). \tag{4.4}$$

Moreover, if we apply the operator $\mathcal{L}$ to a function f, which satisfies (4.4), then we obtain a lower bound of the operator:

$$(\mathcal{L}f)(t,y) \;\geq\; \mathbb{E}[e^{\gamma\int_t^T r(Y(s))ds}] \geq 1, \quad (t,y)\in[0,T]\times(0,\infty), \tag{4.5}$$

which can be derived by noticing that the first term in (4.1) is positive, and by applying the lower estimate (3.10) of the function Q in the second term.

We now turn to the more interesting upper bound of the operator $\mathcal{L}$. We still assume that condition (4.4) holds, which implies that $f(t,y)^{-\frac{\gamma}{1-\gamma}} \leq 1$. By applying the upper estimate (3.10) of the function Q, the estimate (2.11), as well as the representation (2.6), provided that $\psi(\frac{\gamma B}{\lambda}) < \infty$, we obtain the following inequality

$$\begin{aligned}
&(\mathcal{L}f)(t,y)\\
&\leq\; \mathbb{E}\big[(1-\gamma)\int_t^T e^{\gamma A(s-t)+\gamma B\int_t^s Y^{t,y}(u)du}ds + e^{\gamma A(T-t)+\gamma B\int_t^T Y^{t,y}(s)ds}\big]\\
&\leq\; \mathbb{E}\big[(1-\gamma)\int_t^T e^{\gamma A(s-t)+\frac{\gamma B}{\lambda}y+\frac{\gamma B}{\lambda}(L(\lambda s)-L(\lambda t))}ds + e^{\gamma A(T-t)+\frac{\gamma B}{\lambda}y+\frac{\gamma B}{\lambda}(L(\lambda T)-L(\lambda t))}\big]\\
&=\; (1-\gamma)\int_t^T e^{\gamma A(s-t)+\frac{\gamma B}{\lambda}y+\lambda\psi(\frac{\gamma B}{\lambda})(s-t)}ds + e^{\gamma A(T-t)+\frac{\gamma B}{\lambda}y+\lambda\psi(\frac{\gamma B}{\lambda})(T-t)}\\
&\leq\; \big(1+\frac{1-\gamma}{A'}\big)e^{A'(T-t)+B'y},
\end{aligned} \tag{4.6}$$

where we have introduced the constants $A' = \gamma A + \lambda\psi(\frac{\gamma B}{\lambda}) > 0$ and $B' = \frac{\gamma B}{\lambda} \geq 0$.

In the rest of the paper we assume that the following condition holds:

(B) $\psi(w) < \infty$ for $w = 2(1+\frac{\gamma}{2})(B' \vee B'_\sigma) + \varepsilon$ and some $\varepsilon > 0$,

where $B'_\sigma = \frac{\gamma B_\sigma}{\lambda} \geq 0$ is defined analogously to B'. This assumption becomes clear in the course of our calculations. It is only needed in section 6 in order to verify the optimality. Notice that the lemmas from this section and section 5 hold under integrability conditions of lower orders.

Let us investigate the operator $\mathcal{L}$ in a more rigorous way. Denote by $\mathcal{C}_e([0,T]\times(0,\infty))$ the space of continuous functions f on $[0,T]\times(0,\infty)$ satisfying

$$1 \leq f(t,y) \leq \big(1+\frac{1-\gamma}{A'}\big)e^{A'(T-t)+B'y}, \quad (t,y)\in[0,T]\times(0,\infty).$$

We equip the space $\mathcal{C}_e([0,T]\times(0,\infty))$ with the norm $||\cdot||$ defined by

$$||f|| = \sup_{(t,y)\in[0,T]\times(0,\infty)} |e^{-\alpha(T-t)-B'y} f(t,y)|, \tag{4.7}$$

for some $\alpha > A'$ to be specified later. The space $(\mathcal{C}_e([0,T]\times(0,\infty)), ||\cdot||)$ is a complete, metric space. Below we state two lemmas dealing with the properties of the operator $\mathcal{L}$.

Lemma 4.1. *The operator $\mathcal{L}$ defines a mapping from $\mathcal{C}_e([0,T]\times(0,\infty))$ into itself.*

Proof. Based on our previous results (4.5) and (4.6) we can conclude that the lower and upper bounds are preserved. It remains to prove continuity of the mapping $(t,y)\mapsto(\mathcal{L}f)(t,y)$.

Due to the time-homogeneity of Y, the operator $\mathcal{L}$ can be represented as

$$(\mathcal{L}f)(t,y) \;=\; \mathbb{E}\big[(1-\gamma)\int_0^{T-t} e^{\gamma\int_0^s Q(Y^{0,y}(u))du} f(s+t,Y^{0,y}(s))^{-\frac{\gamma}{1-\gamma}} ds + e^{\gamma\int_0^{T-t} Q(Y^{0,y}(s))ds}\big].$$

The above representation simplifies proving continuity in the time variable t. Notice that by the growth condition (3.10) and relation (2.11)

$$e^{\gamma\int_0^s Q(Y^{0,y}(u))du} f(s+t,Y^{0,y}(s))^{-\frac{\gamma}{1-\gamma}} \le e^{\gamma\int_0^s Q(Y^{0,y}(s))ds} \le e^{\gamma AT+B'y+B'L(\lambda T)},$$

holds $\mathbb{P}$-a.s., and that the càdlàg mapping $(y,u)\mapsto Y^{0,y}(u)$ is bounded a.s on compact sets. In order to prove continuity in the time variable one can directly apply Lebesgue's dominated convergence theorem and take the limit under the integral. To prove continuity of the mapping $y\mapsto(\mathcal{L}f)(t,y)$ in a fixed point $y_0>0$, define a compact set U around y_0 and take a sequence of points $y_n\in U$ such that $y_n\to y_0$ as $n\to\infty$. In this setting we can find a uniform bound for all $y_n\in U$ and we can apply Lebesgue's dominated convergence theorem. The continuity of $(t,y)\mapsto(\mathcal{L}f)(t,y)$ now follows from the continuity of f and Q and the continuity of the mapping $y\mapsto Y^{0,y}$. □

Lemma 4.2. *The mapping $\mathcal{L}:\mathcal{C}_e([0,T]\times(0,\infty))\to\mathcal{C}_e([0,T]\times(0,\infty))$ is a contraction with respect to the norm (4.7) for $\alpha > A'+\gamma$.*

Proof. Take two functions $\varphi,\xi\in\mathcal{C}_e([0,T]\times(0,\infty))$. Again we invoke (2.10) and (3.10).

Then the following inequalities hold for all $(t,y) \in [0,T] \times (0,\infty)$:

$$
\begin{aligned}
&|e^{-\alpha(T-t)-B'y}\big((\mathcal{L}\varphi)(t,y) - (\mathcal{L}\xi)(t,y)\big)| \\
&\leq \; e^{-\alpha(T-t)-B'y}\mathbb{E}\big[(1-\gamma)\int_t^T e^{\gamma\int_t^s Q(Y^{t,y}(u))du}\big|\varphi(s,Y^{t,y}(s))^{-\frac{\gamma}{1-\gamma}} - \xi(s,Y^{t,y}(s))^{-\frac{\gamma}{1-\gamma}}\big|ds\big] \\
&\leq \; \gamma e^{-\alpha(T-t)-B'y}\mathbb{E}\big[\int_t^T e^{\gamma\int_t^s Q(Y^{t,y}(u))du}\big|\varphi(s,Y^{t,y}(s)) - \xi(s,Y^{t,y}(s))\big|ds\big] \\
&\leq \; \gamma e^{-\alpha(T-t)-B'y}\mathbb{E}\big[\int_t^T e^{\gamma\int_t^s Q(Y^{t,y}(u))du+\alpha(T-s)+B'Y^{t,y}(s)}\big]||\varphi-\xi|| \\
&\leq \; \gamma e^{-\alpha(T-t)-B'y}\mathbb{E}\big[\int_t^T e^{\gamma A(s-t)+B'(y+L(\lambda s)-L(\lambda t)-Y^{t,y}(s))+\alpha(T-s)+B'Y^{t,y}(s)}\big]||\varphi-\xi|| \\
&= \; \gamma\int_t^T e^{-\alpha(s-t)+\gamma A(s-t)+\lambda\psi(B')(s-t)}ds||\varphi-\xi|| \\
&\leq \; \frac{\gamma}{\alpha-A'}||\varphi-\xi||,
\end{aligned}
$$

where the mean-value theorem has been applied in line 3. We conclude that

$$||\mathcal{L}\phi - \mathcal{L}\xi|| \leq q||\varphi-\xi||, \quad q<1,$$

which proves that the operator $\mathcal{L}$ defines a contraction mapping. □

The main result of this section is the following proposition, which is a consequence of Banach's fixed point theorem.

Proposition 4.3. *The equation*

$$(\mathcal{L}f)(t,y) = f(t,y) \tag{4.8}$$

has a unique solution $\hat{f} \in \mathcal{C}_e([0,T]\times(0,\infty))$.

5 Differentiability of the solution

In this section we investigate the differentiability of the function $\hat{f}$. In order to apply a classical verification theorem we have to prove that $\hat{f}$ is continuously differentiable in the time and in the space variable.

We assume that $B' > 0$. In the case when the function Q is uniformly bounded in y; i.e., $B = 0$, an arbitrary, strictly positive (small) constant $B > 0$ can be chosen, so that the proofs from this section remain true. We remark that our arguments can be modified in order to handle the special case of $B = 0$ and to derive sharper bounds. We would like to point out that the main theorem of our paper, Theorem 6.2, holds true even for $B = 0$.

Recall that the ODE

$$\frac{d\phi}{dt}(t) + (\gamma - \lambda)\phi(t) + \lambda a = 0, \quad \phi(T) = a, \tag{5.1}$$

has the unique, smooth and strictly positive solution in the class $\mathcal{C}^1([0,T])$ given by

$$\phi(t) = a + \gamma \int_t^T \phi(s) e^{-\lambda(s-t)} ds,$$

with constant $a > 0$.

The idea for establishing differentiability in the space variable is to construct a sequence of functions $(f_n)_{n\in\mathbb{N}}$, which converge to $\hat{f}$ and which share some desirable properties.

Lemma 5.1. *Define $A'' = \gamma A + \lambda\psi(B'') > 0$ and $B'' = B'(1+\frac{\gamma}{4}) > 0$ with $B' = \gamma B/\lambda > 0$ and A, B as in (3.10). Choose a function $f_1 \in \mathcal{C}_e([0,T]\times(0,\infty)) \cap \mathcal{C}^{0,1}([0,T]\times(0,\infty))$ such that*

$$\big|\frac{\partial f_1}{\partial y}(t,y)\big| \le \phi(t) e^{A''(T-t)+B''y}, \quad (t,y) \in [0,T]\times(0,\infty), \tag{5.2}$$

where ϕ solves (5.1) with $a = \frac{1}{\lambda}\big(1 + \frac{1-\gamma}{A''}\big)\big(\frac{4D}{B'} \vee C\gamma\big) > 0$.
Let the sequence $(f_n)_{n\in\mathbb{N}}$ be constructed recursively as $f_{n+1} = \mathcal{L}f_n$ with $\mathcal{L}$ as defined in (4.1). Then for all $n \in \mathbb{N}$,

$$f_n \in \mathcal{C}_e([0,T]\times(0,\infty)) \cap \mathcal{C}^{0,1}([0,T]\times(0,\infty)),$$

and

$$\big|\frac{\partial f_n}{\partial y}(t,y)\big| \le \phi(t) e^{A''(T-t)+B''y}, \quad (t,y) \in [0,T]\times(0,\infty). \tag{5.3}$$

Proof. Recall from (4.1) that

$$f_2(t,y) \;=\; \mathbb{E}\big[(1-\gamma)\int_t^T e^{\gamma\int_t^s Q(Y^{t,y}(u))du} f_1(s, Y^{t,y}(s))^{-\frac{\gamma}{1-\gamma}} ds + e^{\gamma\int_t^T Q(Y^{t,y}(s))ds}\big].$$

First we prove that the mapping $(t,y) \mapsto \frac{\partial f_2}{\partial y}(t,y)$ is continuous.

We expect that the derivative equals

$$\begin{aligned}\frac{\partial f_2}{\partial y}(t,y) \;=\; & \mathbb{E}\big[(1-\gamma)\int_t^T \gamma e^{\gamma\int_t^s Q(Y^{t,y}(u))du} f_1(s,Y(s))^{-\frac{\gamma}{1-\gamma}} \int_t^s \frac{dQ}{dy}(Y^{t,y}(u)) e^{-\lambda(u-t)} du ds \\ & - \int_t^T \gamma e^{\gamma\int_t^s Q(Y^{t,y}(u))du} f_1(s,Y^{t,y}(s))^{-\frac{1}{1-\gamma}} \frac{\partial f_1}{\partial y}(s,Y^{t,y}(s)) e^{-\lambda(s-t)} ds \\ & + \gamma e^{\gamma\int_t^T Q(Y^{t,y}(s))ds} \int_t^T \frac{dQ}{dy}(Y^{t,y}(s)) e^{-\lambda(s-t)} ds\big]. \end{aligned} \tag{5.4}$$

This will follow from Lebesgue's dominated convergence theorem provided that it can be applied. Below we establish three estimates which allow us to interchange differentiation and integration. We point out that the interchange is justified, if we can bound the derivative by an integrable function. The estimates are also used later to establish (5.3).

We recall that in order to find a uniform bound, one can take a limit $y_n \to y_0$ as $n \to \infty$, over a sequence of points $y_n \in U$, where U is a compact set around a fixed point $y_0 > 0$.

Notice that by invoking the simple inequality $a + by \leq (\frac{1}{\epsilon} \vee a)e^{b\epsilon y}$ for all $\epsilon > 0$, we obtain that

$$\begin{aligned} \big|\frac{\partial}{\partial y}\big(Q(Y^{t,y}(u))\big)\big| &= \big|\frac{dQ}{dy}(Y^{t,y}(u))\frac{\partial}{\partial y}Y^{t,y}(u)\big| \leq (C + DY^{t,y}(u))e^{-\lambda(u-t)} \\ &\leq \left(\frac{4D}{\gamma B'} \vee C\right) e^{\frac{\gamma B'}{4}Y^{t,y}(u)}e^{-\lambda(u-t)} \leq \left(\frac{4D}{\gamma B'} \vee C\right) e^{\frac{\gamma B'}{4}(y+L(\lambda s)-L(\lambda t))}e^{-\lambda(u-t)} \end{aligned} \tag{5.5}$$

holds $\mathbb{P}$- a.s., for $0 \leq t \leq u \leq s \leq T$. Based on the estimate (5.5) we derive that

$$\begin{aligned} \big|\frac{\partial}{\partial y}\left(e^{\gamma\int_t^s Q(Y^{t,y}(u))du}\right)\big| &= \big|\gamma e^{\gamma\int_t^s Q(Y^{t,y}(u)du}\frac{\partial}{\partial y}\left(\int_t^s Q(Y^{t,y}(u))du\right)\big| \\ &= \big|\gamma e^{\gamma\int_t^s Q(Y^{t,y})(u)du}\int_t^s \frac{\partial}{\partial y}\big(Q(Y^{t,y}(u))\big)\,du\big| \\ &\leq \left(\frac{4D}{B'} \vee C\gamma\right) e^{\gamma A(s-t)+B'(y+L(\lambda s)-L(\lambda t))}e^{\frac{\gamma B'}{4}(y+L(\lambda s)-L(\lambda t))}\int_t^s e^{-\lambda(u-t)}du \\ &\leq \frac{1}{\lambda}\left(\frac{4D}{B'} \vee \gamma C\right) e^{\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t))} \end{aligned} \tag{5.6}$$

holds $\mathbb{P}$-a.s., for $0 \leq t \leq s \leq T$. We would like to point out that indeed we are allowed to interchange integration and differentiation in the first line of (5.6), as the bound (5.5) is integrable $\mathbb{P}$-a.s. on $[t, s]$.

Based on (5.2) and (5.6) we obtain the third estimate

$$\begin{aligned} &\big|\frac{\partial}{\partial y}\left((1-\gamma)e^{\gamma\int_t^s Q(Y^{t,y}(u))du}f_1(s,Y(s))^{-\frac{\gamma}{1-\gamma}}\right)\big| \\ &= \big|(1-\gamma)\frac{\partial}{\partial y}\left(e^{\gamma\int_t^s Q(Y^{t,y}(u))du}\right)f_1(s,Y(s))^{-\frac{\gamma}{1-\gamma}} \\ &\quad -\gamma e^{\gamma\int_t^s Q(Y^{t,y}(u))du}f_1(s,Y^{t,y}(s))^{-\frac{1}{1-\gamma}}\frac{\partial f_1}{\partial y}(s,Y^{t,y}(s))\frac{\partial}{\partial y}\big(Y^{t,y}(s)\big)\big| \\ &\leq \frac{(1-\gamma)}{\lambda}\left(\frac{4D}{B'} \vee C\gamma\right) e^{\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t))} \\ &\quad +\gamma e^{\gamma A(s-t)+\gamma B\int_t^s Y^{t,y}(u)du}\big|\frac{\partial f_1}{\partial y}(s,Y^{t,y}(s))\big|e^{-\lambda(s-t)} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(1-\gamma)}{\lambda}\left(\frac{4D}{B'}\vee C\gamma\right)e^{\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t))}\\
&\quad +\gamma e^{\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t)-Y^{t,y}(s))}\phi(s)e^{A''(T-s)+B''Y^{t,y}(s)}e^{-\lambda(s-t)}\\
&= \frac{1-\gamma}{\lambda}\left(\frac{4D}{B'}\vee C\gamma\right)e^{\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t))}\\
&\quad +\gamma\phi(s)e^{-\lambda(s-t)}e^{A''(T-s)+\gamma A(s-t)+B''y+B''(L(\lambda s)-L(\lambda t))},\quad \mathbb{P}-a.s.. \qquad (5.7)
\end{aligned}
$$

As the derived bound (5.7) is a càdlàg mapping, hence is a.s. integrable, we have that

$$
\begin{aligned}
&\frac{\partial}{\partial y}\int_t^T\left((1-\gamma)e^{\gamma\int_t^s Q(Y^{t,y}(u))du}f_1(s,Y(s))^{-\frac{\gamma}{1-\gamma}}\right)ds\\
&= \int_t^T\frac{\partial}{\partial y}\left((1-\gamma)e^{\gamma\int_t^s Q(Y^{t,y}(u))du}f_1(s,Y(s))^{-\frac{\gamma}{1-\gamma}}\right)ds,\quad \mathbb{P}-a.s..
\end{aligned}
$$

Finally, taking the derivative under the expectation is also justified as by condition (B) we have $\psi(B'')<\infty$. Consequently, we have shown that the derivative (5.4) holds.

The continuity of the mapping $(t,y)\mapsto\frac{\partial f_2}{\partial y}(t,y)$ follows again from Lebesgue's dominated convergence theorem, by applying the estimates (5.6) and (5.7), and from the continuity of the functions f_1 and Q, as well as their derivatives (compare with the proof of the continuity in Lemma (4.1)).

We still have to prove that the bound (5.3) holds for $n=2$. By combining (5.6) and (5.7) we can estimate

$$
\begin{aligned}
\left|\frac{\partial f_2}{\partial y}(t,y)\right| &\leq \mathbb{E}\big[\frac{1-\gamma}{\lambda}\left(\frac{4D}{B'}\vee C\gamma\right)\int_t^T e^{\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t))}ds\\
&\quad +\gamma\int_t^T\phi(s)e^{-\lambda(s-t)}e^{A''(T-s)+\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t))}ds\\
&\quad +\frac{1}{\lambda}\left(\frac{4D}{B'}\vee C\gamma\right)e^{\gamma A(T-t)+B''(y+L(\lambda T)-L(\lambda t))}\big]\\
&= \frac{1-\gamma}{\lambda}\left(\frac{4D}{B'}\vee C\gamma\right)\int_t^T e^{A''(s-t)+B''y}ds+e^{A''(T-t)+B''y}\gamma\int_t^T\phi(s)e^{-\lambda(s-t)}ds\\
&\quad +\frac{1}{\lambda}\left(\frac{4D}{B'}\vee C\gamma\right)e^{A''(T-t)+B''y}\\
&\leq \frac{1-\gamma}{\lambda A''}\left(\frac{4D}{B'}\vee C\gamma\right)e^{A''(T-t)+B''y}+e^{A''(T-t)+B''y}\gamma\int_t^T\phi(s)e^{-\lambda(s-t)}ds\\
&\quad +\frac{1}{\lambda}\left(\frac{4D}{B'}\vee C\gamma\right)e^{A''(T-t)+B''y}\\
&= e^{A''(T-t)+B''y}\big(a+\gamma\int_t^T\phi(s)e^{-\lambda(s-t)}ds\big)=\phi(t)e^{A''(T-t)+B''y},
\end{aligned}
$$

where we have invoked the solution of the ODE (5.1) with the appropriate constant. Repeating the calculations recursively concludes the proof. □

From the properties of the constructed sequence $(f_n)_{n\in\mathbb{N}}$ we can deduce an important property of the function $\hat{f}$.

Proposition 5.2. *The function $\hat{f}$ belongs to the class $\mathcal{C}_e([0,T]\times(0,\infty))\cap\mathcal{C}^{0,1}([0,T]\times(0,\infty))$. Moreover its derivative satisfies*

$$\big|\frac{\partial\hat{f}}{\partial y}(t,y)\big|\leq\phi(t)e^{A''(T-t)+B''y},\quad (t,y)\in[0,T]\times(0,\infty),\tag{5.8}$$

for A'', B'' and ϕ as in Lemma 5.1.

Proof. The result follows if we show that the sequences $(\frac{\partial f_n}{\partial y})_{n\in\mathbb{N}}$, constructed in Lemma 5.1, converges uniformly, at least on compact subsets of $[0,T]\times(0,\infty)$.

Choose $n\geq m$ and $\rho>\alpha\vee A$". Using the definition of the derivative (5.4) we have

$$\begin{aligned}
&e^{-\rho(T-t)-2B''y}\big|\frac{\partial f_{n+1}}{\partial y}(t,y)-\frac{\partial f_{m+1}}{\partial y}(t,y)\big|\\
&\leq\ \mathbb{E}\big[(1-\gamma)\int_t^T\gamma e^{\gamma\int_t^s Q(Y^{t,y}(u))du}|f_n(s,Y^{t,y}(s))^{-\frac{\gamma}{1-\gamma}}-f_m(s,Y^{t,y}(s))^{-\frac{\gamma}{1-\gamma}}|\\
&\quad\times\int_t^s\big|\frac{\partial Q}{\partial y}(Y^{t,y}(u))\big|e^{-\lambda(u-t)}(u)duds\big]e^{-\rho(T-t)-2B''y}\\
&\quad+\mathbb{E}\big[\int_t^T\gamma e^{\gamma\int_t^s Q(Y^{t,y}(u))du}|f_n(s,Y^{t,y}(s))^{-\frac{1}{1-\gamma}}-f_m(s,Y^{t,y}(s))^{-\frac{1}{1-\gamma}}|\\
&\quad\times\frac{\partial f_m}{\partial y}(s,Y^{t,y}(s))e^{-\lambda(s-t)}ds\big]e^{-\rho(T-t)-2B''y}\\
&\quad+\mathbb{E}\big[\int_t^T\gamma e^{\gamma\int_t^s Q(Y^{t,y}(u))du}|\frac{\partial f_n}{\partial y}(s,Y^{t,y}(s))-\frac{\partial f_m}{\partial y}(s,Y^{t,y}(s))|\\
&\quad\times f_n(s,Y^{t,y}(s))^{-\frac{1}{1-\gamma}}e^{-\lambda(s-t)}ds\big]e^{-\rho(T-t)-2B''y}\\
&=:\ M_1+M_2+M_3.
\end{aligned}\tag{5.9}$$

We first find an upper bound for M_1. Applying the estimate (5.5) and the mean-value theorem we find

$$\begin{aligned}
M_1\ &\leq\ \gamma^2\mathbb{E}\big[\int_t^T e^{\gamma A(s-t)+B'(y+L(\lambda s)-L(\lambda t)-Y^{t,y}(s))}e^{-\alpha(T-s)-B'Y^{t,y}(s)}e^{\rho(T-s)+B'Y^{t,y}(s)}\\
&\quad\times|f_n(s,Y^{t,y}(s))-f_m(s,Y^{t,y}(s))|\left(\frac{4D}{\gamma B'}\vee C\right)e^{\frac{\gamma B'}{4}(y+L(\lambda s)-L(\lambda t))}\int_t^s e^{-\lambda(u-t)}duds\big]\\
&\quad\times e^{-\rho(T-t)-2B''y}\\
&\leq\ \frac{1}{\lambda}\left(\frac{4D\gamma}{B'}\vee C\gamma^2\right)||f_n(t,y)-f_m(t,y)||\int_t^T e^{\left(\gamma A+\lambda\psi(B'')-\rho\right)(s-t)}ds\\
&\leq\ K_1||f_n(t,y)-f_m(t,y)||.
\end{aligned}\tag{5.10}$$

Similarly, we have

$$
\begin{aligned}
M_2 \leq & \frac{\gamma}{1-\gamma}\mathbb{E}\big[\int_t^T e^{\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t)-Y^{t,y}(s))}|f_n(s,Y^{t,y}(s))-f_m(s,Y^{t,y}(s))| \\
& \times\phi(s)e^{A''(T-s)+B''Y^{t,y}(s)}e^{-\alpha(T-s)-B'Y^{t,y}(s)}e^{\rho(T-s)+B'Y^{t,y}(s)}e^{-\lambda(s-t)}ds\big]e^{-\rho(T-t)-2B''y} \\
\leq & \frac{\gamma}{1-\gamma}||f_n(t,y)-f_m(t,y)||\mathbb{E}\big[\int_t^T e^{\gamma A(s-t)+B''(y+L(\lambda s)-L(\lambda t)-Y^{t,y}(s))} \\
& \times\phi(s)e^{A''(T-s)+B''Y^{t,y}(s)}e^{\rho(T-s)+B'(y+L(\lambda s)-L(\lambda t))}e^{-\lambda(s-t)}ds\big]e^{-\rho(T-t)-2B''y} \\
\leq & \frac{\gamma}{1-\gamma}e^{A''T}\sup_{t\in[0,T]}\{\phi(t)\}||f_n(t,y)-f_m(t,y)||\int_t^T e^{\big(\gamma A+\lambda\psi(2B'')-\rho-\lambda\big)(s-t)}ds \\
\leq & K_2||\phi_n(t,y)-\phi_m(t,y)||, \qquad (5.11)
\end{aligned}
$$

where we have used the bound (5.3) for the sequence of derivatives $\big(\frac{\partial f_n}{\partial y}\big)_{n\in\mathbb{N}}$. Finally, we obtain a bound for M_3:

$$
\begin{aligned}
M_3 \leq & \mathbb{E}\big[\int_t^T \gamma e^{\gamma A(s-t)+2B''(y+L(\lambda s)-L(\lambda t)-Y^{t,y}(s))}|\frac{\partial f_n}{\partial y}(s,Y^{t,y}(s))-\frac{\partial f_m}{\partial y}(s,Y^{t,y}(s))| \\
& \times e^{-\rho(T-s)-2B''Y^{t,y}(s)}e^{\rho(T-s)+2B''Y^{t,y}(s)}e^{-\lambda(s-t)}ds\big]e^{-\rho(T-t)-2B''y} \\
\leq & \sup_{(t,y)\in[0,T]\times(0,\infty)}|e^{-\rho(T-t)-2B''y}\left(\frac{\partial f_n}{\partial y}(t,y)-\frac{\partial f_m}{\partial y}(t,y)\right)| \\
& \times\gamma\int_t^T e^{\big(\gamma A+\lambda\psi(2B'')-\rho-\lambda\big)(s-t)}ds \\
\leq & K_3\sup_{(t,y)\in[0,T]\times(0,\infty)}|e^{-\rho(T-t)-2B''y}\left(\frac{\partial f_n}{\partial y}(t,y)-\frac{\partial f_m}{\partial y}(t,y)\right)|, \qquad (5.12)
\end{aligned}
$$

where ρ should have been chosen such that $K_3=\frac{\gamma}{\rho-\gamma A-\lambda\psi(2B'')+\lambda}<1$.

Notice that by the contraction property of the operator $\mathcal{L}$ proved in Lemma 4.2 we have

$$
\begin{aligned}
||f_n(t,y)-f_m(t,y)|| \leq & \left(\frac{\gamma}{\alpha-A'}\right)^{m-1}||f_{n-m+1}(t,y)-f_1(t,y)|| \\
\leq & 2\left(\frac{\gamma}{\alpha-A'}\right)^{m-1}\left(1+\frac{1-\gamma}{A'}\right). \qquad (5.13)
\end{aligned}
$$

By combining (5.10)-(5.13) we get

$$
\begin{aligned}
& |e^{-\rho(T-t)-2B''y}\left(\frac{\partial f_{n+1}}{\partial y}(t,y)-\frac{\partial f_{m+1}}{\partial y}(t,y)\right)| \\
& \leq 2\big(K_1+K_2\big)\left(\frac{\gamma}{\alpha-A'}\right)^{m-1}\left(1+\frac{1-\gamma}{A'}\right) \\
& \quad +K_3\sup_{(t,y)\in[0,T]\times(0,\infty)}|e^{-\rho(T-t)-2B''y}\left(\frac{\partial f_n}{\partial y}(t,y)-\frac{\partial f_m}{\partial y}(t,y)\right)|
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\big(K_1+K_2\big)\left(\frac{\gamma}{\alpha-A'}\right)^{m-1}\left(1+\frac{1-\gamma}{A'}\right)\frac{1-K_3^{m-1}}{1-K_3}\\
&\quad +K_3^{m-1}\sup_{(t,y)\in[0,T]\times(0,\infty)}|e^{-\rho(T-t)-2B''y}\left(\frac{\partial f_{n-m+1}}{\partial y}(t,y)-\frac{\partial f_1}{\partial y}(t,y)\right)|\\
&\leq 2\big(K_1+K_2\big)\left(\frac{\gamma}{\alpha-A'}\right)^{m-1}\left(1+\frac{1-\gamma}{A'}\right)\frac{1-K_3^{m-1}}{1-K_3}\\
&\quad +2K_3^{m-1}\sup_{t\in[0,T]}\{\phi(t)\},
\end{aligned}
$$

from which we can conclude that the sequence $(\frac{\partial f_n}{\partial y}(t,y))_{n\in\mathbb{N}}$ converges uniformly on compact sets. □

We now turn to the question of differentiability in the time variable. We first show that the function $\hat{f}(t,y)$ belongs for every fixed $t\in[0,T]$ to the domain of the infinitesimal generator of the process Y, see Chapter 1.3 in Øksendal & Sulem (2005).

As the mapping $y\mapsto\hat{f}(t,y)$ is continuously differentiable on $(0,\infty)$, one can apply Itô's formula and show that the following limit relation holds

$$
\begin{aligned}
&\lim_{s\to 0}\frac{\mathbb{E}[\hat{f}(t,Y^{0,y}(s))]-\hat{f}(t,y)}{s}\\
&\quad =\ -\frac{\partial\hat{f}}{\partial y}(t,y)\lambda y+\int_{z>0}\big(\hat{f}(t,y+z)-\hat{f}(t,y)\big)\nu(dz),
\end{aligned}
\tag{5.14}
$$

provided that for $s>0$

$$
\mathbb{E}\big[\int_0^s\int_{z>0}\Big(\hat{f}(t,Y^{0,y}(u-)+z)-\hat{f}(t,Y^{0,y}(u-))\Big)\tilde{N}(du\times dz)\big]=0,
\tag{5.15}
$$

where $\tilde{N}(du\times dz):=N(du\times dz)-\nu(dz)du$ is the compensated Poisson random measure from (2.5). It is well known, see e.g. Theorem 4.2.3 in Applebaum (2004), that condition (5.15) is equivalent to

$$
\mathbb{E}\big[\int_0^s\int_{z>0}\big|\hat{f}(t,Y^{0,y}(u-)+z)-\hat{f}(t,Y^{0,y}(u-))\big|^2\nu(dz)du\big]<\infty.
\tag{5.16}
$$

The mean-value theorem and the bound (5.8) imply

$$
\begin{aligned}
&\mathbb{E}\big[\int_0^s\int_{z>0}\big|\hat{f}(t,Y^{0,y}(u-)+z)-\hat{f}(t,Y^{0,y}(u-))\big|^2\nu(dz)du\big]\\
&\quad\leq\ \mathbb{E}\big[\int_0^s\int_{z>0}\phi^2(t)e^{2A''(T-t)+2B''(Y^{0,y}(u)+z)}z^2\nu(dz)du\big]\\
&\quad\leq\ \mathbb{E}\big[\int_0^s\int_{z>0}\phi^2(t)e^{2A''(T-t)+2B''y+2B''L(\lambda T)}e^{2B''z}z^2\nu(dz)du\big]\\
&\quad\leq\ Ke^{2B''}\int_{0<z<1}z^2\nu(dz)+K\int_{z\geq 1}e^{2B''z}z^2\nu(dz)
\end{aligned}
\tag{5.17}
$$

for some positive constant K, which is finite since $\psi(2B'') < \infty$. The first term in (5.17) is clearly finite. We show that the second term is finite as well. By applying the inequality $z \leq \frac{4}{\gamma B'} e^{\frac{\gamma B'}{4} z}$ and the assumption (B) we find that

$$\int_{z\geq 1} e^{2B''z} z^2 \nu(dz) \leq \left(\frac{4}{\gamma B'}\right)^2 \int_{z\geq 1} e^{(2B''+\frac{\gamma B'}{2})z} dz = \left(\frac{4}{\gamma B'}\right)^2 \int_{z\geq 1} e^{2B'(1+\frac{\gamma}{2})z} dz < \infty.$$

Before stating the next lemma, we would like to remark that the mapping $(t,y) \mapsto \int_{z>0} \big(\hat{f}(t,y+z) - \hat{f}(t,y)\big)\nu(dz)$ is continuous on $[0,T] \times (0,\infty)$. This follows from the inequality

$$\big|\hat{f}(t,y+z) - \hat{f}(t,y)\big| \leq \phi(t) e^{A''(T-t)+B''(y+z)} z,$$

and Lebesgue's dominated convergence theorem.

Proposition 5.3. *The function $\hat{f}$ satisfies the partial integro-differential equation*

$$\begin{aligned} 0 \;=\; & \frac{\partial \hat{f}}{\partial t}(t,y) - \frac{\partial \hat{f}}{\partial y}(t,y)\lambda y + \lambda \int_{z>0} \big(\hat{f}(t,y+z) - \hat{f}(t,y)\big)\nu(dz) \\ & + \hat{f}(t,y)\gamma Q(y) + (1-\gamma)(\hat{f}(t,y))^{-\frac{\gamma}{1-\gamma}}, \quad \hat{f}(T,y) = 1, \end{aligned} \tag{5.18}$$

in the classical sense. In particular, the mapping $(t,y) \mapsto \frac{\partial \hat{f}}{\partial t}(t,y)$ is continuous on $[0,T) \times (0,\infty)$.

Proof. The idea of the proof is similar as in the proof of Proposition 5.5 in Benth *et al.* (2003). We will calculate the limit in (5.14) explicitly by using the representation of $\hat{f}$.

Consider a fixed $t \in [0,T)$. Notice that by time-homogeneity of Y the equivalent representation holds

$$\begin{aligned} \hat{f}(t,y) : \;=\; & \mathbb{E}\big[(1-\gamma)\int_s^{T-t+s} e^{\gamma\int_s^u Q(Y(w))dw} (\hat{f}(u+t-s,Y(u)))^{-\frac{\gamma}{1-\gamma}} du \\ & + e^{\gamma\int_s^{T-t+s} Q(Y(w))dw} \mid Y(s) = y\big], \end{aligned} \tag{5.19}$$

for $s \geq 0$. Let $\sigma((Y^{0,y}(s))$ denote the σ-algebra generated by the random variable $Y^{0,y}(s)$ defined as in (2.7). We have that

$$\begin{aligned} \hat{f}(t,Y^{0,y}(s)) \;=\; & \mathbb{E}\big[(1-\gamma)\int_s^{T-t+s} e^{\gamma\int_s^u Q(Y(w))dw} (\hat{f}(u+t-s,Y(u)))^{-\frac{\gamma}{1-\gamma}} du \\ & + e^{\gamma\int_s^{T-t+s} Q(Y(w))dw} \mid \sigma((Y^{0,y}(s))\big], \end{aligned}$$

holds $\mathbb{P}$-a.s.. Applying the law of iterated expectations we obtain

$$\begin{aligned} \mathbb{E}[\hat{f}(t,Y^{0,y}(s))] \;=\; & \mathbb{E}\big[(1-\gamma)\int_s^{T-t+s} e^{\gamma\int_s^u Q(Y(w))dw} (\hat{f}(u+t-s,Y(u)))^{-\frac{\gamma}{1-\gamma}} du \\ & + e^{\gamma\int_s^{T-t+s} Q(Y(w))dw} \mid Y(0) = y\big]. \end{aligned}$$

Consider now the difference $\mathbb{E}[\hat{f}(t, Y^{0,y}(s))] - \hat{f}(t,y)$ for some $s > 0$ in the neighborhood of 0. By simple algebraic manipulations we find

$$\begin{aligned}
&\frac{1}{s}\big(\mathbb{E}[\hat{f}(t, Y^{0,y}(s))] - \hat{f}(t,y)\big) \\
&= \mathbb{E}\big[(1-\gamma)\int_s^{T-t+s} \big(\hat{f}(u+t-s, Y^{0,y}(u))\big)^{-\frac{\gamma}{1-\gamma}} e^{\gamma\int_0^u Q(Y^{0,y}(w))dw} \\
&\quad \times \frac{1}{s}\big(e^{-\gamma\int_0^s Q(Y^{0,y}(w))dw} - 1\big)du\big] \\
&\quad -\frac{1}{s}\mathbb{E}\big[(1-\gamma)\int_0^s \big(\hat{f}(u+t-s, Y^{0,y}(u))\big)^{-\frac{\gamma}{1-\gamma}} e^{\gamma\int_0^u Q(Y^{0,y}(w)dw}du\big] \\
&\quad +\mathbb{E}\big[e^{\gamma\int_0^{T-t+s} Q(Y^{0,y}(w))dw}\frac{1}{s}\big(e^{-\gamma\int_0^s Q(Y^{0,y}(w))dw} - 1\big)\big] + \frac{1}{s}\big(\hat{f}(t-s,y) - \hat{f}(t,y)\big) \\
&=: M_1(s) + M_2(s) + M_3(s) + M_4(s).
\end{aligned}$$

Notice that

$$\begin{aligned}
e^{\gamma\int_0^s Q(Y^{0,y}(w))dw} &\leq e^{\gamma AT + B'y + B'L(\lambda T)}, \\
\frac{1}{s}\big(1 - e^{-\gamma\int_0^s Q(Y^{0,y}(w))dw}\big) &\leq \sup_{s\geq 0}\{\gamma Q(Y^{0,y}(s))\} \leq \gamma AT + \gamma B \sup_{s\geq 0}\{Y^{0,y}(s)\} \\
\leq \gamma AT + \gamma By + \gamma BL(\lambda T) &\leq (\lambda \vee \gamma AT)e^{B'y + B'L(\lambda T)} \qquad (5.20)
\end{aligned}$$

hold $\mathbb{P}$-a.s. for $0 < s \leq T$. The above estimates (5.20) ensure that we can apply Lebesgue's dominated convergence theorem to obtain the following limits

$$\begin{aligned}
\lim_{s\to 0} M_1(s) &= -\gamma Q(y)\mathbb{E}\big[(1-\gamma)\int_0^{T-t} e^{\gamma\int_0^u Q(Y^{0,y}(w))du}\big(\hat{f}(u+t, Y^{0,y}(u))\big)^{-\frac{\gamma}{1-\gamma}}du\big], \\
\lim_{s\to 0} M_2(s) &= -(1-\gamma)(\hat{f}(t,y))^{-\frac{\gamma}{1-\gamma}}, \\
\lim_{s\to 0} M_3(s) &= -\gamma Q(y)\mathbb{E}\big[e^{\gamma\int_0^{T-t} Q(Y^{0,y}(w))dw}\big].
\end{aligned}$$

Moreover, $\lim_{s\to 0}(M_1(s) + M_3(s)) = -\gamma Q(y)\hat{f}(t,y)$ holds, and by combining these calculations with (5.14) we arrive at

$$\begin{aligned}
&\lim_{s\to 0} M_4(s) = -\lim_{s\to 0}\frac{\hat{f}(t,y) - \hat{f}(t-s,y)}{s} \\
&= -\frac{\partial\hat{f}}{\partial y}(t,y)\lambda y + \int_{z>0}\big(\hat{f}(t,y+z) - \hat{f}(t,y)\big)\nu(dz) + \hat{f}(t,y)\gamma Q(y) + (1-\gamma)(\hat{f}(t,y))^{-\frac{\gamma}{1-\gamma}}.
\end{aligned}$$

We conclude that the derivative $\frac{\partial\hat{f}}{\partial t}$ exists and that $\hat{f}$ satisfies the partial integro-differential equation (5.18). Moreover, the mapping $(t,y) \mapsto \frac{\partial\hat{f}}{\partial t}(t,y)$ is continuous on $[0,T)\times(0,\infty)$, by continuity of all terms on the right hand side of (5.18). □

We can also conclude that the function $\hat{f}$ is the only classical solution of the partial integro-differential equation (3.14), as for any such solution the Feynman-Kac representation must hold, see Becherer & Schweizer (2005) for a similar argument.

6 Optimality of the solution

In this section we verify that the solution derived in previous sections is optimal.

In principal, one can prove a classical verification theorem. We find it more convenient to prove optimality by applying a localizing sequence; cf. Benth *et al.* (2003).

Theorem 6.1. *Let* $v \in \mathcal{C}^{1,2,1}([0,T)\times(0,\infty)\times(0,\infty)) \cap \mathcal{C}([0,T]\times(0,\infty)\times(0,\infty)$ *satisfy for every* $(c,\pi)\in\mathcal{A}$

$$\begin{aligned} 0 \;\geq\; & c^\gamma + \frac{\partial v}{\partial t}(t,x,y) + \frac{\partial v}{\partial x}(t,x,y)\,(\pi x(\mu(y)-r(y)) + xr(y) - c) \\ & + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x,y)\pi^2x^2\sigma^2(y) - \frac{\partial v}{\partial y}(t,x,y)\lambda y \\ & + \lambda\int_{z>0}\big(v(t,x,y+z)-v(t,x,y)\big)\nu(dz), \end{aligned} \tag{6.1}$$

for all $(t,x,\lambda)\in[0,T]\times(0,\infty)\times(0,\infty)$*, with*

$$v(T,x,y) = x^\gamma, \quad (x,y)\in(0,\infty)\times(0,\infty). \tag{6.2}$$

Assume also that

$$\mathbb{E}^{0,x,y}\big[\int_0^T\int_{z>0}|v(t,X^{c,\pi}(t),Y(t-)+z) - v(t,X^{c,\pi}(t),Y(t-))|^2\nu(dz)dt\big] < \infty, \tag{6.3}$$

and that $\{v^-(\tau,X^{c,\pi}(\tau),Y(\tau))\}_{0<\tau\leq T}$ *is uniformly integrable for all* $\mathbb{F}$*-stopping times* τ*. Then*

$$v(t,x,y) \geq V(t,x,y), \quad (t,x,y)\in[0,T]\times(0,\infty)\times(0,\infty). \tag{6.4}$$

Moreover, if there exists an admissible control $(\hat{c},\hat{\pi})$ *such that*

$$\begin{aligned} 0 \;=\; & \hat{c}^\gamma + \frac{\partial v}{\partial t}(t,x,y) + \frac{\partial v}{\partial x}(t,x,y)\,(\hat{\pi}x(\mu(y)-r(y)) + xr(y) - \hat{c}) \\ & + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x,y)\hat{\pi}^2x^2\sigma^2(y) - \frac{\partial v}{\partial y}(t,x,y)\lambda y \\ & + \lambda\int_{z>0}\big(v(t,x,y+z)-v(t,x,y)\big)\nu(dz)\}, \end{aligned} \tag{6.5}$$

for all $(t,x,y)\in[0,T]\times(0,\infty)\times(0,\infty)$*, and that* $\{v(\tau,X^{\hat{c},\hat{\pi}}(\tau),Y(\tau))\}_{0<\tau\leq T}$ *is uniformly integrable for all* $\mathbb{F}$*-stopping times* τ*, then*

$$v(t,x,y) = V(t,x,y), \quad (t,x,y)\in[0,T]\times(0,\infty)\times(0,\infty), \tag{6.6}$$

and $(\hat{c},\hat{\pi})$ *is the optimal strategy for* (3.3)*.*

Proof. Fix $(t,x,y) \in [0,T) \times (0,\infty) \times (0,\infty)$ and ϵ such that $0 < \epsilon < T-t$. Define a sequence of stopping times

$$t_n = \inf\{s \in (t,T]; |X^{c,\pi}(s) - x| + |Y(s) - y| > n\}. \tag{6.7}$$

We apply Itô's formula for discontinuous semimartingales to a function v on the time interval $[t, t_n \wedge (T-\epsilon)]$, see Theorems 4.4.7 and 4.4.10 in Applebaum (2004) for details. We obtain

$$\begin{aligned}
&\mathbb{E}^{t,x,y}\Big[v(t_n \wedge (T-\epsilon), X^{c,\pi}(t_n \wedge (T-\epsilon)), Y(t_n \wedge (T-\epsilon))) - v(t,x,y)\Big] \\
&= \mathbb{E}^{t,x,y}\Big[\int_{(t,T-\epsilon]} \mathbf{1}\{t_n \geq s\} \frac{\partial v}{\partial t}(s, X^{c,\pi}(s), Y(s-))ds \\
&\quad + \int_{(t,T-\epsilon]} \mathbf{1}\{t_n \geq s\} \frac{\partial v}{\partial x}(s, X^{c,\pi}(s), Y(s-))dX^{c,\pi}(s) \\
&\quad + \int_{(t,T-\epsilon]} \mathbf{1}\{t_n \geq s\} \frac{\partial v}{\partial y}(s, X^{c,\pi}(s), Y(s-))dY(s) \\
&\quad + \frac{1}{2}\int_{(t,T-\epsilon]} \mathbf{1}\{t_n \geq s\} \frac{\partial^2 v}{\partial x^2}(s, X^{c,\pi}(s), Y(s-))\pi^2(s)(X^{c,\pi}(s))^2\sigma^2(Y(s-))ds \\
&\quad + \sum_{\triangle L(s) \neq 0, t < s \leq T-\epsilon} \mathbf{1}\{t_n \geq s\}\big(v(s, X^{c,\pi}(s), Y(s-) + \triangle L(s)) \\
&\quad - v(s, X^{c,\pi}(s), Y(s-)) - \triangle L(s)\frac{\partial v}{\partial y}(s, X^{c,\pi}(s), Y(s-))\Big] \qquad (6.8) \\
&= \mathbb{E}^{t,x,y}\Big[\int_t^{T-\epsilon} \mathbf{1}\{t_n \geq s\}\Big\{\frac{\partial v}{\partial t}(s, X^{c,\pi}(s), Y(s-)) \\
&\quad + \frac{\partial v}{\partial x}(s, X^{c,\pi}(s), Y(s-))\big(\pi(s)X^{c,\pi}(s-)(\mu(Y(s-)) - r(Y(s-))) \\
&\quad + X^{c,\pi}(s)r(Y(s-)) - c(s)\big) \\
&\quad + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(s, X^{c,\pi}(s), Y(s-))\pi^2(s)(X^{c,\pi}(s))^2\sigma^2(Y(s-)) \\
&\quad - \frac{\partial v}{\partial y}(s, X^{c,\pi}(s), Y(s-))\lambda Y(s-) \\
&\quad + \lambda\int_{z>0}\big(v(s, X^{c,\pi}(s), Y(s-) + z) - v(s, X^{c,\pi}(s), Y(s-))\big)\nu(dz)\Big\}ds \\
&\quad + \int_t^{T-\epsilon} \mathbf{1}\{t_n \geq s\}\frac{\partial v}{\partial x}(s, X^{c,\pi}(s), Y(s-))\pi(s)X^{c,\pi}(s)\sigma(Y(s-))dW(s) \\
&\quad + \int_{(t,T-\epsilon]}\int_{z>0} \mathbf{1}\{t_n \geq s\}\big(v(s, X^{c,\pi}(s), Y(s-) + z) \\
&\quad - v(s, X^{c,\pi}(s), Y(s-))\big)\tilde{N}(ds \times dz)\Big]. \qquad (6.9)
\end{aligned}$$

Note that the expectations of the stochastic integrals with respect to the Brownian motion and the compensated Poisson measure are equal to zero as they are martingales. The next

steps are rather standard. First, apply inequality (6.1) to the right hand-side of (6.9) and, then, take the limit as $n \to \infty$ and $\epsilon \to 0$. Inequality (6.4) follows from Fatou's lemma. In order to obtain equality (6.6) one should choose the strategy satisfying (6.5) and, when taking the limit, apply Lebesque's dominated convergence theorem. We refer to Theorem 3.1 in Øksendal & Sulem (2004) for more details. □

We conclude with the following theorem, which states that our solution satisfies the conditions of the verification theorem.

Theorem 6.2. *Assume that the conditions* $(A1)-(A3)$ *and* (B) *hold. Define the investment strategy*

$$\hat{\pi}(t) = \arg\max_{\pi\in[0,1]}\{\pi\big(\mu(Y(t-)) - r(Y(t-))\big) - \frac{1}{2}\pi^2(1-\gamma)\sigma^2(Y(t-))\}, \tag{6.10}$$

and the consumption rate

$$\hat{c}(t) = X^{\hat{c},\hat{\pi}}(t)(\hat{f}(t,Y(t-)))^{-\frac{1}{1-\gamma}}, \tag{6.11}$$

where the function $\hat{f}$ *is the unique solution of the fixed point equation* (4.8) *in the space* $\mathcal{C}^{1,1}([0,T)\times(0,\infty))\cap\mathcal{C}_e([0,T]\times(0,\infty))$

$$f(t,y) \;=\; \mathbb{E}\big[(1-\gamma)\int_t^T e^{\gamma\int_t^s Q(Y^{t,y}(u))du} f\big(s,Y^{t,y}(s)\big)^{-\frac{\gamma}{1-\gamma}}ds + e^{\gamma\int_t^T Q(Y^{t,y}(s))ds}\big],$$

and $X^{\hat{c},\hat{\pi}}$ *is the wealth process of the agent under* $(\hat{c},\hat{\pi})$*, defined as*

$$\begin{aligned} X^{\hat{c},\hat{\pi}}(t) \;=\;& x e^{\int_0^t\left(\hat{\pi}(s)(\mu(Y(s-))-r(Y(s-)))+r(Y(s-))-(\hat{f}(s,Y(s-)))^{-\frac{1}{1-\gamma}}\right)ds} \\ & e^{-\frac{1}{2}\int_0^t(\hat{\pi}(s))^2\sigma^2(Y(s-))ds+\int_0^t\hat{\pi}(s)\sigma(Y(s-))dW(s)}. \end{aligned} \tag{6.12}$$

Then the pair $(\hat{c},\hat{\pi})$ *is the optimal strategy for the consumption and investment problem* (3.2).

Proof. The pair $(\hat{c},\hat{\pi})$ as defined in (6.10) and (6.11) is admissible. The strategy is adapted and progressively measurable as it is left-continuous, the investment strategy $\hat{\pi}$ takes values in $[0,1]$, and the consumption rate $\hat{c}$ is positive and integrable as it is bounded $\mathbb{P}$-a.s..

Consider the stochastic differential equation (3.1) under the strategy $(\hat{c},\hat{\pi})$. Its dynamics is given by

$$\begin{aligned} dX^{\hat{c},\hat{\pi}}(t) \;=\;& \hat{\pi}(t)X^{\hat{c},\hat{\pi}}(t)\big(\mu(Y(t-))dt + \sigma(Y(t-))dW(t)\big) \\ & +\big(1-\hat{\pi}(t)\big)X^{\hat{c},\hat{\pi}}(t)r(Y(t-)dt - X^{\hat{c},\hat{\pi}}(t)(\hat{f}(t,Y(t-)))^{-\frac{1}{1-\gamma}}dt. \end{aligned} \tag{6.13}$$

The equation (6.13) has process Lipschitz coefficients, which guarantees the existence of a unique solution; see Theorem V.7 in Protter (2004). The process defined in (6.12) is the solution to the stochastic differential equation (6.13).

Our solution to the Hamilton-Jacobi-Bellman equation of the form $x^\gamma \hat{f}(t,y)$ is smooth as required. The conditions (6.1), (6.2) and (6.5) are satisfied by the method of constructing the solution. In order to show that condition (6.3) also holds, notice that in (6.3) we can replace the deterministic time T by a stopping time t_n as defined in (6.7). We find

$$\begin{aligned}
&\mathbb{E}^{0,x,y}\big[\int_0^{t_n}\int_{z>0}|v(t,X^{c,\pi}(t),Y(t-)+z)-v(t,X^{c,\pi}(t),Y(t-))|^2\nu(dz)dt\big]\\
&\leq\ \mathbb{E}^{0,x,y}\big[\int_0^{t_n}\int_{z>0}(X^{c,\pi}(t))^\gamma|\hat{f}(t,Y(t-)+z)-\hat{f}(t,Y(t-))|^2\nu(dz)dt\big]\\
&\leq\ (x+n)^\gamma\mathbb{E}^{0,y}\big[\int_0^{T}\int_{z>0}|\hat{f}(t,Y(t-)+z)-\hat{f}(t,Y(t-))|^2\nu(dz)dt\big]
\end{aligned}$$

which is finite as a consequence of (5.16).

It remains to prove uniform integrability of the value function. Uniform integrability of the negative part of the value function holds trivially. We show that

$$\mathbb{E}^{0,x,y}[v(\tau,X^{\hat{c},\hat{\pi}}(\tau),Y(\tau))^{1+\epsilon}]<\infty,$$

holds for every stopping time $\tau\leq T$ and some positive $\epsilon>0$, to be specified later.

We have to prove that the following two expectations are finite:

$$\mathbb{E}^{0,x,y}[(X^{\hat{c},\hat{\pi}}(\tau))^{2\gamma(1+\epsilon)}]\quad\text{and}\quad\mathbb{E}^{0,x,y}[(\hat{f}(\tau,Y(\tau)))^{2(1+\epsilon)}].\tag{6.14}$$

Firstly, we deal with the second expectation. By applying the estimate (4.6) we find

$$\begin{aligned}
\mathbb{E}^{0,x,y}[(\hat{f}(\tau,Y(\tau)))^{2(1+\epsilon)}]\ &\leq\ \left(1+\frac{1-\gamma}{A'}\right)^{2(1+\epsilon)}\mathbb{E}^{0,y}[e^{2(1+\epsilon)(A'(T-\tau)+B'Y(\tau))}]\\
&\leq\ \left(1+\frac{1-\gamma}{A'}\right)^{2(1+\epsilon)}e^{2(1+\epsilon)(A'T+B'y)}\mathbb{E}[e^{2(1+\epsilon)B'L(\lambda T)}],
\end{aligned}$$

which is finite, by condition (B), when choosing $\epsilon\leq\frac{\gamma}{2}$.

Consider now the process $M:=(M(t),0\leq t\leq T)$ given by

$$M(t)=e^{\int_0^{t\wedge\tau}2\gamma(1+\epsilon)\hat{\pi}(s)\sigma(Y(s-))dW(s)-\frac{1}{2}\int_0^{t\wedge\tau}4\gamma^2(1+\epsilon)^2(\hat{\pi}(s))^2\sigma^2(Y(s-))ds}.$$

By Novikov's condition M is a martingale, since

$$\begin{aligned}
&\mathbb{E}^{0,x,y}[e^{\frac{1}{2}\int_0^T4\gamma^2(1+\epsilon)^2(\hat{\pi}(s))^2\sigma^2(Y(s))ds}]\\
&\leq\ e^{2\gamma^2(1+\epsilon)^2A_\sigma T}\mathbb{E}^{0,y}[e^{2\gamma^2(1+\epsilon)^2B_\sigma\int_0^T Y(s)ds}]\\
&\leq\ e^{2\gamma^2(1+\epsilon)^2A_\sigma T+2\gamma(1+\epsilon)^2B'_\sigma y}\mathbb{E}[e^{2\gamma(1+\epsilon)^2B'_\sigma L(\lambda T)}],
\end{aligned}$$

which is finite, when choosing $\epsilon > 0$ such that $\gamma(1+\epsilon)^2 \leq 1 + \frac{\gamma}{2}$.

Define a measure $\tilde{\mathbb{P}}$ by the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\big|\mathcal{F}_T = M(T).$$

The measure $\tilde{\mathbb{P}}$ is a probability measure, equivalent to $\mathbb{P}$, and the dynamics of the Lévy process L under this measure remains unchanged by the independence of L and W and Girsanov's theorem; see Chapter 1.4 in Øksendal & Sulem (2005).

Denote by $\tilde{\mathbb{E}}$ the expectation taken with respect to the measure $\tilde{\mathbb{P}}$. The finiteness of the first expectation in (6.14) can now be proved in the following way. We calculate

$$\begin{aligned}
&\mathbb{E}^{0,x,y}[(X^{\hat{c},\hat{\pi}}(\tau))^{2\gamma(1+\epsilon)}] \\
&= x^{2\gamma(1+\epsilon)}\mathbb{E}^{0,y}\Bigg[\exp\Bigg\{\int_0^\tau 2\gamma(1+\epsilon)\big(\hat{\pi}(s)\big(\mu(Y(s)) - r(Y(s))\big) + r(Y(s)) \\
&\quad -(\hat{f}(s,Y(s)))^{-\frac{1}{1-\gamma}}\big)ds - \frac{1}{2}\int_0^\tau 2\gamma(1+\epsilon)(\hat{\pi}(s))^2\sigma^2(Y(s))ds \\
&\quad + \int_0^\tau 2\gamma(1+\epsilon)\hat{\pi}(s)\sigma(Y(s))dW(s)\Bigg\}\Bigg] \\
&\leq x^{2\gamma(1+\epsilon)}\mathbb{E}^{0,y}\Bigg[\exp\Bigg\{\int_0^\tau 2\gamma(1+\epsilon)\big(\hat{\pi}(s)(\mu(Y(s)) - r(Y(s))) + r(Y(s)) \\
&\quad -\frac{1}{2}(\hat{\pi}(s))^2(1-\gamma)\sigma^2(Y(s-))\big)ds - \gamma^2(1+\epsilon)\int_0^\tau (\hat{\pi}(s))^2\sigma^2(Y(s))ds \\
&\quad + \int_0^\tau 2\gamma(1+\epsilon)\hat{\pi}(s)\sigma(Y(s))dW(s) - \frac{1}{2}\int_0^\tau 4\gamma^2(1+\epsilon)^2(\hat{\pi}(s))^2\sigma^2(Y(s))ds \\
&\quad + \int_0^\tau 2\gamma^2(1+\epsilon)^2(\hat{\pi}(s))^2\sigma^2(Y(s))ds\Bigg\}\Bigg] \\
&= x^{2\gamma(1+\epsilon)}\mathbb{E}^{0,y}\big[M(T)e^{2\gamma(1+\epsilon)\int_0^\tau Q(Y(s))ds + \gamma^2(1+\epsilon)(1+2\epsilon)\int_0^\tau(\hat{\pi}(s))^2\sigma^2(Y(s))ds}\big] \\
&= x^{2\gamma(1+\epsilon)}\tilde{\mathbb{E}}^{0,y}\big[e^{2\gamma(1+\epsilon)\int_0^\tau Q(Y(s))ds + \gamma^2(1+\epsilon)(1+2\epsilon)\int_0^\tau(\hat{\pi}(s))^2\sigma^2(Y(s))ds}\big] \\
&\leq x^{2\gamma(1+\epsilon)}e^{2\gamma(1+\epsilon)AT + \gamma^2(1+\epsilon)(1+2\epsilon)A_\sigma T}\tilde{\mathbb{E}}^{0,y}\big[e^{\left(2\gamma(1+\epsilon)B + \gamma^2(1+\epsilon)(1+2\epsilon)B_\sigma\right)\int_0^T Y(s)ds}\big] \\
&\leq x^{2\gamma(1+\epsilon)}e^{2\gamma(1+\epsilon)AT + \gamma^2(1+\epsilon)(1+2\epsilon)A_\sigma T}\tilde{\mathbb{E}}\big[e^{\left(2(1+\epsilon)B' + \gamma(1+\epsilon)(1+2\epsilon)B'_\sigma\right)(y+L(\lambda T))}\big] \\
&\leq K\tilde{\mathbb{E}}\big[e^{(B'\vee B'_\sigma)\left(2(1+\epsilon)+\gamma(1+\epsilon)(1+2\epsilon)\right)L(\lambda T)}\big] \\
&= K\tilde{\mathbb{E}}\big[e^{\left(2(B'\vee B'_\sigma)(1+\frac{\gamma}{2})+(B'\vee B'_\sigma)(2\gamma\epsilon^2+3\gamma\epsilon+2\epsilon)\right)L(\lambda T)}\big],
\end{aligned}$$

which is finite by condition (B), when choosing $\epsilon > 0$ such that $(B'\vee B'_\sigma)(2\gamma\epsilon^2+3\gamma\epsilon+2\epsilon) \leq \varepsilon$. This completes the proof. □

We would like to point out that we have proved optimality of the strategy under a weaker integrability assumption concerning the Lévy measure ν than Benth *et al.* (2003).

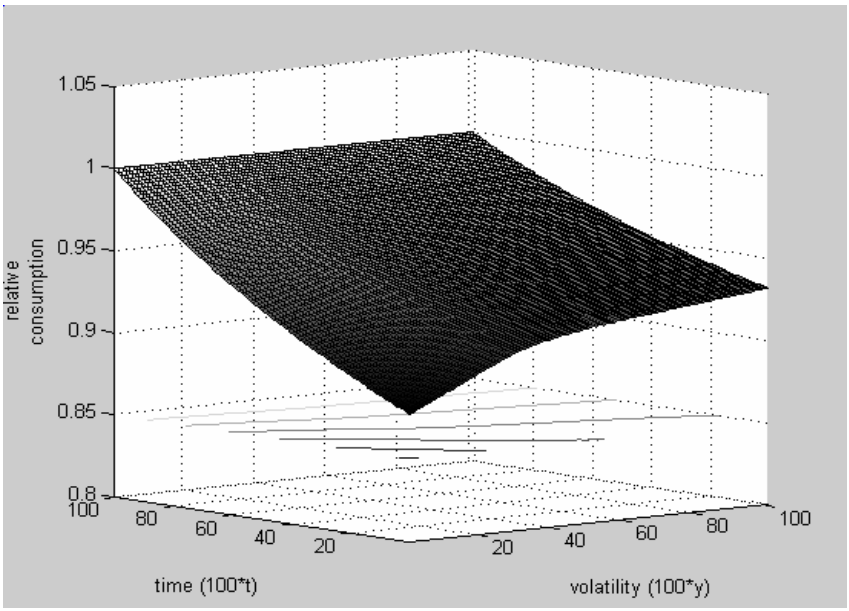

Figure 1: The optimal relative consumption rate as the function of time to maturity and the volatility level; see also the text in Example 6.3.

Example 6.3. We consider a financial model of Barndorff-Nielsen & Shephard type as in (2.4). Choose $T = 1, \gamma = 0.75, \lambda = 1/6, \mu(y) = 0.1 - 0.167y, \sigma^2(y) = y$ and $r(y) = 0$. Let the subordinator L be a compound Poisson process with jumps intensity 0.5 and exponentially distributed jump sizes with expectation $1/15$. We set the initial volatility level to $Y(0) = 0.2$. We want to remark that the expected long-term volatility equals 0.2 as well. As noted in Benth *et al.* (2003), in the considered model, the marginal distribution of the log returns is left skewed.

We have solved the nonlinear partial integro-differential equation (5.18) numerically by applying an explicit finite difference method. As we are dealing with a first-order integro-differential equation and the Lévy measure is finite, the explicit scheme is more efficient than the implicit scheme, see Chapter 12.4 Cont & Tankov (2005) for details. We point out that the finite difference method has been applied to the transformed equation to which the solution is $\hat{f}(t, y)e^{-\kappa y}$. The exponential scaling has been done in order to assign conditions at the boundaries of the bounded domain. The parameter κ should be chosen sufficiently large so that $\lim_{y\to\infty} \hat{f}(t, y)e^{-\kappa y} = 0$ holds.

Based on (3.8), see also Benth *et al* (2003), we can state that the optimal investment

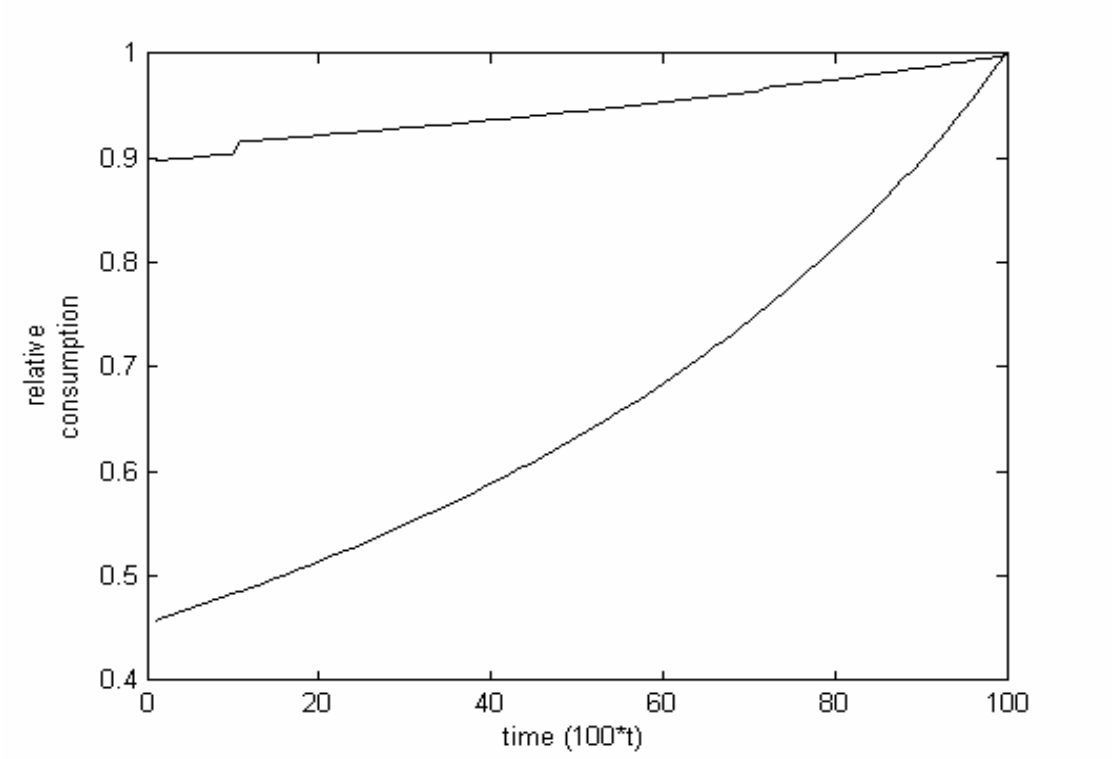

Figure 2: The optimal relative consumption rate in the stochastic volatility model (upper curve) and in the model with constant volatility (lower curve); see also the text in Example 6.3.

strategy is given by

$$\hat{\pi}(y) = \begin{cases} 1 & y < 0.24, \\ \dfrac{0.1 - 0.167y}{0.25y} & y \in [0.24, 0.6), \\ 0 & y \geq 0.6, \end{cases}$$

whereas the optimal relative consumption rate is given in Figure 1.

The first observation, which is common in optimal investment-consumption models, is that the optimal consumption rate is an increasing function of the time t. In our model it is interesting to note that the optimal consumption rate is an increasing function of the volatility level. The result agrees with our intuition as the higher level of the volatility leads to a lower return and to a higher variability, so an investor should consume more in order not to lose wealth.

It has already been stated in Benth *et al.* (2003) that stochastic volatility modelling can change the investment strategy significantly. Notice that the long-term expected volatility level 0.2 corresponds to holding all wealth in the risky asset. We have simulated one path: the volatility jumped at $t = 0.1$ by 0.14 and at $t = 0.7$ by 0.08. For this sample path the

investment strategy is indeed different, since from time $t = 0.1$ it is optimal to invest in the risky asset strictly less than 100% of the wealth; see also the simulation results in Benth *et al.* (2003). The optimal allocation in the risky asset after the first jump decreases to 52%, then increases up to 65% and after the second jump decreases immediately to 38%. The optimal consumption pattern is also significantly different, when comparing with the case of a constant volatility; see Figure 2. Under the stochastic volatility model it is optimal to consume much higher proportions of the wealth as the jump in the volatility may cause a decrease in the portfolio value. Notice at $t = 0.1$ the discontinuity in the consumption strategy, occurring in the upper curve in Figure 2, caused the jump in the volatility. The second discontinuity at $t = 0.7$ is much less visible. □

Finally, we would like to present the solution for the optimization problem in the case of a logaritheoremic utility. Logaritheoremic utility is investigated in depths in Goll & Kallsen (2000), where a general financial market is considered, consisting of stocks, whose prices are driven by semimartingales. The solution is stated in terms of the semimartingales characteristics, which are not straightforward to find in our model.

We make an ansatz with a value function of the form $v(t, x, y) = g(t) \log x + h(t, y)$. This yields the following equations

$$0 = \frac{dg(t)}{dt} + 1, \quad g(T) = 1, \tag{6.15}$$

and

$$\begin{aligned} 0 \;=\; & \frac{\partial h}{\partial t}(t,y) - \frac{\partial h}{\partial y}(t,y)\lambda y + \lambda \int_{z>0} \big(h(t, y+z) - h(t,y)\big)\nu(dz) \\ & + g(t)Q^0(y) - \log g(t) - 1 = 0, \qquad h(T,y) = 0, \end{aligned} \tag{6.16}$$

where

$$Q^0(y) = \max_{\pi \in [0,1]} \{\pi(\mu(y) - r(y)) - \frac{1}{2}\pi^2\sigma^2(y)\} + r(y)$$

is the analogue of (3.9).

We can prove the following theorem.

Theorem 6.4. *Assume that the conditions (A1)-(A3) hold and that the Lévy measure ν of L satisfies*

(C) $\int_{z>1} z^{1+\varepsilon}\nu(dz) < \infty$, *equivalently* $\mathbb{E}[L(1)^{1+\varepsilon}] < \infty$, *for some* $\varepsilon > 0$.

Define the investment strategy

$$\hat{\pi}(t) = \arg\max_{\pi\in[0,1]}\{\pi\big(\mu(Y(t-)) - r(Y(t-))\big) - \frac{1}{2}\pi^2\sigma^2(Y(t-))\},$$

and the consumption rate

$$\hat{c}(t) = \frac{X^{\hat{c},\hat{\pi}}(t)}{1+T-t},$$

where $X^{\hat{c},\hat{\pi}}$ is the wealth process of the agent under $(\hat{c},\hat{\pi})$, defined as

$$\begin{aligned} X^{\hat{c},\hat{\pi}}(t) &= xe^{\int_0^t \left(\hat{\pi}(s)(\mu(Y(s-))-r(Y(s-)))+r(Y(s-))-\frac{1}{1+T-s}\right)ds} \\ &\quad e^{-\frac{1}{2}\int_0^t(\hat{\pi}(s))^2\sigma^2(Y(s-))ds+\int_0^t\hat{\pi}(s)\sigma(Y(s-))dW(s)}. \end{aligned}$$

Then the pair $(\hat{c},\hat{\pi})$ is the optimal strategy for the consumption and investment problem under a logaritheoremic utility function.

As we are facing the linear equation (6.16), existence and smoothness of a solution can be proved much more easily by combining the results from this paper and from Benth *et al.* (2003). It is well known, see Merton (1971), that the optimal relative consumption rate in the case of a logaritheoremic utility does not depend on the financial coefficients.

7 Conclusions

In this paper we have solved a consumption and investment problem for an agent, who invests in a Black-Scholes market with stochastic coefficients driven by a non-Gaussian Ornstein-Uhlenbeck process. We have proved that the candidate value function is the classical solution of the corresponding Hamilton-Jacobi-Bellman equation. In particular, we have provided a classical solution to a non-linear first-order partial integro-differential equation.

The optimal investment strategy has been explicitly calculated, while the optimal consumption rate depends on the function, which solves the partial integro-differential equation. The conclusion from the simulation study is that under stochastic volatility the optimal consumption strategy is significantly different when comparing with the case of a constant volatility.

In Benth *et al.* (2003) a multivariate Ornstein-Uhlenbeck process driven by independent subordinators was considered, while in Lindberg (2006) a financial market consisting of n stocks was investigated. We would like to point out that our results can be extended to both settings.

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References

[1] Applebaum, D. (2004) *Lévy Processes and Stochastic Calculus.* Cambridge University Press, Cambridge.

[2] Bensoussan, A. and Lions, J.L. (1984) *Impulse Control and Quasi-variational Inequalities.* Bordas, Paris.

[3] Bäuerle, N. and Rieder, U. (2005) Portfolio optimization with unobservable Markov modulated drift process. *J. Appl. Prob.* **2**, 362-378.

[4] Becherer, D. and Schweizer, M. (2005) Classical solutions to reaction-diffusion systems for hedging problems with interacting Itó and point processes. *Ann. Appl. Prob.* **15**, 1111-1155.

[5] Benth, F., Karlsen, K. and Reikvam, K. (2003) Merton's portfolio optimization problem in Black-Scholes market with non-Gaussian stochastic volatility of Ornstein-Uhlenbeck type. *Math. Finance* **13**, 215-244.

[6] Barndorff-Nielsen, O.E. and Shephard, N. (2001) Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial mathematics. *J. Royal Stat. Soc.* **B 63**, 167-241.

[7] Bertoin, J. (1996) *Lévy Processes.* Cambridge University Press, Cambridge.

[8] Castañeda-Leyva, N. and Hernández-Hernández, D. (2005) Optimal consumption investment problems in incomplete markets with stochastic coefficient. *SIAM J. Control Opt.* **44**, 1322-1344.

[9] Cont, R. and Tankov, P. (2004) *Financial Modelling with Jump Processes.* Chapman and Hall/CRC Press.

[10] Delong, Ł. (2007) Optimal investment and consumption in the presence of default in a financial market driven by a Lévy process. *Ann. UMCS Mathematica*, to appear.

[11] Fleming, W. and Hernández-Hernández, D. (2003) An optimal consumption model with stochastic volatility. *Finance and Stochastics* **7**, 245-262.

[12] Fleming, W. and Rishel, R.W. (1975) *Deterministic and Stochastic Optimal Control.* Springer, New York.

[13] Goll, T. and Kallsen, J. (2000) Optimal portfolios for logarithmic utility. *Stoch. Proc. Appl.* **89**, 31-48.

[14] Hernández-Hernández, D. and Schied, A. (2006) Robust utility maximization in a stochastic factor model. Preprint.

[15] Kraft, H. and Steffensen, M. (2006) Portfolio problems stopping at first hitting time with applications to default risk. *Math. Meth. Oper. Res.* **63**, 123-150.

[16] Lindberg, C. (2006) News-generated dependency amd optimal portfolios for n stocks in a market of Barndorff-Nielsen and Shephard type. *Math. Finance 16*, 549-568.

[17] Merton, R. (1971) Optimal consumption and portfolio rules in continuous time model. *J. Econ. Theory* **3**, 373-413.

[18] Øksendal, B,. and Sulem, A. (2005) *Applied Stochastic Control of Jump-Diffusions.* Springer, Berlin.

[19] Pham, H. (1998) Optimal stopping of controlled jump-diffusions: a viscosity solution approach. *J. Math. Sys. Estim. and Control* **1**, 1-27.

[20] Pham, H. and Quenez, M.C. (2001) Optimal portfolio in partially observed stochastic volatility models. *Ann. Appl Prob.* **11**, 210-238.

[21] Protter, P. (2004) *Stochastic Integration and Differential Equations.* Springer, New York.

[22] Sato, K.-I. (1999) *Lévy Processes and Infinite Divisibility* Cambridge University Press, Cambridge.

[23] Zariphopoulou, T. (2001) A solution approach to valuation with unhedgeable risk. *Finance and Stochastics* **5**, 61-82.