Time Consistency of Multi-Period Distortion Measures

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Abstract

Dynamic risk measures play an important role for the acceptance or non-acceptance of risks in a bank portfolio. Dynamic consistency and weaker versions like conditional and sequential consistency guarantee that acceptability decisions remain consistent in time. An important set of static risk measures are so-called distortion measures. We extend these risk measures to a dynamic setting within the framework of the notions of consistency as above. As a prominent example, we present the Tail Value-at-Risk (TVaR).

Keywords: acceptability measure, conditional consistency, coherence, distortion measure, dynamic consistency, risk measure, sequential consistency, time-consistency, Tail-Value-at-Risk.

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1 Introduction

Risk measures are used by risk managers and regulators to calculate the risk capital of a company, that is the amount of capital which has to be safely invested to compensate for the risk of holding assets and liabilities. Risk modeling is required under the supervisory frameworks of Basel III for banks, and Solvency II for insurances. An influential approach to measure risk was the definition of the axiomatic system of *coherent risk measures* by Artzner et al. (1999). One of the most popular risk measures is the Value-at-Risk (VaR), which is unfortunately not coherent; in general, it lacks subadditivity. A natural extension of the VaR are *distortion measures*, which are again coherent and contain the popular Tail-Value-at-Risk (TVaR) also known as Expected Shortfall used in the Swiss Solvency Test for the determination of the so called target capital. Distortion measures were introduced by Denneberg (1990, 1994) and Wang et al. (1997), respectively. They are essentially the same as the spectral risk measures of Wang (1996) and Acerbi (2002).

In the last ten years, starting with the work of Wang (1996) and Wang (1999), dynamic risk measures have become more and more important, because there is a need to update the risk capital if new information is available such that the risk capital can be computed for more than one moment in time. This is necessary, in particular, if the risk is measured over longer time horizons. Dynamic risk measures, which are an extension of static (one-period) risk measures, calculate the risk at every time step until a terminal time T, taking into account the information available at that time.

Similarly to the one-period case, where static risk measures satisfy some axioms as coherence and convexity, respectively, there is also an axiomatic system for dynamic risk measures. Besides the discussion on convex and coherent dynamic risk measures as in the static case, the most important axiom for dynamic risk measures is *consistency*, which means that the acceptability of a risk shall be consistent in some way over time. The most popular definition of consistency is *dynamic consistency* sometimes called *time-consistency*. Dynamically consistent coherent risk measures in discrete time have been discussed by Roorda et al. (2005), Roorda and Schumacher (2007) and Artzner et al. (2002, 2007); dynamically consistent convex risk measures are studied in Detlefsen and Scandolo (2005), Cheridito and Kupper (2006), Pflug and Römisch (2007) and Jobert and Rogers (2008) to name only a few; see Acciaio and Penner (2011) for an overview. Stadje (2010) extends static convex risk measures from a particular discrete time market to continuous time and shows that for coherent risk measures on a large time horizon scaling is necessary.

Not to confuse the notation, the expression "dynamic" is also used for risk measures of processes, which describe random cashflows and evaluate processes at time 0, e.g., Cheridito et al. (2004, 2005, 2006). In particular, Cherny and Madan (2009) define performance measures satisfying a set of axioms by distortion measures and apply them in Madan and Cherny (2010) to model the cone of marked cash flows of traders and providers of liquid assets. This work is extended in Madan et al. (2010) to construct dynamically consistent bid and ask price sequences by Markov chains. However, in the present paper we investigate dynamic coherent risk measures in a discrete time set-up.

Dynamic consistency is a strong assumption. There exist several examples where static risk measures cannot be extended (*updated*) to dynamic risk measures in a consistent way. For example

Kupper and Schachermayer (2009) show that the only law-invariant dynamically consistent and relevant risk measures are the dynamic entropic risk measures (cf. also Example 3.6 and 3.7 in Schied (2007)). Hence, also weaker notions of consistency are necessary. Various alternative definitions of consistency have been proposed in the literature; see for instance Tutsch (2006), Roorda and Schumacher (2007), Penner (2007).

In this paper we also use the weaker consistency axioms of *conditional* and *sequential consistency* as introduced in Roorda and Schumacher (2007). To the best of our knowledge there is not much literature about conditionally consistent risk measures except for Roorda and Schumacher (2007, 2010). Roorda and Schumacher (2010) deduce that coherent risk measures can always be updated in a conditionally consistent way. However, the VaR in general does not allow for a conditionally consistent update. In contrast sequential consistency corresponds to the notion of *weak time consistency* in Burgert (2005) and Föllmer and Penner (2006), a combination of *acceptance and rejection consistency* in Weber (2006), and *weak acceptance consistency* in Tutsch (2006), where updating is always possible. Dynamic consistency is the strongest version of the consistency conditions investigated in this paper implying sequential consistency and conditional consistency. Conditional consistency can also be deduced from sequential consistency under some mild assumptions.

In the context of dynamic risk measurement as studied here it is more convenient and established to work with *acceptability measures* (*risk adjusted valuations*) ϕ of financial positions instead of risk measures ρ, which are negative risk measures, i.e. $\phi(X) = -\rho(X)$. Then $\phi(X)$ describes the maximum amount of money, which can be subtracted from the current position keeping it acceptable.

The paper is structured in the following way. First, we start with preliminaries on single and multi-period acceptability measures and distortion measures in Section 2. In particular, this includes the different definitions of time-consistency for coherent multi-period acceptability measures. To prepare the ground for the new results to be presented in Section 3 we show examples to illustrate the different notions, their limitations and differences. Section 3 contains the main results of this paper. We derive conditionally, sequentially and dynamically consistent versions of multi-period distortion measures and present representation theorems with global test sets (sets of probability measures). In particular, the TVaR is a special case. Both, our conditionally and dynamically consistent versions are more conservative than the sequential version. However, it is not possible to compare the conditionally consistent and dynamically consistent multi-period acceptance measures in general. For different distortion measures we present examples of financial positions, which are acceptable in the conditional (static) case, but not in the dynamic case and vice versa. Finally, the Appendix contains some proofs.

Throughout the paper we use the following notation. Let $T \in \mathbb{N}$ denote a finite time horizon. All financial positions are defined on the probability space $(S^T, \mathcal{P}(S^T), \mathbb{P})$, where S is a finite set and $\mathcal{P}(S^T)$ is the power set on S^T. We assume that any scenario $\omega \in S^T$ has a positive probability, i.e., $\mathbb{P}(\omega) := \mathbb{P}(\{\omega\}) > 0$. Then the set of financial positions $\mathcal{X}$ is the collection of all random variables on $(S^T, \mathcal{P}(S^T), \mathbb{P})$. For $X \in \mathcal{X}$ we interpret $X(\omega)$ as the discounted net worth of a position at the end of the holding period T, if scenario $\omega \in S^T$ happens. Note that for convenience elements of S^T will always be denoted by ω, whereas for $t \in \{0, \dots, T\}$ elements of S^t will be denoted by ω_t reflecting the information available until t with $S^0 := \{0\}$. Finally, $\mathfrak{P}$ denotes the collection of all probability measures on $(S^T, \mathcal{P}(S^T))$ and $\mathfrak{P}_S$ is the collection of all probability measures on $(S, \mathcal{P}(S))$.

2 Preliminaries

2.1 Single-period acceptability measures

First, we give a short introduction into single-period acceptability measures. We deal with future values of financial positions as random variables defined on $(S, \mathcal{P}(S))$ where $T = 1$.

Definition 2.1 *A map $\phi : \mathcal{X} \to [-\infty, \infty)$ is called a* coherent acceptability measure*, if the following conditions are satisfied:*

(i) Translation-invariance: For any $X \in \mathcal{X}$ and $\eta \in \mathbb{R}$ we have $\phi(X + \eta) = \phi(X) + \eta$.

(ii) Monotonicity: If $X,\ Y \in \mathcal{X}$ with $X(\omega) \leq Y(\omega)\ \forall\ \omega \in S$ then $\phi(X) \leq \phi(Y)$.

(iii) Superadditivity: For any $X,\ Y \in \mathcal{X}$ we have $\phi(X + Y) \geq \phi(X) + \phi(Y)$.

(iv) Positive homogeneity: For any $X \in \mathcal{X}$ and $\lambda \geq 0$ we have $\phi(\lambda X) = \lambda\phi(X)$.

A position $X \in \mathcal{X}$ is called acceptable *if $\phi(X) \geq 0$.*

Translation-invariance gives the interpretation of $\phi(X)$ as capital reserve. It is also called cash-invariance. Monotonicity postulates that if a position X pays not more than Y, then Y should be considered at least as valuable as X. The positive homogeneity and superadditivity axioms may be relaxed to a concavity axiom (cf. Föllmer and Schied (2004)). We do not consider this generalization here. A general motivation of the above axiomatic system is provided by Artzner et al. (1999).

In our setting, a coherent acceptability measure has the following representation.

Proposition 2.2 (Artzner et al. (1999), Proposition 4.1) *A map $\phi : \mathcal{X} \mapsto (-\infty, \infty]$ is a coherent acceptability measure if and only if there exists a set of probability measures $\mathfrak{Q} \subseteq \mathfrak{P}$ such that*

$$\phi(X) = \min_{\mathbb{Q} \in \mathfrak{Q}} \mathbb{E}_{\mathbb{Q}}(X) \quad \text{for } X \in \mathcal{X}.$$

2.2 Distortion measures

Single-period distortion measures are a subclass of coherent acceptability measures.

Definition 2.3
(a) An increasing concave function $\psi : [0,1] \to [0,1]$ such that $\psi(0) = 0$, $\psi(1) = 1$ and $\lim_{\varepsilon \to 0} \psi(\varepsilon) = 0$ is called distortion function.
(b) Let ψ be a distortion function. Define

$$\mathfrak{Q}_\psi = \{\mathbb{Q} \in \mathfrak{P} : \mathbb{Q}(B) \leq \psi(\mathbb{P}(B))\ \forall\ B \subseteq S\}. \tag{2.1}$$

Then the induced distortion measure ϕ_ψ *is defined as*

$$\phi_\psi(X) = \min_{\mathbb{Q}\in\mathfrak{Q}_\psi} \mathbb{E}_\mathbb{Q}(X) \quad \textit{for } X \in \mathcal{X}.$$

Remark 2.4

(a) A direct conclusion from Proposition 2.2 is that ϕ_ψ is a coherent acceptability measure.

(b) The definition of ϕ_ψ is equivalent to

$$\phi_\psi(X) = \int_0^\infty (1 - \psi(\mathbb{P}(X < x)))\, dx - \int_{-\infty}^0 \psi(\mathbb{P}(X < x))\, dx \quad \text{for } X \in \mathcal{X};$$

see Föllmer and Schied (2004), Theorem 4.88.

(c) The TVaR_λ is a typical example for a distortion measure with $\psi(z) = \min(z/\lambda, 1)$ (see Föllmer and Schied (2004), Example 4.65). Also the minmaxVaR with $\psi(z) = 1 - (1 - z^{\frac{1}{1+\gamma}})^{1+\gamma}$ for some $\gamma > 0$ used in conic finance (cf. Madan and Cherny (2010)) belongs to that class. Further examples are given in Example 3.9 below. □

Properties of distortion measures on non-atomic probability spaces, e.g., coherence, can be found in Föllmer and Schied (2004), Chapter 4.6, and on atomic probability spaces in Denneberg (1994).

The objective is to derive a formula for the explicit calculation of the distortion measure ϕ_ψ. To this end, we fix a position $X \in \mathcal{X}$. Obviously, for any $\mathbb{Q} \in \mathfrak{P}$ the expected value of X with respect to $\mathbb{Q}$ is uniquely determined by the distribution of X with respect to $\mathbb{Q}$, denoted as $\mathbb{Q}_X$, which is a probability measure on the measurable space $(\mathfrak{X}, \mathcal{P}(\mathfrak{X}))$ with $\mathfrak{X} = \{X(\omega) : \omega \in S\}$, i.e. $\mathbb{E}_\mathbb{Q}(X) = \int_\mathfrak{X} x\, \mathbb{Q}_X(dx)$.

Proposition 2.5 *Let $X \in \mathcal{X}$. Then the probability measure $\mathbb{Q}^*_X$ defined as*

$$\mathbb{Q}^*_X(x) := \psi(\mathbb{P}(X \le x)) - \psi(\mathbb{P}(X < x)) \qquad \textit{for } x \in \mathfrak{X},$$

is an element of $\mathfrak{Q}_{\psi,X} := \{\mathbb{Q}_X \in \mathfrak{P}_\mathfrak{X} :\ \mathbb{Q}_X(B) \le \psi(\mathbb{P}_X(B))\ \forall\ B \subseteq \mathfrak{X}\}$ and

$$\mathbb{E}_{\mathbb{Q}^*}(X) = \min_{\{\mathbb{Q}\in\mathfrak{P}:\, \mathbb{Q}_X\in\mathfrak{Q}_{\psi,X}\}} \mathbb{E}_\mathbb{Q}(X).$$

For the proof of Proposition 2.5 the reader is referred to the Appendix. A similar theorem is given in Carlier and Dana (2003) and Föllmer and Schied (2004), Corollary 4.74, for non-atomic probability spaces.

At this point, we want to illustrate the above theorem by the TVaR. We will come back to this example in Section 3, where it will be the leading example.

Example 2.6 (Tail Value-at-Risk (TVaR)) Assume that $S = \{u_1, u_2, u_3, u_4\}$ and for $\omega \in S$,

$$\begin{aligned} \mathbb{P}(\omega) &= 0.552\ \delta_{u_1}(\omega) + 0.028\ \delta_{u_2}(\omega) + 0.4\ \delta_{u_3}(\omega) + 0.02\ \delta_{u_4}(\omega), \\ X(\omega) &= 1000\ \delta_{u_1}(\omega) + 100\ \delta_{u_2}(\omega) + 100\ \delta_{u_3}(\omega) - 100\ \delta_{u_4}(\omega). \end{aligned}$$

Moreover, let the distortion function be $\psi(z) = \min(z/0.05, 1)$ which is the distortion function of the $\text{TVaR}_{0.05}$. To be able to apply Proposition 2.5, note that $\mathfrak{X} = \{-100,\ 100,\ 1000\}$ and

$$\mathbb{P}_X(x) = 0.02\ \delta_{-100}(x) + 0.428\ \delta_{100}(x) + 0.552\ \delta_{1000}(x) \quad \text{for } x \in \mathfrak{X}.$$

Then, we obtain

$$\mathbb{Q}^*_X(x) = 0.4\ \delta_{-100}(x) + 0.6\ \delta_{100}(x) \quad \text{for } x \in \mathfrak{X},$$

and hence, it follows that

$$\text{TVaR}_{0.05}(X) = 0.4 \cdot (-100) + 0.6 \cdot 100 = 2 > 0\,.$$

By Definition 2.1 it follows that X is acceptable. $\square$

2.3 Multi-period acceptability measures

In this section we give an overview of the main results on multi-period acceptability measures, which are required for the construction of multi-period extensions of distortion measures. Single-period acceptability measures have the disadvantage that they cannot take into account new information which arrives over time. Multi-period acceptability measures consider this shortcoming by measuring the risk at every time step $\{0, \dots, T\}$ over the time horizon from 0 to T.

Throughout the paper we need some further notation. For $\omega \in S^T$ and $t \in \{0, \dots, T\}$ let $\omega|_t$ be the temporal restriction of ω, i.e. the sequence of the first t elements of ω reflecting the information available at time t. The set of scenarios starting with $\omega_t \in S^t$ is defined as $F(\omega_t) = \{\omega \in S^T : \omega|_t = \omega_t\}$, whereas $F(\omega_0) = F(0) = S^T$. This means that $F(\omega_t)$ is the set of evolutions of the state of the world until time T if we have at time t the state ω_t. Moreover, we denote by $\mathcal{X}(F(\omega_t))$ the collection of all random variables on $(F(\omega_t), \mathcal{P}(F(\omega_t)))$ with $\mathcal{X}(F(\omega_0)) = \mathcal{X}$, which are the possible outcomes if we start in t with information ω_t, and we define the σ-algebra $\mathcal{F}_t = \sigma(F(\omega_t) : \omega_t \in S^t)$ for $t \in \{0, \dots, T\}$. Finally, for $\omega_t \in S^t$ and $\alpha \in S$ we have $(\omega_t, \alpha) \in S^{t+1}$. For simplicity we write $F(\omega_t, \alpha)$ instead of $F((\omega_t, \alpha))$.

Definition 2.7 *A coherent multi-period acceptability measure ϕ consists of a sequence of mappings $(\phi_t)_{t\in\{0,\dots,T\}}$ where $\phi_t : \mathcal{X} \times S^t \to [-\infty, \infty)$ and $(X, \omega_t) \mapsto \phi_t(X, \omega_t)$ such that for any $t \in \{0, \dots, T\}$ and $\omega_t \in S^t$ the following holds:*

(i) $\phi_t(\cdot, \omega_t)$ is a coherent acceptability measure on $\mathcal{X}(F(\omega_t))$,

(ii) $\phi_t(X, \omega_t) = \phi_t(X\mathbb{1}_{F(\omega_t)}, \omega_t)\ \forall\ X \in \mathcal{X}$ (soundness property),

(iii) if $X \leq 0$ and $X\mathbb{1}_{F(\omega_t)} \neq 0$ then $\phi_t(X, \omega_t) < 0$ (relevance property).

One interprets $\phi_t(X, \omega_t)$ as the risk assessment at date t under the information ω_t for the holding period T. Definition 2.7 immediately implies that $\phi_T(X, \omega) = X(\omega)$ for all $\omega \in S^T$. Hence, it is sufficient to define multi-period acceptability measures for $t \in \{0, \dots, T-1\}$.

In the definition of coherent multi-period acceptability measures, the single period acceptability measures $\phi_t(\cdot, \omega|_t)$, $t \in \{0, \ldots, T\}$, are not related over time and thus, "consistency over time" makes no sense. The idea behind the notion of time consistency is now that risk-adjusted values shall not contradict one another across time. We will use the following three kinds of time consistency.

Definition 2.8 *Let ϕ be a coherent multi-period acceptability measure.*

(i) If for any $t \in \{0, \ldots, T\}$, $\omega_t \in S^t$ and $X \in \mathcal{X}$ the following holds:

$$\phi_t(X, \omega_t) \geq 0 \quad \Longleftrightarrow \quad \phi_0(X \mathbb{1}_{F(\omega_t)}, 0) \geq 0.$$

Then we call ϕ conditionally consistent.

(ii) ϕ is called sequentially consistent *if the following both conditions are satisfied:*

- *For any $t \in \{0, \ldots, T\}$, $\omega_t \in S^t$ and $X \in \mathcal{X}$ with $\phi_t(X, \omega_t) \geq 0$ there exist $\alpha_{t+1}, \ldots, \alpha_T \in S$ such that $\phi_{t+s}(X, (\omega_t, \alpha_{t+1}, \ldots, \alpha_{t+s})) \geq 0$ for any $s \in \{1, \ldots, T-t\}$.*
- *For any $t \in \{0, \ldots, T\}$, $\omega_t \in S^t$ and $X \in \mathcal{X}$ with $\phi_t(X, \omega_t) \leq 0$ there exist $\alpha_{t+1}, \ldots, \alpha_T \in S$ such that $\phi_{t+s}(X, (\omega_t, \alpha_{t+1}, \ldots, \alpha_{t+s})) \leq 0$ for any $s \in \{1, \ldots, T-t\}$.*

(iii) If for any $t \in \{0, \ldots, T\}$, $\omega_t \in S^t$ and $X,\ Y \in \mathcal{X}$ with $\phi_{t+1}(X, (\omega_t, \alpha)) = \phi_{t+1}(Y, (\omega_t, \alpha))$ for every $\alpha \in S$ we can conclude that $\phi_t(X, \omega_t) = \phi_t(Y, \omega_t)$, then we call ϕ dynamically consistent.

This definition of dynamic consistency was introduced in Roorda et al. (2005), whereas conditional and sequential consistency were introduced in Roorda and Schumacher (2007). Note that dynamic consistency implies conditional and sequential consistency (see Roorda and Schumacher (2007), Theorem 5.1). Moreover, under the additional condition of strong relevance, sequential consistency implies conditional consistency (see Roorda and Schumacher (2007), Theorem 5.1).

The intuition behind the three notions of consistency goes as follows:

(i) Let $t \in \{0, \ldots, T\}$, $\omega_t \in S^t$ and $X \in \mathcal{X}$. Define $\widetilde{X} = X \mathbb{1}_{F(\omega_t)}$. Then *conditional consistency* says that the risk $\widetilde{X}$ at the initial time point $t = 0$ without any information is acceptable if and only if X at time t with information ω_t is acceptable. Conditional consistency implies also the lower bound $\min_{\alpha \in S} \phi_t(X, (\omega, \alpha))$ for $\phi_t(X, \omega)$ (cf. Roorda and Schumacher (2007), Lemma 4.4).

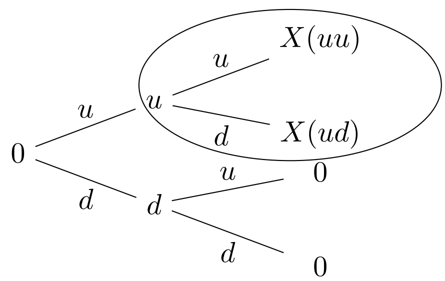

Figure 1: *The figure shows a two-periodic binomial tree. Let at $t = 1$ the information $\omega_1 = u$ be given. Then conditional consistency means that the evaluation of the encircled information at $t = 1$ leads to the same decision as the evaluation at time $t = 0$ of the complete tree.*

(ii) On the other hand, *sequential consistency* means that, if a position is acceptable given the information ω_t at time t, then there exists a succeeding path, where it remains acceptable at all nodes.

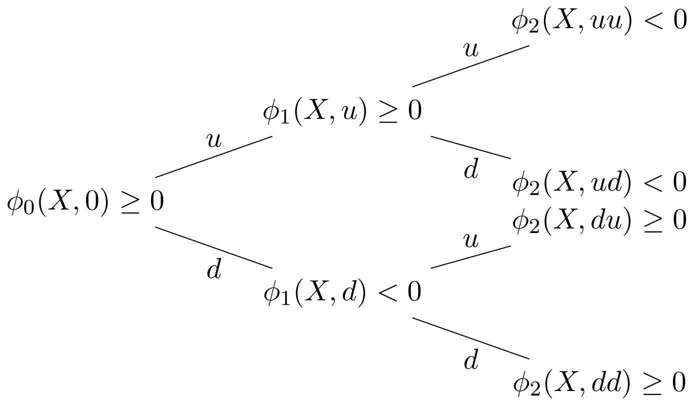

Figure 2: *The figure shows a counterexample for sequential consistency in a two-periodic binomial tree. There exists no path starting at* 0 *such that at any node the evaluation of* X *is non-negative and hence, acceptable.*

(iii) Finally, the idea behind *dynamic consistency* is that the acceptability measure of X with information ω_t at time t depends completely on the evaluations of X at the succeeding points. An equivalent definition is given by the recursion $\phi_t(X, \omega_t) = \phi_t(\phi_{t+1}(X, \cdot|_{t+1}), \omega_t)$ (see Roorda and Schumacher (2007), Theorem 4.1) widely used in the literature.

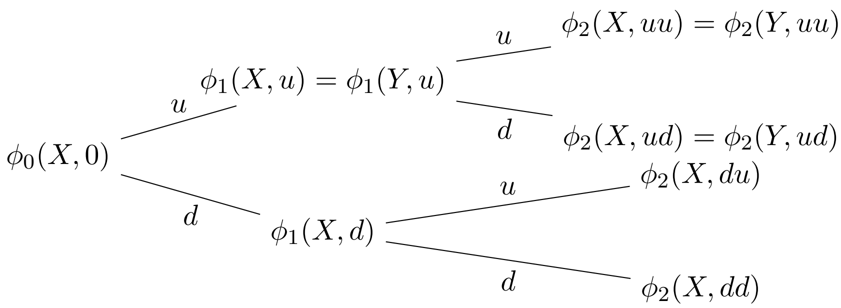

Figure 3: *The figure considers once again the two-periodic binomial tree with* $X, Y \in \mathcal{X}$ *and* $\phi_2(X, \omega) = \phi_2(Y, \omega)$ *for* $\omega \in \{uu, ud\}$*. Then a consequence of dynamic consistency is that* $\phi_1(X, u) = \phi_1(Y, u)$*.*

Now we present equivalent definitions for sequential and dynamic consistency used in this paper.

Proposition 2.9 (Roorda and Schumacher (2007), Theorem 4.2) *A coherent multi-period acceptability measure is sequentially consistent if and only if for any* $t \in \{0, \ldots, T-1\}$, $\omega_t \in S^t$ *and* $X \in \mathcal{X}$ *with* $\phi_{t+1}(X, (\omega_t, \alpha)) = 0$ *for any* $\alpha \in S$ *we conclude that* $\phi_t(X, \omega_t) = 0$.

In the single-period case, coherent acceptability measures have the representation as given in Proposition 2.2. Roorda et al. (2005) proved an analog representation theorem for dynamically consistent coherent acceptability measures. For this, we introduce further notation.

For a measure $\mathbb{Q} \in \mathfrak{P}$, $t \in \{0, \ldots, T-1\}$ and $\omega_t \in S^t$ the *single-step probability measure* is defined as

$$\mathbb{Q}^s_{t,\omega_t}(C) = \frac{\mathbb{Q}(\{\omega \in S^T : \exists\, \alpha \in C \text{ such that } \omega|_{t+1} = (\omega_t, \alpha)\})}{\mathbb{Q}(F(\omega_t))} \quad \text{for } C \subseteq S,$$

which is an element of $\mathfrak{P}_S$. The next definition was introduced in Roorda et al. (2005).

Definition 2.10 *For any $t \in \{0, \ldots, T-1\}$ and $\omega_t \in S^t$ let $\mathfrak{Q}^s_{t,\omega_t}$ be a set of probability measures on $(S, \mathcal{P}(S))$. Then the collection of probability measures*

$$\mathfrak{Q} = \{\mathbb{Q} \in \mathfrak{P} |\ \forall\, t \in \{0, \ldots, T-1\},\ \omega_t \in S^t \text{ with } \mathbb{Q}(F(\omega_t)) > 0 : \mathbb{Q}^s_{t,\omega_t} \in \mathfrak{Q}^s_{t,\omega_t}\}$$

is called of product type. *We shortly write $\mathfrak{Q}$ is generated by $\{\mathfrak{Q}^s_{t,\omega_t} : \omega_t \in S^t,\ t \in \{0, \ldots, T-1\}\}$.*

Now we are able to present a representation theorem for multi-period coherent and dynamically consistent acceptability measures in analogy to the static case of Proposition 2.2.

Proposition 2.11 (Roorda et al. (2005), Theorem 2.2)
Let $\mathfrak{Q}$ be generated by $\{\mathfrak{Q}^s_{t,\omega_t} : \omega_t \in S^t,\ t \in \{0, \ldots, T-1\}\}$. Define $\phi_T(X, \omega) := X(\omega)$ for $\omega \in S^T$ and

$$\phi_t(X, \omega_t) := \min_{\{\mathbb{Q} \in \mathfrak{Q}:\ \mathbb{Q}(F(\omega_t)) > 0\}} \mathbb{E}_{\mathbb{Q}}(X \mid F(\omega_t)) \quad \text{for } \omega_t \in S^t,\ t \in \{0, \ldots, T-1\},\ X \in \mathcal{X}.$$

Furthermore, define the random variable $\varphi_{t,\omega_t}(X) : S \to [-\infty, \infty)$ as $\alpha \mapsto \phi_{t+1}(X, (\omega_t, \alpha))$ on $(S, \mathcal{P}(S))$. Then

$$\phi_t(X, \omega_t) = \min_{\mathbb{Q}^s \in \mathfrak{Q}^s_{t,\omega_t}} \mathbb{E}_{\mathbb{Q}^s}(\varphi_{t,\omega_t}(X)) \quad \text{for } \omega_t \in S^t,\ t \in \{0, \ldots, T-1\},\ X \in \mathcal{X}\ .$$

Furthermore, ϕ is a coherent and dynamically consistent multi-period acceptability measure.

In our setting of a finite scenario set the *stability property* introduced in Artzner et al. (2007) is equivalent to product type and hence, to dynamic consistency (cf. Artzner et al. (2007), Theorem 5.1).

3 Multi-period distortion measures

In this section, we present conditionally, sequentially and dynamically consistent versions of multi-period distortion measures and derive representation theorems similar to Proposition 2.11, where we characterize completely the set of probability measures. Throughout, ψ will denote a distortion function.

3.1 Conditionally consistent multi-period distortion measures

Theorem 3.1 *Let $\mathfrak{Q}_\psi$ be given as in (2.1). Then the measure ϕ^C_ψ defined as $\phi^C_{\psi,T}(X, \omega) := X(\omega)$ for $\omega \in S^T$ and*

$$\phi^C_{\psi,t}(X, \omega_t) := \min_{\{\mathbb{Q} \in \mathfrak{Q}_\psi : \mathbb{Q}(F(\omega_t)) > 0\}} \mathbb{E}_{\mathbb{Q}}(X \mid F(\omega_t)) \quad \text{for } \omega_t \in S^t,\ t \in \{0, \ldots, T-1\},\ X \in \mathcal{X},$$

is a coherent multi-period acceptability measure and conditionally consistent.

Proof. Let $X \leq 0$ and $X\mathbb{1}_{F(\omega_t)} \neq 0$, i.e. there exists an $\omega^* \in F(\omega_t)$ such that $X(\omega^*) < 0$. Since $\mathbb{P} \in \mathfrak{Q}_\psi$ and $\mathbb{P}(F(\omega_t)) \geq \mathbb{P}(\omega^*) > 0$ it follows that $\phi_t(X, \omega_t) \leq \mathbb{E}_\mathbb{P}(X \mid F(\omega_t)) < 0$. Hence, $\phi^C_{\psi,t}$ is relevant. Obviously, ϕ^C_ψ is a coherent multi-period acceptability measure. The global representation of ϕ^C_ψ yields conditional consistency by Roorda and Schumacher (2007), Theorem 7.1. $\square$

This means that at the beginning of the planing horizon in $t = 0$ the conditionally consistent multi-period distortion measure ϕ^C_ψ and the single-period distortion measure ϕ_ψ demand for the same risk capital. The dynamic structure of the model has no influence on the risk valuation in 0.

The acceptability measure ϕ^C_ψ is in general neither sequentially nor dynamically consistent as the following proposition shows. However, it is not possible to make this conclusion in general since, e.g., for the trivial case that $\psi(\mathbb{P}(\omega)) = \mathbb{P}(\omega)$ for every $\omega \in S^T$, the set $\mathfrak{Q}_\psi$ contains only $\mathbb{P}$ and hence, ϕ^C_ψ is sequentially consistent.

Proposition 3.2 *Let the two-periodic binomial tree $S^2 = \{uu, ud, du, dd\}$ be given and $\psi(z) = \min(z/\lambda, 1)$ be the distortion measure of the TVaR$_\lambda$ for some $\lambda \in (0,1)$. Suppose that*

$$\psi(\mathbb{P}(ud)) > \mathbb{P}(ud) \quad \text{and} \quad \psi(\mathbb{P}(uu)) + \psi(\mathbb{P}(ud)) + \psi(\mathbb{P}(dd)) < 1 \ .$$

Then ϕ^C_ψ is neither sequentially nor dynamically consistent.

Proof. Let $\mathbb{Q}_u, \mathbb{Q}_d \in \mathfrak{Q}_\psi$ with $\mathbb{Q}_u(F(u)) > 0$ and $\mathbb{Q}_d(F(d)) > 0$. For any $\mathbb{Q}^*_S \in \mathfrak{P}_S$ define

$$\mathbb{Q}^*(\cdot) = \mathbb{Q}^*_S(u)\mathbb{Q}_u(\cdot \mid F(u)) + \mathbb{Q}^*_S(d)\mathbb{Q}_d(\cdot \mid F(d)). \tag{3.1}$$

If we find $\mathbb{Q}_u, \mathbb{Q}_d \in \mathfrak{Q}_\psi$ such that for any $\mathbb{Q}^*_S \in \mathfrak{P}_S$ the probability measure $\mathbb{Q}^*$ is not contained in $\mathfrak{Q}_\psi$, then $\mathfrak{Q}_\psi$ is not a juncted test set (see Definition 6.4 in Roorda and Schumacher (2007)) and hence, by Roorda and Schumacher (2007), Theorem 7.1, neither sequentially nor dynamically consistent.

We define the following two probability measures

$$\mathbb{Q}_u(\omega) \quad = \quad \begin{cases} \psi(\mathbb{P}(uu)) & \text{for } \omega = uu, \\ \mathbb{P}(ud) & \text{for } \omega = ud, \\ \mathbb{P}(du) & \text{for } \omega = du, \\ 1 - \psi(\mathbb{P}(uu)) - \mathbb{P}(ud) - \mathbb{P}(du) & \text{for } \omega = dd, \end{cases}$$

and

$$\mathbb{Q}_d(\omega) \quad = \quad \begin{cases} \psi(\mathbb{P}(uu)) & \text{for } \omega = uu, \\ \psi(\mathbb{P}(ud)) & \text{for } \omega = ud, \\ 1 - \psi(\mathbb{P}(uu)) - \psi(\mathbb{P}(ud)) - \psi(\mathbb{P}(dd)) & \text{for } \omega = du, \\ \psi(\mathbb{P}(dd)) & \text{for } \omega = dd. \end{cases}$$

Indeed $\mathbb{Q}_u, \mathbb{Q}_d \in \mathfrak{Q}_\psi$ (in the case of a TVaR it holds that $\mathfrak{Q}_\psi = \{\mathbb{Q} \in \mathfrak{P} \ : \ \mathbb{Q}(\omega) \leq \psi(\mathbb{P}(\omega)) \, \forall \ \omega \in S^T\}$), and $\mathbb{Q}_u(F(u)) > 0$ and $\mathbb{Q}_d(F(d)) > 0$, respectively. Let $\mathbb{Q}^*_S \in \mathfrak{P}_S$ and $\mathbb{Q}^*$ as in (3.1). The

following two cases are possible:
Case 1: Let $\mathbb{Q}_S^*(u) > \psi(\mathbb{P}(uu)) + \mathbb{P}(ud)$. Then

$$\mathbb{Q}^*(uu) = \mathbb{Q}_S^*(u)\frac{\psi(\mathbb{P}(uu))}{\psi(\mathbb{P}(uu)) + \mathbb{P}(ud)} > \psi(\mathbb{P}(uu)),$$

which means $\mathbb{Q}^* \notin \mathfrak{Q}_\psi$.
Case 2: Let $\mathbb{Q}_S^*(u) \leq \psi(\mathbb{P}(uu)) + \mathbb{P}(ud)$. Then

$$1 - \mathbb{Q}_S^*(u) \geq 1 - \psi(\mathbb{P}(uu)) - \mathbb{P}(ud) > 1 - \psi(\mathbb{P}(uu)) - \psi(\mathbb{P}(ud)).$$

Thus,

$$\mathbb{Q}^*(dd) = (1 - \mathbb{Q}_S^*(u))\frac{\psi(\mathbb{P}(dd))}{1 - \psi(\mathbb{P}(uu)) - \psi(\mathbb{P}(ud))} > \psi(\mathbb{P}(dd)).$$

Again $\mathbb{Q}^* \notin \mathfrak{Q}_\psi$. □

Example 3.3 If $\mathbb{P}(\omega) = \frac{1}{4}$ for $\omega \in S^2$ and $\psi(z) = \min(\frac{8z}{7}, 1)$, then the assumptions of Proposition 3.2 are satisfied. □

3.2 Sequentially consistent multi-period distortion measures

Our next goal is to modify $\mathfrak{Q}_\psi$ such that we obtain a sequentially consistent acceptability measure.

Theorem 3.4 *Let*

$$\mathfrak{Q}_\psi^S = \{\mathbb{Q} \in \mathfrak{P} : \mathbb{Q}(B \cap F(\omega_s)) \leq \mathbb{Q}(F(\omega_s))\psi(\mathbb{P}(B|F(\omega_s)))\ \forall\ B \subseteq S^T, \forall\ \omega_s \in S^s, \forall\ s \in \{0, \dots, T-1\}\}.$$

Then the measure ϕ_ψ^S defined as $\phi_{\psi,T}^S(X,\omega) := X(\omega)$ for $\omega \in S^T$ and

$$\phi_{\psi,t}^S(X,\omega_t) := \min_{\{\mathbb{Q}\in\mathfrak{Q}_\psi^S:\ \mathbb{Q}(F(\omega_t))>0\}} \mathbb{E}_\mathbb{Q}(X \mid F(\omega_t)) \quad \text{for } \omega_t \in S^t,\ t \in \{0, \dots, T-1\}, \quad X \in \mathcal{X}$$

is a coherent multi-period acceptability measure and sequentially consistent.

Proof. As in the proof of Theorem 3.1 we can show that ϕ_ψ^S is a coherent and relevant multi-period acceptability measure.

We will prove the alternative characterization of sequential consistency as stated in Proposition 2.9. Hence, let $t \in \{0, \dots, T-1\}$, $\omega_t \in S^t$ and $X \in \mathcal{X}$ such that $\phi_{\psi,t+1}^S(X,(\omega_t,\alpha)) = 0$ for any $\alpha \in S$. Then by the definition of $\phi_{\psi,t+1}^S$ we have on the one hand,

$$\mathbb{E}_\mathbb{Q}(X \mid F(\omega_t,\alpha)) \geq 0 \qquad \text{for } \mathbb{Q} \in \mathfrak{Q}_\psi^S,\ \mathbb{Q}(F(\omega_t,\alpha)) > 0,\ \alpha \in S,$$

and on the other hand, that for any $\alpha \in S$ there exists a measure $\widetilde{\mathbb{Q}}_{\omega_t,\alpha} \in \mathfrak{Q}_\psi^S$ with $\widetilde{\mathbb{Q}}_{\omega_t,\alpha}(F(\omega_t,\alpha)) > 0$ and

$$\phi_{\psi,t+1}^S(X,(\omega_t,\alpha)) = \mathbb{E}_{\widetilde{\mathbb{Q}}_{\omega_t,\alpha}}(X \mid F(\omega_t,\alpha)) = 0. \tag{3.2}$$

Step 1. First, we have to show that $\mathbb{E}_{\mathbb{Q}}(X \mid F(\omega_t)) \geq 0$ for any $\mathbb{Q} \in \mathfrak{Q}^S_\psi$ with $\mathbb{Q}(F(\omega_t)) > 0$. But since ϕ^S_ψ has a global representation and is relevant it follows that ϕ^S_ψ is conditionally consistent. Therefore we can conclude from Lemma 4.4 from Roorda and Schumacher (2007) that

$$\phi^S_{\psi,t}(X, \omega_t) \geq \min_{\alpha \in S} \phi^S_{\psi,t+1}(X, (\omega_t, \alpha)) = 0,$$

i.e. $\mathbb{E}_{\mathbb{Q}}(X \mid F(\omega_t)) \geq 0$ for any $\mathbb{Q} \in \mathfrak{Q}^S_\psi$ with $\mathbb{Q}(F(\omega_t)) > 0$.

Step 2. Now, we are left to prove that there exists a $\widetilde{\mathbb{Q}}_{\omega_t} \in \mathfrak{Q}^S_\psi$ with $\widetilde{\mathbb{Q}}_{\omega_t}(F(\omega_t)) > 0$ and $\mathbb{E}_{\widetilde{\mathbb{Q}}_{\omega_t}}(X \mid F(\omega_t)) = 0$. Therefore we define $\widetilde{\mathbb{Q}}_{\omega_t}$ as

$$\widetilde{\mathbb{Q}}_{\omega_t}(\cdot) := \sum_{\alpha \in S} \mathbb{P}(F(\omega_t, \alpha)) \widetilde{\mathbb{Q}}_{\omega_t,\alpha}(\cdot \mid F(\omega_t, \alpha)) + \mathbb{P}(F(\omega_t)^c \cap \cdot),$$

where $\widetilde{\mathbb{Q}}_{\omega_t,\alpha}$ satisfies (3.2) for any $\alpha \in S$. Note that $\widetilde{\mathbb{Q}}_{\omega_t} \in \mathfrak{P}$, $\widetilde{\mathbb{Q}}_{\omega_t}(F(\omega_t)) = \mathbb{P}(F(\omega_t)) > 0$ and

$$\mathbb{E}_{\widetilde{\mathbb{Q}}_{\omega_t}}(X \mid F(\omega_t)) = \sum_{\alpha \in S} \frac{\mathbb{P}(F(\omega_t, \alpha))}{\mathbb{P}(F(\omega_t))} \, \mathbb{E}_{\widetilde{\mathbb{Q}}_{\omega_t,\alpha}}(X \mid F(\omega_t, \alpha)) = 0.$$

Let $B \subseteq S^T$ and $\omega_s \in S^s$, $s \in \{0, \ldots, T-1\}$. Only the following three cases are possible:

Case 1: $F(\omega_s) \subset F(\omega_t)$. Then there exists an $\alpha^* \in S$ such that $F(\omega_s) \subseteq F(\omega_t, \alpha^*)$. Hence, $\widetilde{\mathbb{Q}}_{\omega_t}(F(\omega_s)) = \mathbb{P}(F(\omega_t, \alpha^*)) \widetilde{\mathbb{Q}}_{\omega_t,\alpha^*}(F(\omega_s) \mid F(\omega_t, \alpha^*))$ and

$$\begin{aligned} \widetilde{\mathbb{Q}}_{\omega_t}(B \mid F(\omega_s)) &= \frac{\mathbb{P}(F(\omega_t, \alpha^*)) \widetilde{\mathbb{Q}}_{\omega_t,\alpha^*}(B \cap F(\omega_s) \mid F(\omega_t, \alpha^*))}{\mathbb{P}(F(\omega_t, \alpha^*)) \widetilde{\mathbb{Q}}_{\omega_t,\alpha^*}(F(\omega_s) \mid F(\omega_t, \alpha^*))} \\ &= \widetilde{\mathbb{Q}}_{\omega_t,\alpha^*}(B \mid F(\omega_s)) \leq \psi\left(\mathbb{P}(B \mid F(\omega_s))\right), \end{aligned}$$

where for the last inequality we used that $\widetilde{\mathbb{Q}}_{\omega_t,\alpha^*} \in \mathfrak{Q}^S_\psi$.

Case 2: $F(\omega_t) \subseteq F(\omega_s)$. Then $\widetilde{\mathbb{Q}}_{\omega_t}(F(\omega_s)) = \mathbb{P}(F(\omega_s))$ and by the concavity of ψ we have

$$\begin{aligned} \widetilde{\mathbb{Q}}_{\omega_t}(B \mid F(\omega_s)) &= \sum_{\alpha \in S} \frac{\mathbb{P}(F(\omega_t, \alpha))}{\mathbb{P}(F(\omega_s))} \widetilde{\mathbb{Q}}_{\omega_t,\alpha}(B \cap F(\omega_s) \mid F(\omega_t, \alpha)) + \frac{\mathbb{P}(F(\omega_t)^c \cap B \cap F(\omega_s))}{\mathbb{P}(F(\omega_s))} \\ &\leq \sum_{\alpha \in S} \frac{\mathbb{P}(F(\omega_t, \alpha))}{\mathbb{P}(F(\omega_s))} \psi(\mathbb{P}(B \cap F(\omega_s) \mid F(\omega_t, \alpha))) + \frac{\mathbb{P}(F(\omega_t)^c \cap B \cap F(\omega_s))}{\mathbb{P}(F(\omega_s))} \psi(1) \\ &\leq \psi\left(\sum_{\alpha \in S} \frac{\mathbb{P}(F(\omega_t, \alpha))}{\mathbb{P}(F(\omega_s))} \mathbb{P}(B \cap F(\omega_s) \mid F(\omega_t, \alpha)) + \frac{\mathbb{P}(F(\omega_t)^c \cap B \cap F(\omega_s))}{\mathbb{P}(F(\omega_s))} \right) \\ &= \psi(\mathbb{P}(B|F(\omega_s))). \end{aligned}$$

Case 3: $F(\omega_s) \cap F(\omega_t) = \emptyset$. Then

$$\widetilde{\mathbb{Q}}_{\omega_t}(B \mid F(\omega_s)) = \mathbb{P}(B \mid F(\omega_s)) \leq \psi(\mathbb{P}(B \mid F(\omega_s))).$$

Therefore, $\widetilde{\mathbb{Q}}_{\omega_t} \in \mathfrak{Q}^S_\psi$. □

Since $\mathfrak{Q}_\psi^S \subseteq \mathfrak{Q}_\psi$ we have

$$\phi_{\psi,t}^C(X,\omega_t) \leq \phi_{\psi,t}^S(X,\omega_t). \tag{3.3}$$

This means that the conditionally consistent acceptance measure is more conservative than the sequential one.

The set $\mathfrak{Q}_\psi^S$ is a polytope such that our sequentially consistent version of the multi-period distortion measure can be computed via linear programming. Roorda (2010) presents an algorithm for path-independent payoffs.

To end this section, we want to explain why this acceptability measure is in general not dynamically consistent.

Proposition 3.5 *Let the two-periodic binomial tree $S^2 = \{uu, ud, du, dd\}$ be given and $\psi(z) = \min(z/\lambda, 1)$ be the distortion measure of the TVaR$_\lambda$ for some $\lambda \in (0,1)$. Suppose that*

$$\psi(\mathbb{P}(F(u))) < 1.$$

Then ϕ_ψ^S is not dynamically consistent.

Proof. We will show that $\mathfrak{Q}_\psi^S$ is not of product type and hence, ϕ_ψ^S is not dynamically consistent by Roorda and Schumacher (2007), Theorem 7.1.

Define

$$\mathbb{Q}_S^*(\omega) = \begin{cases} \psi(\mathbb{P}(uu)) & \text{for } \omega = uu, \\ \psi(\mathbb{P}(F(u))) - \psi(\mathbb{P}(uu)) & \text{for } \omega = ud, \\ \psi(\mathbb{P}(F(u) + \mathbb{P}(du))) - \psi(\mathbb{P}(F(u))) & \text{for } \omega = du, \\ 1 - \psi(\mathbb{P}(F(u)) + \mathbb{P}(du)) & \text{for } \omega = dd, \end{cases}$$

which is in $\mathfrak{Q}_\psi^S$. Then $\mathbb{Q}_S^*(F(u)) = \psi(\mathbb{P}(F(u)))$. Furthermore, we define for some properly chosen $\mathbb{Q}_u, \mathbb{Q}_d \in \mathfrak{Q}_\psi^S$ the probability measure

$$\mathbb{Q}^*(\cdot) = \mathbb{Q}_S^*(F(u))\mathbb{Q}_u(\cdot \mid F(u)) + \mathbb{Q}_S^*(F(d))\mathbb{Q}_d(\cdot \mid F(d)).$$

If we can show that $\mathbb{Q}^* \notin \mathfrak{Q}_\psi^S$, then $\mathfrak{Q}_\psi^S$ is not of product type and the proof is finished.

To this effect, we distinguish the following two cases:

Case 1: Suppose $\mathbb{P}(F(u)) \leq \psi(\mathbb{P}(uu))$. We define

$$\mathbb{Q}_u(\omega) = \begin{cases} \psi(\mathbb{P}(uu)) & \text{for } \omega = uu, \\ 0 & \text{for } \omega = ud, \\ (\mathbb{P}(F(u)) + \mathbb{P}(du) - \psi(\mathbb{P}(uu)))_+ & \text{for } \omega = du, \\ 1 - \psi(\mathbb{P}(uu)) - (\mathbb{P}(F(u)) + \mathbb{P}(du) - \psi(\mathbb{P}(uu)))_+ & \text{for } \omega = dd \end{cases}$$

in $\mathfrak{Q}_\psi^S$ and $\mathbb{Q}_d \in \mathfrak{Q}_\psi^S$ is arbitrary.

Since $\psi(\mathbb{P}(F(u))) < 1$ and $\mathbb{P}(\omega) > 0$ for any $\omega \in S^2$ it follows by the structure of ψ that $\psi(\mathbb{P}(F(u))) > \psi(\mathbb{P}(uu))$. Hence,

$$\mathbb{Q}^*(uu) = \mathbb{Q}^*_S(F(u))\mathbb{Q}_u(uu \mid F(u)) = \psi(\mathbb{P}(F(u))) > \psi(\mathbb{P}(uu)) \ ,$$

which means $\mathbb{Q}^* \notin \mathfrak{Q}^S_\psi$.

Case 2: Suppose $\mathbb{P}(F(u)) > \psi(\mathbb{P}(uu))$. We define

$$\mathbb{Q}_u(\omega) \quad = \quad \begin{cases} \psi(\mathbb{P}(uu)) & \text{for } \omega = uu, \\ \mathbb{P}(F(u)) - \psi(\mathbb{P}(uu)) & \text{for } \omega = ud, \\ \mathbb{P}(du) & \text{for } \omega = du, \\ \mathbb{P}(dd) & \text{for } \omega = dd, \end{cases}$$

in $\mathfrak{Q}^S_\psi$ and $\mathbb{Q}_d \in \mathfrak{Q}^S_\psi$ is arbitrary. By the structure of ψ and $\psi(\mathbb{P}(F(u))) < 1$ also $\psi(\mathbb{P}(F(u))) > \mathbb{P}(F(u))$. Finally,

$$\mathbb{Q}^*(uu) = \psi(\mathbb{P}(F(u)))\frac{\psi(\mathbb{P}(uu))}{\mathbb{P}(F(u))} > \psi(\mathbb{P}(uu)).$$

Again $\mathbb{Q}^* \notin \mathfrak{Q}^S_\psi$. □

Example 3.6 If $\mathbb{P}(\omega) = \frac{1}{4}$ for $\omega \in S^2$ and $\psi(z) = \min(\frac{3z}{2}, 1)$, then the assumptions of Proposition 3.5 are satisfied. □

3.3 Dynamically consistent multi-period distortion measures

Theorem 3.7 *Let*

$$\mathfrak{Q}^D_\psi = \{\mathbb{Q} \in \mathfrak{P} : \ \mathbb{Q}(B \cap F(\omega_s)) \le \mathbb{Q}(F(\omega_s))\psi(\mathbb{P}(B \mid F(\omega_s))) \ \forall \ B \in \mathcal{F}_{s+1}, \ \forall \ \omega_s \in S^s, \ \forall \ s \in \{0, \ldots, T-1\}\}.$$

Furthermore, define

$$\mathfrak{Q}^{D,s}_{\psi,t,\omega_t} = \{\mathbb{Q}^s \in \mathfrak{P}_S : \ \mathbb{Q}^s(C) \le \psi(\mathbb{P}^s_{t,\omega_t}(C)) \ \forall \ C \subseteq S\} \quad \textit{for } t \in \{0, \ldots, T-1\}, \ \omega_t \in S^t,$$

and

$$\widetilde{\mathfrak{Q}}^D_\psi = \{\mathbb{Q} \in \mathfrak{P} | \ \forall \ s \in \{0, \ldots, T-1\}, \ \forall \ \omega_s \in S^s \textit{ with } \mathbb{Q}(F(\omega_s)) > 0 : \ \mathbb{Q}^s_{s,\omega_s} \in \mathfrak{Q}^{D,s}_{\psi,s,\omega_s} \ \}.$$

Then the following hold:

(a) $\widetilde{\mathfrak{Q}}^D_\psi = \mathfrak{Q}^D_\psi$.

(b) The measure ϕ^D_ψ *defined as* $\phi^D_{\psi,T}(X, \omega) := X(\omega)$ *for* $\omega \in S^T$ *and*

$$\phi^D_{\psi,t}(X, \omega_t) := \min_{\{\mathbb{Q} \in \mathfrak{Q}^D_\psi : \mathbb{Q}(F(\omega_t)) > 0\}} \mathbb{E}_\mathbb{Q}(X \mid F(\omega_t)) \quad \textit{for } \omega_t \in S^t, \ t \in \{0, \ldots, T-1\}, \ X \in \mathcal{X},$$

is a coherent multi-period acceptability measure and dynamically consistent.

Proof.

(a) Let $s \in \{0,\dots,T-1\}$, $\omega_s \in S^s$, $B \in \mathcal{F}_{s+1}$ with $B \cap F(\omega_s) \neq \emptyset$ and let $\mathbb{Q} \in \mathfrak{P}$ such that $\mathbb{Q}(F(\omega_s)) > 0$. Define $C = \{\alpha \in S : B \cap F(\omega_s, \alpha) \neq \emptyset\}$. Note that $C \neq \emptyset$. Then

$$B \cap F(\omega_s) = \bigcup_{\alpha \in C} F(\omega_s, \alpha) = \{\omega \in S^T : \exists\, \alpha \in C \text{ such that } \omega|_{s+1} = (\omega_s, \alpha)\}$$

and hence,

$$\mathbb{Q}(B \mid F(\omega_s)) = \frac{\mathbb{Q}(\{\omega \in S^T : \exists\, \alpha \in C \text{ such that } \omega|_{s+1} = (\omega_s, \alpha)\})}{\mathbb{Q}(F(\omega_s))} = \mathbb{Q}^s_{s,\omega_s}(C).$$

On the other hand, for any $C \subseteq S$ the set $B = \bigcup_{\alpha \in C} F(\omega_s, \alpha)$ is in $\mathcal{F}_{s+1}$. Therefore, for $\mathbb{Q} \in \mathfrak{P}$:

$$\begin{aligned}
& \forall\, s \in \{0,\dots,T-1\},\ \forall\, \omega_s \in S^s :\ \mathbb{Q}(B \cap F(\omega_s)) \leq \mathbb{Q}(F(\omega_s))\psi(\mathbb{P}(B \mid F(\omega_s)))\ \forall\, B \in \mathcal{F}_{s+1} \\
\Leftrightarrow\ & \forall\, s \in \{0,\dots,T-1\},\ \forall\, \omega_s \in S^s \text{ with } \mathbb{Q}(F(\omega_s)) > 0 :\ \mathbb{Q}(B \mid F(\omega_s)) \leq \psi(\mathbb{P}(B \mid F(\omega_s)))\ \forall\, B \in \mathcal{F}_{s+1} \\
\Leftrightarrow\ & \forall\, s \in \{0,\dots,T-1\},\ \forall\, \omega_s \in S^s \text{ with } \mathbb{Q}(F(\omega_s)) > 0 :\ \mathbb{Q}^s_{s,\omega_s}(C) \leq \psi(\mathbb{P}^s_{s,\omega_s}(C))\ \forall\, C \subseteq S \\
\Leftrightarrow\ & \forall\, s \in \{0,\dots,T-1\},\ \forall\, \omega_s \in S^s \text{ with } \mathbb{Q}(F(\omega_s)) > 0 :\ \mathbb{Q}^s_{s,\omega_s} \in \mathfrak{Q}^{D,s}_{\psi,s,\omega_s}.
\end{aligned}$$

Finally, it follows that *(a)* holds.

(b) By *(a)* we have that $\mathfrak{Q}^D_\psi$ is of product type (Definition 2.10). From this we already get dynamic consistency by Proposition 2.11. □

It is not hard to see that our results, which can be applied to the TVaR itself, lead to exactly the same multi-period TVaR as the multiperiod TVaR in Roorda and Schumacher (2007).

The representation of ϕ^D_ψ by the global test set $\mathfrak{Q}^D_\psi$ has the advantage that we are able to compare our sequential and dynamical versions of multi-period acceptance measures. Since $\mathfrak{Q}^S_\psi \subseteq \mathfrak{Q}^D_\psi$ the inequality

$$\phi^D_{\psi,t}(X, \omega_t) \leq \phi^S_{\psi,t}(X, \omega_t) \tag{3.4}$$

holds for any $\omega_t \in S^t$, where $t \in \{0,\dots,T\}$. Consequently the dynamic consistent update is stronger than the sequential consistent update.

An advantage of our dynamic version is that it can be evaluated by dynamic programming as below; cf. also Roorda et al. (2005).

Example 3.8 (Continuation of Example 2.6)

Let $S = \{u, d\}$ and $T = 2$. We consider the dynamically consistent version of the TVaR at the 5% confidence level, i.e. $\psi_1(z) := \min(z/0.05, 1)$. Moreover, let $\mathbb{P}$ and $X \in \mathcal{X}$ be defined as

$$\begin{aligned}
\mathbb{P}(\omega) &= 0.552\ \delta_{uu}(\omega) + 0.028\ \delta_{ud}(\omega) + 0.4\ \delta_{du}(\omega) + 0.02\ \delta_{dd}(\omega), \\
X(\omega) &= 1000\ \delta_{uu}(\omega) + 100\ \delta_{ud}(\omega) + 100\ \delta_{du}(\omega) - 100\ \delta_{dd}(\omega)
\end{aligned} \tag{3.5}$$

for $\omega \in S^2$.

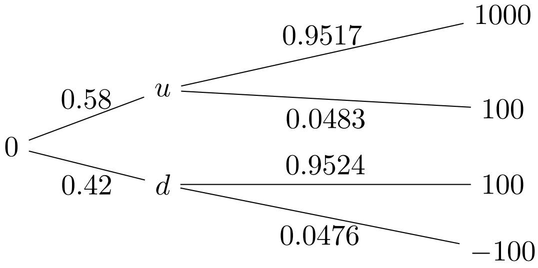

Figure 4: *The graph shows model (3.5).*

The aim is to calculate $\phi^D_{\psi_1,0}(X,0)$. Therefore we use that by Proposition 2.11 and Theorem 3.7

$$\phi_t(X,\omega_t) = \min_{\mathbb{Q}^s \in \mathfrak{Q}^{D,s}_{\psi,t,\omega_t}} \mathbb{E}_{\mathbb{Q}^s}(\varphi_{t,\omega_t}(X)) \qquad \text{for } \omega_t \in S^t,\ t \in \{0,\dots,T-1\},\ X \in \mathcal{X},$$

with $\mathfrak{Q}^{D,s}_{\psi,t,\omega_t}$ as given in Theorem 3.7 and φ_{t,ω_t} as given in Proposition 2.11. First, we calculate $\phi^D_{\psi_1,1}(X,u)$ and $\phi^D_{\psi_1,1}(X,d)$. Thus, define $\varphi_{1,u} : S \to \mathbb{R}$ as $\varphi_{1,u}(u) := 1000$ and $\varphi_{1,u}(d) := 100$. Furthermore,

$$\mathbb{P}_{\varphi_{1,u}}(y) = \mathbb{P}^s_{1,u}(u)\ \delta_{1000}(y) + \mathbb{P}^s_{1,u}(d)\ \delta_{100}(y) = 0.9517\ \delta_{1000}(y) + 0.0483\ \delta_{100}(y).$$

Hence, it follows that

$$\phi^D_{\psi_1,1}(X,u) = \psi_1(\mathbb{P}_{\varphi_{1,u}}(100))\ \varphi_{1,u}(d) + (1-\psi_1(\mathbb{P}_{\varphi_{1,u}}(100)))\ \varphi_{1,u}(u) = 131.03.$$

Analogously, we observe that $\phi^D_{\psi_1,1}(X,d) = -90.48$. In the next step, we define $\varphi_{0,0} : S \to \mathbb{R}$ as $\varphi_{0,0}(u) := 131.03$ and $\varphi_{0,0}(d) := -90.48$. Moreover, it follows that

$$\mathbb{P}_{\varphi_{0,0}}(y) = \mathbb{P}^s_{0,0}(d)\ \delta_{-90.48}(y) + \mathbb{P}^s_{0,0}(u)\ \delta_{131.03}(y) = 0.42\ \delta_{-90.48}(y) + 0.58\ \delta_{131.03}(y),$$

which finally leads to $\phi^D_{\psi_1,0}(X,0) = -90.48$.

The example shows that the information ω_t about the state of the word influences the risk capital of the company. At $t=1$ with information $\omega_1 = u$, the acceptability measure is very high and hence, the position is acceptable. On the other hand, evaluated at d it is not acceptable. This behavior is not surprising, since if d occurs, the position will fall with almost probability 0.05 very low. At the initial time point, however, the acceptability measure is positive and hence, the position is acceptable.

Moreover, we compute the sequentially consistent version $\phi^S_{\psi_1,0}(X,0)$. The global test set $\mathfrak{Q}^S_\psi$ is given by

$$\begin{aligned}
\mathfrak{Q}^S_\psi &= \{\mathbb{Q} \in \mathfrak{P} : \mathbb{Q}(dd) \le 0.4,\ \mathbb{Q}(ud) \le 0.56,\ \mathbb{Q}(ud) \le \mathbb{Q}(u)\frac{28}{29},\ \mathbb{Q}(dd) \le \mathbb{Q}(d)\frac{20}{21}\} \\
&= \{\mathbb{Q} \in \mathfrak{P} : \mathbb{Q}(dd) \le 0.4,\ \mathbb{Q}(ud) \le 0.56,\ \mathbb{Q}(ud) \le 28\mathbb{Q}(uu),\ \mathbb{Q}(dd) \le 20\mathbb{Q}(du)\}.
\end{aligned}$$

The aim is to put as much weight as possible on dd and less weight on uu, where the minimum and maximum, respectively, of X is attained. Thus, a possible choice for the optimal probability measure $\mathbb{Q}^S$ with $\phi^S_{\psi,0}(X,0) = \mathbb{E}_{\mathbb{Q}^S}(X)$ (in this example the probability measure is not unique) is

$$\mathbb{Q}^S(\cdot) = 0.02\delta_{uu}(\cdot) + 0.56\delta_{ud}(\cdot) + 0.02\delta_{du}(\cdot) + 0.4\delta_{dd}(\cdot), \tag{3.6}$$

which gives $\phi^S_{\psi_1,0}(X,0) = 38$.

Furthermore, note, that we are in the setup of Example 2.6 with $u_1 = uu$, $u_2 = ud$, $u_3 = du$, $u_4 = dd$, such that

$$\phi^D_{\psi_1,0}(X,0) < \phi_{\psi_1}(X) = \phi^C_{\psi_1,0}(X,0) < \phi^S_{\psi_1,0}(X,0).$$

We already know (3.3) and (3.4). However, in this example $\phi^D_{\psi_1,0}(X,0)$ is more conservative than $\phi^C_{\psi_1,0}(X)$. The reason is the following.

Let $\mathfrak{X} = \{-100, 100, 1000\}$. The probability measure $\mathbb{Q}^D_X \in \mathfrak{P}_{\mathfrak{X}}$ with $\phi^D_{\psi_1,0}(X,0) = \sum_{x\in\mathfrak{X}} x\mathbb{Q}^D_X(x)$ is given by

$$\mathbb{Q}^D_X(x) = \frac{20}{21}\ \delta_{-100}(x) + \frac{1}{21}\delta_{100}(x) \quad \text{ for } x \in \mathfrak{X}.$$

We know from Example 2.6 that the probability measure $\mathbb{Q}^C_X \in \mathfrak{P}_{\mathfrak{X}}$ with $\phi^C_{\psi_1,0}(X,0) = \phi_{\psi_1}(X) = \sum_{x\in\mathfrak{X}} x\mathbb{Q}^C_X(x)$ is given by

$$\mathbb{Q}^C_X(x) = \frac{2}{5}\ \delta_{-100}(x) + \frac{3}{5}\ \delta_{100}(x) \quad \text{ for } x \in \mathfrak{X}.$$

Obviously, the probability which is assigned to the lowest value -100 plays an important role in the calculation. Comparing the two probabilities we see that $\mathbb{Q}^C_X(-100) = \frac{2}{5} < \frac{20}{21} = \mathbb{Q}^D_X(-100)$. The higher probability of -100 in the dynamic case results in the unacceptability of X.

On the other hand, consider

$$Z(\omega) = 2005\ \delta_{uu}(\omega) - 50\ \delta_{ud}(\omega) + 1055\ \delta_{du}(\omega) - 50\ \delta_{dd}(\omega) \quad \text{ for } \omega \in S^2.$$

Then $\mathfrak{X} = \{-50, 1055, 2005\}$,

$$\phi^D_{\psi_1,0}(Z,0) = 2.619\,, \quad \phi^S_{\psi_1,0}(Z,0) = 13.2 \quad \text{ and } \quad \phi^C_{\psi_1,0}(Z,0) = \phi_{\psi_1}(Z) = -5.8.$$

Hence, Z is considered less risky regarding the dynamic evaluation than in the conditional evaluation. To see the reason for this we once again calculate the probability measure $\mathbb{Q}^D_Z$ which satisfies $\phi^D_{\psi_1,0}(Z,0) = \sum_{x\in\mathfrak{X}} x\mathbb{Q}^D_Z(x)$ and $\mathbb{Q}^C_Z$ which satisfies $\phi^C_{\psi_1,0}(Z) = \sum_{x\in\mathfrak{X}} x\mathbb{Q}^C_Z(x)$, respectively. We obtain

$$\mathbb{Q}^D_Z(x) = \frac{20}{21}\ \delta_{-50}(x) + \frac{1}{21}\delta_{1055}(x) \quad \text{ for } x \in \mathfrak{X},$$

and

$$\mathbb{Q}^C_Z(x) = \frac{24}{25}\ \delta_{-50}(x) + \frac{1}{25}\delta_{1055}(x) \quad \text{ for } x \in \mathfrak{X}.$$

In this case, the lowest value of Z gets a higher probability in the conditional case than in the dynamic case, which is due to the fact that the lowest value of Z is attained in dd and in ud. $\square$

The example of the $\mathrm{TVaR}_{0.05}$ shows that we cannot say in general whether the conditional or the dynamic version of the $\mathrm{TVaR}_{0.05}$ assigns a higher risk capital. The same phenomenon is reflected by other risk measures.

Example 3.9 To see this, let $\psi_2 : [0,1] \to [0,1]$ be the *exponential distortion measure* introduced in Delbaen (1974) as

$$\psi_2(z) = \frac{1 - \exp(-\lambda_2 z)}{1 - \exp(-\lambda_2)} \quad \text{for} \quad z \in [0,1],$$

where $\lambda_2 > 0$ is a constant. Further, let $\psi_3 : [0,1] \to [0,1]$ be given by

$$\psi_3(z) = \Phi(\Phi^{-1}(z) + \lambda_3) \quad \text{for} \quad z \in [0,1],$$

where $\lambda_3 \geq 0$ is a risk-aversion constant, Φ is the the standard normal distribution function with inverse Φ^{-1}. This distortion function was introduced by Wang (2000). If $X \sim \mathcal{N}(\mu, \sigma^2)$ then $-\phi_{\psi_3}(-X) = \mu + \lambda_3\sigma = \mathbb{E}(X) + \lambda_3\sigma(X)$ is the standard deviation premium principle in insurance context. If $Y \sim \log\text{-}\mathcal{N}(\mu, \sigma^2)$ and $X_0 \sim \mathcal{N}(0,1)$ then $\phi_{\psi_3}(Y) = \mathbb{E}(\exp(\mu - \lambda_3\sigma + \sigma X_0))$, the mean of a $\log\text{-}\mathcal{N}(\mu - \lambda_3\sigma, \sigma^2)$ distribution. This means that if stocks are modeled by a log-normal distribution, as in the Black-Scholes model, then measuring the risk via the stock prices or the returns results in a consistent measurement. For stop-loss reinsurance covers, this distortion operator resembles a risk-neutral valuation of financial options.

In our example we choose $\lambda_2 = 45$ and $\lambda_3 = 3$. The parameters are chosen on such a way that the distortion functions are similar (cf. Figure 5).

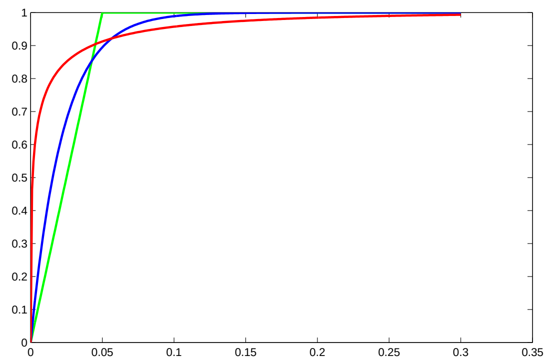

Figure 5: *The distortion functions of the* $TVaR_{0.05}$ *(green), the exponential distortion function with* $\lambda_2 = 45$ *(blue) and Wang's distortion function with* $\lambda_3 = 3$ *(red).*

ψ	ψ_1	ψ_2	ψ_3
$\phi^C_{\psi,0}(X,0)$	2	−18.68	−63.75
$\phi^D_{\psi,0}(X,0)$	−90.48	−76.51	−81.02

ψ	ψ_1	ψ_2	ψ_3
$\phi^C_{\psi,0}(Z,0)$	−5.8	77.43	50.29
$\phi^D_{\psi,0}(Z,0)$	2.62	79.74	51.35

Table 1: *The left table shows the risk assigned to* X *for the conditional (top) and the dynamic (bottom) acceptability measures with distortion function* ψ_1, ψ_2 *and* ψ_3*, respectively. The right table shows the analog for* Z.

A conclusion from Table 1 is that for all three distortion measures ψ_1, ψ_2 and ψ_3, respectively, X is less risky in the conditional evaluation than in the dynamic evaluation. But Z requires a higher risk capital in the conditional case than in the dynamic case. Hence, for the three distortion measures ψ_1, ψ_2 and ψ_3, there exist examples, where the conditional case is more conservative than the dynamic case and vice versa. $\square$

A Appendix

A.1 Proof of Proposition 2.5

We use the following proposition, which can be found in Föllmer and Schied (2004) for atomless probability spaces.

Proposition A.1 *Let ψ be a distortion function, and let ϕ_ψ be the induced distortion measure. Let $X \in \mathcal{X}$ with distribution function F_X. Write $q_X(z) = \inf\{x \in \mathbb{R} : \ F_X(x) \geq z\}$, $z \in (0,1)$ for the quantile function and $q_X^+(z) = \inf\{x \in \mathbb{R} : \ F_X(x) > z\}$, $z \in (0,1)$, for the upper quantile function. Then*

$$\phi_\psi(X) = -\textstyle\int_0^1 q_{-X}(z)\psi'_+(1-z)\ dz = \int_0^1 q_X^+(z)\psi'_+(z)\ dz.$$

Note that ψ'_+ always exists by the concavity of ψ.

Proof. *Step 1.* Let $X \leq 0$. Then

$$-\phi_\psi(X) = \int_{-\infty}^0 \psi(\mathbb{P}(X < x))\ dx = \int_0^\infty \psi(\mathbb{P}(-X > x))\ dx = \int_0^\infty \int_{F_{-X}(x)}^1 \psi'_+(1-z)\ dz\ dx,$$

by properties of concave functions (see Föllmer and Schied (2004), Proposition A.4). In the next step we use Fubini and the fact that $z > F_{-X}(x)$ if and only if $q_{-X}(z) > x$ to observe that

$$-\phi_\psi(X) = \int_0^1 \int_0^\infty \mathbb{1}_{(0,q_{-X}(z))}(x)\ dx\ \psi'_+(1-z)\ dz = \int_0^1 q_{-X}(z)\psi'_+(1-z)\ dz$$

as desired.

Step 2. Let $X \in \mathcal{X}$ be arbitrary. Then there exists a constant $K \in [0,\infty)$ such that $X - K \leq 0$. Thus, by translation-invariance and Step 1

$$-\phi_\psi(X) + K = \int_0^1 q_{-X+K}(z)\psi'_+(1-z)\ dz.$$

Since $q_{-X+K}(z) = q_{-X}(z) + K$ and $\int_0^1 \psi'_+(1-z)\ dz = 1$ we obtain

$$-\phi_\psi(X) + K = \int_0^1 q_{-X}(z)\psi'_+(1-z)\ dz + K,$$

which results in the statement.

For the second equality, note that $-q_{-X}(z) = q_X^+(1-z)$ and hence, the second claim follows by a substitution. $\square$

Proof of Proposition 2.5. We assume without loss of generality that $\mathfrak{X} = \{x_1, \ldots, x_n\}$ with $x_i < x_j$ for $i < j$. Moreover, let F_X denote the distribution function of X with respect to $\mathbb{P}$. Hence,

$$\mathbb{Q}_X^*(x_i) = \begin{cases} \psi(F_X(x_1)) & \text{for } i = 1, \\ \psi(F_X(x_i)) - \psi(F_X(x_{i-1})) & \text{for } 1 < i \leq n. \end{cases}$$

First, we will show that $\mathbb{Q}_X^*$ is a probability measure. Since ψ is increasing it follows immediately that $\mathbb{Q}_X^*(x_i) \geq 0$ for $i \in \{1, \ldots, n\}$. Moreover, since $\psi(1) = 1$ we have

$$\mathbb{Q}_X^*(\mathfrak{X}) = \sum_{i=1}^{n} \mathbb{Q}_X^*(x_i) = \psi(F_X(x_1)) + \sum_{i=2}^{n} \psi\left(F_X(x_i)\right) - \psi\left(F_X(x_{i-1})\right) = \psi(F_X(x_n)) = 1,$$

which proves that $(\mathfrak{X}, \mathcal{P}(\mathfrak{X}), \mathbb{Q}_X^*)$ is a probability space.

Next, we prove $\mathbb{Q}_X^* \in \mathfrak{Q}_{\psi,X}$. Let $B \subseteq \mathfrak{X}$. In particular, B is then finite. The proof goes by induction over the number m of elements in B.

Step 1: $m = 1$. Assume that $B = \{x_i\}$ for some $i \in \{1, \ldots, n\}$. If $B = \{x_1\}$ then it follows by definition that $\mathbb{Q}_X^*(x_1) = \psi(F_X(x_1))$ as desired. Hence, let $i \geq 2$. Then

$$\begin{aligned} \mathbb{Q}_X^*(x_i) &= \psi\left(F_X(x_i)\right) - \psi\left(F_X(x_{i-1})\right) = \psi\left(\textstyle\sum_{j=1}^{i} \mathbb{P}_X(x_j)\right) - \psi\left(\textstyle\sum_{j=1}^{i-1} \mathbb{P}_X(x_j)\right) \\ &= \textstyle\int_0^{\mathbb{P}_X(x_i)} \psi'_+(\sum_{j=1}^{i-1} \mathbb{P}_X(x_j) + z)\ dz \leq \int_0^{\mathbb{P}_X(x_i)} \psi'_+(z)\ dz = \psi(\mathbb{P}_X(x_i)), \end{aligned}$$

since ψ'_+ is decreasing.

Step 2: $m \to m+1$. Assume that $\mathbb{Q}_X^*(B) \leq \psi(\mathbb{P}_X(B))$ holds for any $B \subseteq \mathfrak{X}$ with $|B| = m$. Let $C \subseteq \mathfrak{X}$ with $|C| = m + 1$. Then there exists $k \in \{1, \ldots, n\}$ such that x_k is the largest value of C. We define $B := C \backslash \{x_k\}$. Then $|B| = m$. In particular, this implies $B \subseteq \{x_1, \ldots, x_{k-1}\}$ and

$$\sum_{j=1}^{k-1} \mathbb{P}_X(x_j) \geq \sum_{j=1}^{k-1} \mathbb{P}_X(x_j) \mathbb{1}_B(x_j) = \mathbb{P}_X(B). \tag{A.1}$$

Then we observe by the induction hypothesis and (A.1) that

$$\begin{aligned} \mathbb{Q}_X^*(C) &= \mathbb{Q}_X^*(B) + \mathbb{Q}_X^*(x_k) \leq \psi(\mathbb{P}_X(B)) + \mathbb{Q}_X^*(x_k) \\ &= \psi(\mathbb{P}_X(B)) + \textstyle\int_0^{\mathbb{P}_X(x_k)} \psi'_+(\sum_{j=1}^{k-1} \mathbb{P}_X(x_j) + z)\ dz \\ &\leq \psi(\mathbb{P}_X(B)) + \textstyle\int_0^{\mathbb{P}_X(x_k)} \psi'_+(\mathbb{P}_X(B) + z)\ dz = \psi(\mathbb{P}_X(C)), \end{aligned}$$

which concludes the proof of the induction step.

Finally, we have to show that the infimum is attained in $\mathbb{Q}_X^*$. First, we know by Proposition A.1 that $\phi_\psi(X) = \int_0^1 q_X^+(z)\ \psi'_+(z)\ dz$. Furthermore, $q_X^+(z) = x_i$ for $z \in [F_X(x_{i-1}), F_X(x_i))$. Thus,

$$\begin{aligned} \phi_\psi(X) &= \textstyle\int_0^1 q_X^+(z)\ \psi'_+(z)\ dz = \sum_{i=1}^{n} x_i \int_{[F_X(x_{i-1}), F_X(x_i))} \psi'_+(z)\ dz \\ &= \textstyle\sum_{i=1}^{n} x_i \left(\psi(F_X(x_i)) - \psi(F_X(x_{i-1}))\right) = \sum_{i=1}^{n} x_i \mathbb{Q}_X^*(x_i) = \mathbb{E}_{\mathbb{Q}^*}(X), \end{aligned}$$

which finally proves the theorem. $\square$

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