

CONTINUITY PROPERTIES AND INFINITE DIVISIBILITY OF STATIONARY DISTRIBUTIONS OF SOME GENERALISED ORNSTEIN-UHLENBECK PROCESSES

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ABSTRACT. Properties of the law μ of the integral $\int_0^\infty c^{-N_t-} dY_t$ are studied, where $c > 1$ and $\{(N_t, Y_t), t \geq 0\}$ is a bivariate Lévy process such that $\{N_t\}$ and $\{Y_t\}$ are Poisson processes with parameters a and b , respectively. This is the stationary distribution of some generalised Ornstein-Uhlenbeck process. The law μ is either continuous-singular or absolutely continuous, and sufficient conditions for each case are given. Under the condition of independence of $\{N_t\}$ and $\{Y_t\}$, it is shown that μ is continuous-singular if b/a is sufficiently small for fixed c , or if c is sufficiently large for fixed a and b , or if c is in the set of Pisot-Vijayaraghavan numbers, which includes all integers bigger than 1, for any a and b , and that, for Lebesgue almost every c , μ is absolutely continuous if b/a is sufficiently large. The law μ is infinitely divisible if $\{N_t\}$ and $\{Y_t\}$ are independent, but not in general. Complete characterisation of infinite divisibility is given for μ and for the symmetrisation of μ . Under the condition that μ is infinitely divisible, the continuity properties of the convolution power μ^{t*} of μ are also studied. Some results are extended to the case where $\{Y_t\}$ is an integer valued Lévy process with finite second moment.

1. INTRODUCTION

A generalised Ornstein-Uhlenbeck process $\{V_t, t \geq 0\}$ with initial condition V_0 is defined as

$$V_t = e^{-\xi t} \left(V_0 + \int_0^t e^{\xi s-} d\eta_s \right),$$

where $\{(\xi_t, \eta_t), t \geq 0\}$ is a bivariate Lévy process, independent of V_0 . See Carmona et al. [3], [4] for basic properties. Such processes arise in a variety of situations such as risk theory (e.g. Paulsen [17]), option pricing (e.g. Yor [25]) or financial time series (e.g. Klüppelberg et al. [12]), to name just a few. They also constitute a natural continuous time analogue of random recurrence equations, as studied by de Haan and Karandikar [11]. Lindner and Maller [15] have shown that a generalised Ornstein-Uhlenbeck process admits a strictly stationary solution which is not degenerate to a constant process with a suitable V_0 if and only if

$$(1.1) \quad \int_0^{\infty-} e^{-\xi s-} dL_s := \lim_{t \rightarrow \infty} \int_0^t e^{-\xi s-} dL_s$$

exists and is finite almost surely and not degenerate to a constant random variable. Here, $\{(\xi_t, L_t), t \geq 0\}$ is another bivariate Lévy process, defined in terms of $\{(\xi_t, \eta_t)\}$

by

$$L_t = \eta_t + \sum_{0 < s \leq t} (e^{-(\xi_s - \xi_{s-})} - 1)(\eta_s - \eta_{s-}) - ta_{\xi, \eta}^{1,2},$$

where $a_{\xi, \eta}^{1,2}$ denotes the $(1, 2)$ -element in the Gaussian covariance matrix of the Lévy-Khintchine triplet of $\{(\xi_t, \eta_t)\}$. Conversely, $\{(\xi_t, \eta_t)\}$ can be reconstructed from $\{(\xi_t, L_t)\}$ by

$$\eta_t = L_t + \sum_{0 < s \leq t} (e^{\xi_s - \xi_{s-}} - 1)(L_s - L_{s-}) + ta_{\xi, L}^{1,2}.$$

The distribution of (1.1) then gives the unique stationary distribution. When the integral (1.1) converges was characterised by Erickson and Maller [6] and generalised by Kondo et al. [13] to the case when $\{(\xi_t, L_t)\}$ is an $\mathbb{R} \times \mathbb{R}^d$ valued Lévy process with $d \in \mathbb{N}$.

Suppose now that $\{(\xi_t, L_t)\}$ is a bivariate Lévy process such that (1.1) converges almost surely and is finite, and denote by

$$\mu := \mathcal{L} \left(\int_0^{\infty-} e^{-\xi_{s-}} dL_s \right)$$

the distribution of the integral. If $\xi_t = t$ is deterministic, then it is well known that μ is self-decomposable, hence is infinitely divisible as well as absolutely continuous (if not degenerate to a constant, which happens only if $\{L_t\}$ is also deterministic). Other cases where μ is self-decomposable include the case where $\{\xi_t\}$ is stochastic, but spectrally negative (cf. Bertoin et al. [1]). On the other hand, as remarked by Samorodnitsky, μ is not infinitely divisible if e.g. $\xi_t = N_t + \alpha t$ with a Poisson process $\{N_t, t \geq 0\}$ and a positive drift $\alpha > 0$ and $L_t = t$ (cf. Klüppelberg et al. [12], p. 408). Continuity properties of μ for general $\{(\xi_t, L_t)\}$ were studied by Bertoin et al. [1], who showed that μ cannot have atoms unless μ is a Dirac measure, with this degenerate case also being characterised. Gjessing and Paulsen [8] derived the distribution of μ in a variety of situations; however, in all cases considered the distribution turned out to be absolutely continuous.

With these results in mind, it is natural to ask whether μ will always be absolutely continuous for general $\{(\xi_t, L_t)\}$, unless μ degenerates to a Dirac measure. That this is not the case, even when $\{\xi_t\}$ and $\{L_t\}$ are assumed to be independent, will be shown in the present article. More precisely, we will study in detail

$$(1.2) \quad \mu = \mathcal{L} \left(\int_0^{\infty-} e^{-(\log c)N_{s-}} dY_s \right) = \mathcal{L} \left(\int_0^{\infty-} c^{-N_{s-}} dY_s \right),$$

where c is a constant greater than 1 and $\{N_t\}$ and $\{Y_t\}$ are both Poisson processes with parameters a and b , respectively, and with $\{(N_t, Y_t)\}$ being a bivariate Lévy process. The integral in (1.2) is then an improper Stieltjes integral pathwise. From the strong law of large numbers, we see that the integral exists and is finite. Let T be the first jump time of $\{N_t\}$. Then

$$\int_0^{\infty-} c^{-N_{s-}} dY_s = Y_T + \int_T^{\infty-} c^{-N_{s-}} dY_s = Y_T + c^{-1} \int_0^{\infty-} c^{-N'_{s-}} dY'_s,$$

where $\{(N'_t, Y'_t)\}$ is an independent copy of $\{(N_t, Y_t)\}$. Hence, letting $\rho = \mathcal{L}(Y_T)$, we obtain

$$(1.3) \quad \widehat{\mu}(z) = \widehat{\rho}(z) \widehat{\mu}(c^{-1}z), \quad z \in \mathbb{R},$$

where $\widehat{\mu}(z)$ and $\widehat{\rho}(z)$ denote the characteristic functions of μ and ρ . It follows that

$$\widehat{\mu}(z) = \widehat{\mu}(c^{-k}z) \prod_{n=0}^{k-1} \widehat{\rho}(c^{-n}z), \quad k \in \mathbb{N},$$

and hence

$$(1.4) \quad \widehat{\mu}(z) = \prod_{n=0}^{\infty} \widehat{\rho}(c^{-n}z).$$

In general, if a distribution μ satisfies (1.3) with some distribution ρ , then μ is called c^{-1} -decomposable. Our study of the law μ is based on this c^{-1} -decomposability. The expression (1.4) shows that the law ρ controls μ . The properties of c^{-1} -decomposable distributions are studied by Wolfe [24], Bunge [2], Watanabe [22] and others. In particular, it is known that any non-degenerate c^{-1} -decomposable distribution is either continuous-singular or absolutely continuous (Wolfe [24]). A distribution μ is *self-decomposable* if and only if μ is c^{-1} -decomposable for all $c > 1$. In this case μ and ρ are infinitely divisible. In general if a distribution μ satisfies (1.3) with ρ being infinitely divisible, then μ is called c^{-1} -semi-selfdecomposable. For the law μ in (1.2), we are interested in the Hausdorff dimension $\dim(\mu)$ of μ , defined as the infimum of the Hausdorff dimensions of E over all Borel sets E satisfying $\mu(E) = 1$ (in some papers including [22], this is called upper Hausdorff dimension and denoted by $\dim^*(\mu)$). Watanabe [22] shows that, if a distribution μ is c^{-1} -decomposable satisfying (1.3) with a discrete distribution ρ , then $\dim(\mu) \leq H(\rho)/\log c$, where $H(\rho)$ is the entropy of ρ . It follows that the law μ in (1.2) is continuous-singular if $H(\rho)/\log c < 1$.

In Section 2 we will concentrate on the case where $\{N_t\}$ and $\{Y_t\}$ are independent. The law μ is determined by c and $q = b/(a+b)$, and so it is denoted by $\mu_{c,q}$. We will show that $\mu_{c,q}$ is continuous-singular if q is sufficiently small for fixed c , or if c is sufficiently large for fixed q , or if c is a Pisot-Vijayaraghavan (P.V.) number for any q . Further we will show that if c^{-1} is a Peres-Solomyak (P.S.) number, then $\mu_{c,q}$ is absolutely continuous for q sufficiently close to 1.

In Section 3 we treat the case where the independence of $\{N_t\}$ and $\{Y_t\}$ is not assumed. The Lévy process $\{(N_t, Y_t)\}$ then has Lévy measure concentrated on the three points $(1, 0)$, $(0, 1)$ and $(1, 1)$ and the amounts of the measure of these points are denoted by u , v and w . Letting $p = u/(u+v+w)$, $q = v/(u+v+w)$ and $r = w/(u+v+w)$, we will see that μ is determined by c , q and r , and ρ is by q and r , and hence denote $\mu = \mu_{c,q,r}$ and $\rho = \rho_{q,r}$. We call r the *dependence parameter* of $\{(N_t, Y_t)\}$, since $r = 0$ is equivalent to independence of $\{N_t\}$ and $\{Y_t\}$ and $r = 1$ means $\{N_t\} = \{Y_t\}$. If $r = 0$, then $\rho = \mathcal{L}(Y_T)$ is infinitely divisible as is seen from subordination theory, and hence μ is also infinitely divisible. But, if $r > 0$, the situation is more complicated. We will give complete description of the condition of infinite divisibility of $\mu_{c,q,r}$ and $\rho_{q,r}$ in terms of their parameters. It will turn out that infinite divisibility

of $\mu_{c,q,r}$ does not depend on c . It is shown in Niedbalska-Rajba [16] that there exists a c^{-1} -decomposable infinitely divisible distribution μ that satisfies (1.3) with a non-infinitely-divisible ρ . But, in our case, it will turn out that $\mu_{c,q,r}$ is infinitely divisible if and only if $\rho_{q,r}$ is so. We also address the problem of infinite divisibility of the symmetrisations μ^{sym} and ρ^{sym} of μ and ρ . Infinite divisibility of a distribution implies that of its symmetrisation, but there is a non-infinitely-divisible distribution whose symmetrisation is infinitely divisible, which is pointed out in pp.81–82 in Gnedenko and Kolmogorov [9]. Complete description of infinite divisibility of μ^{sym} and ρ^{sym} will be given, which provides new examples of this phenomenon in [9]. In the proof of non-infinite-divisibility in Section 3, we use three methods: (1) Katti's condition for distributions on nonnegative integers; (2) Lévy-Khintchine type representation of characteristic functions with signed measures in place of Lévy measures; (3) representation of the Laplace transforms of infinitely divisible distributions on $[0, \infty)$ in the form $e^{-\varphi(\theta)}$ with $\varphi'(\theta)$ being completely monotone.

In the latter half of Section 2 we consider a more general bivariate Lévy process $\{(N_t, Y_t)\}$ with independent components, where $\{N_t\}$ is a Poisson process and $\{Y_t\}$ is an integer valued Lévy process with finite second moment. The law μ in (1.2) with $c > 1$ is still c^{-1} -semi-selfdecomposable and infinitely divisible. We will study the convolution power μ^{t^*} of μ , that is, the distribution at time t of the Lévy process associated with μ . We will show that μ^{t^*} is continuous-singular if t is sufficiently small for fixed c , or if c is sufficiently large for fixed t , or if c is a P.V. number for any t , and that μ^{t^*} is absolutely continuous if c^{-1} is a P.S. number and t is sufficiently large. Thus the present paper provides a new class of examples of Lévy processes with distribution changing from continuous-singular to absolutely continuous as time passes. See Section 27 in Sato [20] and Watanabe's survey [23] for such time evolution of Lévy processes. We emphasise that here the distribution μ arises naturally as the stationary distribution of a generalised Ornstein-Uhlenbeck process.

We remark that Theorem 3.2 of Kondo et al. [13] on the law μ of the form (1.2) with $c = e$, $\{N_t\}$ Poisson and $\{N_t\}$ and $\{Y_t\}$ independent can be extended to general $c > 1$ without any change of the proof, and that Remark 3.3 of the same paper points out that such a distribution is either continuous-singular or absolutely continuous. This was the starting point of our research.

Throughout the paper, the set of all positive integers will be denoted by $\mathbb{N} = \{1, 2, 3, \dots\}$, while we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The set of integers is denoted by \mathbb{Z} . The Dirac measure at a point x will be denoted by δ_x . For general definitions and properties regarding Lévy processes and infinitely divisible distributions, we refer to Sato [20].

2. CONTINUITY PROPERTIES IN THE INDEPENDENT CASE

The first task in this section is to establish the c^{-1} -decomposability of the law in (1.2). Since this property prevails in a wider range, we formulate a more general result.

Proposition 2.1. *Suppose that $\{(\xi_t, L_t)\}$ is a bivariate Lévy process having the following properties: $\xi_t = (\log c)\tilde{\xi}_t$ with $c > 1$ and with $\{\tilde{\xi}_t\}$ being a compound Poisson*

process with Lévy measure supported on $\{\dots, -2, -1\} \cup \{1\}$ and $E\tilde{\xi}_t > 0$ for $t > 0$, and $\{L_t\}$ is a Lévy process with finite log-moment. Let μ be defined as

$$(2.1) \quad \mu := \mathcal{L} \left(\int_0^{\infty-} e^{-\xi_s} dL_s \right) = \mathcal{L} \left(\int_0^{\infty-} c^{-\tilde{\xi}_s} dL_s \right).$$

Then μ is c^{-1} -decomposable satisfying (1.3) with

$$(2.2) \quad \rho = \mathcal{L} \left(\int_0^T c^{-\tilde{\xi}_s} dY_s \right),$$

where T is the hitting time of 1 for $\{\tilde{\xi}_t\}$.

Proof. Let $\{\mathcal{F}_t, t \geq 0\}$ be the natural completed filtration of $\{(\xi_t, L_t), t \geq 0\}$. Then T is a stopping time with respect to $\{\mathcal{F}_t, t \geq 0\}$. The process $\{\tilde{\xi}_t\}$ moves by jumps of the height $\in \{\dots, -2, -1\} \cup \{1\}$ and satisfies $\xi_t \rightarrow \infty$ as $t \rightarrow \infty$. Hence T is finite a.s. Existence and finiteness of the integral in (2.1) follows from Theorem 2 of Erickson and Maller [6]. From the Lévy process version of the strong Markov property, we obtain

$$\begin{aligned} Z &:= \int_0^{\infty-} c^{-\tilde{\xi}_s} dL_s = \int_0^T c^{-\tilde{\xi}_s} dL_s + c^{-1} \int_{T+}^{\infty-} c^{-(\tilde{\xi}_s - \tilde{\xi}_T)} d(L_s - L_T)_s \\ &= W + c^{-1} Z', \quad \text{say,} \end{aligned}$$

where Z and Z' both have law μ , W is \mathcal{F}_T -measurable and Z' is independent of \mathcal{F}_T . This implies (1.3) with $\rho = \mathcal{L}(W)$, that is, with ρ of (2.2). \square

As in Watanabe [22], we use two classes of numbers, namely Pisot-Vijayaraghavan (P.V.) numbers (sometimes called Pisot numbers) and Peres-Solomyak (P.S.) numbers. A number $c > 1$ is called a *P.V. number* if there exists a polynomial $F(x)$ with integer coefficients with leading coefficient 1 such that c is a simple root of $F(x)$ and all other roots have modulus less than 1. Every positive integer greater than 1 is a P.V. number, but also $(1 + \sqrt{5})/2$ or the unique real root of $x^3 - x - 1 = 0$ are non-trivial examples. There exist countably infinitely many P.V. numbers which are not integers. See Peres, Schlag and Solomyak [18] for related information. On the other hand, following Watanabe [22], we call c^{-1} a *P.S. number* if $c > 1$ and if there are $p \in (1/2, 1)$ and $k \in \mathbb{N}$ such that the k th power of the characteristic function of the distribution of $\sum_{n=0}^{\infty} c^{-n} U_n$, where $\{U_n\}$ is i.i.d. with $P[U_n = 0] = 1 - P[U_n = 1] = p$, is integrable. Watanabe [22] pointed out that the paper [19] of Peres and Solomyak contains the proof that the set of P.S. numbers in the interval $(0, 1)$ has Lebesgue measure 1. However, according to [22], an explicit example of a P.S. number is not known so far. As follows from the results of [22], the set of P.V. numbers and the set of reciprocals of P.S. numbers are disjoint. The *entropy* $H(\rho)$ of a discrete probability measure ρ on \mathbb{R} is given by

$$H(\rho) := - \sum_{a \in C} \rho(\{a\}) \log \rho(\{a\}),$$

where C is the carrier of μ . Here, as in [20], a measure ρ on \mathbb{R} is said to be *discrete* if there is a countable set C such that $\rho(\mathbb{R} \setminus C) = 0$. The *carrier* of a discrete measure is the set of points with positive mass.

Now we can formulate one of our main results.

Theorem 2.2. *Assume that $\{N_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are independent Poisson processes with parameters $a > 0$ and $b > 0$, respectively. Let*

$$(2.3) \quad q := b/(a + b)$$

and

$$(2.4) \quad \mu_{c,q} := \mathcal{L} \left(\int_0^{\infty-} c^{-N_s-} dY_s \right)$$

with $c > 1$. Let

$$(2.5) \quad h(q) := \frac{1}{1-q} \log \frac{1}{1-q} - \frac{q}{1-q} \log \frac{q}{1-q}.$$

Then the following are true:

(a) *The Hausdorff dimension of $\mu_{c,q}$ is estimated as*

$$(2.6) \quad \dim(\mu_{c,q}) \leq \frac{h(q)}{\log c} \quad \text{for all } c > 1 \text{ and } q \in (0, 1).$$

(b) *If c is a P.V. number, then $\mu_{c,q}$ is continuous-singular for all $q \in (0, 1)$.*

(c) *If c^{-1} is a P.S. number, then there are constants $0 < q_0 \leq q_1 < 1$ such that $\mu_{c,q}$ is continuous-singular for all $q \in (0, q_0)$, absolutely continuous without bounded continuous density for all $q \in (q_0, q_1)$ if $q_0 < q_1$, and absolutely continuous with bounded continuous density for all $q \in (q_1, 1)$.*

The estimate (2.6) is meaningful only when $h(q)/\log c < 1$, as the Hausdorff dimension of any measure on the line is less than or equal to 1. In Theorem 2.2 the law μ in (1.2) depends only on c and q , as will be seen in the proof. The function $h(q)$ is continuous and strictly increasing on $(0, 1)$ and tends to 0 as $q \downarrow 0$ in the speed $h(q) \sim q \log(1/q)$. Under the same assumption we have the following consequences.

Corollary 2.3. (a) *Fix $c > 1$. If $q \in (0, 1)$ is such that $h(q) < \log c$, then $\mu_{c,q}$ is continuous-singular.*

(b) *Let $c > 2$. If*

$$0 < q < 1 - (\log 2 / \log c),$$

then $\mu_{c,q}$ is continuous-singular.

(c) *Fix $q > 0$. If $c > e^{h(q)}$, then $\mu_{c,q}$ is continuous-singular.*

For example of (b), in the case $c = e$, $\mu_{e,q}$ is continuous-singular if

$$q < 1 - \log 2 \approx 0.30685.$$

In the case $q = 1/2$, (c) says that $\mu_{c,1/2}$ is continuous-singular if $c > 4$.

Proof of Corollary 2.3. Assertions (a) and (c) are clear from (2.6), since probability measures with Hausdorff dimension < 1 are singular and since $\mu_{c,q}$ is continuous, as will be seen in the proof of Theorem 2.2. Assertion (b) is a consequence of (a), since

$$h(q) = \frac{1}{1-q} \left((1-q) \log \frac{1}{1-q} + q \log \frac{1}{q} \right) \leq \frac{1}{1-q} \log 2$$

by concavity of the function $\log x$. \square

Proof of Theorem 2.2. Notice that, by Proposition 2.1, $\mu_{c,q}$ is c^{-1} -decomposable and T is now the first jump time of $\{N_t\}$. The law of T is exponential with parameter a and the law ρ in (2.2) equals $\mathcal{L}(Y_T)$. This law $\rho = \mathcal{L}(Y_T)$ is geometric with parameter p , i.e.,

$$\rho(\{k\}) = pq^k, \quad k \in \mathbb{N}_0,$$

where $p = 1 - q = a/(a + b)$. Indeed, using the independence of $\{N_t\}$ and $\{Y_t\}$,

$$\begin{aligned} \rho(\{k\}) &= P(Y_T = k) = \int_0^\infty P(Y_t = k) a e^{-at} dt \\ &= \int_0^\infty e^{-bt} \frac{b^k t^k}{k!} a e^{-at} dt = \frac{a}{a+b} \left(\frac{b}{a+b} \right)^k. \end{aligned}$$

Hence ρ is compound Poisson with Lévy measure

$$a_m := \nu_\rho(\{m\}) = q^m/m, \quad m \in \mathbb{N},$$

that is,

$$(2.7) \quad \widehat{\rho}(z) = \exp \left[\sum_{m=1}^{\infty} (e^{imz} - 1) a_m \right].$$

Thus it follows from (1.4) that $\mu = \mu_{c,q}$ is infinitely divisible with characteristic function

$$(2.8) \quad \widehat{\mu}(z) = \exp \left[\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (e^{imc^{-n}z} - 1) a_m \right].$$

The entropy $H(\rho)$ of ρ equals $h(q)$, since

$$(2.9) \quad H(\rho) = -p(\log p) \sum_{k=0}^{\infty} q^k - p(\log q) \sum_{k=0}^{\infty} k q^k = -\log p - \frac{q}{p} \log q.$$

Now we obtain assertion (a) from Theorem 2.2 of Watanabe [22].

Let us prove (b). Assume that c is a P.V. number. The following proof of continuous-singularity of $\mu = \mu_{c,q}$ is based on an idea of Erdős [5] as in [22]. Since μ is c^{-1} -decomposable and non-degenerate, Wolfe's theorem in [24] (or Theorem 27.15 of [20]) tells us that μ is either continuous-singular or absolutely continuous. So it is enough to show that it is not absolutely continuous. Thus, by virtue of the Riemann–Lebesgue theorem, it is enough to find a sequence $z_k \rightarrow \infty$ such that

$$\limsup_{k \rightarrow \infty} |\widehat{\mu}(z_k)| > 0.$$

By the definition of a P.V. number, there is a polynomial $F(x) = x^N + a_{N-1}x^{N-1} + \dots + a_1x + a_0$ such that $a_{N-1}, \dots, a_0 \in \mathbb{Z}$, $F(c) = 0$, and the totality $\{\alpha_1, \dots, \alpha_N\}$ of roots of $F(x)$ satisfies $\alpha_1 = c$ and $|\alpha_j| < 1$ for $2 \leq j \leq N$. Choose $z_k = 2\pi c^k$. Then

$$|\widehat{\mu}(z_k)| = \exp\left(-\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (1 - \cos(2\pi m c^{k-n})) a_m\right) = \exp\left(-\sum_{m=1}^{\infty} (S_m + R_m) a_m\right)$$

with

$$S_m = \sum_{n=0}^k (1 - \cos(2\pi m c^{k-n})), \quad R_m = \sum_{n=k+1}^{\infty} (1 - \cos(2\pi m c^{k-n})).$$

Now

$$S_m = \sum_{n=0}^k (1 - \cos(2\pi m c^n)) = \sum_{n=0}^k \left(1 - \cos\left(2\pi m \sum_{j=2}^N \alpha_j^n\right)\right),$$

since $c^n = \sum_{j=1}^N \alpha_j^n - \sum_{j=2}^N \alpha_j^n$ and $\sum_{j=1}^N \alpha_j^n$ is an integer. The latter is a consequence of the symmetric function theorem in algebra (e.g. Lang [14], Section IV.6), implying that $\sum_{j=1}^N \alpha_j^n$, as a symmetric function of $\alpha_1, \dots, \alpha_N$, can be expressed as a polynomial with integer coefficients in the elementary symmetric functions $\sigma_1, \dots, \sigma_N$, which are integer valued themselves since F has integer coefficients with leading coefficient 1. Choose $0 < \delta < 1$ such that $|\alpha_j| < \delta$ for $j = 2, \dots, N$. Then, with some constants C_1, C_2, C_3 ,

$$S_m \leq C_1 \sum_{n=0}^k \left(m \sum_{j=2}^N \alpha_j^n\right)^2 \leq C_2 m^2 \sum_{n=0}^k \sum_{j=2}^N |\alpha_j|^{2n} \leq C_3 m^2 \sum_{n=0}^k \delta^{2n} \leq C_3 m^2 / (1 - \delta^2).$$

Further, we have

$$R_m \leq C_1 \sum_{n=1}^{\infty} (m c^{-n})^2 = C_1 m^2 / (c^2 - 1).$$

Hence, it follows that

$$|\widehat{\mu}(z_k)| \geq \exp\left[-\sum_{m=1}^{\infty} a_m m^2 \left(\frac{C_3}{1 - \delta^2} + \frac{C_1}{c^2 - 1}\right)\right].$$

This shows that $\limsup_{k \rightarrow \infty} |\widehat{\mu}(z_k)| > 0$, since $\sum_{m=1}^{\infty} a_m m^2 < \infty$. Thus (b) is true.

For the proof of (c), suppose that c^{-1} is a P.S. number, and let $p \in (1/2, 1)$ and $k \in \mathbb{N}$ as in the definition of a P.S. number. The following proof was suggested by an argument of Watanabe [22] p.392-393. Let $K := k |\log(2p - 1)|/2$, which is positive. Then, Equation (2.4) of [22] tells us that

$$(2.10) \quad \int_{-\infty}^{\infty} \exp\left\{\alpha \sum_{n=0}^{\infty} (\cos(c^{-n}u) - 1)\right\} du < \infty \quad \text{whenever } \alpha \geq K.$$

Recall (2.8). Let

$$\alpha_0 = \sum_{m=1}^{\infty} a_m = \sum_{m=1}^{\infty} \frac{q^m}{m} = \log \frac{1}{1 - q}.$$

Then it follows from Jensen's inequality that

$$\begin{aligned} \int_{-\infty}^{\infty} |\widehat{\mu}(z)| dz &= 2 \int_0^{\infty} \exp \left[\frac{1}{\alpha_0} \sum_{m=1}^{\infty} a_m \left(\alpha_0 \sum_{n=0}^{\infty} (\cos(mc^{-n}z) - 1) \right) \right] dz \\ &\leq 2 \int_0^{\infty} \left[\frac{1}{\alpha_0} \sum_{m=1}^{\infty} a_m \exp \left(\alpha_0 \sum_{n=0}^{\infty} (\cos(mc^{-n}z) - 1) \right) \right] dz \\ &= \frac{2}{\alpha_0} \left(\sum_{m=1}^{\infty} \frac{a_m}{m} \right) \int_0^{\infty} \exp \left(\alpha_0 \sum_{n=0}^{\infty} (\cos(c^{-n}u) - 1) \right) du. \end{aligned}$$

The latter is finite whenever $\alpha_0 \geq K$ by (2.10), that is, whenever $q \geq 1 - e^{-K}$. Hence μ is absolutely continuous with bounded continuous density if $q \geq 1 - e^{-K}$. On the other hand we know from (a) that μ is continuous-singular for small enough q . The expression (2.8) shows that μ has Lévy measure

$$(2.11) \quad \nu_{\mu} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{q^m}{m} \delta_{c^{-n}m},$$

which is increasing in $q \in (0, 1)$. Thus $\mu_{c,q'}$ is a convolution factor of $\mu_{c,q}$ if $q' < q$. Thus, recalling Lemma 27.1 of [20], we obtain assertion (c). \square

Keeping the assumptions that $\{N_t\}$ is a Poisson process and that $\{N_t\}$ and $\{Y_t\}$ are independent, we will allow $\{Y_t\}$ more general than in Theorem 2.2. We cannot give the properties of μ itself (except when c is a P.V. number), but we describe the properties of the convolution power μ^{t*} of μ .

Theorem 2.4. *Assume that $\{N_t\}$ is a Poisson process and $\{Y_t\}$ is an integer valued Lévy process, not identically zero, with finite second moment, and that $\{N_t\}$ and $\{Y_t\}$ are independent. Let*

$$(2.12) \quad \mu_c := \mathcal{L} \left(\int_0^{\infty-} c^{-N_s-} dY_s \right),$$

where $c > 1$. Let T be the first jump time of $\{N_t\}$ and let $\rho = \mathcal{L}(Y_T)$. Then the following are true.

(a) *The entropy $H(\rho^{t*})$ is a finite, continuous, strictly increasing function of $t \in [0, \infty)$, vanishing at $t = 0$.*

(b) *It holds true that*

$$(2.13) \quad \dim(\mu_c^{t*}) \leq \frac{H(\rho^{t*})}{\log c} \quad \text{for all } t \geq 0 \text{ and } c > 1.$$

(c) *If c is a P.V. number, then μ_c^{t*} is continuous-singular for all $t > 0$.*

(d) *If c^{-1} is a P.S. number, then there are t_0 and t_1 with $0 < t_0 \leq t_1 < \infty$ such that μ_c^{t*} is continuous-singular for all $t \in (0, t_0)$, absolutely continuous without bounded continuous density for all $t \in (t_0, t_1)$ if $t_0 < t_1$, and absolutely continuous with bounded continuous density for all $t \in (t_1, \infty)$.*

Corollary 2.5. *For each $c > 1$, $\mu_c^{t^*}$ is continuous-singular for all sufficiently small $t > 0$. For each $t > 0$, $\mu_c^{t^*}$ is continuous-singular for all sufficiently large $c > 1$.*

This is an obvious consequence of (a) and (b) of the theorem.

Lemma 2.6. *If ρ is a distribution on \mathbb{Z} with finite absolute moment of order $1 + \varepsilon$ for some $\varepsilon > 0$, then its entropy $H(\rho)$ is finite.*

Proof. Let $\rho = \sum_{m=-\infty}^{\infty} p_m \delta_m$. Then $\sum_{m=-\infty}^{\infty} |m|^{1+\varepsilon} p_m < \infty$. Hence there is a constant $C > 0$ such that $p_m \leq C|m|^{-1-\varepsilon}$. The function $f(x) = x \log(1/x)$ is increasing for $0 \leq x \leq e^{-1}$. Hence,

$$H(\rho) \leq \sum_{|m| \leq m_0} p_m \log(1/p_m) + \sum_{|m| > m_0} C|m|^{-1-\varepsilon} \log((C|m|^{-1-\varepsilon})^{-1}) < \infty$$

with an appropriate choice of m_0 . □

Proof of Theorem 2.4. By Proposition 2.1, μ is c^{-1} -decomposable satisfying (1.3) with $\rho = \mathcal{L}(Y_T)$. Since $\{Y_t\}$ is integer valued, ρ is a distribution on \mathbb{Z} . Since, for some constant $C > 0$, $E[Y_t^2] \leq C(t + t^2)$ for all $t \geq 0$ and since T is exponentially distributed,

$$E[Y_T^2] = \int_{(0, \infty)} E[Y_s^2] P[T \in ds] \leq C \int_{(0, \infty)} (s + s^2) P[T \in ds] < \infty.$$

Lemma 2.6 then shows that $H(\rho) < \infty$. Notice that ρ is the distribution at time 1 of the Lévy process $\{Y_{T_t}\}$ obtained by subordination of $\{Y_t\}$ by an independent gamma subordinator $\{T_t\}$. The process $\{Y_{T_t}\}$ has drift 0, since it is integer valued (use Corollary 24.6 of [20]; another proof is to use Theorem 30.10 of [20]). Thus ρ is compound Poisson. In particular, μ_c is c^{-1} -semi-selfdecomposable. We obtain (a) from Proposition 5.1 of Watanabe [22] or Exercise 29.24 of Sato [20]. The property of being c^{-1} -semi-selfdecomposable is inherited by going to convolution powers, since (1.3) implies

$$\widehat{\mu_c^{t^*}}(z) = \widehat{\rho^{t^*}}(z) \widehat{\mu_c^{t^*}}(c^{-1}z)$$

for any $t \geq 0$. Then, applying Theorem 2.2 of [22] to the law $\mu_c^{t^*}$, we get (b). The characteristic function of ρ is given by

$$\widehat{\rho}(z) = \exp \left(\sum_{m \in \mathbb{Z} \setminus \{0\}} (e^{imz} - 1) a_m \right), \quad z \in \mathbb{R},$$

with some $a_m \geq 0$ satisfying $\sum_{m \in \mathbb{Z} \setminus \{0\}} m^2 a_m < \infty$. The latter is because ρ has finite second moment. Hence the proof of (c) is given entirely in the same way as that of Theorem 2.2(b). If c^{-1} is a P.S. number and t is sufficiently large, then $\mu_c^{t^*}$ is absolutely continuous with bounded continuous density, which is shown in the same way as Theorem 2.2(c), or we can apply Theorem 2.1 of Watanabe [22]. This proves (d). □

In the set-up of Theorem 2.2, the distribution ρ is geometric with parameter $1 - q$. Thus, the estimate of $H(\rho^{t^*})$ for this ρ is of some interest, in connection with the estimate (2.13) of Theorem 2.4.

Proposition 2.7. *If ρ is geometric with parameter $1 - q$, then*

$$(2.14) \quad H(\rho^{t*}) \leq t \left[\frac{1}{p} (1 + 2 \log \frac{1}{p}) + \frac{q}{p} \log \frac{1}{t} \right] \quad \text{for } 0 < t \leq 1,$$

where $p = 1 - q$.

Proof. The distribution ρ^{t*} , $t > 0$, is negative binomial distribution with parameters t and p , i.e.,

$$\rho^{t*}(\{k\}) = \binom{-t}{k} p^t (-q)^k, \quad k \in \mathbb{N}_0.$$

To estimate $H(\rho^{t*})$ from above, observe that, for $0 < t \leq 1$ and $k \in \mathbb{N}$,

$$t p^t q^k / k \leq \rho^{t*}(\{k\}) \leq t q^k,$$

so that

$$\begin{aligned} H(\rho^{t*}) &= - \sum_{k=0}^{\infty} \rho^{t*}(\{k\}) \log \rho^{t*}(\{k\}) \\ &\leq -(\log p^t) + \sum_{k=1}^{\infty} t q^k (\log k - \log(tp^t) - k \log q) \\ &\leq t \left[\log \frac{1}{p} + \frac{1}{p} \log \frac{1}{p} - \frac{q}{p} \log(tp^t) - \frac{q}{p^2} \log q \right], \end{aligned}$$

where we used $\sum_{k=1}^{\infty} k q^k = q/p^2$ and

$$\sum_{k=1}^{\infty} q^k \log k \leq \sum_{k=1}^{\infty} q^k \sum_{n=1}^k \frac{1}{n} = \frac{1}{p} \log \frac{1}{p},$$

cf. Gradshteyn and Ryzhik [10], Formula 1.513.6. Recalling that $p^t \geq p$ since $t \leq 1$, this can be further estimated to

$$H(\rho^{t*}) \leq t \left(\frac{2}{p} \log \frac{1}{p} + \frac{q}{p} \log \frac{1}{t} + \frac{q}{p^2} \log \frac{1}{q} \right).$$

Together with $(q/p) \log(1/q) = (q/p) \log(1 + p/q) \leq 1$, this gives (2.14). \square

Let $g_q(t)$ denote the function on the right-hand side of (2.14). This is continuous and strictly increasing on $(0, 1]$ with $\lim_{t \downarrow 0} g_q(t) = 0$. For $\mu_{c,q}$ of (2.4), we can estimate $\dim(\mu_{c,q}^{t*})$ by $g_q(t)/\log c$ if $0 < t \leq 1$.

The following proposition explains why Theorems 2.2 and 2.4 have similarity. We say that $\{U_q, 0 \leq q < 1\}$ is an *additive process* if it is stochastically continuous, with independent increments, starting at the origin and having cadlag paths.

Proposition 2.8. *Fix $c > 1$. Then there exists an additive process $\{U_q, 0 \leq q < 1\}$ with time parameter $q \in [0, 1)$ such that $\mathcal{L}(U_q)$ equals $\mu_{c,q}$ of (2.4) for $q > 0$ and $\mathcal{L}(U_0) = \delta_0$.*

Proof. As we observed in the proof of Theorem 2.2, $\mu_{c,q}$ has characteristic function (2.8) and its Lévy measure (2.11) is increasing in $q \in (0, 1)$. Furthermore, for each z , $\widehat{\mu}_{c,q}(z)$ is continuous in $q \in (0, 1)$ and tends to 1 as $q \downarrow 0$. Hence we can apply the analogue of Theorem 9.7 of [20] to time parameter running on $[0, 1)$ and find an additive process in law corresponding to $\mu_{c,q}$. It has a cadlag modification by the analogue of Theorem 11.5 of [20]. \square

3. THE DEPENDENT CASE: INFINITE DIVISIBILITY AND CONTINUITY PROPERTIES

In Theorem 2.2 we studied the law $\mu_{c,q}$ of $\int_0^{\infty-} c^{-N_{s-}} dY_s$ in the case where $\{N_t\}$ and $\{Y_t\}$ were independent Poisson processes. In particular, we showed infinite divisibility of $\mu_{c,q}$. In this section we relax the assumption of independence. Suppose that $\{(N_t, Y_t), t \geq 0\}$ is a bivariate Lévy process such that $\{N_t\}$ is a Poisson process with parameter $a > 0$ and $\{Y_t\}$ is a Poisson process with parameter $b > 0$. It then follows easily that $\{(N_t, Y_t)\}$ has no Gaussian part, zero drift, and Lévy measure $\nu_{(N,Y)}$ concentrated on the set $\{(1, 0), (0, 1), (1, 1)\}$, consisting of three points (e.g. [20], Proposition 11.10). Denote

$$u := \nu_{(N,Y)}(\{(1, 0)\}), \quad v := \nu_{(N,Y)}(\{(0, 1)\}), \quad \text{and} \quad w := \nu_{(N,Y)}(\{(1, 1)\}).$$

Then $u, v, w \geq 0$, $u + w = a$ and $v + w = b$. Let

$$p := \frac{u}{u + v + w}, \quad q := \frac{v}{u + v + w}, \quad r := \frac{w}{u + v + w},$$

so that $p, q, r \in [0, 1]$, $p + q + r = 1$, $p + r > 0$ and $q + r > 0$. If $r = 0$, then $\{N_t\}$ and $\{Y_t\}$ are independent, the case which was treated in Theorem 2.2. If $r = 1$, then $\{N_t\} = \{Y_t\}$. So we call r the *dependence parameter* of $\{(N_t, Y_t)\}$. For $c > 1$ denote

$$(3.1) \quad \mu_{c,q,r} := \mathcal{L} \left(\int_0^{\infty-} c^{-N_{s-}} dY_s \right),$$

where the almost sure convergence to a finite random variable follows again from Erickson and Maller [6] or directly from the strong law of large numbers. If $r = 0$, then $\mu_{c,q,r}$ equals $\mu_{c,q}$ of (2.4) in Theorem 2.2. If $r = 1$ (i.e. $p = q = 0$), then it follows from $\{N_t\} = \{Y_t\}$ that

$$\int_0^{\infty-} c^{-N_{s-}} dY_s = \int_0^{\infty-} c^{-N_{s-}} dN_s = \sum_{j=0}^{\infty} c^{-j} = \frac{c}{c-1},$$

which is degenerate to a constant. So throughout this section we will assume that $p + q > 0$ in addition to the above mentioned conditions $p + r > 0$ and $q + r > 0$. That is, $p, q, r < 1$. All propositions and theorems in this section are in this set-up. By Theorem 2.2 in Bertoin et al. [1], $\mu_{c,q,r}$ will then not degenerate to a Dirac measure, hence will be a continuous distribution. More strongly, since Proposition 2.1 is applicable, $\mu_{c,q,r}$ is c^{-1} -decomposable and, by virtue of Wolfe's theorem in [24], $\mu_{c,q,r}$ is either continuous-singular or absolutely continuous. We define $\rho_{q,r}$ in the following way: if $q > 0$, denote by σ_q a geometric distribution with parameter $1 - q$, i.e. $\sigma_q(\{k\}) = (1 - q)q^k$ for $k = 0, 1, \dots$, and denote

$$(3.2) \quad \rho = \rho_{q,r} := (1 + r/q)\sigma_q - (r/q)\delta_0,$$

so that $\rho_{q,r}$ is a probability distribution concentrated on \mathbb{N}_0 with

$$(3.3) \quad \rho_{q,r}(\{0\}) = (1 + r/q)(1 - q) - (r/q) = 1 - q - r = p;$$

if $q = 0$, let $\rho_{0,r}$ be a Bernoulli distribution with parameter $r \in (0, 1)$, i.e.

$$(3.4) \quad \rho_{0,r}(\{1\}) = 1 - \rho_{0,r}(\{0\}) = r.$$

Proposition 3.1. *We have*

$$(3.5) \quad \widehat{\mu}_{c,q,r}(z) = \widehat{\rho}_{q,r}(z) \widehat{\mu}_{c,q,r}(c^{-1}z), \quad z \in \mathbb{R}.$$

In particular, $\mu_{c,q,r}$ is c^{-1} -decomposable and determined by c , q and r .

Proof. Since we can use Proposition 2.1, we have only to show that $\mathcal{L}(Y_T) = \rho_{q,r}$, where T is the time of the first jump of $\{N_t\}$, i.e. the time of the first jump of $\{(N_t, Y_t)\}$ with size in $\{(1, 0), (1, 1)\}$. Let $S_i \in \mathbb{R}^2$ be the size of the i th jump of $\{(N_t, Y_t)\}$. Then we have for $k \geq 1$

$$Y_T = k \iff [S_1 = \dots = S_{k-1} = (0, 1), S_k = (1, 1)] \\ \text{or } [S_1 = \dots = S_k = (0, 1), S_{k+1} = (1, 0)]$$

as well as

$$Y_T = 0 \iff S_1 = (1, 0).$$

Since

$$P[S_i = (1, 0)] = p, \quad P[S_i = (0, 1)] = q, \quad P[S_i = (1, 1)] = r,$$

it follows that $P(Y_T = 0) = p$ and, for $k \geq 1$, $P(Y_T = k) = q^{k-1}r + q^k p$. From this follows easily that $\mathcal{L}(Y_T) = \rho_{q,r}$ for $q > 0$, while it is a Bernoulli distribution with parameter r for $q = 0$. \square

It is of interest whether $\mu_{c,q,r}$ is infinitely divisible or not. It is also of interest whether the symmetrisation $(\mu_{c,q,r})^{\text{sym}}$ of $\mu_{c,q,r}$ is infinitely divisible or not. Recall that the symmetrisation μ^{sym} of a distribution μ is defined to be the distribution with characteristic function $\widehat{\mu^{\text{sym}}}(z) = |\widehat{\mu}(z)|^2$, $z \in \mathbb{R}$. If X is a random variable such that $\mathcal{L}(X) = \mu$ and X' is an independent copy of X , then $\mu^{\text{sym}} = \mathcal{L}(X - X')$. Thus infinite divisibility of μ implies that of μ^{sym} , but the converse is not true, as is mentioned in the Introduction. Before going to $\mu_{c,q,r}$, we will first settle the question of infinite divisibility of $\rho_{q,r}$ and $(\rho_{q,r})^{\text{sym}}$.

Lemma 3.2. *Assume $q > 0$ and let $\rho = \rho_{q,r}$. Then the following hold true:*

- (a) *If $r \leq pq$, or if $p = 0$, then ρ is infinitely divisible.*
- (b) *If $r > pq$ and $p > 0$, then ρ is not infinitely divisible.*
- (c) *If $r > pq$ and $p > 0$, then ρ^{sym} is infinitely divisible if and only if*

$$(3.6) \quad p \leq qr.$$

We remark that if $0 \leq \alpha \leq 1$, then $(1 - \alpha)\sigma_q + \alpha\delta_0$ is infinitely divisible, since convex combinations of two geometric distributions are infinitely divisible (see pp.379–380 in Steutel and van Harn [21]), and the Dirac measure δ_0 is a limit of geometric distributions. Assertions (a) and (b) of the lemma above show in what extent this fact can be generalised to negative α .

Proof of Lemma 3.2. The characteristic function of ρ is given by

$$(3.7) \quad \widehat{\rho}(z) = (1 + r/q)\widehat{\sigma}_q(z) - r/q = (1 + r/q)\frac{1 - q}{1 - qe^{iz}} - r/q = \frac{p + re^{iz}}{1 - qe^{iz}}, \quad z \in \mathbb{R}.$$

(a) If $p = 0$, then $\rho(\{0\}) = 0$ by (3.3), and $\rho(\{k\}) = (1 + (1 - q)/q)(1 - q)q^k = (1 - q)q^{k-1}$ for $k = 1, 2, \dots$, and thus ρ is a geometric distribution translated by 1, hence infinitely divisible. So assume that $r \leq pq$. Then $p > 0$ (otherwise $p = r = 0$, contradicting $p + r > 0$). Since $p = (1 - q)/(1 + r/p)$, it follows from (3.7) that

$$\widehat{\rho}(z) = \exp \left[\log(1 - q) - \log \left(1 + \frac{r}{p} \right) + \log \left(1 + \frac{r}{p} e^{iz} \right) - \log(1 - qe^{iz}) \right].$$

Hence

$$(3.8) \quad \widehat{\rho}(z) = \exp \left[\sum_{k=1}^{\infty} (e^{ikz} - 1) \frac{q^k}{k} \left(1 - \left(-\frac{r}{pq} \right)^k \right) \right].$$

Recall that $r/(pq) \leq 1$. It follows that ρ is infinitely divisible with Lévy measure $\nu_\rho(\{k\}) = k^{-1}q^k(1 - (-r/(pq))^k)$, $k = 1, 2, \dots$, and drift 0.

(b) Now assume that $r > pq$ and $p > 0$. By Katti's criterion (Corollary 51.2 of Sato [20]), a distribution $\sum_{n=0}^{\infty} p_n \delta_n$ with $p_0 > 0$ is infinitely divisible if and only if there are $q_n \geq 0$, $n = 1, 2, \dots$, such that

$$np_n = \sum_{k=1}^n k q_k p_{n-k}, \quad n = 1, 2, \dots$$

In fact, the equations above determine q_n , $n = 1, 2, \dots$, successively in a unique way. Infinite divisibility of $\sum_{n=0}^{\infty} p_n \delta_n$ is equivalent to nonnegativity of all q_n . Now let $p_n = \rho(\{n\})$. The first two equations are $p_1 = q_1 p_0$ and $2p_2 = q_1 p_1 + 2q_2 p_0$. Hence $q_1 = p_1/p_0 > 0$, but

$$q_2 = \frac{2p_2 - q_1 p_1}{2p_0} = \frac{(1 + r/q)(1 - q)q^2}{2p^2} [1 - q - (r/q)(1 + q)] < 0,$$

since $r > pq$. This shows that ρ is not infinitely divisible.

(c) Assume again that $r > pq$ and $p > 0$. From (3.7) it can be seen that $\widehat{\rho}$ will have a real zero if and only if $p = r$. In that case, also $|\widehat{\rho}|^2$ will have a real zero, and hence ρ^{sym} cannot be infinite divisible, in agreement with the fact that (3.6) is violated for $p = r$. So in the following we assume that $p \neq r$. From (3.7) we have

$$\begin{aligned} \log(|\widehat{\rho}(z)|^2) &= \log |p + re^{iz}|^2 - \log |1 - qe^{iz}|^2 \\ &= \log(p^2 + 2pr \cos z + r^2) - \log(1 - 2q \cos z + q^2). \end{aligned}$$

Write

$$A = \frac{2pr}{p^2 + r^2}, \quad B = \frac{2q}{1 + q^2}, \quad C = \frac{p^2 + r^2}{1 + q^2}.$$

Then $0 < A < 1$, $0 < B < 1$, and $C > 0$ (recall that $0 < q < 1$ and $p \neq r$), and we obtain

$$\log(|\widehat{\rho}(z)|^2) = \log C + \log(1 + A \cos z) - \log(1 - B \cos z)$$

$$\begin{aligned}
&= \log C - \sum_{k=1}^{\infty} k^{-1} (-A)^k \cos^k z + \sum_{k=1}^{\infty} k^{-1} B^k \cos^k z \\
&= \log C + \sum_{k=1}^{\infty} k^{-1} 2^{-k} (-(-A)^k + B^k) \sum_{l=0}^k \binom{k}{l} \cos(k-2l)z,
\end{aligned}$$

since

$$\cos^k z = 2^{-k} (e^{iz} + e^{-iz})^k = 2^{-k} \sum_{l=0}^k \binom{k}{l} e^{ilz - i(k-l)z} = 2^{-k} \sum_{l=0}^k \binom{k}{l} \cos(k-2l)z.$$

Letting $z = 0$, we get

$$0 = \log C + \sum_{k=1}^{\infty} k^{-1} 2^{-k} (-(-A)^k + B^k) \sum_{l=0}^k \binom{k}{l}.$$

Hence

$$\log(|\widehat{\rho}(z)|^2) = \sum_{k=1}^{\infty} k^{-1} 2^{-k} (-(-A)^k + B^k) \sum_{l=0}^k \binom{k}{l} (\cos(k-2l)z - 1).$$

Write

$$(3.9) \quad D_k = k^{-1} 2^{-k} (-(-A)^k + B^k).$$

Then we get

$$(3.10) \quad \log(|\widehat{\rho}(z)|^2) = 2 \sum_{k=1}^{\infty} D_k \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{l} (\cos(k-2l)z - 1),$$

where $\lfloor (k-1)/2 \rfloor$ is the largest integer not exceeding $(k-1)/2$. Since

$$2 \sum_{k=1}^{\infty} |D_k| \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{l} \leq \sum_{k=1}^{\infty} 2^k |D_k| \leq \sum_{k=1}^{\infty} k^{-1} (A^k + B^k) < \infty,$$

we can change the order of summation in the right-hand side of (3.10). First, rewrite, with $m = k - 2l$,

$$\log(|\widehat{\rho}(z)|^2) = 2 \sum_{k=1}^{\infty} D_k \sum_m \binom{k}{(k-m)/2} (\cos mz - 1),$$

where m runs over $k, k-2, \dots, 3, 1$ if k is odd ≥ 1 and over $k, k-2, \dots, 4, 2$ if k is even ≥ 2 . Then

$$(3.11) \quad \log(|\widehat{\rho}(z)|^2) = 2 \sum_{m=1}^{\infty} E_m (\cos mz - 1).$$

Here E_m is a linear sum of $D_m, D_{m+2}, D_{m+4}, \dots$ with positive integer coefficients. More precisely,

$$(3.12) \quad E_m = \sum_{h=0}^{\infty} D_{m+2h} \binom{m+2h}{h}.$$

Hence

$$(3.13) \quad \log(|\widehat{\rho}(z)|^2) = \int_{\mathbb{R}} (e^{ixz} - 1 - ixz 1_{(-1,1)}(x)) \nu(dx),$$

where ν is the symmetric signed measure

$$(3.14) \quad \nu = \sum_{m=1}^{\infty} E_m (\delta_m + \delta_{-m}).$$

Let $F = r/(pq)$. Then $F > 1$. A simple calculation then shows that $A \leq B$ if and only if $F - 1 \leq q^2(F^2 - F)$, which is equivalent to $1 \leq q^2F$, that is, (3.6). Now, if (3.6) holds, then $A \leq B$ and hence $D_k \geq 0$ for all k , which implies $E_m \geq 0$ for all m and ρ^{sym} is infinitely divisible with Lévy-Khintchine representation (3.13) and (3.14). If (3.6) does not hold, then $A > B$, $D_k < 0$ for all even k , and $E_m < 0$ for all even m , which implies, by (3.13) and (3.14), that ρ^{sym} is not infinitely divisible (see Exercise 12.3 of [20]). \square

We can now give criteria when $\rho_{q,r}$ and $\mu_{c,q,r}$ and their symmetrisations are infinitely divisible. Observe that infinite divisibility of $\rho_{q,r}$ implies that of $\mu_{c,q,r}$. Similarly, infinite divisibility of $(\rho_{q,r})^{\text{sym}}$ implies that of $(\mu_{c,q,r})^{\text{sym}}$. The converse of these two implications are by no means clear, as we know Niedbalska-Rajba's example mentioned in the Introduction. However the following theorem will say that the converse is true for $\mu_{c,q,r}$ and $\rho_{q,r}$ and for $(\mu_{c,q,r})^{\text{sym}}$ and $(\rho_{q,r})^{\text{sym}}$. Another remarkable consequence is that $(\mu_{c,q,r})^{\text{sym}}$ can be infinitely divisible without $\mu_{c,q,r}$ being infinitely divisible and that $(\rho_{q,r})^{\text{sym}}$ can be infinitely divisible without $\rho_{q,r}$ being infinitely divisible.

Theorem 3.3. *Let $\{(N_t, Y_t), t \geq 0\}$ be a bivariate Lévy process such that $\{N_t\}$ and $\{Y_t\}$ are Poisson processes, and let the parameters p, q, r of $\{(N_t, Y_t)\}$ satisfy $p, q, r < 1$. Let $c > 1$. Let $\mu_{c,q,r}$ be defined as in (3.1) and $\rho_{q,r}$ as in (3.2) and (3.4). Then the following hold true:*

- (a) *If $p = 0$, then $\rho_{q,r}$ and $\mu_{c,q,r}$ are infinitely divisible.*
- (b) *If $p > 0$ and $q > 0$, then the following conditions are equivalent:*
 - (i) *$\mu_{c,q,r}$ is infinitely divisible.*
 - (ii) *$\rho_{q,r}$ is infinitely divisible.*
 - (iii) *$r \leq pq$.*
- (c) *If $p > 0$, $q > 0$ and $r > pq$, then the following conditions are equivalent:*
 - (i) *$(\mu_{c,q,r})^{\text{sym}}$ is infinitely divisible.*
 - (ii) *$(\rho_{q,r})^{\text{sym}}$ is infinitely divisible.*
 - (iii) *$p \leq qr$.*
- (d) *If $q = 0$, then none of $\rho_{q,r}$, $\mu_{c,q,r}$, $(\rho_{q,r})^{\text{sym}}$ and $(\mu_{c,q,r})^{\text{sym}}$ is infinitely divisible.*

Proof. Write $\mu = \mu_{c,q,r}$ and $\rho = \rho_{q,r}$. (a) Suppose $p = 0$. Then ρ is infinitely divisible by Lemma 3.2, and hence so is μ by (3.5).

(b) Suppose that $p, q > 0$. Under these conditions, the equivalence of (ii) and (iii) follows from Lemma 3.2. Further, (ii) implies (i) by (3.5), so that it remains to show that (i) implies (iii). For that, suppose that $r > pq$, and in order to show that μ is not infinitely divisible, we will distinguish three cases: $p = r$, $p > r$ and $p < r$. The

first case is easy, but in the second and third cases, we have to use rather involved arguments resorting to different conditions that guarantee non-infinite-divisibility.

Case 1: Suppose that $p = r$. Then $\hat{\rho}$ will have a real zero as argued in the proof of Lemma 3.2 (c). By (3.5), also $\hat{\mu}$ will have a real zero, so that μ cannot be infinitely divisible.

Case 2: Suppose that $p > r$. Then $\hat{\rho}$ can be expressed as in (3.8) (with the same derivation). Together with (3.5) and (1.4) this implies

$$(3.15) \quad \hat{\mu}(z) = \exp \left[\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (e^{imc^{-n}z} - 1) \frac{1}{m} (q^m - (-r/p)^m) \right], \quad z \in \mathbb{R}.$$

Absolute convergence of this double series follows from $q < 1$ and $r/p < 1$. Define the real numbers a_m , $m \in \mathbb{N}$, and the signed measure ν by

$$a_m := \frac{1}{m} (q^m - (-r/p)^m) \quad \text{and} \quad \nu := \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_m \delta_{c^{-n}m}.$$

It follows that $\hat{\mu}$ in (3.15) has the same form as the Lévy-Khintchine representation with the signed measure ν in place of a Lévy measure, so that infinite divisibility of μ is equivalent to the signed measure ν having negative part 0; see Exercise 12.3 in Sato [20]. Thus, to show that μ is not infinitely divisible, we will show that there is some even integer m such that $\nu(\{m\}) < 0$.

Since $r/p > q$, it follows that $a_m < 0$ if m is even and that $a_m > 0$ if m is odd. For even m , denote

$$(3.16) \quad \begin{aligned} G_m &:= \{(n', m') \in \mathbb{N}_0 \times \mathbb{N} : c^{-n'} m' = m, \quad m' \text{ odd}\}, \\ H_m &:= \{(n', m') \in \mathbb{N}_0 \times \mathbb{N} : c^{-n'} m' = m, \quad m' \text{ even}\}. \end{aligned}$$

Then

$$(3.17) \quad \nu(\{m\}) = \sum_{(n', m') \in G_m \cup H_m} a_{m'} \leq a_m + \sum_{(n', m') \in G_m} a_{m'}.$$

Denote

$$k_0 := \inf\{k \in \mathbb{N} : c^k \text{ is rational}\}.$$

If there is some even m such that $G_m = \emptyset$, then $\nu(\{m\}) < 0$ by (3.17), and we are done. So suppose from now on that $G_m \neq \emptyset$ for every even m . As a consequence, $k_0 < \infty$. Write

$$c^{k_0} = \alpha/\beta$$

with $\alpha, \beta \in \mathbb{N}$ such that α and β have no common divisor, and denote

$$f := \max\{t \in \mathbb{N}_0 : 2^t | \beta\},$$

i.e. f is the largest integer t such that 2^t divides β . For m even, denote

$$g(m) := \max\{t \in \mathbb{N} : 2^t | m\}.$$

Let m be even and let $(n', m') \in G_m$. Then $c^{n'} = m'/m$ and it follows that $k_0 | n'$. Write $l := n'/k_0$. Then

$$c^{n'} = \left(\frac{\alpha}{\beta}\right)^l = \frac{m'}{m},$$

so that $m(\alpha/\beta)^l = m'$ is odd. It follows that $2|\beta$, so that α is odd, and hence that m/β^l is odd, implying that

$$(3.18) \quad g(m) = lf.$$

Since $g(m)$ and f are completely determined by m and c , these determine l and hence (n', m') uniquely. In particular, $|G_m| \leq 1$, and even $|G_m| = 1$ since we assumed G_m to be non-empty.

Now let $j \in \mathbb{N}$ and let m be an even number of the form

$$m = m_j = 2^{jf}$$

(recall that $f \geq 1$ as just shown, so that m is indeed even). Then $g(m_j) = jf$. By (3.18), it follows that the unique element (n'_j, m'_j) in G_{m_j} is given by $n'_j = jk_0$, and that

$$\frac{m'_j}{m_j} = c^{jk_0}.$$

Noting that $0 < q < r/p < 1$ and $c > 1$, choose j so large that $q^{m_j} \leq 2^{-1}(r/p)^{m_j}$ and $m'_j = m_j c^{jk_0} > 2m_j$. Then

$$a_{m_j} = \frac{1}{m_j}(q^{m_j} - (r/p)^{m_j}) \leq -\frac{1}{2m_j}(r/p)^{m_j}$$

and

$$a_{m'_j} = \frac{1}{m'_j}(q^{m'_j} + (r/p)^{m'_j}) \leq \frac{3}{2m'_j}(r/p)^{m'_j} < \frac{3}{4m_j}(r/p)^{2m_j}.$$

Thus

$$\nu(\{m_j\}) \leq a_{m_j} + a_{m'_j} \leq \frac{1}{2m_j}(r/p)^{m_j}(-1 + (3/2)(r/p)^{m_j}) < 0$$

for large enough j , showing that μ is not infinitely divisible under the conditions of Case 2.

Case 3: Suppose that $p < r$ and, by way of contradiction, assume that μ is infinitely divisible. Denote by $L_\mu(\theta) = \int_{\mathbb{R}} e^{-\theta x} \mu(dx)$, $\theta \geq 0$, the Laplace transform of μ . Then $L_\mu(\theta) = e^{-\varphi(\theta)}$ where φ has a completely monotone derivative $\psi(\theta)$ on $(0, \infty)$, that is, $(-1)^n \psi^{(n)}(\theta) \geq 0$ on $(0, \infty)$ for $n = 0, 1, \dots$ (see Feller [7], p. 450). By (3.5) and (3.7) we have

$$(3.19) \quad \varphi(\theta) = -\log L_\mu(\theta) = -\sum_{n=0}^{\infty} \log \frac{p + r f_n(\theta)}{1 - q f_n(\theta)}$$

where $f_0(\theta) = e^{-\theta}$ and $f_n(\theta) = \exp(-c^{-n}\theta) = f_0(c^{-n}\theta)$, $n = 1, 2, \dots$. Convergence of the summation in (3.19) is easily established. Since $\psi = \frac{d}{d\theta} \varphi$ is completely monotone,

so is $\theta \mapsto c^{-1}\psi(c^{-1}\theta) = \frac{d}{d\theta}(\varphi(c^{-1}\theta))$. As a consequence,

$$\begin{aligned}\psi_0(\theta) &:= \frac{1}{1 - qf_0(\theta)} - \frac{p}{p + rf_0(\theta)} \\ &= \frac{d}{d\theta} \left(-\log \frac{p + rf_0(\theta)}{1 - qf_0(\theta)} \right) = \frac{d}{d\theta}(\varphi(\theta) - \varphi(c^{-1}\theta))\end{aligned}$$

is the difference of two completely monotone functions. The function

$$\frac{1}{1 - qf_0(\theta)} = \sum_{k=0}^{\infty} (qf_0(\theta))^k = \sum_{k=0}^{\infty} (qe^{-\theta})^k = \int_{[0, \infty)} e^{-\theta x} \left(\sum_{k=0}^{\infty} q^k \delta_k \right) (dx)$$

is completely monotone, showing that the function $\xi(\theta)$ on $(0, \infty)$ defined by

$$\theta \mapsto \xi(\theta) = \frac{p}{p + re^{-\theta}}$$

is the difference of two completely monotone functions. Applying Bernstein's Theorem, there must exist a signed measure σ on $[0, \infty)$ (finite on compacts), such that

$$\xi(\theta) = \int_{[0, \infty)} e^{-\theta x} \sigma(dx), \quad \forall \theta \in (0, \infty),$$

with the integral being absolutely convergent for every $\theta > 0$. However, introducing the signed measure $\tau := \sum_{k=0}^{\infty} (-r/p)^k \delta_k$, we have

$$(3.20) \quad \xi(\theta) = \sum_{k=0}^{\infty} \left(-\frac{r}{p} e^{-\theta} \right)^k = \int_{[0, \infty)} e^{-\theta x} \tau(dx)$$

if θ is so large that $e^{-\theta} < p/r$. Thus there is $\theta_0 > 0$ such that $e^{-\theta_0 x} \sigma(dx)$ and $e^{-\theta_0 x} \tau(dx)$ have a common Laplace transform. Now from the uniqueness theorem in Laplace transform theory (p. 430 of Feller [7]) combined with Hahn-Jordan decomposition of signed measures, it follows that $e^{-\theta_0 x} \sigma(dx) = e^{-\theta_0 x} \tau(dx)$, that is, $\sigma = \tau$. But the integral in (3.20) does not converge for $0 < \theta < \log(r/p)$, contradicting the corresponding property of σ . Hence we get a contradiction, and the proof of (b) is finished.

(c) Suppose that $p, q > 0$ and that $r > pq$. The equivalence of (ii) and (iii) then follows from Lemma 3.2, and (ii) implies (i) by (3.5), so that it remains to show that (i) implies (iii). Since $|\widehat{\rho}|^2$ and hence $|\widehat{\mu}|^2$ will have real zeros if $p = r$ as shown in the proof of Lemma 3.2 (c), μ^{sym} cannot be infinitely divisible if $p = r$, in accordance with the fact that condition (iii) is violated in that case. So suppose that $p \neq r$. With A, B, D_k and E_m as in the proof of Lemma 3.2 (c), it follows from $|\widehat{\mu}(z)|^2 = \prod_{n=0}^{\infty} |\widehat{\rho}(c^{-n}z)|^2$ and (3.11) that

$$(3.21) \quad \log(|\widehat{\mu}(z)|^2) = 2 \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} E_m(\cos(mc^{-n}z) - 1).$$

Since

$$\begin{aligned}
& 2 \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} |E_m| |\cos(mc^{-n}z) - 1| \\
&= \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} |D_k| \sum_{l=0}^k \binom{k}{l} |\cos((k-2l)c^{-n}z) - 1| \\
&\leq \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} |D_k| 2^k (kc^{-n}z)^2 \leq z^2 \sum_{n=0}^{\infty} c^{-2n} \sum_{k=1}^{\infty} k(A^k + B^k) < \infty,
\end{aligned}$$

we can consider the right-hand side of (3.21) as an integral with respect to a signed measure. Thus

$$(3.22) \quad \log(|\hat{\mu}(z)|^2) = \int_{\mathbb{R}} (e^{ixz} - 1 - ixz 1_{(-1,1)}(x)) \tilde{\nu}(dx),$$

where $\tilde{\nu}$ is the symmetric signed measure

$$(3.23) \quad \tilde{\nu} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} E_m (\delta_{mc^{-n}} + \delta_{-mc^{-n}}).$$

Now suppose that $p > qr$. As observed in the proof of Lemma 3.2 (c), this is equivalent to $A > B$. In order to show that μ is not infinitely divisible, we use Exercise 12.3 of [20] again. We need to show that $\tilde{\nu}$ has a non-trivial negative part, i.e. that there is some $x_0 \in \mathbb{R}$ such that $\tilde{\nu}(\{x_0\}) < 0$. For that, we will first estimate E_m . Recall that $E_m > 0$ for all odd m and $E_m < 0$ for all even m . Since $\binom{m+2h}{h} \leq 2^{m+2h}$ for all h , it follows from (3.9) and (3.12) that

$$(3.24) \quad |E_m| \leq \sum_{h=0}^{\infty} \frac{1}{m+2h} 2A^{m+2h} \leq \frac{2A^m}{m(1-A^2)}, \quad m \in \mathbb{N}.$$

Choose $\gamma \in (0, 1)$ such that $A/\gamma < 1$, and choose $\alpha \in \mathbb{N}$ such that $(\alpha+1/2)/(\alpha+1) \geq \gamma$. By Stirling's formula, there exists a constant $d_1 > 0$ such that for every $m \in \mathbb{N}$,

$$\begin{aligned}
\binom{m+2\alpha m}{\alpha m} &\geq d_1 \left(\frac{m+2\alpha m}{(m+\alpha m)\alpha m} \right)^{1/2} \frac{(m+2\alpha m)^{m+2\alpha m}}{(m+\alpha m)^{m+\alpha m} (\alpha m)^{\alpha m}} \\
&\geq \frac{d_1}{(\alpha m)^{1/2}} (2\gamma)^{m+2\alpha m}.
\end{aligned}$$

Since $D_k < 0$ for every even k , we conclude

$$\begin{aligned}
(3.25) \quad |E_m| &\geq |D_{m+2\alpha m}| \binom{m+2\alpha m}{\alpha m} \\
&\geq \frac{d_1}{(\alpha m)^{1/2} (m+2\alpha m)} (A\gamma)^{m+2\alpha m} (1 - (B/A)^{m+2\alpha m})
\end{aligned}$$

for every even $m \geq 2$. Let G_m, H_m, k_0 and f be defined as in the proof of (b)-Case 2. If $G_m = \emptyset$ for some even m , then $\tilde{\nu}(\{m\}) \leq E_m < 0$, as shown in (3.17). So suppose that $G_m \neq \emptyset$ for all even $m \geq 2$. As seen in the proof of (b), this implies $|G_m| = 1$,

and if m is of the form $m = m_j = 2^{jf}$ with $j \in \mathbb{N}$, then the unique element (n'_j, m'_j) in G_{m_j} satisfies $m'_j/m_j = c^{jk_0}$. Recall that m'_j is odd by the definition of G_m . For large j , we then have $m'_j/2 > m_j + 2\alpha m_j$, and from (3.24) and (3.25) it follows that there exists some constant $d_2 > 0$ such that

$$\frac{E_{m'_j}}{|E_{m_j}|} \leq d_2 \sqrt{m'_j} (A/\gamma)^{m'_j/2} \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

so that $\nu(\{m_j\}) \leq E_{m_j} + E_{m'_j} < E_{m_j}/2 < 0$ for large j , finishing the proof of (c).

(d) Suppose $q = 0$. By Proposition 3.1, $\rho = \rho_{0,r}$ is Bernoulli distributed with parameter r . Further, $\mu = \mu_{c,0,r}$ is the distribution of $\sum_{n=0}^{\infty} c^{-n} U_n$, where $\{U_n, n \in \mathbb{N}\}$ is an i.i.d. sequence with distribution ρ . The support of μ is then a subset of $[0, c/(c-1)]$. It follows that also ρ^{sym} and μ^{sym} have bounded support. Since Dirac measures are the only infinitely divisible distributions with bounded support, $\rho, \mu, \rho^{\text{sym}}$ and μ^{sym} are not infinitely divisible. \square

We remark that the proof that (i) implies (iii) in Theorem 3.3 (b) and (c) can be simplified if $c > 1$ is a transcendental number. For, in that case, the set G_m appearing in (3.16) can be easily seen to be empty for every even m , so that $\nu(\{m\}) \leq a_m < 0$ and $\tilde{\nu}(\{m\}) \leq E_m < 0$, respectively.

Example 3.4. (a) If $p = q > 0$, then $\rho_{q,r}, \mu_{c,q,r}, (\rho_{q,r})^{\text{sym}}$ and $(\mu_{c,q,r})^{\text{sym}}$ will all be infinitely divisible if $r \in [0, 3 - 2\sqrt{2}]$ and all fail to be infinitely divisible if $r > 3 - 2\sqrt{2} \approx 0.17157$. Recall that r is the dependence parameter.

(b) Let $2p = q > 0$. Then $\rho_{q,r}$ and $\mu_{c,q,r}$ will be infinitely divisible for $r \in [0, (13 - 3\sqrt{17})/4]$ and fail to be infinitely divisible for $r > (13 - 3\sqrt{17})/4 \approx 0.15767$. On the other hand, $(\rho_{q,r})^{\text{sym}}$ and $(\mu_{c,q,r})^{\text{sym}}$ are infinitely divisible if and only if $r \in [0, (13 - 3\sqrt{17})/4] \cup [1/2, 1)$.

Let us study continuity properties of $\mu_{c,q,r}$ and its convolution power. If $q = 0$, then the proof Theorem 3.3 shows that

$$\hat{\mu}(z) = \prod_{n=0}^{\infty} \left[(1-r) + r e^{ic^{-n}z} \right], \quad z \in \mathbb{R},$$

so that μ is an *infinite Bernoulli convolution* (usage of this word is not fixed; here we follow Watanabe [22]). The question of singularity/absolute continuity of infinite Bernoulli convolutions has been investigated by many authors but, even if $r = 1/2$ (i.e. the measure $\mu_{c,0,1/2}$ with $u = w$), characterisation of all $c > 1$ for which the distribution is absolutely continuous is an open problem. See Peres et al. [18], Peres and Solomyak [19], Watanabe [22] and the references therein. In the following, we shall exclude the case $q = 0$ in our considerations and formulate results which are analogues to Theorems 2.2 and 2.4.

Theorem 3.5. *Let $\mu_{c,q,r}$ be defined as in (3.1), with $c > 1$ and $0 < q < 1$. Let*

$$(3.26) \quad h(q, r) := (q+r) \left(\log \frac{1}{1-q} + \frac{1}{1-q} \log \frac{1}{q} - \log \frac{q+r}{q} \right) + p \log \frac{1}{p},$$

where $p = 1 - q - r$ and $p \log(1/p)$ is understood zero for $p = 0$. Then the following are true:

(a) The Hausdorff dimension of $\mu_{c,q,r}$ is estimated as

$$(3.27) \quad \dim(\mu_{c,q,r}) \leq \frac{h(q,r)}{\log c}.$$

Thus, for each $c > 1$, there exists a constant $C_1 = C_1(c) > 0$ such that $\mu_{c,q,r}$ is continuous-singular whenever $p \geq C_1 \max\{q, r\}$.

(b) Fix q and r . Then there exists a constant $C_2 = C_2(q, r) > 0$ such that $\mu_{c,q,r}$ is continuous-singular whenever $c \geq C_2$.

(c) If c is a P.V. number, then $\mu_{c,q,r}$ is continuous-singular whenever $p > 0$ and $r \leq pq$.

(d) Fix $c > 1$ such that c^{-1} is a P.S. number. Then there exists $\varepsilon = \varepsilon(c) \in (0, 1)$ such that $\mu_{c,q,r}$ is absolutely continuous with bounded continuous density whenever $p > 0$, $r \leq pq \leq 1$ and $q \geq 1 - \varepsilon$. In particular, there exist constants $C_3 = C_3(c) > 0$ and $C_4 = C_4(c) > 0$ such that $\mu_{c,q,r}$ is absolutely continuous with bounded continuous density whenever $q \geq C_3 p \geq C_4 r$.

Proof. First, note that $0 < q < 1$ implies $p < 1$ and $r < 1$. Recall that $\rho = \rho_{q,r} = (1 + r/q)\sigma_q - (r/q)\delta_0$, where σ_q is a geometric distribution with parameter $1 - q$. The entropy of ρ can then be readily calculated as

$$H(\rho) = (1 + r/q)(H(\sigma_q) + (1 - q)\log(1 - q) - q\log(1 + r/q)) - p\log p,$$

which equals $h(q, r)$ since $H(\sigma_q)$ equals $h(q)$ of (2.5). Using again Theorem 2.2 of Watanabe, it follows that the Hausdorff dimension of $\mu = \mu_{c,q,r}$ is estimated as in (3.27). To see the latter half of (a), notice that, since μ is continuous by [1], it must be continuous-singular if its Hausdorff dimension is less than 1. Suppose that $p \geq C_1 \max\{q, r\}$. Then

$$p = \frac{1}{1 + q/p + r/p} \geq \frac{1}{1 + 2/C_1},$$

which tends to 1 as $C_1 \rightarrow \infty$. Hence $q \rightarrow 0$ and $r \rightarrow 0$ as $C_1 \rightarrow \infty$. Thus

$$\begin{aligned} (q + r) & \left(\log \frac{1}{1 - q} + \frac{1}{1 - q} \log \frac{1}{q} - \log \frac{q + r}{q} \right) \\ & = (q + r) \left(\log \frac{1}{1 - q} + \frac{q}{1 - q} \log \frac{1}{q} + \log \frac{1}{q + r} \right) \rightarrow 0 \end{aligned}$$

and hence

$$\sup\{h(q, r) : p \geq C_1 \max\{q, r\}\} \rightarrow 0, \quad C_1 \rightarrow \infty.$$

This shows (a).

(b) For given q, r , take any $C_2 > e^{h(q,r)}$. Then $h(q, r)/\log c < 1$ whenever $c \geq C_2$.

(c) Proof is the same as that of Theorem 2.2(b) since, by (3.8) and (1.4), $\hat{\mu}(z)$ has representation (2.8), with $a_m = \nu_\rho(\{m\}) = m^{-1}q^m(1 - (-r/(pq))^m)$.

(d) Suppose that $p > 0$ and $r \leq pq$. Again we use (2.8) with $a_m = m^{-1}q^m(1 - (-r/(pq))^m)$. We have $a_m \geq m^{-1}q^m$ for m odd, and it follows that $\alpha_0 := \sum_{m=1}^{\infty} a_m$ converges to ∞ as $q \uparrow 1$. The proof of the first half of (d) then follows in complete

analogy to that of Theorem 2.2 (c). To see the second half, suppose that $q \geq C_3 p \geq C_4 r$. Then $1 \geq C_3 p/q \geq C_4 r/q$ and

$$\begin{aligned} q &= \left(1 + \frac{p}{q} + \frac{r}{q}\right)^{-1} \geq \left(1 + \frac{p}{q} + \frac{C_3 p}{C_4 q}\right)^{-1} \\ &\geq \left(1 + \left(1 + \frac{C_3}{C_4}\right) \frac{1}{C_3}\right)^{-1} = \left(1 + \frac{1}{C_3} + \frac{1}{C_4}\right)^{-1}. \end{aligned}$$

Hence, $q \geq 1 - \varepsilon(c)$ if C_3 and C_4 are large enough. We also have

$$\frac{r}{pq} = \frac{r}{p} \left(1 + \frac{p}{q} + \frac{r}{q}\right) \leq \frac{C_3}{C_4} \left(1 + \frac{1}{C_3} + \frac{1}{C_4}\right).$$

Hence $r/(pq) \leq 1$ if C_3 is fixed and C_4 is large. Thus there are C_3 and C_4 such that $q > 1 - \varepsilon(c)$ and $r/(pq) \leq 1$ whenever $q \geq C_3 p \geq C_4 r$. \square

Theorem 3.6. *Let $\mu = \mu_{c,q,r}$ be defined as in (3.1) with $c > 1$ and $0 < q < 1$. If $p \neq 0$, assume additionally that $r \leq pq$. Let $\rho = \rho_{q,r}$ be defined as in (3.2) and (3.4). Then the same assertions as in Theorem 2.4 (a) – (d) hold true with $\mu_{c,q,r}$ and $\rho_{q,r}$ in place of μ_c and ρ .*

Proof. Observe that under our assumption μ is infinitely divisible (recall Theorem 3.3), so that μ^{t*} is definable for all nonnegative real t . If $p > 0$ and $r \leq pq$, it follows from (3.8) that ρ is a compound Poisson distribution, concentrated on \mathbb{N}_0 , with finite second moment. If $p = 0$, then ρ is a geometric distribution shifted by 1. In both cases, $H(\rho) < \infty$ by Lemma 2.6. We have

$$(3.28) \quad \widehat{\mu}^{t*}(z) = \prod_{n=0}^{\infty} \widehat{\rho}^{t*}(c^{-n}z) = \exp\left(it\gamma_p^0 z + t \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (e^{imc^{-n}z} - 1)a_m\right),$$

where $a_m = \nu_\rho(\{m\})$, and $\gamma_p^0 = 0$ for $p > 0$ and $\gamma_p^0 = \sum_{n=0}^{\infty} c^{-n} = c/(c-1)$ for $p = 0$. If $p > 0$, then the proof of assertions (a) – (d) is done in complete analogy to Theorem 2.4 (a) – (d). If $p = 0$, then the result is the same as in the case of ρ being geometric distribution, since shifts do not change entropy, Hausdorff dimension, continuous-singularity and absolute continuity. \square

Remark 3.7. If t is restricted to be an integer in part (d) of the theorem above, then the convolution power μ^{t*} can still be defined even if the condition $r \leq pq$ guaranteeing infinite divisibility of μ is violated. In that case, the same proof shows that if c^{-1} is a P.S. number, then μ^{t*} will be absolutely continuous with bounded continuous density for large enough integer t .

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