Bayesian inference for multivariate copulas using pair-copula constructions

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Abstract
This article provides a Bayesian analysis of pair-copula constructions (Aas et al., 2007 *Insurance Math. Econom.*) for modeling multivariate dependence structures. These constructions are based on bivariate $t-$copulas as building blocks and can model the nature of extremal events in bivariate margins individually. According to recent empirical studies (Fischer et al. (2007) and Berg and Aas (2007)) pair-copula constructions (PCC's) outperform many other multivariate copula constructions in fitting multivariate financial data. Parameter estimation in multivariate copulas is generally performed using maximum likelihood. However confidence intervals for parameters of PCC's are not easy to obtain and therefore statistical inference in these models has not been addressed so far. In this article we develop a Markov chain Monte Carlo (MCMC) algorithm which allows for interval estimation by means of credible intervals. Our MCMC algorithm can reveal unconditional as well as conditional independence in the data which can simplify resulting PCC's. In applications we consider Norwegian financial returns and Euro swap rates and are able to identify meaningful conditional independencies in both data sets. For the Norwegian financial returns data our findings support the view of Norway as a healthy economy, while for the Euro swap rates data they explain the nature of small twists in the yield curve.

Keywords: Bayesian inference; Euro swap rates; financial returns; Markov chain Monte Carlo methods; Metropolis-Hastings algorithm; multivariate copula; pair-copula construction; vine.

JEL classification: C11, C51, C52

Introduction

Multivariate data usually exhibit a complex pattern of dependence. One increasingly popular approach for constructing high dimensional dependence is based on copulas. Copulas are multivariate distribution functions with uniform margins which allow to represent a joint distribution function as a function of marginal distributions and a copula (Sklar (1959)). Copulas nowadays are used in different fields of applied sciences and especially heavily in economics, finance and risk management. We mention some related important papers, however our list does not pretend to be complete. Embrechts et al. (2003) demonstrated the usefulness of copulas in insurance and market risk management and illustrated how crucial the choice of the copula may be for expected losses and portfolio returns. Patton (2004) showed the importance of choosing copulas with appropriate asymmetric and skewness properties for optimal asset allocation. Nolte (2008) used a four dimensional Gaussian copula to model price changes, transaction volumes, bid-ask spreads and intertrade durations jointly and came to the conclusion that information contained in the transaction volume and the bid-ask processes prevails the information contained in the trade arrival process.

The class of copulas for bivariate data is very rich in comparison to the one for $d-$dimensional data with $d \geq 3$. Until recently mostly Gaussian and $t-$copulas or, more generally, elliptical copulas have been in use for multivariate data (see Frahm et al. (2003)). The generalization of bivariate copulas to multivariate copulas of dimension larger than 2 is not straightforward. There is one simple generalization for Archimedean copulas known as exchangeable Archimedean copulas (see Frey and McNeil (2003)). It should be noted that not all bivariate Archimedean copulas have a corresponding multivariate exchangeable version (see e.g. Nelsen (1999)). In last years different approaches for constructing multivariate Archimedean copulas of dimension more than 2 have been developed by Joe (1997), Embrechts et al. (2003), Whelan (2004), McNeil et al. (2006), Savu and Trede (2006) and McNeil (2007). For other general constructions of multivariate copulas we refer the readers to Fischer et al. (2007) and Fischer and Köck (2007).

Joe (1996) and Bedford and Cooke (2001, 2002) constructed flexible higher-dimensional copulas by using only bivariate copulas as building blocks which they called vines. The book by Kurowicka and Cooke (2006) discusses these vine constructions in detail but restricts the statistical inference to the case of Gaussian vines. Aas et al. (2007) first appreciated the general construction principle for deriving multivariate copulas. They used more general bivariate copulas than the Gaussian copula and applied these construction methods to financial risk data using more appropriate pair-copulas such as the bivariate t, Clayton and Gumbel copulas. Further they considered statistical parameter inference. According to recent empirical investigations of Berg and Aas (2007) and Fischer et al. (2007), the vine constructions based on bivariate $t-$copulas provide a better fit to multivariate financial data than other multivariate copula constructions such as a hierarchical Archimedean construction (Savu and Trede (2006)), a generalized multiplicative Archimedean construction (Morillas (2005) and Liebscher (2006)), a multiplicative construction of Liebscher (2006), a construction of Koehler and Symanowski (1995) and a multivariate t-copula. Estimation of copula parameters, in general, is often based on classical maximum likelihood (ML) and its variations. The most common approach is a semiparametric one where the margins are fitted empirically and the dependence parameters are fitted by ML. The asymptotic properties of these semiparametric estimates have been rigorously investigated by Genest et al. (1995). However confidence intervals for dependence parameters are difficult to obtain since determination of the asymptotic variance is not a simple task in general. Therefore data analyses often are exclusively based on point estimates of copula parameters.

Alternatively the Bayesian inference approach has become very popular during the last two decades. It gives solutions for many difficult problems which are not simple to solve in a classical ML framework. This is due to Markov chain Monte Carlo (MCMC) algorithm introduced by Metropolis et al. (1953) and Hastings (1970). However the Bayesian literature on copulas is very poor. Pitt et al. (2006) deal with Gaussian copula regression. The main difficulty here encountered is to sample a positive definite correlation matrix. Pitt et al. (2006) solved by employing covariance selection prior of Wong et al. (2003). Dalla Valle (2007) proposes Bayesian inference based on MCMC for multivariate Gaussian and $t-$ copulas using the inverse Wishart distribution as a prior for the correlation matrix .

In this paper we develop Bayesian inference for pair-copula constructions (PPC's) of Aas et al. (2007) based on bivariate $t-$copulas. As they pointed out, their method allows to model tail dependence between two chosen margins individually, while multivariate Gaussian and $t-$copulas have the same tail dependence structure for any two chosen margins. A tail dependence coefficient (see e.g. Embrechts et al. (2002)) is accounting for extremal events of margins occurring simultaneously. It is one of most important characteristics of financial data since it contains information on heavy-tailedness of multivariate financial data. Parameters of the considered PCC are association and degrees of freedom (df) parameters of bivariate $t-$copulas. Thus the problem of proposing a correlation matrix is here no longer relevant since the association parameters are only restricted to be between -1 and 1. Further the Bayesian approach solves the problem of interval estimation in ML framework by means of credible intervals. Credible intervals for parameters of a PCC can simplify this PCC if they detect conditional and unconditional independence between pairs of variables. Based on the MCMC iterates of the parameters of PCC's, we can also construct credible intervals for quantities derived from the parameters such as Kendall's τ, the tail dependence coefficient , the $\lambda-$function (see Genest and Rivest (1993)) and many others.

The paper is organized as follows. In Sections 1 and 2 we review copulas and PPC's , respectively. The MCMC sampling scheme for a PCC with bivariate $t-$copula as building blocks is given in Section 3. Section 4 contains two applications of our Bayesian algorithm. In the first application we revisit Norwegian financial returns data from Aas et al. (2007) while in the second application we deal with Euro swap rates data. Credible intervals for data sets showed a good agreement with empirical estimates of Kendall's τ, the tail dependence coefficient and the $\lambda-$function indicating a good fit of PCC models. Finally, Section 5 summarizes, discusses results and considers further research.

1 Copulas

Copulas are d-dimensional multivariate distributions with uniformly distributed marginal distributions on $[0, 1]$. They are very useful for modeling a dependence structure of multivariate data. Let $\mathbf{X} = (X_1, \ldots, X_d)^t$ be a $d-$dimensional random vector with joint distribution function $F(x_1, \ldots, x_d)$ and marginal distributions $F_i(x_i)$, $i = 1, \ldots, d$. Now according to Sklar's theorem (see Sklar (1959)) there exist a copula $C(u_1, \ldots, u_d)$ such that

$$F(x_1, \ldots, x_d) = C(F_1(x_1), F_2(x_2), ..., F_d(x_d)) \tag{1.1}$$

and the copula $C(u_1, \ldots, u_d)$ is unique if the marginal distributions are continuous. More details can be found in the books by Joe (1997) and Nelsen (1999). From now on we consider only absolutely continuous distributions with a joint density function $f(x_1, \ldots, x_d)$ and marginal density functions $f_i(x_i)$ for $i = 1, \ldots, d$.

Different multivariate distributions may have the same copula which fully describes their dependence structure. For example consider the following simple bivariate distributions. The first distribution is a bivariate normal distribution with zero means, unit variances and zero correlation, while the other one is the product of two independent exponential distributions with unit rate. Thus these two distributions have the same dependence (namely independence) but different margins. This cannot be detected by the scatter plot of observed data $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ with $\mathbf{X}_i = (X_{i1}, \ldots, X_{in})'$ (see top row of Figure 1) but only by the corresponding copula data $\mathbf{U} := (\mathbf{U}_1, \mathbf{U}_2) = (F_1(\mathbf{X}_1), F_2(\mathbf{X}_2))$, where $F_i(\mathbf{X}_i) := (F_i(X_{i1}), \ldots, F_i(X_{in}))'$ for $i = 1, 2$. In Figure 1 we used $n = 200$.

The copula $C(u_1, \ldots, u_d)$ of a multivariate distribution $F(x_1, \ldots, x_d)$ with margins $F_i(x_i)$, $i = 1, \ldots, d$ is given by

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_d^{-1}(u_d))$$

and the copula density is given by

$$c(u_1, \ldots u_d) = \frac{f\left(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_d^{-1}(u_d)\right)}{f_1\left(F_1^{-1}(u_1)\right) \cdots f_n\left(F_d^{-1}(u_d)\right)},$$

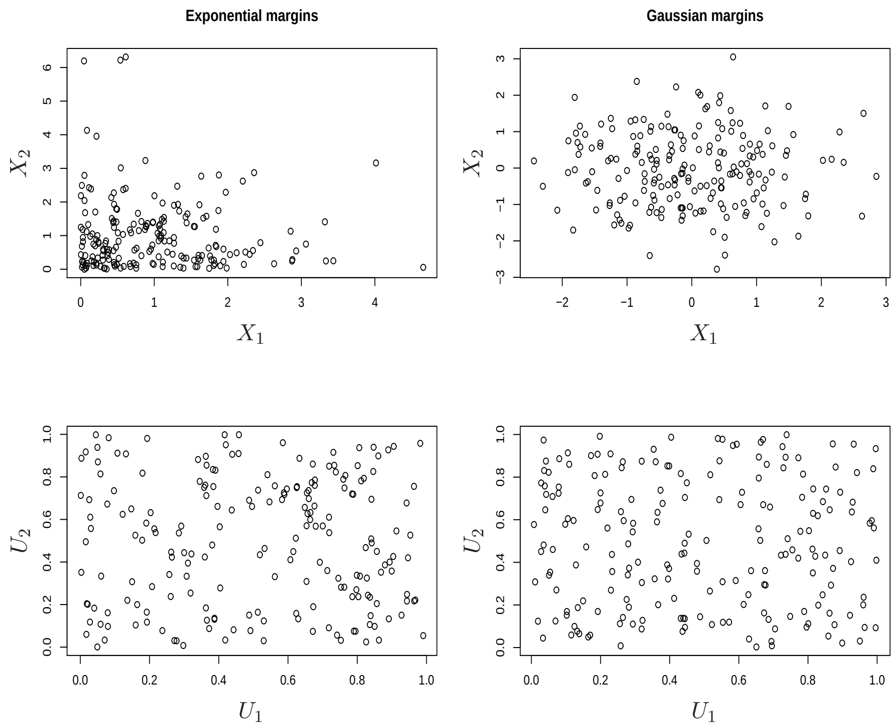

Figure 1: Scatter plot of $\mathbf{X}_1$ versus $\mathbf{X}_2$ (top row) and $\mathbf{U}_1$ versus $\mathbf{U}_2$ (bottom row) for independent exponential (first column) and normal (second column) margins , respectively

where $F_i^{-1}(u_i)$ is the inverse of the margins $F_i(x_i)$ for $i = 1, \ldots, d$. Using (1.1), the multivariate density $f(x_1, \ldots, x_d)$ is a product of the corresponding copula density with marginal densities $f_i(x_i)$, $i = 1, \ldots, d$ and is given by

$$f(x_1, \ldots, x_d) = c(F_1(x_1), \ldots F_d(x_d)) \cdot f_1(x_1) \cdots f_d(x_d), \tag{1.2}$$

thus separating the dependence structure from the marginal structure.

2 PCC's for multivariate distributions

Using (1.1) multivariate distributions with given margins can be easily constructed. However this general approach does not give a solution for the construction of flexible multivariate distributions which fit desired data well. In this section we give such a construction proposed first by Joe (1996), organized by Bedford and Cooke (2002) and applied to Gaussian copulas only. Later Aas et al. (2007) used bivariate Gaussian, t, Gumbel and Clayton copulas as building blocks.

Let $f(x_1, \ldots, x_d)$ be a $d-$dimensional density function and $c(u_1, \ldots, u_d)$ be the corresponding copula density function. In the sequel we denote by $f_{j|\mathbf{i}}(x_j|\mathbf{x_i})$ the conditional density of x_j given $\mathbf{x_i} := (X_{i1}, \ldots, X_{ik})'$ for $\mathbf{i} := (i_1, \ldots, i_k)'$. It is well known that the density $f(x_1, \ldots, x_d)$ can be factorized as

$$f(x_1, \ldots, x_d) = f_d(x_d) \cdot f_{d-1|d}(x_{d-1}|x_d) \cdot f_{d-2|(d-1)d}(x_{d-2}|x_{d-1}, x_d) \cdot \ldots \cdot f_{1|2\cdots d}(x_1|x_2, \ldots, x_d). \tag{2.1}$$

The above factorization is a simple consequence from the definition of conditional densities and is invariant with respect to permutation of the variables.

The second factor $f_{d-1|d}(x_{d-1}|x_d)$ on the right hand side of (2.1) can be represented as a product of a copula density and the marginal density $f_d(x_d)$ in the following way. Consider the bivariate density function $f_{(d-1)d}(x_{d-1}, x_d)$ with marginal densities $f_{d-1}(x_{d-1})$ and $f_d(x_d)$, respectively. Using (1.2) for $d = 2$, we have that the conditional density $f_{d-1|d}(x_{d-1}|x_d)$ is given by

$$\begin{aligned} f_{d-1|d}(x_{d-1}|x_d) &= \frac{f_{(d-1)d}(x_{d-1}, x_d)}{f_d(x_d)} \\ &= c_{(d-1)d}(F_{d-1}(x_{d-1}), F_d(x_d)) \cdot f_{d-1}(x_{d-1}). \end{aligned}$$

Similarly, the conditional density $f_{d-2|(d-1)d}(x_{d-2}|x_{d-1}, x_d)$ is given by

$$\begin{aligned} f_{d-2|(d-1)d}(x_{d-2}|x_{d-1}, x_d) &= \frac{f_{(d-2)(d-1)|d}(x_{d-2}, x_{d-1}|x_d)}{f_{d-1|d}(x_{d-1}|x_d)} \\ &= \frac{c_{(d-2)(d-1)|d}(F_{d-2|d}(x_{d-2}|x_d), F_{d-1|d}(x_{d-1}|x_d)) \cdot f_{d-2|d}(x_{d-2}|x_d) \cdot f_{d-1|d}(x_{d-1}|x_d)}{f_{d-1|d}(x_{d-1}|x_d)} \\ &= c_{(d-2)(d-1)|d}(F_{d-2|d}(x_{d-2}|x_d), F_{d-1|d}(x_{d-1}|x_d)) \cdot c_{(d-2)d}(x_{d-2}, x_d) \cdot f_{d-2}(x_{d-2}). \end{aligned} \tag{2.2}$$

Copula density $c_{(d-2)(d-1)|d}(\cdot, \cdot)$ is the conditional copula density corresponding to the conditional distribution $F_{(d-2)(d-1)|d}(x_{d-2}, x_{d-1}|x_d)$. Further $F_{d-i|d}(x_{d-i}|x_d)$ is the conditional distribution function of x_{d-i} given x_d for $i = 1, 2$. Note that in general the conditional copula density $c_{(d-2)(d-1)|d}(F_{d-2|d}(x_{d-2}|x_d), F_{d-1|d}(x_{d-1}|x_d))$ depends on the given conditioning value x_d.

Relation (2.2) can be generalized for a conditioning vector $\mathbf{v}$ of dimension k $(1 < k < d-1)$. Here the starting point is

$$f_{xv_j|\mathbf{v}_{-j}}(x|\mathbf{v}) = c_{xv_j|\mathbf{v}_{-j}}(F_{x|\mathbf{v}_{-j}}(x|\mathbf{v}_{-j}), F_{v_j|\mathbf{v}_{-j}}(v_j|\mathbf{v}_{-j})) \cdot f_{xv_j|\mathbf{v}_{-j}}(x|\mathbf{v}_{-j}),$$

where v_j is an arbitrary chosen component of $\mathbf{v}$ and the (k-1)-dimensional vector $\mathbf{v}_{-j}$ is the vector $\mathbf{v}$ without the component v_j. Finally we can represent each conditional density term on the right hand side of (2.1) as the product of the corresponding marginal density and copula density terms. This shows that $f(x_1, \ldots, x_d)$ is the product of marginal densities and pair-copula density terms. The pair-copula density terms are unconditional copulas evaluated at marginal distribution function values or conditional copulas evaluated at univariate conditional distribution function values. The above construction was defined in Aas et al. (2007) and was called the pair copula construction (PCC) for multivariate distributions. Joe (1996, p. 125) showed that the conditional distribution function $F_{u|\mathbf{v}}(u|\mathbf{v})$ appearing in the PCC are partial derivatives with respect to the second argument of the conditional copula given by

$$F_{x|\mathbf{v}}(x|\mathbf{v}) = \frac{\partial\, C_{x,v_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j}))}{\partial F(v_j|\mathbf{v}_{-j})}. \tag{2.3}$$

Here $C_{xv_j|\mathbf{v}_{-j}}(\cdot, \cdot)$ is a bivariate copula distribution function.

It is clear that there are many pair copula constructions for a random vector $\mathbf{X}$. In order to systemize PCC's, Bedford and Cooke (2001, 2002) introduced tree representations called regular vines. In this paper we consider only a particular regular vines, namely $D-$vines (see Kurowicka and Cooke (2004) or Aas et al. (2007)). For the convenience of the reader we give here the construction of D-vines for d random variables.

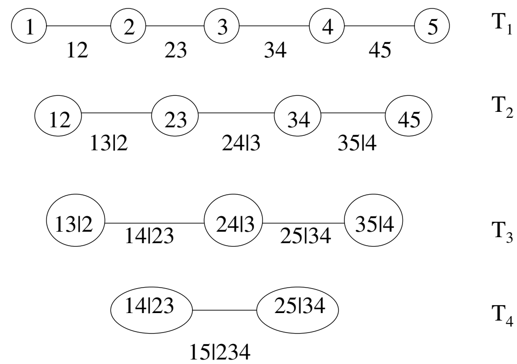

Figure 2: Tree representation of a five dimensional D-vine

First of all the random variables should be labeled from 1 to d and this labeling should remain fixed. The D-vine consists of $d-1$ trees T_i, $i=1,\ldots,d-1$. Figure 2 displays the tree representation of a D-vine for 5 variables on which the the construction of D-vine is below illustrated. The first tree consists of the d labeled nodes. The nodes are placed along a line one after another according to the value of their labels. Further $d-1$ edges connect neighboring nodes. Now each edge of the first tree gets its label. The edge label are elements of the symmetric difference of labels of neighboring nodes this edge connects. The symmetric difference of two sets A and B is defined by $A \triangle B := (A \backslash B) \cup (B \backslash A)$. For example the symmetric difference of the sets $\{1\}$ and $\{2\}$ is the set $\{1,2\}$ and this illustration corresponds to the nodes 1 and 2 of the tree T_1 connected with the edge 12 from Figure 2. In the second tree edges of the first tree become nodes. Thus the second tree has $d-1$ nodes. The nodes in the second tree are connected with an edge if the corresponding edges in the first tree shared a node. There are altogether $d-2$ edges in the second tree. From now on, the edge labels consist of two label sets separated by a vertical line "|". The first label set before the vertical line is the symmetric difference of labels of neighboring nodes the edge connects. The second label set is made of common labels of the neighboring nodes sharing the edge. Thus the symmetric difference of the sets $\{1,2\}$ and $\{2,3\}$ is the set $\{1,3\}$ and their intersection is the set $\{2\}$. This example corresponds to the nodes 12 and 23 of the tree T_2 connected with the edge 13|2 in Figure 2. In general, the ith tree T_i consists of $d+1-i$ nodes. Nodes are connected with an edge if the corresponding edges in tree T_{i-1} share a node. There are altogether $d-i$ edges. Edges are labeled according to the rule from the second tree treating the two label sets of a node as one set. Thus ignoring the vertical line the labels of the nodes in the third tree T_3 of Figure 2 are obtained as follows. The symmetric difference of the labels of the first two nodes 13|2 and 24|3 in Tree T_3 is the set $\{1,4\}$ and their interaction is the set $\{2,3\}$. Therefore the connecting edge has label 14|23. The last $(d-1)$-th tree consists of two nodes with labels $1(d-1)|2,\ldots,(d-2)$ and $2d|3,\ldots,(d-1)$, respectively. These two nodes are connected with one edge labeled as $1d|2,\ldots,(d-1)$. Now each edge corresponds to a pair-copula density and its label indicates the subindex of the pair-copula. Note that the presence of the vertical line in the edge label indicates that the corresponding copula is conditional. Further the second label set after the vertical line corresponds to the set of conditioning variables, while the first label set corresponds to the two variables which will be conditioned. For a $d-$dimensional density $f(x_1,\ldots,x_d)$ the PCC of the D-vine is given in Aas et al. (2007) as follows

$$f(x_1,\ldots,x_d) = \prod_{k=1}^{d} f_k(x_k) \prod_{j=1}^{d-1}\prod_{i=1}^{d-j} c_{i,i+j|i+1,\ldots,i+j-1}\left(F(x_i|x_{i+1},\ldots,x_{i+j-1}), F(x_{i+j}|x_{i+1},\ldots,x_{i+j-1})\right). \tag{2.4}$$

For simplicity we have dropped the subindex of the conditional distribution functions of type $F(x_i|x_{i+1},\ldots,x_{i+j-1})$. Thus, the PCC representation for D-vines given in (2.4) is the product of d marginal densities and $d(d-1)/2$ bivariate copulas.

3 Bayesian inference for PCC models based on bivariate $t-$copula pairs

From now on, we specify the building pair-copulas of the PCC model (2.4) as bivariate $t-$copulas. However the methodology is generic and applies much more widely. Further we assume that the margins of $\mathbf{X}$ are uniform. This is motivated by the standard semiparametric copula estimation procedure suggested by Genest et al. (1995), where approximate uniform margins are obtained by applying the empirical probability integral transformation to multivariate data.

The bivariate $t-$copula (see e.g. Embrechts et al. (2003)) has 2 parameters: the association parameter $\rho \in (-1,1)$ and the df parameter $\nu \in (0,\infty)$ and its density is given by

$$c(u_1,u_2|\nu,\rho) = \frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{1-\rho^2}\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^2}\cdot\frac{\left(\left[1+\frac{\left(t_\nu^{-1}(u_1)\right)^2}{\nu}\right]\left[1+\frac{\left(t_\nu^{-1}(u_2)\right)^2}{\nu}\right]\right)^{\frac{\nu+1}{2}}}{\left(1+\frac{\left(t_\nu^{-1}(u_1)\right)^2+\left(t_\nu^{-1}(u_2)\right)^2-2\rho t_\nu^{-1}(u_1)t_\nu^{-1}(u_2)}{\nu(1-\rho^2)}\right)^{\frac{\nu+2}{2}}},$$

where $t_\nu^{-1}(\cdot)$ is a quantile function of a $t-$distribution with ν degrees of freedom. Specifying the pair-copulas and assuming uniform margins, the conditional distribution function in (2.3) for $x = u_1$ and a scalar $\mathbf{v} = u_2$ takes the following form

$$h(u_1|u_2,\rho,\nu) = t_{\nu+1}\left(\frac{t_\nu^{-1}(u_1)-\rho\, t_\nu^{-1}(u_2)}{\sqrt{\frac{\left(\nu+\left(t_\nu^{-1}(u_2)\right)^2\right)(1-\rho^2)}{\nu+1}}}\right) \tag{3.1}$$

and it is called the $h-$function for the $t-$copula with parameters ρ and ν (see Aas et al. (2007)). For general $\mathbf{v}$ the arguments u_1 and u_2 of the function $h(\cdot|\cdot,\rho,\nu)$ are just nested compositions of the $h-$functions for bivariate $t-$copulas and this is illustrated below.

Let $\mathbf{U}^N := (\mathbf{U}_1', \mathbf{U}_2', \ldots, \mathbf{U}_N')'$ be the concatenated random vector of an i.i.d. sample $\mathbf{U}_n = (U_{1,n}, U_{2,n}\ldots, U_{d,n})'$ for $n = 1,\ldots,N$ from a D-vine specified in (2.4) with bivariate $t-$copulas as the building pair-copulas and with uniform margins. Therefore the unknown $d(d-1)$ dimensional parameter vector $\boldsymbol{\theta}$ is given by

$$\boldsymbol{\theta} = (\rho_{1,2}, \nu_{1,2}, \rho_{2,3}, \nu_{2,3}, \ldots, \rho_{1,d|2,\ldots,d-1}, \nu_{1,d|2,\ldots,d-1})^t.$$

Hence the log likelihood $l(\mathbf{u}^N|\boldsymbol{\theta})$ of the D-vine copula for a realization $\mathbf{u}^N$ of $\mathbf{U}^N$ is given by

$$
\begin{aligned}
l(\mathbf{u}^N|\boldsymbol{\theta}) &= \sum_{n=1}^{N}\Big[\sum_{i=1}^{d-1}\log\left(c\left(u_{i,n},u_{i+1,n}|\rho_{i,i+1},\nu_{i,i+1}\right)\right) \\
&+ \sum_{j=2}^{d-1}\sum_{i=1}^{d-j}\log\left(c\left(v_{j-1,2i-1,n},v_{j-1,2i,n}|\rho_{i,i+j|i+1,\ldots,i+j-1},\nu_{i,i+j|i+1,\ldots,i+j-1}\right)\right)\Big],
\end{aligned}
\tag{3.2}
$$

where for $n=1,\ldots,N$

$$
\begin{aligned}
v_{1,1,n} &:= h(u_{1,n}|u_{2,n},\rho_{1,2},\nu_{1,2}) \\
v_{1,2i,n} &:= h(u_{i+2,n}|u_{i+1,n},\rho_{i+1,i+2},\nu_{i+1,i+2}) \quad \text{for} \quad i=1,\ldots,d-3, \\
v_{1,2i+1,n} &:= h(u_{i+1,n}|u_{i+2,n},\rho_{i+1,i+2},\nu_{i+1,i+2}) \quad \text{for} \quad i=1,\ldots,d-3, \\
v_{1,2d-4,n} &:= h(u_d|u_{d-1},\rho_{d,d-1},\nu_{d,d-1}), \\
v_{j,1,n} &:= h(v_{j-1,1,n}|v_{j-1,2,n},\rho_{1,1+j|2,\ldots,j},\nu_{1,1+j|2,\ldots,j}) \quad \text{for} \quad j=2,\ldots,d-2, \\
v_{j,2i,n} &:= h(v_{j-1,2i+2,n}|v_{j-1,2i+1,n},\rho_{i,i+j|i+1,\ldots,i+j-1},\nu_{i,i+j|i+1,\ldots,i+j-1}) \quad \text{for} \quad d>4, \\
& j=2,\ldots,d-3 \quad \text{and} \quad i=1,\ldots,d-j-2 \\
v_{j,2i+1,n} &:= h(v_{j-1,2i+1,n}|v_{j-1,2i+2,n},\rho_{i,i+j|i+1,\ldots,i+j-1},\nu_{i,i+j|i+1,\ldots,i+j-1}) \quad \text{for} \quad d>4, \\
& j=2,\ldots,d-3 \quad \text{and} \quad i=1,\ldots,d-j-2 \\
v_{j,2d-2j-2,n} &:= h(v_{j-1,2d-2j,n}|v_{j-1,2d-2j-1,n},\rho_{d-j,d|d-j+1,\ldots,d-1},\nu_{d-j,d|d-j+1,\ldots,d-1}) \\
&\qquad \text{for} \quad j=2,\ldots,d-2.
\end{aligned}
$$

Since we are following a Bayesian approach the statistical model has to be completed by specifying the prior distributions for all model parameters. We specify a uniform $(-1,1)$ prior for the association parameter ρ of a $t-$copula pair and a uniform $(1,U)$ prior for the corresponding df parameter ν since in general we have little prior information available. Here the lower cut value 1 is chosen instead of 0 to avoid numerical instabilities in evaluating a quantile function of the bivariate $t-$distribution. The upper cut value U can be chosen by the data analyst to assess the closeness to the bivariate Gaussian copula. Finally we assume that prior distributions for ρ and ν are independent within each pair and independent over all pairs.

Markov chain Monte Carlo (MCMC) methods (see e.g. Chib (2001) for a comprehensive overview) are necessary to approximate the posterior distribution of the joint parameter vector $\boldsymbol{\theta}$ for the PCC specified in (3.2). Since full conditionals are not available a Gibbs sampler cannot be applied. Instead we use the Metropolis-Hasting (MH) algorithm (see Hastings (1970) and Metropolis et al. (1953)). Individual MH steps for each (ρ,ν) pair of $\boldsymbol{\theta}$ are performed using a symmetric normal random walk proposal. Variances of the normal proposals are tuned to achieve parameter acceptance rates between 20%-80% as suggested by Besag et al. (1995).

4 Application

4.1 Norwegian financial returns

In our first application we consider the data from Aas et al. (2007). It records daily returns from January 1, 1999 to July 8, 2003 of the Norwegian stock index (T), the MSCI world stock index (M), the Norwegian bond index (B) and the SSBWG hedged bond index (S). Since the margins are time series data and certainly not i.i.d., Aas et al. (2007) applied an AR(1)-GARCH(1,1)-model separately for each of the 4 margins and transformed the corresponding standardized residuals using the empirical probability integral transformation to achieve approximate i.i.d. uniform margins. To facilitate comparison to the models considered in Aas et al. (2007) we now investigate the following PCC:

$$c(u_S, u_M, u_T, u_B) = c_{SM} \cdot c_{MT} \cdot c_{TB} c_{ST|M} \cdot c_{MB|T} \cdot c_{SB|MT}, \tag{4.1}$$

where the parameter dependence of each bivariate $t-$copula and their arguments are dropped to keep the expression short.

We run the MH algorithm specified in Section 3 for 10000 iterations using independent (1,1000) uniform priors for each df parameter. Proposal variances were determined in pilot runs and resulted in acceptance rates between 26%-82% for all parameters after 10000 iterations. Autocorrelations among the MCMC iterates suggested sub-sampling to reduce these correlations and each 20-th iteration was recorded. Table 1 summarizes the estimated posterior distributions for all parameters based on the recorded iterations. For comparison we also include the corresponding maximum likelihood estimates (MLE) given in Aas et al. (2007). As Bayesian counterpart of the MLE we consider the posterior mode of the estimated kernel density (see Figure 3) to the thinned Markov chain.

Table 1: Estimated posterior mode, mean, median, 2.5% and 97.5% quantiles and MLE for the transformed Norwegian financial returns data

	2.5% Quantile	Est. Median	97.5% Quantile	Est. Post. Mean	Est. Post. Mode	MLE
ρ_{SM}	-0.316	-0.254	-0.184	-0.253	-0.25	-0.25
ν_{SM}	3.483	4.658	7.263	4.849	4.40	4.34
ρ_{MT}	0.422	0.466	0.508	0.465	0.47	0.47
ν_{MT}	13.304	228.006	948.340	326.388	108.61	16.26
ρ_{TB}	-0.224	-0.170	-0.106	-0.168	-0.17	-0.17
ν_{TB}	12.829	321.556	973.567	383.984	148.08	13.17
$\rho_{ST\|M}$	-0.163	-0.104	-0.047	-0.103	-0.11	-0.11
$\nu_{ST\|M}$	106.098	560.460	964.869	550.005	672.33	300.00
$\rho_{MB\|T}$	-0.033	0.031	0.090	0.029	0.03	0.03
$\nu_{MB\|T}$	45.736	514.904	972.857	510.722	632.56	45.59
$\rho_{SB\|MT}$	0.226	0.281	0.337	0.282	0.28	0.28
$\nu_{SB\|MT}$	13.557	285.269	958.333	366.153	127.36	15.04

Figure 3 shows that estimated posterior densities for the association parameters ρ_{SM}, ρ_{MT}, ρ_{TB}, $\rho_{ST|M}$, $\rho_{MB|T}$ and $\rho_{SB|MT}$ are quite symmetric and unimodal. Therefore the difference between posterior mode, mean and median estimate is negligible. For posterior distributions of the df parameters we observe nonsymmetric distribution. We see that the posterior MCMC iterations visit almost all regions of the prior support (1, 1000). This indicates that there is little information in the data to estimate the df parameters precisely.

This also explains why most of the 95%-credible intervals for the df parameters in Table 1 are very wide while the 95%-credible intervals for the association parameters are quite narrow.

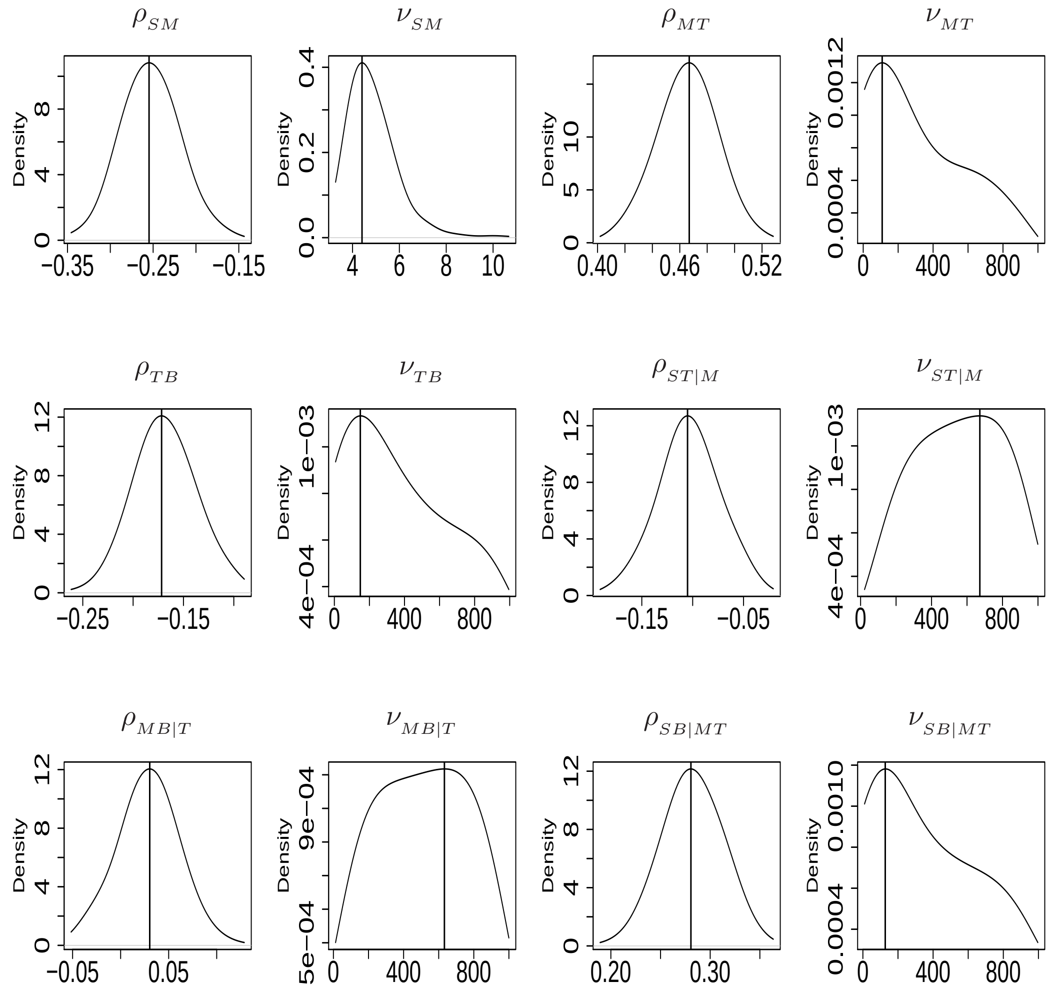

Figure 3: Estimated posterior densities of the parameters for Norwegian return data (vertical line indicates the posterior mode).

The difference between a $t-$copula and a Gaussian copula having the same moderate (< 0.5) association parameter ρ is very small if the df is larger than 10. Therefore the estimation procedure for the df parameter is very unstable. On the other hand Dakovic and Czado (2008) demonstrate that the variance of the ML estimate $\hat{\nu}$ for the df parameter derived from the central limit theorem increases very fast with respect to the magnitude of the df parameter. For example, in a 2-dimensional t-copula with moderate association parameter $\rho = 0.4$ and sample size 500, this variance increases from 0.08 to 25.41 as true value of df increases from 2.1 to 9.5. Therefore we expect confidence intervals for the df parameter to be wide when the true underlying parameter value is larger than 10. Note that almost all MLE's of the df parameters are contained in the corresponding 95%-credible intervals. Further the posterior mode of any of the association parameters is robust with respect to any meaningful selected kernel density bandwidth.

The 95% credible interval for the association parameter $\rho_{MB|T}$ contains 0 and the corresponding credible interval for the df parameter is far away to contain 10. This implies that conditional independence between M and B given T can be assumed and the PCC specified in (4.1) reduces to

$$c(u_S, u_M, u_T, u_B) \quad = \quad c_{SM} \cdot c_{MT} \cdot c_{TB} \cdot c_{ST|M} \cdot c_{SB|MT}.$$

The above conditional independence could be a result of stability of the Norwegian economy and therefore for a given value of the Norwegian stock index there is no influence of the MSCI world stock index on the Norwegian bond index. Another reason could be that Norway is one of the largest oil and gas exporters and their export value lie approximately between 40% and 50% of Norway's total exports. For example the crude oil export in 2006 was accounting for 41% of total exports.

The Bayesian estimation procedure allows to estimate posterior distributions of functions of the parameters using MCMC iterates. We illustrate this first with Kendall's τ (see Kruskal (1958)) which is an alternative dependence measure to the linear correlation coefficient . It is preferred over the linear correlation coefficient since it is invariant with respect to strictly increasing nonlinear transformations and does not require the existence of second moments. For shortcomings and pitfalls of the correlation coefficient we refer to Embrechts et al. (2002). A key role plays here the following relationship

$$\tau \quad = \quad \frac{2}{\pi} \cdot \arcsin \rho \tag{4.2}$$

proven by Lindskog et al. (2003) for elliptical distributions with continuous margins. We apply (4.2) to all recorded MCMC iterates for any of the association parameter to sample from the posterior distribution of τ. Table 2 contains the summary statistics for the estimated posterior distribution of τ for the pairs SM, MT and TB, respectively. We compare the posterior mode estimate to the empirical estimate of Kendall's τ (see Kruskal (1958)) and see good matching, indicating good model fit.

Table 2: Estimated posterior mode, mean, median, 2.5% and 97.5% quantiles and empirical estimate of Kendall's τ for the transformed Norwegian financial returns data

	2.5% Quantile	50% Quantile	97.5% Quantile	Est. Post. Mean	Est. Post Mode	Empirical τ
τ_{SM}	-0.204	-0.164	-0.1179	-0.163	-0.164	-0.158
τ_{MT}	0.277	0.308	0.3393	0.308	0.309	0.313
τ_{TB}	-0.144	-0.109	-0.0677	-0.108	-0.110	-0.110

In financial applications one is especially interested to measure upper and lower tail dependence (see Embrechts et al. (2003)). For the symmetric $t-$copula the upper and the lower tail dependence coefficients coincides and is given by

$$\lambda \quad = \quad 2t_\nu \left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}} \right). \tag{4.3}$$

In contrast the Gaussian copula has zero tail dependence. Since only the posterior mode estimate of the df parameter for the pair SM is small (≤ 10) we are interested in determining the tail dependence for this pair. The posterior mode estimate for λ_{SM} based on the MCMC iterates is 0.023 with a 95% credible interval given by [0.005, 0.0466]. The empirical estimate of the lower (upper) tail dependence coefficient based on the first 34 order statistics (based on the last 34 order statistics) is 0.02941 (0.06). We would like to note that empirical estimates for the tail dependence coefficients in this data set are quite instable. The Bayesian approach is favored here since confidence intervals for tail dependence coefficients are difficult to construct.

For a final model check we consider now the $\lambda-$ function for a bivariate copula $C(u_1, u_2, \boldsymbol{\psi})$ with parameter vector $\boldsymbol{\psi}$ which is defined as

$$\lambda(z, \boldsymbol{\psi}) := z - K(z, \boldsymbol{\psi}),$$

where $K(z, \boldsymbol{\psi}) := P(C(u_1, u_2, \boldsymbol{\psi}) \leq z)$. The K-function $K(z, \boldsymbol{\psi})$ has a simple form for Archimedean copulas (see Genest and Rivest (1993)). For large data sets the $\lambda-$function can be easily estimated empirically by using the empirical copula function. Figure 4 shows the posterior mode estimates (solid line) together with pointwise 95% credible intervals (dashed line) for all unconditional and conditional pairs. For comparison the empirical $\lambda-$function (dotted line) is added. Here the data for the conditional pairs is generated by using the posterior mode parameter

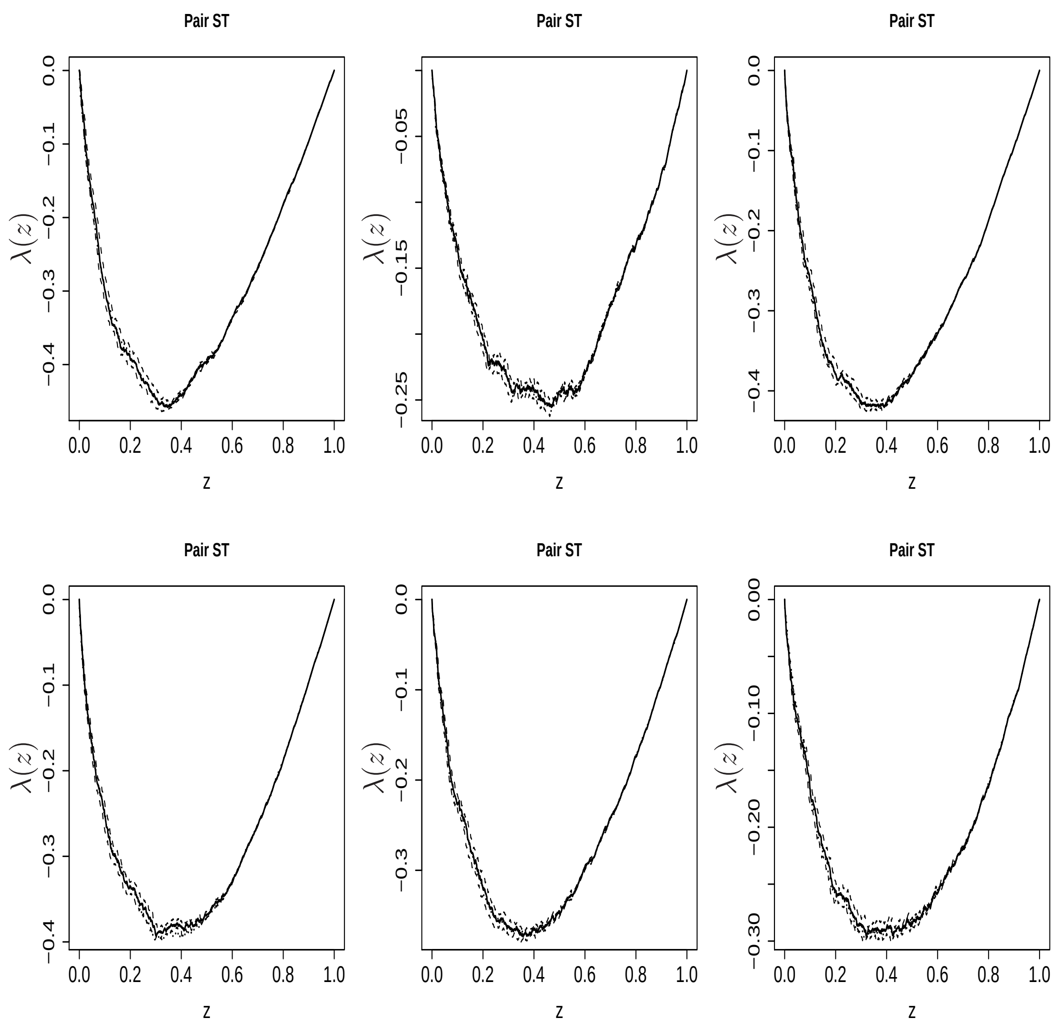

Figure 4: Posterior mode estimate of $\lambda-$function (solid) with pointwise 95% credible intervals (dashed) and its empirical estimate (dotted) for Norwegian return data.

estimates of the copula-pairs involved. For example for the pair ST|M use the transformed data

$$(u_i^{S|M}, u_i^{T|M}) := \left(h(u_i^S | u_i^M, \hat{\rho}_{SM}, \hat{\nu}_{SM}), h(u_i^T | u_i^M, \hat{\rho}_{TM}, \hat{\nu}_{TM}) \right),$$

where the $h-$function for a bivariate $t-$copula is given in (3.1) and u_i^A is the original transformed data for $A = S, M, T$, respectively. From these plots we see no lack of fit for the PCC model.

4.2 Euro swap rates

Factor models are commonly applied to model the term structure of interest rates (see Ingersoll (1983), Litterman and Scheinkman (1993), Bliss (1997) and Audrino et al. (2005)). For effective risk management of interest rates the stability of the factor loadings has to be assumed. Recently Audrino et al. (2005) showed that this stability cannot be assumed in the daily factor structure of interest rates. To overcome this problem they suggest filtering the data using a nonparametric VAR type model in connection with functional gradient descent estimation. They show that the filter is successful in removing autocorrelation and cross correlations present in interest rate levels. This motivates our study of daily swap rates to investigate the dependence structure present in the data. A precise characterization of the dependence structure of daily swap rates will also be helpful for risk management of swap rates portfolios. For example it will allow to construct realistic Monte Carlo simulations to determine risk measures such as VAR and expected short fall in a swap rate portfolio. Our starting point are daily Euro swap rates with maturity of 2, 3, 5, 7 and 10 years, respectively. The swap rates are based on annually compounded zero coupon swaps. The data investigated covers 3182 days starting from December 7, 1998 until May 21, 2001 and is presented in Figure 5. An ARMA(1,1)-GARCH(1,1) model applied to each of the 5 time series corresponding to different maturity are found sufficient to achieve independent margins while an AR(1)-GARCH(1,1) was insufficient. The corresponding standardized residuals of the margins are transformed to achieve approximate uniform margins. These uniform margins corresponding to 2, 3, 5, 7 and 10 years maturity are denoted by S_2, S_3, S_5, S_7 and S_{10}, respectively. Figure 6 displays pair-plots of S_2, S_3, S_5, S_7 and S_{10} showing the expected high cross correlation for close maturities.

Joint tail behavior is crucial for VAR and expected shortfall calculations, therefore we fit bivariate $t-$copulas to each pair. The association parameter ρ is estimated based on (4.2) and the empirical estimate of Kendall's τ. Holding ρ fixed, we estimate the df parameter using maximum likelihood. The tail dependence coefficient λ is then computed using (4.3) with the above estimated parameters. Table 3 gives the corresponding estimated df's, ρ's and tail dependence coefficients for all bivariate pairs of S_2, S_3, S_5, S_7 and S_{10} . We used the tail dependence coefficients from Table 3 to label the variables in the top tree of the D-vine. They are ordered in such a way that neighboring variables have the highest tail dependence coefficient among all possible orderings if tail dependence coefficients are compared componentwise from the left to the right. This gives the order $S_2, S_3, S_5, S_7, S_{10}$ for the top tree of the D-vine. Further we see that the estimated df's are quite small, i.e. displaying strong tail dependence between the pairs. However they are varying between 2.07 and 5.23, thus demonstrating that a multivariate $t-$copula model with a single df parameter is not sufficient. We also see that the pair with short adjacent maturities $((S_2, S_3))$ is less tail dependent than with long maturities $((S_7, S_{10}))$.

The corresponding PCC contains 10 pair-copulas and is given by

$$\begin{aligned} c(u_{S_2}, u_{S_3}, u_{S_5}, u_{S_7}, u_{S_{10}}) &= c_{S_2 S_3} c_{S_3 S_5} c_{S_5 S_7} c_{S_7 S_{10}} c_{S_2 S_5|S_3} c_{S_3 S_7|S_5} c_{S_5 S_{10}|S_7} \\ &\times c_{S_2 S_7|S_3 S_5} c_{S_3 S_{10}|S_5 S_7} c_{S_2 S_{10}|S_3 S_5 S_7}, \end{aligned} \tag{4.4}$$

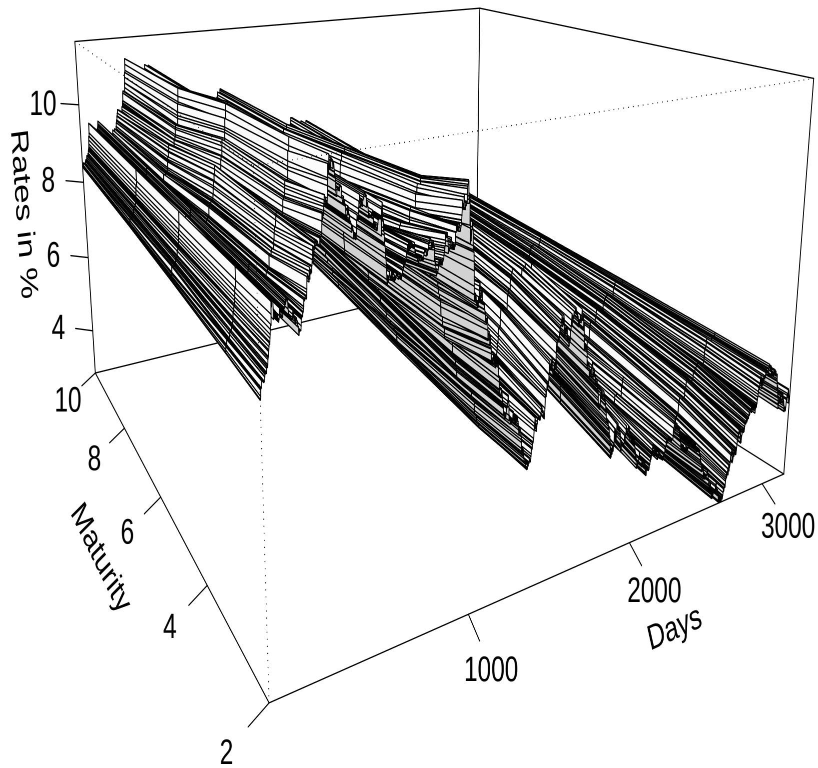

Figure 5: Three dimensional plot of the Euro swap rates data

where the arguments and the parameters of the pair-copulas are dropped for simplicity. We run 10000 iterations MCMC and the first 1000 iterations were discarded as burn-in. The acceptance rates range between 14% and 57%. Autocorrelation functions showed that the chain should be sub-sampled and each 20-th iteration was recorded.

Figure 7 displays the estimated posterior kernel density for parameters of PCC (4.4) based on the sub-sampled MCMC iterations and using the Gaussian kernel. Similarly as for the Norwegian returns data, the estimated posterior densities for the association parameters are symmetric and strict unimodal indicating stable behavior of the MCMC iterations. In contrast to the Norwegian returns data, the swap rates data contains more information on df parameters resulting in strong unimodality of the corresponding estimated posterior densities. Nevertheless the estimated posterior densities for $\nu_{S_2S_7|S_3S_5}$ and $\nu_{S_2S_{10}|S_3S_5S_7}$ in Figure 7 are not symmetric. Here we would expect that the Bayesian estimates as well as the ML estimates are larger than 10 and Table 4 justifies this premise. In the trace plots for $\nu_{S_2S_7|S_3S_5}$ and $\nu_{S_2S_{10}|S_3S_5S_7}$ we have observed that from time to time the MCMC iterations run far away from the posterior mode estimate. The frequency of such a deviation and its size depends on the difference between the posterior mode estimate and 10. The larger the difference, the higher is the deviation and the MCMC iterations move away from the posterior mode estimate more frequently.

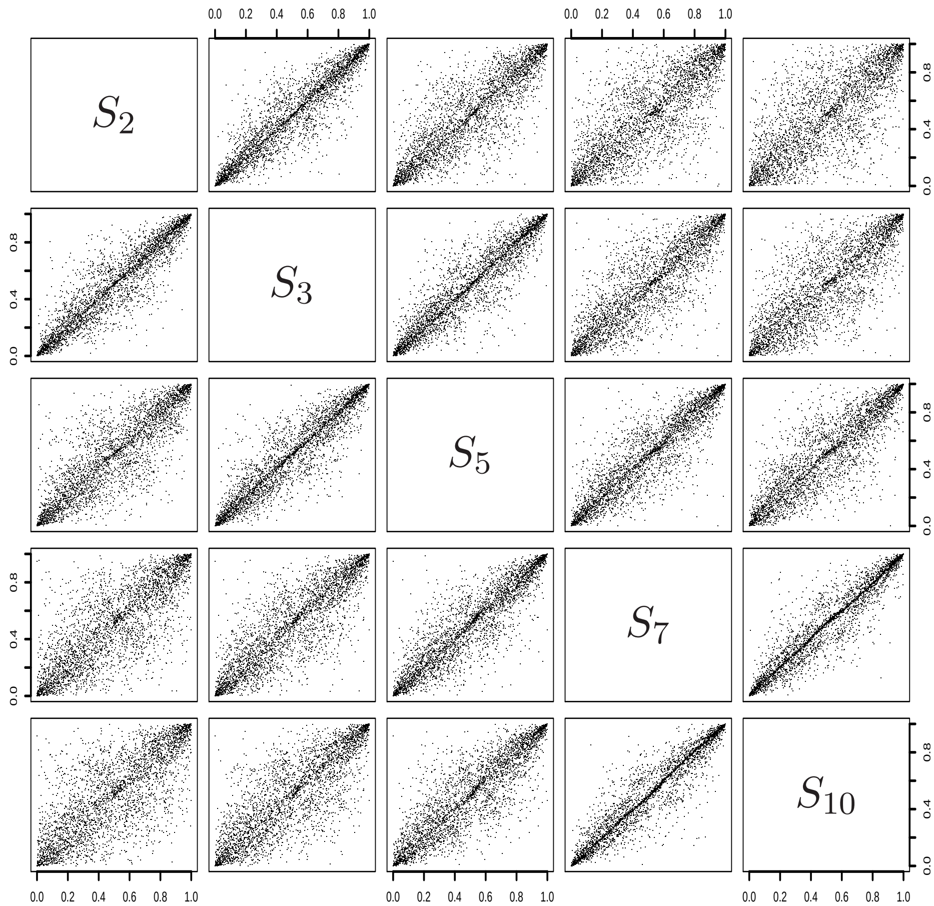

Figure 6: Scatter plots of the transformed to uniform margins standardized residuals for the swap rate data

Table 4 summarizes the estimated posterior distributions for all parameters. For comparison we also present the corresponding ML estimates. Here the difference between Bayesian and ML estimates for df parameters is much smaller compared to the Norwegian financial returns data. It seems that the estimation procedure involving the df parameters in this data is very stable. The 95% credible intervals for $\rho_{S_2S_{10}|S_3S_5S_7}$ and $\rho_{S_2S_5|S_3}$ contain 0 and indicate that the corresponding variables are conditionally uncorrelated. While the estimated posterior mode for $\nu_{S_2S_5|S_3}$ is small, the estimated posterior mode for $\nu_{S_2S_{10}|S_3S_5S_7}$ is large, indicating that in addition to uncorrelatedness we can assume that swap rates with maturity of 2 and 10 years are independent given swap rates with intermediate maturities of 3, 5 and 7 years. Because of high-cross correlation of interest levels, the knowledge of swap rates with the intermediate maturities may determine the swap rates with maturity of 2 and 10 years with high probability and the lowest and highest maturities move independently. However it is expected that the movement size (variance) is small and small twists can occur. A twist denotes a change in the shape of the yield curve and means that interest rates of bonds for some maturities change differently than interest rates of bonds for other maturities. In general, the credible intervals in Table 4 for the association parameter ρ of all conditional copulas contain 0 or are close to contain 0. Further

Table 3: Estimated df's and tail dependence coefficients for each pair of the transformed swap rates assuming bivariate t-copula margins.

		S_3	S_5	S_7	S_{10}
S_2	df	2.87	3.62	4.53	5.23
	ρ	0.94	0.88	0.82	0.79
	λ	0.75	0.62	0.50	0.43
S_3	df		2.86	4.19	4.93
	ρ		0.94	0.87	0.84
	λ		0.75	0.58	0.51
S_5	df			3.00	3.56
	ρ			0.92	0.89
	λ			0.71	0.64
S_7	df				2.07
	ρ				0.96
	λ				0.83

the corresponding posterior mode estimates are also very close to 0. The difference between the bivariate independence copula density and a bivariate $t-$copula density with $\rho = 0$ and $\nu = 5$ is negligible in the center region of the unit square support $[0, 1]^2$. A difference is observed only in thin strips around the support borders and especially at the four corners of the support. To illustrate this point, we simulate two bivariate samples $(\mathbf{U}_1, \mathbf{U}_2)$ of length 2000 from the above copula models and Figure 8 displays the scatter plot of these samples. As one can see in Figure 8 there is no difference in the pattern of scatter plots if the border strip regions are excluded (cf. scatter plots in the dashed square). Therefore the conditional uncorrelatedness could be here interpreted as the corresponding conditional independence in the center region. Thus it seems that PCC model specification (4.4) for the Euro swap rates could be reduced in the center by all its conditional pair-copulas and the reduced central PCC is given by

$$c(u_{S_2}, u_{S_3}, u_{S_5}, u_{S_7}, u_{S_{10}}) \quad \approx \quad c_{S_2S_3}c_{S_3S_5}c_{S_5S_7}c_{S_7S_{10}} \qquad \text{for central} \qquad \mathbf{u} \in [0, 1]^5.$$

This reduction would also be achieved if one assumes a first order Markov dependence between swap rates with adjacent maturities. Note that an independence between swap rates of 2 and 10 year maturities would fit with the market segmentation theory (see Culbertson (1957)) which says that individuals have strong maturity preferences resulting that bonds of different maturities are traded in separate markets. However our analysis confirms only a conditional independence in the central region given swap rates with intermediate maturities. For the tail dependence we note that λ varies between 0.058 and 0.009 when ν is between 5 and 16 for $\rho = 0$, thus only moderate tail dependence is present for the conditional copula pairs.

We apply (4.3) to all recorded MCMC iterates to sample from the posterior distribution of the tail dependence coefficient λ. Table 5 displays the estimated posterior distribution of λ as well as its empirical upper and lower tail dependence coefficients $\hat{\lambda}^{upper}$ and $\hat{\lambda}^{lower}$ for the pairs S_2S_3, S_3S_5, S_5S_7 and S_7S_{10}, respectively. The empirical estimate $\hat{\lambda}^{upper}$ ($\hat{\lambda}^{lower}$) is computed based on upper (lower) order statistics and the exact number of those is chosen to achieve relative stability of the estimate. The 95% credible intervals contain the corresponding

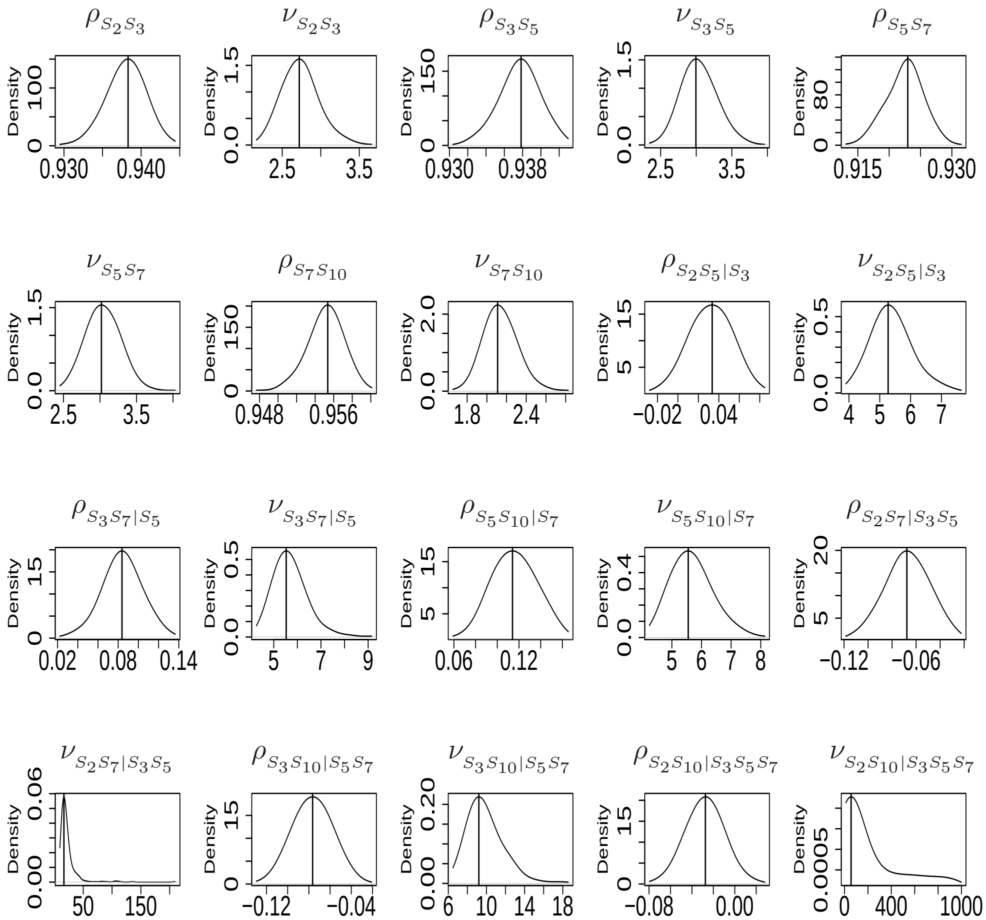

Figure 7: Estimated posterior densities of the parameters for the swap rate data

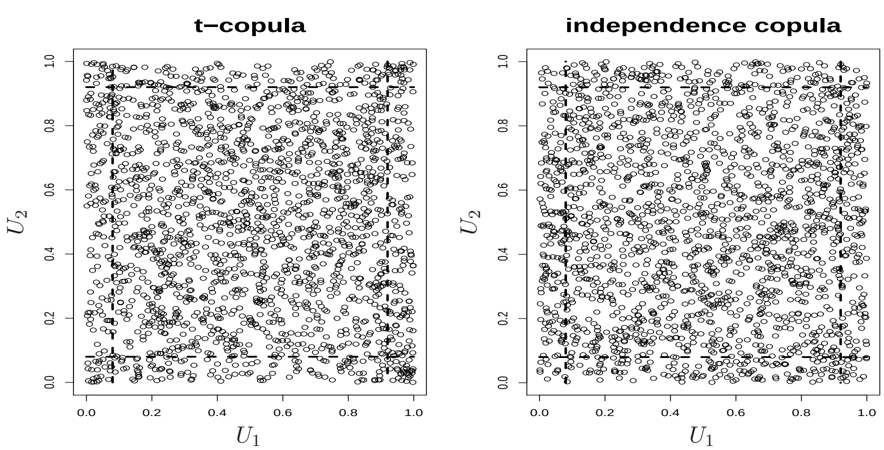

Figure 8: Scatter plot of $\mathbf{U}_1$ versus $\mathbf{U}_2$ for t-copula with $\rho = 0$ and $\nu = 5$ (left) and independence copula (right). The dashed square is the central region of the support $[0, 1]^2$, where the scatter plots look similar to each other.

Table 4: Estimated posterior mode, mean, median, 2.5% and 97.5% quantiles and MLE for the transformed swap rate data

	2.5% Quantile	50% Quantile	97.5% Quantile	Est. Post. Mean	Est. Post. Mode	MLE
$\rho_{S_2S_3}$	0.93	0.94	0.94	0.94	0.94	0.94
$\nu_{S_2S_3}$	2.33	2.73	3.26	2.74	2.72	2.49
$\rho_{S_3S_5}$	0.93	0.94	0.94	0.94	0.94	0.94
$\nu_{S_3S_5}$	2.62	3.03	3.54	3.04	3.00	3.08
$\rho_{S_5S_7}$	0.92	0.92	0.93	0.92	0.92	0.92
$\nu_{S_5S_7}$	2.62	3.03	3.48	3.04	3.02	2.87
$\rho_{S_7S_{10}}$	0.95	0.96	0.96	0.96	0.96	0.96
$\nu_{S_7S_{10}}$	1.86	2.12	2.50	2.14	2.11	2.27
$\rho_{S_2S_5\|S_3}$	-0.01	0.03	0.07	0.03	0.03	0.03
$\nu_{S_2S_5\|S_3}$	4.19	5.32	7.00	5.40	5.27	5.08
$\rho_{S_3S_7\|S_5}$	0.04	0.08	0.12	0.08	0.08	0.08
$\nu_{S_3S_7\|S_5}$	4.63	5.63	7.36	5.70	5.53	5.30
$\rho_{S_5S_{10}\|S_7}$	0.08	0.12	0.15	0.12	0.11	0.12
$\nu_{S_5S_{10}\|S_7}$	4.64	5.62	7.22	5.67	5.55	5.58
$\rho_{S_2S_7\|S_3S_5}$	-0.10	-0.07	-0.03	-0.07	-0.07	-0.07
$\nu_{S_2S_7\|S_3S_5}$	10.84	18.12	55.27	22.21	16.48	11.94
$\rho_{S_3S_{10}\|S_5S_7}$	-0.11	-0.08	-0.04	-0.08	-0.08	-0.08
$\nu_{S_3S_{10}\|S_5S_7}$	7.10	9.51	14.58	9.90	9.20	7.69
$\rho_{S_2S_{10}\|S_3S_5S_7}$	-0.06	-0.03	0.01	-0.03	-0.03	-0.02
$\nu_{S_2S_{10}\|S_3S_5S_7}$	15.37	64.51	954.21	223.08	55.84	16.30

Table 5: Estimated posterior mode, mean, median, 2.5% and 97.5% quantiles and empirical estimate of the lower and upper tail dependence coefficient for different pairs of the transformed swap rate data

	2.5% Quantile	50% Quantile	97.5% Quantile	Est. Post. Mean	Est. Post. Mode	Empirical $\hat{\lambda}^{upper}$	Empirical $\hat{\lambda}^{lower}$
$\lambda_{S_2S_3}$	0.73	0.76	0.77	0.76	0.76	0.76	0.74
$\lambda_{S_3S_5}$	0.72	0.74	0.76	0.74	0.74	0.75	0.74
$\lambda_{S_5S_7}$	0.69	0.71	0.73	0.71	0.71	0.73	0.71
$\lambda_{S_7S_{10}}$	0.80	0.81	0.83	0.81	0.81	0.82	0.82

empirical estimate of λ and therefore the PCC model captures well the tail dependence present in the data.

5 Conclusion and Discussion

The paper considers the flexible modeling of multivariate copulas with PCC models introduced by Aas et al. (2007) in the Bayesian framework. It is well known that financial data are usually heavy tailed and therefore we base the PCC models on bivariate $t-$copulas. In our MCMC analysis we have also used mixtures of uniform distributions as prior for df parameters and the results were quite similar to ones presented here. The sampling algorithm was independent of initial values.

Credible intervals for parameters of PCC's are obtained and they reveal conditional independence between variables. It was found in the Norwegian financial returns data that the MSCI world stock index (M) and the Norwegian bond index (B) are conditionally independent given the Norwegian stock index (T). This could indicate that the Norway has a healthy independent economy and the Norwegian bond index does not depend on the MSCI world stock index if the Norwegian stock index is given. This could also be the consequence having large oil and gas reservoirs, which dominate their export balance. The Bayesian analysis of the swap rate data based on PCC shows that interest rates could be treated as conditionally independent. This is a consequence of the high cross-correlation of the interest rate levels. The knowledge of one particular swap rate contains most information on the swap rates of neighboring maturities years and only small twists of the yield curve may take place. Further, in contrast to the market segmentation theory, our findings support only conditional independence between swap rates of 2 and 10 years maturities given swap rates with intermediate maturities.

We constructed credible intervals for Kendall's τ and tail dependence coefficient λ based on MCMC iterates for parameters of PCC . They illustrate that models based on PCC models fit the data well and provide a good agreement with model free estimates. In a similar manner credible intervals for many other characteristics of random vectors can be constructed whose confidence is difficult to derive.

We have seen that if independence or conditional independence are present in data then the PCC model can be substantially reduced. Therefore the model selection problem becomes extremely important for PCC's. In particular the choice of a decomposition and the choice of a pair-copula from a catalogue of bivariate copulas including the independence copula needs to be addressed. Recently Bayesian model selection procedures have found their applications in a number of complex problems. Here model choice and parameter estimation problems have to be solved simultaneously. From our point of view, there are two most used methods due Green (1995) and Carlin and Chib (1995). The method of Green (1995) requires a model jump mechanism while the method of Carlin and Chib (1995) treats a product space of parameters of all models. Since even for moderate dimensions like 10 or 20, the number of different models becomes very huge, we prefer the reversible jump MCMC of Green (1995). The derivation and implementation of appropriate reversible jump MCMC algorithms for high dimensional data under PCC model specifications is under current construction and its results will be reported in a further paper. The method of Carlin and Chib (1995), by its construction, can only be recommended to the case where only a small set of models is to be compared.

Another open problem is the joint estimation of marginal AR(ARMA)-GARCH and PCC parameters in the Bayesian framework. Recently Kim et al. (2007) have shown that a separate estimation of the marginal parameters may have an essential influence on the parameter estimation of multivariate copulas and inference based on joint estimates might be lead to quite different results compared to the inference ignoring estimation errors in the marginal parameters.

We expect similar results for PCC's and this is a topic of future research.

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