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ON THE RUIN PROBABILITY OF THE GENERALISED ORNSTEIN-UHLENBECK PROCESS IN THE CRAMÉR CASE†

DAMIEN BANKOVSKY,* *Australian National University*

CLAUDIA KLÜPPELBERG,** *Technische Universität München*

ROSS MALLER,*** *Australian National University*

Abstract

For a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ and initial value V_0 define the Generalised Ornstein-Uhlenbeck (GOU) process

$$V_t := e^{\xi_t}\Big(V_0 + \int_0^t e^{-\xi_{s-}}\,\mathrm{d}\eta_s\Big), \quad t \geq 0,$$

and the associated stochastic integral process

$$Z_t := \int_0^t e^{-\xi_{s-}}\,\mathrm{d}\eta_s, \quad t \geq 0.$$

Let $T_z := \inf\{t > 0 : V_t < 0 \mid V_0 = z\}$ and $\psi(z) := P(T_z < \infty)$ for $z \geq 0$ be the ruin time and infinite horizon ruin probability of the GOU. Our results extend previous work of Nyrhinen (2001) and others to give asymptotic estimates for $\psi(z)$ and the distribution of T_z as $z \to \infty$, under very general, easily checkable, assumptions, when ξ satisfies a Cramér condition.

Keywords: exponential functionals of Lévy processes; generalised Ornstein-Uhlenbeck process; ruin probability; stochastic recurrence equation

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1. Introduction

Let $(\xi, \eta) = (\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process on a filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$ and define a generalised Ornstein-Uhlenbeck (GOU) process by

$$V_t := e^{\xi_t}\Big(V_0 + \int_0^t e^{-\xi_{s-}}\,\mathrm{d}\eta_s\Big),\ t \geq 0, \tag{1.1}$$

and the associated stochastic integral process $Z = (Z_t)_{t\geq 0}$ by

$$Z_t := \int_0^t e^{-\xi_{s-}}\,\mathrm{d}\eta_s. \tag{1.2}$$

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* Postal address: Mathematical Sciences Institute, Australian National University, Canberra, Australia, email: Damien.Bankovsky@anu.edu.au
** Postal address: Center for Mathematical Sciences, and Institute for Advanced Study, Technische Universität München, 85747 Garching, Germany, email: cklu@ma.tum.de
*** Postal address: Mathematical Sciences Institute, and School of Finance and Applied Statistics, Australian National University, Canberra, Australia, email: Ross.Maller@anu.edu.au

V_0 is a random variable (r.v.), not necessarily independent of $(V_t)_{t>0}$. To avoid trivialities, assume that neither ξ nor η are identically zero.

Such processes have attracted attention over the last decade as continuous time analogues of solutions to stochastic recurrence equations (SRE); cf. Carmona, Petit and Yor [7, 8], Erickson and Maller [13]. The link between SREs and the GOU was made in de Haan and Karandikar [11]. GOU processes turn up naturally in stochastic volatility models (e.g., the continuous time GARCH model of Klüppelberg, Lindner and Maller [22]), but most prominently as insurance risk models for perpetuities in life insurance or when the insurance company receives some stochastic return on investment; such investigations started with Dufresne [12] and Paulsen [29]. More references are given later.

This paper is intended to fill a gap left between Bankovsky [2] and Bankovsky and Sly [3], where more details on the insurance background can be found. Define

$$T_z := \inf\{t > 0 : V_t < 0 \mid V_0 = z\},\ z \geq 0,$$

(with the convention throughout that $\inf \emptyset = \infty$), and let

$$\psi(z) := P\Big(\inf_{t>0} V_t < 0 \mid V_0 = z\Big) = P\Big(\inf_{t>0} Z_t < -z\Big) = P\left(T_z < \infty\right),\ z \geq 0, \tag{1.3}$$

be the *infinite horizon ruin probability* for the GOU. Note that $\psi(z)$ is a nonincreasing function of z, and we can ask how fast it decreases as $z \to \infty$.

Our main result, Theorem 2.1, provides a very general asymptotic result for $\psi(z)$ as $z \to \infty$ for the case when $\lim_{t\to\infty} Z_t$ exists as an a.s. finite r.v. and shows that, under a Cramér-like condition on ξ, $\psi(z)$ decreases approximately like a power law. This is an extension of a similar asymptotic result of Nyrhinen [28], who, like us, utilises a discrete time result of Goldie [16] for proof. We use more recent developments in the theory of discrete time perpetuities and the continuous time GOU to update Nyrhinen's results. In Section 3 we provide some examples which cannot be dealt with by the prior results but satisfy the conditions of our theorem.

To conclude this introduction, we describe some previous literature relating to the GOU and its ruin probability, beginning with those papers which examine the GOU in its full generality. The process appears implicitly in the work of de Haan and Karandikar [11] as a continuous generalisation of an SRE. Basic properties are given by Carmona et al. [8]. A general survey of the GOU and its applications is given by Maller, Müller and Szimayer [26]. Exact conditions for no ruin ($\psi(z) = 0$ for some $z \geq 0$) are given by Bankovsky and Sly [3] whilst conditions for certain ruin ($\psi(z) = 1$ for some $z \geq 0$) are examined by Bankovsky [2].

The study of the GOU is closely related to the study of integrals of the form Z, defined in (1.2). It is shown in Lindner and Maller [25] that stationarity of V is related to convergence of a stochastic integral constructed from (ξ, η) in a similar way to Z.

Among the few papers dealing with Z in its full generality, Erickson and Maller [13] give necessary and sufficient conditions for the almost sure convergence of Z_t to a r.v. Z_∞ as $t \to \infty$, and Bertoin, Lindner and Maller [4] present necessary and sufficient conditions for the continuity of the distribution of Z_∞, when it exists. Fasen [14], using point process methods, gives an account of the extremal behaviour of a GOU process.

There are a larger number of papers dealing with V and Z when (ξ, η) is subject to restrictions. We discuss a selection of those papers which are relevant to ruin probability. Harrison [18] presents results on the ruin probability of V when ξ is a linear deterministic function and η is a Lévy process with finite variance. His approach is based on an exponential martingale argument, which corresponds to the Cramér case. The heavy-tailed case is investigated in Klüppelberg and Stadtmüller [23] and extended by Asmussen [1]. See also Maulik and Zwart [27] and Konstantinides and Mikosch [24].

Paulsen [29] generalises Harrison's results, and presents new ruin probability results for V, when ξ and η are independent with finite activities. This independent case is also treated in Kalashnikov and Norberg [20] and Paulsen [30, 31]. Chiu and Yin [9] generalise some of Paulsen's results to the case in which η is a jump-diffusion process. Cai [6] and Yuen et al. [36] present results when η is a compound Poisson process.

Most relevant works containing restrictions on (ξ, η) focus on the case when Z_t converges to Z_∞ as $t \to \infty$; cf. Yor [35] and Carmona et al. [7]. Gjessing and Paulsen [15] study the distribution of Z_∞ when ξ and η are independent with finite activity, and obtain exact distributions in some special cases. Hove and Paulsen [19] use Markov chain Monte Carlo methods to find the distribution of Z_∞ in some special cases. Klüppelberg and Kostadinova [21] and Brokate et al. [5] provide results on the tail of the distribution of Z_∞ when η is a compound Poisson process plus drift, independent of ξ.

2. Main Results

Our main results apply under a Cramér-like condition on ξ: assume that

$$Ee^{-w\xi_1} = 1 \text{ for some } w > 0. \tag{2.1}$$

The following consequences of (2.1) are well known and easily verified. Condition (2.1) implies that $E\xi_1$ is well defined, with $E\xi_1^- < \infty$, $E\xi_1^+ \in (0, \infty]$, and $E\xi_1 \in (0, \infty]$, and so $\lim_{t\to\infty} \xi_t = \infty$ a.s. Further, $Ee^{-\alpha\xi_1}$ is finite and nonzero for all $\alpha \in [0, w]$, and $c(\alpha) := \ln Ee^{-\alpha\xi_1}$ is finite at least for all $\alpha \in [0, w)$. The derivatives $c'(\alpha)$ and $c''(\alpha)$ are finite at least for all $\alpha \in [0, w)$, and $c''(\alpha) \in (0, \infty]$ for all $\alpha \geq 0$. So $c(\alpha)$ is strictly convex for $\alpha \in [0, \infty)$ and $\mu^* := c'(w) = -E[\xi_1 e^{-w\xi_1}] \in (0, \infty]$.

We will need the *Fenchel-Legendre transform* of c, defined as

$$c^*(v) := \sup\{\alpha v - c(\alpha) : \alpha \in \mathbb{R}\},\ v \in \mathbb{R}. \tag{2.2}$$

Next, let

$$\alpha_0 := \sup\{\alpha \in \mathbb{R} : c(\alpha) < \infty, E|Z_1|^\alpha < \infty\} \in [0, \infty], \tag{2.3}$$

and define the constant

$$x_0 := \lim_{\alpha \to \alpha_0 -} (1/c'(\alpha)) \in [0, \infty]. \tag{2.4}$$

A distribution is *spread out* if it has a convolution power with an absolutely continuous component.

Theorem 2.1. *Suppose that the following conditions hold:*
Condition A: $\psi(z) > 0$ *for all* $z \geq 0$,
Condition B: *there exists* $w > 0$ *such that* $Ee^{-w\xi_1} = 1$ *(i.e. (2.1) holds),*
Condition C: *there exist* $\varepsilon > 0$ *and* $p, q > 1$ *with* $1/p + 1/q = 1$ *such that*

$$E[e^{-\max\{1, w+\varepsilon\}p\xi_1}] < \infty \quad \textit{and} \quad E[|\eta_1|^{\max\{1, w+\varepsilon\}q}] < \infty. \tag{2.5}$$

Then $0 \le x_0 < 1/\mu^* < \infty$*, the function*

$$R(x) := \begin{cases} xc^*(1/x) & \text{for } x \in (x_0, 1/\mu^*), \\ w & \text{for } x \ge 1/\mu^*, \end{cases}$$

is finite and continuous on (x_0, ∞) *and strictly decreasing on* $(x_0, 1/\mu^*)$*, and we have*

$$\lim_{z\to\infty} (\ln z)^{-1} \ln P(T_z \le x \ln z) = -R(x) \tag{2.6}$$

for every $x > x_0$*. In addition,*

$$\lim_{z\to\infty} (\ln z)^{-1} \ln \psi(z) = -w. \tag{2.7}$$

If, further, the distribution of ξ_1 *is spread out, then there exist constants* $C_- > 0$ *and* $\kappa > 0$ *such that*

$$z^w \psi(z) = C_- + o(z^{-\kappa}) \quad \text{as } z \to \infty. \tag{2.8}$$

Remark 2.1. (i) $\psi(z) > 0$ for all $z \ge 0$ is of course a logical assumption to make in the context of Theorem 2.1, though not necessarily easy to verify. Necessary and sufficient conditions for it in terms of the Lévy measure of (ξ, η) are given in [3]. The moment conditions in Theorem 2.1 are also easily expressed in terms of the Lévy measure of (ξ, η), cf. Sato [33], p. 159. They imply that $E[\sup_{0\le t\le 1} |Z_t|^{\max\{1,w+\varepsilon\}}] < \infty$ (see Lemma 5.1 below). We also have $E[\ln(\max\{1, |\eta_1|\}] < \infty$ in Theorem 2.1, and $\lim_{t\to\infty} \xi_t = \infty$ a.s., so Z_t converges a.s. to a finite r.v. Z_∞ as $t \to \infty$ by Proposition 2.4 of [25] or Theorem 2 of [13].
(ii) Let $\overline{Z}_t := Z_t - \inf_{0\le s\le t} Z_s$ be the process reflected in its minimum, and set

$$(M, Q, \overline{L}) := \big(e^{-\xi_1}, Z_1, -e^{\xi_1}\overline{Z}_1\big). \tag{2.9}$$

Then the value C_- in (2.8) is given by the formula in (2.19) of Goldie [16], namely

$$C_- = \frac{1}{w\mu^*} E\Big[\Big(Q + M \min\big\{\overline{L}, \inf_{t>0} Z_t\big\}^-\Big)^w - \Big(\big(M \inf_{t>0} Z_t\big)^-\Big)^w\Big]. \tag{2.10}$$

When ξ and η are independent, it was pointed out by Paulsen [31] that this constant can be written in a slightly different form, which, by Theorem 4 of [3], is also true in the dependent case. Namely, let $G(z) := P(Z_\infty \le z)$, $h(z) := E[G(-V_{T_z}) \mid T_z < \infty] \in [0, 1]$, and $h := \lim_{z\to\infty} h(z)$. Then

$$C_- = \frac{1}{w\mu^* h} E\Big[\, \Big((Q + MZ_\infty)^-\Big)^w - \Big((MZ_\infty)^-\Big)^w \,\Big].$$

(iii) The requirement that ξ_1 is spread out can be replaced with the less restrictive requirement that ξ_T be spread out, where T is uniformly distributed on $[0, 1]$ and independent of ξ. We omit details of this, which can be carried out as in [31].

3. Examples

In this section we provide examples of Lévy processes for which Conditions A, B and C of Theorem 2.1 are satisfied. Note that conditions B and C only involve the marginal processes ξ and η and they apply to all examples treated in the literature so far; cf. Klüppelberg and Kostadinova [21] for detailed references. The only condition which may involve dependence between ξ and η is Condition A.

We denote the characteristic triplet of (ξ,η) by $((\tilde\gamma_\xi,\tilde\gamma_\eta),\Sigma_{\xi,\eta},\Pi_{\xi,\eta})$. The characteristic triplet of the marginal process ξ is denoted by $(\gamma_\xi,\sigma_\xi^2,\Pi_\xi)$, where

$$\gamma_\xi=\tilde\gamma_\xi+\int_{\{|x|<1\}\cap\{x^2+y^2\geq 1\}} x\Pi_{\xi,\eta}(\mathrm{d}(x,y)), \tag{3.1}$$

and σ_ξ^2 is the upper left entry in the matrix $\Sigma_{\xi,\eta}$. Similarly for η. The random jump measure and Brownian motion components of (ξ,η) will be denoted respectively by $N_{\xi,\eta}$ and (B_ξ,B_η); see Section 1.1 of [3] for further details.

Example 1. [Bivariate compound Poisson process with drift]
Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda>0$, and, independent of it, $(X_i,Y_i)_{i\in\mathbb{N}}$ an iid sequence of random 2-vectors. For $\gamma_\xi,\gamma_\eta\in\mathbb{R}$ set

$$(\xi_t,\eta_t):=(\gamma_\xi,\gamma_\eta)\,t+\sum_{i=1}^{N_t}(X_i,Y_i),\quad t\geq 0,$$

with $E|X_1|<\infty$ and λ, γ_ξ and EX_1 such that $\gamma_\xi+\lambda EX_1>0$. For this process,

$$c(\alpha)=\ln Ee^{-\alpha\xi_1}=-\alpha\gamma_\xi-\lambda\big(1-Ee^{-\alpha X_1}\big)<\infty$$

for $\alpha\in\mathbb{R}$ such that $Ee^{-\alpha X_1}$ is finite, with $c'(0)=-\gamma_\xi-\lambda EX_1<0$.

We consider the special case where (X_1,Y_1) is bivariate Gaussian with mean (m_X,m_Y) and positive definite covariance matrix

$$\Sigma_{X,Y}:=\begin{pmatrix}\sigma_X^2 & \sigma_{X,Y}\\ \sigma_{X,Y} & \sigma_Y^2\end{pmatrix}.$$

Then Condition C obviously holds. For Condition B, note that

$$c(\alpha)=-\alpha\gamma_\xi-\lambda\big(1-e^{-m_X\alpha+\sigma_X^2\alpha^2/2}\big)\to\infty\ \text{ as } \alpha\to\infty. \tag{3.2}$$

Consequently, a Lundberg coefficient exists and Condition B is satisfied. To establish Condition A we note that (ξ,η) is a finite variation process and invoke Remark 2(2) of [3], also using the notation from that paper. In fact, by that Remark 2(2), $\psi(z)=0$ for some $z>0$ would imply that $P_{X,Y}(A_3)=P(X_1\leq 0,Y_1\leq 0)=0$, which obviously is not the case. So Condition A holds.

Example 2. A Brownian motion with drift, i.e., with

$$(\xi_t,\eta_t)=(\gamma_\xi,\gamma_\eta)\,t+(B_{\xi,t},B_{\eta,t}),\quad t\geq 0,$$

where $\gamma_\xi>0$ and $(B_\xi,B_\eta)_t$ is bivariate Brownian motion with mean 0 and positive definite covariance matrix, is easily seen to satisfy Conditions A, B, C.

Example 3. [Jump diffusion ξ and Brownian motion η]
Let $(B_t)_{t\geq 0}$ be Brownian motion with mean zero and variance σ^2, $(N_t)_{t\geq 0}$ a Poisson process with intensity $\lambda>0$, and $(X_i)_{i\in\mathbb{N}}$ iid r.v.s, all independent. Set

$$(\xi_t,\eta_t)=(\gamma_\xi,\gamma_\eta)t+\Big(B_t+\sum_{i=1}^{N_t}X_i,B_t\Big),\quad t\geq 0,$$

where $\gamma_\xi > 0$, and assume that $\gamma_\xi + \lambda E X_1 > 0$. Condition A holds, since the Gaussian covariance matrix of (ξ, η) is of the form

$$\Sigma_{\xi,\eta} := \begin{pmatrix} \sigma^2 + \lambda E X_1^2 & 1 \\ 1 & \sigma^2 \end{pmatrix}, \tag{3.3}$$

and, hence, is not of the form excluded by Theorem 1 of [3]. Moreover, $c(\alpha)$ is the same as in (3.2) with the addition of a term $\alpha^2\sigma^2/2$, so again $c'(0) = -\gamma_\xi - \lambda E X_1 < 0$.

(a) Now assume that X_1 is, as in the Merton model, normally distributed with mean m_X and variance σ_X. Then Conditions B and C are satisfied just as in Example 1.

(b) The picture changes slightly when we consider Laplace distributed X with density $f(x) = \rho e^{-\rho|x|}/2$ for $x \in \mathbb{R}$, $\rho > 0$. Then $Ee^{-\alpha X} = \rho\left((\rho+\alpha)^{-1} + (\rho-\alpha)^{-1}\right)/2$ for $-\rho < \alpha < \rho$ with singularities at $-\rho$ and ρ. Moreover,

$$c'(\alpha) = -\gamma_\xi + \alpha\sigma^2 + \lambda\frac{\rho}{2}\Big(\frac{1}{(\rho-\alpha)^2} - \frac{1}{(\rho+\alpha)^2}\Big),$$

implying that $c'(0) = -\gamma_\xi < 0$. So a Lundberg coefficient $w > 0$ exists. Since the normal r.v. B_1 has absolute moments of every order, for Condition C to hold it suffices that $w < \rho$, which is guaranteed, since ρ is a singularity of c.

Example 4. [Subordinated Brownian motion ξ and spectrally positive η]
Let $(B_t)_{t\ge 0}$ be a standard Brownian motion and $(S_t)_{t\ge0}$ a driftless subordinator with $\Pi_S\{\mathbb{R}\} = \infty$. For constants μ, γ_ξ, γ_η, define

$$(\xi_t, \eta_t) = (\gamma_\xi, \gamma_\eta)t + (B(S_t) + \mu S_t, S_t), \quad t \ge 0.$$

Subordinated Brownian motions play an important role in financial modeling; cf. Cont and Tankov [10], Ch. 4. The bivariate process above has joint Laplace transform

$$\begin{aligned} e^{(\alpha_1\gamma_\xi+\alpha_2\gamma_\eta)t} E\big[e^{\alpha_1(B(S_t)+\mu S_t)+\alpha_2 S_t}\big] &= e^{(\alpha_1\gamma_\xi+\alpha_2\gamma_\eta)t} E\big[e^{\Psi_B(\alpha_1)S_t+(\alpha_1\mu+\alpha_2)S_t}\big] \\ &= e^{t[\Psi_S(\Psi_B(\alpha_1)+\alpha_1\mu+\alpha_2)+\alpha_1\gamma_\xi+\alpha_2\gamma_\eta]}, \end{aligned}$$

where Ψ_B and Ψ_S are the Laplace exponents of B and S, respectively. Thus $\Psi_B(\alpha) = -\alpha^2/2$. By setting $\alpha_2 = 0$ and $t = 1$ we obtain

$$c(\alpha) = \ln Ee^{-\alpha\xi_1} = \Psi_S(\Psi_B(-\alpha) - \alpha\mu) - \alpha\gamma_\xi = \Psi_S(-\alpha^2/2 - \alpha\mu) - \alpha\gamma_\xi.$$

Consider the variance gamma model with parameters $c, \lambda > 0$, where S is a gamma subordinator with Lévy density $\rho(x) = cx^{-1}e^{-\lambda x}$ for $x > 0$ and Laplace transform $Ee^{-uS_t} = (1 + u/\lambda)^{-ct}$. Assume $\gamma_\xi + c\mu/\lambda > 0$ and $\gamma_\eta \le 0$. Now, $\Psi_S(u) = -c\ln(1 - u/\lambda)$, giving

$$c(\alpha) = -\alpha\gamma_\xi - c\ln\Big(1 + \frac{\alpha\mu}{\lambda} - \frac{\alpha^2}{2\lambda}\Big).$$

$c(\alpha)$ is well defined for $\alpha \in (\mu - \sqrt{\mu^2+2\lambda}, \mu + \sqrt{\mu^2+2\lambda})$, which includes 0, and $c'(0) = -\gamma_\xi - c\mu/\lambda < 0$. Then, since $c(\mu + \sqrt{\mu^2+2\lambda}) = \infty$, the Lundberg coefficient w exists.

In order to check Condition A, we have, in the notation of Theorem 1 of [3], $\Pi_{\xi,\eta}(A_2) = \Pi_{\xi,\eta}(A_3) = 0$, since η has only positive jumps, and $\theta_2 = 0$. Now with

$u \geq 0$, $A_4^u = \{x \leq 0, y \geq 0 : y < u(e^{-x} - 1)\} = \{x \geq 0, y \geq 0 : y < u(e^x - 1)\}$. Since $\Pi_\eta(\mathbb{R}) = \infty$, η has jumps arbitrarily close to 0, and we have $\Pi_{\xi,\eta}(A_4^u) > 0$ for $u > 0$, while $\Pi_{\xi,\eta}(A_4^0) = 0$. Thus $\theta_4 := \inf\{u \geq 0 : \Pi_{\xi,\eta}(A_4^u) > 0\} = 0$. There is no Gaussian component, so $\sigma_\xi^2 = 0$, which puts us in the situation of the second item of Theorem 1 of [3], and to verify that $\psi(z) > 0$ for all $z \geq 0$ we only need (since $\theta_2 = \theta_4 = 0$)

$$g(0) = \widetilde{\gamma}_\eta - \int_{x^2+y^2 \leq 1} y \Pi_{\xi,\eta}(\mathrm{d}x, \mathrm{d}y) < 0. \tag{3.4}$$

But by (3.1),

$$\widetilde{\gamma}_\eta = \gamma_\eta - \int_{0 \leq y \leq 1, x^2+y^2 > 1} y \Pi_{\xi,\eta}(\mathrm{d}x, \mathrm{d}y) \leq \gamma_\eta,$$

thus $g(0) < \gamma_\eta \leq 0$, since we chose $\gamma_\eta \leq 0$. Hence Condition A holds in this model.

4. Discrete Time Background and Preliminaries

Our continuous time asymptotic results will be transferred across from discrete time versions, and our first task in the present section is to show how $(V_t)_{t \geq 0}$ can be expressed as a solution of one of two SREs, and give the associated discrete stochastic series for $(Z_t)_{t \geq 0}$. Earlier papers in this area also adopted this approach and we will tap into some of their results in proving Theorem 2.1.

We begin by describing the discrete time setup we use. For $n \in \mathbb{N}$ consider the SRE

$$Y_n = A_n Y_{n-1} + B_n, \tag{4.1}$$

where $(A_n, B_n)_{n \in \mathbb{N}}$ is an iid sequence of $\mathbb{R}^2$-valued random vectors independent of an initial r.v. Y_0. The recursion in (4.1) can be solved in the form

$$Y_n = Y_0 \prod_{j=1}^n A_j + \sum_{i=1}^n \prod_{j=i+1}^n A_j B_i \tag{4.2}$$

(with $\prod_{j=n+1}^n = 1$). From (1.1) we can write, for $n \in \mathbb{N}$

$$V_n = e^{\xi_n - \xi_{n-1}} \Big(e^{\xi_{n-1}} \big(V_0 + \int_0^{n-1} e^{-\xi_{s-}} \mathrm{d}\eta_s \big) \Big) + e^{\xi_n} \int_{(n-1)+}^n e^{-\xi_{s-}} \mathrm{d}\eta_s. \tag{4.3}$$

Thus, if we let $Y_0 = V_0$ and define the $\mathbb{R}^2$-valued random vectors

$$(A_n, B_n) := \Big(e^{\xi_n - \xi_{n-1}}, e^{\xi_n} \int_{(n-1)+}^n e^{-\xi_{s-}} \mathrm{d}\eta_s \Big), \tag{4.4}$$

then V_n satisfies (4.1). An alternative formulation considers for $n \in \mathbb{N}$ the SRE

$$Y_n = C_n Y_{n-1} + C_n D_n\,, \tag{4.5}$$

where $(C_n, D_n)_{n \in \mathbb{N}}$ is an iid sequence independent of Y_0. The solution is

$$Y_n = Y_0 \prod_{j=1}^n C_i + \sum_{i=1}^n \prod_{j=i}^n C_j D_i. \tag{4.6}$$

Using (4.3) it is clear that V_n is a solution of (4.5) if we let $V_0 = Y_0$ and define

$$(C_n, D_n) := \Big(e^{\xi_n - \xi_{n-1}}, e^{\xi_{n-1}} \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s\Big). \tag{4.7}$$

Then it is easily verified that

$$Z_n = \sum_{i=1}^{n} \prod_{j=1}^{i-1} C_j^{-1} D_i \tag{4.8}$$

(with $\prod_{j=1}^{0} = 1$). Note that even when ξ and η are independent, the r.v.s A_n and B_n may be dependent, and similarly for C_n and D_n. But we have

Lemma 4.1. *$(A_n, B_n)_{n\in\mathbb{N}}$ and $(C_n, D_n)_{n\in\mathbb{N}}$ are iid sequences.*

Proof. We begin by proving that the sequence $(C_n, D_n)_{n\in\mathbb{N}}$ is iid. Fix $n \in \mathbb{N}$ and define the new Lévy process $(\bar{\xi}_s, \bar{\eta}_s) := (\xi_{n-1+s} - \xi_{n-1}, \eta_{n-1+s} - \eta_{n-1})$ for $s \geq 0$. Thus $(\bar{\xi}_s, \bar{\eta}_s)_{s\geq 0} =_D (\xi_s, \eta_s)_{s\geq 0}$. Note that we can bring the term $e^{\xi_{n-1}}$ through the integral sign in (4.7) and write $D_n = \int_{(n-1)+}^{n} e^{-(\xi_{s-} - \xi_{n-1})} \mathrm{d}\eta_s$. (ξ, η) has independent increments, so (C_n, D_n) is independent of (C_m, D_m) for every $n \neq m$. Now

$$\begin{aligned}(C_n, D_n) &= \Big(e^{\xi_n - \xi_{n-1}}, \int_{(n-1)+}^{n} e^{-(\xi_{s-} - \xi_{n-1})} \mathrm{d}\eta_s\Big) \\ &= \Big(e^{\bar{\xi}_1}, \int_{0+}^{1} e^{-\bar{\xi}_{s-}} \mathrm{d}\bar{\eta}_s\Big) =_D \Big(e^{\xi_1}, \int_{0+}^{1} e^{-\xi_{s-}} \mathrm{d}\eta_s\Big) = (C_1, D_1).\end{aligned}$$

Thus we have proved that $(C_n, D_n)_{n\in\mathbb{N}}$ is an iid sequence. This implies that $(C_n, C_n D_n)$ is also an iid sequence, and then $(A_n, B_n)_{n\in\mathbb{N}}$ is also an iid sequence since

$$(C_n, C_n D_n) = \Big(e^{\xi_n - \xi_{n-1}}, e^{\xi_n} \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s\Big) = (A_n, B_n). \qquad \square$$

In order to directly access particular results from previous papers, when discretizing V we will use the approach via the recursion (4.1) and the sequence (4.2), whereas when discretizing Z we will use the approach via the series (4.8). There has been significant attention paid to sequences of the form (4.2) and (4.8), and they are linked via the fixed point of the same SRE, see Vervaat [34] and Goldie and Maller [17].

Next we describe two important papers relating to the GOU and its ruin time. In them, ξ and η are general Lévy processes, possibly dependent. The relevant papers are Nyrhinen [28] and Paulsen [31], which are very closely related to Theorem 2.1.

Nyrhinen [28] contains asymptotic ruin probability results for the GOU, in which (ξ, η) is allowed to be an arbitrary bivariate Lévy process. He discretizes the stochastic integral process Z and deduces asymptotic results in the continuous time setting from similar discrete time results. We describe Nyrhinen's results in some detail, and then make some comments.

Let $(M_n, Q_n, L_n)_{n\in\mathbb{N}}$ be iid random vectors with $P(M > 0) = 1$ and $(M, Q, L) \equiv (M_1, Q_1, L_1)$. Define the sequence $(X_n)_{n\in\mathbb{N}}$ by

$$X_n = \sum_{i=1}^{n} \prod_{j=1}^{i-1} M_j Q_i + \prod_{j=1}^{n} M_j L_n, \text{ with } X_0 = 0. \tag{4.9}$$

For $u > 0$ define the passage time $\tau_u^X := \inf\{n \in \mathbb{N} : X_n > u\}$ and the function $c_M(\alpha) := \ln EM^\alpha$. Assume there is a $w^+ > 0$ such that $EM^{w^+} = 1$. Define

$$\alpha_0^+ := \sup\left\{\alpha \in \mathbb{R} : c_M(\alpha) < \infty,\ E|Q|^\alpha < \infty,\ E(ML^+)^\alpha < \infty\right\} \in [0, \infty]. \tag{4.10}$$

Also let

$$\bar{y} := \sup\Big\{y \in \mathbb{R} : P\big(\sup_{n\in\mathbb{N}} X_n > y\big) > 0\Big\} \in (-\infty, \infty]. \tag{4.11}$$

Nyrhinen provides asymptotic results for X_n under the following

Hypothesis H: Suppose that $0 < w^+ < \alpha_0^+ \leq \infty$ and $\bar{y} = \infty$.

Under Hypothesis H, and assuming that $P(M > 1) > 0$, the following quantities are well-defined: $\mu^+ := 1/c'_M(w) \in (0, \infty)$ and $x_0^+ := \lim_{t\to\alpha_0^+ -}(1/c'_M(t)) \in [0, \infty)$. Let $c_M^*(v)$ be the Fenchel-Legendre transform of c_M as in (2.2). Define the function $R : (x_0^+, \infty) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$R(x) := \begin{cases} xc_M^*(1/x) & \text{for } x \in (x_0^+, 1/\mu^+), \\ w & \text{for } x \geq 1/\mu^+. \end{cases}$$

In our situation, R is finite and continuous on (x_0^+, ∞) and strictly decreasing on $(x_0^+, 1/\mu^+)$.

Proposition 4.1. [Nyrhinen's main discrete results, [28], Theorems 2 and 3]
Assume Hypothesis H. Then the following hold.
(i) For every $x > x_0$,

$$\lim_{u\to\infty} (\ln u)^{-1} \ln P(\tau_u^X \leq x \ln u) = -R(x) \tag{4.12}$$

and

$$\lim_{u\to\infty} (\ln u)^{-1} \ln P(\tau_u^X < \infty) = -w. \tag{4.13}$$

(ii) If the distribution of $\ln M$ is spread out, there are constants $C_+ > 0$ and $\kappa > 0$ such that

$$u^{w^+} P(\tau_u^X < \infty) = C_+ + o(u^{-\kappa}), \text{ as } u \to \infty. \tag{4.14}$$

C_+ can be obtained from the formula in Theorem 6.2 and (2.18) of Goldie [16]. Nyrhinen continues in his Theorem 3 to give equivalences for the condition $\bar{y} = \infty$, but they are difficult to verify, as he admits. We discuss these more fully later.

Nyrhinen's continuous result is obtained by applying his discrete results to the case

$$\begin{aligned} (M_n, Q_n) \quad &= \quad \Big(e^{-(\xi_n - \xi_{n-1})}, e^{\xi_{n-1}} \int_{(n-1)+}^n e^{-\xi_{s-}} \mathrm{d}\eta_s\Big) = (C_n^{-1}, D_n) \quad \text{(cf. (4.7))}, \\ \text{and } L_n : \quad &= \quad e^{\xi_n}\Big(\sup_{n-1<t\leq n} \int_{(n-1)+}^t e^{-\xi_{s-}} \mathrm{d}\eta_s - \int_{(n-1)+}^n e^{-\xi_{s-}} \mathrm{d}\eta_s\Big). \end{aligned} \tag{4.15}$$

$(M_n, Q_n, L_n)_{n\in\mathbb{N}}$ is an iid sequence, as follows by an easy extension of our proof of Lemma 4.1. With these allocations Z_n can be written via (4.8) in the form

$$Z_n = \sum_{i=1}^n \prod_{j=1}^{i-1} M_j Q_i = X_n - L_n \prod_{j=1}^n M_j. \tag{4.16}$$

Nyrhinen proves the following result with equality in distribution:

Proposition 4.2. *Let* (M_n, Q_n, L_n) *and* Z_n *be as defined in* (4.15), (**??**) *and* (4.16). *Define* X_n *as in* (4.9). *Then*

$$\sup_{n-1<t\le n} Z_t = X_n \quad \textit{and} \quad \sup_{0\le t\le n} Z_t = \max_{m=1,\dots,n} X_m.$$

Proof. For $n \in \mathbb{N}$ we have

$$\begin{aligned}
\sup_{n-1<t\le n} Z_t &= Z_{n-1} + \sup_{n-1<t\le n} \int_{(n-1)+}^{t} e^{-\xi_{s-}} \mathrm{d}\eta_s \\
&= Z_{n-1} + \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s + e^{-\xi_n} L_n \\
&= X_n - \prod_{j=1}^{n} M_j L_n + e^{-\xi_n} L_n \;=\; X_n.
\end{aligned}$$

This further implies that $\sup_{0\le t\le n} Z_t = \max_{m=1,\dots,n} X_m$. □

Define the first passage time of Z above $u > 0$ by $\tau_u^Z := \inf\{t \ge 0 : Z_t > u\}$. Then Proposition 4.2 implies that for all $t > 0$,

$$P(\tau_u^Z \le t) = P(\tau_u^X \le t) \quad \text{and} \quad P(\tau_u^Z < \infty) = P(\tau_u^X < \infty).$$

So (4.12) and (4.13) hold with τ_u^X replaced by τ_u^Z, when Hypothesis H is satisfied for the associated values of (M_n, Q_n, L_n). If, further, the distribution of $\ln M$ is spread out, then (4.14) holds with τ_u^X replaced by τ_u^Z. This is the content of Theorem 4 and Corollary 5 of [28].

Remark 4.1. We make some comments on Nyrhinen [28].

(i) We begin with the discrete results. Firstly, the sequence X_n defined in (4.9) converges as $n \to \infty$ a.s. to a finite r.v. under Hypothesis H. To see this, note that if we choose $L_n = L$ then X_n is the inner iteration sequence $I_n(L)$ for the random equation $\phi(t) = Mt + Q$. Goldie and Maller [17] prove that $I_n(L)$ converges a.s. to a finite r.v. iff $\prod_{j=1}^n M_j \to 0$ a.s. as $n \to \infty$ and $I_{M,Q} < \infty$, where $I_{M,Q}$ is an integral involving the marginal distributions of M and Q. Since these conditions have no dependence on the distribution of L, it is clear that they are precisely those under which X_n converges a.s. for iid (M_n, Q_n, L_n). We now show that these conditions are in fact satisfied under Hypothesis H, and thus the sequences X_n and $\sum_{i=1}^n \prod_{j=1}^{i-1} M_j Q_i$ converge a.s., and to the same finite r.v..

Under Hypothesis H and our assumption $P(M = 0) = 0$, $E \ln M$ is well-defined and $E \ln M \in [-\infty, 0)$. Hence the random walk $S_n := \sum_{j=1}^n (-\ln M_j) = -\ln \prod_{j=1}^n M_j$ drifts to ∞ a.s., and it follows that $\prod_{j=1}^n M_j \to 0$ a.s. as $n \to \infty$. Since $\alpha_0^+ > 0$ there exists $s > 0$ such that $E|Q|^s < \infty$, thus $E \ln^+ |Q| < \infty$. Hence Corollary 4.1 of [17] implies that the integral condition $I_{M,Q} < \infty$ is satisfied and the sequence $\sum_{i=1}^n \prod_{j=1}^{i-1} M_j Q_i$ converges a.s.

(ii) Nyrhinen transfers his discrete results into continuous time, but the corresponding results are difficult to apply in general. The most problematic assumption is his condition $\bar{y} = \infty$ (see (4.11)). In our notation, this is equivalent to the condition $\psi(z) > 0$ for all $z \ge 0$. Theorem 1 of [3] gives necessary and sufficient conditions on the Lévy measure of (ξ,η) for this, which are amenable to verification in special cases, as we showed in Section 3. Verifying Nyrhinen's condition $0 < w^+ < \alpha_0^+ \le \infty$ requires finiteness of powers of $E|Z_1|$ and $E[\sup_{0<t\le 1}|Z_t|]$. These conditions would be more conveniently stated in terms of the characteristic triplet of (ξ,η) or (at least) the marginal distributions of ξ and η. In the special case that ξ and η are independent Lévy processes, Theorem 3.2 of Paulsen [31] does exactly that. However, problems remain. In [31], the condition $\bar{y} = \infty$ is assumed to be true whenever ξ and η are independent and η is not a subordinator. However, this claim is false†. (It does hold if extra conditions are imposed, in line with Remark 2(3) of [3].) Finally, it would be desirable to remove the finite mean assumption for ξ in [31] and replace the moment conditions in [31], which are sufficient for convergence of Z_t, with the precise necessary and sufficient conditions given in Goldie and Maller [17]. Our Theorem 2.1 addresses all of the above concerns in the most general setting.

5. Proof of Theorem 2.1

The proof requires the following lemma, which was stated but not proved in [2].

Lemma 5.1. *Suppose there exist $r > 0$ and $p,q > 1$ with $1/p + 1/q = 1$ such that $Ee^{-\max\{1,r\}p\xi_1} < \infty$ and $E|\eta_1|^{\max\{1,r\}q} < \infty$. Then*

$$E\Big[\sup_{0\le t\le 1}|Z_t|^{\max\{1,r\}}\Big] = E\Big[\sup_{0\le t\le 1}\Big|\int_0^t e^{-\xi_{s-}}\,\mathrm{d}\eta_s\Big|^{\max\{1,r\}}\Big] < \infty. \tag{5.1}$$

Proof. For ease of notation let $k := \max\{1,r\}$. Assume there exists $r > 0$ and $p,q > 1$ with $1/p + 1/q = 1$ such that $Ee^{-kp\xi_1} < \infty$ and $E|\eta_1|^{kq} < \infty$. We prove the lemma first for the case in which $E\eta_1 = 0$. Since η is a Lévy process this implies that η is a càdlàg martingale. Since ξ is càdlàg $e^{-\xi}$ is a locally bounded process and hence Z is a local martingale for $\mathbb{F}$ by the construction of the stochastic integral (see e.g. Protter [32]). Since additionally $Z_0 = 0$, the Burkholder-Davis-Gundy inequalities ensure that for our choices of p, q and k there exists $b > 0$ such that

$$\begin{aligned} E\Big[\sup_{0\le t\le 1}\Big|\int_0^t e^{-\xi_{s-}}\,\mathrm{d}\eta_s\Big|^k\Big] &\le bE\Big[\Big[\int_0^z e^{-\xi_{s-}}\,\mathrm{d}\eta_s, \int_0^z e^{-\xi_{s-}}\,\mathrm{d}\eta_s\Big]_{z=1}^{z=k/2}\Big] \\ = bE\Big[\Big(\int_0^1 e^{-2\xi_{s-}}\,\mathrm{d}[\eta,\eta]_s\Big)^{k/2}\Big] &\le bE\Big[\Big(\int_0^1 \sup_{0\le t\le 1} e^{-2\xi_t}\,\mathrm{d}[\eta,\eta]_s\Big)^{k/2}\Big], \end{aligned}$$

where in the second inequality recall that $[\eta,\eta]_s$ is increasing. (The notation $[\cdot,\cdot]$ denotes the quadratic variation process.) The last expression equals

$$bE\Big[\sup_{0\le t\le 1} e^{-k\xi_t}[\eta,\eta]_1^{k/2}\Big] \le b\Big(E\Big[\sup_{0\le t\le 1} e^{-pk\xi_t}\Big]\Big)^{1/p}\Big(E\big[[\eta,\eta]_1^{qk/2}\big]\Big)^{1/q},$$

where the inequality follows for our choices of p and q by Hölder's inequality. Since $k \ge 1$, $q > 1$, the Burkholder-Davis-Gundy inequalities give the existence of $c > 0$ such

†To see this, let $(\xi,\eta)_t := (t + N_t, -t)$ where N is a Poisson process with jump times $0 < \tau_0 < \tau_1 < \cdots$. This example trivially satisfies all the conditions in Paulsen's Theorem 3.2. However, using Ito's formula for semi-martingales and some simple manipulation we obtain $Z_t = -1 + (e - 1)\sum_{i=1}^{N_t} e^{-\tau_i - i} + e^{-t-N_t}$, and hence $\inf_{t>0} Z_t \ge -1$ a.s.

that (using Doob's inequality for the second inequality)

$$E\left[[\eta,\eta]_1^{qk/2}\right] \le \frac{1}{c}E\Big[\sup_{0\le t\le 1}|\eta_t|^{qk}\Big] \le \frac{8}{c}E\left[|\eta_1|^{qk}\right] < \infty.$$

Thus it suffices to prove $E\left[\sup_{0\le t\le 1} e^{-pk\xi_t}\right] < \infty$. Now $Y_t := e^{-pk\xi_t}/c^t$, where $c := Ee^{-pk\xi_1} \in (0,\infty)$ is a non-negative martingale, and it follows by Doob's maximal inequality that

$$E\Big[\sup_{0\le t\le 1} e^{-pk\xi_t}\Big] \le \max\{1,c\}E\Big[\sup_{0\le t\le 1}\frac{e^{-pk\xi_t}}{c^t}\Big] \le \max\Big\{\frac{1}{c},1\Big\}\Big(\frac{pk}{pk-1}\Big)^{pk}Ee^{-pk\xi_1} < \infty.$$

Hence the lemma is proved for the case in which $E(\eta_1)=0$. In general, write

$$\begin{aligned}
&E\Big[\sup_{0\le t\le 1}\Big|\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s\Big|^k\Big] = E\Big[\sup_{0\le t\le 1}\Big|\int_0^t e^{-\xi_{s-}}\mathrm{d}(\eta_s - sE\eta_1 + sE\eta_1)\Big|^k\Big]\\
&\le E\Big[\Big(\sup_{0\le t\le 1}\Big|\int_0^t e^{-\xi_{s-}}\mathrm{d}(\eta_s - sE\eta_1)\Big| + |E\eta_1|\sup_{0\le t\le 1}\Big|\int_0^t e^{-\xi_{s-}}\mathrm{d}s\Big|\Big)^k\Big],
\end{aligned}$$

in which the first term on the right-hand side is finite by the first part of the proof. An application of Minkowski's inequality to the second term on the right-hand side completes the proof. □

Remark 5.1. If ξ and η are independent, then Hölder's inequality is not required in the proof of Lemma 5.1, and a simpler independence argument shows that (5.1) holds if $Ee^{-\max\{1,r\}\xi_1} < \infty$ and $E|\eta_1|^{\max\{1,r\}} < \infty$ for some $r > 0$. We can put further restrictions on ξ and η, such as in the example in Section 3 of Nyrhinen [28], which assumes ξ is continuous and η is compound Poisson plus drift, which render the use of the Burkholder-Davis-Gundy inequalities unnecessary and further simplify the conditions. For general Lévy (ξ,η) the above inequality is the sharpest we have found.

Proof of Theorem 2.1: We aim to use Proposition 4.1 for passage below rather than above. We can do this by replacing η by $-\eta$. Note that for $z > 0$,

$$T_z = \inf\{t>0 : Z_t < -z\} = \inf\{t>0 : -Z_t > z\} = \inf\{t>0 : \widehat{Z}_t > z\},$$

where we denote Z_t, when η is replaced by $-\eta$, by $\widehat{Z}_t$ and similarly for the other quantities. Thus $\widehat{Z}_t = -Z_t$, and it is easily checked that, with (M_n,Q_n) as in (4.15), $(\widehat{M}_n,\widehat{Q}_n) = (M_n,-Q_n)$, and, with L_n as in (4.15), $\widehat{L}_n = -\overline{L}_n$, where

$$\overline{L}_n := -e^{\xi_n}\Big(\int_{(n-1)+}^n e^{-\xi_{s-}}\mathrm{d}\eta_s - \inf_{n-1<t\le n}\int_{(n-1)+}^t e^{-\xi_{s-}}\mathrm{d}\eta_s\Big). \qquad (5.2)$$

From (4.9) we get $\widehat{X}_n(\widehat{L}_n) = -X_n(\overline{L}_n)$. Then Proposition 4.1 ensures that (2.6) and (2.7) hold, if we can prove that the relevant conditions are satisfied for $(\widehat{M},\widehat{Q},\widehat{L})$; i.e., we must show that Hypothesis H holds for the hat variables.

The corresponding $\widehat{\overline{y}}$ (see (4.11)) is

$$\sup\Big\{y\in\mathbb{R} : P\Big(\sup_{n\in\mathbb{N}}\widehat{X}_n(\widehat{L}_n) > y\Big) > 0\Big\} \quad = \quad \inf\Big\{z\in\mathbb{R} : P\Big(\inf_{t>0} Z_t < -z\Big) > 0\Big\},$$

so $\widehat{\overline{y}} = \infty$ if and only if $\psi(z) > 0$ for all $z \geq 0$, which we have assumed.

We need a $w^+ > 0$ such that $E\widehat{M}^{w^+} = 1$, and this is the case with $w^+ = w$ under (2.1) since $\widehat{M} = M = e^{-\xi_1}$. Also, $\widehat{c}_M(\alpha) = \ln E\widehat{M}^\alpha = c(\alpha)$, so that α_0^+ in (4.10) here equals α_0 as defined in (2.3). Note that the extra term $E(\widehat{M}\widehat{L}^+)^\alpha = EM\overline{L}^-)^\alpha$ required in (2.3) is superfluous here, since $E(M\overline{L}^-)^\alpha = E\overline{Z}_1^\alpha$, and this is finite for $\alpha \geq 0$ if and only if $E|Z_1|^\alpha < \infty$.

Under the moment conditions of Theorem 2.1, the conditions of Lemma 5.1 hold with $r = w^+ + \varepsilon$, so $E|Z_1|^\alpha < \infty$ for $\alpha = \max\{1, w+\varepsilon\}$, and hence $\alpha_0^+ \geq w^+ + \varepsilon > w^+$. Thus indeed Hypothesis H is fulfilled in the present situation and Proposition 4.1 applies to give (2.6) and (2.7). Also $\alpha_0^+ \geq w^+ + \varepsilon > w^+$ implies $c'(\alpha_0-) > c'(w) = \mu^* = -E\xi_1 e^{-w\xi_1}$, and this is finite since $Ee^{-(w+\varepsilon)\xi_1}$ is. So $0 \leq \alpha_0 < 1/\mu^* < \infty$.

Suppose, further, that ξ_1 is spread out. Then the dual version of (2.8) follows from Nyrhinen's comments in [28], which we expressed as Proposition 4.1. □

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