Optimized Capacity Bounds for the Half-Duplex Gaussian MIMO Relay Channel

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***Abstract*—In this paper, the optimization of the cut-set bound (CSB) and the achievable decode-and-forward (DF) rate for the half-duplex Gaussian multiple-input multiple-output (MIMO) relay channel is considered. In particular, it is shown that evaluating the cut-set bound and the maximal achievable DF rate is equivalent to solving convex optimization problems if perfect channel state information (CSI) is available at all nodes. Our work therefore extends results for the full-duplex relay channel, which we also revisit here, and it demonstrates that it is possible to efficiently determine (generally loose) bounds on the capacity of the half-duplex MIMO relay channel.**

***Index Terms*—Relay channel, MIMO, half-duplex, cut-set bound, decode-and-forward, convex optimization.**

I. Introduction

Relaying has attracted a lot of interest in recent years due to the ability to provide increased throughput and extended coverage to a growing number of mobile users. Scenarios in which relaying may be employed include for example multihop wireless networks and sensor networks where nodes have limited transmit power. In this work, we consider the simplest relay channel where one source transmits information to one destination with the help of a single relay. It is assumed that this relay has no own information to transmit so that its only purpose is to assist the communication between the source and the terminal. A model for this particular relay channel was first studied by van der Meulen as early as 1971 [1], but its general capacity is still unknown.

However, substantial advances towards the information theoretic understanding of the relay channel were made by Cover and El Gamal. In [2], they derived upper and lower bounds on the capacity of the full-duplex relay channel based on a then new cut-set bound (CSB) as well as two fundamental coding schemes that are now referred to as decode-and-forward (DF) and compress-and-forward (CF), respectively. The DF strategy requires the relay to decode the whole source message, which is then re-encoded and sent to the destination. When the relay uses CF, it reliably forwards an estimate, a compressed version of its receive signal, to the destination. In [3], these two basic strategies were generalized to various relay channel models that include multiple sources, relays, or destinations. Another relaying strategy of lower complexity than both DF and CF is called amplify-and-forward (AF), which is considered in [4] for example. When the relay uses AF, it is restricted to perform linear operations on its receive signal.

All information theoretic results that have been derived for the relay channel hold whether the nodes are equipped with one or multiple antennas. However, multi-antenna nodes may employ precoding, which devices having a single antenna are not capable of. The additional degrees of freedom in the multiple-input multiple-output (MIMO) relay channel are of course reflected in the corresponding optimization problems. Assuming Gaussian channel inputs, maximizing the achievable DF rate for the MIMO relay channel requires to solve an optimization problem with respect to the source covariance matrix $\boldsymbol{R}_{\mathrm{S}}$, the relay covariance matrix $\boldsymbol{R}_{\mathrm{R}}$, and their cross covariance matrix $\boldsymbol{R}_{\mathrm{SR}}$, for example. If all nodes have a single antenna, on the other hand, the same objective is maximized with respect to a scalar correlation coefficient $\rho \in [0, 1]$.

Surprisingly, there are only few contributions that have focussed on optimizing bounds on the capacity of the MIMO relay channel. In [5], it is proved that Gaussian input distributions maximize the CSB and the achievable DF rate for the full-duplex relay channel. Furthermore, the authors provide an upper bound on the CSB that is loose in general. Suboptimal lower bounds are also given based on point-to-point transmission (source to destination) and the cascaded relay channel (source to relay, relay to destination). Two partial decode-and-forward (PDF) strategies, where the relay only decodes part of the source message, using superposition or dirty-paper coding [6] at the source are presented in [7]. While the achievable rates are shown to improve on the lower bounds of [5], the authors only formulate but do not solve the corresponding rate maximization problems for the general case.

We were able to show that, for the full-duplex case, the cut-set bound as well as the maximal achievable DF rate are obtained as the solutions of convex optimization problems [8]. In this paper, we extend these results to the half-duplex relay channel, where a practical half-duplex constraint is imposed on all nodes. Similar work (for both the full-duplex and the half-duplex case) is presented in [9]. However, it can be verified that the expressions resulting from those derivations are only upper bounds to the optimal solutions.

The remainder of this paper is organized as follows. The system models for the full-duplex and the half-duplex MIMO relay channel are introduced in Section II. Section III revisits the optimization of the CSB for the full-duplex case and shows how these results can be extended to the half-duplex

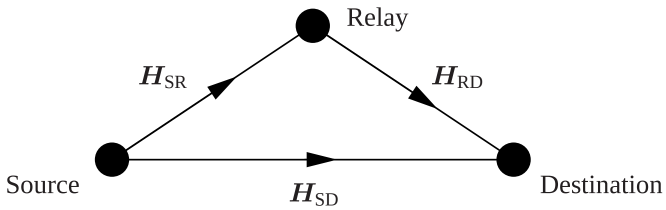


Figure 1. Full-duplex MIMO relay channel.

case. The maximization of the achievable DF rate for both the full-duplex and the half-duplex relay channel is addressed in Section IV. Numerical results and a discussion of these results are presented in Section V before we conclude in Section VI.

II. System Model

In the full-duplex relay channel, which is illustrated in Figure 1, the signals received at the relay and the destination can be expressed as

$$\begin{aligned} \boldsymbol{y}_{\mathrm{R}} &= \boldsymbol{H}_{\mathrm{SR}}\boldsymbol{x}_{\mathrm{S}} + \boldsymbol{n}_{\mathrm{R}}, \\ \boldsymbol{y}_{\mathrm{D}} &= \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{x}_{\mathrm{S}} + \boldsymbol{H}_{\mathrm{RD}}\boldsymbol{x}_{\mathrm{R}} + \boldsymbol{n}_{\mathrm{D}}, \end{aligned} \tag{1}$$

where $\boldsymbol{H}_{\mathrm{SR}} \in \mathbb{C}^{N_{\mathrm{R}}\times N_{\mathrm{S}}}, \boldsymbol{H}_{\mathrm{SD}} \in \mathbb{C}^{N_{\mathrm{D}}\times N_{\mathrm{S}}}, \boldsymbol{H}_{\mathrm{RD}} \in \mathbb{C}^{N_{\mathrm{D}}\times N_{\mathrm{R}}}$ represent the channel gain matrices assumed to be perfectly known at all nodes, and $\boldsymbol{n}_{\mathrm{R}} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I}_{N_{\mathrm{R}}}), \boldsymbol{n}_{\mathrm{D}} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I}_{N_{\mathrm{D}}})$ denote complex white Gaussian noise of unit variance. Let $\boldsymbol{R}_{\mathrm{S}}$ and $\boldsymbol{R}_{\mathrm{R}}$ denote the covariance matrices of the zero-mean source and relay inputs, then the source and the relay are subject to transmit power constraints given by $\mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}) \leq P_{\mathrm{S}}$ and $\mathrm{tr}(\boldsymbol{R}_{\mathrm{R}}) \leq P_{\mathrm{R}}$, respectively. Furthermore, the joint transmit covariance matrix of the source and relay inputs is determined by

$$\boldsymbol{R} = \mathrm{E}\left[\begin{bmatrix}\boldsymbol{x}_{\mathrm{S}}\\ \boldsymbol{x}_{\mathrm{R}}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{\mathrm{S}}\\ \boldsymbol{x}_{\mathrm{R}}\end{bmatrix}^{\mathrm{H}}\right] = \begin{bmatrix}\boldsymbol{R}_{\mathrm{S}} & \boldsymbol{R}_{\mathrm{SR}}\\ \boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}} & \boldsymbol{R}_{\mathrm{R}}\end{bmatrix}. \tag{2}$$

Note that, by defining the two selection matrices

$$\boldsymbol{D}_{\mathrm{S}} = \begin{bmatrix}\boldsymbol{I}_{N_{\mathrm{S}}} & \mathbf{0}_{N_{\mathrm{S}}\times N_{\mathrm{R}}}\end{bmatrix}, \quad \boldsymbol{D}_{\mathrm{R}} = \begin{bmatrix}\mathbf{0}_{N_{\mathrm{R}}\times N_{\mathrm{S}}} & \boldsymbol{I}_{N_{\mathrm{R}}}\end{bmatrix}, \tag{3}$$

both the source and the relay transmit covariance matrices can be expressed as linear functions of $\boldsymbol{R}$:

$$\boldsymbol{R}_{\mathrm{S}} = \boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}, \quad \boldsymbol{R}_{\mathrm{R}} = \boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}. \tag{4}$$

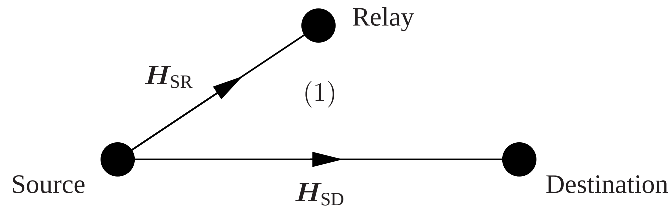
Let us now consider the more practical case where a half-duplex constraint is imposed on all nodes. In the half-duplex relay channel, which is depicted in Figure 2, a relay receive phase is followed by a relay transmit phase. These two phases are specified by

$$\begin{aligned} \boldsymbol{y}_{\mathrm{R}}^{(1)} &= \boldsymbol{H}_{\mathrm{SR}}\boldsymbol{x}_{\mathrm{S}}^{(1)} + \boldsymbol{n}_{\mathrm{R}}^{(1)}, \\ \boldsymbol{y}_{\mathrm{D}}^{(1)} &= \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{x}_{\mathrm{S}}^{(1)} + \boldsymbol{n}_{\mathrm{D}}^{(1)}, \end{aligned} \tag{5}$$

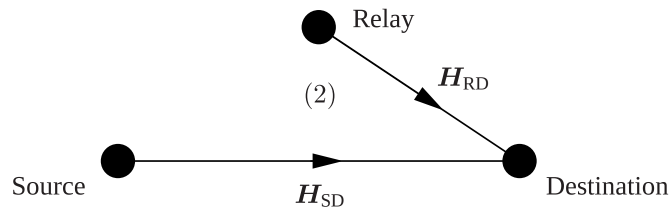
and

$$\boldsymbol{y}_{\mathrm{D}}^{(2)} = \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{x}_{\mathrm{S}}^{(2)} + \boldsymbol{H}_{\mathrm{RD}}\boldsymbol{x}_{\mathrm{R}}^{(2)} + \boldsymbol{n}_{\mathrm{D}}^{(2)}, \tag{6}$$

where the noise $\boldsymbol{n}_{\mathrm{R}}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I}_{N_{\mathrm{R}}}), \boldsymbol{n}_{\mathrm{D}}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I}_{N_{\mathrm{D}}})$, and $\boldsymbol{n}_{\mathrm{D}}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I}_{N_{\mathrm{D}}})$ is independent of $\boldsymbol{n}_{\mathrm{D}}^{(1)}$. Note that the



(a) Relay receive phase.



(b) Relay transmit phase.

Figure 2. Half-duplex MIMO relay channel.

source transmits and the destination listens during both phases, which means that the half-duplex constraint basically affects only the relay. Furthermore, without loss of generality $\boldsymbol{H}_{\mathrm{SD}}$ is assumed to be the same for both phases, and all channels are assumed to be perfectly known at all nodes again. Since only the source transmits in the first phase, we have

$$\boldsymbol{R}^{(1)} = \mathrm{E}\left[\boldsymbol{x}_{\mathrm{S}}^{(1)}\boldsymbol{x}_{\mathrm{S}}^{(1),\mathrm{H}}\right] = \boldsymbol{R}_{\mathrm{S}}^{(1)}. \tag{7}$$

The joint transmit covariance matrix of the source and relay inputs for the relay transmit phase is given by

$$\boldsymbol{R}^{(2)} = \mathrm{E}\left[\begin{bmatrix}\boldsymbol{x}_{\mathrm{S}}^{(2)}\\ \boldsymbol{x}_{\mathrm{R}}^{(2)}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{\mathrm{S}}^{(2)}\\ \boldsymbol{x}_{\mathrm{R}}^{(2)}\end{bmatrix}^{\mathrm{H}}\right] = \begin{bmatrix}\boldsymbol{R}_{\mathrm{S}}^{(2)} & \boldsymbol{R}_{\mathrm{SR}}^{(2)}\\ \boldsymbol{R}_{\mathrm{SR}}^{(2),\mathrm{H}} & \boldsymbol{R}_{\mathrm{R}}^{(2)}\end{bmatrix}. \tag{8}$$

Moreover, we have

$$\boldsymbol{R}_{\mathrm{S}}^{(2)} = \boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}^{(2)}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}, \quad \boldsymbol{R}_{\mathrm{R}}^{(2)} = \boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}^{(2)}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}. \tag{9}$$

Like in the full-duplex case, the source and the relay are subject to power constraints of the form $\mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}^{(1)}) \leq P_{\mathrm{S}}^{(1)}$, $\mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}^{(2)}) \leq P_{\mathrm{S}}^{(2)}$, and $\mathrm{tr}(\boldsymbol{R}_{\mathrm{R}}^{(2)}) \leq P_{\mathrm{R}}^{(2)}$, respectively.

III. Cut-Set Bound

First, let us revisit the optimization of the cut-set bound for the full-duplex case. Cover and El Gamal [2] proved that the capacity of the full-duplex relay channel is upper bounded by

$$C_{\mathrm{CSB}}^{\mathrm{FD}} = \max\min\left\{I(X_{\mathrm{S}};Y_{\mathrm{R}}Y_{\mathrm{D}}|X_{\mathrm{R}}), I(X_{\mathrm{S}}X_{\mathrm{R}};Y_{\mathrm{D}})\right\}, \tag{10}$$

where the maximization is with respect to the joint distribution of the source and relay signals. Furthermore, it is known that the source and relay inputs that optimize the CSB are Gaussian [5]. With $\boldsymbol{x}_{\mathrm{S}} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{R}_{\mathrm{S}})$ and $\boldsymbol{x}_{\mathrm{R}} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{R}_{\mathrm{R}})$, it follows that

$$I(X_{\mathrm{S}};Y_{\mathrm{R}}Y_{\mathrm{D}}|X_{\mathrm{R}}) = \log\det\left(\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{R}_{\mathrm{S}|\mathrm{R}}\boldsymbol{H}_1^{\mathrm{H}}\right), \tag{11}$$

$$I(X_{\mathrm{S}}X_{\mathrm{R}};Y_{\mathrm{D}}) = \log\det\left(\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{R}\boldsymbol{H}_2^{\mathrm{H}}\right), \tag{12}$$

where $\boldsymbol{H}_1 = \left[\boldsymbol{H}_{\mathrm{SR}}^{\mathrm{H}} \quad \boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right]^{\mathrm{H}}, \boldsymbol{H}_2 = \left[\boldsymbol{H}_{\mathrm{SD}} \quad \boldsymbol{H}_{\mathrm{RD}}\right]$, and $\boldsymbol{R}_{\mathrm{S|R}} = \boldsymbol{R}_{\mathrm{S}} - \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}}$ is the conditional covariance matrix of X_{S} given X_{R}. Here $\boldsymbol{R}_{\mathrm{R}}^{\dagger}$ denotes the Moore-Penrose pseudoinverse of $\boldsymbol{R}_{\mathrm{R}}$.

Remark 1: Note that $\boldsymbol{R}_{\mathrm{S|R}}$ is equal to the covariance matrix of the random vector $X_{\mathrm{S}} - \hat{X}_{\mathrm{S}}(X_{\mathrm{R}})$ in this case, where $\hat{X}_{\mathrm{S}}(X_{\mathrm{R}}) = \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}X_{\mathrm{R}}$ is the linear minimum mean square error (MMSE) estimator of X_{S} given X_{R}. However, this is only because we have Gaussian inputs and the linear MMSE estimator is equal to the MMSE estimator for Gaussian random vectors. For general inputs, it does not hold that the conditional covariance matrix of X_{S} given X_{R} is equal to $\boldsymbol{R}_{\mathrm{S}} - \boldsymbol{R}_{\mathrm{SR}}\boldsymbol{R}_{\mathrm{R}}^{\dagger}\boldsymbol{R}_{\mathrm{SR}}^{\mathrm{H}}$.

We see that computing the cut-set bound for the full-duplex MIMO relay channel requires to solve the nonconvex optimization problem

$$\begin{aligned} &\max_{\boldsymbol{R}\succeq\boldsymbol{0}} C_{\mathrm{CSB}}^{\mathrm{FD}} \\ \text{s.t.}\quad & C_{\mathrm{CSB}}^{\mathrm{FD}} \le \log\det\left(\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{R}_{\mathrm{S|R}}\boldsymbol{H}_1^{\mathrm{H}}\right), \\ & C_{\mathrm{CSB}}^{\mathrm{FD}} \le \log\det\left(\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{R}\boldsymbol{H}_2^{\mathrm{H}}\right), \\ & \mathrm{tr}(\boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}) \le P_{\mathrm{S}}, \quad \mathrm{tr}(\boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}) \le P_{\mathrm{R}}, \end{aligned} \tag{13}$$

where $\boldsymbol{D}_{\mathrm{S}}$ and $\boldsymbol{D}_{\mathrm{R}}$ denote the selection matrices defined in (3). Observe that the nonconvexity is caused only by the conditional covariance matrix $\boldsymbol{R}_{\mathrm{S|R}}$ in the first inequality constraint. However, by means of introducing a slack variable $\boldsymbol{Q}$ and applying the Schur complement condition for positive semi-definite matrices [10, A.5.5], it is shown in [8] that the following is an equivalent optimization problem:

$$\begin{aligned} C_{\mathrm{CSB}}^{\mathrm{FD}} = &\max_{\boldsymbol{Q},\boldsymbol{R}} \min\left\{\log\det\left(\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{Q}\boldsymbol{H}_1^{\mathrm{H}}\right),\right. \\ &\qquad\left.\log\det\left(\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{R}\boldsymbol{H}_2^{\mathrm{H}}\right)\right\} \\ \text{s.t.}\quad & \mathrm{tr}(\boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}) \le P_{\mathrm{S}}, \quad \mathrm{tr}(\boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}) \le P_{\mathrm{R}}, \\ & \boldsymbol{Q}\succeq\boldsymbol{0}, \quad \boldsymbol{R} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{D}_{\mathrm{S}} \succeq \boldsymbol{0}. \end{aligned} \tag{14}$$

Note that the last two constraints imply $\boldsymbol{R}\succeq\boldsymbol{0}$. As the objective function is the pointwise minimum of two concave functions (in $\boldsymbol{Q},\boldsymbol{R}\succeq\boldsymbol{0}$), it is also concave [10]. Furthermore, all constraints are affine so that this optimization problem, which determines the cut-set bound, is convex.

Now, consider the optimization of the cut-set bound for the half-duplex case. In [11], it was shown that the CSB for the half-duplex relay channel reads as

$$\begin{aligned} C_{\mathrm{CSB}}^{\mathrm{HD}} = \max\min\left\{\tau_1 I(X_{\mathrm{S}}^{(1)}; Y_{\mathrm{R}}^{(1)}Y_{\mathrm{D}}^{(1)}) + \tau_2 I(X_{\mathrm{S}}^{(2)}; Y_{\mathrm{D}}^{(2)}|X_{\mathrm{R}}^{(2)}),\right. \\ \left.\tau_1 I(X_{\mathrm{S}}^{(1)}; Y_{\mathrm{D}}^{(1)}) + \tau_2 I(X_{\mathrm{S}}^{(2)}X_{\mathrm{R}}^{(2)}; Y_{\mathrm{D}}^{(2)})\right\}, \end{aligned} \tag{15}$$

where τ_1 and τ_2 denote the durations of the relay receive and the relay transmit phase, respectively. The same arguments as in the full-duplex case may be put forward to show that $C_{\mathrm{CSB}}^{\mathrm{HD}}$ is also maximized by Gaussian source and relay inputs. Assuming $\boldsymbol{x}_{\mathrm{S}}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{S}}^{(1)}), \boldsymbol{x}_{\mathrm{S}}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{S}}^{(2)})$, and $\boldsymbol{x}_{\mathrm{R}}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{R}}^{(2)})$, the optimized cut-set bound is hence given by

$$\begin{aligned} &\max_{\substack{\boldsymbol{R}^{(1)},\boldsymbol{R}^{(2)}\succeq\boldsymbol{0},\\ \tau_1,\tau_2\ge 0,\tau_1+\tau_2\le 1}} C_{\mathrm{CSB}}^{\mathrm{HD}} \quad \text{s.t.} \\ & C_{\mathrm{CSB}}^{\mathrm{HD}} \le \tau_1\log\left|\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{R}^{(1)}\boldsymbol{H}_1^{\mathrm{H}}\right| + \tau_2\log\left|\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{R}_{\mathrm{S|R}}^{(2)}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right|, \\ & C_{\mathrm{CSB}}^{\mathrm{HD}} \le \tau_1\log\left|\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{R}^{(1)}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right| + \tau_2\log\left|\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{R}^{(2)}\boldsymbol{H}_2^{\mathrm{H}}\right|, \\ & \mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}^{(1)}) \le P_{\mathrm{S}}^{(1)}, \mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}^{(2)}) \le P_{\mathrm{S}}^{(2)}, \mathrm{tr}(\boldsymbol{R}_{\mathrm{R}}^{(2)}) \le P_{\mathrm{R}}^{(2)}. \end{aligned} \tag{16}$$

This is again a nonconvex optimization problem, where the nonconvexity is caused by the conditional covariance matrix $\boldsymbol{R}_{\mathrm{S|R}}^{(2)} = \boldsymbol{R}_{\mathrm{S}}^{(2)} - \boldsymbol{R}_{\mathrm{SR}}^{(2)}\boldsymbol{R}_{\mathrm{R}}^{(2),\dagger}\boldsymbol{R}_{\mathrm{SR}}^{(2),\mathrm{H}}$ in the first constraint on $C_{\mathrm{CSB}}^{\mathrm{HD}}$. However, after reformulating this problem using the slack variable $\boldsymbol{Q}$ and the properties of the generalized Schur complement, we obtain the following equivalent optimization problem:

$$\begin{aligned} &\max_{\boldsymbol{R}^{(1)},\boldsymbol{R}^{(2)},\boldsymbol{Q},\tau_1,\tau_2} C_{\mathrm{CSB}}^{\mathrm{HD}} \quad \text{s.t.} \\ & C_{\mathrm{CSB}}^{\mathrm{HD}} \le \tau_1\log\left|\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{R}^{(1)}\boldsymbol{H}_1^{\mathrm{H}}\right| + \tau_2\log\left|\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{Q}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right|, \\ & C_{\mathrm{CSB}}^{\mathrm{HD}} \le \tau_1\log\left|\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{R}^{(1)}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right| + \tau_2\log\left|\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{R}^{(2)}\boldsymbol{H}_2^{\mathrm{H}}\right|, \\ & \boldsymbol{Q}\succeq\boldsymbol{0}, \boldsymbol{R}^{(1)}\succeq\boldsymbol{0}, \boldsymbol{R}^{(2)} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{D}_{\mathrm{S}} \succeq \boldsymbol{0}, \\ & \mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}^{(1)}) \le P_{\mathrm{S}}^{(1)}, \mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}^{(2)}) \le P_{\mathrm{S}}^{(2)}, \mathrm{tr}(\boldsymbol{R}_{\mathrm{R}}^{(2)}) \le P_{\mathrm{R}}^{(2)}, \\ & \tau_1 \ge 0, \tau_2 \ge 0, \tau_1+\tau_2 \le 1. \end{aligned} \tag{17}$$

Since the nonnegative weighted sum of concave functions is concave [10], we can follow the same arguments as for the full-duplex case and conclude that this problem is convex (in $\boldsymbol{Q}, \boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)} \succeq \boldsymbol{0}$, where the latter is implied by $\boldsymbol{Q}\succeq\boldsymbol{0}$ and $\boldsymbol{R}^{(2)} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{D}_{\mathrm{S}} \succeq \boldsymbol{0}$) for fixed τ_1, τ_2. However, joint convexity in $\boldsymbol{Q}, \boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)}, \tau_1$, and τ_2 cannot be claimed in this formulation. Nevertheless, letting $\tau_1 = \tau$ and $\tau_2 = 1-\tau$, the optimal solution can be found by sampling over one scalar parameter $\tau\in[0,1]$ and convex programming.

Beyond that, it is also possible to formulate an equivalent optimization problem that is jointly convex in all parameters. To this end, we introduce the new variables

$$\boldsymbol{C}^{(1)} = \tau_1\boldsymbol{R}^{(1)}, \boldsymbol{C}^{(2)} = \tau_2\boldsymbol{R}^{(2)}, \boldsymbol{Q}' = \tau_2\boldsymbol{Q}. \tag{18}$$

The resulting optimization problem hence reads as

$$\begin{aligned} &\max_{\boldsymbol{C}^{(1)},\boldsymbol{C}^{(2)},\boldsymbol{Q}',\tau_1,\tau_2} C_{\mathrm{CSB}}^{\mathrm{HD}} \quad \text{s.t.} \\ & C_{\mathrm{CSB}}^{\mathrm{HD}} \le \tau_1\log\left|\boldsymbol{I} + \boldsymbol{H}_1\frac{\boldsymbol{C}^{(1)}}{\tau_1}\boldsymbol{H}_1^{\mathrm{H}}\right| + \tau_2\log\left|\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SD}}\frac{\boldsymbol{Q}'}{\tau_2}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right|, \\ & C_{\mathrm{CSB}}^{\mathrm{HD}} \le \tau_1\log\left|\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SD}}\frac{\boldsymbol{C}^{(1)}}{\tau_1}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right| + \tau_2\log\left|\boldsymbol{I} + \boldsymbol{H}_2\frac{\boldsymbol{C}^{(2)}}{\tau_2}\boldsymbol{H}_2^{\mathrm{H}}\right|, \\ & \boldsymbol{Q}'\succeq\boldsymbol{0}, \boldsymbol{C}^{(1)}\succeq\boldsymbol{0}, \boldsymbol{C}^{(2)} - \boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}\boldsymbol{Q}'\boldsymbol{D}_{\mathrm{S}} \succeq \boldsymbol{0}, \\ & \mathrm{tr}(\boldsymbol{C}_{\mathrm{S}}^{(1)}) \le \tau_1 P_{\mathrm{S}}^{(1)}, \mathrm{tr}(\boldsymbol{C}_{\mathrm{S}}^{(2)}) \le \tau_2 P_{\mathrm{S}}^{(2)}, \mathrm{tr}(\boldsymbol{C}_{\mathrm{R}}^{(2)}) \le \tau_2 P_{\mathrm{R}}^{(2)}, \\ & \tau_1 \ge 0, \tau_2 \ge 0, \tau_1+\tau_2 \le 1. \end{aligned} \tag{19}$$

Note that $g : (\boldsymbol{C}^{(1)}, \tau_1) \mapsto \tau_1\log|\boldsymbol{I} + \boldsymbol{H}_1\frac{\boldsymbol{C}^{(1)}}{\tau_1}\boldsymbol{H}_1^{\mathrm{H}}|$ defined on $\mathrm{dom}\, g = \left\{(\boldsymbol{C}^{(1)}, \tau_1) : \boldsymbol{C}^{(1)} \succeq \boldsymbol{0}, \tau_1 > 0\right\}$ is the perspective of the function $f : \boldsymbol{C}^{(1)} \mapsto \log|\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{C}^{(1)}\boldsymbol{H}_1^{\mathrm{H}}|$ defined on the cone of positive semi-definite matrices. Furthermore, f is concave in $\boldsymbol{C}^{(1)} \succeq \mathbf{0}$. The convexity preserving property of the perspective operation [10, Sec. 3.2.6], which was used in [9] to convexify a very similar problem, thus implies that g is jointly convex in $\boldsymbol{C}^{(1)}$ and τ_1 on its entire domain. Analogous results hold for the other log-det functions. In addition, observe that all other constraints are affine. Therefore, we have obtained an optimization problem that is jointly convex in all parameters $\boldsymbol{Q}', \boldsymbol{C}^{(1)}, \boldsymbol{C}^{(2)}, \tau_1, \tau_2$.

Remark 2: The perspective function is defined only for positive real numbers. However, if we assume the convention that $0 \log \frac{x}{0} = 0$ for all $x \in \mathbb{R}$, which is commonly used in information theory based on continuity arguments [12, p. 19], then it is not necessary to exclude the cases $\tau_1 = 0$ and $\tau_2 = 0$ as the problematic terms vanish.

IV. Achievable Decode-and-Forward (DF) Rate

A lower bound on the capacity of the relay channel is given by the rate that can be achieved with the decode-and-forward protocol derived by Cover and El Gamal [2]. Before we address the half-duplex case, let us first revisit the full-duplex relay channel again. If the relay uses DF, all achievable rates in the full-duplex case satisfy

$$R_{\text{DF}}^{\text{FD}} \leq \max \min\left\{ I(X_{\text{S}}; Y_{\text{R}} | X_{\text{R}}), I(X_{\text{S}} X_{\text{R}}; Y_{\text{D}}) \right\}, \tag{20}$$

where the maximization is again with respect to the joint distribution of the source and relay signals. Observe that $R_{\text{DF}}^{\text{FD}}$ differs from $C_{\text{CSB}}^{\text{FD}}$ only in the first mutual information term, whereas the second one is the same.

Beyond that, it has also been proved that Gaussian input distributions optimize the achievable DF rate [5]. Therefore, letting $\boldsymbol{x}_{\text{S}} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{R}_{\text{S}})$ and $\boldsymbol{x}_{\text{R}} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{R}_{\text{R}})$, it follows that

$$I(X_{\text{S}}; Y_{\text{R}} | X_{\text{R}}) = \log\det\left(\boldsymbol{I} + \boldsymbol{H}_{\text{SR}} \boldsymbol{R}_{\text{S}|\text{R}} \boldsymbol{H}_{\text{SR}}^{\text{H}} \right), \tag{21}$$

where the only difference to (11) is that $\boldsymbol{H}_1 = \begin{bmatrix} \boldsymbol{H}_{\text{SR}}^{\text{H}} & \boldsymbol{H}_{\text{SD}}^{\text{H}} \end{bmatrix}^{\text{H}}$ is replaced by $\boldsymbol{H}_{\text{SR}}$. Using the same arguments as for the calculation of the cut-set bound, it is thus shown in [8] that

$$\begin{aligned} R_{\text{DF}}^{\text{FD}} \leq \max_{\boldsymbol{Q}, \boldsymbol{R}} \min\big\{ &\log\det\left(\boldsymbol{I} + \boldsymbol{H}_{\text{SR}} \boldsymbol{Q} \boldsymbol{H}_{\text{SR}}^{\text{H}} \right), \\ &\log\det\left(\boldsymbol{I} + \boldsymbol{H}_2 \boldsymbol{R} \boldsymbol{H}_2^{\text{H}} \right) \big\} \\ \text{s.t.} \quad &\operatorname{tr}(\boldsymbol{D}_{\text{S}} \boldsymbol{R} \boldsymbol{D}_{\text{S}}^{\text{H}}) \leq P_{\text{S}}, \quad \operatorname{tr}(\boldsymbol{D}_{\text{R}} \boldsymbol{R} \boldsymbol{D}_{\text{R}}^{\text{H}}) \leq P_{\text{R}}, \\ &\boldsymbol{Q} \succeq \mathbf{0}, \quad \boldsymbol{R} - \boldsymbol{D}_{\text{S}}^{\text{H}} \boldsymbol{Q} \boldsymbol{D}_{\text{S}} \succeq \mathbf{0}. \end{aligned} \tag{22}$$

Consequently, the maximal achievable DF rate for the full-duplex MIMO relay channel is also obtained as the solution of a convex optimization problem.

A rather straightforward application of the decode-and-forward scheme to the half-duplex relay channel yields that the best DF rate is given by

$$\begin{aligned} R_{\text{DF}}^{\text{HD}} = \max \min\big\{ &\tau_1 I(X_{\text{S}}^{(1)}; Y_{\text{R}}^{(1)}) + \tau_2 I(X_{\text{S}}^{(2)}; Y_{\text{D}}^{(2)} | X_{\text{R}}^{(2)}), \\ &\tau_1 I(X_{\text{S}}^{(1)}; Y_{\text{D}}^{(1)}) + \tau_2 I(X_{\text{S}}^{(2)} X_{\text{R}}^{(2)}; Y_{\text{D}}^{(2)}) \big\}, \end{aligned} \tag{23}$$

where τ_1 and τ_2 again denote the lengths of the time slots allocated to the relay receive and the relay transmit phase, respectively.

Remark 3: This assumes that the source may transmit new information to the destination in the relay transmit phase. As the relay does not receive (and thus decode) all the information the source communicates due to the imposed half-duplex constraint, this strategy is sometimes termed a partial DF scheme in the literature, including [9] for example. We agree with [11], however, and consider this as a DF strategy since the relay decodes everything it can receive.

Like in the full-duplex case, $C_{\text{CSB}}^{\text{HD}}$ and $R_{\text{DF}}^{\text{HD}}$ differ only in one term. For Gaussian source and relay inputs $\boldsymbol{x}_{\text{S}}^{(1)} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{R}_{\text{S}}^{(1)}), \boldsymbol{x}_{\text{S}}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{R}_{\text{S}}^{(2)})$, and $\boldsymbol{x}_{\text{R}}^{(2)} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{R}_{\text{R}}^{(2)})$, the achievable DF rate is hence maximized by

$$\begin{aligned} &\max_{\boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)}, \boldsymbol{Q}, \tau_1, \tau_2} R_{\text{DF}}^{\text{HD}} \quad \text{s.t.} \\ &R_{\text{DF}}^{\text{HD}} \leq \tau_1 \log\left| \boldsymbol{I} + \boldsymbol{H}_{\text{SR}} \boldsymbol{R}^{(1)} \boldsymbol{H}_{\text{SR}}^{\text{H}} \right| + \tau_2 \log\left| \boldsymbol{I} + \boldsymbol{H}_{\text{SD}} \boldsymbol{Q} \boldsymbol{H}_{\text{SD}}^{\text{H}} \right|, \\ &R_{\text{DF}}^{\text{HD}} \leq \tau_1 \log\left| \boldsymbol{I} + \boldsymbol{H}_{\text{SD}} \boldsymbol{R}^{(1)} \boldsymbol{H}_{\text{SD}}^{\text{H}} \right| + \tau_2 \log\left| \boldsymbol{I} + \boldsymbol{H}_2 \boldsymbol{R}^{(2)} \boldsymbol{H}_2^{\text{H}} \right|, \\ &\boldsymbol{Q} \succeq \mathbf{0}, \boldsymbol{R}^{(1)} \succeq \mathbf{0}, \boldsymbol{R}^{(2)} - \boldsymbol{D}_{\text{S}}^{\text{H}} \boldsymbol{Q} \boldsymbol{D}_{\text{S}} \succeq \mathbf{0}, \\ &\operatorname{tr}(\boldsymbol{R}_{\text{S}}^{(1)}) \leq P_{\text{S}}^{(1)}, \operatorname{tr}(\boldsymbol{R}_{\text{S}}^{(2)}) \leq P_{\text{S}}^{(2)}, \operatorname{tr}(\boldsymbol{R}_{\text{R}}^{(2)}) \leq P_{\text{R}}^{(2)}, \\ &\tau_1 \geq 0, \tau_2 \geq 0, \tau_1 + \tau_2 \leq 1, \end{aligned} \tag{24}$$

where we have already introduced the slack variable $\boldsymbol{Q}$ and reformulated the problem. This optimization problem is basically obtained by replacing $\boldsymbol{H}_1$ in (17) by $\boldsymbol{H}_{\text{SR}}$, and like problem (17), it is convex in $\boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)}$, and $\boldsymbol{Q}$ for fixed time allocation parameters τ_1 and τ_2.

If we further introduce the variables $\boldsymbol{C}^{(1)}, \boldsymbol{C}^{(2)}$, and $\boldsymbol{Q}'$ as defined in (18), then, not surprisingly, it is also possible to express the maximal achievable DF rate for the half-duplex MIMO relay channel as the solution of an optimization problem that is jointly convex in all parameters. We state it here for reasons of completeness:

$$\begin{aligned} &\max_{\boldsymbol{C}^{(1)}, \boldsymbol{C}^{(2)}, \boldsymbol{Q}', \tau_1, \tau_2} R_{\text{DF}}^{\text{HD}} \quad \text{s.t.} \\ &R_{\text{DF}}^{\text{HD}} \leq \tau_1 \log\left| \boldsymbol{I} + \boldsymbol{H}_{\text{SR}} \frac{\boldsymbol{C}^{(1)}}{\tau_1} \boldsymbol{H}_{\text{SR}}^{\text{H}} \right| + \tau_2 \log\left| \boldsymbol{I} + \boldsymbol{H}_{\text{SD}} \frac{\boldsymbol{Q}'}{\tau_2} \boldsymbol{H}_{\text{SD}}^{\text{H}} \right|, \\ &R_{\text{DF}}^{\text{HD}} \leq \tau_1 \log\left| \boldsymbol{I} + \boldsymbol{H}_{\text{SD}} \frac{\boldsymbol{C}^{(1)}}{\tau_1} \boldsymbol{H}_{\text{SD}}^{\text{H}} \right| + \tau_2 \log\left| \boldsymbol{I} + \boldsymbol{H}_2 \frac{\boldsymbol{C}^{(2)}}{\tau_2} \boldsymbol{H}_2^{\text{H}} \right|, \\ &\boldsymbol{Q}' \succeq \mathbf{0}, \boldsymbol{C}^{(1)} \succeq \mathbf{0}, \boldsymbol{C}^{(2)} - \boldsymbol{D}_{\text{S}}^{\text{H}} \boldsymbol{Q}' \boldsymbol{D}_{\text{S}} \succeq \mathbf{0}, \\ &\operatorname{tr}(\boldsymbol{C}_{\text{S}}^{(1)}) \leq \tau_1 P_{\text{S}}^{(1)}, \operatorname{tr}(\boldsymbol{C}_{\text{S}}^{(2)}) \leq \tau_2 P_{\text{S}}^{(2)}, \operatorname{tr}(\boldsymbol{C}_{\text{R}}^{(2)}) \leq \tau_2 P_{\text{R}}^{(2)}, \\ &\tau_1 \geq 0, \tau_2 \geq 0, \tau_1 + \tau_2 \leq 1. \end{aligned} \tag{25}$$

V. Numerical Results and Discussion

In this section, we provide numerical results for the cut-set bound and the best achievable DF rate for both the full-duplex and the half-duplex case and several antenna configurations. Considering the full-duplex relay channel, we compare the CSB to the upper bound that was derived in [5], and we discuss conditions under which this generally loose upper bound is equal to the cut-set bound. For the half-duplex relay channel, we compare our results to those presented in [9]. Finally, we comment on why it is not possible to directly apply standard

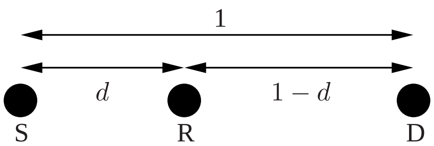


Figure 3. Line Network.

semi-definite program (SDP) solvers like SDPT3 or Sedumi to the half-duplex optimization problems.

The example scenario we consider here is the line network depicted in Figure 3. This is a commonly used geometry where $d_{\text{SD}} = 1$ is fixed and the relay is positioned on the line connecting the source and the destination such that $d_{\text{SR}} = |d|$ and $d_{\text{RD}} = |1-d|$. The entries of the channel gain matrices $\boldsymbol{H}_{\text{SR}}, \boldsymbol{H}_{\text{SD}}$, and $\boldsymbol{H}_{\text{RD}}$ are assumed to be independent and identically distributed complex Gaussian random variables with zero mean and variance $d_{\text{SR}}^{-\alpha}$, $d_{\text{SD}}^{-\alpha}$, and $d_{\text{RD}}^{-\alpha}$, respectively, where the pass loss exponent is chosen as $\alpha = 2$. Note that all numerical results are averaged over a number of independent channel realizations, where perfect CSI at all nodes is assumed for every realization.

The upper bound for the full-duplex MIMO relay channel presented in [5] is given by

$$\begin{aligned} C_{\text{UB}}^{\text{FD}} = \max_{\boldsymbol{R}_{\text{S}},\boldsymbol{R}_{\text{R}},\rho} \min\Big\{ &\log\det\left(\boldsymbol{I} + (1-\rho^2)\boldsymbol{H}_1\boldsymbol{R}_{\text{S}}\boldsymbol{H}_1^{\text{H}}\right), \\ &\inf_{a>0}\log\det\Big(\boldsymbol{I} + (1+\tfrac{\rho^2}{a})\boldsymbol{H}_{\text{SD}}\boldsymbol{R}_{\text{S}}\boldsymbol{H}_{\text{SD}}^{\text{H}} \\ &\quad + (1+a)\boldsymbol{H}_{\text{RD}}\boldsymbol{R}_{\text{R}}\boldsymbol{H}_{\text{RD}}^{\text{H}}\Big)\Big\} \\ \text{s.t.}\quad &\text{tr}(\boldsymbol{R}_{\text{S}}) \le P_{\text{S}}, \quad \text{tr}(\boldsymbol{R}_{\text{R}}) \le P_{\text{R}}, \\ &\boldsymbol{R}_{\text{S}} \succeq \boldsymbol{0}, \quad \boldsymbol{R}_{\text{R}} \succeq \boldsymbol{0}, \quad 0 \le \rho \le 1. \end{aligned} \tag{26}$$

In order to arrive at this result, the authors introduced the scalar parameter $\rho \in [0,1]$ to capture the cross correlation between the source and relay inputs (instead of the matrix $\boldsymbol{R}_{\text{SR}}$) and subsequently made smart use of matrix inequalities. Although the derivation is very elegant, the bound suffers from several restrictions that are pointed out in [9]. Most importantly, this bound is loose in general. Moreover, it is only valid for antenna configurations that satisfy $N_{\text{S}} \le N_{\text{R}}$, which will most likely be violated if the source is a base station and the relay is some device of lower complexity for example. However, in order to compare our results to this upper bound, we restrict our scenarios to those which satisfy this condition.

For the simplest case of single antenna nodes, i.e., when all channels are scalars, the cross covariance matrix $\boldsymbol{R}_{\text{SR}}$ boils down to a scalar. As a consequence, $C_{\text{UB}}^{\text{FD}}$ is equal to $C_{\text{CSB}}^{\text{FD}}$ in this case [5]. Figure 4 thus compares only the cut-set bound with the achievable DF rate given that $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 1$. We can observe the well known result that the DF scheme achieves the CSB when the relay is very close to the source, i.e., when the probability that the source-relay link is the bottleneck goes to zero. The value of ρ that maximizes $C_{\text{UB}}^{\text{FD}}$ (and thus $C_{\text{CSB}}^{\text{FD}}$) is close to 1 in this case, which means that source and relay can realize multi-antenna transmission.

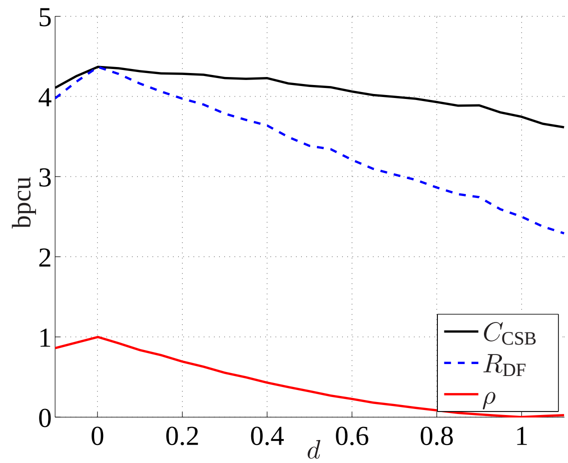


Figure 4. Comparison of cut-set bound and maximal achievable DF rate for full-duplex relay channel, $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 1$ and $P_{\text{S}} = P_{\text{R}} = 10$ (averaged over 5000 realizations).

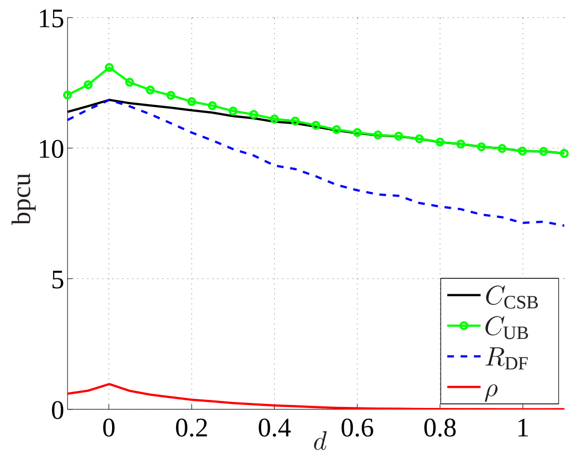


Figure 5. Comparison of cut-set bound, upper bound (from [5]), and maximal achievable DF rate for full-duplex relay channel, $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 2$ and $P_{\text{S}} = P_{\text{R}} = 10$ (averaged over 1000 realizations).

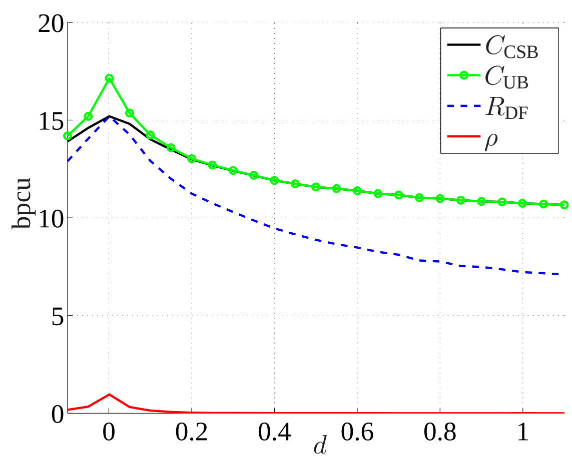


Figure 6. Comparison of cut-set bound, upper bound (from [5]), and maximal achievable DF rate for full-duplex relay channel, $N_{\text{S}} = N_{\text{R}} = 2$, $N_{\text{D}} = 3$, and $P_{\text{S}} = P_{\text{R}} = 10$ (averaged over 1000 realizations).

When some nodes are equipped with multiple antennas, on the other hand, then $C_{\mathrm{UB}}^{\mathrm{FD}}$ is not tight for all channel conditions. This is illustrated in Figures 5 and 6 for $N_{\mathrm{S}} = N_{\mathrm{R}} = N_{\mathrm{D}} = 2$ as well as $N_{\mathrm{S}} = N_{\mathrm{R}} = 2, N_{\mathrm{D}} = 3$, respectively. We see that $C_{\mathrm{UB}}^{\mathrm{FD}} = C_{\mathrm{CSB}}^{\mathrm{FD}}$ only if $\rho = 0$. In fact, it can be shown that the upper bound is equal to the cut-set bound if the optimal solution requires that the source and relay inputs are independent, i.e., if $\rho = 0$ and $\boldsymbol{R}_{\mathrm{SR}} = \boldsymbol{0}$ are optimizers of the respective optimization problems. This is because the structure of the optimal correlation is completely captured by the scalar $\rho = 0$ in this case. In general, this does not hold, which explains why the upper bound is not always tight.

For the half-duplex MIMO relay channel, no such upper bound that relies on the introduction of a scalar correlation parameter and utilizes matrix inequalities has been derived. However, a different upper bound is presented in [9], which reads as

$$\max_{\substack{\boldsymbol{R}^{(1)},\boldsymbol{R}^{(2)}\succeq\boldsymbol{0},\\ \tau_1,\tau_2\geq 0,\tau_1+\tau_2\leq 1}} C_{\mathrm{UB}}^{\mathrm{HD}} \quad \text{s.t.}$$

$$C_{\mathrm{UB}}^{\mathrm{HD}} \leq \tau_1 \log\left|\boldsymbol{I} + \boldsymbol{H}_1\boldsymbol{R}^{(1)}\boldsymbol{H}_1^{\mathrm{H}}\right| + \tau_2 \log\left|\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{R}_{\mathrm{S}}^{(2)}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right|,$$

$$C_{\mathrm{UB}}^{\mathrm{HD}} \leq \tau_1 \log\left|\boldsymbol{I} + \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{R}^{(1)}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}\right| + \tau_2 \log\left|\boldsymbol{I} + \boldsymbol{H}_2\boldsymbol{R}^{(2)}\boldsymbol{H}_2^{\mathrm{H}}\right|,$$

$$\mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}^{(1)}) \leq P_{\mathrm{S}}^{(1)}, \mathrm{tr}(\boldsymbol{R}_{\mathrm{S}}^{(2)}) \leq P_{\mathrm{S}}^{(2)}, \mathrm{tr}(\boldsymbol{R}_{\mathrm{R}}^{(2)}) \leq P_{\mathrm{R}}^{(2)}. \tag{27}$$

Remark 4: Note that a bound of the same structure was also presented for the full-duplex case in [9]. However, as the differences to our results are of the same kind, we discuss all properties on the basis of the half-duplex relay channel.

Observe that the conditional covariance matrix $\boldsymbol{R}_{\mathrm{S|R}}^{(2)}$ in (16) must be replaced by $\boldsymbol{R}_{\mathrm{S}}^{(2)}$ to obtain (27), which is the only difference between these two optimization problems. Because $\log\det(\boldsymbol{I}+\boldsymbol{H}\boldsymbol{R}\boldsymbol{H}^{\mathrm{H}})$ is an increasing function in the transmit covariance matrix $\boldsymbol{R}$ given the channel gain matrix $\boldsymbol{H}$, and since $\boldsymbol{R}_{\mathrm{S|R}}^{(2)} = \boldsymbol{R}_{\mathrm{S}}^{(2)} - \boldsymbol{R}_{\mathrm{SR}}^{(2)}\boldsymbol{R}_{\mathrm{R}}^{(2),\dagger}\boldsymbol{R}_{\mathrm{SR}}^{(2),\mathrm{H}} \preceq \boldsymbol{R}_{\mathrm{S}}^{(2)}$, it follows that $C_{\mathrm{CSB}}^{\mathrm{HD}} \leq C_{\mathrm{UB}}^{\mathrm{HD}}$. Therefore, $C_{\mathrm{UB}}^{\mathrm{HD}}$ is loose in general, and $C_{\mathrm{CSB}}^{\mathrm{HD}} = C_{\mathrm{UB}}^{\mathrm{HD}}$ only if the optimal solution of (16) requires that $\boldsymbol{R}_{\mathrm{SR}}^{(2)} = \boldsymbol{0}$, i.e., if the optimal source and relay inputs are independent. Note that this condition for the tightness of the upper bound is the same as for the bound derived in [5] for the full-duplex case, although the reasons for the bounds not being tight in general are completely different.

Irrespective of these properties, (16) poses a much more complicated optimization problem than (27). In particular, (27) is already convex in the transmit covariance matrices for fixed τ_1 and τ_2. It is hence not necessary to introduce a slack variable and to apply the Schur complement condition. Rather, only the convexity preserving property of the perspective function is needed to reformulate (27) as a convex optimization problem.

Figures 7–9 compare $C_{\mathrm{CSB}}^{\mathrm{HD}}, C_{\mathrm{UB}}^{\mathrm{HD}}$, and $R_{\mathrm{DF}}^{\mathrm{HD}}$ for different antenna configurations, which are actually the same as for the full-duplex case. First, observe that there is no antenna configuration for which $C_{\mathrm{UB}}^{\mathrm{HD}} = C_{\mathrm{CSB}}^{\mathrm{HD}}$ in general, not even the single antenna case. In fact, the relative gap between $C_{\mathrm{CSB}}^{\mathrm{HD}}$

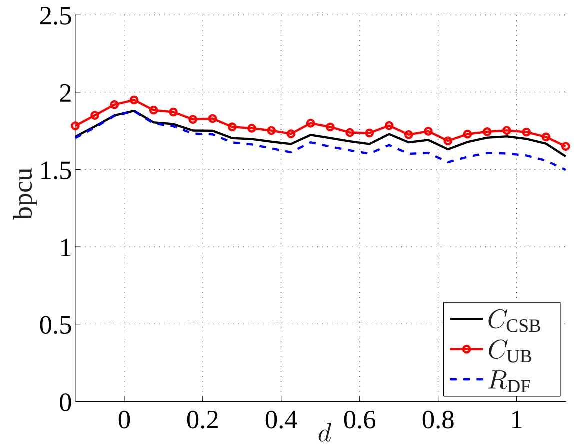


Figure 7. Comparison of cut-set bound, upper bound (from [9]), and maximal achievable DF rate for half-duplex relay channel, $N_{\mathrm{S}} = N_{\mathrm{R}} = N_{\mathrm{D}} = 1$ and $P_{\mathrm{S}}^{(1)} = P_{\mathrm{S}}^{(2)} = P_{\mathrm{R}}^{(2)} = 10$ (averaged over 500 realizations).

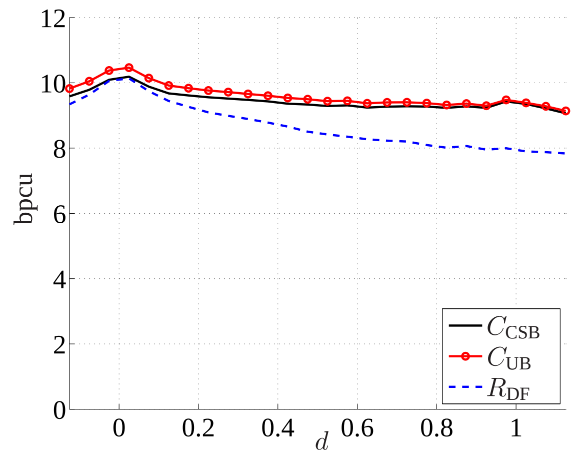


Figure 8. Comparison of cut-set bound, upper bound (from [9]), and maximal achievable DF rate for half-duplex relay channel, $N_{\mathrm{S}} = N_{\mathrm{R}} = N_{\mathrm{D}} = 2$ and $P_{\mathrm{S}}^{(1)} = P_{\mathrm{S}}^{(2)} = P_{\mathrm{R}}^{(2)} = 10$ (averaged over 200 realizations).

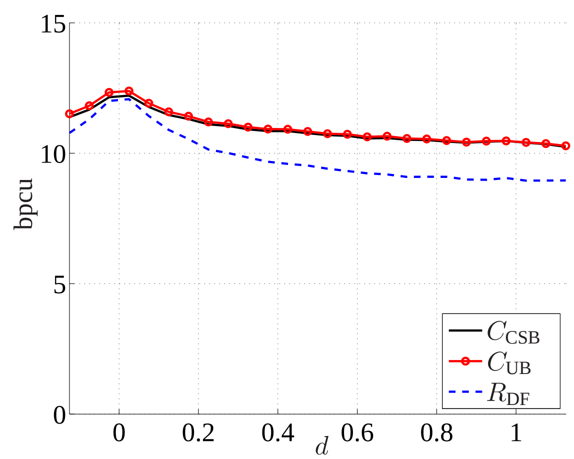


Figure 9. Comparison of cut-set bound, upper bound (from [9]), and maximal achievable DF rate for half-duplex relay channel, $N_{\mathrm{S}} = N_{\mathrm{R}} = 2$, $N_{\mathrm{D}} = 3$, and $P_{\mathrm{S}}^{(1)} = P_{\mathrm{S}}^{(2)} = P_{\mathrm{R}}^{(2)} = 10$ (averaged over 200 realizations).

and $C_{\text{UB}}^{\text{HD}}$ among the three configurations considered here is largest for $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 1$. Furthermore, it can be seen that the gap between the cut-set bound and the upper bound decreases with d. From this, we can conclude that the better the relay-destination link is compared to the source-destination link, the less correlated the optimal source and relay inputs must be. Moreover, the simulation results shown in Figures 8 and 9 suggest that the gap between $C_{\text{CSB}}^{\text{HD}}$ and $C_{\text{UB}}^{\text{HD}}$ decreases faster with d for larger N_{D} if the number of source and relay antennas is not changed, which means that more degrees of freedom at the destination also favor less correlated source and relay input distributions.

Comparing the full-duplex to the half-duplex results, it stands out that the relative gap between $C_{\text{CSB}}^{\text{HD}}$ and $R_{\text{DF}}^{\text{HD}}$ increases less than that between $C_{\text{CSB}}^{\text{FD}}$ and $R_{\text{DF}}^{\text{FD}}$ when the relay is moved closer to the destination, especially for the single antenna case. This may be explained by the fact that in the half-duplex case the optimization is with respect to the source and relay inputs and the time-sharing between the relay receive and the relay transmit phase. The optimal time allocation for the achievable DF rate need not be the same as for the cut-set bound. Since the source transmits during both phases and the source-destination channel does not change, a weak source-relay channel can thus partly be compensated for by prolonging the relay receive phase. In the full-duplex case, this is not possible as both the source and the relay transmit the whole time.

Across-the-board, all simulation results show that the rates achieved using the DF scheme approach the CSB when the relay is close to the source. Furthermore, it can be seen that substantial rate gains can be achieved when multiple antennas are used at each node without increasing the power at the source or the relay. These results are certainly not surprising as they are in accordance with previous knowledge about decode-and-forward relaying and MIMO channels.

Before we conclude this section, a few comments about problems (19) and (25) are in order. Note that these two convex optimization problems do not satisfy the ruleset of *disciplined convex programming* (DCP) [13]. Problems that adhere to this ruleset can automatically be verified as convex and converted to solvable form, which allows to directly use standard SDP solvers. This does not apply to problems that violate the ruleset, even if they are convex. That is not to say that there does not exist a suitable reformulation of such problems. However, if no applicable reformulation can be found, other solution methods need to be considered.

VI. Conclusion

In this paper, we extend previous results for the full-duplex MIMO relay channel and show that the cut-set bound and the achievable DF rate can also be obtained as the solutions of convex optimization problems in the half-duplex case. It is therefore possible to efficiently determine these (generally loose) upper and lower bounds on the capacity of the MIMO relay channel, which may serve as benchmarks for achievable rates of various relay strategies for example. Additionally, we discuss how our results for the cut-set bound compare to the upper bounds presented in [5] and [9]. In particular, we point out why these upper bounds are not tight in general, and we identify cases in which they are equal to the CSB. Finally, simulation results for various antenna configurations demonstrate the importance of our work as they reveal non-negligible gaps between the CSB and the addressed upper bounds.

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