Technische Universität München
Lehrstuhl M12

L^1-algebras on commutative hypergroups: Structure and properties arising from harmonic analysis

Eva Perreiter

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

Vorsitzender:		Univ.-Prof. Dr. Dr. Jürgen Richter-Gebert
Prüfer der Dissertation:	1.	Univ.-Prof. Dr. Rupert Lasser
	2.	Univ.-Prof. Dr. Eberhard Kaniuth (em.), Universität Paderborn
	3.	Univ.-Prof. Dr. Michael Voit, Technische Universität Dortmund (schriftliche Beurteilung)

Die Dissertation wurde am 26.07.2011 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 08.12.2011 angenommen.

Contents

Introduction

W. Bloom and H. Heyer state in [7, p. 69] that the idea of hypergroups can be traced back quite formally to the work of F. Marty and H. S. Wall in the 1930ies. Nowadays there are three axiom schemes for hypergroups advanced independently by C. F. Dunkl [13], R. I. Jewett [24] and R. Spector [50] during the 1970ies. We follow [7] which is based on the commonly used axiomatic framework of Jewett.

Hypergroups generalize the class of locally compact groups. Roughly speaking, a hypergroup is a locally compact Hausdorff space K, endowed with a probability measure-valued convolution $\omega : K \times K \to M^1(K)$ (generalizing the group operation) and an involution $\tilde{}: K \to K$ (in the group case given by the group inversion).

For example, hypergroups arise naturally as double coset spaces, spaces of conjugacy classes or orbit spaces [24, Ch. 8]. Important examples are also given on $\mathbb{R}_0^+$ and $\mathbb{N}_0$ (with operations different from the ones inherited by the group operations on $\mathbb{R}$ and $\mathbb{Z}$) by Sturm-Liouville and polynomial hypergroups, respectively.

Polynomial hypergroups as defined by R. Lasser in [29] constitute our main examples: In their definition linearization coefficients derived from orthogonal polynomial systems are used to induce commutative hypergroup structures on $\mathbb{N}_0$ (similar to the system $(x^n)_{n\in\mathbb{Z}}$ 'inducing' the group structure $(\mathbb{Z}, +)$). On the one hand, polynomial hypergroups play the role of comparatively simple examples of commutative hypergroups (like $(\mathbb{Z}, +)$ for locally compact abelian groups). On the other hand, the class of polynomial hypergroups has proven to be a rich source of examples and counterexamples.

In this thesis we consider the Banach algebra $L^1(K)$ with respect to the Haar measure on a commutative hypergroup K. For general hypergroups it is not clear if there always exists a Haar measure with respect to the hypergroup translations, whereas for compact or commutative hypergroups the existence of a Haar measure is known; cf. [24, Thm. 7.2A] and [51]. Furthermore, polynomial hypergroups which serve as our main examples are always commutative. For these reasons the present thesis is concerned with commutative hypergroups only.

Still, commutative hypergroups display a number of properties not familiar from the case of locally compact abelian groups. Examples include the facts that the translations are in general not isometries but only contractions, the characters need not be of modulus one and the character space rarely carries a natural dual hypergroup structure. While there still are an inverse Fourier and Plancherel transformation, the Plancherel measure does not have to have full support in the character space; all these facts can be found in [7].

The topic of the present thesis are structural properties as well as properties arising from harmonic analysis of L^1-algebras defined on a commutative hypergroup.

Properties of commutative hypergroups are discussed in Chapter 1.

Chapters 2 and 5 are concerned with some properties of L^1-algebras arising from harmonic analysis: Chapter 2 approaches questions relating to amenability of l^1-algebras on polynomial hypergroups, while Chapter 5 considers regularity of the L^1-algebras on commutative hypergroups.

Chapters 3 and 4 deal with structural properties of L^1-algebras: In Chapter 3 we have a look at embeddings and isomorphisms of l^1-algebras on polynomial hypergroups.

Chapter 4 comments on the action of the L^1-algebra of a commutative hypergroup on the corresponding L^p-spaces. The results of both sections have mostly been published, see [34, 41].

Chapter 1 introduces commutative hypergroups: After defining commutative hypergroups we turn to basic notions in their harmonic analysis. Special attention is directed to their dual objects. As examples we consider the class of polynomial hypergroups in Chapter 1.2. We conclude the chapter by introducing the slightly more general notion of signed polynomial hypergroups which we need in Chapter 3.

In Chapter 2 we are concerned with amenability and weak amenability of the l^1-algebra of a polynomial hypergroup structure on $\mathbb{N}_0$ and an observation on the α-amenability of a general commutative Banach algebra A.

The definition of an amenable Banach algebra is motivated by Johnson's Theorem [48, Thm. 2.1.8, Ch. 2.5], dating from the 1970ies, which characterizes the amenability of a locally compact group G by the amenability of $L^1(G)$. M. Skantharajah already remarks in [49, Pro. 4.9, Ex. 4.10] that Johnson's Theorem is not valid for hypergroups: For a hypergroup the amenability of its L^1-algebra is not equivalent to but only implies the amenability of the hypergroup. For an illustration consider the family of ultraspherical polynomial hypergroups: Each member of this family is amenable (since every commutative hypergroup is amenable [49, Ex. 3.3.(a)]) whereas only the l^1-algebra induced by the Chebyshev polynomials of the first kind is amenable. This has been shown by R. Lasser in [32, Cor. 3] using a sufficient condition for the existence of an approximate diagonal [32, Thm. 4]. More general, if the Haar weight $(h_n)_n$ of a polynomial hypergroup tends to infinity, then $l^1(h)$ is not amenable [32, Thm. 3]; this demonstrates that there is quite a gap between amenability of hypergroups and amenability of their L^1-algebras.

Thus, for hypergroups we are led to study well known properties of Banach algebras weaker than amenability; weak amenability has been introduced in the 1980ies by W. Bade, P. Curtis and H. Dales (compare [48, Ch. 4.5]), whereas α-amenability was defined only recently by E. Kaniuth, A. T. Lau and J. Pym in [26]. Weak amenability of the l^1-algebras of polynomial hypergroups has been studied by R. Lasser in [32]. In particular, for the family of ultraspherical polynomial hypergroups with nonnegative parameter he proves that the l^1-algebra is not even weakly amenable [32, Cor. 1]. The α-amenability of l^1-algebras of polynomial hypergroups has been studied in [16, 33] by F. Filbir, R. Lasser and R. Szwarc.

In Chapter 2.1 we consider amenability of $l^1(h)$ by studying the possible forms of approximate diagonals for $l^1(h)$. Making use of a simple form with symmetric Gelfand transforms we obtain sufficient conditions on the growth of the Haar weight $(h_n)_n$ for $l^1(h)$ to be amenable (Proposition 2.12, Corollary 2.13). In Chapter 2.2 we first treat weak amenability of $l^1(h)$. We derive two characterizations of weak amenability, one of them in a way similar to approximate diagonals characterizing amenability of $l^1(h)$ (Proposition 2.18). Unfortunately, the results on amenability and weak amenability remain rather theoretical; we have not been able to provide examples yet. Afterwards we make an observation on α-amenability of a general commutative Banach algebra A (Proposition 2.22); we show that α-amenability for all $\alpha \in \Delta(A)$ implies that $\Delta(A)$ is discrete with respect to the weak topology $\sigma(A^*, A^{**})$.

In Chapter 3 we consider isomorphisms between l^1-algebras of (signed) polynomial hypergroups.

Isomorphisms of hypergroups have been studied by W. Bloom and M. Walter in [8], their main focus lying on isometric isomorphisms. Isometric isomorphisms between l^1-algebras are quite rare since, in general, the translation operators are not unitary and the characters are not of modulus 1; we only consider non-isometric isomorphisms. The results are formulated for signed polynomial hypergroups which include the class of polynomial hypergroups.

In Chapter 3.1 we derive sufficient conditions for the existence of homomorphisms and isomorphisms between the l^1-algebras of two (signed) polynomial hypergroups. Those are conditions imposed on the connection coefficients between the two inducing systems of orthogonal polynomials (Theorems 3.2 and 3.8). On our way we show in Corollary 3.6 that the l^1-algebra carrying the convolution structure of the semigroup $\mathbb{N}_0$ is continuously embedded in the l^1-algebra of every polynomial hypergroup $\mathbb{N}_0$.

In Chapter 3.2 we apply the constructed class of homomorphisms to transfer amenability and related properties from one l^1-algebra to another. These properties are usually hard to verify directly, whereas the approach via inheritance under homomorphisms turns out to be a practicable alternative. As examples we consider the Bernstein-Szegő polynomials of the first and the second kind, as well as the Jacobi and the Associated Legendre polynomials. In particular, we show that all l^1-algebras w.r.t. Bernstein-Szegő polynomials of the first and the second kind are isomorphic to the l^1-algebras w.r.t. Chebyshev polynomials of the first and the second kind, respectively. This in turn implies that the l^1-algebras w.r.t. Bernstein-Szegő polynomials of the first kind are all amenable.

Almost all results of Chapter 3 have already been published in [34] (with the noteworthy exception of the Bernstein-Szegő polynomials of the second kind).

In Chapter 4 we consider the spectra of the convolution operators

$$T_f = T_{f,p} : \ L^p(K) \to L^p(K), \ T_f(g) = f * g,$$

for $f \in L^1(K)$ on commutative hypergroups K. We are interested in how, for fixed $f \in L^1(K)$, the spectra $\sigma_p(T_f)$ of $T_{f,p}$ vary with p. The starting point for our investigation is B. Barnes' article [4] where he treats this problem for locally compact groups.

In Chapter 4.1 we obtain that, as for locally compact amenable groups [4, Pro. 3], for any commutative hypergroup K and $f \in L^1(K)$, the inclusion $\sigma_q(T_f) \subseteq \sigma_p(T_f)$ is true whenever $p \leq q \leq 2$ or $2 \leq q \leq p$ (Proposition 4.6).

In Chapter 4.2 we mainly characterize those commutative hypergroups where for each L^1-convolution operator all its p-spectra coincide (Theorem 4.12). We prove that $\sigma_p(T_f)$ is independent of $p \in [1, \infty]$ for all $f \in L^1(K)$ exactly when the Plancherel measure is supported on the whole character space. A reformulation of this characterization reads: $\sigma_p(T_f)$ is independent of $p \in [1, \infty]$ for all $f \in L^1(K)$ exactly when $L^1(K)$ is symmetric and for every $\alpha \in \hat{K}$ Reiter's condition P_2 holds true. This is similar to [4, Thm. 6] where p-independence for locally compact groups G is characterized by $L^1(G)$ being symmetric and G being amenable. For groups, Barnes' assumption of amenability is equivalent to Reiter's condition P_2 (in $\alpha \equiv 1$) whereas for hypergroups the various properties which characterize amenability (including the P_2-condition) in the group case are not equivalent. On our way to prove Theorem 4.12 we obtain in Corollary 4.9 that each connected component of the character space intersects the support of the Plancherel measure (modulo $L^1(K)$ being unital).

In Chapter 4.3 we explicitly determine the spectra $\sigma_p(T_{\varepsilon_1})$, $p \in [1, \infty]$, for the family of Karlin-McGregor polynomial hypergroups and the generating elements ε_1 of their l^1-algebras for all parameters $\alpha, \beta \geq 2$. These spectra turn out to equal the set of both square roots of certain ellipses in the complex plane (Theorem 4.15). By [4, Thm. 6] for abelian locally compact groups the spectra $\sigma_p(T_f)$ coincide for all $p \in [1, \infty]$. Our examples demonstrate that this is not true for commutative hypergroups since here $\sigma_q(T_{\varepsilon_1}) \subsetneq \sigma_p(T_{\varepsilon_1})$ whenever $q > p$, $q, p \in [1, 2]$, and $(\alpha, \beta) \neq (2, 2)$. This also shows that the inclusion relation of Chapter 4.1 can not be improved for general commutative hypergroups. Furthermore, as a byproduct we obtain the shape of the complex set where the Karlin-McGregor polynomials are uniformly bounded; for a polynomial hypergroup this set is the customary representation of its corresponding character space.

The results contained in this chapter have been published in [41].

In Chapter 5 we are concerned with regularity of $L^1(K)$, i.e., the question of whether there are enough Gelfand transforms to separate points from closed subsets in the structure space.

Regularity of a commutative Banach algebra was first introduced (via the notion of the hull-kernel topology on the maximal ideal space) during the 1940ies by I. Gelfand, G. Shilov and N. Jacobson; cf. [25, Ch. 4.9]. It is well-known that $L^1(G)$ is regular for all locally compact abelian groups G, see [25, Thm. 4.4.14]; this is not true for commutative hypergroups. In fact, in this chapter we first observe the probably well known fact that the regularity of $L^1(K)$ implies that the Plancherel measure is supported on the whole character space, which demonstrates that not all L^1-algebras on commutative hypergroups are regular.

Since hypergroups are in general not equipped with a natural dual convolution structure (induced by pointwise multiplication of characters), an approach to the problem analogous to the group case is not possible. In [17, Thm. 2.1], L. Gallardo and O. Gebuhrer show regularity of $L^1(K)$ for commutative hypergroups K whose Haar measure is of polynomial growth. Their proof uses J. Dixmier's functional calculus based on [11, Lem. 7] for certain Lie groups.

In this chapter we extend this functional calculus to functions in a certain Beurling algebra on $\mathbb{R}$ (Theorem 5.3) which allows us to slightly improve the result of Gallardo and Gebuhrer beyond polynomial growth of the Haar measure (Theorem 5.6).

1 Commutative Hypergroups

Hypergroups have been defined independently by Dunkl [13], Jewett [24] and Spector [50] during the 1970ies; we follow [7] which is based on the axiomatic framework of Jewett. This thesis is only concerned with commutative hypergroups.

Commutative Hypergroups generalize the class of locally compact abelian groups (LCAG). Roughly speaking, a commutative hypergroup is a locally compact Hausdorff space K, endowed with a commutative, probability measure-valued convolution $\omega: K \times K \to M^1(K)$ and an involution $\tilde{}: K \to K$. The convolution generalizes the group operation and the involution in the group case is given by the group inversion.

In Chapter 1.1 we define commutative hypergroups and state some basic notions in their harmonic analysis. Special attention is directed to the dual objects. As examples we consider the rich class of polynomial hypergroups in Chapter 1.2. The scope of Chapter 1.3 is to introduce the slightly more general notion of signed polynomial hypergroups which we need in Chapter 3.

1.1 Definition and Basic Properties

An extensive reference for the following basics on hypergroups (as well as a guide for our presentation) is the monograph by W. R. Bloom and H. Heyer [7]. First we define commutative hypergroups, before turning to basic notions in their harmonic analysis. In particular detail we consider three natural dual objects of a commutative hypergroup.

In order to state the exact definition of a hypergroup we first have to lay down some notation and get to know the Michael topology on the set of compact subsets of K. Let K be a nonvoid locally compact Hausdorff space. By $M^b(K)$ we denote the complex, bounded Radon measures on K. We refer to the probability measures by $M^1(K) \subset M^b(K)$. For the point measure at $x \in K$ we write $\varepsilon_x \in M^1(K)$. As usual, $C_c(K), C_0(K)$ and $C_b(K)$ denote the spaces of continuous functions which have compact support, vanish at infinity or are bounded, respectively.

Definition 1.1. *Let K be a locally compact Hausdorff space. Denote by $\mathcal{C}(K)$ the nonvoid compact subsets of K. The Michael topology on $\mathcal{C}(K)$ is the topology generated by the subbasis $\{\mathcal{U}_{U,V} : U, V$ open subsets of $K\}$, where $\mathcal{U}_{U,V} := \{C \in \mathcal{C}(K) : C \cap U \neq \emptyset$ and $C \subset V\}$.*

If K is metrizable then the Michael topology is stronger than the topology on $\mathcal{C}(K)$ which is induced by the Hausdorff metric [7, 1.1.1].

Definition 1.2 (Commutative Hypergroup)**.** *Let K be a nonvoid locally compact Hausdorff space. The triple $(K, *, \tilde{})$ will be called a commutative hypergroup if the following conditions are satisfied.*

(HG1) *The vector space $(M^b(K), +)$ admits a (second) operation $*: M^b(K) \times M^b(K) \to M^b(K)$ under which it is an algebra.*

(HG2) *For $x, y \in K$, $\varepsilon_x * \varepsilon_y \in M^1(K)$ and supp $(\varepsilon_x * \varepsilon_y)$ is compact.*

(HG3) *The mapping $K \times K \to M^1(K)$, $(x, y) \mapsto \varepsilon_x * \varepsilon_y$, is continuous w.r.t. the weak-$*$-topology on $M^1(K)$.*

(HG4) *The mapping* $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto$ *supp* $(\varepsilon_x * \varepsilon_y)$, *is continuous w.r.t. the Michael topology on* $\mathcal{C}(K)$.

(HG5) *There exists a (necessarily unique) element* $e \in K$ *such that* $\varepsilon_x * \varepsilon_e = \varepsilon_e * \varepsilon_x = \varepsilon_x$ *for all* $x \in K$.

(HG6) *There exists a (necessarily unique) involution* $\tilde{}: K \to K$ *(a homeomorphism with the property* $\tilde{\tilde{x}} = x$ *for all* $x \in K$*) such that* $(\varepsilon_x * \varepsilon_y)\tilde{}(B) := (\varepsilon_x * \varepsilon_y)(\tilde{B}) = (\varepsilon_{\tilde{y}} * \varepsilon_{\tilde{x}})(B)$ *for all* $x, y \in K$ *and all Borel sets* $B \subset K$.

(HG7) *For* $x, y \in K$, $e \in$ *supp* $(\varepsilon_x * \varepsilon_y)$ *if and only if* $x = \tilde{y}$.

(HGC) *The convolution* $*$ *is commutative, i.e.,* $\varepsilon_x * \varepsilon_y = \varepsilon_y * \varepsilon_x$ *for all* $x, y \in K$.

Since the span of the point measures is weakly-$*$-dense in $M^b(K)$, it is clear that the convolutions $\varepsilon_x * \varepsilon_y,\ x, y \in K$, determine the entire hypergroup structure. In particular, it is sufficient to check associativity and commutativity with respect to the point measures. We frequently consider the convolution restricted to K and write $\omega: K \times K \to M^1(K),\ \omega(x, y) = \varepsilon_x * \varepsilon_y$.

Having defined commutative hypergroups we now turn to their basic properties w.r.t. harmonic analysis. For $y \in K$, the translation of a function $f \in C_c(K)$ by y is given by

$$L_y f(x) = \int_K f d\omega(y, x) = \int_K f d(\varepsilon_x * \varepsilon_y). \tag{1.1}$$

(Compare $L_y f(x) = \int_K f d(\varepsilon_{xy}) = \int_K f d(\varepsilon_x * \varepsilon_y)$ for LCAG.) For commutative hypergroups, a Haar measure m corresponding to these translations exists; further it is unique up to normalization [7, Thm. 1.3.15, 1.3.22].

For $p \in [1, \infty)$ we consider the spaces $L^p(K) := L^p(K, m)$. The above translations extend to the space $L^p(K)$ for all $p \in [1, \infty]$. But note that, contrasting the group case, these translation operators in general are not isometric but only contractive, i.e. $\|L_y f\|_p \leq \|f\|_p$.

For $p \in [1, \infty]$ the convolution of $f \in L^1(K)$ and $g \in L^p(K)$ is defined by

$$f * g(x) = \int_K L_{\tilde{y}} f(x) g(y) dm(y). \tag{1.2}$$

This convolution obeys $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ [7, Ch. 1.4], thus turning $L^p(K)$ into a symmetric Banach-$L^1(K)$-bimodule. For $p = 1$, $L^1(K)$ is a commutative Banach-$*$-algebra with respect to the above convolution and the isometric $*$-operation $f^*(x) = \overline{f(\tilde{x})}$. The algebra $L^1(K)$ will be of particular interest to us.

Next we get to know three dual objects of the Banach-$*$-algebra $L^1(K)$, namely its structure space, $*$-structure space and the structure space of its representation as convolution operators on the Hilbert space $L^2(K)$. Note that these dual objects of commutative hypergroups rarely carry a natural hypergroup structure of their own, compare [7, Ch. 2.4, Ex. 3.3.13]. They even need not coincide, see [7, Ex. 2.2.49]. These facts contrast the case of LCAG, where all three dual objects equal the dual group. For the following facts on the dual objects see Chapter 2.2 in [7], in particular Thm. 2.2.4.

The character space $\chi_b(K)$ is given by

$$\chi_b(K) = \{\alpha \in C_b(K) : \ \alpha(e) = 1, \ L_y\alpha(x) = \alpha(x)\alpha(y) \text{ for all } x, y \in K\}. \tag{1.3}$$

$\chi_b(K)$ (endowed with the compact-open topology) is homeomorphic to the structure space $\Delta(L^1(K))$ via $\hat{f}(\alpha) = \int_K f\overline{\alpha}dm$. Note that in contrast to the case of LCAG the characters are in general not of modulus one; compare (1.13) and subsequent remarks. $L^1(K)$ is semisimple.

Furthermore we define the closed subset of hermitian characters $\hat{K}$ of $\chi_b(K)$ by

$$\hat{K} = \left\{\alpha \in \chi_b(K) : \ \alpha(\tilde{x}) = \overline{\alpha(x)} \text{ for all } x \in K\right\}.$$

$\hat{K}$ is homeomorphic to the $*$-structure space $\Delta^*(L^1(K))$ via the above identification $\hat{f}(\alpha) = \int_K f\overline{\alpha}dm$. For more information about $*$-structure spaces of Banach-$*$-algebras we refer to [44, Ch. IV.2]. An important feature of $\hat{K}$ is that the space $\widehat{L^1(K)}|_{\hat{K}}$ is dense in $C_0(\hat{K})$ [7, Thm. 2.2.4 (ix)]; the same need not be true of $\chi_b(K)$.

For $f \in L^1(K)$ define the bounded operator $T_{f,2} : \ L^2(K) \to L^2(K), \ T_{f,2}g = f * g$. A third dual object associated with $L^1(K)$ is the structure space S of the algebra of bounded operators $T_{f,2}$, i.e., the representation of $L^1(K)$ as convolution operators on the Hilbert space $L^2(K)$. S can be defined as the following closed subset of $\hat{K}$:

$$S = \left\{\alpha \in \hat{K} : \ |\hat{f}(\alpha)| \leq \|T_{f,2}\| \text{ for all } f \in L^1(K)\right\}. \tag{1.4}$$

Theorem 1.3 below states that S equals the support of the Plancherel measure π on $\hat{K}$.

We know that $\hat{K} = \chi_b(K)$ if and only if $L^1(K)$ is symmetric, i.e., if and only if the spectra $\sigma_{L^1(K)}(f) \subset \mathbb{R}$ for all $f = f^* \in L^1(K)$. Moreover, $S = \hat{K}$ if and only if Reiter's condition P_2 holds for every $\alpha \in \hat{K}$, see [15, Thm. 3.1]. A character $\alpha \in \hat{K}$ satisfies the P_2-condition if for each $\varepsilon > 0$ and every compact subset $C \subset K$ there exists some compactly supported continuous function $g \in C_c(K)$ such that $\|g\|_2 = 1$ and

$$\|L_{\tilde{y}}g - \overline{\alpha(y)}g\|_2 < \varepsilon \quad \text{ for all } y \in C. \tag{1.5}$$

All in all: $S = \chi_b(K)$ if and only if $L^1(K)$ is symmetric and for every $\alpha \in \hat{K}$ Reiter's condition P_2 holds true.

While the behavior of the dual objects contrasts the group case, the Plancherel isometry and inverse Fourier transform still work analogously to the case of LCAG, see [7, Thm. 2.2.13, 2.2.22, Prop. 2.2.19 and Thm. 2.2.32, 2.2.36]:

Theorem 1.3 (Levitan-Plancherel)**.** *Let K be a commutative hypergroup. There exists a unique nonnegative measure π on $\hat{K}$ such that*

$$\int_K |f|^2 dm = \int_{\hat{K}} |\hat{f}|^2 d\pi$$

for all $f \in L^1(K) \cap L^2(K)$. Furthermore, supp $\pi = S$. The continuous extension of the Gelfand transform $f \mapsto \hat{f}$ from $L^1(K) \cap L^2(K)$ to $L^2(K)$ is called Plancherel isomorphism.

Theorem 1.4. *Let K be a commutative hypergroup. The inverse Fourier transform $\check{g} \in C_0(K)$ of $g \in L^1(\hat{K})$ is defined by*

$$\check{g}(x) = \int_{\hat{K}} g(\alpha)\alpha(x)d\pi(\alpha), \quad x \in K.$$

If $f \in L^1(K)$ such that $\hat{f} \in L^1(\hat{K})$, then $f = (\hat{f})\check{}$.

The following useful fact about approximate identities for $L^1(K)$ can be found in [7, Thm. 2.2.28].

Theorem 1.5. *Let K be a commutative hypergroup with neutral element $e \in K$. The algebra $L^1(K)$ admits a bounded approximate identity $(e_\lambda)_\lambda$ satisfying $e_\lambda \in C_c(K)$, $e_\lambda \geq 0$, $\|e_\lambda\|_1 = 1$, $\lim_\lambda$ supp $e_\lambda = \{e\}$, $\hat{e}_\lambda \in L^1_+(\hat{K})$ and $\lim_\lambda \hat{e}_\lambda = 1$ uniformly on compact subsets of $\hat{K}$.*

1.2 Example: The Class of Polynomial Hypergroups

In this chapter we consider the rich class of polynomial hypergroups. For their definition certain orthogonal polynomial systems are used to induce hypergroup structures on $\mathbb{N}_0$. On the one hand, polynomial hypergroups play the role of comparatively simple examples of commutative hypergroups. On the other hand, they have proven to be a rich source of examples and counterexamples. The class of polynomial hypergroups was defined by R. Lasser in [29]; further references are [30] or [7, Ch. 3.2]. After stating the definition we have a look at the basic notions in their harmonic analysis. We conclude this part with the Jacobi polynomial hypergroups as an example.

First we consider a sequence $\{R_n\}_{n\in\mathbb{N}_0}$ of real polynomials, deg $R_n = n$, orthogonal with respect to a probability measure $\pi^R \in M^1(\mathbb{R})$ which has compact and infinite support, i.e.

$$\int_{\mathbb{R}} R_n R_m d\pi^R = \delta_{nm} h_n^{-1} \quad \text{for } m, n \in \mathbb{N}_0. \tag{1.6}$$

Here $h_n^{-1} = \|R_n\|^2_{L^2(\mathbb{R},\pi^R)} > 0$. We assume that $R_n(1) \neq 0$ for all n such that the normalization

$$R_n(1) = 1 \quad \text{for all } n \in \mathbb{N}_0 \tag{1.7}$$

is possible. This implies a recurrence relation of the following form:

$$\begin{aligned} R_0 = 1,\ R_1(x) &= \frac{1}{a_0}(x - b_0), \\ R_1 R_n &= a_n R_{n+1} + b_n R_n + c_n R_{n-1}, \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{1.8}$$

where $a_n,\ b_n,\ c_n \in \mathbb{R}$ and $a_n \neq 0,\ c_n \neq 0$. This three term recurrence can be extended to the following product formula

$$R_m R_n = \sum_{k=|n-m|}^{n+m} g(m,n,k) R_k, \quad m, n \in \mathbb{N}_0, \tag{1.9}$$

where all $g(m,n,k)$ are real. Furthermore, the orthogonality implies that $g(m,n,|n-m|) \neq 0$, $g(m,n,n+m) \neq 0$ and because of our normalization we get $\sum_{k=|n-m|}^{n+m} g(m,n,k) = 1$.

If $g(m,n,k) \geq 0$ for $k,n,m \in \mathbb{N}_0$, then a (commutative) polynomial hypergroup structure is induced on $\mathbb{N}_0$: Denoting by ε_k the point measure at $k \in \mathbb{N}_0$ we define the convolution $\omega : \ \mathbb{N}_0 \times \mathbb{N}_0 \to M^b(\mathbb{N}_0)$ by letting

$$\omega(n,m) = \varepsilon_n * \varepsilon_m = \sum_{k=|n-m|}^{n+m} g(m,n,k)\varepsilon_k \quad \text{for } n,m \in \mathbb{N}_0.$$

Since we assume positive linearization coefficients, the measures $\varepsilon_n * \varepsilon_m$ are probability measures with finite support. The point measure ε_0 is the neutral element for this convolution. Taking the identity mapping as involution we actually obtain a hypergroup structure on $\mathbb{N}_0$.

Having defined polynomial hypergroups we now turn to their harmonic analysis. The translations on the space of finitely supported sequences $l^{\text{fin}}(\mathbb{N}_0)$ are given by

$$T_m : \ l^{\text{fin}}(\mathbb{N}_0) \to l^{\text{fin}}(\mathbb{N}_0), \ T_m f(n) = \sum_{\mathbb{N}_0} f d\omega(n,m) = \sum_{k=|n-m|}^{n+m} g(m,n,k)f(k). \tag{1.10}$$

The Haar measure with respect to these translations on $\mathbb{N}_0$ is given by the sequence $(h_n)_{n\in\mathbb{N}_0}$ defined by $h_n = (\|R_n\|^2_{L^2(\mathbb{R},\pi^R)})^{-1}$. Note that the Haar measure is normalized such that $h_0 = 1$ and that $h_n \geq 1$ due to the normalization $R_n(1) = 1$.

Using these translations, the convolution on $l^1(\mathbb{N}_0, h)$ is given by

$$f * g(n) = \sum_{k=0}^{\infty} T_n f(k) g(k) h_k. \tag{1.11}$$

Since $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, this convolution turns $l^1(\mathbb{N}_0, h)$ into a commutative Banach-$*$-algebra with unit ε_0, if we define $f^* = \overline{f}$.

The structure space $\Delta(l^1(\mathbb{N}_0,h)) \cong \chi_b(\mathbb{N}_0)$ of $l^1(\mathbb{N}_0,h)$ is homeomorphic to the compact set $D \subset \mathbb{C}$ which can be characterized in the following two ways:

$$\begin{aligned} D &= \{z \in \mathbb{C} : \ |R_n(z)| \leq C \ \text{ for all } n \in \mathbb{N}_0 \text{ and some } C > 0\} \\ &= \{z \in \mathbb{C} : \ |R_n(z)| \leq 1 \ \text{ for all } n \in \mathbb{N}_0\}. \end{aligned} \tag{1.12}$$

The homeomorphism is given by

$$D \to \chi_b(\mathbb{N}_0), \ z \mapsto (R_n(z))_{n\in\mathbb{N}_0}. \tag{1.13}$$

Note that the characters $(R_n(z))_{n\in\mathbb{N}_0}$ rarely are of modulus one. Moreover note that $1 \in D$ due to our normalization and that $(R_n(1))_{n\in\mathbb{N}_0} = (1)_{n\in\mathbb{N}_0}$ is the constant one-character. Under the above identification, $\hat{\mathbb{N}}_0 = \chi_b(\mathbb{N}_0) \cap \mathbb{R} = D \cap \mathbb{R}$. Furthermore, the support of the Plancherel measure S is a closed subset of $D \cap \mathbb{R}$.

The Gelfand transform is thus given by

$$\mathcal{F}(f)(z) = \hat{f}(z) = \sum_{n=0}^{\infty} f_n R_n(z) h_n, \quad f \in l^1(\mathbb{N}_0, h), \ z \in D. \tag{1.14}$$

Since $\mathbb{N}_0$ is discrete, the point measures ε_n have a density w.r.t. the Haar measure $(h_n)_n$ and thus lie in $l^1(\mathbb{N}_0, h)$: They can be written as $\varepsilon_n = \frac{\delta_n}{h_n}$, where $\delta_n \in l^1(\mathbb{N}_0, h)$ is the sequence $\delta_n := (\delta_{nk})_{k\in\mathbb{N}_0}$ peaking at n. Their Gelfand transform (1.14) obeys $\hat{\varepsilon}_n = R_n|_D$.

Since the Gelfand transform is a homomorphism, $A(D) := \mathcal{F}(l^1(\mathbb{N}_0, h))$ is an algebra w.r.t. pointwise multiplication. Endowed with the original norm $\|\hat{v}\| := \|v\|_1$ this algebra becomes a Banach algebra which is called the Wiener algebra corresponding to $l^1(\mathbb{N}_0, h)$.

The Plancherel measure is exactly the orthogonalization measure π^R of the polynomial system $(R_n)_n$; in particular, $S = \text{supp } \pi^R$. The Plancherel isomorphism is given by

$$\hat{g} = \sum_{n=0}^{\infty} g_n R_n|_S h_n, \quad g \in l^2(\mathbb{N}_0, h).$$

Since $l^1(\mathbb{N}_0, h) \subset l^2(\mathbb{N}_0, h)$, the Plancherel isomorphism extends the Gelfand transform (1.14); we will use the same notation for both transforms.

The inverse Fourier transform $L^1(S, \pi^R) \to c_0(\mathbb{N}_0)$ is defined by

$$\check{g}(n) = \int_S g(\alpha) R_n(\alpha) d\pi^R(\alpha).$$

Since $S = \text{supp } \pi^R$ is compact, $C(S) \subset L^1(S, \pi^R)$. Thus for all $f \in l^1(\mathbb{N}_0, h)$, $\hat{f} \in L^1(S, \pi^R)$ and $f = (\hat{f})\check{}$ by Theorem 1.4.

Example: Jacobi Polynomial Hypergroups. For $\alpha, \beta > -1$ the Jacobi polynomials $(P_n^{(\alpha,\beta)})_n$ are orthogonal with respect to the measure

$$d\pi^{(\alpha,\beta)}(x) = C_{(\alpha,\beta)}(1-x)^\alpha(1+x)^\beta dx \quad \text{ on supp } \pi^{(\alpha,\beta)} = [-1, 1],$$

where $C_{(\alpha,\beta)}$ is a constant such that $\pi^{(\alpha,\beta)}$ is a probability measure. The normalization $R_n(1) = 1$, $n \in \mathbb{N}_0$, is possible, see for example [2, Equ. (3)]. Let $a = \alpha+\beta+1,\ b = \alpha-\beta$ and

$$V = \{(\alpha, \beta) : \ \alpha \geq \beta > -1,\ a(a+5)(a+3)^2 \geq (a^2 - 7a - 24)b^2\}. \tag{1.15}$$

If $(\alpha, \beta) \in V$, then all $g(m, n, k) \geq 0$ in (1.9), see [18, Thm. 1]. So for $(\alpha, \beta) \in V$ the Jacobi polynomials $(P_n^{(\alpha,\beta)})_n$ induce a polynomial hypergroup; they were among the first examples of polynomial hypergroups in [29].

1.3 Excursion: The Class of Signed Polynomial Hypergroups

The scope of this part is to introduce the notion of signed polynomial hypergroups which we need in Chapter 3. Signed hypergroups generalize the concept of hypergroups mainly by weakening the assumption of positivity of the convolution of point measures. In the literature several slightly different definitions of (general) signed hypergroups coexist. We will neither give the definitions nor work out the differences; for this the reader is for example referred to [45], [46] or [39] and the survey [47]. Instead, we give a rather simple definition restricted to the special case of signed polynomial hypergroups and note that these signed polynomial hypergroups are signed hypergroups in the sense of all the definitions cited above. Since signed polynomial hypergroups are not the primary focus of this thesis, we confine ourselves to a description of the differences to the case of polynomial hypergroups and note the slightly generalized notions concerning their harmonic analysis. Signed polynomial hypergroups have been considered in [14].

As in the case of polynomial hypergroups we begin by considering a sequence $\{R_n\}_{n\in\mathbb{N}_0}$ of real polynomials fulfilling (1.6)-(1.8). Instead of positive linearization we assume that the linearization coefficients in (1.9) obey

$$\sum_{k=|n-m|}^{n+m} |g(m,n,k)| \leq M \quad \text{for } n,m\in\mathbb{N}_0 \text{ and some } M>0. \tag{1.16}$$

Denoting by ε_k the point measure at $k\in\mathbb{N}_0$ we define the convolution $\omega:\ \mathbb{N}_0\times\mathbb{N}_0\to M^b(\mathbb{N}_0)$ (as in the case of polynomial hypergroups) by letting

$$\omega(n,m)=\varepsilon_n*\varepsilon_m=\sum_{k=|n-m|}^{n+m} g(m,n,k)\varepsilon_k \quad \text{for } n,m\in\mathbb{N}_0.$$

By (1.16), the measures $\varepsilon_n*\varepsilon_m$ are uniformly bounded measures with finite support. Again, the point measure ε_0 is the neutral element for this convolution. We call the set $\mathbb{N}_0$ (endowed with this convolution and the identity mapping as involution) a *signed polynomial hypergroup*. We note that signed polynomial hypergroups are (commutative) signed hypergroup structures on $\mathbb{N}_0$ for the various definitions in [45] and [47].

Polynomial hypergroups are signed polynomial hypergroups with bound $M=1$, since in their case $\sum_{k=|n-m|}^{n+m}|g(m,n,k)|=\sum_{k=|n-m|}^{n+m}g(m,n,k)=1$.

Similar to the case of polynomial hypergroups, the translations are defined by (1.10). Again we obtain a Haar measure by setting $h_n=(\|R_n\|^2_{L^2(\mathbb{R},\pi^R)})^{-1}$. It is normalized such that $h_0=1$. Note that $0<h_n<1$ is possible for signed polynomial hypergroups.

Next we endow $l^1(\mathbb{N}_0,h)$ with the norm

$$\|f\|_1=M\cdot\sum_{n=0}^{\infty}|f(n)|h_n.$$

Together with (1.16) this definition ensures that the convolution defined by (1.11) turns $l^1(\mathbb{N}_0,h)$ into a Banach algebra. In fact,

$$\begin{aligned}
\|f*g\|_1 &= M\cdot\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}T_nf(k)g(k)h_k\right|h_n \leq M\cdot\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=|n-k|}^{n+k}|g(n,k,j)||f(j)||g(k)|h_kh_n\\
&= M\cdot\sum_{k=0}^{\infty}|g(k)|h_k\cdot\sum_{j=0}^{\infty}\sum_{n=|k-j|}^{k+j}|g(n,k,j)|h_n|f(j)|\\
&= M\cdot\sum_{k=0}^{\infty}|g(k)|h_k\cdot\sum_{j=0}^{\infty}\sum_{n=|k-j|}^{k+j}|g(j,k,n)|h_j|f(j)|\\
&\leq M\cdot\sum_{k=0}^{\infty}|g(k)|h_k\cdot\sum_{j=0}^{\infty}Mh_j|f(j)|=\|g\|_1\|f\|_1,
\end{aligned}$$

where we used that $g(n,k,j)h_n = g(j,k,n)h_j$. $l^1(\mathbb{N}_0,h)$ becomes a Banach-$*$-algebra with the involution $f^* = \overline{f}$.

The structure space $D \subset \mathbb{C}$ of $l^1(\mathbb{N}_0,h)$ is now characterized by

$$\begin{aligned} D =& \{z \in \mathbb{C} : \ |R_n(z)| \leq C \ \text{ for all } n \in \mathbb{N}_0 \text{ and some } C > 0\} \\ =& \{z \in \mathbb{C} : \ |R_n(z)| \leq M \ \text{ for all } n \in \mathbb{N}_0\} . \end{aligned}$$

This differs from (1.12) for polynomial hypergroups in M replacing 1 in the second characterization.

Again, the point measures ε_n can be viewed as elements of $l^1(\mathbb{N}_0,h)$: They can be written as $\varepsilon_n = \frac{\delta_n}{h_n}$, where $\delta_n \in l^1(\mathbb{N}_0,h)$ is the sequence $\delta_n := (\delta_{nk})_{k\in\mathbb{N}_0}$ peaking at n. But the norm of the point measures in $l^1(\mathbb{N}_0,h)$ need not equal one anymore: $\|\varepsilon_n\|_1 = M \cdot \frac{1}{h_n} h_n = M$. In particular, the norm of the unit ε_0 does not equal 1 in general. Observe that still $\hat{\varepsilon}_n = R_n|_D$.

Example: Jacobi Signed Polynomial Hypergroups. As in Chapter 1.2 above we consider the Jacobi polynomials $(P_n^{(\alpha,\beta)})_n$ for $\alpha, \beta > -1$. Let

$$W = \{(\alpha,\beta) : \ \alpha \geq \beta > -1, \ \alpha \geq -\frac{1}{2}\}. \tag{1.17}$$

If $(\alpha,\beta) \in W$, then the coefficients in (1.9) fulfill (1.16), see [2, Thm. 1]. So for $(\alpha,\beta) \in W$ the Jacobi polynomials $(P_n^{(\alpha,\beta)})_n$ induce a signed polynomial hypergroup. Furthermore, if $(\alpha,\beta) \in W \setminus V$, V as in (1.15), then the induced signed polynomial hypergroup is not a hypergroup, see [18, Thm. 1].

2 Amenability of $l^1(h)$ and related properties

Amenability and weak amenability of the l^1-algebras of polynomial hypergroups have been studied in [32]. Results include that if the Haar weight tends to infinity then $l^1(h)$ is not amenable, as well as a sufficient condition for the existence of an approximate diagonal. Furthermore, if the l^1-algebra is induced by the Chebyshev polynomials of the first kind it is amenable, if it is induced by the ultraspherical polynomials with parameter greater than zero it is not even weakly amenable. The α-amenability of l^1-algebras of polynomial hypergroups has been studied in [16, 33]. Amenability and related properties of Banach algebras are for example treated extensively in the monographs [48, 10].

In Chapter 2.1 we consider amenability of $l^1(h)$ by studying the possible forms of approximate diagonals for $l^1(h)$. Making use of a simple form, we obtain sufficient conditions on the growth of the Haar weight $(h_n)_n$ for $l^1(h)$ to be amenable.

In Chapter 2.2 we first treat weak amenability of $l^1(h)$. We give two characterizations of weak amenability, one of them by dropping an assumption on the special approximate diagonals characterizing amenability of $l^1(h)$ in Proposition 2.9. Afterwards we consider the α-amenability of a general commutative Banach algebra A. We touch on the relation of α-amenability to amenability on the one hand and to $\Delta(A)$ being discrete on the other hand.

Unfortunately, the results on amenability and weak amenability remain rather theoretical; we have not been able to provide examples yet. Methods already providing examples are discussed in Chapter 3.2.

We start with some notions which we need in order to define amenability properties [48, 10].

Definition 2.1. *Let A be a commutative Banach algebra. A Banach space X is called a Banach-A-bimodule if there are two bilinear maps $A \times X \to X,\ (a,x) \mapsto ax$, and $A \times X \to X,\ (a,x) \mapsto xa$, such that*

$$a(bx) = (ab)x, \quad (xa)b = x(ab) \quad \text{and} \quad a(xb) = (ax)b, \qquad a,b \in A,\ x \in X,$$

and such that there is $\kappa > 0$ with

$$\|ax\| \leq \kappa \|a\|\, \|x\| \quad \text{and} \quad \|xa\| \leq \kappa \|a\|\, \|x\|, \qquad a \in A,\ x \in X.$$

X is called symmetric if

$$ax = xa, \qquad a \in A,\ x \in X.$$

X^* with left and right module operations defined as follows is called a dual Banach-A-bimodule:

$$\langle ax^*, x\rangle := \langle x^*, xa\rangle \quad \text{and} \quad \langle x^*a, x\rangle := \langle x^*, ax\rangle\,, \qquad a \in A,\ x \in X,\ x^* \in X^*.$$

Furthermore, a linear map $D:\ A \to X$ is called a derivation if

$$D(ab) = a(Db) + (Da)b, \qquad a,b \in A.$$

A derivation is called inner if for some $x_0 \in X$ it is of the form

$$D_{x_0}(a) = ax_0 - x_0a, \qquad a \in A.$$

Before turning to amenability in Chapter 2.1 we set down some notation for operations on the projective tensor product.

Definition 2.2. *Let A be a commutative Banach algebra. By $A\hat{\otimes}A$ we denote the projective tensor product which is the completion of $A \otimes A$ with respect to the projective norm*

$$\|m\|_\pi := \inf\left\{\sum_{i=1}^{j} \|a_1^{(i)}\| \cdot \|a_2^{(i)}\| : \; m = \sum_{i=1}^{j} a_1^{(i)} \otimes a_2^{(i)}\right\}, \qquad m \in A \otimes A.$$

By $\cdot : A\hat{\otimes}A \times A \to A\hat{\otimes}A$ we denote the left module action of the projective tensor product which is determined by $(a\otimes b)\cdot c = a\otimes(bc)$. Analogously we define the corresponding right module action of the Banach-A-bimodule $A\hat{\otimes}A$.
By $\cdot : A\hat{\otimes}A \times A\hat{\otimes}A \to A\hat{\otimes}A$ we denote the algebra product on $A\hat{\otimes}A$ determined by $(a\otimes b)\cdot(c\otimes d) = (ac)\otimes(bd)$.
By π_A we denote the bounded linear map $\pi_A : A\hat{\otimes}A \to A$ determined by $\pi_A(a\otimes b) := ab$.

In the following we consider the case $A = l^1(h)$. We adopt an easier way of looking at the projective tensor product $l^1(h)\hat{\otimes}l^1(h)$ and the operations thereon. To that end we note that $\mathbb{N}_0 \times \mathbb{N}_0$ is a hypergroup in the canonical way of [7, 1.5.28]. In fact, the convolution

$$\omega((n,m),(k,l)) = \varepsilon_{(n,m)} * \varepsilon_{(k,l)} := (\varepsilon_n * \varepsilon_k) \otimes (\varepsilon_m * \varepsilon_l)$$

together with the identity mapping as involution and unit $(0,0)$ defines a hypergroup structure on $\mathbb{N}_0 \times \mathbb{N}_0$. The Haar measure H thus is $H(n,m) = h(n)h(m)$. In particular, $l^1(\mathbb{N}_0\times\mathbb{N}_0, H)$ is a Banach algebra with respect to the corresponding convolution with unit ε_{00}. The Gelfand transform restricted to the support of the Plancherel measure $S\times S \subset \mathbb{R}^2$ reads $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H) \to C_0(S\times S)$,

$$\widehat{\sum_{n,m} \alpha_{nm}\varepsilon_{nm}}(x,y) = \sum_{n,m} \alpha_{nm} R_n(x) R_m(y), \quad x,y \in S;$$

in particular it obeys $\widehat{\varepsilon_{nm}}(x,y) = R_n(x)R_m(y),\ x,y \in S$.

In the following lemma we state the well-known fact that, as in the case of locally compact groups, $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ can be identified with the projective tensor product $l^1(h)\hat{\otimes}l^1(h)$.

Lemma 2.3. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup. The map*

$$I : \; l^1(h)\hat{\otimes}l^1(h) \to l^1(\mathbb{N}_0 \times \mathbb{N}_0, H), \qquad \text{uniquely determined by } I(\varepsilon_n \otimes \varepsilon_m) := \varepsilon_{nm},$$

is an isometric isomorphism of Banach algebras.

What do the other concepts (beside the algebra product) of Definition 2.2 look like, if $l^1(h)\hat{\otimes}l^1(h)$ is viewed as $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$? In [32] Lasser uses, but does not state the following simple identifications.

Lemma 2.4. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup. Then the following hold:*

a) The induced left module action $\cdot :\ l^1(\mathbb{N}_0 \times \mathbb{N}_0, H) \times l^1(h) \to l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ reads

$$\widehat{g \cdot f}(x,y) = \hat{g}(x,y)\hat{f}(y), \quad x, y \in S;$$

analogously for the right module action.

b) The induced map $\pi_{l^1(h)} = \pi :\ l^1(\mathbb{N}_0 \times \mathbb{N}_0, H) \to l^1(h)$ reads

$$\widehat{\pi(h)}(x) = \hat{h}(x,x), \quad x \in S.$$

Proof. Let I be as in Lemma 2.3. For a) we calculate

$$\begin{aligned} \varepsilon_{nm} \cdot \varepsilon_k &:= I\left((I^{-1}\varepsilon_{nm}) \cdot \varepsilon_k\right) = I\left((\varepsilon_n \otimes \varepsilon_m) \cdot \varepsilon_k\right) = I\left(\varepsilon_n \otimes (\varepsilon_m * \varepsilon_k)\right) \\ &= I\left(\sum_{j=|m-k|}^{m+k} g(m,k,j)\varepsilon_n \otimes \varepsilon_j\right) = \sum_{j=|m-k|}^{m+k} g(m,k,j)\varepsilon_{nj} \end{aligned}$$

and further for $x, y \in S$:

$$\widehat{\varepsilon_{nm} \cdot \varepsilon_k}(x,y) = \sum_{j=|m-k|}^{m+k} g(m,k,j)R_n(x)R_j(y) = R_n(x)R_m(y)R_k(y) = \widehat{\varepsilon}_{nm}(x,y)\widehat{\varepsilon}_k(y).$$

Since $(\varepsilon_{nm})_{n,m}$ is total in $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ and $(\varepsilon_k)_k$ is total in $l^1(h)$ the statement follows. For b) we have

$$\pi(\varepsilon_{nm}) := \pi I^{-1}(\varepsilon_{nm}) = \pi(\varepsilon_n \otimes \varepsilon_m) = \varepsilon_n * \varepsilon_m.$$

Furthermore, for $x \in S$:

$$\widehat{\pi(\varepsilon_{nm})}(x) = \widehat{\varepsilon_n * \varepsilon_m}(x) = R_n(x)R_m(x) = \widehat{\varepsilon_{nm}}(x,x)$$

and $(\varepsilon_{nm})_{n,m}$ is total in $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$. □

2.1 Amenability of $l^1(h)$

In this chapter we consider amenability of $l^1(h)$. First we define amenability of a commutative Banach algebra A and state the well-known characterization of amenability by the existence of approximate diagonals. Thereafter we study the possible forms of approximate diagonals for $l^1(h)$. Making use of a simple form we obtain sufficient conditions on the growth of the Haar weight $(h_n)_n$ for $l^1(h)$ to be amenable.

Definition 2.5. *Let A be a (commutative) Banach algebra. A is called amenable if every bounded derivation from A into a dual Banach-A-bimodule is inner.*

An approximate diagonal for A is a bounded net $(m_i)_{i \in I}$ in $A \hat{\otimes} A$ such that, for each $a \in A$, we have $\lim_i (m_i \cdot a - a \cdot m_i) = 0$ and $\lim_i \pi_A(m_i)a = a$.

The following theorem can, for example, be found in [10, Thm. 2.9.65].

Theorem 2.6. *Let A be a (commutative) Banach algebra. Then the existence of an approximate diagonal for A is equivalent to the amenability of A.*

Definition 2.7. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup induced by the orthogonal polynomials $(R_n)_{n\in\mathbb{N}_0}$. For the remainder of this chapter we reserve the symbol f for the following particular element:*

$$f \in \ ker\ \pi_{l^1(h)} \subset l^1(\mathbb{N}_0 \times \mathbb{N}_0, H), \quad f := \varepsilon_{1,0} - \varepsilon_{0,1},$$

with $\hat{f}(x,y) = R_1(x) - R_1(y),\ x, y \in S$.

Lemma 2.8. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup. Let $\mathcal{P}$ denote the set of inverse Fourier transforms of all polynomials on the support of the Plancherel measure $S \times S \subset \mathbb{R}^2$. Then $\mathcal{P} * f$ is dense in ker $\pi_{l^1(h)}$.*

Proof. At first we note that by [10, (2.1.16)]

$$\ker\ \pi_{l^1(h)} = \overline{\text{lin}}\{a \otimes b - \varepsilon_0 \otimes (a * b) :\ a, b \in l^1(h)\}.$$

Since $\overline{\text{lin}}\{\varepsilon_k :\ k \in \mathbb{N}_0\} = l^1(h)$, we obtain

$$\ker\ \pi_{l^1(h)} = \overline{\text{lin}}\{\varepsilon_l \otimes \varepsilon_k - \varepsilon_0 \otimes (\varepsilon_l * \varepsilon_k) :\ l, k \in \mathbb{N}_0\} \text{ which reads}$$

$$\ker\ \pi_{l^1(h)} = \overline{\text{lin}}\{\varepsilon_{lk} - \sum_{j=|k-l|}^{k+l} g(k,l,j)\varepsilon_{0j} :\ l, k \in \mathbb{N}_0\}$$

in our notation on $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$. Now the Gelfand transforms equal

$$\begin{aligned}\mathcal{F}\left(\varepsilon_{lk} - \sum_{j=|k-l|}^{k+l} g(k,l,j)\varepsilon_{0j}\right) &= R_l(\cdot)R_k(\times) - \sum_{j=|k-l|}^{k+l} g(k,l,j)R_j(\times) \\ &= (R_l(\cdot) - R_l(\times))R_k(\times)\end{aligned}$$

and thus the inverse Fourier transforms of lin $\{(R_l(\cdot) - R_l(\times))R_k(\times) :\ l, k \in \mathbb{N}_0\}$ are dense in ker $\pi_{l^1(h)}$. Next we note that, if $R_l(x) = \sum_{k=0}^{l} d_k x^k$, then

$$\begin{aligned}R_l(x) - R_l(y) &= \sum_{k=0}^{l} d_k(x^k - y^k) = \sum_{k=1}^{l} d_k \sum_{n=0}^{k-1} x^n y^{k-1-n} \cdot (x-y) \\ &= \sum_{k=1}^{l} d_k \sum_{n=0}^{k-1} x^n y^{k-1-n} a_0 (R_1(x) - R_1(y)) =: \sigma_l(x,y)(R_1(x) - R_1(y)).\end{aligned}$$

Since now there are polynomials σ_l such that $R_l(\cdot) - R_l(\times) = \sigma_l(\cdot,\times)(R_1(\cdot) - R_1(\times)) = \sigma_l(\cdot,\times)\hat{f}(\cdot,\times)$, we also know that $\mathcal{P} * f$ is dense in ker $\pi_{l^1(h)}$. □

The following is similar to [32, Thm. 4]: We only switched to our notation and wrote the second condition with an equality instead of a limit. However, a proof different from the one in [32, Thm. 4] is given which uses Lemma 2.8. In the following, Lemma 2.8 will be the basis of our approach (also to weak amenability in the next chapter).

Proposition 2.9. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup. The Banach algebra $l^1(h)$ is amenable if and only if there is a bounded sequence $(m_n)_{n\in\mathbb{N}_0} \subset l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ such that $\lim_n m_n * f = 0$ and $\pi_{l^1(h)}(m_n) = \varepsilon_0$ for all n. The sequence $(m_n)_n$ then is an approximate diagonal.*

Proof. Assume that $l^1(h)$ is amenable. Then by Theorem 2.6 there is a bounded net $(u_i)_{i\in I} \subset l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ such that, for each $a \in l^1(h)$, we have $\lim_i(u_i \cdot a - a \cdot u_i) = 0$ and $\lim_i \pi_{l^1(h)}(u_i)a = a$. For $a = \varepsilon_0$ choose $j \in I$ such that $\|\pi_{l^1(h)}(u_i) - \varepsilon_0\|_1 < \varepsilon$ for $i \geq j$. Let

$$m_i := \pi_{l^1(h)}(u_i)^{-1} \cdot u_i, \qquad i \geq j.$$

Then we have $\pi_{l^1(h)}(m_i) = \varepsilon_0$ by Lemma 2.4:

$$\widehat{\pi_{l^1(h)}(m_i)}(x) = \hat{m}_i(x,x) = 1 = \hat{\varepsilon}_0(x), \qquad \text{for all } x \in S,$$

and for $g \in l^1(h)$ it follows from $\hat{g}|_S = 0$ that $g = 0$, see Theorem 1.4. Using again Lemma 2.4 and Theorem 1.4 (for $G \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ it follows from $\hat{G}|_{S\times S} = 0$ that $G = 0$), we obtain that $m_i * (\varepsilon_{10} - \varepsilon_{01}) = \varepsilon_1 \cdot m_i - m_i \cdot \varepsilon_1$:

$$\mathcal{F}(m_i * (\varepsilon_{10} - \varepsilon_{01}))(x,y) = \hat{m}_i(x,y)(R_1(x) - R_1(y)) = \mathcal{F}(\varepsilon_1 \cdot m_i - m_i \cdot \varepsilon_1)(x,y).$$

Furthermore, since $\pi_{l^1(h)}(u_i)^{-1} \to \varepsilon_0$ we see that

$$\begin{aligned} m_i * f = m_i * (\varepsilon_{10} - \varepsilon_{01}) &= \varepsilon_1 \cdot m_i - m_i \cdot \varepsilon_1 = \varepsilon_1 \cdot (\pi_{l^1(h)}(u_i)^{-1} \cdot u_i) - (\pi_{l^1(h)}(u_i)^{-1} \cdot u_i) \cdot \varepsilon_1 \\ &= \pi_{l^1(h)}(u_i)^{-1} \cdot (\varepsilon_1 \cdot u_i - u_i \cdot \varepsilon_1) \to 0. \end{aligned}$$

Picking a subsequence of the net $(m_i)_{i\geq j}$ we have found a sequence $(m_n)_n$ as demanded in the statement.
Now conversely suppose that there is a sequence $(m_n)_n$ as demanded in the statement; we show that it is an approximate diagonal meeting the requirements in Theorem 2.6. By definition, $(m_n)_n$ is bounded and for each $a \in l^1(h)$, $\lim_n \pi_{l^1(h)}(m_n)a = \lim_n a = a$. Now let $g = \sum_{j\in\mathbb{N}_0} g_j\varepsilon_j \in l^1(h)$. We have to show that $\lim_n(m_n \cdot g - g \cdot m_n) = 0$. Define $G := \sum_{k\in\mathbb{N}_0} \sum_{l\in\mathbb{N}_0} g_k\varepsilon_{0k} - g_l\varepsilon_{l0} \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$. We obtain $m_n \cdot g - g \cdot m_n = m_n * G$ since

$$m_n \cdot \widehat{g - g} \cdot m_n(x,y) = \hat{m}_n(x,y)(\hat{g}(y) - \hat{g}(x)) = \hat{m}_n(x,y)\hat{G}(x,y) = \widehat{m_n * G}(x,y).$$

Furthermore, $G \in \ker \pi_{l^1(h)}$ because $\hat{G}(x,x) = \hat{g}(x) - \hat{g}(x) = 0$. Let C be the uniform bound $\|m_n\| \leq C$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. By Lemma 2.8 we can choose a polynomial p_G such that $\|G - \check{p}_G * f\| < \frac{\varepsilon}{2C}$. Now choose N such that $\|m_n * f\| < \frac{\varepsilon}{2\|\check{p}_G\|}$ for all $n \geq N$. Then for all $n \geq N$,

$$\begin{aligned} \|m_n \cdot g - g \cdot m_n\| = \|m_n * G\| &\leq \|m_n * (G - \check{p}_G * f)\| + \|m_n * \check{p}_G * f\| \\ &\leq C \cdot \frac{\varepsilon}{2C} + \|\check{p}_G\| \cdot \frac{\varepsilon}{2\|\check{p}_G\|} = \varepsilon. \end{aligned}$$

So the sequence $(m_n)_n$ is an approximate diagonal for $l^1(h)$. $\square$

We look for sequences $(m_n)_{n\in\mathbb{N}_0} \subset l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ meeting the requirements of Proposition 2.9. Clearly $\lim_i \hat{m}_i(x,y) = 0$ for $x \neq y \in S$ and $\hat{m}_i(x,x) = 1$. The following proposition shows that we can choose a *symmetric* sequence $(m_n)_n$, i.e., $\hat{m}_n(x,y) = \hat{m}_n(y,x)$ for $x, y \in S$; this statement is not used in the sequel, but of independent interest.

Proposition 2.10. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup. If there is an approximate diagonal for $l^1(h)$, then there also exists an approximate diagonal $(m_n)_{n\in\mathbb{N}_0}$ such that $\hat{m}_n$ is a symmetric polynomial and $1 - \hat{m}_n = p_n\hat{f}$; here p_n is an antisymmetric polynomial, i.e., $\hat{p}_n(x,y) = -\hat{p}_n(y,x)$ for $x, y \in S$, with $\pi_{l^1(h)}(p_n) = 0$ and $\check{p}_n(k,k) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Let $(u_n)_{n\in\mathbb{N}_0}$ be an approximate diagonal for $l^1(h)$ of the form in Proposition 2.9. Let $(\varepsilon_n)_{n\in\mathbb{N}_0}$ be a sequence of small numbers tending to zero. Since $\varepsilon_{00} - u_n \in \ker \pi_{l^1(h)}$ we can choose a sequence of polynomials $(g_n)_{n\in\mathbb{N}_0}$ such that $\|\varepsilon_{00} - u_n - \check{g}_n * f\|_1 < \varepsilon_n$ for all $n \in \mathbb{N}_0$ by Lemma 2.8. Define

$$\hat{m}_n(x,y) = 1 - \frac{1}{2}(g_n(x,y) - g_n(y,x))(R_1(x) - R_1(y)) =: 1 - p_n\hat{f}(x,y).$$

Then $\hat{m}_n(x,y) = \hat{m}_n(y,x)$ and thus $\hat{m}_n$ is a symmetric polynomial. Moreover, p_n is an antisymmetric polynomial with $p_n(x,y) = -p_n(y,x)$, in particular $p_n(x,x) = 0$, and

$$\begin{aligned}\check{p}_n(k,k) &= \int_S\int_S p_n(x,y)R_k(x)R_k(y)d\pi(x)d\pi(y)\\ &=\frac{1}{2}\int_S\int_S g_n(x,y)R_k(x)R_k(y)d\pi(x)d\pi(y) - \frac{1}{2}\int_S\int_S g_n(y,x)R_k(x)R_k(y)d\pi(x)d\pi(y) = 0.\end{aligned}$$

Furthermore, $(m_n)_{n\in\mathbb{N}_0}$ is an approximate diagonal for $l^1(h)$ by Proposition 2.9: $\hat{m}_n(x,x) = 1$ for all $x \in S$, $n \in \mathbb{N}_0$, which means $\pi_{l^1(h)}(m_n) = \varepsilon_0$ for all $n \in \mathbb{N}_0$. Concerning the second condition we first define the isometric linear map $l^1(\mathbb{N}_0\times\mathbb{N}_0, H) \to l^1(\mathbb{N}_0\times\mathbb{N}_0, H)$, $a \mapsto a^S$, by $\hat{a}^S(x,y) = \hat{a}(y,x)$ which means $a^S(k,l) = a(l,k)$. Then from $u_n * f \to 0$ it follows that $u_n^S * f = -u_n^S * f^S = -(u_n * f)^S \to 0$. Furthermore, from $\|\varepsilon_{00} - u_n - \check{g}_n * f\|_1 \to 0$ it follows that $\|\varepsilon_{00}^S - u_n^S - (\check{g}_n * f)^S\|_1 \to 0$. Since $(\check{g}_n * f)^S = \check{g}_n^S * f^S = -\check{g}_n^S * f$ we obtain $\|\varepsilon_{00} - u_n^S + \check{g}_n^S * f\|_1 \to 0$. Thus,

$$\begin{aligned}&\|m_n * f\|_1 = \|(\varepsilon_{00} - \frac{1}{2}(\check{g}_n - \check{g}_n^S) * f) * f\|_1\\ \leq\ &\frac{1}{2}\|\varepsilon_{00} - \check{g}_n * f - u_n\|_1\|f\|_1 + \frac{1}{2}\|u_n * f\|_1 + \frac{1}{2}\|\varepsilon_{00} + \check{g}_n^S * f - u_n^S\|_1\|f\|_1 + \frac{1}{2}\|u_n^S * f\|_1,\end{aligned}$$

which tends towards zero. □

In the following we look for candidates for approximate diagonals. To that end we need the following notion: A polynomial hypergroup has property (H) if

$$\lim_{n\to\infty} \frac{h_n}{\sum_{k=0}^n h_k} = 0.$$

This property is extensively used in the literature, see for example [33, 22]. Note that a polynomial hypergroup fulfilling property (H) is of subexponential growth and thus the three dual objects coincide ([57, Pro. 2.6 and Rem 2.7] and [59, Thm. 2.17]), i.e., supp $\pi = D$.

Now we define candidates $(m_n)_n$ for approximate diagonals. The reader may want to compare m_n to the elements β_n used in [16, Ch. 4] in the context of x-amenability: $\hat{\beta}_n(y) = \hat{m}_n(x,y)$.

Definition 2.11. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup fulfilling property (H) which is induced by the orthogonal polynomials $(R_n)_{n\in\mathbb{N}_0}$. For $n \in \mathbb{N}_0$ define $v_n \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ by*

$$v_n = \left(\sum_{k=0}^{n} h_k\right)^{-1} \sum_{k=0}^{n} \varepsilon_{kk} h_k \quad \textit{with} \quad \hat{v}_n(x,y) = \left(\sum_{k=0}^{n} h_k\right)^{-1} \sum_{k=0}^{n} R_k(x)R_k(y)h_k,$$

and $w_n \in l^1(h)$ by

$$w_n = \left(\sum_{k=0}^{n} h_k\right)^{-1} \sum_{k=0}^{n} \varepsilon_k * \varepsilon_k h_k \quad \textit{with} \quad \hat{w}_n(x) = \left(\sum_{k=0}^{n} h_k\right)^{-1} \sum_{k=0}^{n} R_k(x)^2 h_k. \tag{2.1}$$

Since $\hat{w}_n|_{supp\ \pi} = \hat{w}_n|_D > 0$, w_n is invertible. We define $m_n \in l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ by

$$m_n = v_n \cdot w_n^{-1} \qquad \textit{with} \qquad \hat{m}_n(x,y) = \frac{\hat{v}_n(x,y)}{\hat{v}_n(y,y)}. \tag{2.2}$$

Considering $\hat{v}_n$ and $\hat{w}_n$, we will need the Christoffel-Darboux formula [52, (3.2.3)]:

$$\sum_{k=0}^{n} R_k(x)R_k(y)h_k = a_0 a_n h_n \frac{R_{n+1}(y)R_n(x) - R_n(y)R_{n+1}(x)}{R_1(y) - R_1(x)}. \tag{2.3}$$

For the use of property (H) in the following proposition also note that by [32, Thm. 3] $h_n \to \infty$ implies that $l^1(h)$ is not amenable.

Proposition 2.12. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup fulfilling property (H). If the sequence $(w_n)_{n\in\mathbb{N}_0} \subset l^1(h)$ defined in (2.1) is boundedly invertible, then $l^1(h)$ is amenable with approximate diagonal $(m_n)_{n\in\mathbb{N}_0}$ defined in (2.2).*

Proof. Define $(m_n)_{n\in\mathbb{N}_0}$ according to (2.2). We immediately see that $\hat{m}_n(x,x) = 1$ for all $x \in S$ and thus $\pi_{l^1(h)}(m_n) = \varepsilon_0$. Furthermore, $\|m_n\|_1 \leq \|v_n\|_1 \|w_n^{-1}\|_1 = \|w_n^{-1}\|_1 \leq C$ is bounded for $n \in \mathbb{N}_0$. In order to apply Proposition 2.9 we still have to consider $\|m_n * f\|_1 \leq \|v_n * f\|_1 \|w_n^{-1}\|_1$. By the Christoffel-Darboux formula (2.3) we know that

$$\begin{aligned} -\hat{v}_n \cdot \hat{f}(x,y) &= \left(\sum_{k=0}^{n} h_k\right)^{-1} \sum_{k=0}^{n} R_k(x)R_k(y)h_k(R_1(y) - R_1(x)) \\ &= \left(\sum_{k=0}^{n} h_k\right)^{-1} a_0 a_n h_n (R_{n+1}(y)R_n(x) - R_n(y)R_{n+1}(x)), \end{aligned}$$

and thus $\|v_n * f\|_1 \leq \left(\sum_{k=0}^{n} h_k\right)^{-1} \cdot 2a_0 a_n h_n$. Since the hypergroup fulfills property (H), $\|v_n * f\|_1 \to 0$. □

Corollary 2.13. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup fulfilling property (H). If for some $0 < \varepsilon < 1$ we have $\sum_{k=0}^{n} h_k \leq \frac{2}{1+\varepsilon}(n+1)$ for all but finitely many $n \in \mathbb{N}_0$, then $l^1(h)$ is amenable.*

Proof. We let $C_n := \sum_{k=0}^{n} h_k$ and calculate

$$\begin{aligned}
w_n - \varepsilon_0 &= \frac{1}{C_n}\left(\sum_{k=0}^{n} \varepsilon_k^2 h_k - C_n\varepsilon_0\right) \\
&= \frac{1}{C_n}\left((1-C_n)\varepsilon_0 + \sum_{k=1}^{n} h_k \sum_{l=0}^{2k} g(k,k,l)\varepsilon_l\right) \\
&= \frac{1}{C_n}\left((1-C_n)\varepsilon_0 + \sum_{l=1}^{2n} \sum_{k=\lceil \frac{l}{2} \rceil}^{n} g(k,k,l)\varepsilon_l h_k + \sum_{k=1}^{n} h_k g(k,k,0)\varepsilon_0\right) \\
&= \frac{1}{C_n}\left((1-C_n+n)\varepsilon_0 + \sum_{l=1}^{2n} \sum_{k=\lceil \frac{l}{2} \rceil}^{n} g(k,k,l)\varepsilon_l h_k\right),
\end{aligned}$$

where we used $h_k^{-1} = g(k,k,0)$ for the forth equality. Thus we can calculate

$$\begin{aligned}
\|w_n - \varepsilon_0\|_1 &= \frac{1}{C_n}\left(|1-C_n+n| + \sum_{l=1}^{2n} \sum_{k=\lceil \frac{l}{2} \rceil}^{n} g(k,k,l) h_k\right) \\
&= \frac{1}{C_n}\left(C_n - n - 1 + \sum_{k=1}^{n} \sum_{l=1}^{2k} g(k,k,l) h_k\right) \\
&= \frac{1}{C_n}\left(C_n - n - 1 + \sum_{k=1}^{n} h_k(1-g(k,k,0))\right) \\
&= \frac{1}{C_n}\left(C_n - n - 1 + C_n - 1 - n\right) \\
&= \frac{2}{C_n}\left(C_n - n - 1\right) \\
&\leq 1 - \varepsilon.
\end{aligned}$$

Thus we can estimate $\|w_n^{-1}\|_1 = \|\varepsilon_0 + \sum_{k=1}^{\infty}(\varepsilon_0 - w_n)^k\|_1 \leq \frac{1}{1-\|w_n-\varepsilon_0\|_1} \leq \varepsilon^{-1}$. □

The norms $\|w_n^{-1}\|_1$ in Proposition 2.12 are very hard to estimate without an explicit inversion. In the above corollary we used the Neumann series which imposes an additional condition $\|w_n - \varepsilon_0\|_1 < 1$ and thus does not yield a very satisfying result: Even the hypergroup induced by the Chebyshev polynomials of the first kind whose Haar measure is constant (and where amenability of its l^1-algebra is already known) does not allow for an application.

2.2 Related properties

In this chapter we first treat weak amenability of $l^1(h)$: We give two characterizations, one of them by dropping an assumption on the special approximate diagonals characterizing amenability of $l^1(h)$ in Proposition 2.9.

Afterwards we consider the α-amenability of a general commutative Banach algebra A. We touch on the relation of α-amenability to amenability on the one hand and to $\Delta(A)$ being discrete on the other hand.

Definition 2.14. *Let A be a commutative Banach algebra. A is weakly amenable if every bounded derivation from A into the dual Banach-A-bimodule $A^\star$ is inner.*

Thus, amenability of A implies weak amenability. The following can be found in [10, Thm. 2.8.73].

Proposition 2.15 (Grønbæk)**.** *Let A be a commutative, unital Banach algebra. Then $(ker\ \pi_A)^2 =\ ker\ \pi_A$ is equivalent to the weak amenability of A.*

Definition 2.16. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup and let $f \in\ ker\ \pi_{l^1(h)} \subset l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ as in Definition 2.7. We say that $l^1(h)$ fulfills property (W) if there exist approximate units for f in ker $\pi_{l^1(h)}$, i.e. for all $\varepsilon > 0$ there is some $u_\varepsilon \in ker\ \pi_{l^1(h)}$ such that $\|f - u_\varepsilon f\|_1 \leq \varepsilon$.*

Proposition 2.17. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup. $l^1(h)$ is weakly amenable if and only if it fulfills property (W).*

Proof. Suppose that $l^1(h)$ is weakly amenable and let $\varepsilon > 0$. By Proposition 2.15 we can find $v, w \in$ ker $\pi_{l^1(h)}$ with $\|f - v * w\|_1 < \frac{\varepsilon}{2}$. Now $v, w \in$ ker $\pi_{l^1(h)}$ and by Lemma 2.8 we can find polynomials $\hat{p}_v, \hat{p}_w$ such that $\|v*w-p_v*p_w*f*f\|_1 < \frac{\varepsilon}{2}$. So $\|f-(p_v*p_w*f)*f\|_1 < \varepsilon$ and $p_v * p_w * f \in$ ker $\pi_{l^1(h)}$ since $f \in$ ker $\pi_{l^1(h)}$.
Now suppose conversely that $l^1(h)$ fulfills property (W). For an application of Proposition 2.15 let $v \in$ ker $\pi_{l^1(h)}$ and $\varepsilon > 0$. We use Lemma 2.8 to choose a polynomial $\hat{p}$ such that $\|v - p * f\|_1 < \frac{\varepsilon}{2}$ and afterwards choose $u \in$ ker $\pi_{l^1(h)}$ such that $\|f - u * f\|_1 < \frac{\varepsilon}{2\|p\|_1}$. Then $p * f$ and u are in ker $\pi_{l^1(h)}$ and

$$\begin{aligned}\|v - (p * f) * u\|_1 &\leq \|v - p * f\|_1 + \|p * f - p * u * f\|_1 \\ &< \frac{\varepsilon}{2} + \|p\|_1 \frac{\varepsilon}{2\|p\|_1} \leq \varepsilon.\end{aligned}$$

Proposition 2.15 tells us that $l^1(h)$ is weakly amenable. □

Now we can characterize weak amenability in a way similar to approximate diagonals characterizing amenability. In fact, the sequence $(m_n)_{n\in\mathbb{N}_0}$ in the following proposition need not be an approximate diagonal since it is not required to be bounded; compare Proposition 2.9.

Proposition 2.18. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup. The Banach algebra $l^1(h)$ is weakly amenable if and only if there is a sequence $(m_n)_{n\in\mathbb{N}_0} \subset l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ such that $\lim_{n\to\infty} m_n * f = 0$ and $\pi_{l^1(h)}(m_n) = \varepsilon_0$ for all n.*

Proof. First we note that $\pi_{l^1(h)}(\varepsilon_{00}) = \varepsilon_0$ by Lemma 2.4:

$$\widehat{\pi_{l^1(h)}(\varepsilon_{00})}(x) = \hat{\varepsilon}_{00}(x,x) = 1 = \hat{\varepsilon}_0(x), \qquad \text{for all } x \in S,$$

and for $g \in l^1(h)$ it follows from $\hat{g}|_S = 0$ that $g = 0$, see Theorem 1.4. Now suppose that $l^1(h)$ is weakly amenable and thus fulfills property (W). Defining $m_n = \varepsilon_{00} - u_{\frac{1}{n}}$ we immediately obtain $\lim_{n\to\infty} m_n * f = \lim_{n\to\infty} f - u_{\frac{1}{n}} * f = 0$ and $\pi_{l^1(h)}(m_n) = \pi_{l^1(h)}(\varepsilon_{00}) - 0 = \varepsilon_0$ for all $n \in \mathbb{N}_0$.
Conversely suppose that the sequence $(m_n)_{n\in\mathbb{N}_0}$ in $l^1(\mathbb{N}_0 \times \mathbb{N}_0, H)$ fulfills $\lim_{n\to\infty} m_n * f = 0$ and $\pi_{l^1(h)}(m_n) = \varepsilon_0$ for all n. Defining $u_n = \varepsilon_{00} - m_n$ for all $n \in \mathbb{N}_0$ we obtain an approximate unit for f in ker $\pi_{l^1(h)}$: $\pi_{l^1(h)}(u_n) = \varepsilon_0 - \varepsilon_0 = 0$ and $f - u_n f = f - f + m_n * f \to 0$. □

For the following proposition we proceed similar to Proposition 2.12 and obtain weaker conditions.

Proposition 2.19. *Let $\mathbb{N}_0$ carry the convolution structure of a polynomial hypergroup fulfilling property (H). If the sequence $(w_n)_{n\in\mathbb{N}_0} \subset l^1(h)$ defined in (2.1) fulfills*

$$\left(\sum_{k=0}^{n} h_k\right)^{-1} a_n h_n \cdot \|w_n^{-1}\|_1 \to 0 \quad \text{for} \quad n \to \infty,$$

then $l^1(h)$ is weakly amenable.

Proof. Define $(m_n)_{n\in\mathbb{N}_0}$ according to (2.2). As in Proposition 2.12 we immediately see that $\hat{m}_n(x,x) = 1$ for all $x \in S$ and thus $\pi_{l^1(h)}(m_n) = \varepsilon_0$. Furthermore, in order to apply Proposition 2.18 we still have to consider $\|m_n * f\|_1 \le \|v_n * f\|_1 \|w_n^{-1}\|_1$. By the Christoffel-Darboux formula (2.3) we know again that

$$\begin{aligned} -\hat{v}_n \cdot \hat{f}(x,y) &= \left(\sum_{k=0}^{n} h_k\right)^{-1} \sum_{k=0}^{n} R_k(x) R_k(y) h_k (R_1(y) - R_1(x)) \\ &= \left(\sum_{k=0}^{n} h_k\right)^{-1} a_0 a_n h_n (R_{n+1}(y) R_n(x) - R_n(y) R_{n+1}(x)), \end{aligned}$$

and thus $\|v_n * f\|_1 \le \left(\sum_{k=0}^{n} h_k\right)^{-1} \cdot 2a_0 a_n h_n$. According to our assumptions, $\|m_n * f\|_1 \le \|v_n * f\|_1 \|w_n^{-1}\|_1 \to 0$. □

Again we face the problem that $\|w_n^{-1}\|_1$ is very hard to estimate; the approach via the Neumann series analogous to Corollary 2.13 would yield so weak results that we refrain from stating them.

Now we come to an observation on α-amenability.

Definition 2.20. *Let A be a commutative Banach algebra and $\alpha \in \Delta(A)$ a character. A is α-amenable if every bounded derivation from A into a dual Banach-A-bimodule X^α such that $a \cdot x = \langle \alpha, a \rangle \cdot x,\ a \in A,\ x \in X^\alpha$, is inner.*

Thus amenability of A implies α-amenability. The following theorem is extracted from [26, Pro. 2.1 and 2.2].

Theorem 2.21. *Let A be a commutative Banach algebra with bounded approximate identity and let $\alpha \in \Delta(A)$ be a character. Then the existence of a bounded approximate identity for ker α is equivalent to the α-amenability of A.*

In the following Proposition a) $\Rightarrow$ d) is due to Gourdeau [20]; the statement has been sharpened by Ülger to a) $\Rightarrow$ c) in [55, Thm. 2.1]. We inserted point b).

Proposition 2.22. *Let A be a commutative Banach algebra. We consider the following properties:*

a) A is amenable,

b) A is α-amenable for all all characters $\alpha \in \Delta(A)$,

c) $\Delta(A)$ is discrete in the weak topology $\sigma(A^*, A^{**})$,

d) $\Delta(A)$ is discrete in the norm-topology of A^*.

Then a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d).

Proof. a) $\Rightarrow$ b) is clearly valid.
b) $\Rightarrow$ c): Let $\alpha \in \Delta(A)$. First we note the easy fact that for all $\beta \neq \alpha \in \Delta(A)$ there is $g_\beta \in$ ker α with $\hat{g}_\beta(\beta) \neq 0$: Denote by $(e_\lambda)_{\lambda \in \Lambda}$ the bounded approximate identity for A which exists according to [37, Thm. 2.3]. Let $\alpha \neq \beta \in \Delta(A)$ and choose $f \in A$ with $\hat{f}(\alpha) \neq \hat{f}(\beta)$. Let $\varepsilon > 0$ and choose $\lambda \in \Lambda$ such that $|\hat{e}_\lambda(\alpha) - 1| < \varepsilon$ and $|\hat{e}_\lambda(\beta) - 1| < \varepsilon$. Then $g_\beta := f - \frac{\hat{f}(\alpha)}{\hat{e}_\lambda(\alpha)} e_\lambda \in A$ fulfills $\hat{g}_\beta(\alpha) = 0$ and for ε small enough $\hat{g}_\beta(\beta) = \hat{f}(\beta) - \hat{f}(\alpha)\frac{\hat{e}_\lambda(\beta)}{\hat{e}_\lambda(\alpha)} \neq 0$. Now let $\alpha \in \Delta(A)$ and $(u_i)_{i \in I}$ denote a bounded approximate identity of ker α which exists according to Theorem 2.21. From

$$\begin{aligned} |\hat{u}_i(\beta) - 1| \cdot |\hat{g}_\beta(\beta)| &= |\hat{u}_i(\beta)\hat{g}_\beta(\beta) - \hat{g}_\beta(\beta)| = |\widehat{u_i * g_\beta}(\beta) - \hat{g}_\beta(\beta)| \\ &\leq \|u_i * g_\beta - g_\beta\|_A \xrightarrow{i} 0 \end{aligned}$$

it follows that $\hat{u}_i(\beta) \xrightarrow{i} 1$ for all $\beta \in \Delta(A) \setminus \{\alpha\}$. This means that for a weak star cluster point $u \in A^{**}$ of $(u_i)_{i \in I}$ holds $\langle u, \alpha \rangle = 0$ and $\langle u, \beta \rangle = 1$ for all $\beta \in \Delta(A) \setminus \{\alpha\}$. In particular, $\{\alpha\}$ is open with respect to the weak topology $\sigma(\Delta(A), A^{**})$.
c) $\Rightarrow$ d) is clearly valid. $\square$

Note that if b) is fulfilled, i.e. if A is α-amenable for all characters $\alpha \in \Delta(A)$, then A is called *character amenable* in [37].

3 Homomorphisms and Isomorphisms

Isomorphisms of hypergroups have been studied by Bloom and Walter in [8], their main focus lying on isometric isomorphisms. In this chapter we consider non-isometric isomorphisms between l^1-algebras of (signed) polynomial hypergroups; isometric isomorphisms between l^1-algebras are quite rare, due to the fact that the translation operators need not be unitary and the characters not of modulus 1.

The results are formulated for signed polynomial hypergroups which include the class of polynomial hypergroups. Some basics on the l^1-algebras of signed polynomial hypergroups are collected in Chapter 1.3.

The purpose of Chapter 3.1 is to derive sufficient conditions for the existence of homomorphisms and isomorphisms between the l^1-algebras of two (signed) polynomial hypergroups. In Chapter 3.2 the results are applied to transfer amenability and related properties from one l^1-algebra to another. As examples the Bernstein-Szegő polynomials of the first and the second kind, as well as the Jacobi and the Associated Legendre polynomials are considered. In particular, all l^1-algebras w.r.t. Bernstein-Szegő polynomials of the first and the second kind are shown to be isomorphic to the l^1-algebras w.r.t. Chebyshev polynomials of the first and the second kind, respectively. Almost all of this chapter has already been published in [34] with R. Lasser (with the noteworthy exception of the Bernstein-Szegő polynomials of the second kind).

This chapter is the continuation of an investigation started in the author's diploma thesis [40]. In particular, the basic statements Lemma 3.1 and Theorem 3.2 have already been derived there (for polynomial hypergroups). Furthermore, special cases of the results for Bernstein-Szegő polynomials of the first kind as well as Jacobi polynomials in Chapter 3.2 have also been obtained in [40] (using significantly more complicated methods of proof).

3.1 Conditions for the Existence of Homomorphisms and Isomorphisms

The first aim of this chapter is to find a homomorphism between l^1-algebras on signed polynomial hypergroups whose connection coefficients fulfill certain requirements. Afterwards we give conditions such that the constructed homomorphism is an isomorphism.

To avoid confusion we frequently write D^R, h_n^R, ε_k^R etc. stressing the dependence on the polynomial system $\{R_n\}_{n\in\mathbb{N}_0}$ that induces the (signed) polynomial hypergroup.

Lemma 3.1. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be polynomial sequences inducing signed polynomial hypergroups. A bounded linear map $S:\ l^1(\mathbb{N}_0,h^R)\to l^1(\mathbb{N}_0,h^P)$ is a homomorphism of Banach algebras if and only if*

(i) $S\varepsilon_0=\varepsilon_0^P$ and

(ii) $S(\varepsilon_1*\varepsilon_n)=S\varepsilon_1*^P S\varepsilon_n$ for all $n\in\mathbb{N}_0$.

Proof. If $S:\ l^1(\mathbb{N}_0,h^R)\to l^1(\mathbb{N}_0,h^P)$ is a homomorphism of Banach algebras, then (i) and (ii) are clearly valid. We show that these conditions are sufficient. From our two assumptions it immediately follows that $S(\varepsilon_0*\varepsilon_n)=S\varepsilon_n=S\varepsilon_0*^P S\varepsilon_n$ and $S(\varepsilon_1*$

$\varepsilon_n) = S\varepsilon_1 *^P S\varepsilon_n$ for all $n \in \mathbb{N}_0$. Let $k \geq 1$ and suppose as induction hypothesis that $S(\varepsilon_j * \varepsilon_n) = S\varepsilon_j *^P S\varepsilon_n$ for all $0 \leq j \leq k$ and $n \in \mathbb{N}_0$. First we obtain

$$\begin{aligned} S(\varepsilon_1 * \varepsilon_k * \varepsilon_n) &= a_n S(\varepsilon_k * \varepsilon_{n+1}) + b_n S(\varepsilon_k * \varepsilon_n) + c_n S(\varepsilon_k * \varepsilon_{n-1}) \\ &= a_n S\varepsilon_k *^P S\varepsilon_{n+1} + b_n S\varepsilon_k *^P S\varepsilon_n + c_n S\varepsilon_k *^P S\varepsilon_{n-1} \\ &= S\varepsilon_k *^P S(\varepsilon_1 * \varepsilon_n) = S\varepsilon_1 *^P S\varepsilon_k *^P S\varepsilon_n \\ &= S(\varepsilon_1 * \varepsilon_k) *^P S\varepsilon_n \end{aligned}$$

and furthermore

$$\begin{aligned} S\varepsilon_{k+1} *^P S\varepsilon_n &= \frac{1}{a_k} \cdot S(\varepsilon_1 * \varepsilon_k - b_k\varepsilon_k - c_k\varepsilon_{k-1}) *^P S\varepsilon_n \\ &= \frac{1}{a_k} \cdot S\left((\varepsilon_1 * \varepsilon_k - b_k\varepsilon_k - c_k\varepsilon_{k-1}) * \varepsilon_n\right) = S(\varepsilon_{k+1} * \varepsilon_n). \end{aligned}$$

Hence we have shown that $S(\varepsilon_k * \varepsilon_n) = S\varepsilon_k *^P S\varepsilon_n$ for all $k, n \in \mathbb{N}_0$. Since S is assumed to be bounded, for $v, w \in l^1(\mathbb{N}_0, h^R)$, $v = \sum_{k=0}^\infty v_k\varepsilon_k,\ w = \sum_{n=0}^\infty w_n\varepsilon_n$, it follows that

$$S(v * w) = \sum_{k=0}^\infty \sum_{n=0}^\infty v_k w_n S(\varepsilon_k * \varepsilon_n) = \sum_{k=0}^\infty \sum_{n=0}^\infty v_k w_n S\varepsilon_k *^P S\varepsilon_n = Sv *^P Sw.$$

□

Given a family of orthogonal polynomials $\{P_k\}_{k\in\mathbb{N}_0}$ and a polynomial R_n of degree n we consider the linear combination $R_n = \sum_{k=0}^n c_{nk}P_k$ with the so-called connection coefficients (c_{nk}). In all of the following we define $c_{nk} = 0$ for $k > n$ which enables us to write $R_n = \sum_{k=0}^\infty c_{nk}P_k$.

The following theorem is more general than [28, Thm. 3.1], since we do not impose conditions on the dual objects.

Theorem 3.2. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be families of orthogonal polynomials inducing signed polynomial hypergroups such that $R_n = \sum_{k=0}^n c_{nk}P_k$. If there is $C > 0$ such that*

$$\sum_{k=0}^n |c_{nk}| \leq C \quad \text{for all } n \in \mathbb{N}_0, \tag{3.1}$$

then the linear operator $S:\ l^1(\mathbb{N}_0, h^R) \to l^1(\mathbb{N}_0, h^P)$ determined by $S\varepsilon_n := \sum_{k=0}^n c_{nk}\varepsilon_k^P$ is a continuous homomorphism of Banach algebras with dense range. Furthermore, $D^P \subseteq D^R$ and $\widehat{Sf} = \hat{f}|_{D^P}$ for all $f \in l^1(\mathbb{N}_0, h^R)$.

Proof. Let $v = \sum_{n=0}^N v_n\varepsilon_n \in l^1(\mathbb{N}_0, h^R)$; then $\|v\|_1 = M^R \sum_{n=0}^N |v_n|$, where M^R is the constant of the signed polynomial hypergroup. By

$$\|Sv\| = \left\| \sum_{n=0}^N \sum_{k=0}^n v_n c_{nk}\varepsilon_k^P \right\| \leq M^P \sum_{n=0}^N \sum_{k=0}^n |v_n||c_{nk}| \leq \frac{M^P}{M^R}\|v\|_1 \cdot C$$

the linear map S is bounded on a dense subset of $l^1(\mathbb{N}_0, h^R)$. Thus it can be uniquely extended to a bounded linear operator on $l^1(\mathbb{N}_0, h^R)$. Condition (3.1) implies $D^P \subseteq D^R$

for the following reason: If $z \in \mathbb{C}$ such that $|P_k(z)| \leq M^P$ for all $k \in \mathbb{N}_0$, we obtain $|R_n(z)| = |\sum_{k=0}^n c_{nk} P_k(z)| \leq \sum_{k=0}^n |c_{nk}| \cdot M^P \leq C \cdot M^P$ for all $n \in \mathbb{N}_0$. (i) in Lemma 3.1 is fulfilled since $R_0 = 1 = P_0$, which means $c_{00} = 1$ and $S\varepsilon_0 = \varepsilon_0^P$. Now consider (ii) in Lemma 3.1. For $n \in \mathbb{N}_0$ we observe that $\widehat{S\varepsilon_n} = \sum_{k=0}^n c_{nk} \hat{\varepsilon}_k^P = \sum_{k=0}^n c_{nk} P_k|_{D^P} = R_n|_{D^P}$. Therefore

$$\begin{aligned} S\widehat{(\varepsilon_1 * \varepsilon_n)} &= a_n \widehat{S\varepsilon_{n+1}} + b_n \widehat{S\varepsilon_n} + c_n \widehat{S\varepsilon_{n-1}} = a_n R_{n+1}|_{D^P} + b_n R_n|_{D^P} + c_n R_{n-1}|_{D^P} \\ &= (R_1 \cdot R_n)|_{D^P} = R_1|_{D^P} \cdot R_n|_{D^P} = \widehat{S\varepsilon_1} \cdot \widehat{S\varepsilon_n} \end{aligned}$$

and $S(\varepsilon_1 * \varepsilon_n) = S\varepsilon_1 *^P S\varepsilon_n$ for all $n \in \mathbb{N}_0$, since $l^1(\mathbb{N}_0, h^P)$ is semisimple. Thus S is a continuous homomorphism of Banach algebras. S has dense range since the polynomials are dense in $A(D^P)$. Finally, for $f = \sum_{k=0}^\infty f_k \varepsilon_k \in l^1(\mathbb{N}_0, h^R)$ it follows that $\widehat{Sf} = \lim_{N\to\infty} \widehat{S\left(\sum_{k=0}^N f_k \varepsilon_k\right)} = \lim_{N\to\infty} \sum_{k=0}^N f_k R_n|_{D^P} = \hat{f}|_{D^P}$, where the limits are w.r.t. $\| \ \|_{D^P,\infty}$. □

Remark 3.3. (i) Condition (3.1) is in particular fulfilled when the connection coefficients are nonnegative; in this case our normalization yields that $\sum_{k=0}^n |c_{nk}| = \sum_{k=0}^n c_{nk} = 1$, $n \in \mathbb{N}_0$. The non-negativity of connection coefficients has for example been studied by Askey and Gasper in [1], Szwarc in [53], Trench in [54] or Wilson in [60].
(ii) If $\operatorname{supp} \pi^R \subset D^P$, then S is injective. In fact, in this case it follows from $\widehat{Sf} = \hat{f}|_{D^P} = 0$ that $\hat{f}|_{\operatorname{supp} \pi^R} = 0$. This is only possible if $f = 0$.

Definition 3.4. *A commutative Banach algebra A is called regular, if for every closed subset V of $\Delta(A)$ and $\alpha \in \Delta(A) \setminus V$ there is $a \in A$ with Gelfand transform $\hat{a}|_V = 0$ and $\hat{a}(\alpha) \neq 0$.*

Note that $l^1(\mathbb{N}_0, h)$ is regular whenever the Haar measure h is of polynomial growth, see [57, 2.8] or [17, Thm. 2.1].

Proposition 3.5. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be families of orthogonal polynomials inducing signed polynomial hypergroups such that (3.1) is fulfilled. Suppose that $l^1(\mathbb{N}_0, h^R)$ is regular. Then S in Theorem 3.2 is injective if and only if $D^P = D^R$. Furthermore $l^1(\mathbb{N}_0, h^P)$ is also regular.*

Proof. Suppose that $D^P \subsetneq D^R$. D^P is closed in D^R. For all $\alpha \in D^R \setminus D^P$ there is $f \in l^1(\mathbb{N}_0, h^R)$ such that $\hat{f}|_{D^P} = 0$ and $\hat{f}(\alpha) \neq 0$. Since $\widehat{Sf} = \hat{f}|_{D^P}$, this means that S is not injective. On the other hand, S is obviously injective for $D^P = D^R$. Now take a closed subset $V \subset D^P$ and $\alpha \in D^P \setminus V$. There is $f \in l^1(\mathbb{N}_0, h^R)$ such that $\hat{f}|_V = 0$ and $\hat{f}(\alpha) \neq 0$. The same is true for $\hat{f}|_{D^P}$, so $l^1(\mathbb{N}_0, h^P)$ is also regular. □

Before turning to isomorphisms let us for a moment consider the semigroup $\mathbb{N}_0$; we show that for each polynomial hypergroup there is a homomorphism $S : l^1(\mathbb{N}_0, 1) \to l^1(\mathbb{N}_0, h^P)$. The induced convolution on $l^1(\mathbb{N}_0, 1)$, where 1 denotes the constant sequence with members one, is determined by $\varepsilon_n * \varepsilon_m = \varepsilon_{n+m}$ for all $n, m \in \mathbb{N}_0$. The structure space $\Delta(l^1(\mathbb{N}_0, 1))$ can be identified with the closed unit disc $\mathbb{D} \subset \mathbb{C}$. The Gelfand transform reads $\mathcal{F} : l^1(\mathbb{N}_0, 1) \to C(\mathbb{D})$, $\hat{v}(x) = \sum_{k=0}^\infty v(k) x^k|_{\mathbb{D}}$ and thus maps $l^1(\mathbb{N}_0, 1)$ onto the space of absolutely convergent Taylor series on $\mathbb{D}$ [10, Example 2.1.13(v)]. In this sense one can say that the semigroup $\mathbb{N}_0$ is induced by the family of polynomials $\{x^n\}_{n\in\mathbb{N}_0}$. The analogue of (1.8) reads $x^1 \cdot x^n = x^{n+1}$, i.e. $a_n = 1$ and $b_n = c_n = 0$ for all $n \in \mathbb{N}_0$.

Corollary 3.6. *Let $\{P_n\}_{n\in\mathbb{N}_0}$ be a family of orthogonal polynomials inducing a polynomial hypergroup. Let furthermore $l^1(\mathbb{N}_0,1)$ carry the convolution structure of the semigroup $\mathbb{N}_0$. Then $S:\ l^1(\mathbb{N}_0,1)\to l^1(\mathbb{N}_0,h^P)$ determined by $\widehat{Sf}=\hat{f}|_{D^P}$ is a continuous homomorphism of Banach algebras with dense range.*

Proof. Lemma 3.1 and Theorem 3.2 also hold true if one replaces $\{R_n\}_{n\in\mathbb{N}_0}$ by $\{x^n\}_{n\in\mathbb{N}_0}$, $l^1(\mathbb{N}_0,h^R)$ by $l^1(\mathbb{N}_0,1)$ and D^R by $\mathbb{D}$; the proofs are exactly the same ones. The connection coefficients in

$$x^n=\sum_{k=0}^{n} d_{nk}P_k \tag{3.2}$$

are all nonnegative: First, we calculate the connection coefficients of $P_1^n=\sum_{k=0}^n c_{nk}P_k$. $P_1^2=a_1P_2+b_1P_1+c_1P_0$ and using the recurrence

$$\begin{aligned}
&c_{n+1,0}=c_{n1}c_1^P,\ c_{n+1,1}=c_{n2}c_2^P+c_{n1}b_1^P+c_{n0},\\
&c_{n+1,k}=c_{n,k+1}c_{k+1}^P+c_{n,k}b_k^P+c_{n,k-1}a_{k-1}^P, \qquad\qquad 2\le k\le n-1,\\
&c_{n+1,n}=c_{n,n}b_n^P+c_{n,n-1}a_{n-1}^P,\ c_{n+1,n+1}=c_{nn}a_n^P,
\end{aligned}$$

the non-negativity of c_{nk} follows by induction. Now from $P_1(x)=\frac{1}{a_0^P}(x-b_0^P)$ it follows that

$$x^n=\sum_{l=0}^{n}\binom{n}{l}(a_0^P)^lP_1(x)^l(b_0^P)^{n-l}=\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}(a_0^P)^l(b_0^P)^{n-l}c_{lk}P_k(x)$$

and thus the connection coefficients in (3.2) are all nonnegative. An application of Theorem 3.2 concludes the proof. □

Next we want to obtain sufficient conditions for S of Theorem 3.2 to be an isomorphism. Let us at first consider two families of orthogonal polynomials $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ inducing signed polynomial hypergroups with $d\pi^P=fd\pi^R$ for some $f\in L^2(D^R,d\pi^R)$. Since both measures are probability measures, $f\ge 0$ π^R-a.e. For the representation $R_n=\sum_{k=0}^n c_{nk}P_k$ one gets that

$$\frac{c_{nk}}{h_k^P}=(\mathcal{F}^P)^{-1}R_n(k)=\int_{\mathbb{R}}P_kR_nd\pi^P=\int_{\mathbb{R}}fP_kR_nd\pi^R=(\mathcal{F}^R)^{-1}(f\cdot P_k)(n)$$

for all $k,n\in\mathbb{N}_0$. In particular we obtain for $k=0$ that the Plancherel transform $\check{f}\in l^2(\mathbb{N}_0,h^R)$ reads $\check{f}(n)=c_{n0},\ n\in\mathbb{N}_0$, and thus $\sum_{n=0}^\infty|c_{n0}|^2h_n^R<\infty$. The converse of this observation is also true in the following sense.

Proposition 3.7. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be families of orthogonal polynomials inducing signed polynomial hypergroups such that $R_n=\sum_{k=0}^n c_{nk}P_k$. Suppose that*

$$\sum_{n=0}^{\infty}|c_{n0}|^2h_n^R<\infty$$

and define $f\in l^2(\mathbb{N}_0,h^R)$ by $f(n)=c_{n0},\ n\in\mathbb{N}_0$. Then *(i)* $(\mathcal{F}^R)^{-1}(\hat{f}\cdot P_k)(n) = \frac{c_{nk}}{h_k^P} = (\mathcal{F}^P)^{-1}R_n(k)$ *for all* $k,n\in\mathbb{N}_0$ *and*

(ii) $d\pi^P = \hat{f}d\pi^R$. *In particular,* $\hat{f}\geq 0$ π^R*-a.e. and supp* $\pi^P \subset$ *supp* π^R.

Proof. Note that $\hat{f}\in L^2(D^R,d\pi^R)\subset L^1(D^R,d\pi^R)$. In order to prove the first statement we have to show that $\frac{c_{nk}}{h_k^P} = \int_{D^P} P_kR_n d\pi^P = \int_{D^R}\hat{f}P_kR_nd\pi^R$ for all $k,n\in\mathbb{N}_0$. Fix $n\in\mathbb{N}_0$. For $k=0$ the equality holds true by definition of f. For $k\in\mathbb{N}_0$ we know that

$$\begin{aligned}\int_{D^R}\hat{f}R_kR_nd\pi^R &= \sum_{j=|k-n|}^{|k+n|} g^R(k,n,j)\int_{D^R}\hat{f}P_0R_jd\pi^R = \sum_{j=|k-n|}^{|k+n|} g^R(k,n,j)c_{j0}\\ &= \sum_{j=|k-n|}^{|k+n|} g^R(k,n,j)\int_{D^P}P_0R_jd\pi^P = \int_{D^P}R_kR_nd\pi^P. \end{aligned}\tag{3.3}$$

Writing $P_m = \sum_{n=0}^k d_{mk}R_k$ it follows that for all $m\in\mathbb{N}_0$,

$$\begin{aligned}\int_{D^R}\hat{f}P_mR_nd\pi^R &= \sum_{k=0}^m d_{mk}\int_{D^R}\hat{f}R_kR_nd\pi^R = \sum_{k=0}^m d_{mk}\int_{D^P}R_kR_nd\pi^P\\ &= \int_{D^P}P_mR_nd\pi^P = \frac{c_{nm}}{h_m^P}.\end{aligned}$$

For the second statement consider the compact set supp $\pi^R\cup$ supp $\pi^P\subset\mathbb{R}$. From the case $k=0$ in (3.3) we obtain that the bounded Borel measures $d\pi^P$ and $\hat{f}d\pi^R$ coincide on the dense subset span $(\{R_n\})$ of $C($supp $\pi^R\cup$ supp $\pi^P)$. This means they have to be equal. □

Theorem 3.8. *Let* $\{R_n\}_{n\in\mathbb{N}_0}$ *and* $\{P_n\}_{n\in\mathbb{N}_0}$ *be families of orthogonal polynomials inducing signed polynomial hypergroups such that* $R_n = \sum_{k=0}^n c_{nk}P_k$. *Suppose that*

(i) $\sum_{k=0}^n |c_{nk}|\leq C$ *for all* $n\in\mathbb{N}_0$,

(ii) $\sum_{n=k}^\infty |c_{nk}|\frac{h_n^R}{h_k^P}\leq\tilde{C}$ *for all* $k\in\mathbb{N}_0$,

(iii) the function $f\in l^1(\mathbb{N}_0,h^R)\subset l^2(\mathbb{N}_0,h^R)$ *defined by* $f(n)=c_{n0}$, $n\in\mathbb{N}_0$, *fulfills* $\hat{f}>0$ *on* D^R *(by Proposition 3.7 we know that* $d\pi^P=\hat{f}d\pi^R$, *where* $\hat{f}$ *is continuous and* $\hat{f}|_{supp\ \pi^R}\geq 0$*).*

Then the operator $S:\ l^1(\mathbb{N}_0,h^R)\to l^1(\mathbb{N}_0,h^P)$, *uniquely determined by* $S\varepsilon_n = \sum_{k=0}^n c_{nk}\varepsilon_k^P$ *(as in Theorem 3.2) is an isomorphism of Banach algebras. Furthermore,* $D^P = D^R$ *and* $\widehat{S(g)} = \hat{g}$ *for all* $g\in l^1(\mathbb{N}_0,h^R)$.

Proof. Applying Theorem 3.2 it suffices to show that there is a constant $C>0$ such that for $P_n = \sum_{k=0}^n d_{nk}R_k$ there holds $\sum_{k=0}^n|d_{nk}| = \frac{1}{M^R}\|(\mathcal{F}^R)^{-1}P_k\|_1\leq C$ for all $n\in\mathbb{N}_0$. Since $\hat{f}>0$, f is invertible in $l^1(\mathbb{N}_0,h^R)$ and $(\|(\mathcal{F}^R)^{-1}P_k\|_1)_{k\in N_0}$ is bounded if and only if $(\|(\mathcal{F}^R)^{-1}(\hat{f}\cdot P_k)\|_1)_{k\in N_0}$ is bounded. By Proposition 3.7 and our second assumption we obtain

$$\frac{1}{M^R}\|(\mathcal{F}^R)^{-1}(\hat{f}\cdot P_k)\|_1 = \sum_{n=0}^{\infty} |(\mathcal{F}^R)^{-1}(\hat{f}\cdot P_k)(n)|h_n^R = \sum_{n=k}^{\infty} |c_{nk}|\frac{h_n^R}{h_k^P} \leq \tilde{C} \text{ for all } k\in\mathbb{N}_0.$$

□

In the case of hypergroups the isomorphism above is isometric if and only if $\hat{f}=1$, i.e. in the trivial case. In fact, it follows from [8, Thm. 4.4] that an isometric isomorphism $l^1(\mathbb{N}_0,h^R)\to l^1(\mathbb{N}_0,h^P)$ maps point measures onto point measures. For our class of isomorphisms this is only possible if $R_n=P_n$ for all $n\in\mathbb{N}_0$.

Let $\{T_n\}_{n\in\mathbb{N}_0}$ be the Chebyshev polynomials of the first kind. There is an abundance of polynomial sequences $\{R_n\}_{n\in\mathbb{N}_0}$ such that the connection coefficients $R_n=\sum_{k=0}^n c_{nk}T_k,\ n\in\mathbb{N}_0$, are non-negative, i.e., the assumption of Theorem 3.2 and the first condition of Theorem 3.8 are fulfilled. For instance, this is true for all polynomial hypergroups where the limits $a=\lim_{n\to\infty}a_n$ and $c=\lim_{n\to\infty}c_n$ of the coefficients in (1.8) exist with $a,c>0$ (their orthogonalization measure is of Nevai class $M(0,1)$), see [35, Ch. 2].

3.2 Application to Amenability-properties

Now we apply the constructed class of homomorphisms to transfer amenability and related properties from one l^1-algebra to another. These properties are usually hard to verify directly, whereas the approach via inheritance under homomorphisms is a practicable alternative. For more references on amenability-properties, in particular with regard to hypergroups, see Chapter 2.

Proposition 3.9. *Let A and B be Banach algebras, and let $\theta:\ A\to B$ be a continuous homomorphism with $\overline{\theta(A)}=B$.*

(i) Suppose that A is amenable. Then B is amenable.

(ii) Suppose that A is commutative and weakly amenable. Then B is weakly amenable.

(iii) Suppose that A is commutative and let $\alpha\in\Delta(B)$. Suppose further that A is $\theta^*\alpha$-amenable. Then B is α-amenable.

For (i) and (ii) see for example [10, Proposition 2.8.64]. A proof of (iii) can be found in [26, Proposition 3.5].

We can now use the homomorphism of Theorem 3.2 to apply Proposition 3.9. In the first two examples we will use the isomorphism of Theorem 3.8 (and subsequently apply Proposition 3.9). Note that for both examples we have supp $\pi^R=D^R$, since the hypergroups are of polynomial growth, see [57, Pro. 2.6 and Rem 2.7] and [59, Thm. 2.17]. In [32, Cor. 3] Lasser has shown that $l^1(\mathbb{N}_0,h^T)$ is amenable, where $\{T_n\}_{n\in\mathbb{N}_0}$ are the Chebyshev polynomials of the first kind. Up to now this was the only example of a polynomial hypergroup with an amenable $l^1(\mathbb{N}_0, h)$.

(i) Bernstein-Szegő polynomials of the first kind: We consider a polynomial $H: \mathbb{C} \to \mathbb{C},\ H(z) = \sum_{k=0}^{r} \alpha_k z^k$, of degree $r \geq 1$, with real coefficients $\alpha_k,\ 0 \leq k \leq r$. We assume that H has no zero for $|z| \leq 1$ and $H(0) > 0$. Define

$$\rho: \ [-1,1] \to \mathbb{R},\ \rho(\cos t) := |H(e^{it})|^2.$$

ρ is strictly positive in $[-1,1]$. The Bernstein-Szegő polynomials of the first kind $\{B_n^\rho\}_{n\in\mathbb{N}_0}$ are defined as the ones orthogonal with respect to the probability measure π^ρ on $[-1,1]$, where

$$d\pi^\rho := c_\rho \cdot \rho(x)^{-1}(1-x^2)^{-\frac{1}{2}}dx = c_\rho \cdot \rho(x)^{-1} d\pi^T.$$

The Chebyshev measure of the first kind π^T has density $c_\rho^{-1} \cdot \rho > 0$ w.r.t. the Bernstein-Szegő measure of the first kind and thus (iii) of Theorem 3.8 is fulfilled with respect to the Bernstein-Szegő polynomials of the first kind $\{B_n^\rho\}_{n\in\mathbb{N}_0}$. Furthermore, it is stated in [52, Chapter 2.6], or more explicitly in [23], that

$$B_n^\rho = \sum_{k=0}^{r} \alpha_k T_{n-k} \quad \text{for } n \geq r.$$

So (i) of Theorem 3.8 is fulfilled. Using this representation it is shown in [14] that for every ρ the Bernstein-Szegő polynomials of the first kind $\{B_n^\rho\}_{n\in\mathbb{N}_0}$ induce a signed polynomial hypergroup. Furthermore, $h_n^\rho = \text{const.}$ for $n \geq r$ [23] and $h_k^T = 2^{-1}$ which yields (ii) of Theorem 3.8. Thus, we can apply Theorem 3.8. In [32, Cor. 3] it is shown that $l^1(\mathbb{N}_0, h^T)$ is amenable, so Proposition 3.9(i) yields the following corollary.

Corollary 3.10. *For every admissible ρ, $l^1(\mathbb{N}_0, h^{B^\rho})$ w.r.t. Bernstein-Szegő polynomials of the first kind is isomorphic to $l^1(\mathbb{N}_0, h^T)$ w.r.t. Chebyshev polynomials of the first kind. In particular, $l^1(\mathbb{N}_0, h^{B^\rho})$ is amenable.*

Special cases of the Bernstein-Szegő polynomials of the first kind for $\rho(x) = 1 - \mu x^2$ are the Grinspun polynomials [29, 3.(g)(ii)].

(ii) Bernstein-Szegő polynomials of the second kind: As for the Bernstein-Szegő polynomials of the first kind we consider a polynomial $H: \ \mathbb{C} \to \mathbb{C},\ H(z) = \sum_{k=0}^{r} \alpha_k z^k$, of degree $r \geq 1$, with real coefficients $\alpha_k,\ 0 \leq k \leq r$. We assume that H has no zero for $|z| \leq 1$ and $H(0) > 0$. Again we define

$$\rho: \ [-1,1] \to \mathbb{R},\ \rho(\cos t) := |H(e^{it})|^2.$$

ρ is strictly positive in $[-1,1]$. The Bernstein-Szegő polynomials of the second kind $\{C_n^\rho\}_{n\in\mathbb{N}_0}$ are defined as the ones orthogonal with respect to the probability measure π^ρ on $[-1,1]$, where

$$d\pi^\rho := c_\rho \cdot \rho(x)^{-1}(1-x^2)^{\frac{1}{2}}dx = c_\rho \cdot \rho(x)^{-1} d\pi^S,$$

see [14, 52]. The Chebyshev measure of the second kind π^S has density $c_\rho^{-1} \cdot \rho > 0$ w.r.t. the Bernstein-Szegő measure of the second kind and thus (iii) of Theorem 3.8 is fulfilled with respect to the Bernstein-Szegő polynomials of the second kind $\{C_n^\rho\}_{n\in\mathbb{N}_0}$. It is shown in [14] that for every ρ the Bernstein-Szegő polynomials of the second kind $\{C_n^\rho\}_{n\in\mathbb{N}_0}$ induce a signed polynomial hypergroup. Furthermore, it is stated in [14] that

$$C_n^\rho = M_n \cdot \sum_{k=0}^{r} \alpha_k \cdot (n-k+1) S_{n-k} \quad \text{for } n \geq r, \quad \text{where } M_n = \left(\sum_{k=0}^{r} \alpha_k \cdot (n-k+1) \right)^{-1}$$

is chosen such that $C_n^\rho(1) = 1$. We show that (i) of Theorem 3.8 is fulfilled: First note that, since $\sum_{k=0}^{r} \alpha_k \neq 0$, there is a constant $M > 0$ such that for n large enough

$$\left| \sum_{k=0}^{r} \alpha_k \cdot (n-k+1) \right| \geq \left| n \left| \sum_{k=0}^{r} \alpha_k \right| - \left| \sum_{k=0}^{r} \alpha_k (k-1) \right| \right| \geq n \cdot M.$$

Let $L := \sum_{k=0}^{r} |\alpha_k|$. Then

$$\sum_{l=0}^{n} |c_{nl}| = \frac{\sum_{k=0}^{r} |\alpha_k| \cdot (n-k+1)}{\left|\sum_{k=0}^{r} \alpha_k \cdot (n-k+1)\right|} \leq \frac{(n+1) \cdot L}{n \cdot M}$$

is uniformly bounded in n. For (ii) of Theorem 3.8 we first note that $h_n^\rho = 2 \cdot (c_\rho \pi M_n^2)^{-1}$ for $n \geq r$ [23] and that $h_k^S = (k+1)^2$. Since $c_{nk} = M_n \alpha_{n-k}(k+1)$ we obtain that

$$\begin{aligned} \sum_{n=k}^{\infty} |c_{nk}| \frac{h_n^\rho}{h_k^S} &= \frac{2}{c_\rho \pi} \frac{1}{k+1} \sum_{n=k}^{k+r} |M_n^{-1} \alpha_{n-k}| = \frac{2}{c_\rho \pi} \frac{1}{k+1} \sum_{n=k}^{k+r} \left| \sum_{l=0}^{r} \alpha_l \cdot (n-l+1) \right| |\alpha_{n-k}| \\ &\leq \frac{2}{c_\rho \pi} \frac{1}{k+1} \sum_{n=k}^{k+r} (n+1) \cdot L \cdot |\alpha_{n-k}| \leq \frac{2}{c_\rho \pi} \frac{1}{k+1} (k+r+1) \cdot L^2 \end{aligned}$$

is uniformly bounded in k. Thus, we can apply Theorem 3.8. In [32, Cor. 1] $l^1(\mathbb{N}_0, h^S) = l^1(\mathbb{N}_0, h^{P^{(\frac{1}{2},\frac{1}{2})}})$ has been shown to be not weakly amenable, where $\{P_n^{(\alpha,\alpha)}\}_{n\in\mathbb{N}_0}$ are ultraspherical polynomials. Now Proposition 3.9(i) yields the following corollary.

Corollary 3.11. *For every admissible ρ, $l^1(\mathbb{N}_0, h^{C^\rho})$ w.r.t. Bernstein-Szegő polynomials of the second kind is isomorphic to $l^1(\mathbb{N}_0, h^S)$ w.r.t. Chebyshev polynomials of the second kind. In particular, $l^1(\mathbb{N}_0, h^{C^\rho})$ is not weakly amenable.*

By [14] special cases of Bernstein-Szegő polynomials of the second kind for $r = 2$ include the Geronimus, Cartier and generalized Soardi polynomials [7, 3.3.15, 3.3.20 and 3.3.36].

(iii) Jacobi polynomials: For $(\alpha, \beta) \in W$, $W = \{(\alpha,\beta) : \ \alpha \geq \beta > -1, \ \alpha \geq -\frac{1}{2}\}$ as in (1.17), the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}_0}$ induce a signed polynomial hypergroup.

In [32, Cor. 1] $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ has been shown to be not weakly amenable, where $\{P_n^{(\alpha,\alpha)}\}_{n\in\mathbb{N}_0}$ are ultraspherical polynomials and $\alpha \geq 0$. Via Theorem 3.2 we can transfer this property to a large region of parameters of Jacobi polynomials.

Corollary 3.12. *The Banach algebra $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta)}})$ w.r.t. Jacobi polynomials is not weakly amenable whenever $\alpha \geq 0$.*

Proof. In [32, Cor. 1] it has been shown that $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ is not weakly amenable for $\alpha \geq 0$. By [5, Theorem 2] we get that the connection coefficients in $P_n^{(\alpha,\beta)} = \sum_{k=0}^n c_{nk} P_k^{(\alpha,\alpha)}$ fulfill the requirements of Theorem 3.2 for $\beta < \alpha$. Proposition 3.9(ii) yields that $l^1(\mathbb{N}_0, P^{(\alpha,\beta)})$ is not weakly amenable whenever $\alpha \geq 0,\ (\alpha,\beta) \in W$. □

Stefan Kahler has recently shown that $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ is weakly amenable whenever $-\frac{1}{2} < \alpha < 0$; his results have not been published yet. From [32, Thm. 3] it follows that $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ is not amenable for those α.

In [16, Example 4.6] it is shown that $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ is not x-amenable for $x \in (-1,1)$ and all $\alpha > -\frac{1}{2}$. This property is also inherited by $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta)}})$:

Corollary 3.13. *The Banach algebra $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta)}})$ w.r.t. Jacobi polynomials is not x-amenable for all $x \in (-1,1)$ whenever $\alpha > -\frac{1}{2}$.*

Proof. In [16, Example 4.6] it is shown that $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ is not x-amenable for $x \in (-1,1)$ for all $\alpha > -\frac{1}{2}$. By [5, Theorem 2] we get that the connection coefficients in $P_n^{(\alpha,\beta)} = \sum_{k=0}^n c_{nk} P_k^{(\alpha,\alpha)}$ fulfill the requirements of Theorem 3.2 for $\beta < \alpha$. Proposition 3.9(iii) yields that $l^1(\mathbb{N}_0, P^{(\alpha,\beta)})$ is not x-amenable for all $x \in (-1,1),\ (\alpha,\beta) \in W$. □

Actually, this result has already been shown in [16, Example 4.6] using different techniques. Notwithstanding the above, $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ is (-1)-amenable for all α, whereas $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta)}})$ lacks this property whenever $\alpha \neq \beta$, see [16, Example 4.6].

<u>(iv) Associated Legendre polynomials:</u> The associated Legendre polynomials $\{P_n^\nu\}_{n\in\mathbb{N}_0}$ are orthogonal on $[-1,1]$ w.r.t. $d\pi^\nu = |{}_2F_1(\frac{1}{2},\nu;\nu+\frac{1}{2};\exp(2i\arccos x))|^{-2}dx$ (${}_2F_1$ is the customary notation for the hypergeometric series). They define polynomial hypergroups whenever $\nu \geq 0$, see [30, Thm. 3.1]. For $\nu = 0$ we obtain the classical Legendre polynomials $(P_n^{(0,0)})_n$ whose l^1-algebra is not weakly amenable by Corollary 3.12. The connection coefficients in

$$P_n^\nu = \sum_{k=0}^n c_{nk} P_n^0$$

are non-negative, see [28]. Therefore, we can apply Theorem 3.2 and Proposition 3.9(ii) to obtain the following corollary:

Corollary 3.14. *The Banach algebra $l^1(\mathbb{N}_0, h^{P^\nu})$ w.r.t. Associated Legendre polynomials is not weakly amenable for all $\nu \geq 0$.*

4 Spectra of L^1-convolution operators

In this chapter we consider the spectra of the convolution operators

$$T_f = T_{f,p}: \; L^p(K) \to L^p(K), \; T_f(g) = f * g, \tag{4.1}$$

for $f \in L^1(K)$ on commutative hypergroups K. Since $\|f * g\|_p \leq \|f\|_1 \|g\|_p$, as remarked after (1.2), the convolution operators T_f are bounded. We are interested in how, for fixed $f \in L^1(K)$, the spectra $\sigma_p(T_f)$ of $T_{f,p}$ vary with p. The results contained in this chapter have already been published in [41].

In Chapter 4.1 we obtain that for any commutative hypergroup K and $f \in L^1(K)$, the inclusion $\sigma_q(T_f) \subseteq \sigma_p(T_f)$ is true whenever $p \leq q \leq 2$ or $2 \leq q \leq p$. Furthermore, we show that this result is sharp in the sense that there is a commutative hypergroup and an element $f \in L^1(K)$, such that $\sigma_q(T_f) \subsetneq \sigma_p(T_f)$ whenever $q > p$, $q, p \in [1, 2]$. In fact, in Chapter 4.3 we will calculate the spectra $\sigma_p(T_{\varepsilon_1})$, $p \in [1, \infty]$, for the family of Karlin-McGregor polynomial hypergroups and the generating element ε_1 of $l^1(h)$; these spectra turn out to fulfill the above strict inclusions. This once more illustrates the significant difference between abelian locally compact groups and commutative hypergroups: For abelian locally compact groups the spectra $\sigma_p(T_f)$ coincide for all $p \in [1, \infty]$, which is a consequence of the following theorem [4, Thm. 6], since abelian locally compact groups are always amenable and symmetric.

Theorem 4.1 (Barnes). *For a locally compact group G, the following are equivalent;*

(i) for all $f \in L^1(G)$, $\sigma_{L^1(G)}(f) = \sigma_2(T_f)$.

(ii) for all $f \in L^1(G)$, $\sigma_p(T_f)$ is independent of $p \in [1, \infty]$.

(iii) for all $f \in L^1(G)$ with $f = f^*$, $r_{L^1(G)}(f) = r_2(T_f)$ (r denotes the spectral radius).

(iv) $L^1(G)$ is symmetric and G is amenable.

Although the spectra $\sigma_p(T_f)$ do not generally coincide for commutative hypergroups, we can characterize those commutative hypergroups for which $\sigma_p(T_f)$ is independent of $p \in [1, \infty]$. In fact, in Chapter 4.2 we prove that, for a commutative hypergroup K, $\sigma_p(T_f)$ is independent of $p \in [1, \infty]$ for all $f \in L^1(K)$ exactly when the Plancherel measure is supported on the whole character space $\chi_b(K)$, i.e., if the three dual objects $\chi_b(K), \hat{K}$ and S (see (1.3) and following) of the commutative hypergroup K coincide.

Moreover, we can formulate our characterization similar to Barnes' Theorem 4.1 above: $\sigma_p(T_f)$ is independent of $p \in [1, \infty]$ for all $f \in L^1(K)$ exactly when $L^1(K)$ is symmetric and for every $\alpha \in \hat{K}$ Reiter's condition P_2 (defined in (1.5)) holds true. For groups, Barnes' assumption of amenability is equivalent to Reiter's condition P_2 (in $\alpha \equiv 1$) [42, Thm. 6.14]. For hypergroups, however, the various properties which characterize amenability (including the P_2-condition) in the group case are not equivalent, see for example [15, 16, 32] and Chapter 2.

4.1 Inclusion relations of p-spectra of L^1-convolution operators

This chapter is mainly devoted to proving that if $p \le q \le 2$ or $2 \le q \le p$, then $\sigma_q(T_f) \subseteq \sigma_p(T_f)$ for all $f \in L^1(K)$, which is formulated as Proposition 4.6. We begin with some basic facts about L^1-convolution operators.

In the following, for $g \in L^p(K)$ and $h \in L^{p'}(K)$, $\frac{1}{p}+\frac{1}{p'}=1$, we write $\langle g,h\rangle = \int_K ghdm$. The scalar product of the Hilbert space $L^2(K)$ is denoted by $\langle g,h\rangle_2 = \int_K g\overline{h}dm$.

Lemma 4.2. *Let $f \in L^1(K)$, $1 \le p < \infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then the adjoint operator fulfills $T_{f,p}^* = T_{\tilde{f},p'}$, where $\tilde{f}$ is given by $\tilde{f}(x) = f(\tilde{x})$. Furthermore, the Hilbert-space adjoint $T_{f,2}^H$ obeys $T_{f,2}^H = T_{\overline{\tilde{f}},2} = T_{f^*,2}$.*

Proof. First we note that for $x, y \in K$,

$$L_y\tilde{f}(\tilde{x}) = \int_K \tilde{f}d\omega(y,\tilde{x}) = \int_K fd\omega(\tilde{y},x) = L_{\tilde{y}}f(x).$$

Let $g \in L^p(K)$ and $h \in L^{p'}(K)$. Using Fubini's theorem we obtain

$$\begin{aligned}
\left\langle T_f^*h, g\right\rangle &= \langle h, T_fg\rangle = \langle h, f*g\rangle = \int_K h(x)f*g(x)dm(x) \\
&= \int_K\int_K h(x)L_{\tilde{y}}f(x)g(y)dm(y)dm(x) \\
&= \int_K\int_K h(x)L_y\tilde{f}(\tilde{x})g(y)dm(x)dm(y) \\
&= \int_K \tilde{f}*h(y)g(y)dm(y) = \left\langle \tilde{f}*h, g\right\rangle. \qquad (4.2)
\end{aligned}$$

Thus $T_{f,p}^* = T_{\tilde{f},p'}$. For the statement concerning the Hilbert-space adjoint T_f^H, we let $g, h \in L^2(K)$ and obtain

$$\begin{aligned}
\left\langle T_f^Hh, g\right\rangle_2 &= \langle h, T_fg\rangle_2 = \int_K h(x)\overline{f*g(x)}dm(x) \\
&= \int_K \overline{\tilde{f}}*h(y)\overline{g(y)}dm(y) = \langle f^**h, g\rangle_2.
\end{aligned}$$

Here we used (4.2) for the third equality. □

Proposition 4.3. *Let $1 \le p < \infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. For $f \in L^1(K)$, $\|T_f\|_{p'} = \|T_f\|_p$ and $\sigma_{p'}(T_f) = \sigma_p(T_f)$. If either $p \le q \le 2$ or $2 \le q \le p$, then $\|T_f\|_q \le \|T_f\|_p$.*

Proof. By

$$\tilde{f}*h = \widetilde{f*\tilde{h}},\ h \in L^{p'}(K), \qquad (4.3)$$

it immediately follows from Lemma 4.2 that $\|T_f\|_{p'} = \|T_{\tilde{f}}\|_{p'} = \|T_f\|_p$, and that $\sigma_{p'}(T_f) = \sigma_{p'}(T_{\tilde{f}}) = \sigma_p(T_f)$. Since q lies between p and p', Riesz's convexity theorem [12, VI.10.11] yields $\|T_f\|_q \le \max(\|T_f\|_p, \|T_f\|_{p'})$, and the second statement follows. □

For our next proposition, we need some auxiliary statements. Recall that for $f \in L^p(K),\ g \in L^{p'}(K)$, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, the convolution is given by

$$f * g(x) = \int_K L_{\tilde{y}} f(x) g(y) dm(y).$$

Then $f * g \in C_0(K)$ is a continuous function vanishing at infinity, fulfilling $\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}$.

Lemma 4.4. *Let $1 < p \leq p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $h \in L^p(K),\ h' \in L^{p'}(K)$. If $\langle h, w\rangle = \langle h', w\rangle$ for all $w \in L^p(K) \cap L^{p'}(K)$, then $h = h'$ m-a.e., i.e. $h = h' \in L^p(K) \cap L^{p'}(K)$.*

Proof. We define $M = \{x \in K : |h(x)| \geq \varepsilon\}$, where $1 \geq \varepsilon > 0$ is chosen such that $m(M) > 0$. Since $h \in L^p(K)$, we know that $m(M) < \infty$. Let

$$\begin{aligned} M_{R,+} &= \{x \in M : \operatorname{Re} h - \operatorname{Re} h' \geq 0\}, \quad & M_{R,-} &= \{x \in M : \operatorname{Re} h - \operatorname{Re} h' \leq 0\}, \\ M_{I,+} &= \{x \in M : \operatorname{Im} h - \operatorname{Im} h' \geq 0\}, \quad & M_{I,-} &= \{x \in M : \operatorname{Im} h - \operatorname{Im} h' \leq 0\}. \end{aligned}$$

The four corresponding characteristic functions lie in $L^p(K) \cap L^{p'}(K)$, since $m(M) < \infty$. Denoting by χ any of these four functions, we know that $\langle h - h', \chi\rangle = 0$. This implies $h|_M = h'|_M$ m-a.e. for the following reason: Consider $\chi_{M_{I,-}}$, where

$$0 = \operatorname{Im}\left\langle h - h', \chi_{M_{I,-}}\right\rangle = \left\langle \operatorname{Im}(h - h'), \chi_{M_{I,-}}\right\rangle,$$

because m is a positive measure and $\chi_{M_{I,-}}$ is a positive function. Since $\operatorname{Im}(h - h') \leq 0$ on $M_{I,-}$, $\operatorname{Im}(h - h')$ has to equal zero m-a.e. on $M_{I,-}$. Analogously, $\operatorname{Im}(h - h') = 0$ m-a.e. on $M_{I,+}$, which means $\operatorname{Im}(h - h') = 0$ on M. In the same way, $\operatorname{Re}(h - h') = 0$ m-a.e. on M. Using $h|_M = h'|_M$ m-a.e. and $p \leq p'$, we can estimate

$$\int_K |h|^{p'} dm = \int_{K \setminus M} |h|^{p'} dm + \int_M |h|^{p'} dm \leq \int_{K \setminus M} |h|^p dm + \int_M |h'|^{p'} dm < \infty,$$

and thus $h \in L^{p'}(K)$. By assumption, the two elements $h, h' \in L^{p'}(K)$ fulfill $\langle h, w\rangle = \langle h', w\rangle$ for all $w \in L^p(K) \cap L^{p'}(K)$, which is dense in $L^{p'}(K)$ for $p' < \infty$. Therefore, $h = h'$ m-a.e. □

Lemma 4.5. *Let $g \in L^p(K),\ 1 < p < \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose that g is related to some bounded operator $T_g \in B(L^p(K))$ via $T_g w = g * w$ for all $w \in L^p(K) \cap L^{p'}(K)$. Then its adjoint $T_g^* \in B(L^{p'}(K))$ is related to g via $T_g^* v = \tilde{g} * v$ for all $v \in C_c(K)$.*

Proof. We proceed similar to the proof of Lemma 4.2. First we note that for $x, y \in K$

$$L_y \tilde{g}(\tilde{x}) = \int_K \tilde{g} d\omega(y, \tilde{x}) = \int_K g d\omega(\tilde{y}, x) = L_{\tilde{y}} g(x).$$

Now let $v \in C_c(K),\ w \in L^p(K) \cap L^{p'}(K)$. By Fubini's theorem we obtain

$$\begin{aligned} \left\langle T_g^* v, w\right\rangle &= \langle v, T_g w\rangle = \langle v, g * w\rangle = \int_K v(x) g * w(x) dm(x) \\ &= \int_K \int_K v(x) L_{\tilde{y}} g(x) w(y) dm(y) dm(x) \\ &= \int_K \int_K v(x) L_y \tilde{g}(\tilde{x}) w(y) dm(x) dm(y) \\ &= \int_K \tilde{g} * v(y) w(y) dm(y) = \langle \tilde{g} * v, w\rangle. \end{aligned}$$

We now know that the results of testing $\tilde{g} * v \in L^p(K)$ and $T_g^* v \in L^{p'}(K)$ on elements of $L^p(K) \cap L^{p'}(K)$ coincide. By Lemma 4.4 this means that $\tilde{g} * v = T_g^* v$ m-a.e., and that $\tilde{g} * v \in L^{p'}(K)$ for all $v \in C_c(K)$. □

Proposition 4.6. *Let K be a commutative hypergroup and $1 < p < \infty$. Suppose that either $p \leq q \leq 2$ or $2 \leq q \leq p$. Then $\sigma_q(T_f) \subseteq \sigma_p(T_f)$ for all $f \in L^1(K)$.*

Proof. It suffices to show that when $I - T_{f,p}$ is invertible on $L^p(K)$ then $I - T_{f,q}$ is invertible on $L^q(K)$.

First assume $f \in L^1(K) \cap L^\infty(K)$. Then $f \in L^s(K)$ for all $1 \leq s \leq \infty$. Since $I - T_{f,p}$ is invertible on $L^p(K)$ there exists $g \in L^p(K)$ such that $(I - T_{f,p})g = -f$. Therefore, for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$(I - T_{f,p})(w - g * w) = w \quad \text{for all } w \in L^p(K) \cap L^{p'}(K).$$

This implies $(I - T_{f,p})[(I - T_{f,p})^{-1}w - (w - g * w)] = 0$, and since $I - T_{f,p}$ is invertible,

$$(I - T_{f,p})^{-1}w = w - g * w \quad \text{for all } w \in L^p(K) \cap L^{p'}(K). \tag{4.4}$$

We define $T_g \in B(L^p(K))$ by $T_g = I - (I - T_{f,p})^{-1}$. By (4.4), $T_{\tilde{g}} \in B(L^p(K))$ and we know that $\|T_{\tilde{g}}\|_p = \|T_g\|_p$. We can apply Lemma 4.5 to $T_{\tilde{g}}$. This yields

$$\|T_g h\|_{p'} = \|(I - (I - T_{f,p})^{-1})h\|_{p'} = \|T_{\tilde{g}}^* h\|_{p'} \leq \|T_g\|_p \cdot \|h\|_{p'} \quad \text{for all } h \in C_c(K).$$

Since $q \in [p, p']$ or $q \in [p', p]$, Riesz's convexity theorem [12, VI.10.11] yields that

$$\|(I - (I - T_{f,p})^{-1})h\|_q = \|T_g h\|_q \leq \max(\|T_g\|_p, \|T_g\|_{p'}) \cdot \|h\|_q = \|T_g\|_p \cdot \|h\|_q$$

for all $h \in C_c(K)$. Thus,

$$\|(I - T_{f,p})^{-1}h\|_q \leq (1 + \|T_g\|_p) \cdot \|h\|_q \leq (2 + \|(I - T_{f,p})^{-1}\|_p) \cdot \|h\|_q \tag{4.5}$$

for all $h \in C_c(K)$. For $h \in L^p(K) \cap L^q(K)$ we choose a sequence $(h_n)_n \subset C_c(K)$ that approximates h simultaneously in $L^p(K)$ and $L^q(K)$ (this is possible, since $L^1(K)$ has a bounded approximate identity of functions with compact support, see Theorem 1.5. By (4.5) we obtain that $(I - T_{f,p})^{-1}h_n$ converges in $L^q(K)$, and of course it converges to $(I - T_{f,p})^{-1}h$ in $L^p(K)$. The limits have to coincide m-a.e., which means that $(I - T_{f,p})^{-1}h \in L^p(K) \cap L^q(K)$. Thus,

$$(I - T_{f,p})^{-1}(I - T_{f,p}) = (I - T_{f,p})(I - T_{f,p})^{-1} = I \text{ on } L^p(K) \cap L^q(K), \tag{4.6}$$

which is dense in $L^q(K)$. $(I - T_{f,q})$ clearly is the bounded extension of $(I - T_{f,p})$ from $L^p(K) \cap L^q(K)$ to $L^q(K)$. Therefore, the bounded extension of $(I - T_{f,p})^{-1}$ on $L^q(K)$ (which exists according to (4.5)) has to be an inverse of $(I - T_{f,q})$ by (4.6).

Now we consider general $f \in L^1(K)$ and assume that $I - T_{f,p}$ is invertible. We choose a sequence $(f_n)_n \subset L^1(K) \cap L^\infty(K)$ such that $\|f_n - f\|_1 \to 0$. Since $L^1(K)$ has a bounded approximate identity with bound 1, $\|f_n - f\|_1 = \|T_{f_n} - T_f\|_1$. By Proposition 4.3 it follows that $I - T_{f_n} \to I - T_f$ in both $B(L^p(K))$ and $B(L^q(K))$. For large enough n, say $n > n_0$, $I - T_{f_n,p}$ is invertible. As inversion is continuous, $(I - T_{f_n,p})^{-1} \to (I - T_{f,p})^{-1}$ and in particular $\|(I - T_{f_n,p})^{-1}\|_p \leq C, \; n > n_0$. As argued above, $I - T_{f_n,q}$ is invertible and according to (4.5),

$$\|(I - T_{f_n,q})^{-1}\|_q \leq 2 + \|(I - T_{f_n,p})^{-1}\|_p \leq 2 + C, \quad n > n_0.$$

Since $I - T_{f_n,q} \to I - T_{f,q}$ with uniformly bounded inverses $(I - T_{f_n,q})^{-1}$, $I - T_{f,q}$ has to be invertible as well. □

The proof of Proposition 4.6 above is modeled on the proof of [4, Pro. 3], which deals with locally compact amenable groups. However, there the statement analogous to (4.5) is obtained by an application of [42, Pro. 18.18]. The proof of [42, Pro. 18.18] uses a certain consequence of amenability which is only available in the group case and which is not valid for commutative hypergroups. We obtained (4.5) by means of Lemma 4.5 instead. Notice that our result brings no news for locally compact abelian groups. This is due to the fact that the amenability assumption in [4, Pro. 3] is automatically fulfilled for commutative groups.

Furthermore, the result in Proposition 4.6 is sharp in the sense that there is a commutative hypergroup and an element $f \in L^1(K)$, such that $\sigma_q(T_f) \subsetneq \sigma_p(T_f)$ whenever $q > p, \; q, p \in [1, 2]$. We will encounter such an example in Chapter 4.3.

For the sake of completeness let us briefly mention that the spectra $\sigma_p(T_f)$ are upper semicontinuous: For $1 < p < 2$, $L^p(K)$ is a complex interpolation space of $L^1(K)$ and $L^2(K)$, see [6, Ch. 5]. Applying the result on interpolated operators [43, Thm. 2.7] we note that the map $(1, 2) \to \mathcal{C}(\mathbb{C}), \; p \mapsto \sigma_p(T_f)$, is upper semicontinuous ($\mathcal{C}(\mathbb{C})$ are the compact subsets of $\mathbb{C}$). Here a map $M : \; (1, 2) \to \mathcal{C}(\mathbb{C})$ is said to be upper semicontinuous if whenever U is open in $\mathbb{C}$, the set $\{p \in (1, 2) : \; M(p) \subset U\}$ is open in X.

4.2 p-independence of the spectrum of L^1-convolution operators

In this chapter, we characterize those hypergroups where for each L^1-convolution operator all its p-spectra coincide. In order to prove our main result, which is Theorem 4.12, we introduce the algebras of convolutors on $L^p(K)$. To that end, we first have a look at the unitization of $L^1(K)$.

If and only if K is discrete, $L^1(K)$ already has an identity, which is the point measure δ_e at the neutral element e of K. If K is not discrete, it is still true that $\delta_e * f = f$ as measures for all $f \in L^1(K)$. We can adjoin the point measure δ_e to $L^1(K)$ within the measure algebra. To unify notation, we use the symbol $L^1(K)_e$ to denote either the unitization $L^1(K) \oplus \mathbb{C} \cdot \delta_e$ if $L^1(K)$ is not unital, or $L^1(K)_e = L^1(K)$ if it is unital. Then,

$$\sigma_{L^1(K)}(f) = \hat{f}(\Delta(L^1(K)_e)), \; f \in L^1(K).$$

If $L^1(K)$ is unital, we identify $\Delta(L^1(K)_e)$ with $\chi_b(K)$. If $L^1(K)$ is not unital, $\Delta(L^1(K)_e)$ is homeomorphic to the one-point compactification $\Delta(L^1(K)) \cup \beta_0$, where the compactification character $\beta_0 \in \Delta(L^1(K)_e)$ is given by

$$\beta_0|_{L^1(K)} = 0, \qquad \beta_0(\delta_e) = 1.$$

Since $\Delta(L^1(K))$ is homeomorphic to $\chi_b(K)$, endowed with the compact-open topology, $\Delta(L^1(K)_e)$ is homeomorphic to the one-point compactification of

$\chi_b(K)$. Therefore, we identify $\Delta(L^1(K)_e)$ with the one-point compactification $\chi_b(K) \cup \beta_0$, where we also denote the point at infinity by β_0. Furthermore, we will identify $\Delta^*(L^1(K)_e)$ with $\hat{K}$ or $\hat{K} \cup \beta_0$ depending on whether or not $L^1(K)$ is unital.

We define the *algebra of convolutors* $C_p \subset B(L^p(K))$ as the norm-closure of $L^1(K)_e$ in $B(L^p(K))$, i.e.

$$C_p = \overline{\{T_f \in B(L^p(K)) : \; f \in L^1(K)_e\}}^{B(L^p(K))}.$$

This subalgebra is easier to handle than $B(L^p(K))$. If either $p \leq q \leq 2$ or $2 \leq q \leq p$, Proposition 4.3 yields $\|T_f\|_q \leq \|T_f\|_p$. Hence, $C_p \subseteq C_q$ in the sense that to every element of $T \in C_p$ there is an element of C_q which coincides with T on $C_c(K)$.

Lemma 4.7. *C_p is a commutative Banach-$*$-algebra with identity.*

Proof. Since $\delta_e * g = g$ for all $g \in L^p(K),\ p \in [1, \infty]$, C_p has the identity $T_{\delta_e} = I$. For $f \in L^1(K),\ h \in L^p(K)$,

$$(f^* + (\mu\delta_e)^*) * h(x) = (\overline{\tilde{f}} + \overline{\mu}\delta_e) * h(x) = \overline{f * \overline{\tilde{h}}(\tilde{x})} + \overline{\mu} h(x) = \overline{(f + \mu\delta_e) * \overline{\tilde{h}}(\tilde{x})}.$$

Thus we know that the $*$-operation in $L^1(K)_e$ is also isometric with respect to the norm of $B(L^p(K))$, and can be extended to a $*$-operation in C_p. □

We have a look at the structure space of C_p, and at $\sigma_{C_p}(T_f)$, the spectrum of T_f with respect to the algebra C_p.

Proposition 4.8. *Let K be a commutative hypergroup. For $1 \leq p \leq \infty$ define*

$$S_p := \{\beta \in \Delta(L^1(K)_e) : \; |\langle \beta, f \rangle| \leq \|T_f\|_p \;\; \text{for all } f \in L^1(K)_e\}.$$

Then,

(i) the mapping $R : \; \Delta(C_p) \to S_p$, given by $\langle R(\alpha), f \rangle = \langle \alpha, T_f \rangle$, is a homeomorphism, and $\sigma_{C_p}(T_f) = \hat{f}(S_p)$. In particular, $S_q \subseteq S_p$ and $\sigma_{C_q}(T_f) \subseteq \sigma_{C_p}(T_f)$ for all $f \in L^1(K)$, whenever $p \leq q \leq 2$ or $2 \leq q \leq p$. Furthermore, $\sigma_{C_p}(T_{f^*}) = \overline{\sigma_{C_p}(T_f)}$,

(ii) the restriction to the set of $*$-characters $R|_{\Delta^*(C_p)} : \; \Delta^*(C_p) \to S_p \cap \Delta^*(L^1(K)_e)$ is also a homeomorphism, and $\sigma^*_{C_p}(T_f) = \hat{f}(S_p \cap \Delta^*(L^1(K)_e))$,

(iii) if $L^1(K)$ is not unital, the compactification character $\beta_0 \in \Delta(L^1(K)_e)$ lies in S_p for all $p \in [1, \infty]$,

(iv) $S_1 = \Delta(L^1(K)_e)$. If the Plancherel measure has full support, i.e., $S = \chi_b(K)$, then $S_2 = S_1$.

Proof. (i): R restricts $\alpha \in \Delta(C_p)$ to the dense subset $A := \{T_f \in B(L^p(K)) : \; f \in L^1(K)_e\}$ of C_p, and identifies A with $L^1(K)_e$. Thus $R(\alpha),\ \alpha \in \Delta(C_p)$, is still a multiplicative functional (and thus bounded) on $L^1(K)_e$. Since α is bounded on C_p, $R(\alpha) \in S_p$ and R is one-to-one. Conversely, $\beta \in S_p$ defines a bounded multiplicative functional on the dense subset A of C_p. Thus there exists a unique bounded extension of β to $\Delta(C_p)$

which is easily checked to be still multiplicative. By its definition, R is continuous with respect to the Gelfand topologies on $\Delta(C_p)$ and $\Delta(L^1(K)_e)$. Because $\Delta(C_p)$ is compact with respect to the Gelfand topology, the bijective map R is a homeomorphism. If either $p \leq q \leq 2$ or $2 \leq q \leq p$, Proposition 4.3 tells us that $\|T_f\|_q \leq \|T_f\|_p$. Thus $S_q \subseteq S_p$ and in addition $\sigma_{C_q}(T_f) \subseteq \sigma_{C_p}(T_f)$. Since C_p is a $*$-algebra, $\sigma_{C_p}(T_{f^*}) = \overline{\sigma_{C_p}(T_f)}$.

(ii): Since a character φ is a $*$-character if and only if $\langle \varphi, x^* \rangle = \overline{\langle \varphi, x \rangle}$ for all x, we obtain for $\alpha \in \Delta^*(C_p)$ that $\langle R(\alpha), f^* \rangle = \langle \alpha, T_{f^*} \rangle = \overline{\langle \alpha, T_f \rangle} = \overline{\langle R(\alpha), f \rangle}$. This in turn means that R maps $\Delta^*(C_p)$ into $S_p \cap \Delta^*(L^1(K)_e)$. Conversely, the unique bounded extension α of $\beta \in S_p \cap \Delta^*(L^1(K)_e)$ to $\Delta(C_p)$ fulfills $\langle \alpha, T^* \rangle = \lim_n \left\langle \alpha, T_{f_n^*} \right\rangle = \lim_n \langle \beta, f_n^* \rangle = \lim_n \overline{\langle \beta, f_n \rangle} = \lim_n \overline{\langle \alpha, T_{f_n} \rangle} = \overline{\langle \alpha, T \rangle}$, which means that $R|_{\Delta^*(C_p)}$ is onto. By (i), R is one-to-one and the statement follows.

(iii): Let $f + \mu\delta_e \in L^1(K)_e$. Since $L^1(K)$ is not unital, K is not discrete and from [31, Thm. 3.4] it follows that S is not compact. Because $f \in L^1(K)$, we know that $\hat{f} \in C_0(S)$. Since S is not compact, for all $\varepsilon > 0$ we can find $\beta_f \in S$ such that $|\hat{f}(\beta_f)| < \varepsilon$. Furthermore, it is well known that $\|T_f + \mu T_{\delta_e}\|_2 = \sup_{\beta \in S} |\mu + \hat{f}(\beta)|$. We obtain

$$\|T_f + \mu T_{\delta_e}\|_p \geq \|T_f + \mu T_{\delta_e}\|_2 \geq \left| |\mu| - |\hat{f}(\beta_f)| \right| \geq |\langle \beta_0, f + \mu\delta_e \rangle| - \varepsilon.$$

For $\varepsilon \to 0$, the above inequality yields $\beta_0 \in S_p$.

(iv): Since $L^1(K)_e$ is unital, we obtain $\|T_f\|_1 = \|f\|_1$ for all $f \in L^1(K)_e$. Hence, $L^1(K)_e$ and C_1 are isometrically isomorphic and the largest structure space $S_1 \cong \Delta(C_1)$ is the whole structure space $\Delta(L^1(K)_e)$. Now suppose that $S = \chi_b(K)$. If $L^1(K)$ is unital, then by the definition of S in (1.4), $S = S_2$. Since $\Delta(L^1(K)_e) = \chi_b(K)$, it follows that $S_2 = S = \chi_b(K) = S_1$. If $L^1(K)$ is not unital, then for $\gamma \in S$,

$$|(\mu\hat{\delta}_e + \hat{f})(\gamma)| \leq \sup_{\beta \in S} |(\mu\hat{\delta}_e + \hat{f})(\beta)| = \|T_f + \mu T_{\delta_e}\|_2, \quad \text{for all } \mu\delta_e + f \in L^1(K)_e.$$

This means that $S \subset S_2$. By (iii) we obtain

$$S \cup \beta_0 \subseteq S_2 \subseteq S_1 = \chi_b(K) \cup \beta_0 = S \cup \beta_0,$$

and thus $S_2 = S_1$. □

In the following, the preceding proposition allows us to identify $\Delta(C_p)$ with $S_p \subseteq \Delta(L^1(K)_e)$.

Applying a result of T. J. Ransford on interpolated operators, we draw a conclusion concerning the structure spaces S_p of the algebras $C_p \subset B(L^p(K))$.

Corollary 4.9. *If either $p \leq q \leq 2$ or $2 \leq q \leq p$, then each connected component of S_p intersects S_q. In particular, each connected component of S_1 intersects S_2.*

Proof. Let C be a connected component of S_p and suppose towards a contradiction that $C \cap S_q = \emptyset$. By Shilov's Idempotent Theorem [25, Thm. 3.5.1] there is $T \in C_p$ such that $\hat{T}(C) = \{1\}$ and $\hat{T}(S_q) \subseteq \hat{T}(S_p \setminus C) = \{0\}$. Since $\sigma_{C_p}(T) = \{0, 1\}$, we know that $\sigma_{C_p}(T) = \sigma_p(T)$ and thus $\{1\}$ is a connected component of $\sigma_p(T)$.

For $p_0 \leq p \leq p_1$, $L^p(K)$ is a complex interpolation space of $L^{p_0}(K)$ and $L^{p_1}(K)$, see [6, Ch. 5]. The proof of Proposition 4.6 shows that for all $f \in L^1(K)$, T_f fulfills the 'uniqueness of resolvent condition'; compare [43, Cor. 2.2 b)]. Thus we can apply [43, Thm. 2.8], whereby it follows that $\{1\} \cap \sigma_q(T) \neq \emptyset$. But $\{1\} \cap \sigma_q(T) = \{1\} \cap \hat{T}(S_q) = \{1\} \cap \{0\} = \emptyset$, a contradiction. $\square$

Since S_1 is essentially the character space $\chi_b(K)$ and S_2 is the support of the Plancherel measure S (except for maybe the compactification character $\beta_0 \in \Delta(L^1(K)_e)$) we can read the above as follows: Each connected component of $\chi_b(K)$ intersects S (modulo $L^1(K)$ being unital).

In the cases $p = 1$ and $p = 2$, the spectra of T_f with respect to the different algebras coincide.

Lemma 4.10. *For all $f \in L^1(K)$,*

$$\sigma_{C_2}(T_f) = \sigma_2(T_f) \quad \text{and} \quad \sigma_{L^1(K)}(f) = \sigma_{C_1}(T_f) = \sigma_1(T_f).$$

Proof. T_f is always a normal operator on the Hilbert space $L^2(K)$ and thus always $\sigma_{C_2}(T_f) = \sigma_2(T_f)$. We consider the 'maximal' case $p = 1$: Since $L^1(K)_e$ and C_1 are isometrically isomorphic, the first equality follows. For the second equality we have to show that if $I - T_f$ is invertible in $B(L^1(K))$, then its inverse is already in C_1. If $I - T_f$ is invertible on $L^1(K)$ there exists $g \in L^1(K)$ such that $(I - T_f)g = -f$. Therefore for any $h \in L^1(K)$, $(I - T_f)(h - g * h) = h$ and thus $(I - T_f)^{-1}h = h - g * h = (T_{\delta_e} - T_g)h$. This argument is taken from [4, Note 2 and Note 1]. $\square$

It is natural to ask if the equality $\sigma_p(T_f) = \sigma_{C_p}(T_f)$ is true for all $p \in [1, \infty]$, i.e., if the algebra C_p is always inverse-closed in $B(L^p(K))$. This seems to be a more difficult problem than one would think at first glance; we have no solution. Clearly, always $\sigma_p(T_f) \subseteq \sigma_{C_p}(T_f)$, and equality holds for example in the special cases where $\sigma_{C_p}(T_f)$ has empty interior or where the resolvent set $\rho_p(T_f)$ is connected, see [25, Lem. 1.2.11 and Thm. 1.2.12].

We need the following version of Hulanicki's Theorem; a proof can be found in [4].

Theorem 4.11. *(Hulanicki) Assume A is a Banach $*$-algebra and B is a $*$-subalgebra of A. Let $f \to T_f$ be a faithful $*$-representation of A on a Hilbert space H. If $r_A(f) = \|T_f\|$ for all $f = f^* \in B$, then $\sigma_A(f) = \sigma(T_f)$ for all $f \in B$.*

We state our main result.

Theorem 4.12. *For a commutative hypergroup K, the following are equivalent;*

(i) for all $f \in L^1(K)$, $\sigma_{L^1(K)}(f) = \sigma_2(T_f)$.

(ii) for all $f \in L^1(K)$, $\sigma_{C_p}(T_f)$ is independent of $p \in [1, \infty]$.

(iii) for all $f \in L^1(K)$, $\sigma_p(T_f)$ is independent of $p \in [1, \infty]$.

(iv) for all $f \in L^1(K)$ with $f = f^*$, $r_{L^1(K)}(f) = r_2(T_f)$.

(v) for the support of the Plancherel measure S holds $S = \chi_b(K)$.

(vi) $L^1(K)$ is symmetric and for every $\alpha \in \hat{K}$ Reiter's condition P_2 holds.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv): Lemma 4.10 implies that $\sigma_{C_1}(T_f) = \sigma_{L^1(K)}(f) = \sigma_2(T_f) = \sigma_{C_2}(T_f)$. Using the inclusions of C_p-spectra of Proposition 4.8(i), we obtain (ii). Proposition 4.6 tells us that $\sigma_2(T_f) \subseteq \sigma_p(T_f)$ for each $p \in [1,\infty]$. Hence,

$$\sigma_p(T_f) \subseteq \sigma_{C_p}(T_f) = \sigma_{C_2}(T_f) = \sigma_2(T_f) \subseteq \sigma_p(T_f),$$

which shows (iii). We use Lemma 4.10 again, and get $\sigma_{L^1(K)}(f) = \sigma_1(T_f) = \sigma_2(T_f)$, which yields (iv).

(iv) $\Rightarrow$ (i): According to Lemma 4.2, the map $f \to T_f$ is a faithful $*$-representation of $L^1(K)$ on the Hilbert space $L^2(K)$. Applying Hulanicki's Theorem we obtain $\sigma_{L^1(K)}(f) = \sigma_2(T_f)$ for all $f \in L^1(K)$.

(i) $\Rightarrow$ (v): If $f \in L^1(K)$ is self-adjoint, then by Lemma 4.2 $T_{f,2}$ is also self-adjoint and therefore has real spectrum. If (i) holds true, $L^1(K)$ is thus symmetric, and we obtain that $\hat{K} = \chi_b(K)$. Furthermore, the space $\widehat{L^1(K)}|_{\hat{K}}$ is dense in $C_0(\hat{K})$ and S is closed in the locally compact Hausdorff space $\hat{K}$. Supposing towards a contradiction that $S \neq \hat{K}$, we choose $g \in C_c(\hat{K})$, supp $g \subseteq \hat{K} \setminus S$, and $f \in L^1(K)$ such that $\|g - \hat{f}\|_{\infty,\hat{K}} < \varepsilon$. For small enough $\varepsilon > 0$ then holds $\hat{f}(\hat{K}) \neq \hat{f}(S)$ and $\hat{f}(\hat{K}) \cup \{0\} \neq \hat{f}(S) \cup \{0\}$. If $L^1(K)$ is unital, it follows that $\sigma_{L^1(K)}(f) = \hat{f}(\hat{K}) \neq \hat{f}(S) = \sigma_2(T_f)$. If $L^1(K)$ is not unital, Lemma 4.10 and Proposition 4.8 yield

$$\sigma_{L^1(K)}(f) = \sigma_{C_1}(T_f) = \hat{f}(S_1) = \hat{f}(\hat{K}) \cup \{0\} \neq \hat{f}(S) \cup \{0\} = \hat{f}(S_2) = \sigma_2(T_f).$$

In either case, we have obtained a contradiction to (i).

(v) $\Rightarrow$ (i): Since $S_1 = S_2$ by Proposition 4.8(iv), Proposition 4.8(i) yields that $\sigma_{C_1}(T_f) = \sigma_{C_2}(T_f) = \sigma_2(T_f)$ for all $f \in L^1(K)$; again by Lemma 4.10 $\sigma_{C_1}(T_f) = \sigma_{L^1(K)}(f)$.

(v) $\Leftrightarrow$ (vi): $\hat{K} = \chi_b(K)$ if and only if $L^1(K)$ is symmetric. Further $S = \hat{K}$ if and only if Reiter's condition P_2 holds true for every $\alpha \in \hat{K}$, see [15, Thm. 3.1]. □

Theorem 4.12 is similar to Theorem 4.1 of [4, Thm. 6] for locally compact groups. We draw a conclusion for quite a large class of commutative hypergroups.

Corollary 4.13. *Let K be a commutative hypergroup whose Haar measure m is of subexponential growth, i.e. for every compact set $C \subset K$ and every $k > 1$ we have $m(C^n) = o(k^n)$ as $n \to \infty$. Then for all $f \in L^1(K)$ and all $p \in [1,\infty]$,*

$$\sigma_{L^1(K)}(f) = \sigma_{C_p}(T_f) = \sigma_p(T_f).$$

Proof. By M. Vogel [57, Pro. 2.6 and Rem 2.7] and M. Voit [59, Thm. 2.17] the support of the Plancherel measure S is equal to $\chi_b(K)$. Then (v) of the preceding theorem is fulfilled which implies the statement. □

If the equivalent properties of Theorem 4.12 are not necessarily fulfilled, we can still say something about the behavior of the spectra $\sigma_p(T_f)$ in special situations.

Proposition 4.14. *Let K be a commutative hypergroup and $f \in L^1(K)$. If $\hat{K} = \chi_b(K)$, or if S is countable, then*

$$\sigma_{C_p}(T_f) = \sigma_p(T_f) = \sigma_2(T_f) \quad \textit{for all } \; 1 < p < \infty. \tag{4.7}$$

Proof. If $\hat{K} = \chi_b(K)$, then for all $f = f^* \in L^1(K)$, $\sigma_p(T_f) \subseteq \sigma_1(T_f) \subset \mathbb{R}$. Thus $\sigma_1(T_f)$ has empty interior and connected complement; in particular, $\sigma_{C_p}(T_f) = \sigma_p(T_f)$. Now the general result on interpolated operators [43, Cor. 3.7 b)] yields that, for $1 < p < \infty$, $\sigma_{C_p}(T_f) = \sigma_p(T_f) = \sigma_2(T_f)$. So $r_{C_p}(T_f) = \|T_f\|_2$ for all $f = f^* \in L^1(K)$. Since $L^1(K)$ is a $*$-subalgebra of C_p, Hulanicki's theorem yields $\sigma_{C_p}(T_f) = \sigma_2(T_f)$ for all $f \in L^1(K)$. From Proposition 4.6 we know that $\sigma_2(T_f) \subseteq \sigma_p(T_f)$. Hence, $\sigma_p(T_f) \subseteq \sigma_{C_p}(T_f) = \sigma_2(T_f) \subseteq \sigma_p(T_f)$, and the statement follows. If S is countable, the statement is a direct consequence of [43, Cor. 3.2]; since the resolvent set $\rho_p(T_f)$ is connected, $\sigma_{C_p}(T_f) = \sigma_p(T_f)$. □

In view of the above proposition, note that for S to be countable it is not necessary that K is compact; for example, S is countable for those polynomial hypergroups (which are never compact) induced by the polynomials considered in [36, Cor. 2]. We remark that the Haar measures of these polynomial hypergroups are of exponential growth such that Corollary 4.13 is not applicable. We further notice that (4.7) is not true for general commutative hypergroups: In the next chapter we give an example of $f \in L^1(K)$ (with $\hat{K} \neq \chi_b(K)$) where $\sigma_q(T_f) \neq \sigma_p(T_f)$ whenever $q \neq p,\ q,p \in [1,2]$.

4.3 p-structure spaces of the Karlin-McGregor polynomial hypergroups

In this chapter, we explicitly determine the spectra $\sigma_p(T_{\varepsilon_1})$ and the structure spaces S_p for the family of Karlin-McGregor polynomial hypergroups for all parameters $\alpha, \beta \geq 2$, and all $p \in [1, \infty]$. Note that S_1 and S_2 agree with the structure spaces $\chi_b(K)$ and S, respectively.

Their use as an example is twofold: Firstly, for a commutative hypergroup K with $\hat{K} = \chi_b(K)$, we have seen in Proposition 4.14 that $\sigma_p(T_f) = \sigma_2(T_f)$ for all $p \in (1, 2]$. In contrast, Theorem 4.15 shows that $\sigma_q(T_{\varepsilon_1}) \subsetneq \sigma_p(T_{\varepsilon_1})$ whenever $q > p,\ q,p \in [1,2]$, and $(\alpha, \beta) \neq (2,2)$. This also shows that Proposition 4.6 is sharp in the sense that there is a commutative hypergroup and an element $f \in L^1(K)$, such that $\sigma_q(T_f) \subsetneq \sigma_p(T_f)$ whenever $q > p,\ q,p \in [1,2]$.

Secondly, the structure spaces also fulfill $S_q \subsetneq S_p$ whenever $q > p,\ q,p \in [1,2],\ (\alpha,\beta) \neq (2,2)$. This is a stark contrast to the case of abelian locally compact groups, whose Plancherel measure always has full support and thus $S_q = S_p$ for all $q,p \in [1,2]$.

The Karlin-McGregor polynomials $(P_n^{(\alpha,\beta)})_{n\in\mathbb{N}_0}$ with normalization $P_n^{(\alpha,\beta)}(1) = 1$ are given by the three-term-recurrence $P_0^{(\alpha,\beta)} = 1,\ P_1^{(\alpha,\beta)}(x) = \frac{1}{a_0}(x - b_0)$,

$$P_1^{(\alpha,\beta)} P_n^{(\alpha,\beta)} = a_n P_{n+1}^{(\alpha,\beta)} + b_n P_n^{(\alpha,\beta)} + c_n P_{n-1}^{(\alpha,\beta)},\ n \geq 1,$$

where

$$\begin{aligned} a_0 &= 1,\ b_0 = 0, \\ a_n &= \begin{cases} \frac{\alpha-1}{\alpha}, & n \text{ is odd}, \\ \frac{\beta-1}{\beta}, & n \text{ is even}, \end{cases} \\ b_n &= 0,\ c_n = 1 - a_n. \end{aligned} \tag{4.8}$$

For $\alpha = \beta = 2$, the Karlin-McGregor polynomials are the Chebyshev polynomials of the first kind.

The Karlin-McGregor polynomials $(P_n^{(\alpha,\beta)})_{n\in\mathbb{N}_0}$ induce a (commutative and unital) polynomial hypergroup on $\mathbb{N}_0$ whenever $\alpha, \beta \geq 2$, see [15] and [27]. From now on we fix $\alpha, \beta \geq 2$. The Haar measure of the corresponding hypergroup (normalized such that $h_0 = 1$) is given by

$$h_n = \begin{cases} \alpha(\alpha-1)^{\frac{n-1}{2}}(\beta-1)^{\frac{n-1}{2}}, & n \text{ is odd,} \\ \beta(\alpha-1)^{\frac{n}{2}}(\beta-1)^{\frac{n-2}{2}}, & n \text{ is even.} \end{cases}$$

In the customary way for polynomial hypergroups, we identify $\chi_b(K)$ with the complex subset $D = \{z \in \mathbb{C} : \ |P_n^{(\alpha,\beta)}(z)| \leq 1 \text{ for all } n \in \mathbb{N}_0\}$ via

$$\hat{f}(z) = \sum_{n=0}^{\infty} f_n P_n^{(\alpha,\beta)}(z) h_n, \quad (f_n)_{n\in\mathbb{N}_0} \in l^1(h).$$

Denote by δ_n the point measure at $n \in \mathbb{N}_0$. We consider the convolution operator T_{ε_1}, $\varepsilon_1 = h_1^{-1}\delta_1 \in l^1(h)$. Since $\hat{\varepsilon}_1(z) = P_1^{(\alpha,\beta)}(z) = z$, we know that $\hat{\varepsilon}_1$ is the identity mapping. Thus $\chi_b(K)$ is identical with $\sigma_{C_1}(T_{\varepsilon_1})$ and furthermore, the subsets S_p of $\chi_b(K)$ are identical with $\sigma_{C_p}(T_{\varepsilon_1})$ for all $1 \leq p \leq \infty$. Under the above identification, $\hat{K} = \chi_b(K) \cap \mathbb{R}$.

The Plancherel measure π for the Karlin-McGregor polynomials is known [27] to be supported on

$$S = \operatorname{supp} \pi = \begin{cases} [-v,-w] \cup [w,v], & \beta \geq \alpha, \\ \{0\} \cup [-v,-w] \cup [w,v], & \beta < \alpha, \end{cases} \tag{4.9}$$

where

$$v = \frac{(\alpha-1)^{\frac{1}{2}} + (\beta-1)^{\frac{1}{2}}}{(\alpha\beta)^{\frac{1}{2}}} \quad \text{and} \quad w = \frac{|(\alpha-1)^{\frac{1}{2}} - (\beta-1)^{\frac{1}{2}}|}{(\alpha\beta)^{\frac{1}{2}}}.$$

In particular, the support of the Plancherel measure $S = S_2$ is identical with $\sigma_{C_2}(T_{\varepsilon_1})$; thus for $p = 2$ the spectrum is already known.

Writing $\varepsilon_n = h_n^{-1}\delta_n$ and noting that $a_n h_n = c_{n+1} h_{n+1}$ for all $n \in \mathbb{N}_0$, the convolution operator T_{ε_1} is given by

$$\begin{aligned} \varepsilon_1 * \delta_0 &= \varepsilon_1 * \varepsilon_0 = \varepsilon_1 = c_1\delta_1, \\ \varepsilon_1 * \delta_n &= h_n \cdot \varepsilon_1 * \varepsilon_n = h_n(a_n\varepsilon_{n+1} + c_n\varepsilon_{n-1}) = c_{n+1}\delta_{n+1} + a_{n-1}\delta_{n-1}, \ n \geq 1. \end{aligned} \tag{4.10}$$

In the following theorem we state the spectra $\sigma_{C_p}(T_{\varepsilon_1})$ which coincide with the structure spaces S_p under the above identification.

Theorem 4.15. *Let $1 \leq p \leq \infty$ and let $\alpha, \beta \geq 2$ be the parameters of a Karlin-McGregor polynomial hypergroup. Define the positive numbers*

$$\begin{aligned} A_p &= \frac{(\alpha-1)^{1-\frac{1}{p}}(\beta-1)^{1-\frac{1}{p}}}{\alpha\beta}, \\ B &= \frac{\alpha+\beta-2}{\alpha\beta}, \\ C_p &= \frac{(\alpha-1)^{\frac{1}{p}}(\beta-1)^{\frac{1}{p}}}{\alpha\beta}, \end{aligned}$$

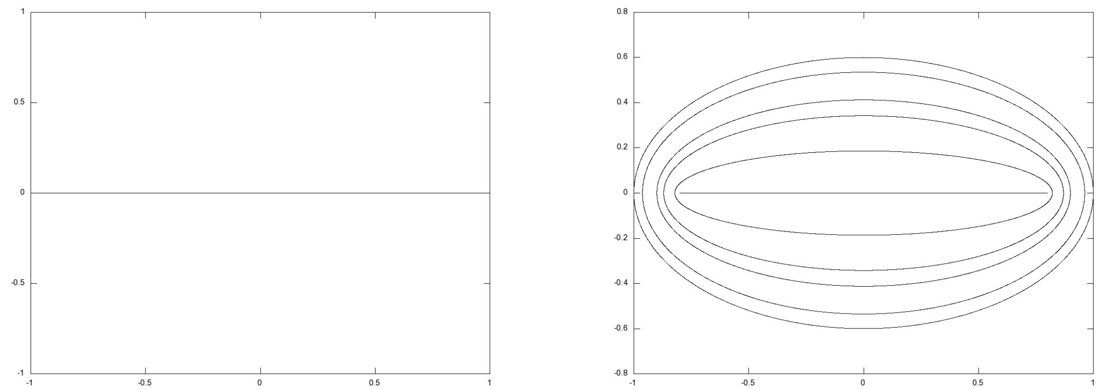

Figure 1: Left: For $\alpha = \beta = 2$, the case of Chebyshev polynomials, it is well known that $S = \chi_b(K) = [-1, 1]$. Right: All structure spaces S_p consist of one connected component if and only if $\beta = \alpha$. S_p is depicted for $p \in \{1, \frac{21}{20}, \frac{7}{6}, \frac{5}{4}, \frac{3}{2}, 2\}$, $\alpha = \beta = 5$ ($p_0 = 2$).

and the (maybe degenerate) ellipse

$$E_p = \left\{ x + iy \in \mathbb{C} : \ \frac{(x - B)^2}{(C_p + A_p)^2} + \frac{y^2}{(C_p - A_p)^2} \leq 1 \right\}.$$

Then, for the structure spaces S_p and the convolution operator $T_{\varepsilon_1} : \ l^p(h) \to l^p(h)$, $T_{\varepsilon_1} g = \varepsilon_1 * g$, we obtain

$$S_p = \sigma_{C_p}(T_{\varepsilon_1}) = \sigma_p(T_{\varepsilon_1}) = \begin{cases} \{\lambda \in \mathbb{C} : \ \lambda^2 \in E_p\}, & \beta \geq \alpha, \\ \{0\} \cup \{\lambda \in \mathbb{C} : \ \lambda^2 \in E_p\}, & \beta < \alpha. \end{cases}$$

For $\alpha = \beta = 2$, the case of Chebyshev polynomials, all ellipses E_p are degenerate, i.e. $C_p - A_p = 0$. We obtain the well-known fact $S = \chi_b(K) = [-1, 1]$. For all other choices of $\alpha, \beta \geq 2$, the ellipses E_p are degenerate if and only if $p = 2$; we obtain the support of the Plancherel measure, which has already been computed in [27].

The ellipse E_p is symmetric in α, β; the same ellipse is obtained when the roles of α and β are interchanged. Hence the structure space S_p remains the same when the roles of α and β are interchanged, except for the point $\{0\}$, if $0 \notin E_p$.

For $p \in [1, 2]$, we obtain that S_p consists of one connected component if and only if $p \in [1, p_0]$, where $p_0 \in [1, 2]$ is

$$p_0 = \begin{cases} 1 + \frac{\ln(\alpha-1)}{\ln(\beta-1)}, & \beta \geq \alpha \geq 2, \ \beta > 2, \\ 1 + \frac{\ln(\beta-1)}{\ln(\alpha-1)}, & 2 \leq \beta \leq \alpha, \ \alpha > 2. \end{cases}$$

If $\beta > \alpha$, then S_p, $p \in (p_0, 2]$, consists of two connected components. If $\beta < \alpha$, then S_p, $p \in (p_0, 2]$, consists of three connected components. In particular, all structure spaces S_p consist of one connected component, i.e. $p_0 = 2$, if and only if $\beta = \alpha$. Furthermore, only $S_1 = \chi_b(K)$ consists of one connected component, i.e. $p_0 = 1$, if and only if either $\beta = 2$ or $\alpha = 2$. See also the Figures 1-3 for a visualization of these different cases.

In order to prove Theorem 4.15 we first establish three auxiliary statements.

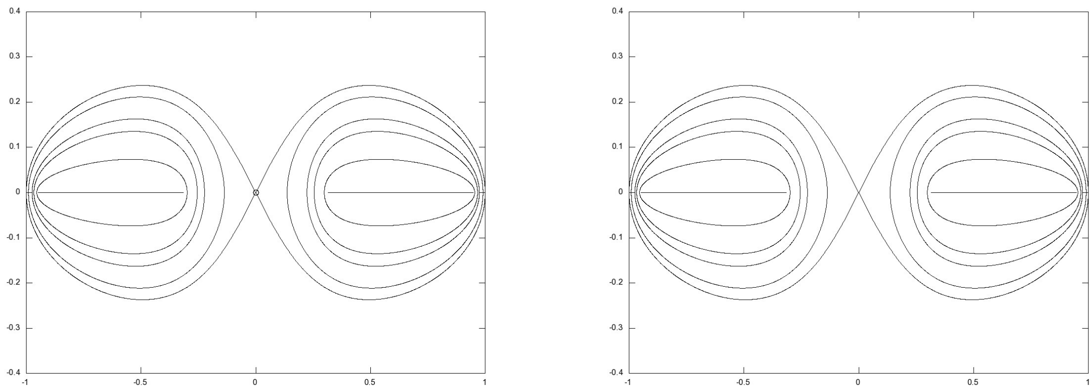

Figure 2: $S_1 = \chi_b(K)$ is the only structure space which consists of one connected component if and only if either $\beta = 2$ or $\alpha = 2$. For interchanged roles of α and β, the structure spaces S_p remain the same except that for $2 = \beta < \alpha$ (left: $\alpha = 5,\ \beta = 2$) all S_p contain the point 0, while for $\beta > \alpha = 2$ (right: $\alpha = 2,\ \beta = 5$) S_p contains 0 if and only if $p = 1$. S_p is depicted for $p \in \{1, \frac{21}{20}, \frac{7}{6}, \frac{5}{4}, \frac{3}{2}, 2\}$ $(p_0 = 1)$.

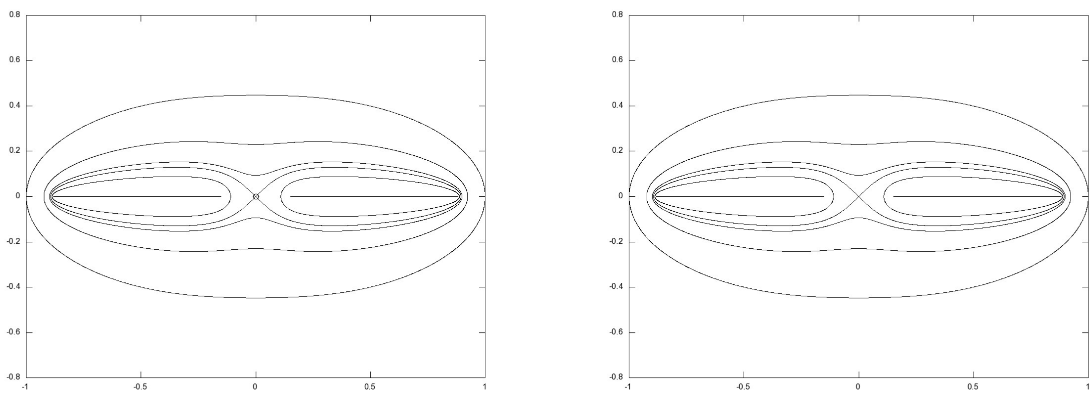

Figure 3: In the general case, $\alpha, \beta > 2$ and $\alpha \neq \beta$, the structure space S_p $(p \in [1, 2])$ consists of one connected component if and only if $p \in [1, p_0]$. For interchanged roles of α and β, the structure spaces S_p remain the same except that for $\beta < \alpha$ (left: $\alpha = 5,\ \beta = 3$) all S_p contain the point 0, while for $\beta > \alpha$ (right: $\alpha = 3,\ \beta = 5$) S_p contains 0 if and only if $p \in [1, p_0]$. S_p is depicted for $p \in \{1, \frac{20}{16}, \frac{23}{16}, \frac{3}{2}, \frac{26}{16}, 2\}$ $(p_0 = \frac{3}{2})$.

Lemma 4.16. *Let $1 \le p \le 2$ and let $\alpha, \beta \ge 2$ be the parameters of a Karlin-McGregor polynomial hypergroup. If $\beta \ge \alpha$, the point spectrum $\sigma_{point,p}(T_{\varepsilon_1 * \varepsilon_1})$ of the operator $T_{\varepsilon_1 * \varepsilon_1} : l^p(h) \to l^p(h)$ is empty. If $\beta < \alpha$, then $\sigma_{point,p}(T_{\varepsilon_1 * \varepsilon_1}) \subseteq \{0\}$ and $0 \in \sigma_p(T_{\varepsilon_1 * \varepsilon_1})$.*

Proof. Suppose that $(\varepsilon_1 * \varepsilon_1 - \lambda \varepsilon_0) * f = 0$ for some $f \in l^p(h) \subset l^2(h)$. Then its Plancherel transform $(x^2 - \lambda)\hat{f}(x) = 0$ for π-almost all $x \in \operatorname{supp} \pi$. In [27] the Plancherel measure is explicitly stated: On the intervals $[-v, -w], [w, v]$ in (4.9), π is a density w.r.t. Lebesgue measure. If $\beta \ge \alpha$, then $x^2 - \lambda \ne 0$ π-a.e.; this means $\hat{f}(x) = 0$ a.e. and thus $f = 0$. If $\beta < \alpha$, $\pi(\{x \in S : \ x^2 - \lambda = 0\}) > 0$ is only possible if $\lambda = 0$. Furthermore, $0 \ne g := \chi_{\{0\}} \in L^2(S)$ with $x^2 g = 0$. Thus $\varepsilon_1 * \varepsilon_1 * \check{g} = 0$, where $0 \ne \check{g} \in l^2(h) \subseteq l^{p'}(h)$ and thus $0 \in \sigma_{p'}(T_{\varepsilon_1 * \varepsilon_1}) = \sigma_p(T_{\varepsilon_1 * \varepsilon_1})$. □

The following proposition is a special case of a result of Gokhberg and Zambitskij [19] which can be also found in [38, Pro. 2].

Proposition 4.17. *Let $1 \le p \le 2$ and denote by R_p, L_p the right and left shift operators $l^p(\mathbb{N}_0) \to l^p(\mathbb{N}_0)$. With the constants A_p, B, C_p from Theorem 4.15, define $Y_p = A_p L_p + C_p R_p + BI$. Then*

$$\sigma(Y_p) = \overline{\sigma(Y_p) \setminus \sigma_{point}(Y_p)} = E_p,$$

where E_p is the ellipse from Theorem 4.15.

Lemma 4.18. *Consider a 'symmetric' polynomial hypergroup, i.e. $b_n = 0$ for all $n \in \mathbb{N}_0$ in the three-term-recurrence (1.8). If $\lambda \in \sigma_p(T_{\varepsilon_1})$, then $-\lambda \in \sigma_p(T_{\varepsilon_1})$.*

Proof. Define the isometric 'switch' isomorphisms $S_o, S_e : \ l^p(h) \to l^p(h)$,

$$S_o f(k) = \begin{cases} -f(k), & k \text{ odd} \\ f(k), & k \text{ even} \end{cases}, \quad S_e f(k) = \begin{cases} f(k), & k \text{ odd} \\ -f(k), & k \text{ even} \end{cases}.$$

$T_{\varepsilon_1} f(k) = a_k f(k+1) + c_k f(k-1)$; thus one can calculate

$$\begin{aligned} S_e(T_{\varepsilon_1} + \lambda I)S_o f(k) &= \begin{cases} -(T_{\varepsilon_1} + \lambda I)S_o f(k), & k \text{ even}, \\ (T_{\varepsilon_1} + \lambda I)S_o f(k), & k \text{ odd}, \end{cases} \\ &= \begin{cases} -(a_k S_o f(k+1) + \lambda S_o f(k) + c_k S_o f(k-1)), & k \text{ even}, \\ a_k S_o f(k+1) + \lambda S_o f(k) + c_k S_o f(k-1), & k \text{ odd}, \end{cases} \\ &= a_k f(k+1) - \lambda f(k) + c_k f(k-1) = (T_{\varepsilon_1} - \lambda I) f(k). \end{aligned}$$

Hence,

$$T_{\varepsilon_1} - \lambda I = S_e(T_{\varepsilon_1} + \lambda I)S_o,$$

and if $T_{\varepsilon_1} + \lambda I$ is invertible on $l^p(h)$, so is $T_{\varepsilon_1} - \lambda I$. □

Proof of Theorem 4.15: We show that for $1 \le p \le 2$,

$$\sigma_p(T_{\varepsilon_1 * \varepsilon_1}) = \begin{cases} E_p, & \beta \ge \alpha, \\ \{0\} \cup E_p, & \beta < \alpha. \end{cases} \tag{4.11}$$

From (4.11), the statement of the theorem then follows: If $\lambda \in \sigma_p(T_{\varepsilon_1 * \varepsilon_1})$, then at least one of its square roots $\lambda_1, \ \lambda_2 = -\lambda_1$ lies in $\sigma_p(T_{\varepsilon_1})$, since $T_{\varepsilon_1 * \varepsilon_1} - \lambda I = (T_{\varepsilon_1} - \lambda_1) \circ (T_{\varepsilon_1} - \lambda_2)$.

(4.8) and Lemma 4.18 yield that both $\lambda_1,\ \lambda_2 = -\lambda_1 \in \sigma_p(T_{\varepsilon_1})$. Furthermore, the resolvent set $\rho_p(T_{\varepsilon_1})$ then is connected and thus $\sigma_p(T_{\varepsilon_1}) = \sigma_{C_p}(T_{\varepsilon_1})$.

The convolution operator $T_{\varepsilon_1 * \varepsilon_1}\ :\ l^p(h) \to l^p(h)$, is easier to study than T_{ε_1}; since the coefficients a_n, c_n alternate depending on whether n is even or odd (4.8), the values $T_{\varepsilon_1}(\delta_n)$ in (4.10) (which determine the convolution operator T_{ε_1}) also are of an alternating form. In contrast, $T_{\varepsilon_1 * \varepsilon_1}(\delta_n)$ is of the same form for all $n \geq 3$:

$$\begin{aligned}
T_{\varepsilon_1 * \varepsilon_1}(\delta_0) &= c_1\delta_0 + c_1c_2\delta_2,\\
T_{\varepsilon_1 * \varepsilon_1}(\delta_1) &= (c_1 + a_1c_2)\delta_1 + c_2c_3\delta_3,\\
T_{\varepsilon_1 * \varepsilon_1}(\delta_n) &= a_{n-1}a_{n-2}\delta_{n-2} + (a_{n-1}c_n + a_nc_{n+1})\delta_n + c_{n+1}c_{n+2}\delta_{n+2},\ n \geq 2.
\end{aligned}$$

In the next step, we find a representation of $T_{\varepsilon_1 * \varepsilon_1}$ which we can compare to the operator Y_p in Proposition 4.17 whose spectrum is the desired ellipse E_p : We use the isometric isomorphisms

$$\begin{aligned}
I_p :\ l^p(h) \to l^p(\mathbb{N}_0), &\qquad I_p(\delta_n) = h_n^{\frac{1}{p}}\delta_n,\\
J_p :\ l^p(\mathbb{N}_0) \to l^p(\mathbb{Z}), &\qquad J_p(\delta_{2n}) = \delta_n,\ J_p(\delta_{2n+1}) = \delta_{-n-1},
\end{aligned}$$

to define

$$Z_p :\ l^p(\mathbb{Z}) \to l^p(\mathbb{Z}),\ Z_p = J_pI_pT_{\varepsilon_1 * \varepsilon_1}(I_p)^{-1}(J_p)^{-1}. \tag{4.12}$$

Clearly, Z_p has the same spectrum as $T_{\varepsilon_1 * \varepsilon_1}$. A straightforward calculation yields for $k \geq 2$:

$$\begin{aligned}
Z_p(\delta_{-k}) &= \left(\frac{h_{2k+1}}{h_{2k-1}}\right)^{\frac{1}{p}} c_{2k}c_{2k+1}\delta_{-k-1} + (a_{2k-2}c_{2k-1} + a_{2k-1}c_{2k})\delta_{-k}\\
&\qquad + \left(\frac{h_{2k-3}}{h_{2k-1}}\right)^{\frac{1}{p}} a_{2k-3}a_{2k-2}\delta_{-k+1}\\
&= C_p\delta_{-k-1} + B\delta_{-k} + A_p\delta_{-k+1},\\
Z_p(\delta_k) &= \left(\frac{h_{2k-2}}{h_{2k}}\right)^{\frac{1}{p}} a_{2k-2}a_{2k-1}\delta_{k-1} + (a_{2k-1}c_{2k} + a_{2k}c_{2k+1})\delta_k\\
&\qquad + \left(\frac{h_{2k+2}}{h_{2k}}\right)^{\frac{1}{p}} c_{2k+1}c_{2k+2}\delta_{k+1}\\
&= A_p\delta_{k-1} + B\delta_k + C_p\delta_{k+1},\\
Z_p(\delta_{-1}) &= \left(\frac{h_3}{h_1}\right)^{\frac{1}{p}} c_2c_3\delta_{-2} + (c_1 + a_1c_2)\delta_{-1} = C_p\delta_{-2} + (B + (\alpha\beta)^{-1})\delta_{-1},\\
Z_p(\delta_0) &= c_1\delta_0 + h_2^{\frac{1}{p}}c_1c_2\delta_1 = (B - (\alpha-2)(\alpha\beta)^{-1})\delta_0 + C_p\left(\frac{\beta}{\beta-1}\right)^{\frac{1}{p}}\delta_1,\\
Z_p(\delta_1) &= \left(\frac{1}{h_2}\right)^{\frac{1}{p}} a_1\delta_0 + (a_1c_2 + a_2c_3)\delta_1 + \left(\frac{h_4}{h_2}\right)^{\frac{1}{p}} c_3c_4\delta_2\\
&= A_p\left(\frac{\beta}{\beta-1}\right)^{1-\frac{1}{p}}\delta_0 + B\delta_1 + C_p\delta_2.
\end{aligned}$$

We denote by $P_{\mathbb{N}_0}$ and $P_{-\mathbb{N}}$ the projections of a $\mathbb{Z}$-indexed sequence to its nonnegatively indexed and negatively indexed part, respectively. Z_p leaves the subspaces $P_{\mathbb{N}_0}(l^p(\mathbb{Z}))$ and $P_{-\mathbb{N}}(l^p(\mathbb{Z}))$ of $l^p(\mathbb{Z})$ invariant; we access the spectrum of Z_p by separately considering its action on these two invariant subspaces. Firstly, we set

$$X_p := Z_p|_{P_{\mathbb{N}_0}(l^p(\mathbb{Z}))} : \ l^p(\mathbb{N}_0) \to l^p(\mathbb{N}_0).$$

Secondly, we use the identification $\tilde{I}_p : \ l^p(-\mathbb{N}) \to l^p(\mathbb{N}_0), \ \delta_{-k} \mapsto \delta_{k-1}$, to define

$$\tilde{X}_p := \tilde{I}_p Z_p|_{P_{-\mathbb{N}}(l^p(\mathbb{Z}))} \tilde{I}_p^{-1} : \ l^p(\mathbb{N}_0) \to l^p(\mathbb{N}_0);$$

the only reason to use the identification $\tilde{I}_p$ is that we think that it is more comfortable to use nonnegative indexing in the following. For some $\rho, \sigma, \psi, \tilde{\rho} \in \mathbb{R}$, X_p and $\tilde{X}_p$ are of the tridiagonal form

$$X_p = \begin{pmatrix} B+\rho & A_p+\sigma & & & \\ C_p+\psi & B & A_p & & \\ & C_p & B & A_p & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \tilde{X}_p = \begin{pmatrix} B+\tilde{\rho} & A_p & \\ C_p & B & \ddots \\ & \ddots & \ddots \end{pmatrix}.$$

This means that both X_p and $\tilde{X}_p$ are a certain slight perturbation of $Y_p = A_pL_p + C_pR_p + BI$ from Proposition 4.17, i.e., for all $g \in l^p(\mathbb{N}_0)$,

$$\begin{aligned} X_p g &= Y_p g + (\rho g_0 + \sigma g_1)\delta_0 + \psi g_0 \delta_1, \\ \tilde{X}_p g &= Y_p g + \tilde{\rho} g_0 \delta_0. \end{aligned} \tag{4.13}$$

Having found a connection between the operators $T_{\varepsilon_1 * \varepsilon_1}$ and Y_p, we next examine how their spectra are related. By (4.12), $\sigma(Z_p) = \sigma_p(T_{\varepsilon_1 * \varepsilon_1})$, and $\sigma_{point}(Z_p) = \sigma_{point,p}(T_{\varepsilon_1 * \varepsilon_1})$. Since Z_p is the direct sum of X_p and $\tilde{X}_p$, $\sigma(Z_p) = \sigma(X_p) \cup \sigma(\tilde{X}_p)$ and $\sigma_{point}(Z_p) = \sigma_{point}(X_p) \cup \sigma_{point}(\tilde{X}_p)$. By (4.11), to complete the proof we have to show that for $1 \le p \le 2$,

$$\sigma(X_p) \cup \sigma(\tilde{X}_p) = \begin{cases} E_p, & \beta \ge \alpha, \\ \{0\} \cup E_p, & \beta < \alpha. \end{cases} \tag{4.14}$$

In order to prove (4.14), we will show that, for $1 \le p \le 2$,

$$\sigma(X_p) \subseteq \sigma(Y_p) \cup \sigma_{point}(Z_p), \quad \sigma(\tilde{X}_p) \subseteq \sigma(Y_p) \cup \sigma_{point}(Z_p), \tag{4.15}$$

$$\sigma(Y_p) \subseteq \sigma(X_p) \cup \sigma_{point}(Y_p), \quad \sigma(Y_p) \subseteq \sigma(\tilde{X}_p) \cup \sigma_{point}(Y_p). \tag{4.16}$$

If we know (4.15) and (4.16), then Proposition 4.17 yields

$$E_p = \overline{\sigma(Y_p) \setminus \sigma_{point}(Y_p)} \subseteq \sigma(X_p) \cup \sigma(\tilde{X}_p) \subseteq \sigma(Y_p) \cup \sigma_{point}(Z_p) = E_p \cup \sigma_{point}(Z_p).$$

Lemma 4.16 then tells us that, in case $\beta \ge \alpha$,

$$E_p \subseteq \sigma(X_p) \cup \sigma(\tilde{X}_p) \subseteq E_p.$$

Furthermore, in case $\beta < \alpha$,

$$E_p \cup \{0\} \subseteq \sigma(X_p) \cup \sigma(\tilde{X}_p) \subseteq E_p \cup \{0\}.$$

Hence (4.14) follows, which completes the proof.

In order to show (4.15) for X_p, suppose $Y_p - \lambda I$ is invertible for some $\lambda \notin \sigma_{point}(Z_p)$. We have to show that $X_p - \lambda I$ is invertible. First we note that $X_p - \lambda I$ is one-to-one: If it were not one-to-one, then $Z_p - \lambda I$ would not be one-to-one either which contradicts $\lambda \notin \sigma_{point}(Z_p)$. We have to prove that $X_p - \lambda I$ is onto. Let $\tilde{h} \in l^p(\mathbb{N}_0)$ and define

$$f := (Y_p - \lambda I)^{-1}\tilde{h}, \;\; g := (Y_p - \lambda I)^{-1}\delta_0, \;\; \tilde{g} := (Y_p - \lambda I)^{-1}\delta_1.$$

From (4.13) we obtain for $s, t \in \mathbb{C}$ that

$$\begin{aligned} (X_p \; - \; & \lambda I)(f + sg + t\tilde{g}) \\ = \; & (Y_p - \lambda I)(f + sg + t\tilde{g}) + (\rho(f_0 + sg_0 + t\tilde{g}_0) + \sigma(f_1 + sg_1 + t\tilde{g}_1))\delta_0 \\ & \qquad + \psi(f_0 + sg_0 + t\tilde{g}_0)\delta_1 \\ = \; & \tilde{h} + [(\rho f_0 + \sigma f_1) + s(1 + \rho g_0 + \sigma g_1) + t(\rho\tilde{g}_0 + \sigma\tilde{g}_1)]\delta_0 \\ & \qquad + [\psi f_0 + s \cdot \psi g_0 + t(1 + \psi\tilde{g}_0)]\delta_1. \end{aligned} \tag{4.17}$$

The matrix

$$Q = \begin{pmatrix} (1 + \rho g_0 + \sigma g_1) & (\rho\tilde{g}_0 + \sigma\tilde{g}_1) \\ \psi g_0 & (1 + \psi\tilde{g}_0) \end{pmatrix}$$

does not depend on $\tilde{h}$ and f. If Q is invertible, there are $(s,t) \in \mathbb{C}^2$ such that $Q(s,t)^T = (-(\rho f_0 + \sigma f_1), -\psi f_0)^T$. By (4.17) this means that $(X_p - \lambda I)(f + sg + t\tilde{g}) = \tilde{h}$ and thus $X_p - \lambda I$ is onto. Towards a contradiction we suppose that Q is not invertible and choose $0 \neq (s_0, t_0) \in \mathbb{C}^2$ such that $Q(s_0, t_0)^T = 0$. For $\tilde{h} = 0$ (and thus $f = 0$) (4.17) yields $(X_p - \lambda I)(s_0 g + t_0 \tilde{g}) = 0$. As shown above, $X_p - \lambda I$ is one-to-one, which implies $s_0 g + t_0 \tilde{g} = 0$. But $Y_p - \lambda I$ is one-to-one and $(Y_p - \lambda I)(s_0 g + t_0 \tilde{g}) = s_0\delta_0 + t_0\delta_1 \neq 0$, which implies $s_0 g + t_0 \tilde{g} \neq 0$, a contradiction.

The remaining inclusion of (4.15) as well as (4.16) are shown in an analogous but easier way. $\square$

5 Regularity of $L^1(K)$

Regularity of a commutative Banach algebra was first introduced (via the notion of the hull-kernel topology on the maximal ideal space) during the 1940ies by Gelfand, Shilov and Jacobson, see [25, Ch. 4.9].

We recall Definition 3.4.

Definition. *A commutative Banach algebra A is called regular, if for every closed subset V of $\Delta(A)$ and $\alpha \in \Delta(A)\backslash V$ there is $a \in A$ with Gelfand transform $\hat{a}|_V = 0$ and $\hat{a}(\alpha) \neq 0$.*

For locally compact abelian groups G it is well-known that $L^1(G)$ is regular, see [25, Thm. 4.4.14]. All proofs known to the author use the Plancherel theorem and, most notably, the convolution on $\hat{G}$. Since hypergroups are in general not equipped with a natural dual convolution structure (induced by pointwise multiplication of characters), there is no obvious approach to the problem for hypergroups. For so-called strong commutative hypergroups K (those hypergroups which carry a dual convolution structure) regularity of $L^1(K)$ is shown in [9, Thm. 2.6] using the familiar proof for locally compact abelian groups.

We recall that the structure space of $L^1(K)$ can be identified with the character space $\chi_b(K)$, see (1.3). The following observation shows that in contrast to the group case not all L^1-algebras on hypergroups are regular; for an example of a hypergroup with supp $\pi \subsetneq \hat{K} \subsetneq \chi^b(K)$ see for example Chapter 4.3.

Lemma 5.1. *Let K be a commutative hypergroup. If $L^1(K)$ is regular, then supp $\pi = \hat{K} = \chi^b(K)$.*

Proof. Suppose that $\alpha \in \chi^b(K) \setminus$ supp π. Since $L^1(K)$ is regular and supp π is closed in $\chi^b(K)$, there is $f \in L^1(K)$ such that $\hat{f}|_{\text{supp }\pi} = 0$ and $\hat{f}(\alpha) \neq 0$. This is impossible because $\hat{f}|_{\text{supp }\pi} = 0$ gives $\check{f} = 0 \in L^1(\hat{K})$, and thus by Theorem 1.4 that $f = (\hat{f})\check{} = 0$. In particular, $\hat{f}(\alpha) = 0$ for all $\alpha \in \chi^b(K)$. $\square$

To the author's knowledge, until now there exist two positive results for hypergroups: In [17, Thm. 2.1], Gallardo and Gebuhrer prove regularity of $L^1(K)$ for commutative hypergroups K whose Haar measure is of polynomial growth (they also supposed $\hat{K} = \chi_b(K)$ which nowadays is known to be automatically fulfilled by [57, Pro. 2.6]). Their proof uses Dixmier's functional calculus based on [11, Lem. 7]. In [58, Cor. 2.8] Vogel proves complete regularity of $L^1(K)$ for central (not necessarily commutative) hypergroups K. He first shows that K is always of polynomial growth which enables him to use Dixmier's functional calculus, too.

In this chapter we extend this functional calculus of Dixmier (Theorem 5.3) which leads to a slight improvement of Gallardo and Gebuhrer's result beyond polynomial growth (Lemma 5.4, Theorem 5.6). We begin with some terminology and facts about Beurling algebras on $\mathbb{R}$.

As before we denote by $L^1(K)_e$ the algebra $L^1(K)$ with adjoint identity δ_e (if $L^1(K)$ is not unital, or equivalently, K is not discrete). $\|f\|$ and $r(f)$ denote the norm and the spectral radius of $f \in L^1(K)_e$, respectively. For every $f \in L^1(K)_e$ and $t \geq 0$ we define

$e^{tf} \in L^1(K)_e$ via the holomorphic functional calculus; this clearly yields the same as the absolutely convergent sum

$$e^{tf} = \sum_{k=0}^{\infty} \frac{t^k f^{*k}}{k!}, \qquad \text{where } f^{*0} = \delta_e.$$

Lemma 5.2. *Let K be a commutative hypergroup and let $f \in L^1(K)$ have real spectrum. Define the weight function $\omega(t) = \|e^{-itf}\|$, $t \in \mathbb{R}$. Then $L^1(\mathbb{R}, \omega)$ is a Beurling algebra. Its structure space can be identified with $\mathbb{R}$ in a way such that the Gelfand transform reads*

$$\hat{\varphi}(s) = \int_{\mathbb{R}} \varphi(t) e^{-ist} dt, \quad \varphi \in L^1(\mathbb{R}, \omega), \; s \in \mathbb{R}. \tag{5.1}$$

Proof. Clearly $\omega(t+s) = \|e^{-i(t+s)f}\| = \|e^{-itf} * e^{-isf}\| \leq \|e^{-itf}\| \cdot \|e^{-isf}\| = \omega(t)\omega(s)$ and further ω is continuous and thus Borel measurable. By [25, Chapter 1.3] $L^1(\mathbb{R}, \omega)$ is a Beurling algebra. Since $f \in L^1(K)$ has real spectrum, we obtain in addition that

$$\begin{aligned} R_+ &:= \inf_{t>0} \{\omega(t)^{\frac{1}{t}}\} = \inf_{t>0} \{\|e^{-itf}\|^{\frac{1}{t}}\} = r(e^{-if}) = 1 \qquad \text{and} \\ R_- &:= \sup_{t>0} \{\omega(-t)^{-\frac{1}{t}}\} = \sup_{t>0} \{\|e^{itf}\|^{-\frac{1}{t}}\} = \frac{1}{\inf_{t>0}\{\|e^{itf}\|^{\frac{1}{t}}\}} = \frac{1}{r(e^{if})} = 1. \end{aligned}$$

Since $R_+ = R_- = 1$ an application of [25, Lem. 2.8.6 and Pro. 2.8.7] yields that the structure space of $L^1(\mathbb{R}, \omega)$ can be identified with $\mathbb{R}$ in the way of (5.1). □

Theorem 5.3. *Let K be a commutative hypergroup and $f \in L^1(K)$ have real spectrum. Let furthermore $\varphi \in L^1(\mathbb{R}, \omega)$ be a complex-valued Beurling function with respect to the weight function $\omega(t) = \|e^{-itf}\|$, i.e.*

$$\int_{\mathbb{R}} |\varphi(t)| \|e^{-itf}\| dt < \infty.$$

Then the Bochner integral

$$\hat{\varphi}(f) := \int_{\mathbb{R}} \varphi(t) e^{-itf} dt$$

exists and thus defines an element $\hat{\varphi}(f) \in L^1(K)_e$. If $\hat{\varphi}(0) = 0$ then $\hat{\varphi}(f) \in L^1(K)$. Furthermore we have

$$\langle \hat{\varphi}(f), \alpha \rangle = \hat{\varphi}(\langle f, \alpha \rangle) \text{ for all } \alpha \in \chi^b(K).$$

Proof. The map $\mathbb{R} \to L^1(K)_e$, $t \mapsto e^{-itf}$, is continuous and thus separably valued. This implies that $\mathbb{R} \to L^1(K)_e$, $t \mapsto e^{-itf}\varphi(t)$, is also separably valued and therefore strongly measurable. Since it is absolutely integrable, the Bochner integral exists (see for example [21, Thm. 7.5.11]). Now suppose that $\hat{\varphi}(0) = 0$. Then $\int_{\mathbb{R}} \varphi(t) dt = 0$ and also $\int_{\mathbb{R}} \delta_e \varphi(t) dt = \delta_e \cdot \int_{\mathbb{R}} \varphi(t) dt = 0$, where $\delta_e \in L^1(K)_e$ is the identity. Then

$$\hat{\varphi}(f) = \int_{\mathbb{R}} e^{-itf} \varphi(t) dt = \int_{\mathbb{R}} (e^{-itf} - \delta_e) \varphi(t) dt.$$

Since $(e^{-its}-1)=0$ for $s=0$ we know from the holomorphic functional calculus that $e^{-itf}-\delta_e \in L^1(K)$ for all $t \in \mathbb{R}$. Thus the integral converges to $\hat{\varphi}(f)$ in $L^1(K)$ which is closed in $L^1(K)_e$. Finally, for $\alpha \in \chi^b(K)$ we obtain

$$\begin{aligned}\langle \hat{\varphi}(f), \alpha \rangle = \left\langle \int_{\mathbb{R}} e^{-itf}\varphi(t)dt, \alpha \right\rangle &= \int_{\mathbb{R}} \left\langle e^{-itf}, \alpha \right\rangle \varphi(t)dt \\ &= \int_{\mathbb{R}} e^{-it\langle f,\alpha\rangle}\varphi(t)dt = \hat{\varphi}(\langle f,\alpha\rangle),\end{aligned}$$

where the equalities are in turn due to a property of the Bochner integral (see for example [21, Cor. after Thm. 7.5.11]), the holomorphic functional calculus and the Gelfand transform according to (5.1). □

The above functional calculus is essentially due to Dixmier [11, Lem. 7] in the case of certain Lie groups. However, he does not show it for all $\varphi \in L^1(\mathbb{R},\omega)$, but only for $\hat{\psi}$ with derivatives of 'high enough' order; his assumptions yield that $\omega(t) = \|e^{-itf}\|$ is of polynomial growth and thus $\psi \in L^1(\mathbb{R},\omega)$.

Nevertheless, the idea of embedding Beurling algebras into Banach algebras in the above way is not new; see for example [56, Lem. 2.4.3], where Beurling algebras are embedded into the algebra of bounded linear operators on a Banach space in the context of C_0-groups of linear operators.

Lemma 5.4. *Let K be a commutative hypergroup, V a closed subset of $\Delta(L^1(K))$ and $\alpha \in \Delta(L^1(K)) \setminus V$. Suppose that there is $f \in L^1(K)$ with real spectrum, $\hat{f}(V) \subset [-\varepsilon,\varepsilon]$, $\hat{f}(\alpha) > \varepsilon$ for some $\varepsilon > 0$, such that*

$$\sum_{n\in\mathbb{N}_0} \frac{\ln \|e^{inf}\|}{1+n^2} < \infty. \tag{5.2}$$

Then there is $g \in L^1(K)$ with Gelfand transform $\hat{g}|_V = 0$ and $\hat{g}(\alpha) = 1$.

Proof. If we can show that

$$\int_{\mathbb{R}} \frac{\ln \|e^{-itf}\|}{1+t^2} dt < \infty, \tag{5.3}$$

then the statement follows; by [25, Lem. 4.7.8] the Beurling algebra $L^1(\mathbb{R},\omega)$ with weight function $\omega(t) = \|e^{itf}\|$ is then regular, since ω is non-quasianalytic. In particular there is a $\varphi \in L^1(\mathbb{R},\omega)$ such that $\hat{\varphi} = 0$ on $[-\varepsilon,\varepsilon]$ and $\widehat{\hat{\varphi}(\hat{f}(\alpha))} = 1$. Thus $g := \hat{\varphi}(f) \in L^1(K)$ of Theorem 5.3 fulfills the wanted properties $\widehat{\hat{\varphi}(f)}(V) = \hat{\varphi}(\hat{f}(V)) = 0$ and $\langle \hat{\varphi}(f), \alpha \rangle = \hat{\varphi}(\langle f,\alpha\rangle) = 1 \neq 0$.

In order to show (5.3) we first note that $f \in L^1(K)$ has real spectrum. Thus, for all $\beta \in \hat{K}$ we have

$$\widehat{e^{-itf}}(\beta) = e^{-it\hat{f}(\beta)} = \overline{e^{it\hat{f}(\beta)}} = \overline{\widehat{e^{itf}}(\beta)} = \widehat{(e^{itf})^*}(\beta).$$

This means $e^{-itf} = (e^{itf})^*$ and thus $\|e^{-itf}\| = \|e^{itf}\|$. Secondly we note that for $t \geq 0$ we can estimate $\|e^{itf}\| \leq \|e^{i\lfloor t\rfloor f}\| \cdot \|e^{i(t-\lfloor t\rfloor)f}\| \leq \|e^{i\lfloor t\rfloor f}\| \cdot e^{\|f\|}$. We obtain

$$\int_{\mathbb{R}} \frac{\ln \|e^{-itf}\|}{1+t^2} dt = 2 \cdot \int_0^\infty \frac{\ln \|e^{itf}\|}{1+t^2} dt \leq 2e^{\|f\|} \cdot \int_0^\infty \frac{\ln \|e^{i\lfloor t\rfloor f}\|}{1+\lfloor t\rfloor^2} dt$$
$$= 2e^{\|f\|} \cdot \sum_{\mathbb{N}_0} \frac{\ln \|e^{inf}\|}{1+n^2} < \infty.$$

This completes the proof. □

For an estimation of $\|e^{inf}\|$ in (5.2) we note that the theory of strongly continuous one-parameter-semigroups of operators makes heavy use of the connection between the growth of the generator's resolvents and the growth of the semigroup. In particular, polynomial growth of $\|e^{inf}\|$, $n \to \pm\infty$, is characterized by a certain growth of $\|(\lambda - f)^{-1}\|$ when λ approaches the spectrum of f [3].

For the following well-known notions see also [7, Def. 2.5.11].

Definition 5.5. *A commutative hypergroup K is said to be of*

a) subexponential growth if for every compact $C \subset K$ and every $k > 1$ we have $m(C^n) = o(k^n)$ as $n \to \infty$,

b) polynomial growth if for every compact $C \subset K$ there exists $k := k(C) \geq 0$ such that $m(C^n) = O(n^k)$ as $n \to \infty$.

The assumption in the following theorem is more general than polynomial growth, but not as general as subexponential growth.

Theorem 5.6. *Let K be a commutative hypergroup whose Haar measure fulfills the following: for every compact set $C \subset K$ there are $\delta > 0$, $k > 1$ and some $0 \leq t < \frac{1}{2}$ such that $m(C^n) \leq \delta \cdot k^{(n^t)}$ for all $n \in \mathbb{N}$. Then $L^1(K)$ is regular.*

Proof. Let V be a closed subset of $\Delta(L^1(K))$ and $\alpha \in \Delta(L^1(K)) \setminus V$. Since the Haar measure is of subexponential growth we know by [7, Thm. 2.5.12 and Cor. 2.5.13] that $\operatorname{supp} \pi = \Delta(L^1(K))$. Because $\widehat{L^1(K)}|_{\hat{K}}$ is dense in $C_0(\hat{K})$ [7, Thm. 2.2.4 (ix)], we can choose $f \in C_C(K)$ with real spectrum $\hat{f}(\Delta(L^1(K))) \subset \mathbb{R}$, $\hat{f}(V) \subset [-\varepsilon, \varepsilon]$, $\hat{f}(\alpha) > \varepsilon$ and $\|f\|_1 \leq 1$. If we can show that

$$\sum_{\mathbb{N}_0} \frac{\ln \|e^{inf}\|}{1+n^2} < \infty,$$

then the needed $g \in L^1(K)$ with $\hat{g}|_V = 0$ and $\hat{g}(\alpha) \neq 0$ would exist by Lemma 5.4. We first observe that $e^{inf} - \delta_e = \sum_{k=1}^\infty \frac{i^k n^k f^{*k}}{k!} \in L^1(K)$. Since $f \in C_C(K)$, f^{*k} also has compact support in $(\operatorname{supp} f)^k$. Thus,

$$\int_{K\setminus(\operatorname{supp} f)^{n^2-1}} |e^{inf} - \delta_e| dm = \int_{K\setminus(\operatorname{supp} f)^{n^2-1}} \left| \sum_{k=n^2}^\infty \frac{i^k n^k f^{*k}}{k!} \right| dm \leq \sum_{k=n^2}^\infty \frac{n^k \|f\|^k}{k!} \leq \sum_{k=n^2}^\infty \frac{n^k}{k!}$$
$$\leq \frac{n^{n^2} e^n}{(n^2)!} = O(n^{n^2} e^n (n^2)^{-n^2} e^{n^2} n^{-1}) = O(n^{-n^2-1} e^{n^2+n}) \to 0$$

for $n \to \infty$. Secondly we note that $|e^{ix} - 1| \le |x|$, $x \in \mathbb{R}$, and thus $|e^{in\hat{f}(\alpha)} - 1| \le n|\hat{f}(\alpha)|$, $\alpha \in \Delta(L^1(K))$. From the Plancherel isomorphism we know that $e^{inf} - \delta_e \in L^2(K)$ and $\|e^{inf} - \delta_e\|_2 \le n\|f\|_2$. The assumed growth of the Haar measure enables us to estimate

$$\int_{(\mathrm{supp}\ f)^{n^2-1}} |e^{inf} - \delta_e| dm \le \|e^{inf} - \delta_e\|_2 \cdot m((\mathrm{supp}\ f)^{n^2-1})^{\frac{1}{2}} \le n\|f\|_2 \cdot (\delta \cdot k^{(n^{2t})})^{\frac{1}{2}}.$$

For large enough $n \in \mathbb{N}$ we obtain that

$$\|e^{inf}\| \le \|e^{inf} - \delta_e\|_1 + 1 \le \tilde{\varepsilon} + n\|f\|_2 \cdot (\delta \cdot k^{(n^{2t})})^{\frac{1}{2}} + 1 \le 2 \cdot n\|f\|_2 \delta^{\frac{1}{2}} \cdot (k^{\frac{1}{2}})^{(n^{2t})}.$$

This shows (5.2) since for large enough $n \in \mathbb{N}$ we now know that

$$\frac{\ln \|e^{inf}\|}{1+n^2} \le \frac{\ln(2\|f\|_2 \delta^{\frac{1}{2}})}{1+n^2} + \frac{\ln n}{1+n^2} + \ln(k^{\frac{1}{2}}) \cdot \frac{n^{2t}}{1+n^2}$$

and $t < \frac{1}{2}$. The proof is complete. □

The first part of our proof is modeled on the one of Gallardo and Gebuhrer [17, Thm. 2.1], namely the use of the fact that $\widehat{L^1(K)}|_{\hat{K}}$ is dense in $C_0(\hat{K})$. In the second part they proceed by using Dixmier's functional calculus while we used Lemma 5.4 (which is due to our Beurling point of view). The method used to estimate $\|e^{inf}\|$ in the second part of the proof is basically the original one used by Dixmier for certain Lie groups in [11, Lem. 6].

A special case of the above theorem or the one in [17, Thm. 2.1] occurs for commutative and compact hypergroups; here $m(C^n)$ is bounded when the compact subsets C and $n \in \mathbb{N}$ vary. For compact hypergroups the structure space is discrete [31]. In this case Shilov's Idempotent Theorem [25, Thm. 3.5.1] then immediately yields regularity of $L^1(K)$.

References

[1] R. Askey and G. Gasper. Jacobi polynomial expansions of Jacobi polynomials with non-negative coefficients. *Proc. Camb. Philos. Soc.*, 70:243–255, 1971.

[2] R. Askey and G. Gasper. Linearization of the product of Jacobi polynomials. III. *Can. J. Math.*, 23:332–338, 1971.

[3] B. A. Barnes. Operators which satisfy polynomial growth conditions. *Pac. J. Math.*, 138(2):209–219, 1989.

[4] B. A. Barnes. When is the spectrum of a convolution operator on L^p independent of p? *Proc. Edinb. Math. Soc., II. Ser.*, 33(2):327–332, 1990.

[5] H. Bavinck. On absolute convergence of Jacobi series. *J. Approx. Theory*, 4:387–400, 1971.

[6] J. Bergh and J. Löfström. *Interpolation Spaces.* Springer, Berlin, 1976.

[7] W. R. Bloom and H. Heyer. *Harmonic analysis of probability measures on hypergroups.* de Gruyter, Berlin, 1995.

[8] W. R. Bloom and M. E. Walter. Isomorphisms of hypergroups. *J. Aust. Math. Soc., Ser. A*, 52(3):383–400, 1992.

[9] A. K. Chilana and K. A. Ross. Spectral synthesis in hypergroups. *Pac. J. Math.*, 76:313–328, 1978.

[10] H. G. Dales. *Banach algebras and automatic continuity.* London Mathematical Society, Oxford, 2000.

[11] J. Dixmier. Opérateurs de rang fini dans les représentations unitaires. *Publ. Math., Inst. Hautes Étud. Sci.*, 6:305–317, 1960.

[12] N. Dunford and J. T. Schwartz. *Linear operators. Part I: General theory.* John Wiley & Sons Ltd., New York, 1988.

[13] C. F. Dunkl. The measure algebra of a locally compact hypergroup. *Trans. Amer. Math. Soc.*, 179:331–348, 1973.

[14] B. Engelhardt. *Representation and Statistical Inference of Random Sequences on Convolution Structures.* Dissertation, Munich University of Technology, 2003.

[15] F. Filbir and R. Lasser. Reiter's condition P_2 and the Plancherel measure for hypergroups. *Ill. J. Math.*, 44(1):20–32, 2000.

[16] F. Filbir, R. Lasser, and R. Szwarc. Reiter's condition P_1 and approximate identities for polynomial hypergroups. *Monatsh. Math.*, 143(3):189–203, 2004.

[17] L. Gallardo and O. Gebuhrer. Marches aleatoires sur les hypergroupes localement compacts et analyse harmonique commutative (preprint). 1985.

[18] G. Gasper. Linearization of the product of Jacobi polynomials. II. *Can. J. Math.*, 22:582–593, 1970.

[19] I. T. Gokhberg and M. K. Zambitskij. On the theory of linear operators in spaces with two norms. *Amer. Math. Soc., Transl., II. Ser.*, 85:145–163, 1969.

[20] F. Gourdeau. Amenability of Banach algebras. *Math. Proc. Camb. Philos. Soc.*, 105(2):351–355, 1989.

[21] E. Hille. *Methods in classical and functional analysis.* Addison-Wesley, Reading, Mass., 1972.

[22] V. Hösel, M. Hofmann, and R. Lasser. Means and Følner condition on polynomial hypergroups. *Mediterr. J. Math.*, 7(1):75–88, 2010.

[23] V. Hösel and R. Lasser. Approximation with Bernstein-Szegő polynomials. *Numer. Funct. Anal. Optim.*, 27:377–389, 2006.

[24] R. I. Jewett. Spaces with an abstract convolution of measures. *Adv. Math.*, 18:1–101, 1975.

[25] E. Kaniuth. *A course in commutative Banach algebras.* Springer, New York, 2009.

[26] E. Kaniuth, A. T. Lau, and J. Pym. On φ-amenability of Banach algebras. *Math. Proc. Camb. Philos. Soc.*, 144(1):85–96, 2008.

[27] S. Karlin and J. McGregor. Random walks. *Ill. J. Math.*, 3:66–81, 1959.

[28] R. Lasser. Fourier-Stieltjes transforms on hypergroups. *Analysis (Munich)*, 2:281–303, 1982.

[29] R. Lasser. Orthogonal polynomials and hypergroups. *Rend. Mat. Appl., VII. Ser.*, 3:185–209, 1983.

[30] R. Lasser. Orthogonal polynomials and hypergroups. II: The symmetric case. *Trans. Amer. Math. Soc.*, 341(2):749–770, 1994.

[31] R. Lasser. On the character space of commutative hypergroups. *Jahresber. Dtsch. Math.-Ver.*, 104:3–16, 2002.

[32] R. Lasser. Amenability and weak amenability of l^1-algebras of polynomial hypergroups. *Studia Math.*, 182(2):183–196, 2007.

[33] R. Lasser. Point derivations on the L^1-algebra of polynomial hypergroups. *Colloq. Math.*, 116(1):15–30, 2009.

[34] R. Lasser and E. Perreiter. Homomorphisms of l^1-algebras on signed polynomial hypergroups. *Banach J. Math. Anal.*, 4(2):1–10, 2010.

[35] R. Lasser and M. Rösler. A note on property (T) of orthogonal polynomials. *Arch. Math. (Basel)*, 60(5):459–463, 1993.

[36] W. Młotkowski and R. Szwarc. Nonnegative linearization for polynomials orthogonal with respect to discrete measures. *Constr. Approx.*, 17(3):413–429, 2001.

[37] M. S. Monfared. Character amenability of Banach algebras. *Math. Proc. Camb. Philos. Soc.*, 144(3):697–706, 2008.

[38] J. I. Nieto. On the essential spectrum of symmetrizable operators. *Math. Ann.*, 178:145–153, 1968.

[39] A. W. Parr. *Signed Hypergroups.* PhD thesis, Graduate School of the University of Oregon, 1997.

[40] E. Perreiter. *Amenability and weak amenability of l^1-algebras of some particular polynomial hypergroups.* Diploma thesis, Munich University of Technology, 2007.

[41] E. Perreiter. Spectra of L^1-convolution operators acting on L^p-spaces of commutative hypergroups. *Math. Proc. Camb. Philos. Soc.*, 151(3):503–519, 2011.

[42] J.-P. Pier. *Amenable locally compact groups.* John Wiley & Sons, New York, 1984.

[43] T. J. Ransford. The spectrum of an interpolated operator and analytic multivalued functions. *Pac. J. Math.*, 121:445–466, 1986.

[44] C. E. Rickart. *General theory of Banach algebras.* Princeton, Van Nostrand, 1960.

[45] M. Rösler. Convolution algebras which are not necessarily positivity-preserving. Connett, William C. (ed.) et al., Applications of hypergroups and related measure algebras. American Mathematical Society, Providence, 299-318, 1995.

[46] M. Rösler. On the dual of a commutative signed hypergroup. *Manuscripta Math.*, 88(2):147–163, 1995.

[47] K. A. Ross. Signed hypergroups - a survey. Connett, William C. (ed.) et al., Applications of hypergroups and related measure algebras. American Mathematical Society, Providence, 319-329, 1995.

[48] V. Runde. *Lectures on amenability.* Springer, Berlin, 2002.

[49] M. Skantharajah. Amenable hypergroups. *Ill. J. Math.*, 36(1):15–46, 1992.

[50] R. Spector. Apercu de la théorie des hypergroupes. Anal. harmon. Groupes de Lie, Semin. Nancy-Strasbourg 1973-75, Lect. Notes Math. 497. Springer, Berlin, 643-673, 1975.

[51] R. Spector. Mesures invariantes sur les hypergroupes. *Trans. Amer. Math. Soc.*, 239:147–165, 1978.

[52] G. Szegő. *Orthogonal polynomials.* American Mathematical Society, Providence, 1975.

[53] R. Szwarc. Connection coefficients of orthogonal polynomials. *Canad. Math. Bull.*, 35(4):548–556, 1992.

[54] W. F. Trench. Nonnegative and alternating expansions of one set of orthogonal polynomials in terms of another. *SIAM J. Math. Anal.*, 4:111–115, 1973.

[55] A. Ülger. Some results about the spectrum of commutative Banach algebras under the weak topology and applications. *Monatsh. Math.*, 121(4):353–379, 1996.

[56] J. van Neerven. *The asymptotic behaviour of semigroups of linear operators.* Birkhäuser, Basel, 1996.

[57] M. Vogel. Spectral synthesis on algebras of orthogonal polynomial series. *Math. Z.*, 194:99–116, 1987.

[58] M. Vogel. Harmonic analysis and spectral synthesis in central hypergroups. *Math. Ann.*, 281(3):369–385, 1988.

[59] M. Voit. Positive characters on commutative hypergroups and some applications. *Math. Z.*, 198(3):405–421, 1988.

[60] M. W. Wilson. Nonnegative expansions of polynomials. *Proc. Amer. Math. Soc.*, 24:100–102, 1970.