Lehrstuhl für Regelungstechnik
Technical Reports on Automatic Control
Vol. TRAC-6, August 2, 2011
Available online at: www.rt.mw.tum.de
Technische Universität München

3D-Modeling of a Robot Balancing on a Ball

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Abstract

This technical report presents a 3D model of a robot that balances on a ball. This new generation of robots, also called Ballbots, has recently been developed and described by several authors. Different from the conventional approach of generalizing 2D behavior to the 3D case, however, this report presents a full 3D model derived from geometrical, mechanical and energetical considerations; moreover, the motor dynamics is included. For control purposes, the model is linearized and analyzed with particular attention to controllability. The valid range of the linearized system is discussed based on a comparison with the original nonlinear model. Finally, the new model is validated using a 2D model from the literature.

Keywords: Robot modeling, 3D-modeling, Nonlinear modeling, controllability.

1 Introduction

During the last couple of years, a new generation of robots, called Ballbots [7] have been designed, developed and controlled by several authors, [1, 8, 9, 10]. In this arrangement, the robot balances upright on a sphere which is the only contact to the ground; the concept of motion might thus be called an "Inverse Mouse-Ball Drive". Similar to an inverse pendulum the Ballbot is an under-actuated system whose center of gravity has to be balanced by a stabilizing feedback controller. Unlike the classical pendulum, however, the Ballbot dynamics is of higher complexity, as the ball is also free to move and has to be modeled by a finite inertia, accordingly. Additional nonholonomic constraints [13] are required to describe the motion relation between ball and robot. Due to that complex dynamical behavior a detailed system description is needed for the purpose of control. While simplified models resulting from a generalization of the 2D to the 3D behavior have been described in former works [8, 9], a detailed 3D model developed from spatial considerations has not been addressed yet, to the best of the authors' knowledge. The aim of this report is therefore to derive a 3D model of a Ballbot which allows the design and testing of modern control methods. The model is developed with the help of Balancino, a new Ballbot designed at the Institute of Automatic Control at the Technische Universität München. Its mechanical construction, shown in Figure 1(b), is similar to the one presented in [8].

Balancino mainly consists of two mechanical parts: body and undercarriage. Three sustaining actuators are placed rotation-symmetrically at 120°, where each of them drives a so-called omnidirectional wheel (omniwheel). The wheels bear on a sphere that rolls on the ground and enables the robot to move in any direction. The omniwheel effect allows to transfer three horizontal forces to the ball (along the circle of latitude), whereas tangential forces (along a meridian)

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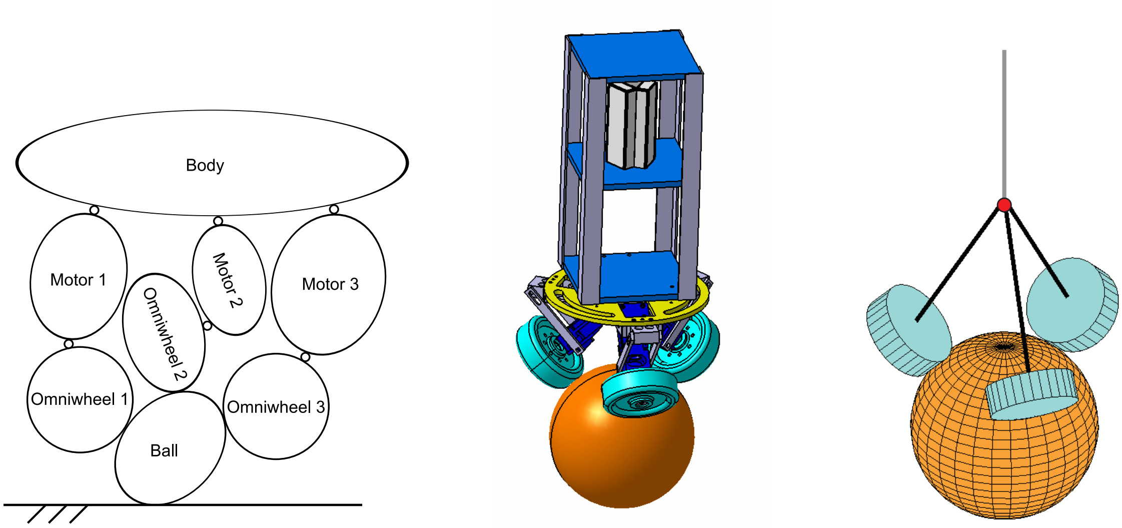
cannot be conveyed. Upon the undercarriage, the body of the robot is mounted. It is shaped like a cuboid and carries batteries, electronics and, as the case may be, loads. For a technical realization, the following hardware is considered: brushless DC electric motors as actuators, three Lithium-ion polymer batteries as power supply, a 32-bit microcontroller and gyroscopic and acceleration sensors, all installed at the body of the robot. In addition, a camera based location system and a digital compass might be considered for navigation purposes.
Based on that technical conditions, the rest of the report is organized as follows: In Section 2, the robot design and its degrees of freedom (DOFs) are detailed. The equations of motion, with particular attention to constraints, energy and non-conservative force derivation, are mathematically derived in Section 3. The nonlinear equations are linearized and a linear state space model is given in Section 4, while in Sections 5 the linear system is analyzed and its controllability conditions are discussed in details. The report contribution is summarized in Section 6.

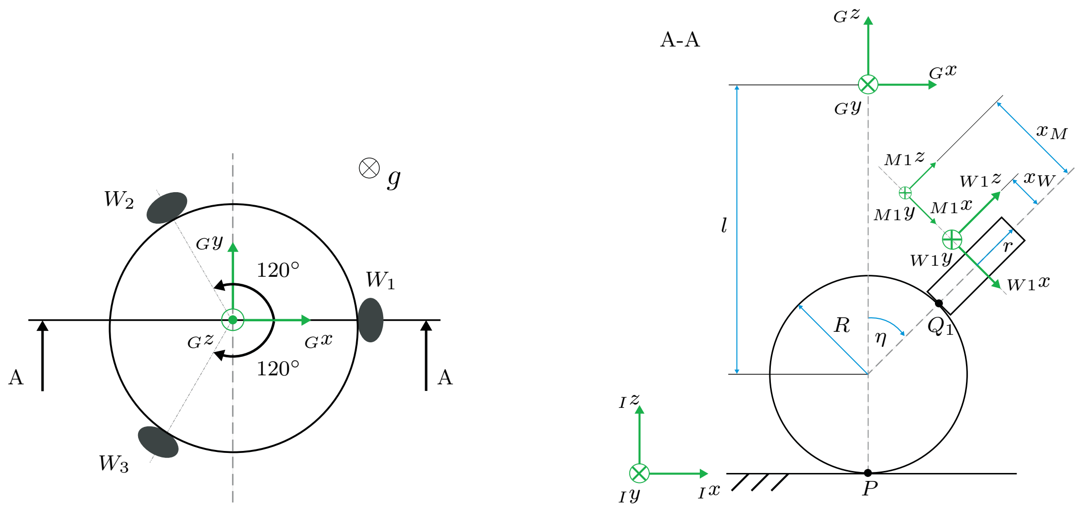
2 Balancino's design and DOFs

The computer-aided design (CAD) model of Balancino is shown in Figure 1(b). The main components of the robot and their topological interconnection to a multi-body system structure are represented in Figure 1(a). It consists of a ball, three omniwheels, three motors and the robot body, which are all considered as rigid bodies even if small deformations may occur, due to the robot weight and motion. In Figure 1(a) 'o' represents joints or fixed connections between the bodies: The three omniwheels, for instance, have direct contact to the ball. These contacts are modeled by slip-free rolling (see Section 3.1), which is a commonly used modeling assumption for such systems [8, 9, 10] and legitimate if the acting forces and momentums stay within reason. Also the Matlab simulation model is shown (Figure 1(c)), where the dynamic characteristics of Balancino's body are summed up in the center of gravity: only the kinetic connections are illustrated. In order to represent the relative position and orientation of a rigid body with respect to another one, coordinate frames are attached to the center of mass of each body as illustrated in Figure 2(b). This Figure is the section view A-A of the topview (Figure 2(a)) and shows one motor-omniwheel combination M_1, W_1. Table 1 resumes the notation for the components and their degrees of freedom (DOFs). In the following, the reference of the coordinate plane is subscripted on the left side of the described quantity; the subscript G, for instance, refers to



(a) Topology (b) CAD model (c) Simulation model

Figure 1: Modeling steps for Balancino



(a) Construction top view

(b) Section view A-A

Figure 2: Structure details

the coordinate plane belonging to the body of the robot. The index i ($i \in 1, 2, 3$) refers to the corresponding omniwheel or motor. The motors and the omniwheels have only one rotational DOF and their translation is fixed to Balancino's body coordinate frame. However, they are not located in the same center of mass, see Figure 2(b).
Coordinate transformations of vectors, which allow to express the kinematic relationship between the components of a multi-body system, can be carried out with the general procedure described e.g. in [5, 14]. The generalized transformation matrix

$$\mathbf{A}_{\mathbf{e},\vartheta} = \begin{bmatrix} e_1^2(1-\cos\vartheta)+\cos\vartheta & e_1e_2(1-\cos\vartheta)+e_3\sin\vartheta & e_1e_3(1-\cos\vartheta)-e_2\sin\vartheta \\ e_1e_2(1-\cos\vartheta)-e_3\sin\vartheta & e_2^2(1-\cos\vartheta)+\cos\vartheta & e_2e_3(1-\cos\vartheta)+e_1\sin\vartheta \\ e_1e_3(1-\cos\vartheta)+e_2\sin\vartheta & e_2e_3(1-\cos\vartheta)-e_1\sin\vartheta & e_3^2(1-\cos\vartheta)+\cos\vartheta \end{bmatrix} \quad (1)$$

describes the rotation ϑ around an axis intersecting the origin. Its direction is given by the normalized vector $\mathbf{e} = \begin{bmatrix} e_1, & e_2, & e_3 \end{bmatrix}^T$. Referring to the above indicated rigid bodies and their coordinate frames, the following transformation operations are needed to complete the kinematic

Table 1: Notation of DOFs and rigid bodies

Rigid body	Index	trans. DOF	rot. DOF	Comments
Ball	K	x_K, y_K, z_K	φ, ν, ψ	Rotation about the inertial axes
Robot body	G	x_G, y_G, z_G	α, β, γ	Cardan angles
Omniwheel	W_i	x_{W_i}, y_{W_i}, z_{W_i}	δ_i	Rotation about y-/z-axis not allowed
Motor	M_i	x_{M_i}, y_{M_i}, z_{M_i}	ϵ_i	Rotation about y-/z-axis not allowed

relationships between the components:

$$\mathbf{A}_{KI} = \mathbf{A}_{\mathbf{e}_b,\psi} \cdot \mathbf{A}_{\mathbf{e}_a,\nu} \cdot \mathbf{A}_{x,\varphi} \tag{2}$$
$$\mathbf{A}_{GI} = \mathbf{A}_{z,\gamma} \cdot \mathbf{A}_{y,\beta} \cdot \mathbf{A}_{x,\alpha} \tag{3}$$
$$\mathbf{A}_{W_1G} = \mathbf{A}_{y,\eta} \tag{4}$$
$$\mathbf{A}_{W_2G} = \mathbf{A}_{y,\eta} \cdot \mathbf{A}_{z,120^\circ} \tag{5}$$
$$\mathbf{A}_{W_3G} = \mathbf{A}_{y,\eta} \cdot \mathbf{A}_{z,240^\circ}, \tag{6}$$

with

$$\mathbf{e}_a = \begin{bmatrix} 0, & \cos\varphi, & -\sin\varphi \end{bmatrix}^T \tag{7}$$
$$\mathbf{e}_b = \begin{bmatrix} -\sin\nu, & \sin\varphi\cdot\cos\nu, & \cos\varphi\cdot\cos\nu \end{bmatrix}^T. \tag{8}$$

The same rotation matrices as for the corresponding omniwheel (equation (4) to (6)) are obtained for the motors. Based on that design preliminaries, the equations of motion are derivable, which is focused in the next Section.

3 Derivation of the Equations of motion

To obtain the equation of motion the Lagrangian approach of the first kind [5], which treats constraints explicitly in an extra term, is considered:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{p}}}\right)^T - \left(\frac{\partial T}{\partial \mathbf{p}}\right)^T + \left(\frac{\partial V}{\partial \mathbf{p}}\right)^T + \left(\frac{\partial D}{\partial \dot{\mathbf{p}}}\right)^T = \mathbf{Q}_{NK} + \mathbf{G}^T\boldsymbol{\lambda}, \tag{9}$$

where T represents the kinetic energy, V the potential energy, D the dissipation energy and $\mathbf{Q}_{NK}$ the non-conservative forces of all bodies. The quantity $\mathbf{G}^T$ is the derivative of the constraint conditions with respect to the bodies' coordinates, which multiplied by the Lagrange multipliers $\boldsymbol{\lambda}$ form the constraint forces. See [5] for a more detailed description of the Lagrangian approach. Because of the relation between the coordinate frames not only holonomic but also nonholonomic constraints [5, 13] are required. The equations of motion are therefore based on redundant coordinates $\mathbf{p}$, meaning that the obtained equations will have more state variables than DOFs exist. However, by considering constraint equations it is possible to reduce the system coordinates to the minimal coordinate configuration $\mathbf{q}$.
In the following Sections the unknown terms of (9) and thereafter the minimal coordinate configuration of the resulting equations of motion are derived. Unless otherwise noted, the terms are given in the inertial coordinate frame wherefore the subscript index I is omitted for the sake of readability. The parameters of the construction (Figure 1(b)) and all quantities required for modeling are summarized in Table 2 in appendix B.

3.1 Constraints

In this Section the contact constraints are discussed. They are required to formulate the physically impossible penetration of two rigid bodies in the model. These constraints characterize and relate the kinematics of a contact point of rigid bodies. In such a way, the DOFs of a component are bordered by adjoining components. In particular, the motion of the ball (contact point to the ground P) and the contact points between the ball and the omniwheels Q_i are considered (Figure 2(b)). The rolling relations of the ball position can be written as follow:

$$\begin{bmatrix} x_K \\ y_K \\ z_K \end{bmatrix} - \begin{bmatrix} R\,\nu \\ -R\,\varphi \\ R \end{bmatrix} = \mathbf{0}, \tag{10}$$

where R represents the radius of the ball. Due to the contact dynamics, the center of mass of Balancino's body is related to the ball position. Thus, the kinematic chain allows to describe the position of the body center of mass by the geometrical constraints given by

$$\begin{bmatrix} x_G \\ y_G \\ z_G \end{bmatrix} - \begin{bmatrix} x_K \\ y_K \\ z_K \end{bmatrix} + \mathbf{A}_{IG} \cdot \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix} = \mathbf{0}, \tag{11}$$

where $\mathbf{A}_{IG}$ represents the frame transformation (from frame G to frame I). Analogously to the body also the geometrical constraints for the i-th omniwheel

$$\begin{bmatrix} x_{W_i} \\ y_{W_i} \\ z_{W_i} \end{bmatrix} - \begin{bmatrix} x_K \\ y_K \\ z_K \end{bmatrix} + \mathbf{A}_{IG}\ \mathbf{A}_{GW_i} \cdot \begin{bmatrix} -x_W \\ 0 \\ R+r \end{bmatrix} = \mathbf{0}, \quad i = \{1,\ 2,\ 3\} \tag{12}$$

and the i-th DC-motor

$$\begin{bmatrix} x_{M_i} \\ y_{M_i} \\ z_{M_i} \end{bmatrix} - \begin{bmatrix} x_K \\ y_K \\ z_K \end{bmatrix} + \mathbf{A}_{IG}\ \mathbf{A}_{GW_i} \cdot \begin{bmatrix} -x_M \\ 0 \\ R+r \end{bmatrix} = \mathbf{0}, \quad i = \{1,\ 2,\ 3\} \tag{13}$$

can be named. The equations (11), (12) and (13) describe a kinematic permanent junction between ball and omniwheels. The axial rotations ϵ_i and δ_i, of each omniwheel and DC-motor, respectively, are coupled by the transmission factor i_G of the planetary gear:

$$\delta_i = \frac{1}{i_G}\epsilon_i. \tag{14}$$

Accordingly, the chosen redundant coordinates are

$$\mathbf{p} = \left[x_K, y_K, \psi, \alpha, \beta, \gamma, \delta_1, \delta_2, \delta_3\right]^T. \tag{15}$$

Next, the relation between the rotation of the omniwheels and the ball is considered. While a description only with holonomic constraints is not possible, additional kinematic chain equations have to be added. The considered slip-free motion leads to the same tangential velocity of the ball and each omniwheel in their contact points Q_i ($i = \{1,\ 2,\ 3\}$) (see Figure 2(b)). These velocities are

$${}_{W_i}\mathbf{v}_{Q_i}^{(K)} = \mathbf{A}_{W_iG}\mathbf{A}_{GI} \cdot \left\{ \underbrace{\begin{bmatrix} -\dot{y}_K/R \\ \dot{x}_K/R \\ \dot{\psi} \end{bmatrix}}_{\boldsymbol{\omega}_{IK}} \times \underbrace{\left(\begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix} + \mathbf{A}_{IG}\mathbf{A}_{GW_i} \cdot \begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix} \right)}_{\mathbf{r}_{PQ_i}} \right\} \tag{16}$$

for the ball and

$${}_{W_i}\mathbf{v}_{Q_i}^{(W_i)} = \mathbf{A}_{W_iG}\ \mathbf{A}_{GI} \cdot \begin{bmatrix} \dot{x}_{W_i} \\ \dot{y}_{W_i} \\ \dot{z}_{W_i} \end{bmatrix} + \left(\begin{bmatrix} \dot{\delta}_i \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix} \right) \tag{17}$$

for the omniwheels, where r represents their radius. As the omniwheels can not rotate around their y- and z-axis (see Figure 3 and Table 1), three constraint equations considering the y-component of the velocities ${}_{W_i}v_{Q_i,y}$ can be given in the velocity domain:

$$\dot{\mathbf{g}}\left(\mathbf{p}, \dot{\mathbf{p}}\right) = \begin{bmatrix} {}_{W_1}v_{Q_1,y}^{(K)} - {}_{W_1}v_{Q_1,y}^{(W_1)} \\ {}_{W_2}v_{Q_2,y}^{(K)} - {}_{W_2}v_{Q_2,y}^{(W_2)} \\ {}_{W_3}v_{Q_3,y}^{(K)} - {}_{W_3}v_{Q_3,y}^{(W_3)} \end{bmatrix} = \mathbf{0}. \tag{18}$$

Note, slip between the omniwheels and the ball can be considered in equation (18) by an additional variable which ensures the equivalence of the tangential velocities. However, it is not always mandatory (see Section 2). By means of equation (18), the projection term of the non-holonomic contribution forces $\mathbf{G}$ along the bodies coordinates can be given by the following expression, according to [5, 12]:

$$\mathbf{G} = \frac{\partial \dot{\mathbf{g}}\left(\mathbf{p},\ \dot{\mathbf{p}}\right)}{\partial \dot{\mathbf{p}}}. \tag{19}$$

The transposed of equation (19) together with the Lagrange multiplier $\boldsymbol{\lambda}$ quantifies the force contribution $\mathbf{G}^T\boldsymbol{\lambda}$ in equation (9).

3.2 Energy

In establishing geometrical constraints, the motion of the robot is described without considering forces and torques producing the motion. The kinetic and potential energies of the multi-body robot system, composed of n_K elements, are needed. Translatory, rotatory and coupling energies are scalar quantities and can be calculated separately in different coordinate frames (prefixed with general indices B, C, D) as follows:

$$T = \sum_{i=1}^{n_K} \left(\underbrace{\frac{1}{2} \cdot m \cdot {}_B\mathbf{v}_{\hat{Q}}^T \cdot {}_B\mathbf{v}_{\hat{Q}}}_{T_{Trans}} + \underbrace{m \cdot {}_C\mathbf{v}_{\hat{Q}}^T \cdot \left({}_C\boldsymbol{\omega}_{\boldsymbol{IC}} \times {}_C\mathbf{r}_{\hat{Q}S} \right)}_{T_{Coup}} + \underbrace{\frac{1}{2} \cdot {}_D\boldsymbol{\omega}_{ID}^T \cdot {}_D\boldsymbol{\theta}_{\hat{Q}} \cdot {}_D\boldsymbol{\omega}_{ID}}_{T_{Rot}} \right)_i . \tag{20}$$

Considering the rotation terms T_{Rot}, it is convenient to describe the energy quantities with respect to the coordinate frame of the corresponding body, because in this coordinate plane the inertia tensor is constant. Concerning the translation terms T_{Trans}, it is convenient to write the equations in the inertial system, because the absolute velocities are easily obtained in this coordinate plane by the derivation of the body centers of mass vectors. As suggested in [12], the general reference points $\hat{Q}_i$ are chosen to the center of mass of each element, in order to set the contribution of the coupling energy to zero ($T_{Coup} = 0$).
Balancino's potential energy in a general coordinate frame E is given by

$$V = \sum_{i=1}^{n_K} \left(-m \cdot {}_E\mathbf{r}_{OS}^T \cdot {}_E\mathbf{g} \right)_i , \tag{21}$$

where m represents the mass of each body and ${}_E\mathbf{g}$ the gravity vector in the coordinate frame E. The vector ${}_E\boldsymbol{r}_{OS}$ denotes the position of the center of mass of the considered component referring the origin O of the inertial frame. For $E = I$ (inertial coordinate frame), the positions of the centers of masses are given by

$$\mathbf{r}_{OS}^{(K)} = \left[x_K, y_K, R\right]^T , \tag{22}$$

$$\mathbf{r}_{OS}^{(G)} = \left[x_K, y_K, R\right]^T + \mathbf{A}_{IG} \cdot \left[0, 0, l\right]^T , \tag{23}$$

$$\mathbf{r}_{OS}^{(W_i)} = \left[x_K, y_K, R\right]^T + \mathbf{A}_{IG}\ \mathbf{A}_{GW_i} \cdot \left[-x_W, 0, R + r\right]^T , \tag{24}$$

$$\mathbf{r}_{OS}^{(M_i)} = \left[x_K, y_K, R\right]^T + \mathbf{A}_{IG}\ \mathbf{A}_{GW_i} \cdot \left[-x_M, 0, R + r\right]^T . \tag{25}$$

Based on that, the absolute and the angular velocity of each of the n_k-bodies and their inertia tensors, required for determining (20), can be calculated as follows: The absolute velocities

$$\mathbf{v}_{OS} = \dot{\mathbf{r}}_{OS} = \frac{\partial \mathbf{r}_{OS}}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial t} = \mathbf{J}_T \cdot \dot{\mathbf{p}}, \tag{26}$$

are derived with a differentiation operation of the center of mass vector, where $\mathbf{J}_T$ represents the Jacobian matrix of translation.
The following four equations

$$
\begin{aligned}
{}_K\boldsymbol{\omega}_{IK} &= \mathbf{A}_{KI}\ {}_I\boldsymbol{\omega}_{IK}, &(27)\\
{}_G\boldsymbol{\omega}_{IG} &= \mathbf{A}_{z,\gamma}\cdot\mathbf{A}_{y,\beta}\cdot\left[\dot{\alpha},0,0\right]^T+\mathbf{A}_{z,\gamma}\cdot\left[0,\dot{\beta},0\right]^T+\left[0,0,\dot{\gamma}\right]^T, &(28)\\
{}_{W_i}\boldsymbol{\omega}_{IW_i} &= \mathbf{A}_{W_iG}\cdot{}_G\boldsymbol{\omega}_{IG}+\left[\dot{\delta}_i,\quad 0,\quad 0\right]^T, &(29)\\
{}_{M_i}\boldsymbol{\omega}_{IM_i} &= \mathbf{A}_{W_iG}\cdot{}_G\boldsymbol{\omega}_{IG}+i_G\cdot\left[\dot{\delta}_i,\quad 0,\quad 0\right]^T, &(30)
\end{aligned}
$$

describe the absolute rotational velocities. The Jacobian matrix of rotation $\mathbf{J}_R$ is

$$
{}_\chi\mathbf{J}_R=\frac{\partial_\chi\boldsymbol{\omega}_{I\chi}}{\partial\dot{\mathbf{p}}}, \tag{31}
$$

where $\chi\in\{K,G,W_i,M_i\}$ is the reference coordinate frame and the required rotational velocities are given by (27) to (30).
Considering the ball as a hollow sphere and approximating the omniwheels as disks with thickness b_W, the inertia tensors

$$
\begin{aligned}
{}_K\boldsymbol{\theta}_S^{(K)} &= \begin{bmatrix}\frac{2}{3}m_KR^2 & 0 & 0\\ 0 & \frac{2}{3}m_KR^2 & 0\\ 0 & 0 & \frac{2}{3}m_KR^2\end{bmatrix}, &(32)\\
{}_{W_i}\boldsymbol{\theta}_S^{(W_i)} &= \begin{bmatrix}\frac{1}{2}m_Wr^2 & 0 & 0\\ 0 & \frac{1}{4}m_W(r^2+\frac{1}{3}b_W^2) & 0\\ 0 & 0 & \frac{1}{4}m_W(r^2+\frac{1}{3}b_W^2)\end{bmatrix} &(33)
\end{aligned}
$$

can be calculated according to [6]. The inertia tensors of Balancino and the motors are obtained from the data sheets of the electronic components resumed in Table 2 and form the computer-aided design model depicted in Figure 1(b):

$$
{}_G\boldsymbol{\theta}_S^{(G)}=\begin{bmatrix}\theta_{xx}^{(G)} & 0 & 0\\ 0 & \theta_{yy}^{(G)} & 0\\ 0 & 0 & \theta_{zz}^{(G)}\end{bmatrix},\qquad {}_{M_i}\boldsymbol{\theta}_S^{(M_i)}=\begin{bmatrix}\theta_{xx}^{(M_i)} & 0 & 0\\ 0 & \theta_{yy}^{(M_i)} & 0\\ 0 & 0 & \theta_{zz}^{(M_i)}\end{bmatrix}. \tag{34}
$$

Because of the symmetry of the construction, these matrices have diagonal form. Combining all the obtained equations, the kinetic (20) and the potential (21) energy terms can be calculated. Equation (20) can be further simplified to a matrix-vector notation by applying the Jacobian matrices of translation and rotation [12], resulting in

$$
T=\frac{1}{2}\,\dot{\mathbf{p}}^T\,\mathbf{M}_{red}\,\dot{\mathbf{p}}, \tag{35}
$$

where the mass matrix of the system, considering the reference frame of the i-th body, can be written as

$$
\mathbf{M}_{red}=\sum_{i=1}^{n_K}\left(m\cdot\mathbf{J}_T^T\mathbf{J}_T+\mathbf{J}_R^T\boldsymbol{\theta}_S\mathbf{J}_R\right)_i. \tag{36}
$$

Spin friction is considered as a viscous damped friction momentum

$$
M_d=-d\cdot\dot{\psi} \tag{37}
$$

between ball and ground, which is proportional (damping factor d) to the rotation velocity $\dot{\psi}$ of the ball around its vertical axis. This friction is needed to assure the transfer of momentums from the motors to the ground which enables Balancino's motion. Thus, the spin friction is required to guarantee the full controllability of the system which is further detailed in Section 5. Such friction terms are integrated in the Lagrange equation (9) by the dissipation energy of the Rayleigh dissipation function [11],

$$D = \frac{1}{2} \cdot d \cdot \dot{\psi}^2. \tag{38}$$

The coefficient d is related to the friction pairing (e.g. gum/asphalt) and needs to be identified experimentally. For the later simulation, d is chosen according to the results in [10] (see Table 2).

3.3 Non-conservative forces and moments

Forces acting on the robot, which are not included in the potential V, are condensed in the vector of the non-conservative forces $\mathbf{Q}_{NK}$. This vector includes damper forces and loads which operate on the robot as external excitations. Here, these terms allow to influence and manipulate Balancino's motion. They are generated by the three motors and build the desired mechanical moments on the ball to stabilize and to drive the robot to a desired position. Based on the torque moments ${}_{W_i}\mathbf{M}_i$ of each motor, $\mathbf{Q}_{NK}$ is given by

$$\mathbf{Q}_{NK} = \sum_{i=1}^{3} \left(\left[\frac{\partial_{W_i} \boldsymbol{\omega}_{IW_i}}{\partial \dot{\mathbf{p}}} \right]^T \cdot {}_{W_i}[M_i, \quad 0, \quad 0]^T \right), \tag{39}$$

where the Jacobian matrix of rotation $\mathbf{J}_R$ (31) is required to project the torque moments into the direction of the redundant coordinates $\mathbf{p}$. As the omniwheels can only rotate around their x-axis, the first element of ${}_{W_i}\mathbf{M}_i$ is non-zero. The corresponding torque moment values (M_1, M_2, M_3) depend on the characteristics of the motors. The dynamics of the DC-motors is defined by their mechanical and electrical time constants due to the rotor inertia and the motor inductance, respectively. The latter can be neglected because of its minor effect. Figure 3 shows an equivalent circuit diagram of a DC-motor.

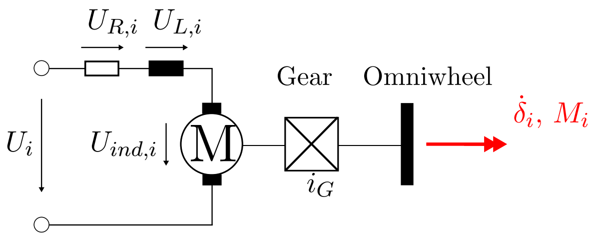


Figure 3: DC motor circuit model

Applying Kirchhoff's voltage law to the circuit, the voltage of each motor can be modeled depending on the mutual induction $U_{ind,i}$, generated by the motor rotation, the potential drop on the resistor $U_{R,i}$ and the inductance $U_{L,i}$ as follows:

$$U_i = U_{R,i} + U_{L,i} + U_{ind,i}, \qquad i = \{1,\ 2,\ 3\}. \tag{40}$$

According to [4], the mutual induction can be substituted by $U_{ind,i} = k_E \cdot \dot{\delta}_i \cdot i_G$. Assuming that the inductance term can be neglected, equation (40) can be reformulated to

$$U_i = R_M \cdot I_i + k_E \cdot \dot{\delta}_i \cdot i_G, \qquad i = \{1,\ 2,\ 3\}. \tag{41}$$

The constant k_E is the ratio between the inducted voltage and the rotating speed and i_G represents the gear transmission ratio. According to the motors' data sheets (parameters given in

Table 2), the motor torque $M_{M,i}$ depends linearly on the currents I_i ($i = 1,\ 2,\ 3$) as follows:

$$\left.\begin{aligned} M_{M,i} &= k_M \cdot I_i \\ M_i &= M_{M,i} \cdot i_G \cdot \eta_M \cdot \eta_G \end{aligned}\right\} M_i = I_i \cdot k_M \cdot i_G \cdot \eta_M \cdot \eta_G, \tag{42}$$

where the term k_M represents the linearity factor. Because of the non-ideality of the torque transmission in (42), a gear and a motor efficiency factor ($\eta_M,\ \eta_G$) are added to the gear transmission ratio i_G. Combining equation (41) and (42), the torque moments operating on the ball can be expressed by

$$M_i = -\frac{k_E k_M i_G^2 \eta_M \eta_G}{R_M} \cdot \dot{\delta}_i + \frac{k_M i_G \eta_M \eta_G}{R_M} \cdot U_i, \quad i = \{1, 2, 3\}. \tag{43}$$

3.4 Nonlinear Dynamical Equations

Using the equations obtained in the previous Sections, the Lagrangian approach of the first kind (9) is solvable. Its solution gives Balancino's equations of motion, which are summarized in a system of two differential algebraic equations (DAEs):

$$\begin{aligned} &\mathbf{M}_{red} \cdot \ddot{\mathbf{p}} = \mathbf{h}(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{u}) + \mathbf{G}^T \boldsymbol{\lambda} \\ &\dot{\mathbf{g}}(\mathbf{p}, \dot{\mathbf{p}}) = \mathbf{0}, \end{aligned} \tag{44}$$

where the term $\mathbf{h}(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{u})$ is

$$\mathbf{h}(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{u}) = -\frac{\partial}{\partial \mathbf{p}}\left(\frac{\partial T}{\partial \dot{\mathbf{p}}}\right)^T \dot{\mathbf{p}} + \left(\frac{\partial T}{\partial \mathbf{p}}\right)^T - \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\mathbf{p}}}\right)^T - \left(\frac{\partial V}{\partial \mathbf{p}}\right)^T - \left(\frac{\partial T}{\partial \dot{\mathbf{p}}}\right)^T - \left(\frac{\partial D}{\partial \dot{\mathbf{p}}}\right)^T + \mathbf{Q_{NK}}(\mathbf{u}), \tag{45}$$

and the input vector $\mathbf{u}$ consists of the actuator voltages

$$\mathbf{u} = \begin{bmatrix} U_1, & U_2, & U_3 \end{bmatrix}^T. \tag{46}$$

For control purposes, a transformation towards minimal coordinates $\mathbf{q}$ is needed. Due to the considered sensors and in order to achieve the stabilization and possible trajectory planing goals, the choice of the following coordinates

$$\mathbf{q} = \begin{bmatrix} x_K, & y_K, & \psi, & \alpha, & \beta, & \gamma \end{bmatrix}^T \tag{47}$$

is advantageous. Thereby, the system dynamics is uniquely described, because $\dot{\delta}_i$ is at hand by equation (18), resulting in a functional relation $\dot{\delta}_i(\mathbf{q}, \dot{\mathbf{q}})$. Based on that, δ_i are given by integration. By

$$\dot{\mathbf{p}}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} \dot{x}_K, & \dot{y}_K, & \dot{\psi}, & \dot{\alpha}, & \dot{\beta}, & \dot{\gamma}, & \dot{\delta}_1(\mathbf{q}, \dot{\mathbf{q}}), & \dot{\delta}_2(\mathbf{q}, \dot{\mathbf{q}}), & \dot{\delta}_3(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}^T \tag{48}$$

the nonholonomic forces and moments $\mathbf{G}^T \boldsymbol{\lambda}$ can be eliminated and a set of differential equations in minimal coordinates is obtained. Therefore, the derivative in time of the vector $\dot{\mathbf{p}}$

$$\ddot{\mathbf{p}} = \underbrace{\frac{\partial \dot{\mathbf{p}}}{\partial \dot{\mathbf{q}}}}_{\mathbf{J}} \cdot \ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{p}}}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} \tag{49}$$

is inserted in equation (44) which is then multiplied from the left by $\mathbf{J}^T$, resulting in the differential equations of motion

$$\underbrace{\mathbf{J}^T \mathbf{M}_{red} \mathbf{J}}_{\mathbf{M}_{min}} \cdot \ddot{\mathbf{q}} = \underbrace{\mathbf{J}^T \cdot \left(\mathbf{h}(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{u}) - \mathbf{M}_{red} \frac{\partial \dot{\mathbf{p}}}{\partial \mathbf{q}} \dot{\mathbf{q}}\right)}_{\mathbf{h}_{min}} + \underbrace{\mathbf{J}^T \mathbf{G}^T}_{=\mathbf{0}} \boldsymbol{\lambda} \tag{50}$$

or rewritten in implicit form (considering only minimal coordinates):

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{u}) = \mathbf{M}_{min}(\mathbf{q}) \cdot \ddot{\mathbf{q}} - \mathbf{h}_{min}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u}) = \mathbf{0}. \tag{51}$$

4 Linearization

Some fundamental analysis, like controllability or observability, can be easily done based on a linear system formulation. Additionally, well-known controller design methods, like LQ-controllers, require linear state space equations. In this Section the nonlinear system equations are linearized at the equilibrium point $\mathbf{0}$, where the robot stands in a steady state. In the linearization point all quantities are labeled with an additional lowered index 0:

$$\begin{aligned}\mathbf{q} &= \mathbf{q}_0 = \begin{bmatrix} x_{K,0}, & y_{K,0}, & \psi_0, & \alpha_0, & \beta_0, & \gamma_0 \end{bmatrix}^T = \mathbf{0}^T \\ \dot{\mathbf{q}} &= \dot{\mathbf{q}}_0 = \mathbf{0}^T \\ \ddot{\mathbf{q}} &= \ddot{\mathbf{q}}_0 = \mathbf{0}^T \\ \mathbf{u} &= \mathbf{u}_0 = \begin{bmatrix} U_{1,0}, & U_{2,0}, & U_{3,0} \end{bmatrix}^T = \mathbf{0}^T. \end{aligned} \tag{52}$$

The nonlinear differential equations of second order (51) are linearized using the first order Taylor series approximation

$$\mathbf{f}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{u}) \approx \underbrace{\mathbf{f}(\tilde{\mathbf{q}}_0,\mathbf{u}_0)}_{=\mathbf{0}} + \underbrace{\left.\frac{\partial \mathbf{f}}{\partial \ddot{\mathbf{q}}}\right|_{\tilde{\mathbf{q}}_0,\mathbf{u}_0}}_{\mathbf{M}_{lin}} \cdot \ddot{\mathbf{q}} + \underbrace{\left.\frac{\partial \mathbf{f}}{\partial \dot{\mathbf{q}}}\right|_{\tilde{\mathbf{q}}_0,\mathbf{u}_0}}_{\mathbf{D}_{lin}} \cdot \dot{\mathbf{q}} + \underbrace{\left.\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right|_{\tilde{\mathbf{q}}_0,\mathbf{u}_0}}_{\mathbf{K}_{lin}} \cdot \mathbf{q} + \underbrace{\left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\tilde{\mathbf{q}}_0,\mathbf{u}_0}}_{\mathbf{Q}_{lin}} \cdot \mathbf{u} = \mathbf{0}, \tag{53}$$

with $\tilde{\mathbf{q}}_0 = [\mathbf{q}_0, \dot{\mathbf{q}}_0, \ddot{\mathbf{q}}_0]^T$, the mass matrices $\mathbf{M}_{lin} \in \mathbf{R}^{6\text{x}6}$, the stiffness matrix $\mathbf{K}_{lin} \in \mathbf{R}^{6\text{x}6}$, the damping matrix $\mathbf{D}_{lin} \in \mathbf{R}^{6\text{x}6}$ and the input matrix $\mathbf{Q}_{lin} \in \mathbf{R}^{6\text{x}3}$. These equations are reformulated with the following definition,

$$\left.\begin{aligned}\widetilde{\mathbf{x}}_1 &= \mathbf{q} \\ \widetilde{\mathbf{x}}_2 &= \dot{\mathbf{q}}\end{aligned}\right\} \begin{aligned}\dot{\widetilde{\mathbf{x}}}_1 &= \widetilde{\mathbf{x}}_2 \\ \dot{\widetilde{\mathbf{x}}}_2 &= -\mathbf{M}_{lin}^{-1}\mathbf{K}_{lin} \cdot \widetilde{\mathbf{x}}_1 - \mathbf{M}_{lin}^{-1}\mathbf{D}_{lin} \cdot \widetilde{\mathbf{x}}_2 - \mathbf{M}_{lin}^{-1}\mathbf{Q}_{lin} \cdot \mathbf{u},\end{aligned} \tag{54}$$

to obtain a set of first order differential equations. For the analysis in Section 5, the position of the ball x_K, y_K and the orientation of the robot γ are chosen as outputs, leading to a linear state space representation calculated to

$$\begin{aligned}\dot{\widetilde{\mathbf{x}}} &= \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_{lin}^{-1}\mathbf{K}_{lin} & -\mathbf{M}_{lin}^{-1}\mathbf{D}_{lin} \end{bmatrix}}_{\widetilde{\mathbf{A}}} \cdot \widetilde{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ -\mathbf{M}_{lin}^{-1}\mathbf{Q}_{lin} \end{bmatrix}}_{\widetilde{\mathbf{B}}} \cdot \mathbf{u} \\ \widetilde{\mathbf{y}} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}}_{\widetilde{\mathbf{C}}} \cdot \widetilde{\mathbf{x}} + \underbrace{\mathbf{0}}_{\widetilde{\mathbf{D}}} \cdot \mathbf{u},\end{aligned} \tag{55}$$

with the system matrix $\widetilde{\mathbf{A}} \in \mathbb{R}^{12\text{x}12}$, the input matrix $\widetilde{\mathbf{B}} \in \mathbb{R}^{12\text{x}3}$, the output matrix $\widetilde{\mathbf{C}} \in \mathbb{R}^{3\text{x}12}$ and the feedthrough matrix $\widetilde{\mathbf{D}} \in \mathbb{R}^{3\text{x}3}$.

Applying the parameter values according to Table 2 to this state space model it can be noticed that the state variables x_K, y_K, γ and ψ have no effects on the dynamics of the system (zero columns). Hence, the state space model could actually be reduced by canceling these state variables. However, because x_K, y_K, γ are interesting control variables (e. g. for position, motion or trajectory control of the robot), only the state ψ is omitted and therefore the state vector is reduced to 11 elements. Moreover, since it is not possible to arbitrarily affect the angles γ and ψ simultaneously to preserve the full controllability of the system, the state ψ has to be canceled.

More details can be found in the following Section 5. The finally obtained state space model is given by:

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -14.29 & 0 & -2.61 & 0 & 0 & 0 & 0.58 & 0 \\ 0 & 0 & 14.29 & 0 & 0 & 0 & -2.61 & 0 & -0.58 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -23.02 & 0 & 0 & 11.93 \\ 0 & 0 & 54.67 & 0 & 0 & 0 & -7.46 & 0 & -1.66 & 0 & 0 \\ 0 & 0 & 0 & 54.67 & 0 & 7.46 & 0 & 0 & 0 & -1.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.20 & 0 & 0 & -2.92 \end{bmatrix}}_{\mathbf{A}} \cdot \mathbf{x} + \\ + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.21 & 0.21 \\ 0.24 & -0.12 & -0.12 \\ 3.47 & 3.47 & 3.47 \\ 0.70 & -0.35 & -0.35 \\ 0 & 0.60 & -0.60 \\ -0.85 & -0.85 & -0.85 \end{bmatrix}}_{\mathbf{B}} \cdot \mathbf{u} \tag{56}$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \cdot \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{D}} \cdot \mathbf{u},$$

where

$$\mathbf{x} = \left[x_K, \quad y_K, \quad \alpha, \quad \beta, \quad \gamma, \quad \dot{x}_K, \quad \dot{y}_K, \quad \dot{\psi}, \quad \dot{\alpha}, \quad \dot{\beta}, \quad \dot{\gamma}\right]^T. \tag{57}$$

5 System Analysis

For the purpose of control, the linear state space model is analyzed in terms of stability, controllability, observability and its dynamical characteristics. The eigenvalues of system (56) are

$$\lambda = \left[-23.13, \quad -9.58, \quad -9.58, \quad -2.80, \quad -0.63, \quad -0.63, \quad 0, \quad 0, \quad 0, \quad +5.95, \quad +5.95\right]^T. \tag{58}$$

As expected, not all eigenvalues are negative and the system is unstable, accordingly. Because the eigenvectors belonging to $\lambda_{7,8,9} = 0$ are $[1, 0, \ldots, 0]^T$, $[0, 1, 0, \ldots, 0]^T$ and $[0, 0, 0, 0, 1, 0, \ldots, 0]^T$, the position of the ball x_K, y_K and the robot's yaw angle γ span the system kernel $\mathcal{N}$ of dimension $\dim(\mathcal{N}) = 3$. All possible equilibria are inside this kernel $\mathcal{N}$ which is plausible, as the dynamics of a robot balancing on a ball has to be independent from its position and orientation. In other words, if Balancino is stabilizable via state feedback control, each position and yaw

angle is identically stabilizable.
For analyzing the system's controllability and observability, the rank of the Kalman controllability and observability matrices are checked, yielding

$$\operatorname{rank}\begin{bmatrix}\mathbf{B}, & \mathbf{AB}, & \ldots, & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} = 11 \quad \text{and} \tag{59a}$$

$$\operatorname{rank}\begin{bmatrix}\mathbf{C}, & \mathbf{CA}, & \ldots, & \mathbf{CA}^{n-1}\end{bmatrix}^T = 11, \tag{59b}$$

which states full controllability and observability of the system.
The importance of eliminating ψ from the final LTI-system (56) in order to obtain complete controllability can be seen using Kalman's controllability matrix according to equation (59a). Calculating its rank using the non-reduced system matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ from (55) results in a rank of 11, which is smaller than the full rank 12. Thus, complete controllability is not given in this case. In addition, the rank of (59a) is also reduced (from 11 to 10) if the spin friction (equations (37) and (38)) is eliminated from the $\mathbf{A}$ matrix. In both mentioned cases of non-controllability, yawing of the robot body and of the ball are coupled in a way which makes it impossible to transfer all of the state variables to zero from any arbitrary initialization. While in the first case the yaw angles are directly coupled, in the second one the non-damped yawing of the ball leads to such a coupling.
In order to clarify this coupling in more detail, the motion of the ball and Balancino's body is reduced to yawing and thus only the above mentioned angles can be considered as minimal coordinates. This implies that the same voltage is applied to all DC-motors; hence the input vector $\mathbf{u}$ according to equation (46) is replaced by a scaler input $u_{yaw} = U_1 = U_2 = U_3$. Considering these restrictions and applying the modeling procedure according to the Sections 2 and 4 with the parameter in Table 2, the following SISO state space representation of yawing is obtained[1]:

$$\begin{aligned}\dot{\mathbf{x}}_{yaw} &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -23.02 & 11.93\\ 0 & 0 & 0.20 & -2.92\end{bmatrix}\cdot \mathbf{x}_{yaw} + \begin{bmatrix}0\\ 0\\ 10.41\\ -2.55\end{bmatrix}\cdot u_{yaw}\\ y_{yaw} &= \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}\cdot \mathbf{x}_{yaw}\end{aligned} \tag{60}$$

with the state space vector

$$\mathbf{x}_{yaw} = \begin{bmatrix}\psi, & \gamma, & \dot{\psi}, & \dot{\gamma}\end{bmatrix}^T. \tag{61}$$

The system eigenvalues are located at

$$\lambda_{yaw} = [-23.13, -2.80, 0, 0], \tag{62}$$

whereby, because of the absence of eigenvalues on the right side of the imaginary-axis, the stability of the system is shown. However, the controllability matrix is not of full rank:

$$\operatorname{rank}\begin{bmatrix}\mathbf{B}, & \mathbf{AB}, & \mathbf{A}^2\mathbf{B}, & \mathbf{A}^3\mathbf{B}\end{bmatrix} = 3 < 4 = n, \tag{63}$$

meaning that the system cannot be led to $\mathbf{x}_{yaw} = \mathbf{0}$ when starting form a perturbed initial state. This effect can be illustrated considering a controller designed by pole placement

$$\det\Big[\lambda\mathbf{I} - \underbrace{\left(\mathbf{A} - \mathbf{b}\cdot\mathbf{k}^T\right)}_{\mathbf{A}_{close}}\Big] \overset{!}{=} p\left(\lambda_{set}\right), \tag{64}$$

[1] considering friction effects

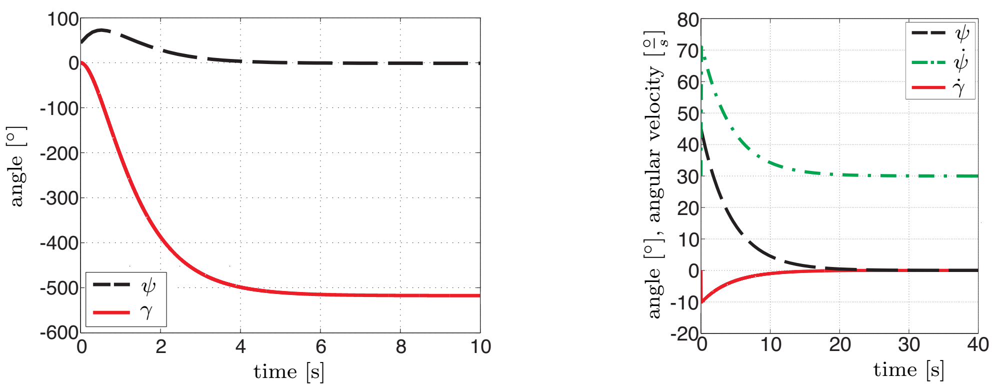
where $\mathbf{k}^T = \begin{bmatrix} k_1, & k_2, & k_3, & k_4 \end{bmatrix}$ is the linear state feedback. The desired eigenvalues λ_{set} of the closed loop system $\mathbf{A}_{close}$ are defined by the desired control polynomial $p(\lambda_{set})$. The resulting eigenvalues of the system (60) can be represented as

$$\lambda = \begin{bmatrix} 0, & \lambda_2\left(\mathbf{k}^T\right), & \lambda_3\left(\mathbf{k}^T\right), & \lambda_4\left(\mathbf{k}^T\right) \end{bmatrix}. \tag{65}$$

It can be recognized, that the zero-eigenvalue cannot be shifted by $\mathbf{k}^T$ towards the desired λ_{set}. For the controlled system $\dot{\mathbf{x}}_{yaw} = \mathbf{A}_{close} \cdot \mathbf{x}_{yaw}$ the general solution can be written as

$$\mathbf{x}(t) = \sum_{i=1}^{n} C_i \cdot \mathbf{v}_i \cdot e^{\lambda_i t}, \tag{66}$$

where C_i depends on the initial values and $\mathbf{v}$ are the eigenvectors, respectively. Choosing $\mathbf{k}^T$ properly leads to a decaying oscillation of three eigenmodes ($\lambda_2, \lambda_3, \lambda_4 < 0$). However, λ_1 stays zero and as the corresponding eigenvector is $[0, 1, 0, 0]^T$, the angle γ reaches a constant value ($\neq 0$) for $t \to \infty$. In Figure 4(a) the behavior of both angles, with an initial perturbation of $\psi_0 = 45°$ is shown. In case of neglecting the friction between ball and ground, the reduction of the state quantity ψ shows a similar effect, because the controllability matrix is also not of full rank. A simulation result initialized with $\mathbf{x}_{yaw} = [45°, 30\frac{°}{s}, 0\frac{°}{s}]$ (ψ is eliminated) is illustrated in Figure 4(b). It can be seen, that in this case $\dot{\psi}$ reaches a constant value which differs from zero. These two simplified examples show the necessity to remove ψ from the state space model in (55) and also to consider the spin friction between ball and ground in order to guarantee full controllability of Balancino.



(a) full system with friction (b) reduced system without friction

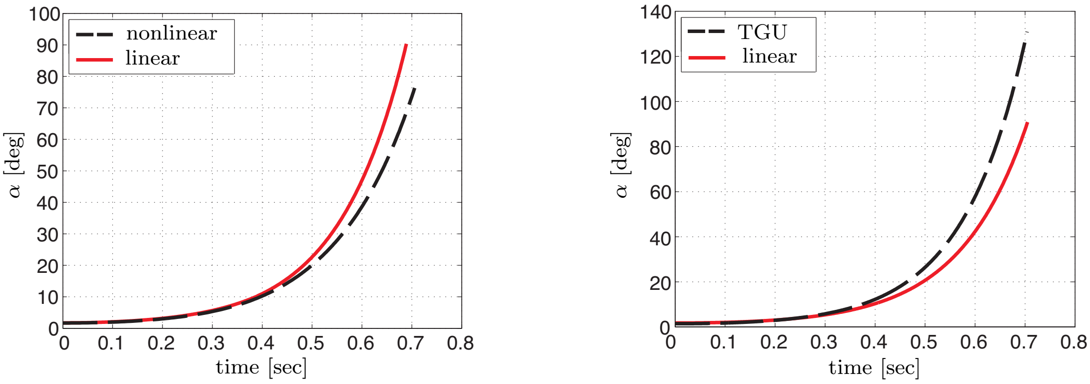
Figure 4: Time responses illustrating the controllability problem

A further system characteristic becomes obvious by the transfer zeros

$$\eta = \begin{bmatrix} -22.21, & -3.72, & -3.72, & +3.72, & +3.72 \end{bmatrix}^T. \tag{67}$$

Because of $\eta_{4,5} > 0$, the system is of non-minimum phase, which is easy to understand when comparing its physical behavior to a simple inverted pendulum. Thus, the robot has to move backwards first for tilting its body forwards into the desired direction of motion.

Finally, the range of validity of the linear model is approximated by comparing it to its nonlinear representation (50) and to the behavior of the plane model presented in [8], named TGU ballbot. A free tip over motion for an initial tilting $\alpha(0) = 1°$ is considered for both comparisons whose results are depicted in Figure 5. As clarified by Figure 5(a), the range of validity can be set to



(a) Compared to the nonlinear system (b) Compared to the 2d-TGU ballbot

Figure 5: Range of validity of the 3D-linearized state space model

$[0, 22°]$, where the tilt angle error between the nonlinear and the linearized model of Balancino's is still below 5%. In order to enable a comparison with the plain TGU ballbot model, its parameters were modified, meaning the mass of the robot body, its moment of inertia and its centroidal distance to the ball were matched with Balancino. Despite that, the tip over motion of the TGU ballbot is faster than the one of Balancino (Figure 5(b)). However, this behavior has been expected due to the higher abstraction level of the TGU model (e.g. missing moments of inertia of the power train).

6 Conclusion and outlook

In this report, a detailed 3D-model of a robot balancing on a ball, named Balancino, has been presented. The nonlinear equations of motion have been derived based on Lagrange's equations of the first kind. In addition to the robot and the ball also the dynamics of the three required actuators was considered. The complete model has been linearized and further analyzed for control purposes. Special attention has been paid to the controllability problem. It has been shown that the yaw angles of the ball and robot are not simultaneously controllable. The necessity to introduce spin friction in order to guarantee Balancino's controllability has been detailed.
In future work, the obtained model will be used to design and test linear and nonlinear controllers as well as switching or adaptive strategies. Also real time experiments for checking the performance of control strategies are of high interest.

Acknowledgment

The authors would like to thank their colleagues Paul De Monte, Tobias Kloiber and their former colleague Dr. Michael Buhl for fruitful discussions. Moreover, the authors thank Thomas Huber for his collaboration in realizing and constructing the robot. The CAD model is due to Paula Reimer, which is gratefully acknowledged.

A Explicit representation of modeling equations

Vectors and matrices used in the modeling Sections are given in the following. This may help for clarification or at least it will ease the modeling procedure for the readers.

The explicit outlines of the transformation matrices are given by

$$\mathbf{A}_{KI} = \begin{bmatrix} \cos\nu\cos\psi & \sin\psi\cos\nu & -\sin\nu \\ \sin\varphi\sin\nu\cos\psi - \cos\varphi\sin\psi & \sin\psi\sin\varphi\sin\nu + \cos\varphi\cos\psi & \sin\varphi\cos\nu \\ \sin\varphi\sin\psi + \cos\varphi\sin\nu\cos\psi & -\sin\varphi\cos\psi + \cos\varphi\sin\psi\sin\nu & \cos\varphi\cos\nu \end{bmatrix}$$

$$\mathbf{A}_{GI} = \begin{bmatrix} \cos\gamma\cos\beta & \sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha & \sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha \\ -\sin\gamma\cos\beta & \cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha & \cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha \\ \sin\beta & -\cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix}$$

$$\mathbf{A}_{W_1G} = \begin{bmatrix} \cos\eta & 0 & -\sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & \cos\eta \end{bmatrix}$$

$$\mathbf{A}_{W_2G} = \begin{bmatrix} -\frac{1}{2}\cos\eta & \frac{\sqrt{3}}{2}\cos\eta & -\sin\eta \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2}\sin\eta & \frac{\sqrt{3}}{2}\sin\eta & \cos\eta \end{bmatrix}$$

$$\mathbf{A}_{W_3G} = \begin{bmatrix} -\frac{1}{2}\cos\eta & -\frac{\sqrt{3}}{2}\cos\eta & -\sin\eta \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2}\sin\eta & -\frac{\sqrt{3}}{2}\sin\eta & \cos\eta \end{bmatrix}$$

The elements $g_{i,j}$ of the projection matrix $\mathbf{G}^T \in \mathbb{R}^{9\times 3}$ are:

$$\begin{aligned}
g_{1,1} &= -\cos\beta\sin\eta\sin\alpha - \cos\gamma\sin\alpha\cos\eta\sin\beta - \sin\gamma\cos\alpha\cos\eta \\
g_{1,2} &= -\cos\beta\sin\eta\sin\alpha + \frac{\sqrt{3}}{2}\sin\gamma\sin\alpha\cos\eta\sin\beta - \frac{\sqrt{3}}{2}\cos\gamma\cos\alpha\cos\eta \\
&\quad + \frac{1}{2}\cos\gamma\sin\alpha\cos\eta\sin\beta + \frac{1}{2}\sin\gamma\cos\alpha\cos\eta \\
g_{1,3} &= -\cos\beta\sin\eta\sin\alpha - \frac{\sqrt{3}}{2}\sin\gamma\sin\alpha\cos\eta\sin\beta + \frac{\sqrt{3}}{2}\cos\gamma\cos\alpha\cos\eta \\
&\quad + \frac{1}{2}\cos\gamma\sin\alpha\cos\eta\sin\beta + \frac{1}{2}\sin\gamma\cos\alpha\cos\eta \\
g_{2,1} &= -\sin\beta\sin\eta + \cos\gamma\cos\eta\cos\beta \\
g_{2,2} &= -\sin\beta\sin\eta - \frac{\sqrt{3}}{2}\sin\gamma\cos\eta\cos\beta - \frac{1}{2}\cos\gamma\cos\eta\cos\beta \\
g_{2,3} &= -\sin\beta\sin\eta + \frac{\sqrt{3}}{2}\sin\gamma\cos\eta\cos\beta - \frac{1}{2}\cos\gamma\cos\eta\cos\beta \\
g_{3,1} &= R\left(\cos\beta\sin\eta\cos\alpha + \cos\gamma\cos\alpha\cos\eta\sin\beta - \sin\gamma\sin\alpha\cos\eta\right) \\
g_{3,2} &= \frac{R}{2}\Big(2\cos\beta\sin\eta\cos\alpha - \sqrt{3}\sin\gamma\cos\alpha\cos\eta\sin\beta - \sqrt{3}\cos\gamma\sin\alpha\cos\eta \\
&\quad - \cos\gamma\cos\alpha\cos\eta\sin\beta + \sin\gamma\sin\alpha\cos\eta\Big) \\
g_{3,3} &= \frac{R}{2}\Big(2\cos\beta\sin\eta\cos\alpha + \sqrt{3}\sin\gamma\cos\alpha\cos\eta\sin\beta + \sqrt{3}\cos\gamma\sin\alpha\cos\eta \\
&\quad - \cos\gamma\cos\alpha\cos\eta\sin\beta + \sin\gamma\sin\alpha\cos\eta\Big) \\
g_{4,1} &= -\left(R + r\right)\left(\sin\beta\sin\eta - \cos\gamma\cos\eta\cos\beta\right)
\end{aligned}$$

$$
\begin{aligned}
g_{4,2} &= -\frac{1}{2}(R+r)\left(2\sin\beta\sin\eta + \sqrt{3}\sin\gamma\cos\eta\cos\beta + \cos\gamma\cos\eta\cos\beta\right)\\
g_{4,3} &= -\frac{1}{2}(R+r)\left(-\sqrt{3}\sin\gamma\cos\eta\cos\beta + 2\sin\beta\sin\eta + \cos\gamma\cos\eta\cos\beta\right)\\
g_{5,1} &= \sin\gamma\cos\eta\,(R+r)\\
g_{5,2} &= \frac{1}{2}(R+r)\cos\eta\left(\sqrt{3}\cos\gamma - \sin\gamma\right)\\
g_{5,3} &= -\frac{1}{2}(R+r)\cos\eta\left(\sin\gamma + \sqrt{3}\cos\gamma\right)\\
g_{6,1} &= -\sin\eta\,(R+r)\\
g_{6,2} &= -\sin\eta\,(R+r)\\
g_{6,3} &= -\sin\eta\,(R+r)\\
g_{7,1} &= -r\\
g_{7,2} &= 0\\
g_{7,3} &= 0\\
g_{8,1} &= 0\\
g_{8,2} &= -r\\
g_{8,3} &= 0\\
g_{9,1} &= 0\\
g_{9,2} &= 0\\
g_{9,3} &= -r
\end{aligned}
$$

B Parameter

In the following the mechanical and electrical parameters used for modeling are given. The DC-motor and the gear characteristics are taken from [2] and [3]. The mass of the modeled omniwheels is composed by the wheel and the binding mechanism, which connects the spindle.

Parameter	Value	Unit	Parameter	Value	Unit
l	0,373	m	$\theta_{xx}^{(G)}$	0,149	kgm^2
R	0,13	m	$\theta_{yy}^{(G)}$	0,149	kgm^2
r	0,062	m	$\theta_{zz}^{(G)}$	0,036	kgm^2
η	45	$\circ$	$\theta_{xx}^{(W)}$	$6{,}942 \cdot 10^{-4}$	kgm^2
x_M	0,132	m	$\theta_{yy}^{(W)}$	$5{,}463 \cdot 10^{-4}$	kgm^2
x_W	0,031	m	$\theta_{zz}^{(W)}$	$5{,}463 \cdot 10^{-4}$	kgm^2
g	9,81	$\frac{m}{s^2}$	$\theta_{xx}^{(M)}$	$2{,}7 \cdot 10^{-7}$	kgm^2
m_K	0,6	kg	$\theta_{yy}^{(M)}$	$4{,}793 \cdot 10^{-5}$	kgm^2
m_G	4,79	kg	$\theta_{zz}^{(M)}$	$4{,}793 \cdot 10^{-5}$	kgm^2
m_W	0,475	kg	η_G	70	%
m_M	0,046	kg	η_M	82	%
d	0,17	$\frac{Nms}{rad}$	U_{max}	12	V
i_G	43	–	k_M	14,5	$\frac{mNm}{A}$
k_E	1,52	$\frac{mV}{rpm}$	R_M	8,71	Ω

Table 2: Values of the parameters

C Energy contribution

In the following the kinetic and the potential energies of Balancino's components are given.

Potential energy

$$
\begin{aligned}
V_K =& m_K g R \\
V_G =& m_G g(R + l \cdot \cos\beta\cos\alpha) \\
V_{W1} =& m_W\,(R - ((\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)\cos\eta - \sin\eta\cos\beta\cos\alpha)x_W + \\
&+ (\sin\eta(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha) + \cos\eta\cos\beta\cos\alpha)(R + r))g \\
V_{W2} =& m_W\,(R - (-1/2\,(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)\cos\eta + 1/2\,(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)\cos\eta\sqrt{3} + \\
&- \sin\eta\cos\beta\cos\alpha)x_W + (-1/2\ \sin\eta(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha) + 1/2\ \sin\eta\sqrt{3}(\cos\gamma\sin\alpha + \\
&+ \sin\gamma\sin\beta\cos\alpha) + \cos\eta\cos\beta\cos\alpha)(R + r))g \\
V_{W3} =& m_W\,(R - (-1/2\,(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)\cos\eta - 1/2\,(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)\cos\eta\sqrt{3} + \\
&- \sin\eta\cos\beta\cos\alpha)x_W + (-1/2\ \sin\eta(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha) - 1/2\ \sin\eta\sqrt{3}(\cos\gamma\sin\alpha \\
&+ \sin\gamma\sin\beta\cos\alpha) + \cos\eta\cos\beta\cos\alpha)(R + r))g \\
V_{M1} =& m_M\,(R - ((\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)\cos\eta - \sin\eta\cos\beta\cos\alpha)x_M + (\sin\eta(\sin\gamma\sin\alpha + \\
&- \cos\gamma\sin\beta\cos\alpha) + \cos\eta\cos\beta\cos\alpha)(R + r))g \\
V_{M2} =& m_M\,(R - (-1/2\,(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)\cos\eta + 1/2\,(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)\cos\eta\sqrt{3} - \\
&+ \sin\eta\cos\beta\cos\alpha)x_M + (-1/2\ \sin\eta(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha) + 1/2\ \sin\eta\sqrt{3}(\cos\gamma\sin\alpha + \\
&+ \sin\gamma\sin\beta\cos\alpha) + \cos\eta\cos\beta\cos\alpha)(R + r))g \\
V_{M3} =& m_M\,(R - (-1/2\,(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)\cos\eta - 1/2\,(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)\cos\eta\sqrt{3} + \\
&- \sin\eta\cos\beta\cos\alpha)x_M + (-1/2\ \sin\eta(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha) - 1/2\ \sin\eta\sqrt{3}(\cos\gamma\sin\alpha + \\
&+ \sin\gamma\sin\beta\cos\alpha) + \cos\eta\cos\beta\cos\alpha)(R + r))g
\end{aligned}
$$

Kinetical energy

$$
\begin{aligned}
T_K =& \frac{1}{2}m_K[\frac{5}{3}(\dot{x}_K^2 + \dot{y}_K^2) + \frac{2}{3}\dot{\psi}^2 R^2], \\
T_G =& \frac{1}{2}\theta_{xx}^{(G)}[\dot{\beta}^2(1 - \cos^2\gamma) + \dot{\alpha}^2\cos^2\beta\cos^2\gamma + 2\dot{\alpha}\dot{\beta}\cos\beta\cos\gamma] + \\
&+ \frac{1}{2}\theta_{yy}^{(G)}[\dot{\beta}^2\cos^2\gamma + \dot{\alpha}^2\cos^2\beta\cdot(1 - \cos^2\gamma) - 2\dot{\alpha}\dot{\beta}\cos\beta\cos\gamma\sin\gamma] + \\
&+ \frac{1}{2}\theta_{zz}^{(G)}[\dot{\gamma}^2 + \dot{\alpha}^2(1 - \cos^2\beta) + 2\dot{\alpha}\dot{\gamma}\sin\beta] + \\
&+ \frac{1}{2}m_G(l^2\dot{\alpha}^2\cos^2\beta + l^2\dot{\beta}^2 + 2\dot{x}_K l\dot{\beta}\cos\beta - 2\dot{y}_K l\dot{\alpha}\cos\alpha\cos\beta + \\
&+ 2\dot{y}_K l\dot{\beta}\sin\alpha\sin\beta + \dot{x}_K^2 + \dot{y}_K^2)
\end{aligned}
$$

$$
\begin{aligned}
T_{W1} =&1/2\, m_W\, ((\dot{x}_K + (-(-\cos\eta\cos\gamma\sin\beta - \sin\eta\cos\beta)x_W + (-\sin\eta\cos\gamma\sin\beta + \cos\eta\cos\beta)(R+\\
&+r))\dot{\beta} + (\cos\eta\sin\gamma\cos\beta x_W - \sin\eta\sin\gamma\cos\beta(R+r))\dot{\gamma})^2 + (\dot{y}_K + (-(\cos\eta(-\sin\gamma\sin\alpha+\\
&+\cos\gamma\sin\beta\cos\alpha) + \sin\eta\cos\beta\cos\alpha)x_W + (\sin\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha)+\\
&-\cos\eta\cos\beta\cos\alpha)(R+r))\dot{\alpha} + (-(\cos\eta\cos\gamma\cos\beta\sin\alpha - \sin\eta\sin\beta\sin\alpha)x_W+\\
&+(\sin\eta\cos\gamma\cos\beta\sin\alpha + \cos\eta\sin\beta\sin\alpha)(R+r))\dot{\beta} + (-\cos\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)x_W+\\
&+\sin\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(R+r))\dot{\gamma})^2 + ((-(\cos\eta(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)+\\
&+\sin\eta\cos\beta\sin\alpha)x_W + (\sin\eta(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha) - \cos\eta\cos\beta\sin\alpha)(R+r))\dot{\alpha}+\\
&+(-(-\cos\eta\cos\gamma\cos\beta\cos\alpha + \sin\eta\sin\beta\cos\alpha)x_W + (-\sin\eta\cos\gamma\cos\beta\cos\alpha+\\
&-\cos\eta\sin\beta\cos\alpha)(R+r))\dot{\beta} + (-\cos\eta(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)x_W+\\
&+\sin\eta(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(R+r))\dot{\gamma})^2) + (1/2\,\cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+\\
&+(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma})+\\
&+(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - 1/2\,\sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+\\
&-\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))+\\
&+1/2\,\dot{\delta}_1)\theta_{W_{xx}}\,(\cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+\\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+\\
&-\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \dot{\delta}_1)+\\
&+(-1/2\,\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + 1/2\,(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma})+\\
&+1/2\,(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))\theta_{W_{yy}}\,(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+\\
&+(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+\\
&+\cos\alpha\cos\beta\dot{\gamma})) + (1/2\,\sin\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+\\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))\\
&+1/2\,\cos\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta}+\\
&+\cos\alpha\cos\beta\dot{\gamma})))\theta_{W_{zz}}\,(\sin\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+\\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))+\\
&+\cos\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma})\\
&+\cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})))
\end{aligned}
$$

$$
\begin{aligned}
T_{W2} =&1/2\, m_W\, ((\dot{x}_K + (-x_W(1/2\, \cos\eta\cos\gamma\sin\beta + 1/2\, \cos\eta\sqrt{3}\sin\gamma\sin\beta - \sin\eta\cos\beta) \\
&+(1/2\, \sin\eta\cos\gamma\sin\beta + 1/2\, \sin\eta\sqrt{3}\sin\gamma\sin\beta + \cos\eta\cos\beta)(R+r))\dot{\beta}+ \\
&+(-x_W(1/2\, \cos\eta\sin\gamma\cos\beta - 1/2\, \cos\eta\sqrt{3}\cos\gamma\cos\beta) + (1/2\, \sin\eta\sin\gamma\cos\beta+ \\
&-1/2\, \sin\eta\sqrt{3}\cos\gamma\cos\beta)(R+r))\dot{\gamma})^2 + (\dot{y}_K + (-(-1/2\, \cos\eta(-\sin\gamma\sin\alpha+ \\
&+\cos\gamma\sin\beta\cos\alpha) + 1/2\, \cos\eta\sqrt{3}(-\cos\gamma\sin\alpha - \sin\gamma\sin\beta\cos\alpha) + \sin\eta\cos\beta\cos\alpha)x_W+ \\
&+(-1/2\, \sin\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha) + 1/2\, \sin\eta\sqrt{3}(-\cos\gamma\sin\alpha+ \\
&-\sin\gamma\sin\beta\cos\alpha) - \cos\eta\cos\beta\cos\alpha)(R+r))\dot{\alpha} + (-(-1/2\, \cos\eta\cos\gamma\cos\beta\sin\alpha+ \\
&-1/2\, \cos\eta\sqrt{3}\sin\gamma\cos\beta\sin\alpha - \sin\eta\sin\beta\sin\alpha)x_W + (-1/2\, \sin\eta\cos\gamma\cos\beta\sin\alpha+ \\
&-1/2\, \sin\eta\sqrt{3}\sin\gamma\cos\beta\sin\alpha + \cos\eta\sin\beta\sin\alpha)(R+r))\dot{\beta} + (-(-1/2\, \cos\eta(\cos\gamma\cos\alpha+ \\
&-\sin\gamma\sin\beta\sin\alpha) + 1/2\, \cos\eta\sqrt{3}(-\sin\gamma\cos\alpha - \cos\gamma\sin\beta\sin\alpha))x_W+ \\
&+(-1/2\, \sin\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) + 1/2\, \sin\eta\sqrt{3}(-\sin\gamma\cos\alpha+ \\
&-\cos\gamma\sin\beta\sin\alpha))(R+r))\dot{\gamma})^2 + ((-(-1/2\, \cos\eta(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)+ \\
&+1/2\, \cos\eta\sqrt{3}(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) + \sin\eta\cos\beta\sin\alpha)x_W + (-1/2\, \sin\eta(\sin\gamma\cos\alpha+ \\
&+\cos\gamma\sin\beta\sin\alpha) + 1/2\, \sin\eta\sqrt{3}(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) - \cos\eta\cos\beta\sin\alpha)(R+r))\dot{\alpha}+ \\
&+(-(1/2\, \cos\eta\cos\gamma\cos\beta\cos\alpha + 1/2\, \cos\eta\sqrt{3}\sin\gamma\cos\beta\cos\alpha + \sin\eta\sin\beta\cos\alpha)x_W+ \\
&+(1/2\, \sin\eta\cos\gamma\cos\beta\cos\alpha + 1/2\, \sin\eta\sqrt{3}\sin\gamma\cos\beta\cos\alpha+ \\
&-\cos\eta\sin\beta\cos\alpha)(R+r))\dot{\beta} + (-(-1/2\, \cos\eta(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)+ \\
&+1/2\, \cos\eta\sqrt{3}(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha))x_W + (-1/2\, \sin\eta(\cos\gamma\sin\alpha+ \\
&+\sin\gamma\sin\beta\cos\alpha) + 1/2\, \sin\eta\sqrt{3}(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha))(R+r))\dot{\gamma})^2)+ \\
&+(-1/4\, \cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+ \\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))+ \\
&+1/4\, \cos\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+ \\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))+ \\
&-1/2\, \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma})+ \\
&+\cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + 1/2\, \dot{\delta}_2)\theta_{W_{xx}}\, (-1/2\, \cos\eta(\cos\gamma\cos\beta(\dot{\alpha}+ \\
&+\sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha+ \\
&-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + 1/2\, \cos\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+ \\
&+(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha+ \\
&+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+ \\
&-\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \dot{\delta}_2)+ \\
&+(-1/4\, \sqrt{3}(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+ \\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))+ \\
&+1/4\, \sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - 1/4\, (\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+ \\
&-\sin\alpha\cos\beta\dot{\gamma}) - 1/4\, (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+ \\
&+\cos\alpha\cos\beta\dot{\gamma}))\theta_{W_{yy}}\, (-1/2\, \sqrt{3}(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha+
\end{aligned}
$$

$$
\begin{aligned}
&+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\sin\gamma\sin\alpha-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+\\
&+\cos\alpha\cos\beta\dot\gamma))+1/2\ \sin\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)-1/2\,(\cos\gamma\cos\alpha+\\
&-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)-1/2\,(\cos\gamma\sin\alpha+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+\\
&+\cos\alpha\cos\beta\dot\gamma))+(-1/4\ \sin\eta(\cos\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+(\sin\gamma\cos\alpha+\\
&+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\sin\gamma\sin\alpha-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+\\
&+\cos\alpha\cos\beta\dot\gamma))+1/4\ \sin\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+(\cos\gamma\cos\alpha+\\
&-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\cos\gamma\sin\alpha+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+\\
&+\cos\alpha\cos\beta\dot\gamma))+1/2\ \cos\eta(\sin\beta(\dot\alpha+\sin\beta\dot\gamma)-\sin\alpha\cos\beta(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+\\
&+\cos\alpha\cos\beta(\sin\alpha\dot\beta+\cos\alpha\cos\beta\dot\gamma)))\theta_{W_{zz}}\,(-1/2\ \sin\eta(\cos\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+\\
&+(\sin\gamma\cos\alpha+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\sin\gamma\sin\alpha+\\
&-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+\cos\alpha\cos\beta\dot\gamma))+1/2\ \sin\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+\\
&+(\cos\gamma\cos\alpha-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\cos\gamma\sin\alpha+\\
&+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+\cos\alpha\cos\beta\dot\gamma))+\cos\eta(\sin\beta(\dot\alpha+\sin\beta\dot\gamma)+\\
&-\sin\alpha\cos\beta(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+\cos\alpha\cos\beta(\sin\alpha\dot\beta+\cos\alpha\cos\beta\dot\gamma)))
\end{aligned}
$$

$$
\begin{aligned}
T_{W3} =&1/2\, m_W\, ((\dot{x}_K + (-x_W(1/2\,\cos\eta\cos\gamma\sin\beta - 1/2\,\cos\eta\sqrt{3}\sin\gamma\sin\beta + \\
&-\sin\eta\cos\beta) + (1/2\,\sin\eta\cos\gamma\sin\beta - 1/2\,\sin\eta\sqrt{3}\sin\gamma\sin\beta + \\
&+\cos\eta\cos\beta)(R+r))\dot{\beta} + (-x_W(1/2\,\cos\eta\sin\gamma\cos\beta + 1/2\,\cos\eta\sqrt{3}\cos\gamma\cos\beta) + \\
&+(1/2\,\sin\eta\sin\gamma\cos\beta + 1/2\,\sin\eta\sqrt{3}\cos\gamma\cos\beta)(R+r))\dot{\gamma})^2 + \\
&+(\dot{y}_K + (-(-1/2\,\cos\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha) - 1/2\,\cos\eta\sqrt{3}(+ \\
&-\cos\gamma\sin\alpha - \sin\gamma\sin\beta\cos\alpha) + \sin\eta\cos\beta\cos\alpha)x_W + \\
&+(-1/2\,\sin\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha) - 1/2\,\sin\eta\sqrt{3}(-\cos\gamma\sin\alpha + \\
&-\sin\gamma\sin\beta\cos\alpha) - \cos\eta\cos\beta\cos\alpha)(R+r))\dot{\alpha} + \\
&+(-(-1/2\,\cos\eta\cos\gamma\cos\beta\sin\alpha + 1/2\,\cos\eta\sqrt{3}\sin\gamma\cos\beta\sin\alpha + \\
&+\sin\eta\sin\beta\sin\alpha)x_M + (-1/2\,\sin\eta\cos\gamma\cos\beta\sin\alpha + \\
&+1/2\,\sin\eta\sqrt{3}\sin\gamma\cos\beta\sin\alpha + \cos\eta\sin\beta\sin\alpha)(R+r))\dot{\beta} + \\
&+(-(-1/2\,\cos\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) - 1/2\,\cos\eta\sqrt{3}(-\sin\gamma\cos\alpha + \\
&-\cos\gamma\sin\beta\sin\alpha))x_M + (-1/2\,\sin\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) + \\
&-1/2\,\sin\eta\sqrt{3}(-\sin\gamma\cos\alpha - \cos\gamma\sin\beta\sin\alpha))(R+r))\dot{\gamma})^2 + \\
&+((-(-1/2\,\cos\eta(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha) - 1/2\,\cos\eta\sqrt{3}(\cos\gamma\cos\alpha + \\
&-\sin\gamma\sin\beta\sin\alpha) + \sin\eta\cos\beta\sin\alpha)x_M + (-1/2\,\sin\eta(\sin\gamma\cos\alpha + \\
&+\cos\gamma\sin\beta\sin\alpha) - 1/2\,\sin\eta\sqrt{3}(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) + \\
&-\cos\eta\cos\beta\sin\alpha)(R+r))\dot{\alpha} + (-(1/2\,\cos\eta\cos\gamma\cos\beta\cos\alpha + \\
&-1/2\,\cos\eta\sqrt{3}\sin\gamma\cos\beta\cos\alpha + \sin\eta\sin\beta\cos\alpha)x_M + \\
&+(1/2\,\sin\eta\cos\gamma\cos\beta\cos\alpha - 1/2\,\sin\eta\sqrt{3}\sin\gamma\cos\beta\cos\alpha + \\
&-\cos\eta\sin\beta\cos\alpha)(R+r))\dot{\beta} + (-(-1/2\,\cos\eta(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha) + \\
&-1/2\,\cos\eta\sqrt{3}(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha))x_M + (-1/2\,\sin\eta(\cos\gamma\sin\alpha + \\
&+\sin\gamma\sin\beta\cos\alpha) - 1/2\,\sin\eta\sqrt{3}(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha))(R+r))\dot{\gamma})^2) + \\
&+(-1/4\,\cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} + \\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \\
&-1/4\,\cos\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} + \\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \\
&-1/2\,\sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \\
&+\cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + 1/2\,\dot{\delta}_3)\theta_{W_{xx}}\,(-1/2\,\cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \\
&+\sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha + \\
&-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - 1/2\,\cos\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + \\
&+(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \\
&+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + \\
&-\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \dot{\delta}_3) + \\
&+(1/4\,\sqrt{3}(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} + \\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \\
&+1/4\,\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - 1/4\,(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} + \\
&-\sin\alpha\cos\beta\dot{\gamma}) - 1/4\,(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \\
&+\cos\alpha\cos\beta\dot{\gamma}))\theta_{W_{yy}}\,(1/2\,\sqrt{3}(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha +
\end{aligned}
$$

$$+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}-\sin\alpha\cos\beta\dot{\gamma})+(\sin\gamma\sin\alpha-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+$$
$$+\cos\alpha\cos\beta\dot{\gamma}))+1/2\ \sin\gamma\cos\beta(\dot{\alpha}+\sin\beta\dot{\gamma})-1/2\,(\cos\gamma\cos\alpha+$$
$$-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}-\sin\alpha\cos\beta\dot{\gamma})-1/2\,(\cos\gamma\sin\alpha+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+$$
$$+\cos\alpha\cos\beta\dot{\gamma}))+(-1/4\ \sin\eta(\cos\gamma\cos\beta(\dot{\alpha}+\sin\beta\dot{\gamma})+(\sin\gamma\cos\alpha+$$
$$+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}-\sin\alpha\cos\beta\dot{\gamma})+(\sin\gamma\sin\alpha-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+$$
$$+\cos\alpha\cos\beta\dot{\gamma}))-1/4\ \sin\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha}+\sin\beta\dot{\gamma})+(\cos\gamma\cos\alpha+$$
$$-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}-\sin\alpha\cos\beta\dot{\gamma})+(\cos\gamma\sin\alpha+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+$$
$$+\cos\alpha\cos\beta\dot{\gamma}))+1/2\ \cos\eta(\sin\beta(\dot{\alpha}+\sin\beta\dot{\gamma})-\sin\alpha\cos\beta(\cos\alpha\dot{\beta}-\sin\alpha\cos\beta\dot{\gamma})+$$
$$+\cos\alpha\cos\beta(\sin\alpha\dot{\beta}+\cos\alpha\cos\beta\dot{\gamma})))\theta_{W_{zz}}\,(-1/2\ \sin\eta(\cos\gamma\cos\beta(\dot{\alpha}+\sin\beta\dot{\gamma})+$$
$$+(\sin\gamma\cos\alpha+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}-\sin\alpha\cos\beta\dot{\gamma})+(\sin\gamma\sin\alpha+$$
$$-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+\cos\alpha\cos\beta\dot{\gamma}))-1/2\ \sin\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha}+\sin\beta\dot{\gamma})+$$
$$+(\cos\gamma\cos\alpha-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}-\sin\alpha\cos\beta\dot{\gamma})+(\cos\gamma\sin\alpha+$$
$$+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+\cos\alpha\cos\beta\dot{\gamma}))+\cos\eta(\sin\beta(\dot{\alpha}+\sin\beta\dot{\gamma})+$$
$$-\sin\alpha\cos\beta(\cos\alpha\dot{\beta}-\sin\alpha\cos\beta\dot{\gamma})+\cos\alpha\cos\beta(\sin\alpha\dot{\beta}+\cos\alpha\cos\beta\dot{\gamma})))$$

$$
\begin{aligned}
T_{M1} =& 1/2\, m_M\, ((\dot{x}_K + (-(-\cos\eta\cos\gamma\sin\beta - \sin\eta\cos\beta)x_M + (-\sin\eta\cos\gamma\sin\beta + \cos\eta\cos\beta)(R+\\
&+r))\dot{\beta} + (\cos\eta\sin\gamma\cos\beta x_M - \sin\eta\sin\gamma\cos\beta(R+r))\dot{\gamma})^2 + (\dot{y}_K + (-(\cos\eta(-\sin\gamma\sin\alpha+\\
&+\cos\gamma\sin\beta\cos\alpha) + \sin\eta\cos\beta\cos\alpha)x_M + (\sin\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha)+\\
&-\cos\eta\cos\beta\cos\alpha)(R+r))\dot{\alpha} + (-(\cos\eta\cos\gamma\cos\beta\sin\alpha - \sin\eta\sin\beta\sin\alpha)x_M+\\
&+(\sin\eta\cos\gamma\cos\beta\sin\alpha + \cos\eta\sin\beta\sin\alpha)(R+r))\dot{\beta} + (-\cos\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)x_M+\\
&+\sin\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(R+r))\dot{\gamma})^2 + ((-(\cos\eta(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)+\\
&+\sin\eta\cos\beta\sin\alpha)x_M + (\sin\eta(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha) - \cos\eta\cos\beta\sin\alpha)(R+r))\dot{\alpha}+\\
&+(-(-\cos\eta\cos\gamma\cos\beta\cos\alpha + \sin\eta\sin\beta\cos\alpha)x_M + (-\sin\eta\cos\gamma\cos\beta\cos\alpha+\\
&-\cos\eta\sin\beta\cos\alpha)(R+r))\dot{\beta} + (-\cos\eta(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)x_M+\\
&+\sin\eta(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(R+r))\dot{\gamma})^2)+\\
&+(1/2\ \cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma})+\\
&+(\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - 1/2\ \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+\\
&-\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))+\\
&+1/2\, i_g\dot{\delta}_1)\theta_{W_{xx}}\, (\cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+\\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+\\
&-\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + i_g\dot{\delta}_1)+\\
&+(-1/2\ \sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + 1/2\,(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma})+\\
&+1/2\,(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))\theta_{W_{yy}}\,(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma})+\\
&+(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta}+\\
&+\cos\alpha\cos\beta\dot{\gamma})) + (1/2\ \sin\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+\\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))\\
&+1/2\ \cos\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta}+\\
&+\cos\alpha\cos\beta\dot{\gamma})))\theta_{W_{zz}}\,(\sin\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta}+\\
&-\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma}))+\\
&+\cos\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma})\\
&+\cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})))
\end{aligned}
$$

$$
\begin{aligned}
T_{M2} =& 1/2\, m_M\, ((\dot{x}_K + (-x_M(1/2\, \cos\eta\cos\gamma\sin\beta + 1/2\, \cos\eta\sqrt{3}\sin\gamma\sin\beta + \\
& -\sin\eta\cos\beta) + (1/2\, \sin\eta\cos\gamma\sin\beta + 1/2\, \sin\eta\sqrt{3}\sin\gamma\sin\beta + \\
& +\cos\eta\cos\beta)(R+r))\dot{\beta} + (-x_M(1/2\, \cos\eta\sin\gamma\cos\beta - 1/2\, \cos\eta\sqrt{3}\cos\gamma\cos\beta) + \\
& +(1/2\, \sin\eta\sin\gamma\cos\beta - 1/2\, \sin\eta\sqrt{3}\cos\gamma\cos\beta)(R+r))\dot{\gamma})^2 + \\
& +(y_{K_d} + (-(-1/2\, \cos\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha) + 1/2\, \cos\eta\sqrt{3}(+ \\
& -\cos\gamma\sin\alpha - \sin\gamma\sin\beta\cos\alpha) + \sin\eta\cos\beta\cos\alpha)x_M + \\
& +(-1/2\, \sin\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha) + 1/2\, \sin\eta\sqrt{3}(-\cos\gamma\sin\alpha + \\
& -\sin\gamma\sin\beta\cos\alpha) - \cos\eta\cos\beta\cos\alpha)(R+r))\dot{\alpha} + \\
& +(-(-1/2\, \cos\eta\cos\gamma\cos\beta\sin\alpha - 1/2\, \cos\eta\sqrt{3}\sin\gamma\cos\beta\sin\alpha + \\
& -\sin\eta\sin\beta\sin\alpha)x_M + (-1/2\, \sin\eta\cos\gamma\cos\beta\sin\alpha + \\
& -1/2\, \sin\eta\sqrt{3}\sin\gamma\cos\beta\sin\alpha + \cos\eta\sin\beta\sin\alpha)(R+r))\dot{\beta} + \\
& +(-(-1/2\, \cos\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) + 1/2\, \cos\eta\sqrt{3}(-\sin\gamma\cos\alpha + \\
& -\cos\gamma\sin\beta\sin\alpha))x_M + (-1/2\, \sin\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) + \\
& +1/2\, \sin\eta\sqrt{3}(-\sin\gamma\cos\alpha - \cos\gamma\sin\beta\sin\alpha))(R+r))\dot{\gamma})^2 + \\
& +((-(-1/2\, \cos\eta(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha) + 1/2\, \cos\eta\sqrt{3}(\cos\gamma\cos\alpha + \\
& -\sin\gamma\sin\beta\sin\alpha) + \sin\eta\cos\beta\sin\alpha)x_M + (-1/2\, \sin\eta(\sin\gamma\cos\alpha + \\
& +\cos\gamma\sin\beta\sin\alpha) + 1/2\, \sin\eta\sqrt{3}(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) + \\
& -\cos\eta\cos\beta\sin\alpha)(R+r))\dot{\alpha} + (-(1/2\, \cos\eta\cos\gamma\cos\beta\cos\alpha + \\
& +1/2\, \cos\eta\sqrt{3}\sin\gamma\cos\beta\cos\alpha + \sin\eta\sin\beta\cos\alpha)x_M + \\
& +(1/2\, \sin\eta\cos\gamma\cos\beta\cos\alpha + 1/2\, \sin\eta\sqrt{3}\sin\gamma\cos\beta\cos\alpha + \\
& -\cos\eta\sin\beta\cos\alpha)(R+r))\dot{\beta} + (-(-1/2\, \cos\eta(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha) + \\
& +1/2\, \cos\eta\sqrt{3}(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha))x_M + (-1/2\, \sin\eta(\cos\gamma\sin\alpha + \\
& +\sin\gamma\sin\beta\cos\alpha) + 1/2\, \sin\eta\sqrt{3}(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha))(R+r))\dot{\gamma})^2) + \\
& +(-1/4\, \cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} + \\
& -\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \\
& +1/4\, \cos\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} + \\
& -\sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \\
& -1/2\, \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \\
& +\cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + 1/2\, i_g\delta_{2_d})\theta_{W_{xx}}\, (-1/2\, \cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \\
& +\sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha + \\
& -\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + 1/2\, \cos\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + \\
& +(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \\
& +\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + \\
& -\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + i_g\dot{\delta}_2) + \\
& +(-1/4\, \sqrt{3}(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} + \\
& -\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \\
& +1/4\, \sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - 1/4\, (\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} + \\
& -\sin\alpha\cos\beta\dot{\gamma}) - 1/4\, (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} +
\end{aligned}
$$

$$+\cos\alpha\cos\beta\dot\gamma))\theta_{W_{yy}}\,(-1/2\,\sqrt{3}(\cos\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+(\sin\gamma\cos\alpha+$$
$$+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\sin\gamma\sin\alpha-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+$$
$$+\cos\alpha\cos\beta\dot\gamma))+1/2\,\sin\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)-1/2\,(\cos\gamma\cos\alpha+$$
$$-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)-1/2\,(\cos\gamma\sin\alpha+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+$$
$$+\cos\alpha\cos\beta\dot\gamma))+(-1/4\,\sin\eta(\cos\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+(\sin\gamma\cos\alpha+$$
$$+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\sin\gamma\sin\alpha-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+$$
$$+\cos\alpha\cos\beta\dot\gamma))+1/4\,\sin\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+(\cos\gamma\cos\alpha+$$
$$-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\cos\gamma\sin\alpha+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+$$
$$+\cos\alpha\cos\beta\dot\gamma))+1/2\,\cos\eta(\sin\beta(\dot\alpha+\sin\beta\dot\gamma)-\sin\alpha\cos\beta(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+$$
$$+\cos\alpha\cos\beta(\sin\alpha\dot\beta+\cos\alpha\cos\beta\dot\gamma)))\theta_{W_{zz}}\,(-1/2\,\sin\eta(\cos\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+$$
$$+(\sin\gamma\cos\alpha+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\sin\gamma\sin\alpha+$$
$$-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+\cos\alpha\cos\beta\dot\gamma))+1/2\,\sin\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot\alpha+\sin\beta\dot\gamma)+$$
$$+(\cos\gamma\cos\alpha-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+(\cos\gamma\sin\alpha+$$
$$+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot\beta+\cos\alpha\cos\beta\dot\gamma))+\cos\eta(\sin\beta(\dot\alpha+\sin\beta\dot\gamma)+$$
$$-\sin\alpha\cos\beta(\cos\alpha\dot\beta-\sin\alpha\cos\beta\dot\gamma)+\cos\alpha\cos\beta(\sin\alpha\dot\beta+\cos\alpha\cos\beta\dot\gamma)))$$

$$
\begin{aligned}
T_{M3} =& 1/2\, m_M\, ((\dot{x}_K + (-x_M(1/2\, \cos\eta\cos\gamma\sin\beta - 1/2\, \cos\eta\sqrt{3}\sin\gamma\sin\beta +\\
& -\sin\eta\cos\beta) + (1/2\, \sin\eta\cos\gamma\sin\beta - 1/2\, \sin\eta\sqrt{3}\sin\gamma\sin\beta +\\
& +\cos\eta\cos\beta)(R+r))\dot{\beta} + (-x_M(1/2\, \cos\eta\sin\gamma\cos\beta + 1/2\, \cos\eta\sqrt{3}\cos\gamma\cos\beta) +\\
& +(1/2\, \sin\eta\sin\gamma\cos\beta + 1/2\, \sin\eta\sqrt{3}\cos\gamma\cos\beta)(R+r))\dot{\gamma})^2 +\\
& +(y_{K_d} + (-(-1/2\, \cos\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha) - 1/2\, \cos\eta\sqrt{3}(+\\
& -\cos\gamma\sin\alpha - \sin\gamma\sin\beta\cos\alpha) + \sin\eta\cos\beta\cos\alpha)x_M +\\
& +(-1/2\, \sin\eta(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha) - 1/2\, \sin\eta\sqrt{3}(-\cos\gamma\sin\alpha +\\
& -\sin\gamma\sin\beta\cos\alpha) - \cos\eta\cos\beta\cos\alpha)(R+r))\dot{\alpha} +\\
& +(-(-1/2\, \cos\eta\cos\gamma\cos\beta\sin\alpha + 1/2\, \cos\eta\sqrt{3}\sin\gamma\cos\beta\sin\alpha +\\
& +\sin\eta\sin\beta\sin\alpha)x_M + (-1/2\, \sin\eta\cos\gamma\cos\beta\sin\alpha +\\
& +1/2\, \sin\eta\sqrt{3}\sin\gamma\cos\beta\sin\alpha + \cos\eta\sin\beta\sin\alpha)(R+r))\dot{\beta} +\\
& +(-(-1/2\, \cos\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) - 1/2\, \cos\eta\sqrt{3}(-\sin\gamma\cos\alpha +\\
& -\cos\gamma\sin\beta\sin\alpha))x_M + (-1/2\, \sin\eta(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) +\\
& -1/2\, \sin\eta\sqrt{3}(-\sin\gamma\cos\alpha - \cos\gamma\sin\beta\sin\alpha))(R+r))\dot{\gamma})^2 +\\
& +((-(-1/2\, \cos\eta(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha) - 1/2\, \cos\eta\sqrt{3}(\cos\gamma\cos\alpha +\\
& -\sin\gamma\sin\beta\sin\alpha) + \sin\eta\cos\beta\sin\alpha)x_M + (-1/2\, \sin\eta(\sin\gamma\cos\alpha +\\
& +\cos\gamma\sin\beta\sin\alpha) - 1/2\, \sin\eta\sqrt{3}(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha) +\\
& -\cos\eta\cos\beta\sin\alpha)(R+r))\dot{\alpha} + (-(1/2\, \cos\eta\cos\gamma\cos\beta\cos\alpha +\\
& -1/2\, \cos\eta\sqrt{3}\sin\gamma\cos\beta\cos\alpha + \sin\eta\sin\beta\cos\alpha)x_M +\\
& +(1/2\, \sin\eta\cos\gamma\cos\beta\cos\alpha - 1/2\, \sin\eta\sqrt{3}\sin\gamma\cos\beta\cos\alpha +\\
& -\cos\eta\sin\beta\cos\alpha)(R+r))\dot{\beta} + (-(-1/2\, \cos\eta(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha) +\\
& -1/2\, \cos\eta\sqrt{3}(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha))x_M + (-1/2\, \sin\eta(\cos\gamma\sin\alpha +\\
& +\sin\gamma\sin\beta\cos\alpha) - 1/2\, \sin\eta\sqrt{3}(-\sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha))(R+r))\dot{\gamma})^2) +\\
& +(-1/4\, \cos\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} +\\
& -\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) +\\
& -1/4\, \cos\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} +\\
& -\sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) +\\
& -1/2\, \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) +\\
& +\cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + 1/2\, i_g\dot{\delta}_3)\theta_{W_{xx}}\, (-1/2\, \cos\eta(\cos\gamma\cos\beta(\dot{\alpha} +\\
& +\sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha +\\
& -\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - 1/2\, \cos\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) +\\
& +(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha +\\
& +\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - \sin\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) +\\
& -\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + i_g\dot{\delta}_3) +\\
& +(1/4\, \sqrt{3}(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} +\\
& -\sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) +\\
& +1/4\, \sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - 1/4\, (\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} +\\
& -\sin\alpha\cos\beta\dot{\gamma}) - 1/4\, (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} +\\
& +\cos\alpha\cos\beta\dot{\gamma}))\theta_{W_{yy}}\, (1/2\, \sqrt{3}(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha +\\
& +\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} +
\end{aligned}
$$

$$
\begin{aligned}
&+\cos\alpha\cos\beta\dot{\gamma})) + 1/2\ \sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - 1/2\,(\cos\gamma\cos\alpha +\\
&-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) - 1/2\,(\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} +\\
&+\cos\alpha\cos\beta\dot{\gamma})) + (-1/4\ \sin\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\sin\gamma\cos\alpha +\\
&+\cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} +\\
&+\cos\alpha\cos\beta\dot{\gamma})) - 1/4\ \sin\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) + (\cos\gamma\cos\alpha +\\
&-\sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} +\\
&+\cos\alpha\cos\beta\dot{\gamma})) + 1/2\ \cos\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) - \sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) +\\
&+\cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})))\theta_{W_{zz}}\,(-1/2\ \sin\eta(\cos\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) +\\
&+(\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\sin\gamma\sin\alpha +\\
&-\cos\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) - 1/2\ \sin\eta\sqrt{3}(-\sin\gamma\cos\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) +\\
&+(\cos\gamma\cos\alpha - \sin\gamma\sin\beta\sin\alpha)(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + (\cos\gamma\sin\alpha +\\
&+\sin\gamma\sin\beta\cos\alpha)(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})) + \cos\eta(\sin\beta(\dot{\alpha} + \sin\beta\dot{\gamma}) +\\
&-\sin\alpha\cos\beta(\cos\alpha\dot{\beta} - \sin\alpha\cos\beta\dot{\gamma}) + \cos\alpha\cos\beta(\sin\alpha\dot{\beta} + \cos\alpha\cos\beta\dot{\gamma})))
\end{aligned}
$$

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