Combinatorial Characterization of Interference Coupling in Wireless Systems

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***Abstract*—We provide a combinatorial characterization of interference coupling in wireless systems, with the intent of obtaining a better insight into interference coordination and management. We introduce two bipartite graphs, namely the *power graph* and *interference graph*. We utilize these graphs and *global dependency matrix* containing only binary (0 and 1) entries to capture the effects of interference coupling in communication systems. We show that the irreducibility of the *global dependency* matrix G is related to the connectivity of the *power graph* and the irreducibility of the matrix GG^T is related to the connectivity of the *interference graph*. We prove that for *strictly positive* and *strictly log-convex* interference functions, the irreducibility of the matrices G and GG^T are necessary and sufficient conditions for the considered utility sets to be strictly convex. In this case there exists a unique optimizer for the problem of maximizing the product of utilities. We show that an *interference balancing* function is *strictly log-convex*, if and only if matrices G and GG^T are irreducible. We provide a simple yet comprehensive combinatorial characterization of interference coupled systems which abstracts away certain complexities of the physical layer.**

***Index Terms*—Interference coupling, power graph, interference graph, global dependency matrix.**

I. Introduction

THE performance of the users in a multi-point to multi-point communication system is often influenced by interference coupling of the involved users. Investigating interference management and coordination could provide a means for delivering better performance to the users and in turn increasing the revenue of the operator. Furthermore, interference management and coordination for indoor wireless systems has not as yet been completely addressed.

We present a combinatorial characterization of interference coupling in wireless systems. A combinatorial characterization could be comprehensive and still a fairly simplistic framework to capture the effects of interference coupling in wireless networks. A combinatorial characterization has a strong intuitive graph theoretic appeal and could provide another insight into existent power control algorithms, e.g. uniqueness of optimal solution outcome and convergence speed of the power control algorithms.

Consider a wireless system with $K \geq 2$ users from an index set $\mathcal{K} := \{1, \ldots, K\}$. If the users are coupled by interference, then there is a general trade-off between the utilities $\boldsymbol{u} = [u_1, \ldots, u_K]^T$. We utilize an interference function framework [1] (defined in Section II-B) to capture the effects of interference coupling in wireless networks. The performance (utility) of each user is characterized by its signal-to-interference ratio (SIR), which is an important performance indicator in wireless systems [2]. The set of all feasible SIR levels with all users being active concurrently is called the *feasible SIR region*. We denote the feasible SIR region or more generally the utility region by $\mathbf{U}$.

It is known that in general the feasible SIR region is not convex, e.g. SIR region in the case of linear receivers, certain rate regions for cellular systems without full coordination, etc. [3]–[5]. Standard properties of the utility sets, like convexity and compactness cannot be taken for granted. However, the region can be "convexified" if the SIRs are expressed in the logarithmic scale, thereby giving rise to the notion of a *feasible log-SIR* region. The convexity property of the feasible log-SIR region is a crucial element in the development of power control strategies.

We investigate the structure of certain utility sets (SIR regions) for the case without power constraints. This not only simplifies the problem but it has the advantage of clearly elucidating the effects of interference coupling on the feasible SIR region and the resulting proportional fair operating point. Analyzing the problem for high signal-to-noise-and-interference-ratio (SINR) often helps better understand its analytical structure [6], providing a basis for future extensions, where noise and power constraints are included. The axiomatic framework is useful for characterizing coupling effects.

As an example, consider the function

$$\sum_{k\in\mathcal{K}} \omega_k \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k}, p_k = \exp(s_k), \boldsymbol{\omega} > \mathbf{0}, \qquad \sum_{k\in\mathcal{K}} \omega_k = 1, \quad (1)$$

where $\boldsymbol{V}$ is the coupling matrix and $[\boldsymbol{V}]_{kl}$ is the coupling coefficient between the l^{th} transmitter and k^{th} receiver. This is the weighted sum of inverse SIRs. We can picture (1) as a loss function. Hence, we would like to minimize this function and we would be interested in exploring the convexity

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properties of such a function. The results in [7] states that (1) is jointly convex with respect to $\boldsymbol{s}$, if and only if $\boldsymbol{V}\boldsymbol{V}^T$ is irreducible. $\boldsymbol{V}$ is the link gain matrix for a wireless system. However, the convexity of the function defined in (1) is not sufficient to ensure *strict convexity* of the log-SIR region. *Strict convexity* is a very useful property from an optimization perspective. Strict convexity along with certain additional natural properties of interference functions (shall be discussed in detail in Section II-D) allow the existence of a minimal/maximal element. It can be observed, that the joint convexity of (1) does not guarantee that a minimizer to the optimization problem exists, i.e. the infimum need not be equal to the minimum.

We impose an additional property of *strict convexity* of the *interference balancing* functions (introduced in Section II-B) to ensure that the log-SIR region is *strictly convex*. There is no connection (as yet) between the strict convexity of (1) and the *strict convexity* of the log-SIR region. In [8] only a subset of the family of *strict convex* sets, specifically a subset of the family of *strictly convex* log-SIR regions was considered. The considered sets possessed the property, that $\max_{\boldsymbol{u}\in\mathbf{U}}\prod_{k\in\mathcal{K}}u_k$ has a strictly positive solution, where $\mathbf{U}$ is the utility region or the feasible SIR region. It is not immediately obvious that these sets are connected with the interference function framework.

In this paper we shall establish this connection between the *strict convexity* of the log-SIR regions and certain properties of the interference function framework for the case of no power constraints. The contributions and outline of our paper are as follows:

1) We show that under certain conditions for interference functions with the following two properties:
   a) *strict log-convexity* on their respective *dependency* sets, and
   b) and *strict positivity* on their respective *dependency* sets (introduced in Section II-D)

   we obtain strictly convex sets (strictly convex log-SIR regions). In this case there exists a unique optimizer for the product of utilities maximization problem for these sets, i.e. the infimum is equal to the minimum. We are able to construct the corresponding power vector.
2) In Section III we introduce two graph theoretical representations of interference coupling:
   a) the *power graph*, and
   b) the *interference graph*.

   We investigate the relationship between the connectivity of the *power graph* and the irreducibility of the *global dependency* matrix $\boldsymbol{G}$ (explained in Section II-C).
3) We investigate the relationship between the connectivity of the *interference graph* and the irreducibility of the matrix $\boldsymbol{G}\boldsymbol{G}^T$. We show that the irreducibility of both $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$, along with the two properties of the interference function described above are *necessary and sufficient* conditions, to show the existence of a unique optimizer for the family of strictly convex sets (see Section IV).

We provide a complete characterization of the log-SIR region and extend the results of *strict convexity* of the feasible log-SIR region from linear interference functions to a broader class of log-convex interference functions.

The main theorem in this paper has applications to social choice theory and specifically to the existence of a *maximal element*[1]. Existence of the *maximal element* follows from the strict convexity of the utility set. The *maximal element* could be the Nash bargaining solution (NBS) (the NBS is discussed briefly in Section II-A). We provide a constructive proof of the results. We show that we require the reducibility of the *global dependency* matrix $\boldsymbol{G}$ (explained in detail in Section II-C) to investigate the weak *Pareto optimal* points [2]. We state the consequences of irreducibility via the combinatorial characterization in Section IV.

II. Interference Coupled Wireless Systems

Before we begin to describe our system model and present the relevant definitions, we describe certain preliminaries in Section II-A below.

A. Notation and Preliminaries

Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the l^{th} component. Likewise let $G_{mn} = [\boldsymbol{G}]_{mn}$ be a component of the matrix $\boldsymbol{G}$. Let $\boldsymbol{y} \geq 0$ imply that $y_l \geq 0$ for all components l. Let $\boldsymbol{x} \gneqq \boldsymbol{y}$ imply component-wise inequality with strict inequality for at least one component. Corresponding definitions hold for the reverse directions. Let $\boldsymbol{x} \neq \boldsymbol{y}$ imply that the vector differs in at least one component. Let $\mathbf{X}$ represent a set. Let $\mathcal{X}$ represent a family of sets. Let the set of non-negative real numbers be denoted as $\mathbb{R}_+$. Let the set of positive real numbers be denoted as $\mathbb{R}_{++}$. Let the set of all natural numbers be denoted as $\mathbb{N}$. Let $\underline{\boldsymbol{e}}(r)$ be a K-dimensional vector, which is the all-zero vector with the r^{th} component set to one, i.e. $[\underline{\boldsymbol{e}}(r)]_k = \begin{cases} 1 & k = r \\ 0 & k \neq r. \end{cases}$ A set $\mathbf{U} \subset \mathbb{R}^K_{++}$ is said to be (downward)-comprehensive if for all $\boldsymbol{u} \in \mathbf{U}$ and $\hat{\boldsymbol{u}} \in \mathbb{R}^K_{++}$, $\mathbf{0} < \hat{\boldsymbol{u}} \leq \boldsymbol{u}$ implies $\hat{\boldsymbol{u}} \in \mathbf{U}$. In this paper, compact and closed are defined relatively[3] in $\mathbb{R}^K_{++}$. A bargaining solution is a unanimous agreement on certain utilities $\boldsymbol{u} = [u_1, \ldots, u_K]$ from a utility set $\mathbf{U}$. The Nash bargaining solution (NBS) corresponds to a *Pareto optimal* point $\Phi(\mathbf{U})$ characterized by a set of axioms (Nash axioms)[9], [10]. Let the region $\mathbf{U} \subset \mathbb{R}^K_{++}$ be compact, convex and comprehensive. Then, the unique NBS fulfilling Nash's axioms is obtained by maximizing the product of the utilities, i.e. $\max_{\boldsymbol{u}\in\mathbf{U}}\prod_{k\in\mathcal{K}}u_k$. Since, $\log\max\prod_k u_k = \max\log\prod_k u_k = \max\sum_k \log u_k$, an equivalent optimizer can be found solving $\max_{\boldsymbol{u}\in\mathbf{U}}\sum_{k\in\mathcal{K}}\log u_k$. This strategy is related to proportional fairness (PF) [11]. If the set $\mathbf{U}$ is not compact and convex then it is a-priori unclear whether the

[1]Let $(\mathbf{V}, \leq)$ be a partially ordered set and $\mathbf{U} \subset \mathbf{V}$. Then, $\boldsymbol{u} \in \mathbf{U}$ is a *maximal element* of $\mathbf{U}$, if for all $\hat{\boldsymbol{u}} \in \mathbf{U}$, $\boldsymbol{u} \leq \hat{\boldsymbol{u}}$ implies that $\boldsymbol{u} = \hat{\boldsymbol{u}}$.

[2]For every set compact, comprehensive set $\mathbf{U} \subset \mathbb{R}^K_{++}$, the set of weak *Pareto optimal* points is defined as follows: $W(\mathbf{U}) := \{\boldsymbol{u}^{(1)} \in \mathbf{U}$: such that there is no $\boldsymbol{u}^{(2)} \in \mathbf{U}$ with $\boldsymbol{u}^{(2)} > \boldsymbol{u}^{(1)}\}$.

[3]A set $\mathbf{U} \subset \mathbb{R}^K_{++}$ is said to be relatively closed in $\mathbb{R}^K_{++}$ if there exists a closed set $\mathbf{A} \subset \mathbb{R}^K$ such that $\mathbf{U} = \mathbf{A} \cap \mathbb{R}^K_{++}$. This is the case when all the users are active. It is possible to extend this framework to cases where all users need not be active.

maximum $\max_{\boldsymbol{u}\in\mathbf{U}} \sum_{k\in\mathcal{K}} \log u_k$ exists. If it exists then it is unclear whether the optimum is really the NBS.

Definition 1. We say that a set $\mathbf{U} \subset \mathbb{R}^K_{++}$ is log-convex, if the image set $\mathcal{L}og(\mathbf{U}) = \{\boldsymbol{q} = \log(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbf{U}\}$ is convex.

Definition 2. By $\mathcal{ST}$ we denote the family of all closed comprehensive utility sets $\mathbf{U} \in \mathbb{R}^K_{++}$ such that $\mathcal{L}og(\mathbf{U})$ is a strictly convex set in $\mathbb{R}^K$.

By $\mathcal{ST}_c$ we denote the family of all $\mathbf{U} \in \mathcal{ST}$ that are additionally upper bounded.

For bounded sets from $\mathcal{ST}_c$, it was shown in [12] that the unique optimizer fulfilling the Nash axioms is always the optimizer of $\max_{\boldsymbol{u}\in\mathbf{U}} \prod_{k\in\mathcal{K}} u_k$. In this paper we consider a possibly unbounded set $\mathbf{U} \in \mathcal{ST}$, for which the results in [8] cannot be applied directly. We now present the interference function framework, which is used to capture interference coupling in wireless systems.

B. Interference Functions

In a wireless system, the users' utilities can strongly depend on the underlying physical layer. The utility under consideration is $\text{SIR}_k(\boldsymbol{p}) = \frac{p_k}{\mathcal{I}_k(\boldsymbol{p})}$, for a user $k \in \mathcal{K}$. The function $\mathcal{I}_k(\boldsymbol{p})$ yields the interference power experienced by the k^{th} user. It is function of the transmission powers $\boldsymbol{p} = [p_1, \dots, p_K]^T$. Many other performance measures have a direct relationship with SIR. Utility can represent certain arbitrary performance measure, which depends on the SIR by a strictly monotone and continuous function ϕ defined on $\mathbb{R}_+$. The utility of user k is $u_k(\boldsymbol{p}) = \phi_k\big(\text{SIR}_k(\boldsymbol{p})\big)$, where user $k \in \mathcal{K}$. An example of the above case is capacity: $\phi(x) = \log(1+x)$. Note that the effective path gain of user k can be incorporated in the function $\mathcal{I}_k(\boldsymbol{p})$ as an additional scaling factor. In this case, $\text{SIR}_k(\boldsymbol{p})$ is the ratio of the received power to the interference power. In order to model the interference, we use the axiomatic framework proposed in [1].

Definition 3. We say that $\mathcal{I} : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ is an *interference function* if the following axioms are fulfilled:

- $A1$ positivity: $\exists \boldsymbol{p} > \mathbf{0}$with $\mathcal{I}(\boldsymbol{p}) > 0$
- $A2$ scale invariance: $\mathcal{I}(\alpha\boldsymbol{p}) = \alpha\mathcal{I}(\boldsymbol{p})$ for all $\alpha \geq 0$
- $A3$ monotonicity: $\mathcal{I}(\boldsymbol{p}) \geq \mathcal{I}(\hat{\boldsymbol{p}})$ if $\boldsymbol{p} \geq \hat{\boldsymbol{p}}$.

The axiomatic framework $A1$ to $A3$ is related with the framework of *standard interference functions* [13]. The details about the relationship between the model $A1$-$A3$ and Yates' *standard interference functions* were discussed in [1], [14]. For the purpose of this paper it is sufficient to be aware that there exists a connection between these two models and the results of this paper are applicable to *standard interference functions*.

We now introduce the function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$, known as the *interference balancing* function. The function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$ is an indicator of the feasibility of SIR values $\boldsymbol{\gamma} = [\gamma_1, \dots, \gamma_K]^T$ for users $k = 1, \dots, K$, respectively.

$$C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}) = \inf_{\boldsymbol{p}>\mathbf{0}} \max_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} \tag{2}$$

We have the following expression $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}) \leq 1$, if and only if for any $\epsilon > 0$ there exists a power vector $\boldsymbol{p}_\epsilon > \mathbf{0}$ such that $\gamma_k(\boldsymbol{p}_\epsilon) \geq \gamma_k - \epsilon$, for all $k \in \mathcal{K}$. The feasible SIR region is the sub-level set

$$\mathbf{S} = \{\gamma \in \mathbb{R}^K_{++} \mid C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}) \leq 1\}. \tag{3}$$

Boundary points of $\mathbf{S}$ are characterized by $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}) = 1$.

C. Interference Coupling

The structure of the SIR region depends on the interference coupling. For general interference functions it is not obvious as to what would be an appropriate system to define interference coupling. We define the system as "coupled", if there is some arbitrary power vector $\boldsymbol{p}$ such that $[\boldsymbol{D}_\mathcal{I}(\boldsymbol{p})]_{kr} = 1$, where for a given k, r we have

$$[\boldsymbol{D}_\mathcal{I}(\boldsymbol{p})]_{kr} = \begin{cases} 1 & \text{if there exists a } \delta_r(\boldsymbol{p}) > 0 \\ & \text{ such that the function} \\ & f_r(\delta, \boldsymbol{p}) = \mathcal{I}_k(\underline{\boldsymbol{p}} - \delta\underline{\boldsymbol{e}}(r)) \\ & \text{is strictly monotone decreasing} \\ & \text{for } 0 \leq \delta \leq \delta_r(\boldsymbol{p}) \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\boldsymbol{D}$ is called the *dependency* matrix. Thereby, we obtain the *global dependency* matrix, which is independent of the choice of the power vector $\boldsymbol{p}$ as follows.

Definition 4. *Global dependency* matrix: $\boldsymbol{G}_\mathcal{I}$ is the *global dependency* matrix, given by

$$[\boldsymbol{G}_\mathcal{I}]_{kr} = \begin{cases} 1 & \text{if there exists a } \boldsymbol{p} \geq \mathbf{0} \text{ such that} \\ & [\boldsymbol{D}_\mathcal{I}(\boldsymbol{p})]_{kr} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The non-zero entries in $\boldsymbol{G} := \boldsymbol{G}_\mathcal{I}$ mark the transmitter/receiver pairs, which are coupled by interference. A zero entry implies that no interference is received, irrespective of the magnitude of the transmission power.

Example 1. Consider that users are assigned to different orthogonal resources separated by adaptive interference rejection techniques. This coupling model includes the widely used concept of a "link gain matrix" as a special case.

We assume $\boldsymbol{G}$ to be an *irreducible* matrix [15], pp. 360-361. This implies that each user is interfered by at least one other user. *Irreducibility* of $\boldsymbol{G}$ is equivalent to strong connectivity of the graph $\mathcal{G}(\boldsymbol{G})$, where $\mathcal{G}(\boldsymbol{G})$ is defined to be the directed graph of $\mathcal{K}$ nodes, in which there is a directed edge leading from node $r \in \mathcal{K}$ to $k \in \mathcal{K}$ if and only if $[\boldsymbol{G}]_{kr} > 0$. The matrix $\boldsymbol{G}$ is called the adjacency matrix of the graph $\mathcal{G}(\boldsymbol{G})$. This graph is said to be strongly connected if for each pair of nodes (k, r), there is an uninterrupted sequence of directed edges leading from r to k.

Remark 1. Note that the direction matters in the definition of strong connectivity for directed graphs.

We now introduce the *dependency* set.

Definition 5. *Dependency* set: L_k is the *dependency* set for user k, if $L_k = \{r \in \mathcal{K} \mid [\boldsymbol{G}_\mathcal{I}]_{kr} = 1\}$, where $\boldsymbol{G}$ is given by (4).

This is the set of transmitters, which have an impact on user k. We now present an example below, which display the following remark.

Remark 2. Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be linear interference functions, e.g. $\mathcal{I}_k(\boldsymbol{p}) = \sum_{j\in\mathcal{K}} v_{kj}p_j$. Then, this assumption is not always sufficient to ensure that the infimum in (1) is achieved.

Example 2. Consider the case of linear interference functions $\mathcal{I}_k(\boldsymbol{p})$. Hence we have that the link gain matrix $\boldsymbol{V} = \boldsymbol{G}$, where $\boldsymbol{G}$ is the *global dependency* matrix. Consider the case of a system with 4 users. Let

$$\boldsymbol{V} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let $p_1, p_2, p_3, p_4 > 0$ be arbitrary powers of user 1, 2, 3 and 4 respectively. Then we have that $\sum_{k=1}^4 \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} = \log \frac{p_2}{p_1} + \log \frac{p_1}{p_2} + \log \frac{p_1+p_2+p_4}{p_3} + \log \frac{p_1+p_2+p_3}{p_4} = \log \frac{(p_1+p_2+p_4)(p_1+p_2+p_3)}{p_3 p_4} > \log \frac{p_3 p_4}{p_3 p_4} = 0$. Let us choose the powers of user 3 and user 4 as $p_3, p_4 > 0$ and the powers of user 1 and user 2 as $p_1^{(n)} = p_2^{(n)} = 1/n$ for $n \in \mathbb{N}$. With the choice of power vector from the previous sentence, we have that $\inf_{\boldsymbol{p}>\boldsymbol{0}} \sum_{k=1}^4 \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} \leq \log \frac{(\frac{2}{n}+p_4)(\frac{2}{n}+p_3)}{p_3 p_4}$, for all choices of $n \in \mathbb{N}$. We now have that $0 \leq \inf_{\boldsymbol{p}>\boldsymbol{0}} \sum_{k=1}^4 \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} \leq \inf_{n\in\mathbb{N}} \log \frac{(\frac{2}{n}+p_4)(\frac{2}{n}+p_3)}{p_3 p_4} = 0$. We can observe that $\inf_{\boldsymbol{p}>\boldsymbol{0}} \sum_{k=1}^K \log \frac{[\boldsymbol{V}\boldsymbol{p}]_k}{p_k} = 0$. Let us now assume that there exists a power vector $\tilde{\boldsymbol{p}} > \boldsymbol{0}$ such that $0 = \sum_{k=1}^4 \log \frac{[\boldsymbol{V}\hat{\boldsymbol{p}}]_k}{\hat{p}_k}$. Then we have that $0 = \log \frac{(\hat{p}_1+\hat{p}_2+\hat{p}_4)(\hat{p}_1+\hat{p}_2+\hat{p}_3)}{\hat{p}_3\hat{p}_4} = \log\big(1 + \frac{(\hat{p}_1+\hat{p}_2)(\hat{p}_3+\hat{p}_4)}{\hat{p}_3\hat{p}_4} + \frac{\hat{p}_1+\hat{p}_2}{\hat{p}_3\hat{p}_4}\big) > \log 1 = 0$. Hence, we conclude that the infimum is not achieved.

Hence, we impose certain additional properties of interference functions, so that we can achieve the desired infimum. These properties are introduce in Section II-D below.

D. Additional Properties of Interference Functions

We begin by introducing the property of *strict monotonicity*.

Definition 6. *Strict monotonicity*: An interference function is $\mathcal{I}_k(\boldsymbol{p})$ is said to be *strictly monotonic* on its *dependency* set L_k, if $\boldsymbol{p}^{(1)} \geq \boldsymbol{p}^{(2)}$ with $p_r^{(1)} > p_r^{(2)}$ for some $r \in L_k$, then $\mathcal{I}_k(\boldsymbol{p}^{(1)}) > \mathcal{I}_k(\boldsymbol{p}^{(2)})$.

Example 3. Consider the interference function $\mathcal{I}_k(\boldsymbol{p}) = \sum_{r\in L_k} v_{kr}p_r$, where $v_{kr} \in \mathbb{R}_+$ is the link gain between transmitter r and receiver k. All users $r \in L_k$, where L_k is the dependency set interfere with user k. *Strict monotonicity* of $\mathcal{I}_k$ implies, that there exists at least one user $r \in L_k$ such that $\mathcal{I}_k(\boldsymbol{p})$ is strictly increasing with respect to p_r.

Definition 7. *Strict positivity*: An interference function $\mathcal{I}_k(\boldsymbol{p})$ for $k \in \mathcal{K}$ is said to be *strictly positive* on its *dependency* set L_k, if for $\boldsymbol{p} \geq \boldsymbol{0}$, $\boldsymbol{p} \neq [0, \ldots, 0]^T$ and $p_l > 0$ for some $l \in L_k$, we have that $\mathcal{I}_k(\boldsymbol{p}) > 0$.

From a practical point of view *strict positivity* seems very natural in wireless systems. None the less, it is an important mathematical restriction whose impact shall be observed in Section IV.

Remark 3. *Strict positivity* of the interference function is a stronger condition than $A1$. *Strict positivity* of the interference function $\mathcal{I}_k(\boldsymbol{p})$ is defined with respect to its respective *dependency* set L_k.

Example 4. Consider the following examples: $\mathcal{I}(\boldsymbol{p}) = \sum_{k=1}^K v_k p_k$, $\boldsymbol{v} > \boldsymbol{0}$, $\boldsymbol{v} \neq [0, \ldots, 0]^T$ is a *strict positive* interference function; $\mathcal{I}(\boldsymbol{p}) = \prod_k p_k^{\omega_k}$, $\boldsymbol{\omega} \geq \boldsymbol{0}$, $\sum_{k=1}^K \omega_k = 1$ is not a *strict positive* interference function.

Definition 8. *Log-convex* interference function: An interference function $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is said to be a *log-convex* interference function if $A1$-$A3$ are fulfilled and $\mathcal{I}(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^K$.

Let $f(\boldsymbol{s}) := \mathcal{I}(\exp\{\boldsymbol{s}\})$. The function $f : \mathbb{R}^K \mapsto \mathbb{R}_+$ is log-convex on $\mathbb{R}^K$ if and only if $\log f$ is convex or equivalently $f(\boldsymbol{s}(\lambda)) \leq f(\boldsymbol{s}^{(1)})^{1-\lambda} f(\boldsymbol{s}^{(2)})^{1-\lambda}$, for all $\lambda \in (0,1)$, $\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)} \in \mathbb{R}^K$, where $\boldsymbol{s}(\lambda) = (1-\lambda)\boldsymbol{s}^{(1)} + \lambda\boldsymbol{s}^{(2)}$, $\lambda \in (0,1)$. Note that the change of variable $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$ was already used in [16], [17].

Definition 9. *Strict log-convexity*: A log-convex interference function $\mathcal{I}_k$ is said to be *strictly log-convex* on its *dependency* set if for all $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(1)} \in \mathbb{R}_{++}^K$, with $p_l^{(1)} \neq p_l^{(2)}$ for at least one $l \in L_k$, we have $\mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big) < \big(\mathcal{I}_k(\boldsymbol{p}^{(1)})\big)^{1-\lambda}\big(\mathcal{I}_k(\boldsymbol{p}^{(2)})\big)^{1-\lambda}$, where $\boldsymbol{p}(\lambda) = \boldsymbol{p}^{(1)^{1-\lambda}} \boldsymbol{p}^{(2)^{\lambda}}$.

From Remark 2 we have that linear interference functions always satisfy definitions 6, 7, 8. However, they need not satisfy definition 9. The property of *strict log-convexity* of interference functions on their respective *dependency* sets will be used to show that the infimum can be achieved. We introduce a table of notation, for quickly reviewing all the properties of interference functions introduced in the paper.

$A1$ (Positivity)	There exists a $\boldsymbol{p} > \boldsymbol{0}$ with $\mathcal{I}(\boldsymbol{p}) > 0$
$A2$ (Scale Invariance)	$\mathcal{I}(\alpha\boldsymbol{p}) = \alpha\mathcal{I}(\boldsymbol{p})$ for all $\alpha \geq 0$
$A3$ (Monotonicity)	$\mathcal{I}(\boldsymbol{p}) \geq \mathcal{I}(\hat{\boldsymbol{p}})$ if $\boldsymbol{p} \geq \hat{\boldsymbol{p}}$
Strict Monotonicity	$\boldsymbol{p}^{(1)} \geq \boldsymbol{p}^{(2)}$, $p_r^{(1)} > p_r^{(2)}$ for some $r \in L_k$ then $\mathcal{I}_k(\boldsymbol{p}^{(1)}) > \mathcal{I}_k(\boldsymbol{p}^{(2)})$
Strict Positivity	$\boldsymbol{p} \geq \boldsymbol{0}$, $\boldsymbol{p} \neq [0, \ldots, 0]^T$ and $p_l > 0$ for $l \in L_k$ then $\mathcal{I}_k(\boldsymbol{p}) > 0$
Log-convexity	$f(\boldsymbol{s}) := \mathcal{I}(\exp\{\boldsymbol{s}\})$ and $\log f$ is convex
Strict Log-convexity	for all $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(1)} \in \mathbb{R}_{++}^K$, with $p_l^{(1)} \neq p_l^{(2)}$ for at least one $l \in L_k$, we have $\mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big) < \big(\mathcal{I}_k(\boldsymbol{p}^{(1)})\big)^{1-\lambda}$ $\big(\mathcal{I}_k(\boldsymbol{p}^{(2)})\big)^{1-\lambda}$ where $\boldsymbol{p}(\lambda) = \big(\boldsymbol{p}^{(1)}\big)^{1-\lambda}\big(\boldsymbol{p}^{(2)}\big)^{\lambda}$

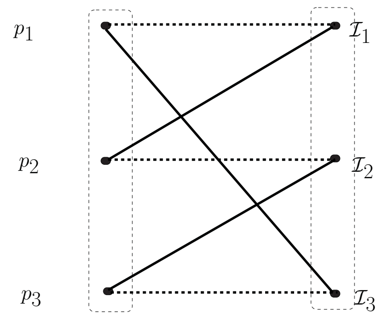

Figure 1. Bipartite graph; shaded edges correspond to connections between power node p_k and interference node $\mathcal{I}_k$ of user k; *Power graph* for 3 users displays full connectivity, corresponding to irreducible matrix $\boldsymbol{G}$.

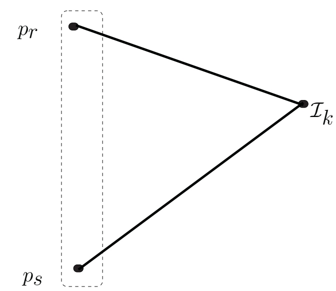

Figure 2. *Elementary power step* in a *Power graph*; p_r and p_s are in $L_k^{'}$.

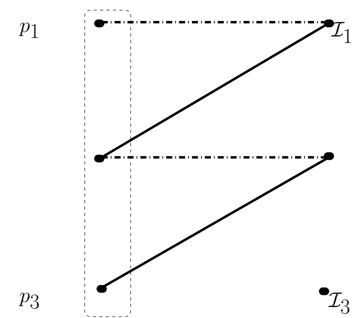

Figure 3. Example of a *power graph*, which is not fully connected; $L_k^{'} = \{k\} \cup L_k$.

III. Combinatorial Characterization of Interference Coupling

In this section we present a combinatorial characterization of interference coupling and its relation to the irreducibility of the matrices $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$ respectively. For this purpose we shall construct a new graphical representation. We know that the *global dependency* matrix $\boldsymbol{G}$ is irreducible, if and only if the "classical graph" is connected. Before introducing the power and interference bipartite graphs, we give certain intuition as to what certain concepts in these definitions could look like.

Let $\mathbf{P}$ and $\mathbf{I}$ represent the set power nodes and the set of interference nodes, respectively. In the definition below, a power node $p_k \in \mathbf{P}$ could be thought of as an abstraction of the transmit power of user k. Similarly, an interference node $\mathcal{I}_k \in \mathbf{I}$ could be thought of as an abstraction of the interference experienced by user k. Power nodes are not connected to each other. Interference nodes are not connected to each other. If node p_k has an undirected edge to certain other interference nodes, e.g. $\mathcal{I}_{k_1}, \mathcal{I}_{k_2}$, where $k, k_1, k_2 \in \mathcal{K}$, then it implies that user k has the ability to cause interference to users k_1 and k_2. Having provided this basic intuition we are now in a position to introduce the power and interference bipartite graphs.

Definition 10. For any given matrix $\boldsymbol{G} \in \mathbb{R}_+^{K\times K}$, let $\tilde{\mathcal{G}}(\boldsymbol{G})$ be the (undirected) bipartite graph of $2K$ nodes divided into two disjoint sets $\mathbf{P}$ and $\mathbf{I}$ (each of cardinality K) such that, there is no edge between nodes within each of these groups and there is an undirected edge between $p_l \in \mathbf{P}$ and $\mathcal{I}_k \in \mathbf{I}$ if and only if $[\boldsymbol{G}]_{kl} > 0$.

It is worth pointing out that if the power nodes are labeled $\mathbf{P} = \{p_1, \ldots, p_K\}$ and the interference nodes are labeled $\mathbf{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_K)$ then the adjacency matrix of $\tilde{\mathcal{G}}(\boldsymbol{G})$ is a $2K\times 2K$ partitioned matrix of the form $\begin{pmatrix} \mathbf{0} & \boldsymbol{G}^T \\ \boldsymbol{G} & \mathbf{0} \end{pmatrix}$ [18], page 15.

Definition 11. *Connectivity of* $\tilde{\mathcal{G}}(\boldsymbol{G})$: For any $\boldsymbol{G}$, two nodes k_1 and k_2 in $\tilde{\mathcal{G}}(\boldsymbol{G})$ are said to be connected, if and only if there exists a sequence of edges $(i, j) \in \mathbf{P}\times\mathbf{I}$ or $(i, j) \in \mathbf{P}\times\mathbf{I}$ such that $\{(l_0, l_1), (l_1, l_2), \ldots, (l_{N-2}, l_{N-1}), (l_{N-1}, l_N)\}$ with $l_0 = k_0$ and $l_N = k_2$.

Stated informally, the definition says that $\tilde{\mathcal{G}}$ is connected, if and only if every two distinct nodes are linked by a sequence of undirected edges, each of which now connects a power nodes with an interference node. Consider the following matrix

$$\boldsymbol{G} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The bipartite graph for matrix $\boldsymbol{G}$ is displayed in Fig. 1. The dotted edges in Fig. 1 corresponds to an artificial connection in this graph. It does not immediately imply that we have self interference. The presence of self interference shall be discussed in detail in Example 3). We now introduce the *power graph* and the *interference graph*. These graphs will help abstract away certain physical layer complications, while capturing interference coupling effects in wireless systems.

A. Power Graph

We begin by explaining, what we imply by *elementary power step*. Here we introduce the *dependency* set $L_k^{'}$, where $L_k^{'} = \{k\} \cup L_k$.

Definition 12. *Elementary power step*: An *elementary power step* is a connection between any two power nodes in the bipartite graph through an interference node.

We can construct an *elementary power step* between two power nodes p_t and p_s if and only if $t, s \in L_k^{'}$, where $L_k^{'} = \{k\} \cup L_k$ (see Fig. 2).

Definition 13. *Power graph*: We call a bipartite graph of $2K$ nodes divided into two disjoint sets $\mathbf{P}$ and $\mathbf{I}$ (each of cardinality K) a *power graph* if it consists only of *elementary*

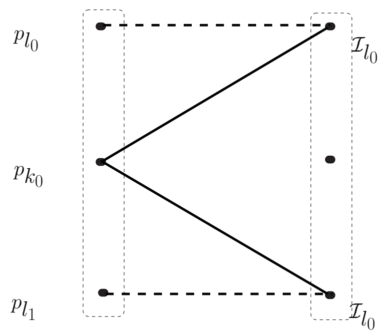

Figure 4. Power of user k_0, p_{k_0} can be decreased and in turn power of all users in L_{k_0} can be decreased.

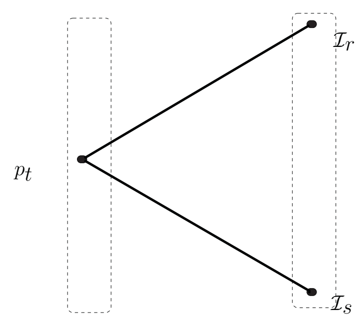

Figure 5. *Interference graph*; $t \in L_r$ and $t \in L_s$; *Elementary interference step* between interference nodes $\mathcal{I}_r$ and $\mathcal{I}_s$.

power steps. A *power path* is a path in the *power graph* made up one or more *elementary power steps*.

If the *power graph* is fully connected then each power node can be reached by every other power node. An example of a fully connected *power graph* is displayed in Fig. 1. An example of a *power graph*, which is not fully connected is displayed in Fig. 3. We now mention a remark, which will be utilized while proving one of the main results of the paper.

Remark 4. Let for any user k_0 the following inequality $p_{k_0} > \gamma_{k_0}\mathcal{I}_{k_0}(\boldsymbol{p})$ hold. Then, we can reduce the power of user k_0 and in turn reduce the power p_l of all users with $k_0 \in L_l$. Let $L^{(1)}_{k_0} = \{k \mid k_0 \in L_k\}$. In the next step we can reduce all the powers p_l when there exists a user $k \in L^{(1)}_{k_0}$ such that $k \in L_l$. Similarly we can construct the *dependency* set $L^{(2)}_{k_0} = \{k \mid \exists k_1 \in L^{(1)}_{k_0}, k_1 \in L_k\}$. Using induction we can construct the *dependency* set $L^{(v+1)}_{k_0} = \{k \mid \exists k_1 \in L^{(v)}_{k_0}, k_1 \in L_k\}$. The *power graph* is connected, if and only if for each $k_0 \in \mathcal{K}$ we can find a v_0 such that $L^{(v_0)}_{k_0} = k$.

This is a formal outline of how a power control procedure is regularly carried out in wireless systems. As can be observed, the combinatorial characterization helps to get a better insight into the fashion in which users are coupled by interference, preparing us for interference management and coordination. The irreducibility of $\boldsymbol{G}$ implies the *Pareto optimality* of the boundary points.

B. Interference Graph

In this section we discuss the *interference graph* and the corresponding irreducibility of the matrix $\boldsymbol{G}\boldsymbol{G}^T$. We begin by defining an *elementary interference step*.

Definition 14. *Elementary interference step*: An *elementary interference step* is a connection between any two interference nodes in the bipartite graph through a power node.

An *elementary interference step* between interference nodes $\mathcal{I}_r$ and $\mathcal{I}_s$ can be constructed if and only if there exists a t, $t \in \mathcal{K}$ such that $t \in L_r \cap L_s$ (see Fig. 5).

Definition 15. *Interference graph*: We call a bipartite graph of $2K$ nodes divided into two disjoint sets P_r and I_c (each of cardinality K) a *interference graph* if it consists of only *elementary interference steps*. An *interference path* is a path in the *interference graph* made up of one or more *elementary interference steps*.

Based on the above definitions we shall prove certain results, which display the relation between the connectivity of the *power graph* and the irreducibility of the matrix $\boldsymbol{G}$; and the relation between the connectivity of the *interference graph* and the irreducibility of the matrix $\boldsymbol{G}\boldsymbol{G}^T$ in Section IV below. In particular the importance of the *interference graph* and its interplay with interference functions and the matrix $\boldsymbol{G}\boldsymbol{G}^T$ will be visible in the second part of the proof of Theorem 1.

IV. Combinatorial Characterization based on Interference Functions

We begin by presenting a lemma, which displays the positivity of function $C(\boldsymbol{\gamma}, \mathcal{I}) > 0$, which is used frequently in our analysis. Here $\boldsymbol{\gamma} := \boldsymbol{\gamma}(\boldsymbol{u}) = [\gamma_1(u_1), \ldots, \gamma_K(u_K)]^T$ is the SIR vector associated with a utility vector $\boldsymbol{u} \in \mathbf{U}$.

Lemma 1. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be strictly positive interference functions on their dependency sets $L_1, \ldots, L_K$, respectively. Then, for all $\gamma_1, \ldots, \gamma_K > 0$ we have that*

$$C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}) = \inf_{\boldsymbol{p}>\mathbf{0}} \max_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\boldsymbol{p})}{p_k} > 0.$$

Proof: We achieve this proof by contradiction. Let $\gamma_k > 0$, for all $k \in \mathcal{K}$ be such that $C(\boldsymbol{\gamma}) = 0$. Then for all $\epsilon > 0$ there exists a vector $\boldsymbol{p}(\epsilon)$, $\boldsymbol{p}(\epsilon) > \mathbf{0}$, $\|\boldsymbol{p}\|_\infty = 1$ (chosen for compactness; it does not alter the problem, since, we are in the case without power constraints) and $\big(\gamma_k \mathcal{I}_k(\boldsymbol{p}(\epsilon))\big)/p_k(\epsilon) < \epsilon$, for $k \in \mathcal{K}$, i.e. $\gamma_k \mathcal{I}_k\big(\boldsymbol{p}(\epsilon)\big) < \epsilon p_k(\epsilon)$. Let l_0 be an index with p_{l_0} with $p_{l_0}(\epsilon) = 1$. Let k_0 be an index such that $\mathcal{I}_{k_0}$ is an interference function with $l_0 \in L_{k_0}$. Then we have that $0 < c_3 := \gamma_{k_0}\mathcal{I}_{k_0}\big(\boldsymbol{e}(l_0)\big) \le \gamma_{k_0}\mathcal{I}_{k_0}(\boldsymbol{p}(\epsilon)) < \epsilon p_{k_0}(\epsilon) = \epsilon$. We have chosen $\epsilon > 0$. Therefore, we have that $0 < c_3 \le \epsilon$ and $c_3 = 0$, which gives us our required contradiction. ■

Next, we present a lemma, which insures that *strictly log-convex* interference functions on their respective *dependency* sets are also *strictly monotonic* on their respective dependency sets. This result will be required while investigating the relation between the interference function framework and the family of feasible utility sets.

Lemma 2. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be strictly log-convex interference functions on their dependency sets $L_1, \ldots, L_K$, respectively.*

Then, $\mathcal{I}_1, \ldots, \mathcal{I}_K$ are strictly monotonic on their dependency sets $L_1, \ldots, L_K$, respectively.

Proof: Refer to [12, Proof of Lemma 5]. ■

Our main resulted presented below, depicts the connection between the family of feasible utilities sets and the interference function framework. We begin with interference functions with the natural properties of *strict log-convexity* and *strict positivity* on their respective *dependency* sets. Then the corresponding *global dependency matrix* $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$ are irreducible (i.e. the users in the systems are fully coupled by interference), if and only if the SIR region $\mathbf{S}$ is in the family of strict convex sets $\mathcal{ST}$. In this case there exists an unique optimum for the problem $\prod_{k\in\mathcal{K}} u_k$.

Theorem 1. *Let $\mathbf{S} = \{\boldsymbol{\gamma} \in \mathbb{R}^K_{++} \mid C(\boldsymbol{\gamma}) \le 1\}$, where $C(\boldsymbol{\gamma}) = \inf_{\boldsymbol{p}>\mathbf{0}} \max_{k\in\mathcal{K}} \big(\gamma_k \mathcal{I}_k(\boldsymbol{p})/p_k\big)$. Let $\mathcal{I}_k$ for $k \in \mathcal{K}$ be strictly log-convex and strictly positive interference functions on their respective dependency sets. Then, $\mathbf{S}$ is in the family of sets $\mathcal{ST}$, if and only if $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$ are irreducible. For this case there exists a unique optimizer $\hat{\boldsymbol{p}} > \mathbf{0}$ for $PF(\boldsymbol{\mathcal{I}}) = \inf_{\boldsymbol{p}\in\mathcal{P}} \sum_{k=1}^K \log \frac{\mathcal{I}_k(\boldsymbol{p})}{p_k}$, with an associated SIR vector $\hat{\boldsymbol{\gamma}}$, where $\mathcal{P}$ is the set of power vectors.*

Proof: "$\Longleftarrow$": This direction has been proved in [8, Theorem 8].

"$\Longrightarrow$": We have to show that the irreducibility of $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$ together are necessary and sufficient conditions. We carry out the proof in two steps. Let us assume that $\boldsymbol{G}$ is not irreducible. Then, we show that there are boundary points of $\mathbf{S}$ defined by $\mathbf{S} = \{\boldsymbol{\gamma} \in \mathbb{R}^K_{++} \mid C(\boldsymbol{\gamma}) \le 1\}$, which are weak *Pareto optimal*, but not *Pareto optimal*, i.e. there exists $\boldsymbol{\gamma}^{(1)} \ge \boldsymbol{\gamma}^{(2)}$, $\boldsymbol{\gamma}^{(1)} \neq \boldsymbol{\gamma}^{(2)}$ with $1 = C(\boldsymbol{\gamma}^{(1)}) = C(\boldsymbol{\gamma}^{(2)})$. Since $\boldsymbol{G}$ is reducible, through a permutation of the rows and columns of $\boldsymbol{G}$ we have $\boldsymbol{G} = \begin{pmatrix} \boldsymbol{G}_{(1)} & \mathbf{0} \\ \boldsymbol{G}_{(1,2)} & \boldsymbol{G}_{(2)} \end{pmatrix}$, where the matrix $\boldsymbol{G}_{(1)}$ is irreducible. Matrix $\boldsymbol{G}_{(1)}$ is a $k_1 \times k_1$-dimensional matrix. We assume that interference functions $\mathcal{I}_k$ for all $k \in \mathcal{K}$ are in the required form. So interference function $\mathcal{I}_k$ for all $1 \le k \le k_1$ is dependent only on p_l for $1 \le l \le k_1$. We now analyze the following equations: $C(\boldsymbol{\gamma}) = \inf_{\boldsymbol{p}>\mathbf{0}} \max_{k\in\mathcal{K}} \gamma_k \mathcal{I}_k(\boldsymbol{p})/p_k$ and $C(\boldsymbol{\gamma}^{(1)}) = \inf_{\boldsymbol{p}^{(1)}\in\mathbb{R}^{k_1}_{++}} \max_{1\le k\le k_1} \gamma_k \mathcal{I}_k(\boldsymbol{p})/p_k$, with $\boldsymbol{p}^{(1)} = [p_1, \ldots, p_{k_1}]^T$. We also have that $C(\boldsymbol{\gamma}^{(2)}) = \inf_{\boldsymbol{p}>\mathbf{0}, \boldsymbol{p}\in\mathbb{R}^K_+} \max_{k_1<k\le K+1} \gamma_k \mathcal{I}_k(\boldsymbol{p})/p_k$. Then, from Lemma 1 we know that $C(\boldsymbol{\gamma}^{(1)}) > 0$, $C(\boldsymbol{\gamma}^{(2)}) > 0$ and $C(\boldsymbol{\gamma}) = \max\big(C(\boldsymbol{\gamma}^{(1)}), C(\boldsymbol{\gamma}^{(2)})\big) > 0$. We choose $\boldsymbol{\gamma}$ such that $C(\boldsymbol{\gamma}) = C(\boldsymbol{\gamma}^{(1)})$. Therefore, we have that $0 < C(\boldsymbol{\gamma}^{(2)}) \le 1$. For $0 < \lambda < 1$ we choose $\boldsymbol{\gamma}(\lambda)$ with $\gamma_k(\lambda) = \gamma_k$, for $1 \le k \le k_1$ and $\gamma_k(\lambda) = \lambda\gamma_k$, for $k_1 \le k \le K+1$. Then we have that $C(\boldsymbol{\gamma}(\lambda)) \ge C(\boldsymbol{\gamma}^{(1)}(\lambda)) = C(\boldsymbol{\gamma}^{(1)}) = 1$ and $C(\boldsymbol{\gamma}(\lambda)) = \inf_{\boldsymbol{p}>\mathbf{0}} \max_{k\in\mathcal{K}} \big(\gamma_k(\lambda)\mathcal{I}_k(\boldsymbol{p})\big)/p_k \le \inf_{\boldsymbol{p}>\mathbf{0}} \max_{k\in\mathcal{K}} \big(\gamma_k \mathcal{I}_k(\boldsymbol{p})\big)/p_k = 1$ and $C(\boldsymbol{\gamma}(\lambda)) = 1$. Therefore, $\boldsymbol{\gamma}(\lambda)$ is a boundary point. We know that $\boldsymbol{\gamma}(\lambda) \le \boldsymbol{\gamma}$ and $\boldsymbol{\gamma}(\lambda) \neq \boldsymbol{\gamma}$ for $\lambda \neq 1$. Hence, $\boldsymbol{\gamma}(\lambda)$ is only weak *Pareto optimal*. We also have for $\boldsymbol{\gamma}(\lambda) = (1-\lambda)\boldsymbol{\gamma}^{(1)} + \lambda\boldsymbol{\gamma}^{(2)}$ that $C(\boldsymbol{\gamma}) = 1$, hence $\mathbf{S} \notin \mathcal{ST}$. Therefore, if $\mathbf{S} \in \mathcal{ST}$, then $\boldsymbol{G}$ is irreducible.

Let us assume that matrix $\boldsymbol{G}\boldsymbol{G}^T$ is not irreducible, i.e. $\boldsymbol{G}\boldsymbol{G}^T$ is reducible. Now we have to show that $\mathbf{S} \notin \mathcal{ST}$. Let $\mathcal{I}_k$, $k \in \mathcal{K}$ be an interference function and $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)}$ be any two power vectors with $p_k^{(1)} \neq p_k^{(2)}$. Then we have that $\mathcal{I}_k(\boldsymbol{p}(\lambda)) < \big(\mathcal{I}_k(\boldsymbol{p}^{(1)})\big)^{1-\lambda}\big(\mathcal{I}_k(\boldsymbol{p}^{(2)})\big)^{\lambda}$. Therefore when $\mathcal{I}_k(\boldsymbol{p}(\lambda)) = \big(\mathcal{I}_k(\boldsymbol{p}^{(1)})\big)^{1-\lambda}\big(\mathcal{I}_k(\boldsymbol{p}^{(2)})\big)^{\lambda}$, then we must have that $p_k^{(1)} = \mu_k p_k^{(2)}$. Let k_1 and k_2 be any two indices with $L_{k_1} \cap L_{k_2} \neq \emptyset$. Then $\mu_{k_1} = \mu_{k_2}$ and μ_k is a constant for all $k \in \mathcal{K}$. When $\boldsymbol{G}\boldsymbol{G}^T$ is irreducible, then for all $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)}$, $\boldsymbol{p}^{(1)} \neq \boldsymbol{p}^{(2)}$ we can find an index k_0 such that $\mathcal{I}_{k_0}(\boldsymbol{p}(\lambda_0)) < \big(\mathcal{I}_k(\boldsymbol{p}^{(1)})\big)^{1-\lambda_0}\big(\mathcal{I}_k(\boldsymbol{p}^{(2)})\big)^{\lambda_0}$ for some λ_0 (hence for all $\lambda \in (0,1)$). This implies that $p_{k_0}(\lambda_0)/\mathcal{I}_k(\lambda_0) > \big(p_{k_0^{(1)}}/\mathcal{I}_{k_0}(\boldsymbol{p}^{(1)})\big)^{1-\lambda_0}\big(p_{k_0^{(2)}}/\mathcal{I}_{k_0}(\boldsymbol{p}^{(2)})\big)^{\lambda_0}$. This implies that $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)}$ such that $\boldsymbol{p}^{(1)} \neq \boldsymbol{p}^{(2)}$ such that $\gamma_k(\boldsymbol{p}(\lambda)) = \big(\gamma_k(\boldsymbol{p}^{(1)})\big)^{1-\lambda}\big(\gamma_k(\boldsymbol{p}^{(2)})\big)^{\lambda}$.

Assume that $\boldsymbol{G}$ is irreducible. We can find power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)}$, $\boldsymbol{p}^{(1)} \neq \boldsymbol{p}^{(2)}$ such that

$$\gamma_k^{(l)} = \gamma_k(\boldsymbol{p}^{(l)}) = \frac{p_k^l}{\mathcal{I}_k(\boldsymbol{p}^{(l)})}, \tag{5}$$

for $l = \{1,2\}$. Let $\boldsymbol{\gamma}^{(1)}$ and $\boldsymbol{\gamma}^{(2)}$ be boundary points. Then $\gamma_k(\boldsymbol{p}(\lambda)) = \big(\gamma_k(\boldsymbol{p}^{(1)})\big)^{1-\lambda}\big(\gamma_k(\boldsymbol{p}^{(2)})\big)^{\lambda}$, for $k \in \mathcal{K}$. Then $\boldsymbol{\gamma}(\lambda)$ is a boundary point. Hence $\mathbf{S} \notin \mathcal{ST}$. So if $\mathbf{S} \in \mathcal{ST}$ then $\boldsymbol{G}\boldsymbol{G}^T$ is irreducible. ■

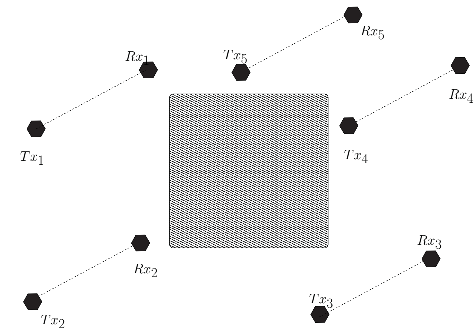

Figure 6. Example of an ad-hoc wireless network with 5 point to point links, where $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$ are irreducible matrices.

We now present two corollaries, which display the relationship between the irreducibility of the *global dependency* matrix $\boldsymbol{G}$ and the matrix $\boldsymbol{G}\boldsymbol{G}^T$ with the connectivity of the *power graph* and the *interference graph* respectively.

Example 5. Consider an ad-hoc wireless network with 5 point to point links surrounding a building (any object blocking the ability of certain links to interfere with certain other links) as shown in Fig. 6, where Tx_k and Rx_k represent the transmitter and receivers for the k^{th} link, respectively. The dependency sets of users are as follows: $L_1 = \{2,5\}$, $L_2 = \{1,3\}$, $L_3 = \{2,4\}$, $L_4 = \{3,5\}$ and $L_5 = \{4,1\}$. We can construct the matrices $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$ and check, that they are irreducible matrices. If the above case fulfills the conditions of Theorem 1, then we observe, that even though all the links do not interfere with all the other links, the following statements hold:

- all the boundary points of the resulting SIR set are strictly convex, and
- the proportional fairness resource allocation strategy $PF(\boldsymbol{\mathcal{I}})$ (defined in the statement of Theorem 1) has a unique solution.

Corollary 1. *The global dependency matrix $\boldsymbol{G}$ is irreducible, if and only if we can connect all the power nodes through power paths in the power graph.*

Proof: The *global dependency* matrix $\boldsymbol{G}$ is irreducible if and only if we can find for each pair (l, k) a sequence of natural numbers $k_0, k_1, \ldots, k_m$ such that $k_0 = l$, $k_m = k$ and $[\boldsymbol{G}]_{k_{r+1}k_r} > 0$ for all $r = 0, \ldots, m-1$. Then we can connect all power nodes p_l and p_k in the *power graph*. We illustrate such a connection in the chain below (refer to Remark 4 and first part of proof of Theorem 1 for detailed explanation of the chain).

$$p_l \xrightarrow{[\boldsymbol{G}]_{k_1 l}} \mathcal{I}_{k_1} \longrightarrow p_{k_1} \xrightarrow{[\boldsymbol{G}]_{k_2 l}} \mathcal{I}_{k_2} \ldots \xrightarrow{[\boldsymbol{G}]_{kl}} \mathcal{I}_k \longrightarrow p_k \quad (6)$$

To prove the opposite direction let us assume that we can connect all the power nodes through the *power graph*. Then we have that for a pair (l, k) of power nodes we can find a path according to (6). Then we have that $[\boldsymbol{G}]_{k_0 l} > 0$, $[\boldsymbol{G}]_{k_1 k_0} > 0, \ldots, [\boldsymbol{G}]_{k_{r+1}k_r} > 0$ for all $r = 0, \ldots, m-1$ and $[\boldsymbol{G}]_{kk_{m-1}} > 0$. Hence, we have that the *global dependency* matrix is irreducible. ■

Corollary 2. *Matrix $\boldsymbol{G}\boldsymbol{G}^T$ is irreducible, if and only if we can connect all interference nodes through interference paths in the interference graph.*

The next corollary is a direct application of Corollary 1 and Corollary 2 to the case of full self interference. In this special case we display the equivalence between the irreducibility of the *global dependency* matrix $\boldsymbol{G}$ and the matrix $\boldsymbol{G}\boldsymbol{G}^T$.

Corollary 3. *Consider a system with full self interference, i.e. $k \in L_k$, for all $k \in \mathcal{K}$. For such a system the global dependency matrix $\boldsymbol{G}$ is irreducible, if and only if the matrix $\boldsymbol{G}\boldsymbol{G}^T$ is irreducible.*

In fact in the case of full self interference, the *power graph* and the *interference graph* are equivalent. The dotted edges in the *power graph* are now solid edges, corresponding to self interference. Under the restriction of full self interference Corollary 3 has shown the equivalence between the irreducibility of the matrices $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$. This equivalence does not hold in the general case. These insights could better prepare us to design algorithms for interference coordination and management and serve as a building block to study the general case.

We would now like to analyze the case, when the *global dependency* matrix $\boldsymbol{G}$ is reducible. For a certain SIR vector $\boldsymbol{\gamma} > \boldsymbol{0}$ with the corresponding *interference balancing* function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}) = 0$ we consider the following set of power vectors

$$\mathbf{P}(\boldsymbol{\gamma}) := \{\boldsymbol{p} \in \mathbb{R}^K_{++} \mid \frac{p_k}{\mathcal{I}_k(\boldsymbol{p})} \geq \gamma_k, \forall k \in \mathcal{K}\}. \quad (7)$$

There is a possibility that the set of powers $\mathbf{P}(\boldsymbol{\gamma}) = \emptyset$. We analyze this case, when the power set $\mathbf{P}(\boldsymbol{\gamma}) \neq \emptyset$. The next result shows that for *strictly positive* and *strictly log-convex* interference functions on their respective *dependency* sets, the irreducibility of the matrices $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$ are necessary and sufficient conditions for the considered utility sets to be strictly convex. In this case we have shown that there exists a unique optimum for the product of utilities maximization problem.

Theorem 2. *Let there exist a SIR vector $\boldsymbol{\gamma}$ such that the power set (defined in (7)) $\mathbf{P}(\boldsymbol{\gamma}) \neq \emptyset$. Then, there exists a set $\mathcal{K}_1 \subseteq \mathcal{K} = \{1, \ldots, K\}$ and a power vector $\tilde{\boldsymbol{p}} \in \mathbf{P}(\boldsymbol{\gamma})$ such that $\frac{p_k}{\mathcal{I}_k(\boldsymbol{p})} = \gamma_k$, for all users $k \in \mathcal{K}_1$ and for all power vectors $\boldsymbol{p} \in \mathbf{P}(\boldsymbol{\gamma})$, $\frac{\tilde{p}_k}{\mathcal{I}_k(\tilde{\boldsymbol{p}})} > \gamma_k$, for all users $k \in \mathcal{K}_2 = \mathcal{K} \backslash \mathcal{K}_1$. Then, we have that $\cup_{k \in \mathcal{K}_1} L_k = \mathcal{K}_1$.*

Proof: The proof of the prerequisite conditions is discussed in [19], Appendix J, pages 5488-5489. To prove that $\cup_{k \in \mathcal{K}_1} L_k = \mathcal{K}_1$ we begin by showing that $\cup_{k \in \mathcal{K}_1} L_k \subseteq \mathcal{K}_1$. For the sake of obtaining a contradiction assume that $\cup_{k \in \mathcal{K}_1} L_k \subseteq \mathcal{K}_1$ is not true. Then there exists an index k_0 such that $k_0 \in \cup_{k \in \mathcal{K}_1}$ and $k_0 \notin \mathcal{K}_1$. Let us choose a set $\mathcal{K}_2$ such that $k_0 \in \mathcal{K}_2$. We can construct a new power vector such that

$$\hat{p}_{k_0} < \tilde{p}_{k_0}, \quad \frac{\hat{p}_{k_0}}{\mathcal{I}_k(\hat{\boldsymbol{p}})} > \gamma_{k_0}. \quad (8)$$

For an index $k \neq k_0$ we define a power vector $\hat{\boldsymbol{p}}$, such that

$$\hat{p}_k = \tilde{p}_k, \quad \hat{\boldsymbol{p}} \leq \tilde{\boldsymbol{p}}. \quad (9)$$

Then from (8), (9) and $A2$ we have that $\frac{\tilde{p}_{k_0}}{\mathcal{I}_{k_0}(\tilde{\boldsymbol{p}})} > \frac{\hat{p}_{k_0}}{\mathcal{I}_{k_0}(\hat{\boldsymbol{p}})} > \gamma_{k_0}$ and $\frac{\hat{p}_k}{\mathcal{I}_k(\hat{\boldsymbol{p}})} > \gamma_k$. Now we choose an index $k_1 \in \mathcal{K}_1$ and $k_0 \notin \mathcal{K}_1$. Since we have that $\mathcal{I}_{k_0}(\tilde{\boldsymbol{p}}) > \mathcal{I}_{k_1}\hat{\boldsymbol{p}}$, we can reduce the power $\tilde{p}_{k_1}$ in the following manner. We choose a power vector $\hat{\boldsymbol{p}}^{(1)}$ such that $\hat{p}_{k_1} > \gamma_{k_1}\mathcal{I}_{k_1}(\tilde{\boldsymbol{p}})$. We choose $\hat{p}^{(1)}_k = \tilde{p}_k$ for $k \neq k_1$ such that $\hat{\boldsymbol{p}}^{(1)} \leq \tilde{\boldsymbol{p}}$ and power vector $\hat{\boldsymbol{p}}^{(1)} \in \mathbf{P}(\boldsymbol{\gamma})$. Then we have that $\hat{p}_{k_1} > \gamma_{k_1}\mathcal{I}_{k_1}(\hat{\boldsymbol{p}}^{(1)})$, for all $k_1 \in \mathcal{K}_1$, which is in contradiction with $\cup_{k \in \mathcal{K}_1} L_k \subseteq \mathcal{K}_1$.

To prove the opposite direction we have to show that $\mathcal{K}_1 \subseteq \cup_{k \in \mathcal{K}_1} L_k$. Assume for the sake of obtaining a contradiction that $\mathcal{K}_1 \subseteq \cup_{k \in \mathcal{K}_1} L_k$ is not true. Then there exists an index $k_0 \in \mathcal{K}_1$ with $k_0 \notin \cup_{k \in \mathcal{K}_1} L_k$. We have that $\tilde{p}_{k_0} = \gamma_{k_0}\mathcal{I}_{k_0}(\tilde{\boldsymbol{p}})$. Since we know that $k_0 \notin \cup_{k \in \mathcal{K}_1} L_k$ we have that the power of user k_0, $\tilde{p}_{k_0}$ has no influence on the interference function $\mathcal{I}_k$ of user k with $k \in \mathcal{K}_1$. However we are aware that p_{k_0} has an impact on certain interference functions $\mathcal{I}_k$ with $k \in \mathcal{K}_2$. For $k \in \mathcal{K}_2$ we have that $\gamma_k < \frac{\tilde{p}_k}{\mathcal{I}_k(\tilde{\boldsymbol{p}})}$. We can now increase the power $\tilde{p}_{k_0}$ of user k_0 such that $\tilde{\boldsymbol{p}} \in \mathbf{P}(\boldsymbol{\gamma})$, $\hat{p}_{k_0} > \tilde{p}_{k_0}$, $\hat{p}^{(2)}_k = \tilde{p}_k$, for $k \in \mathcal{K} \backslash \{k_0\}$ and $\gamma_k < \frac{\hat{p}^{(2)}_k}{\mathcal{I}_k(\hat{\boldsymbol{p}}^{(2)})}$, for all $k \in \mathcal{K}_2$. Now we have that $\frac{\tilde{p}^{(2)}_k}{\mathcal{I}_k(\hat{\boldsymbol{p}}^{(2)})} > \gamma_k, k \in \mathcal{K}_2 \cup \{k_0\}$, which is in contradiction to $\frac{p_k}{\mathcal{I}_k(\boldsymbol{p})} = \gamma_k$, for all users $k \in \mathcal{K}_1$ and for all power vectors $\boldsymbol{p} \in \mathbf{P}(\boldsymbol{\gamma})$ and $k_0 \in \mathcal{K}_1$. This implies that $\cup_{k \in \mathcal{K}_1} L_k \subseteq \mathcal{K}_1$. From $\cup_{k \in \mathcal{K}_1} L_k \subseteq \mathcal{K}_1$ and $\mathcal{K}_1 \subseteq \cup_{k \in \mathcal{K}_1} L_k$ we have the desired result. ■

We now return to the point (5) in the proof of Theorem 1. Recollect that, we are in the case when the *global dependency* matrix $\boldsymbol{G}$ is reducible. In the result below, we show that for all power vectors in the set $\mathbf{P}(\boldsymbol{\gamma})$, where $\mathbf{P}(\boldsymbol{\gamma})$ is not empty, the power vectors can be constructed as scaled versions of each other for all users in $\mathcal{K}_1$, where $\frac{p_k}{\mathcal{I}_k(\boldsymbol{p})} = \gamma_k$, for all users $k \in \mathcal{K}_1$ (for more details on $\mathcal{K}_1$ refer to the statement of Theorem 2).

Corollary 4. *For all power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} \in \mathbf{P}(\boldsymbol{\gamma})$, there exists a scalar $\mu > 0$ with $p^{(2)}_k = \mu p^{(1)}_k$, for all $k \in \mathcal{K}_1$.*

Proof: Choose two arbitrary power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(1)} \in \mathbf{P}(\boldsymbol{\gamma})$. For $0 < \lambda < 1$, we choose $\mathbf{P}(\lambda)$ such that $p_k(\lambda) =$

$(p_k^{(1)})^{1-\lambda}(p_k^{(1)})^{\lambda}$, for all $k \in \mathcal{K}$. Then we have that $\boldsymbol{p}(\lambda) \in \mathbf{P}(\lambda)$ for all $0 \leq \lambda \leq 1$. From Theorem 2 we have for all $k \in \mathcal{K}_1$ that $\mathcal{I}_k(\boldsymbol{p}(\lambda)) = (\mathcal{I}_k(\boldsymbol{p}^{(1)}))^{1-\lambda}(\mathcal{I}_k(\boldsymbol{p}^{(2)}))^{\lambda}$. Then there exists a scalar $\mu > 0$ such that for all $l \in \mathcal{K}_1 = \cup_{k\in\mathcal{K}_1} L_k$ we have that $p_l^{(2)} = \mu p_l^{(1)}$. This shows that for an arbitrary choice of power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} \in \mathbf{P}(\boldsymbol{\lambda})$ there exists a scalar $\mu > 0$ such that $p_l^{(2)} = \mu p_l^{(1)}$ for all $l \in \mathcal{K}_1$. ∎

We have provided a combinatorial characterization of interference coupling in terms of the interference functions. We know that the *interference balancing* function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$ is itself an interference function. If interference functions $\mathcal{I}_k$ for all $k \in \mathcal{K}$ are log-convex then the corresponding *interference balancing* function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$ is log-convex. We now briefly discuss the combinatorial characterization of interference coupling in terms of *interference balancing* functions and its relation to the property of *strict log-convexity*.

Corollary 5. *The interference balancing function* $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$ *is strict log-convex if and only if* $\boldsymbol{G}$ *and* $\boldsymbol{G}\boldsymbol{G}^T$ *are irreducible matrices, where* $\boldsymbol{G}$ *is the global dependency matrix.*

The result shows that the *interference balancing* function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$ is *strictly monotonic* with respect to the SIR vector $\boldsymbol{\gamma}$, if and only if the *global dependency* matrix $\boldsymbol{G}$ is irreducible. We know that strict *log-convexity* of an interference function implies *strict monotonicity* of the interference function. However, we cannot conclude the converse. If we have *strict monotone interference balancing* function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$ and *strict convexity* of $\log(\mathcal{I}_k(\exp(\boldsymbol{s}))/\exp(s_k))$, then both conditions together are equivalent to the *strict convexity* of the log-SIR region. This follows from the fact that the first condition is equivalent to the irreducibility of the *global dependency* matrix $\boldsymbol{G}$ and that the second condition is equivalent to the irreducibility of the matrix $\boldsymbol{G}\boldsymbol{G}^T$.

Corollary 6. *Interference balancing function* $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$ *is strictly log-convex if and only if the corresponding log-SIR set is strictly convex.*

Example 6. For a 2 user system, let $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}})$ be a *strictly monotonic interference balancing* function, which does not possess the property of being *strictly log-convex*. Let $\mathcal{I}_1(\boldsymbol{p}) = p_2$ and $\mathcal{I}_2(\boldsymbol{p}) = p_1$. Since we have linear interference functions the corresponding link gain matrix $\boldsymbol{V} = \boldsymbol{G}$ is given by $\boldsymbol{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Interference balancing* function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}) = \gamma_1\gamma_2$ and is *strictly monotonic*. Let SIR of user 1 be $\gamma_1 = \exp(q_1)$. Then we have that $\log C(\exp(\boldsymbol{q}), \boldsymbol{\mathcal{I}}) = q_1 + q_2$.

V. Conclusions

We have provided a combinatorial characterization of interference coupling in wireless systems. We have introduced the *power graph* and *interference graph* and displayed the relation between their connectivity and the irreducibility of the *global dependency* matrix $\boldsymbol{G}$ and the matrix $\boldsymbol{G}\boldsymbol{G}^T$ respectively. We have proved that for *strict positive* and *strict log-convex* interference functions on their respective *dependency* sets the irreducibility of the matrices $\boldsymbol{G}$ and $\boldsymbol{G}\boldsymbol{G}^T$ are necessary and sufficient conditions for the considered utility sets to be strict convex. We have provided a characterization of the log-SIR region, extending results of *strict convexity* of the feasible log-SIR region from linear interference functions to a broader class of log-convex interference functions and have elucidated the importance of the property of *log-convexity*.

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