On the Association of Power-flow with Circuit Terminals

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Abstract — **In classical circuit theory, the question of how much power flows into a circuit through a set of terminals only makes sense if these terminals form a port, i.e., if the sum of the terminal currents is zero all of the time. On the other hand, looking from a field theory perspective, it can make sense to associate a power flow even to a set of terminals which do not form a port, especially to single terminals. This can be done by integrating the Poynting vector over those parts of a closed surface around the circuit, which points are geometrically more close to one terminal than any other terminal. We argue that, under certain conditions, the product of terminal current with the *difference* of the terminal potential and the *average* potential of all port terminals can serve as power assigned to this terminal in the field theoretic sense stated above.**

1 INTRODUCTION

Terminals are the points through which a circuit can be interconnected with its environment. For their circuit theoretic description one needs two quantities: the electric current flowing through and the electric potential observed at the terminal. A number of terminals is said to make a port, in case that the sum of its terminal currents is zero all the time [1]. Thus,

$$\sum_{n=1}^{N} i_n \varphi_n \tag{1}$$

is *independent* of the reference point of the potentials φ_n, as long as the N terminals form a port, for

$$\sum_{n=1}^{N} i_n \left(\varphi_n + \varphi\right) = \sum_{n=1}^{N} i_n \varphi_n + \varphi \sum_{n=1}^{N} i_n = \sum_{n=1}^{N} i_n \varphi_n, \tag{2}$$

with any offset φ in potential, provided that the sum of the N terminal currents i_n vanishes identically. The physical meaning of (1) is the power which flows through the port into the circuit. This can easily be seen by choosing one terminal of the port as the potential reference, and connecting $(N-1)$ voltage or current sources (whichever is appropriate for the circuit) between the remaining terminals and the reference terminal. Clearly, (1) is identical to the sum of the powers delivered by the sources. The terminals being lossless, this total power must be delivered into the circuit. The individual terms

$$i_n \varphi_n \tag{3}$$

on the other hand, do not seem to have any physical meaning, for the potential φ_n is unique only up to an additive constant, such that (3) could have any value at all. Consequently, (3) *cannot* be interpreted as the power associated with the n-th terminal of the port. Only the sum in (1) over all the terminals of the port represents the power flow. Thus, it can be said, that »terminals are for interconnection, [but] ports are for energy transfer« [2].

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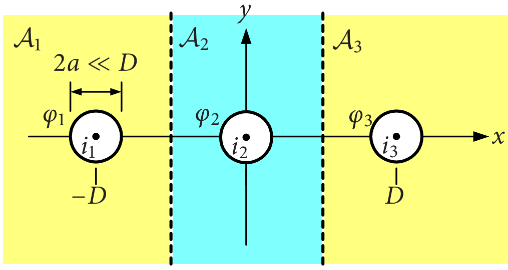

Figure 1: Cross section of 3 thin, parallel, and long wires.

2 POWER-FLOW AND TERMINALS

The previous discussion ignores the interesting possibility that the potential offset φ in (2) is a function of the terminal potentials. In contrast to (3), the term

$$i_n \left(\varphi_n - \frac{1}{N}\sum_{i=1}^{N} \varphi_i\right), \tag{4}$$

is *independent* from the reference point chosen for the potentials φ_i. Does it have a physical meaning?

To this end, consider 3 parallel long wires lined up with the z-axis of a rectangular coordinate system, as shown in a cross section in Figure 1. The diameter of the wires is assumed to be small compared to the distance between the wires. Let there flow the electric currents i_1, i_2 and i_3 through the wires. We assume the frequency is low enough such that the electric field can be described accurately enough as the gradient of the scalar potential. The potentials of the wires are denoted by φ_1, φ_2, and φ_3, respectively. Let us think of the cross section in Figure 1 as the terminals of a 3-terminal circuit. Because of the Kirchhoff current law, the sum of these currents must vanish identically ($i_1 + i_2 + i_3 \equiv 0$), such that the 3 terminals form a port. In the following, we will derive the result:

$$i_n \left(\varphi_n - \frac{1}{N}\sum_{i=1}^{N} \varphi_i\right) = \epsilon_0 c^2 \int_{\mathcal{A}_n} \left(\vec{E} \times \vec{B}\right) \cdot \vec{e}_z \, \mathrm{d}A, \tag{5}$$

where $\mathcal{A}_n$ is the part of the x-y plane, which points are nearest to the n-th wire, as indicated in Figure 1. For (5) is the integral of the Poynting vector [3] over the area which is nearer to the n-th terminal's wire than to any other terminal's wire, (4) can, at least in this case, be interpreted as the *power which is associated with the n-th terminal.*

3 WORKING ASSUMPTIONS

The computation of the electric field is complicated by the fact that the distribution of charge on the surfaces of the wires is unknown apriori and depends on the distribution of charge on the other wires. However, the situation simplifies tremendously when we assume that the diameter of the wires is very small compared to the distance of adjacent wires. In the cross section shown in Figure 1, the points on the circumference of one wire then all have almost the *same* distance to a given charge on the surface of one of the other wires. Hence, the charge on a wire will be distributed (almost) *uniformly* on the circumference. If, in addition, the wires extend long enough in front and behind the x-y plane, there will be almost no dependence on the z-coordinate. In total, this makes the wires resemble line charges which carry the charges q_1, q_2, and q_3, per unit of length. We assume perfectly electrically conducting (PEC) wires.

Additionally, we will make the calculations for DC only. This makes the electric field the gradient of a scalar potential φ, the wires' surfaces having the respective potentials φ_1, φ_2 and φ_3. These potentials are assumed to be set up and maintained by batteries which are connected to the far end of the wires. These batteries are responsible for the redistribution of charge on the *initially uncharged* wires. Consequently, $q_1 + q_2 + q_3 = 0$, due to conservation of charge. In summary:

$$a/D \to 0, \quad f = 0, \quad \sum_i q_i = 0, \quad \text{long PEC wires}, \tag{6}$$

are our working assumptions. The results are expected to apply also approximately for AC provided that the frequency f is low enough such that the wavelength $\lambda = c/f$ is much larger than the aperture $2D$ of the considered 3-wire system (see Figure 1).

4 THE ELECTRIC FIELD

The electric field $\vec{\boldsymbol{E}}$ is the sum of the electric fields $\vec{\boldsymbol{E}}_i$ contributed by the charges on each of the 3 wires:

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_1 + \vec{\boldsymbol{E}}_2 + \vec{\boldsymbol{E}}_3. \tag{7}$$

With (6) the wires behave like line charges. With Figure 1, we then have from Coulomb's law:

$$\vec{\boldsymbol{E}}_2 = \frac{q_2}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \begin{bmatrix} x \\ y \\ z-\zeta \end{bmatrix} \frac{\mathrm{d}\zeta}{\left(x^2+y^2+(z-\zeta)^2\right)^{3/2}}, \tag{8}$$

and similar results for $\vec{\boldsymbol{E}}_1$ and $\vec{\boldsymbol{E}}_3$. Performing the integrations, we obtain:

$$\vec{\boldsymbol{E}}_m = \begin{bmatrix} x+(2-m)D \\ y \\ 0 \end{bmatrix} \frac{q_m/(2\pi\epsilon_0)}{\left(x+(2-m)D\right)^2+y^2}, \tag{9}$$

where $m \in \{1,2,3\}$, and (x,y,z) are the Cartesian coordinates of a point P in space where we want to know the electric field. Note that P must be *outside* any wire. As the wires are PEC, the electric field inside the wires is zero. Because of $\textbf{rot}\,\vec{\boldsymbol{E}} = \boldsymbol{0}$, we have

$$\vec{\boldsymbol{E}} = -\textbf{grad}\,\varphi, \tag{10}$$

where φ is the scalar potential. Referring to Figure 1:

$$\int_{-D+a}^{-a} E_x \mathrm{d}x = \varphi_1 - \varphi_2, \qquad \int_{a}^{D-a} E_x \mathrm{d}x = \varphi_2 - \varphi_3, \tag{11}$$

where E_x is the x-component of the electric field $\vec{\boldsymbol{E}}$. With (9) and (7) E_x depends linearly on q_1, q_2 and q_3. With

$$q_1 + q_2 + q_3 = 0, \tag{12}$$

we therefore have three linear equations to uniquely determine the three unknown charges per unit length as a function of the potentials:

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix} \boldsymbol{C}, \tag{13}$$

with the capacitance matrix $\boldsymbol{C}$ per unit length given by

$$\boldsymbol{C} = \frac{2\pi\epsilon_0}{3\alpha+2\beta-\beta^2/\alpha} \begin{bmatrix} 2 & -1-\beta/\alpha & -1+\beta/\alpha \\ -1-\beta/\alpha & 2+2\beta/\alpha & -1-\beta/\alpha \\ -1+\beta/\alpha & -1-\beta/\alpha & 2 \end{bmatrix}, \tag{14}$$

where the parameters α and β are defined as:

$$\alpha = \ln\left(-1+D/a\right), \quad \beta = \ln\left(\left(2D-a\right)/\left(D+a\right)\right). \tag{15}$$

As $a/D \to 0$, it follows that $\alpha \to \ln(D/a)$, and $\beta \to \ln 2$. Thus, $\alpha \gg \beta$, and:

$$\boldsymbol{C} \to \frac{2\pi\epsilon_0}{3\ln(D/a)} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad \text{as } a/D \to 0. \tag{16}$$

By defining the average potential:

$$\overline{\varphi} = \frac{1}{3}\sum_{i=1}^{3} \varphi_i, \tag{17}$$

it therefore follows from (13) and (16) that

$$q_m = \frac{2\pi\epsilon_0}{\ln(D/a)} \left(\varphi_m - \overline{\varphi}\right), \quad m \in \{1,2,3\}. \tag{18}$$

Note that q_m is proportional to the difference $(\varphi_m - \overline{\varphi})$, which is *independent* of the arbitrarily chosen reference point for the potentials. By using (18) in (9), and the latter in (7), we have arrived at an analytical expression for the electric field *outside* the wires, parametrized by the potentials φ_1, φ_2, and φ_3.

5 THE MAGNETIC FIELD

The magnetic field $\vec{B}$ is the sum of the magnetic fields $\vec{B}_k$ which are generated by the currents i_k:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3. \tag{19}$$

As we are looking at DC, the fields $\vec{B}_k$ can be easily determined from Ampère's law:

$$\textbf{rot}\,\vec{B}_k = \frac{1}{\epsilon_0 c^2}\vec{j}_k,$$

where $\vec{j}_k$ is the current density vector for the k-th wire. Thus, the magnetic fields $\vec{B}_k$ form circles around their respective wires. Integrating the magnetic field along such a concentric circle with radius r then yields $B_k 2\pi r$. Using Stoke's law, this must then equal the integral of the current density over the area of the circle divided by $(\epsilon_0 c^2)$. For $r > a$, that is, for positions *outside* the wire, this simply yields $i_k/(\epsilon_0 c^2)$. Thus, $B_k = i_k/(2\pi\epsilon_0 r c^2)$. Referring to Figure 1, the currents i_k are positive when they flow in the direction of the positive z-axis. The right-hand rule then yields the desired magnetic fields *outside the wires*:

$$\vec{B}_k = \begin{bmatrix} -y \\ x + (2-k)D \\ 0 \end{bmatrix} \frac{i_k/(2\pi\epsilon_0 c^2)}{\left(x + (2-k)D\right)^2 + y^2}, \tag{20}$$

where $k \in \{1, 2, 3\}$. While the magnetic field also exists inside the wires, it is not of much interest for our goal of determining power flow, for the electric field is zero, thus, no power flows inside the wires. Note that the Kirchhoff current law

$$i_1 + i_2 + i_3 = 0, \tag{21}$$

from circuit theory is indeed required to maintain (12). Substituting (20) into (19) then yields the desired magnetic field *outside* the wires.

6 THE PARTIAL POWER FLOW

The power flow is computed from Poynting's vector [3]:

$$\vec{S} = \epsilon_0 c^2 \vec{E} \times \vec{B}. \tag{22}$$

Because neither the electric field nor the magnetic field have a z-component, the Poynting vector points into the direction of the z-axis:

$$\vec{S} = \vec{e}_z S_z, \quad S_z = \left(E_x B_y - E_y B_x\right)\epsilon_0 c^2. \tag{23}$$

From (7), (9), (18), (19), (20), and (23) then follows:

$$S_z = \frac{1}{2\pi\ln(D/a)} \sum_{m=1}^{3}\sum_{k=1}^{3} A_{m,k}\, i_k (\varphi_m - \overline{\varphi}), \tag{24}$$

where

$$A_{m,k} = \frac{\left(x + (2-m)D\right)\left(x + (2-k)D\right) + y^2}{\left(\left(x + (2-m)D\right)^2 + y^2\right)\left(\left(x + (2-k)D\right)^2 + y^2\right)}. \tag{25}$$

The *partial power* which flows through the area $\mathcal{A}_n$ shown in Figure 1, is then obtained as:

$$P_n = \int_{\mathcal{A}_n} S_z \,\mathrm{d}A. \tag{26}$$

Because the points belonging to $\mathcal{A}_n$ are all nearer to the n-th wire than to any other wire, one can interpret the partial power P_n, as the power which is *associated* with the n-th wire, or, in terms of circuit theory, associated with the n-th terminal.

It turns out that all parts in (24), except for $m = k = n$, do *not* contribute to the integral in (26) as $a/D \to 0$. To see this, consider the case $n = 2$, that is, the center wire in Figure 1. Note that $\mathcal{A}_2$ consists of the rectangle $|x| \le D/2$, $|y| < \infty$, excepting the circle in the origin with radius a. For the case $m = k = 1$, we see that

$$A_{1,1} = \frac{1}{(x + D)^2 + y^2}$$

is *regular* and *positive* in $|x| \le D/2$, $|y| < \infty$. Thus,

$$\left|\int_{\mathcal{A}_2} A_{1,1}\mathrm{d}x\mathrm{d}y\right| < \int_{\substack{|x| \le D/2 \\ |y| < \infty}} A_{1,1}\mathrm{d}x\mathrm{d}y = \pi\ln 3. \tag{27}$$

Obviously, after division by $\ln(D/a)$, (see (24)), the integral of $A_{1,1}$ over $\mathcal{A}_2$ does *not* contribute to P_2, as $a/D \to 0$. Exactly the same also holds for $A_{3,3}$. Now

$$A_{1,3} = \frac{x^2 + y^2 - D^2}{\left((x + D)^2 + y^2\right)\left((x - D)^2 + y^2\right)}$$

is also regular in $|x| \le D/2$, $|y| < \infty$. Integration over the whole rectangle yields:

$$\int_{\substack{|x| \le D/2 \\ |y| < \infty}} A_{1,3}\mathrm{d}x\mathrm{d}y = 0,$$

while the integral of $A_{1,3}$ over the circle $x^2 + y^2 \le a$ yields:

$$\int_{r=0}^{a}\int_{\phi=0}^{2\pi} \frac{r(r^2 - D^2)\mathrm{d}\phi\mathrm{d}r}{D^4 + r^4 - 2D^2r^2\cos 2\phi} = -\pi\ln\left(1 + a^2/D^2\right).$$

Thus,

$$\int_{\mathcal{A}_2} A_{1,3}\,\mathrm{d}x\mathrm{d}y = \pi\ln\left(1 + a^2/D^2\right). \tag{28}$$

This is a finite value which approaches zero as $a/D \to 0$. Consequently, after division by $\ln(D/a)$, the integral of $A_{1,3}$ does *not* contribute to P_2 as $a/D \to 0$. Exactly the same is true for $A_{3,1}$, for $A_{m,k} = A_{k,m}$. Now consider

$$A_{1,2} = \frac{(x + D)x + y^2}{\left((x + D)^2 + y^2\right)\left(x^2 + y^2\right)}.$$

Notice that $\lim_{x\to 0}(\lim_{y\to 0} A_{1,2}) \ne \lim_{y\to 0}(\lim_{x\to 0} A_{1,2})$. Hence, $A_{1,2}$ is indeterminate in the origin, making its integration more subtle. We do it in two parts:

$$\int_{\mathcal{A}_2} A_{1,2}\,\mathrm{d}x\mathrm{d}y = \int_{\substack{a \le |x| \le D/2 \\ |y| < \infty}} A_{1,2}\,\mathrm{d}x\mathrm{d}y + \int_{\substack{|x| \le a \\ |y| > \sqrt{a^2 - x^2}}} A_{1,2}\,\mathrm{d}x\mathrm{d}y. \tag{29}$$

The first integral on the right hand side of (29) makes no problems, for the integrand is regular inside this region and can be integrated elementary. One obtains:

$$\int_{a \le |x| \le D/2,\ |y| < \infty} A_{1,2}\,\mathrm{d}x\mathrm{d}y \;=\; 0. \tag{30}$$

The second integral on the right of (29) is the subtle one:

$$\int_{|x| \le a,\ |y| > \sqrt{a^2 - x^2}} A_{1,2}\,\mathrm{d}x\mathrm{d}y = \int_{-a}^{a} F(x)\mathrm{d}x, \tag{31}$$

where

$$F(x) = \int_{-\infty}^{-\sqrt{a^2-x^2}} A_{1,2}\,\mathrm{d}y + \int_{\sqrt{a^2-x^2}}^{\infty} A_{1,2}\,\mathrm{d}y$$

$$= \frac{-\arctan\left(\frac{\sqrt{a^2-x^2}}{x}\right) - \arctan\left(\frac{\sqrt{a^2-x^2}}{(x+D)}\right) + \pi\frac{1+\mathrm{sign}(x)}{2}}{x + D/2}.$$

Since the largest value returned by $\arctan(\cdot)$ is $\pi/2$, it follows:

$$|F(x)| \le \frac{2\pi}{|x + D/2|}. \tag{32}$$

Using this upper bound in (31), we obtain:

$$\left|\int_{|x| \le a,\ |y| > \sqrt{a^2 - x^2}} A_{1,2}\,\mathrm{d}x\mathrm{d}y\right| \;\le\; 2\pi \ln\left(\frac{D + 2a}{D - 2a}\right) \to 0, \tag{33}$$

as $a/D \to 0$. From (33), (30) and (29) we therefore have:

$$\int_{\mathcal{A}_2} A_{1,2}\,\mathrm{d}x\mathrm{d}y \;\to\; 0, \quad \text{as } a/D \to 0. \tag{34}$$

In this way, $A_{1,2}$ does *not* contribute to P_2 as $a/D \to 0$. Obviously, the same is true for $A_{2,1}$. Moreover, also $A_{3,2}$ and $A_{2,3}$ do not contribute, for the difference to the case of $A_{1,2}$ merely is that D is replaced by $-D$ in the integrand, which does not change the argument. Consequently, the remaining term, $A_{2,2}$, is the *only* term in (24) which can contribute to P_2 as $a/D \to 0$:

$$P_2 = \frac{i_2\left(\varphi_2 - \overline{\varphi}\right)}{2\pi \ln(D/a)} \int_{\mathcal{A}_2} \frac{\mathrm{d}x\mathrm{d}y}{x^2 + y^2}. \tag{35}$$

Again we split the integral into two parts. For the first one we have

$$\int_{x=-D/2}^{-a} \mathrm{d}x \int_{y=-\infty}^{\infty} \frac{\mathrm{d}y}{x^2 + y^2} + \int_{x=a}^{D/2} \mathrm{d}x \int_{y=-\infty}^{\infty} \frac{\mathrm{d}y}{x^2 + y^2} \;=\; 2\pi \ln \frac{D}{2a},$$

by elementary calculation. The second part yields:

$$\int_{|x| \le a,\ |y| > \sqrt{a^2 - x^2}} \frac{\mathrm{d}x\mathrm{d}y}{x^2 + y^2} = 4 \int_{x=0}^{a} \mathrm{d}x \int_{y=\sqrt{a^2-x^2}}^{\infty} \frac{\mathrm{d}y}{x^2 + y^2} \;=\; 2\pi \ln 2.$$

Thus, we obtain

$$\int_{\mathcal{A}_2} \frac{\mathrm{d}x\mathrm{d}y}{x^2 + y^2} = 2\pi \ln(D/a). \tag{36}$$

Using this result in (35) then shows that:

$$P_2 = i_2\left(\varphi_2 - \overline{\varphi}\right). \tag{37}$$

With (26) and (22) the result (37) then shows that

$$i_2\left(\varphi_2 - \overline{\varphi}\right) = \epsilon_0 c^2 \int_{\mathcal{A}_2} \left(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\right) \cdot \vec{\mathbf{e}}_z\,\mathrm{d}A. \tag{38}$$

Now this establishes the *physical meaning* of the term $i_2\left(\varphi_2 - \overline{\varphi}\right)$. With a similar development, one can show that an equivalent result also holds for the other two wires.

7 CONCLUSION

In classical circuit theory, terminals can be used to interconnect circuits, but computation of power flow requires the notion of a port. We suggest, however, that it nevertheless might make sense to associate power flow with single terminals by computing the product of the terminal current with the difference of the terminal potential and the average potential over all terminals. This comes about because, under certain conditions, this product can be interpreted, from a field theory point of view, as the power which flows through those parts of a perpendicular reference plain, which are nearer to the given terminal than to any other terminal of the circuit. In this way, the proposed product can be interpreted as the power associated with a single terminal.

References

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