Non-parametric Estimation of Elliptical Copulae With Application to Credit Risk

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Abstract

This paper develops a method for statistical estimation of the dependence structure of financial assets. As we are interested mainly in applications to credit risk, our approach focuses directly on the copula function of a random vector and works independently of any marginal assumptions. We use the class of elliptical copulas, which provide a natural extension to the standard for the practice Gaussian copula and a flexible model for joint extreme events. We calibrate the linear correlation coefficients using the whole sample of observations and the non-linear (tail) dependence coefficients using only the extreme observations. We provide theoretical as well as numerical support for our method.

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The author is supported financially by a scholarship from DFG (Deutsche Forschungsgemeinschaft) through the Graduate Program "Applied Algorithmic Mathematics".

AMS 2000 Mathematics Subject Classification: 62G32, 62H12, 62H20, 91B28

JEL Classification: G11, G21

Keywords: elliptical copula estimation, heavy-tailed risk factors, portfolio credit risk, tail dependence

1 Introduction

During the last decade, the dependencies between the financial assets have increased due to globalization effects and relaxed market regulation. The standard industrial methodologies like RiskMetrics (see [27]) and CreditMetrics (see [19]) model the dependence structure in the derivatives or credit portfolio by assuming multivariate normality of the underlying risk factors. It has been well recognized (see Mandelbrot [26] for a classical, or Cont [5] for a recent study), that many financial assets exhibit a number of features which contradict the normality assumption – namely asymmetry, skewness and heavy tails. However, asset return data suggests also a dependence structure which is quite different from the Gaussian (see Fortin and Kuzmics [12]). In particular, empirical studies like Junker and May [22] and Malevergne and Sornette [25] indicate that especially during highly volatile and bear markets the probability for joint extreme events leading to simultaneous losses in a portfolio could be seriously underestimated under the normality assumption. Theoretically, Embrechts et al. [10] show that the traditional dependence measure (linear correlation) is not always suited for a proper understanding of dependency in financial markets. When it comes to measuring the dependence between extreme losses, other measures (e.g. the tail dependence) are more appropriate. In the credit risk framework, Frey et al. [16] provide examples and insight on the impact of a violated Gaussian assumption on the tail of the credit portfolio loss distribution. Holding the marginal loss distributions of the individual credits fixed and introducing tail dependence through heavy-tailed risk factors, Frey et al. [16] conclude that the overall portfolio risk increases drastically compared to the Gaussian (tail-independent) case. Clearly, appropriate multivariate models and corresponding estimation methods suited for extreme events are needed.

This paper focuses on the statistical estimation of the dependence structure of financial assets. We are interested mainly in applications to portfolio credit risk, where the observable financial assets (for instance stock returns) enter the model only to introduce dependency between the credits. Mathematically, the whole information for the dependence in a random vector is in its copula function.

Definition 1.1. *For $d \geq 2$ a d-dimensional distribution function with marginals uniformly distributed on $[0,1]$ is called a* copula.

By means of Sklar's theorem (see Sklar [32]), if H is a d-dimensional distribution function (d.f.) with continuous marginals $F_1, \ldots, F_d$, then there exists a unique copula C such that for all $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$

$$H(y_1, \ldots, y_d) = C(F_1(y_1), \ldots, F_d(y_d)) \,.$$

Conversely, if C is a copula and $F_1, \ldots, F_d$ are d.f.s, then the function H defined as above is a d-dimensional d.f. with marginals $F_1, \ldots, F_d$.

The problem we want to analyse in this paper can be formulated as follows: given a sample $(Y_1^{(k)}, \ldots, Y_d^{(k)}),\ k = 1, \ldots, n$, of i.i.d. observations with d.f. H, estimate the copula C regardless of the marginals $F_1, \ldots, F_d$.

We use the class of elliptical copulas, which provide a natural extension to the standard for the practice Gaussian copula and, at the same time, a flexible model for joint extreme events. In Section 2 we define these copulas and present some classical and more recent results on them. The main estimation method is described in Section 3. We calibrate the linear correlation coefficients using the whole sample of observations and the non-linear (tail) dependence coefficients using only the extreme observations. In Section 4 we give numerical examples and investigate the accuracy and robustness of the estimation. In Section 5 we present a modification of the method which decreases the variance of the estimates of the tail dependence matrix. In Section 6, we introduce a portfolio credit risk model. We provide further results which enable the application of our method in the estimation of some of the crucial parameters of the model. We conclude with a real data example for a credit portfolio under the influence of business sector common factors.

2 Model and preliminary results

The copula most frequently used in practice is the Gaussian

$$C_\Sigma(u_1, \ldots, u_d) = N_\Sigma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)),$$

which is the copula of the standard d-dimensional Gaussian distribution with correlation matrix Σ. We use a more general class of copulas, generated from the class of elliptically distributed random vectors (see Fang et al. [11] for a detailed overview on elliptical distributions).

Definition 2.1. *If Y is a d-dimensional random vector and, for some vector $\mu \in \mathbb{R}^d$, some non-negative definite symmetric $d \times d$ matrix Σ and some function $\phi : [0, \infty) \to \mathbb{R}$, the characteristic function $\varphi_{Y-\mu}$ is of the form $\varphi_{Y-\mu}(t) = \phi(t'\Sigma t)$, we say that Y has an* elliptical distribution *with parameters μ, Σ and ϕ. The function ϕ is referred to as the* characteristic generator *of Y.*

When $d = 1$, the class of elliptical distributions coincides with the symmetric ones. For elliptically distributed random vectors, we have the representation

$$Y = \mu + RAU, \tag{2.1}$$

where R is a non-negative random variable (r.v.), A is a deterministic $d \times k$ matrix with $AA' = \Sigma$ ($k := \mathrm{rank}\Sigma$) and U is a k-dimensional random vector uniformly distributed on the unit hyper-sphere $S_k = \left\{z \in \mathbb{R}^k \,:\, z'z = 1\right\}$, independent of R. The r.v. R in (2.1) is referred to as the spectral variable. When $R^2 \in \chi^2_d$ (chi-square distributed with d degrees of freedom), we obtain the Gaussian distribution $Y \in N_d(\mu, \Sigma)$; when $R^2/d \in F(d, \nu)$ (F-distributed with d and ν degrees of freedom), we obtain the t-distribution with ν degrees of freedom $Y \in T_d(\nu, \mu, \Sigma)$.

A useful fact about elliptical distributions is that their marginals of any dimension $k < d$ are also elliptical with the same characteristic generator, and so are any linear combinations $a + BY$, $a \in \mathbb{R}^m$, $B \in \mathbb{R}^m \times \mathbb{R}^d$, see Fang et al. [11], Theorems 2.6 and 2.16.

Next we denote $\rho_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$ and we note that it is equal to the usual linear correlation coefficient if $\mathrm{var}(Y_i), \mathrm{var}(Y_j) < \infty$. However, for elliptical distributions ρ_{ij} is well defined even when linear correlation is not (see Example 4.1 for such a model). In the sequel we mean by *correlation* the constant ρ_{ij}.

Using (2.1) one can show that the elliptical copula is uniquely determined by the d.f. of the spectral variable R and the correlation matrix $[\rho_{ij}]_{i,j=1,\ldots,d}$ (see for instance Embrechts et al. [9], Lemma 5.1 and the comments after). These are the parameters which we are interested in.

Among the numerous possible statistical procedures for elliptical distributions in general, only few are designed to work on elliptical copulas regardless of the marginals (see Demarta and McNeil [7]). One of the possible methods is by means of pseudo - maximum likelihood (see Genest et al. [17]), which is briefly explained in the sequel. Denote

$$U_j^{(k)} = F_j^E(Y_j^{(k)}),\ j = 1, \ldots, d,\ k = 1, \ldots, n\,,$$

where F_j^E, $j = 1, \ldots, d$, is the empirical d.f.. Assume also that the d.f. of the spectral r.v. R as in (2.1) belongs to some parametric family with parameter vector Ψ, i.e. $P(R < x) = F(x, \Psi)$ (for example, one may take the F-family with (ν, d) degrees of freedom, leading to the t-copula). Then one may estimate the correlation matrix $[\rho_{ij}]_{i,j=1,\ldots,d}$ and the parameter vector Ψ by maximizing the pseudo-log-likelihood

$$\log L(\Psi, [\rho_{ij}], U) = \sum_{k=1}^{n} \log L(\Psi, [\rho_{ij}], U_1^{(k)}, \ldots, U_d^{(k)})\,,$$

where L is the likelihood function of the elliptical copula. As typically the likelihood function is available in terms of d-dimensional integrals (see Demarta and McNeil [7]), in practice some numerical issues arise, in particular when $d > 2$. A different approach overcoming this problem is taken in Lindskog et al. [24]. It is based on the Kendall's tau.

Definition 2.2. Kendall's tau *for a bivariate random vector* (Y_1, Y_2) *is defined as*

$$\tau := P((\widehat{Y_1} - \widetilde{Y_1})(\widehat{Y_2} - \widetilde{Y_2}) > 0) - P((\widehat{Y_1} - \widetilde{Y_1})(\widehat{Y_2} - \widetilde{Y_2}) < 0)\,, \tag{2.2}$$

where $(\widehat{Y_1}, \widehat{Y_2})$ *and* $(\widetilde{Y_1}, \widetilde{Y_2})$ *are independent copies of* (Y_1, Y_2).

Kendall's tau is a copula property in the sense that it is invariant under increasing transformations of the marginal random variables, see for instance Embrechts et al. [9], Theorem 3.3. The relation between Kendall's tau and the linear correlation coefficient is well known for bivariate normally distributed random vectors. There is in fact a more general relation between Kendall's tau and the correlations ρ_{ij} for elliptically distributed random vectors with absolutely continuous marginals, namely

$$\tau(Y_i, Y_j) = \frac{2}{\pi} \arcsin \rho_{ij}, \quad i, j = 1, \ldots, d\,, \tag{2.3}$$

see Lindskog et al. [24] for a proof.

The result (2.3) provides a robust method to determine the correlations in random vectors with arbitrary continuous marginals and elliptical copula. Using the observations for each pair $(Y_i^{(k)}, Y_j^{(k)})$, $k = 1, \ldots, n,\ i, j = 1, \ldots, d$, from such a random vector $(Y_1, \ldots, Y_d)$, one may consistently estimate Kendall's tau matrix by

$$\widehat{\tau_{ij}}^{n} = \begin{pmatrix} n \\ 2 \end{pmatrix}^{-1} \sum_{k>l} \operatorname{sign}[(Y_i^{(k)} - Y_i^{(l)})(Y_j^{(k)} - Y_j^{(l)})],\ i, j = 1, \ldots, m\,, \tag{2.4}$$

and then use $\widehat{\rho_{ij}}^{n} = \sin(\frac{\pi}{2}\widehat{\tau_{ij}}^{n}),\ i, j = 1, \ldots, d$, see Lindskog et al. [24] for asymptotic properties and numerical examples. Then, in order to estimate the remaining parameters Ψ in the distribution of the spectral variable R, one maximizes the pseudo-log-likelihood

$$\log L(\Psi, [\widehat{\rho_{ij}}^{n}], U) = \sum_{k=1}^{n} \log L(\Psi, [\widehat{\rho_{ij}}^{n}], U_1^{(k)}, \ldots, U_d^{(k)}).$$

As already discussed, the main reason why we are interested in copulas different from the Gaussian in that we need better models for the dependency between extreme events. In this sense both of the above mentioned statistical approaches have the drawback that they infer the parameters using the whole sample of observations. Thus, they provide a good fit on the empirical copula for the center of the distribution, but they might be misleading when it comes to joint extreme events.

The tail dependence coefficient relates to the amount of dependence in the lower-left-quadrant tail of a bivariate distribution, i.e. it is relevant for the study of dependence between extreme events.

Definition 2.3. *The* lower tail dependence coefficient *for a bivariate random vector* (Y_1, Y_2) *with marginals* F_1 *and* F_1 *and copula* C_{12} *is given by*

$$\lambda(Y_1, Y_2) := \lim_{u \to 0} P(Y_1 < F_1^{-1}(u) \,|\, Y_2 < F_2^{-1}(u)) = \lim_{u \to 0} \frac{C_{12}(u, u)}{u}\,, \tag{2.5}$$

if the limit exists. When $\lambda = 0$*, we speak of* $\text{tail} - \text{independence}$*, otherwise* $0 < \lambda \leq 1$ *and we speak of* $\text{tail} - \text{dependence}$.

From (2.5) we see that the coefficient is a copula property and hence invariant under strictly increasing transformations of the marginals Y_1 and Y_2. There are various statistical methods for detection and estimation of tail dependence (see Frahm et al. [13], also Schlather and Tawn [30]). For instance,

$$\widehat{\lambda_{12}}^{n,s} = 2 - \frac{1}{s}\sum_{k=1}^{n} 1_{\left\{F_1^E(Y_1^{(k)})<\frac{s}{n}\right\}\bigcup\left\{F_2^E(Y_2^{(k)})<\frac{s}{n}\right\}}, \ 1 < s < n\,, \tag{2.6}$$

where $(Y_1^{(k)}, Y_2^{(k)})$, $k = 1, \ldots, n$ are i.i.d. observations from (Y_1, Y_2), F_1^E and F_2^E denote the empirical d.f.s, $s = s(n) \to \infty$ and $\frac{s(n)}{n} \to 0$ as $n \to \infty$ is a consistent estimate of the tail dependence coefficient, see Schmidt and Stadtmüller [29].

The next proposition relates the tail dependence coefficient for an elliptical random vector to the tail behaviour of the spectral r.v. R as given in (2.1).

Proposition 2.4. *Let $Y = \mu + RAU$ be a d-dimensional elliptically distributed random vector with absolutely continuous marginals with support on the whole of $\mathbb{R}$. The following statements are equivalent*

(1) For some $\alpha > 0$, all pairs (Y_i, Y_j) are tail-dependent with coefficient

$$\lambda(Y_i, Y_j) = \frac{\int_{\frac{\pi}{4}(1-\tau(Y_i,Y_j))}^{\frac{\pi}{2}} \cos^\alpha t\, dt}{\int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt}, \ i, j = 1, \ldots, d, \tag{2.7}$$

where $\tau(Y_i, Y_j)$ denotes Kendall's tau.

(2) The tail of the spectral variable R is regularly varying at infinity with index $\alpha > 0$, i.e.

$$\lim_{x\to\infty} \frac{P(R > qx)}{P(R > x)} = q^{-\alpha}$$

for every $q > 0$. In this case the the pairwise tail dependence coefficients are given by (2.7).

Proof. See Hult and Lindskog [20], Theorem 4.3. □

From the proposition above we may conclude that the bivariate marginals of an elliptically distributed random vector Y have tail dependence if and only if the spectral r.v. R in (2.1) is regularly varying. Kendall's tau τ_{ij} only affects the magnitude of tail dependence. As a consequence of this proposition, r.v.s with a Gaussian copula are tail-independent, whereas the t-copula with ν degrees of freedom leads to tail dependence with $\alpha = \nu$.

3 Estimation methodology

In the next lemma we analyze further the function on the right-hand side of (2.7).

Lemma 3.1. *Setting $x = \frac{\pi}{2}(1-\tau)$ in (2.7), we define*

$$\lambda(\alpha, x) = \frac{\int_x^{\frac{\pi}{2}} \cos^\alpha t\, dt}{\int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt},\ \alpha \geq 0,\ x \in [0, \frac{\pi}{2}]\,.$$

It satisfies the following properties.

(1) $\lambda(\alpha, x)$ is continuous and differentiable in $\alpha > 0$, $x \in (0, \frac{\pi}{2})$.

(2) $0 < \lambda(\alpha, x) < 1$.

(3) Let $x \in (0, \frac{\pi}{2})$ be fixed. Then $\lambda(\alpha, x)$ is strictly decreasing in $\alpha > 0$. Furthermore,

$$\lim_{\alpha \to 0} \lambda(\alpha, x) = 1 - \frac{2x}{\pi}, \quad \lim_{\alpha \to \infty} \lambda(\alpha, x) = 0\,.$$

(4) Let $\alpha > 0$ be fixed. Then $\lambda(\alpha, x)$ is strictly decreasing in $x \in (0, \frac{\pi}{2})$.

Proof. The function $\cos^\alpha t$ for $t \in (0, \frac{\pi}{2})$ is continuous and differentiable, and so are $\int_x^{\frac{\pi}{2}} \cos^\alpha t\, dt$ and $\int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt$ as functions of $\alpha > 0$ and of $x \in (0, \frac{\pi}{2})$. Furthermore, $\cos^\alpha t > 0$ for $t \in (0, \frac{\pi}{2})$, hence $0 < \int_x^{\frac{\pi}{2}} \cos^\alpha t\, dt < \int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt$, therefore we obtain (2).

To prove (3) we differentiate with respect to α

$$\begin{aligned} \frac{\partial}{\partial \alpha} \lambda(\alpha, x) &= \frac{\int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt \int_x^{\frac{\pi}{2}} \log(\cos t) \cos^\alpha t\, dt - \int_x^{\frac{\pi}{2}} \cos^\alpha t\, dt \int_0^{\frac{\pi}{2}} \log(\cos t) \cos^\alpha t\, dt}{\left(\int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt\right)^2} \\ &= \frac{D(\alpha, x)}{\left(\int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt\right)^2}\,. \end{aligned}$$

We will prove that $D(\alpha, x) < 0$ for any fixed $0 < x < \frac{\pi}{2}$ and $\alpha > 0$.

First we note that $D(\alpha, 0) = D(\alpha, \frac{\pi}{2}) = 0$. Then we differentiate with respect to x:

$$\begin{aligned} \frac{\partial}{\partial x} D(\alpha, x) &= -\log(\cos x) \cos^\alpha x \int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt + \cos^\alpha x \int_0^{\frac{\pi}{2}} \log(\cos t) \cos^\alpha t\, dt \\ &= \cos^\alpha x \left(-\log(\cos x) \int_0^{\frac{\pi}{2}} \cos^\alpha t\, dt + \int_0^{\frac{\pi}{2}} \log(\cos t) \cos^\alpha t\, dt \right) \\ &= C(\alpha, x) \cos^\alpha x\,. \end{aligned}$$

Note that

$$C(\alpha, 0) = \int_0^{\frac{\pi}{2}} \log(\cos t) \cos^\alpha t\, dt < 0$$

and that

$$\lim_{x \to \frac{\pi}{2}} C(\alpha, x) = \infty$$

and that $C(\alpha, x)$ is strictly increasing in x for $x \in (0, \frac{\pi}{2})$, as $-\log(\cos x)$ is strictly increasing. Therefore there exists a unique point y, $0 < y < \frac{\pi}{2}$, such that $C(\alpha, y) = 0$. Furthermore, $\frac{\partial}{\partial x} D(\alpha, x) = C(\alpha, x) \cos^\alpha x < 0$ for $x \in (0, y)$ and $\frac{\partial}{\partial x} D(\alpha, x) = C(\alpha, x) \cos^\alpha x > 0$

for $x \in (y, \frac{\pi}{2})$, so $D(\alpha, x)$ is strictly decreasing for $x \in (0, y)$ (i.e decreasing from $D(\alpha, 0) = 0$ to $D(\alpha, y) < 0$) and $D(\alpha, x)$ is strictly increasing for $x \in (y, \frac{\pi}{2})$ (i.e. increasing from $D(\alpha, y) < 0$ to $D(\alpha, \frac{\pi}{2}) = 0$). Therefore $D(\alpha, x) < 0$ for any $x \in (0, \frac{\pi}{2})$ and $\alpha > 0$. Therefore $\frac{\partial}{\partial \alpha} \lambda(\alpha, x) < 0$ for any $x \in (0, \frac{\pi}{2})$ and $\alpha > 0$, which proves that $\lambda(\alpha, x)$ is strictly decreasing. Furthermore,

$$\lim_{\alpha \to 0} \lambda(\alpha, x) = \lim_{\alpha \to 0} \frac{\int_x^{\frac{\pi}{2}} \cos^\alpha t \, dt}{\int_0^{\frac{\pi}{2}} \cos^\alpha t \, dt} = 1 - \frac{2x}{\pi} \, .$$

Taking some $0 < \epsilon < x$ and using the fact that $\cos^\alpha t$ is strictly decreasing in t for every $\alpha > 0$ we obtain

$$\begin{aligned} \frac{1}{\lambda(\alpha, x)} &= \frac{\int_0^{\frac{\pi}{2}} \cos^\alpha t \, dt}{\int_x^{\frac{\pi}{2}} \cos^\alpha t \, dt} \\ &= \frac{\int_0^{\epsilon} \cos^\alpha t \, dt}{\int_x^{\frac{\pi}{2}} \cos^\alpha t \, dt} + \frac{\int_\epsilon^{\frac{\pi}{2}} \cos^\alpha t \, dt}{\int_x^{\frac{\pi}{2}} \cos^\alpha t \, dt} \\ &\geq \frac{\epsilon \cos^\alpha \epsilon}{(\frac{\pi}{2} - x) \cos^\alpha x} + 1 \, . \end{aligned}$$

Since $\left(\frac{\cos \epsilon}{\cos x}\right)^\alpha \to \infty, \ \alpha \to \infty$, we obtain

$$\lim_{\alpha \to \infty} \lambda(\alpha, x) = 0 \, .$$

As $\cos^\alpha t > 0$ for $t \in (0, \frac{\pi}{2})$, we have also the monotonicity of $\int_x^{\frac{\pi}{2}} \cos^\alpha t \, dt$, i.e. (4). □

Let $X = (X_1, \ldots, X_d)$ be a random vector with absolutely continuous marginals with support on the whole of $\mathbb{R}$ and an elliptical copula equal to the copula of the random vector $Y = \mu + RAU$. Assume that the spectral random variable R has a regularly varying tail at infinity with index $0 < \alpha^* < \infty$ as in Proposition 2.4 (2).

Let $|| \cdot ||$ be the L^2 norm defined on the space $\mathbb{R}^d \times \mathbb{R}^d$, i.e.

$$||A|| = ||[A_{ij}]|| = \sum_{i,j=1}^{d} A_{ij}^2, \ A \in \mathbb{R}^d \times \mathbb{R}^d \, .$$

Denote $\widehat{\Lambda}^n = [\widehat{\lambda_{ij}}^n]_{i,j=1,\ldots,d} \in \mathbb{R}^d \times \mathbb{R}^d$, where $(\widehat{\lambda_{ij}}^n)_{n \in \mathbb{N}}$, for $i, j = 1, \ldots, d$, is a sequence of (weakly) consistent estimates of the pairwise tail-dependence coefficients λ_{ij}^* of X_i and X_j, i.e.

$$\widehat{\lambda_{ij}}^n \xrightarrow{P} \lambda_{ij}^*, \ n \to \infty \, . \tag{3.1}$$

For example, (2.6) provides such a sequence, see e.g. Schmidt and Stadtmüller [29]. Denote the true tail dependence coefficients matrix $\Lambda^* = [\lambda_{ij}^*] \in \mathbb{R}^d \times \mathbb{R}^d$.

Denote $\widehat{\tau}^n = [\widehat{\tau_{ij}}^n]_{i,j=1,\dots,d} \in \mathbb{R}^d \times \mathbb{R}^d$, where $(\widehat{\tau_{ij}}^n)_{n\in\mathbb{N}}$, for $i, j = 1, \dots, d$, is a sequence of consistent estimates of the Kendall's tau coefficients τ_{ij}^* of X_i and X_j, i.e.

$$\widehat{\tau_{ij}}^n \xrightarrow{P} \tau_{ij}^*,\ n \to \infty\,. \tag{3.2}$$

For example, (2.4) provides such a sequence, see e.g. Lindskog et al. [24]. Denote the true Kendall's tau matrix by $\tau^* = [\tau_{ij}^*] \in \mathbb{R}^d \times \mathbb{R}^d$.

Furthermore, for $\alpha > 0$ and $\tau \in \mathbb{R}^d \times \mathbb{R}^d$ with $\tau_{ij} \in (-1, 1)$ denote by

$$L(\alpha, \tau) = [\lambda(\alpha, \frac{\pi}{4}(1 - \tau_{ij}))] \in \mathbb{R}^d \times \mathbb{R}^d\,, \tag{3.3}$$

where $\lambda(\alpha, x)$ is the function from Lemma 3.1.

Proposition 3.2. *Let $X = (X_1, \dots, X_d)$ be a random vector with absolutely continuous marginals with support on the whole of $\mathbb{R}$ and an elliptical copula such that the spectral random variable R in (2.1) has a regularly varying tail with index $0 < \alpha^* < \infty$. Let $\widehat{\Lambda}^n = [\widehat{\lambda_{ij}}^n] \in \mathbb{R}^d \times \mathbb{R}^d$ and $\widehat{\tau}^n = [\widehat{\tau_{ij}}^n] \in \mathbb{R}^d \times \mathbb{R}^d$ satisfy (3.1) and (3.2). In addition, let $\widehat{\tau_{ij}}^n = \widehat{\tau_{ji}}^n \in (-1, 1)$ a.s. and $\widehat{\lambda_{ij}}^n = \widehat{\lambda_{ji}}^n \in (0, \frac{1+\widehat{\tau_{ij}}^n}{2})$ a.s. for every $n \in \mathbb{N}$, $i, j = 1, \dots, d$ (we set $\widehat{\tau_{ij}}^n = \widehat{\lambda_{ii}}^n = 1$). Denote*

$$\widehat{\alpha}^n = \arg\min_{\alpha>0} ||L(\alpha, \widehat{\tau}^n) - \widehat{\Lambda}^n||. \tag{3.4}$$

Then

(1) $\widehat{\alpha}^n$ exists and is unique a.s. for every $n \in \mathbb{N}$.

(2) $\widehat{\alpha}^n$ is a consistent estimate of α^*, i.e.

$$\widehat{\alpha}^n \xrightarrow{P} \alpha^*,\ n \to \infty.$$

(3) Denote by $\widehat{\theta}^n$ the vector, composed of all $\widehat{\tau_{ij}}^n, \widehat{\lambda_{ij}}^n$, $i = 1, \dots, d$, $j = i+1, \dots, d$ and by θ^* the corresponding vector with the true Kendall's tau and tail dependence coefficients. If

$$\sqrt{n}\left(\widehat{\theta}^n - \theta^*\right) \xrightarrow{d} N(0, \Sigma),\ n \to \infty,$$

for some non-degenerate $2d(d-1) \times 2d(d-1)$ covariance matrix Σ, then

$$\sqrt{n}\,(\widehat{\alpha}^n - \alpha^*) \xrightarrow{d} N(0, \sigma),\ n \to \infty, \tag{3.5}$$

where $\sigma > 0$ is explicitly specified in (3.9).

Proof. (1) The following arguements are valid a.s. We note that by means of Lemma 3.1 (2), for every $\alpha > 0$ we have

$$0 \le ||L(\alpha, \widehat{\tau}^n) - \widehat{\Lambda}^n|| \le 4d^2.$$

Also, using Lemma 3.1 (3) and the fact that

$$\lim_{\alpha\to 0}\lambda(\alpha,\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))=\frac{1+\widehat{\tau_{ij}}^n}{2}>\widehat{\lambda_{ij}}^n$$

and

$$\lim_{\alpha\to\infty}\lambda(\alpha,\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))=0<\widehat{\lambda_{ij}}^n$$

we have for $i,j=1,\dots,d$ and for any $n\in\mathbb{N}$ a unique solution of the equation

$$\lambda(\alpha,\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))=\widehat{\lambda_{ij}}^n,$$

which we denote by $\alpha_{ij}\geq 0$. Let

$$\alpha_{max}=\max_{i,j=1,\dots,d}\alpha_{ij},\ \ \alpha_{min}=\min_{i,j=1,\dots,d}\alpha_{ij}.$$

Due to the monotonicity of $\lambda(\alpha,\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))$ as a function of α (Lemma 3.1 (3)), for any $\alpha>\alpha_{max}$ we have

$$\begin{aligned}\lambda(\alpha,\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))-\widehat{\lambda_{ij}}^n &< \lambda(\alpha_{max},\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))-\widehat{\lambda_{ij}}^n\\ &\leq \lambda(\alpha_{ij},\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))-\widehat{\lambda_{ij}}^n=0\,,\end{aligned}$$

therefore $|\lambda(\alpha,\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))-\widehat{\lambda_{ij}}^n|>|\lambda(\alpha_{max},\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))-\widehat{\lambda_{ij}}^n|$ and hence

$$||L(\alpha,\widehat{\tau}^n)-\widehat{\Lambda}^n||>||L(\alpha_{max},\widehat{\tau}^n)-\widehat{\Lambda}^n||\,.$$

By analogy for any $\alpha<\alpha_{min}$

$$||L(\alpha,\widehat{\tau}^n)-\widehat{\Lambda}^n||>||L(\alpha_{min},\widehat{\tau}^n)-\widehat{\Lambda}^n||\,.$$

Therefore, either $\widehat{\alpha}^n=\alpha_{min}=\alpha_{max}$ or $||L(\alpha,\widehat{\tau}^n)-\widehat{\Lambda}^n||$ is bounded on the compact interval $[\alpha_{min},\alpha_{max}]$, hence $\widehat{\alpha}^n$ exists.

To prove uniqueness, assume for some $n\in\mathbb{N}$ that there are $\alpha_1\neq\alpha_2$ which are both minimizers of $||L(\alpha,\widehat{\tau}^n)-\widehat{\Lambda}^n||$. As $L(\alpha,\widehat{\tau}^n)-\widehat{\Lambda}^n$ is a symmetric matrix, we may concentrate on the upper triangle of the matrix, i.e. the same α_1,α_2 minimize also

$$G(\alpha)=\sum_{k=1}^{d(d-1)}(g_k(\alpha))^2\,, \tag{3.6}$$

where

$$g_k(\alpha)=\lambda(\alpha,\frac{\pi}{4}(1-\widehat{\tau_{ij}}^n))-\widehat{\lambda_{ij}}^n,\ \ i=1,\dots,d,j=i+1,\dots,d,$$

i.e. $k=1,\dots,d(d-1)$. Next define

$$H(\alpha,w)=\sum_{k=1}^{d(d-1)}w_kg_k(\alpha)$$

where w is a $d(d-1)$-dimensional non-random vector with non-negative components.

Without loss of generality assume that $\alpha_1 < \alpha_2$, which implies by 3.1 (3) that $g_k(\alpha_1) > g_k(\alpha_2),\ k = 1, \dots, d(d-1)$. From the fact that α_2 is a minimizer of $G(\alpha)$ we obtain

$$\sum_{k=1}^{d(d-1)} \left(\frac{\partial}{\partial \alpha} g_k(\alpha_2) \right) g_k(\alpha_2) = 0$$

Since $\frac{\partial}{\partial \alpha} g_k(\alpha_2)$ are all negative (Lemma 3.1 (3)), there are only two cases (a) and (b).

(a) $g_k(\alpha_2) = 0,\ k = 1, \dots, d(d-1)$. From this we obtain immediately that $G(\alpha_2) = 0$. However, as $g_k(\alpha_1) > 0,\ k = 1, \dots, d(d-1)$, we have $G(\alpha_1) > 0 = G(\alpha_2)$, which is a contradiction.

(b) There exists some index j for which $g_j(\alpha_2) > 0$. Therefore we have also $g_j(\alpha_1) > g_j(\alpha_2) > 0$, and, for every $w_j \geq 0$, $R(w_j) = w_j(g_j(\alpha_1) - g_j(\alpha_2))$ is a positive and increasing function in w_j. Therefore we may always find a vector w such that

$$H(\alpha_1, w) > H(\alpha_2, w)$$

by selecting its j-th component sufficiently large.

Fix an $\epsilon > 0$ such that $\alpha_2 - \alpha_1 > \epsilon$. As $H(\alpha, w)$ is continuous in α (Lemma 3.1 (1)), we may find w such that

$$H(\alpha, w) > H(\alpha_2, w) \tag{3.7}$$

for every $\alpha_1 - \epsilon < \alpha < \alpha_1 + \epsilon$. We define a function

$$F(\alpha) = \begin{cases} (1 - Q(\alpha)G(\alpha) + Q(\alpha)H(\alpha, w) & \alpha_1 - \epsilon < \alpha < \alpha_1 + \epsilon\,, \\ H(\alpha, w) & \alpha > \alpha_1 + \epsilon\,, \end{cases}$$

where

$$Q(\alpha) = 2\left(\frac{\alpha - \alpha_1}{\epsilon}\right)^4 - \left(\frac{\alpha - \alpha_1}{\epsilon}\right)^8 .$$

Note that $F(\alpha)$ is continuous on its domain, as $Q(\alpha_1 + \epsilon) = 1$. The derivative of $F(\alpha)$ for $\alpha < \alpha_1 + \epsilon$ is given by

$$F'(\alpha) = Q'(\alpha)(H(\alpha, w) - G(\alpha)) + (1 - Q(\alpha))G'(\alpha) + Q(\alpha)H'(\alpha, w)\,.$$

Since $Q'(\alpha_1 + \epsilon) = 0$, $F(\alpha)$ is also differentiable on its domain. Note also that α_1 is a strict local minimizer of F, because $F'(\alpha_1) = G'(\alpha_1) = 0$ and

$$\begin{aligned} F''(\alpha) &= Q''(\alpha)(H(\alpha, w) - G(\alpha)) + 2Q'(\alpha)(H'(\alpha, w) - G'(\alpha)) + \\ &\quad + Q(\alpha)H''(\alpha, w) + (1 - Q(\alpha))G''(\alpha)\,, \end{aligned}$$

therefore $F''(\alpha_1) = G''(\alpha_1) > 0$. Therefore, for a sufficiently small ϵ we have $F(\alpha)$ is increasing in $\alpha \in (\alpha_1, \alpha_1 + \epsilon]$. On the other hand, by (3.7) we have $F(\alpha_1 + \epsilon) > F(\alpha_2)$, which implies that there exists a point $\alpha_3 \in [\alpha_1+\epsilon, \alpha_2)$ such that $F'(\alpha_3) = 0$, therefore we have $\sum_{k=1}^{d(d-1)} w_k \frac{\partial}{\partial\alpha} g_k(\alpha_3) = 0$, which is a contradiction to the monotonicity of $g_k,\ k = 1,\ldots,d(d-1)$ (Lemma 3.1 (3)). This proves (1).

(2) As $\lambda(\alpha, \frac{\pi}{4}(1-\tau))$ is continuous in τ (Lemma 3.1 (1)) by the continuous mapping theorem we have for every $\alpha > 0$

$$L(\alpha, \widehat{\tau}^n) \xrightarrow{P} L(\alpha, \tau^*),\ n \to \infty.$$

On the other hand $\widehat{\Lambda}^n \xrightarrow{P} \Lambda^*$, hence, by Proposition 2.4 we have

$$L(\alpha^*, \widehat{\tau}^n) - \widehat{\Lambda}^n \xrightarrow{P} \overline{0},\ n \to \infty\,,$$

where $\overline{0} \in \mathbb{R}^d \times \mathbb{R}^d$ is the zero matrix. By monotonicity of $\lambda(\alpha, x)$ in α (Lemma 3.1 (3)), for every $\alpha \neq \alpha^*$ we have

$$L(\alpha, \tau_n) - \Lambda_n \xrightarrow{P} \overline{A}(\alpha) \in \mathbb{R}^d \times \mathbb{R}^d,\ n \to \infty\,,$$

where $\overline{A}(\alpha) \neq \overline{0}$, therefore $||\overline{A}(\alpha)|| > 0$. Hence we have

$$\widehat{\alpha}^n = \arg\min_{\alpha>0} ||L(\alpha, \widehat{\tau}^n) - \widehat{\Lambda}^n|| \xrightarrow{P} \alpha^*,\ n \to \infty\,.$$

(3) To prove asymptotic normality, we use the delta method. We consider the function $G(\alpha) = G(\alpha, \widehat{\theta}^n)$ defined in (3.6). By the definition of $\widehat{\alpha}^n$, we have

$$0 \;=\; \frac{\partial}{\partial\alpha} G(\widehat{\alpha}^n, \widehat{\theta}^n).$$

By Taylor expansion of $\frac{\partial}{\partial\alpha} G(\widehat{\alpha}^n, \widehat{\theta}^n)$ around α^* and we get

$$0 = \frac{\partial}{\partial\alpha} G(\alpha^*, \widehat{\theta}^n) + \frac{\partial^2}{\partial\alpha^2} G(\widetilde{\alpha}^n, \widehat{\theta}^n)(\widehat{\alpha}^n - \alpha^*)$$

where $\widetilde{\alpha}^n$ lies between $\widehat{\alpha}^n$ and α^* a.s. for every $n \in \mathbb{N}$. Therefore

$$\widehat{\alpha}^n - \alpha^* = -\left(\frac{\partial^2}{\partial\alpha^2} G(\widetilde{\alpha}^n, \widehat{\theta}^n)\right)^{-1} \frac{\partial}{\partial\alpha} G(\alpha^*, \widehat{\theta}^n). \tag{3.8}$$

As $\widetilde{\alpha}^n$ lies between $\widehat{\alpha}^n$ and α^*, by Lemma 3.1 (1), the definition of G and the continuous mapping theorem we have

$$\frac{\partial^2}{\partial\alpha^2} G(\widetilde{\alpha}^n, \widehat{\theta}^n) \xrightarrow{P} \frac{\partial^2}{\partial\alpha^2} G(\alpha^*, \theta^*),\ n \to \infty.$$

Since $\alpha^* = \arg\min_\alpha G(\alpha, \theta^*)$, we obtain $\frac{\partial^2}{\partial\alpha^2} G(\alpha^*, \theta^*) < 0$.

Next we use a Taylor expansion around θ^* of the function $\frac{\partial}{\partial\alpha}G(\alpha^*,\theta)$:

$$\frac{\partial}{\partial\alpha}G(\alpha^*,\widehat{\theta}^n) \quad = \quad \frac{\partial}{\partial\alpha}G(\alpha^*,\theta^*) + \nabla_\theta\left(\frac{\partial}{\partial\alpha}G(\alpha^*,\widetilde{\theta}^n)\right)(\widehat{\theta}^n - \theta^*)^{\mathrm{T}},$$

where $\widetilde{\theta}^n$ lies componentwise between $\widehat{\theta}^n$ and θ^*. Since $\frac{\partial}{\partial\alpha}G(\alpha^*,\theta^*) = 0$ and

$$\nabla_\theta\left(\frac{\partial}{\partial\alpha}G(\alpha^*,\widetilde{\theta}^n)\right) \xrightarrow{P} \nabla_\theta\left(\frac{\partial}{\partial\alpha}G(\alpha^*,\theta^*)\right),\ n\to\infty,$$

by the continuous mapping theorem we have

$$\sqrt{n}\frac{\partial}{\partial\alpha}G(\alpha^*,\widehat{\theta}^n) \xrightarrow{d} N(0,\nabla_\theta\left(\frac{\partial}{\partial\alpha}G(\alpha^*,\theta^*)\right)\Sigma\nabla_\theta\left(\frac{\partial}{\partial\alpha}G(\alpha^*,\theta^*)\right)^{\mathrm{T}}).$$

Going back to (3.8) we obtain

$$\sqrt{n}\,(\widehat{\alpha}^n - \alpha^*) \xrightarrow{d} N(0,\sigma),\ n\to\infty,$$

where

$$\sigma = \frac{\nabla_\theta\left(\frac{\partial}{\partial\alpha}G(\alpha^*,\theta^*)\right)\Sigma\nabla_\theta\left(\frac{\partial}{\partial\alpha}G(\alpha^*,\theta^*)\right)^{\mathrm{T}}}{\frac{\partial^2}{\partial\alpha^2}G(\alpha^*,\theta^*)}\,. \tag{3.9}$$

□

We return to our original problem for copula parameter estimation of a random vector $X = (X_1,\dots,X_d)$ with arbitrary continuous marginals and elliptical copula, equal to the copula of the elliptical vector $Y = RAU$ as in (2.1). Recall that the parameters of the copula which we are interested in are the correlation matrix $[\rho_{ij}]_{i,j=1,\dots,d}$ and the distribution of the spectral r.v. R. Due to Proposition 2.4, the only parameter in the distribution of R which has significant influence on the joint extremes is the tail index α. Therefore we focus on $[\rho_{ij}]_{i,j=1,\dots,d}$ and α only. Proposition 3.2 suggests the following algorithm for estimation of these parameters.

Algorithm 3.3. *(1) Estimate Kendall's tau matrix by $\widehat{\tau}^n = [\widehat{\tau_{i,j}}^n]_{i,j=1,\dots,d}$ as in (2.4).*

(2) Estimate the correlation matrix by $[\widehat{\rho_{ij}}^n]_{i,j=1,\dots,d}$ using (2.3) and Kendall's tau estimates $\widehat{\tau}^n$.

(3) Estimate the lower tail dependence coefficients by $\widehat{\Lambda}^n = [\widehat{\lambda_{ij}}^n]_{i,j=1,\dots,d}$ as in (2.6), i.e. using only the extreme observations.

(4) Estimate the tail index of the spectral r.v. R by $\widehat{\alpha}^n$ as in (3.4).

(5) In order to quantify the extremal dependence implied by the estimated $\widehat{\tau}^n$ and α, compute the implied tail dependence matrix $L(\widehat{\alpha}^n,\widehat{\tau}^n)$, where $L \in \mathbb{R}^d \times \mathbb{R}^d$ is defined in (3.3).

	mean estimate	true value	m.s.e.	std
α	5.0366	5	22.1086	4.7042
ρ_{12}	0.3001	0.3	0.0010	0.0321
ρ_{13}	0.3969	0.4	0.0009	0.0301
ρ_{23}	0.6002	0.6	0.0005	0.0226
λ_{12}(implied by α)	0.1618	0.1224	0.0092	0.0877
λ_{13}(implied by α)	0.1983	0.1559	0.0108	0.0950
λ_{23}(implied by α)	0.3031	0.2666	0.0125	0.1059

Table 1: Estimation of the t-copula for the model from Example 4.1 with sample size $n = 1000$. The estimators of correlation and tail dependence perform equally well across the different marginals. The correlation estimates are accurate (low empirical standard deviation (std) and mean square errors (m.s.e.)). The estimators of the tail index α of the copula and the tail dependence coefficents have high empirical variance and m.s.e.

4 Numerical examples

In a simulation study we examine the accuracy of the copula estimation when the marginals are heavy-tailed and / or non-symmetric.

Example 4.1. We consider the following model: The random vector (X_1, X_2, X_3) has t-copula with $\nu = 5$ degrees of freedom and a correlation matrix $\rho_{12} = 0.3$, $\rho_{13} = 0.4$, $\rho_{23} = 0.6$. The tail index for this copula is $\alpha = \nu$, see Daul et al. [6], Section 1. For the marginals we use $X_1 \in N(2, 4)$ (normal with mean 2 and variance 4), $X_2 \in GPD(0.3, 10)$ (generalized Pareto distribution with shape parameter $\xi = 0.3$ and scale parameter 10, see Embrechts et al. [8], Definition 3.4.9), and $X_3 \in t(1)$ (t-distribution with $\nu = 1$ degrees of freedom (Cauchy distribution)). Note that the d.f. of X_2 is not symmetric (not elliptical) and the covariance matrix does not exist as $\mathrm{var} X_3 = \infty$.

Our goal is to assess the performance of Algorithm 3.3 when applied to this model. For this reason we simulate $n = 1000$ i.i.d copies of the vector (X_1, X_2, X_3) and estimate the copula parameters using the algorithm. The simulation is repeated 1000 times. In Table 1 we summarize the results.

In Figure 1 we present the results on Kendall's tau matrix estimated by (2.4). We observe that the estimation is accurate even in the cases of asymmetric / heavy-tailed marginals, i.e. we obtain narrow confidence bounds and the histograms of the estimates are symmetric around the true values. This accuracy transfers also to the estimated correlation matrix, see Figure 1.

In Figure 2 we present the results on the tail dependence matrix estimated by (2.6). We observe that the estimation is not very accurate (high empirical variance, wide confidence bounds), which is due to the small sample of observations (1000) on which it is based.

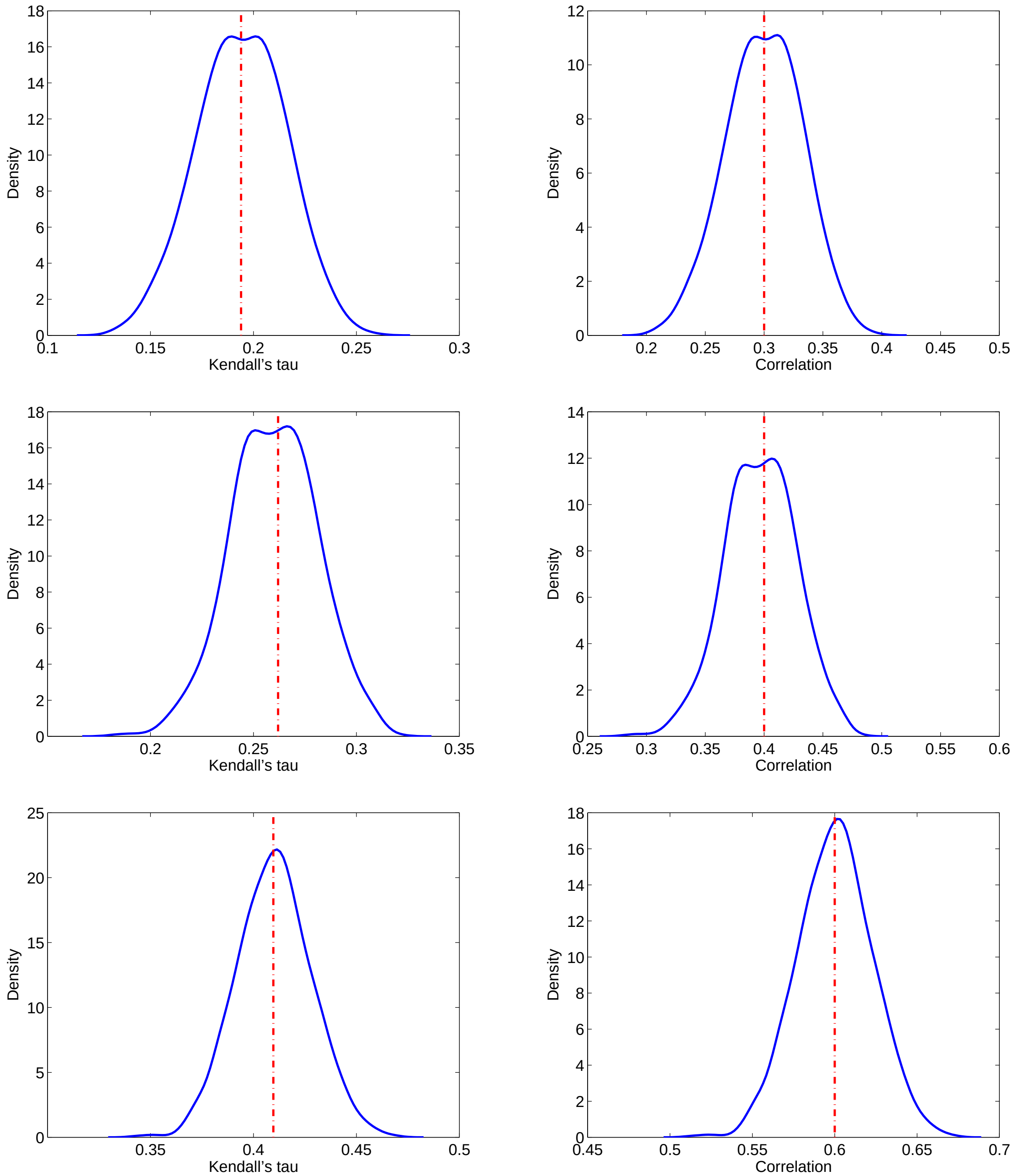

Figure 1: Left column (τ_{12}, τ_{13}, τ_{23}): the histograms of the Kendall's tau estimates compared to the true values for Example 4.1 with sample size $n = 1000$. The histograms are symmetric arround the true values and the empirical variance is low. Right column (ρ_{12}, ρ_{13}, ρ_{23}): the histograms of correlation estimates compared to the true values for Example 4.1 with sample size $n = 1000$. Again, histograms are symmetric arround the true values and the estimation is accurate even in cases when the covariance does not exist (ρ_{13}, ρ_{23}).

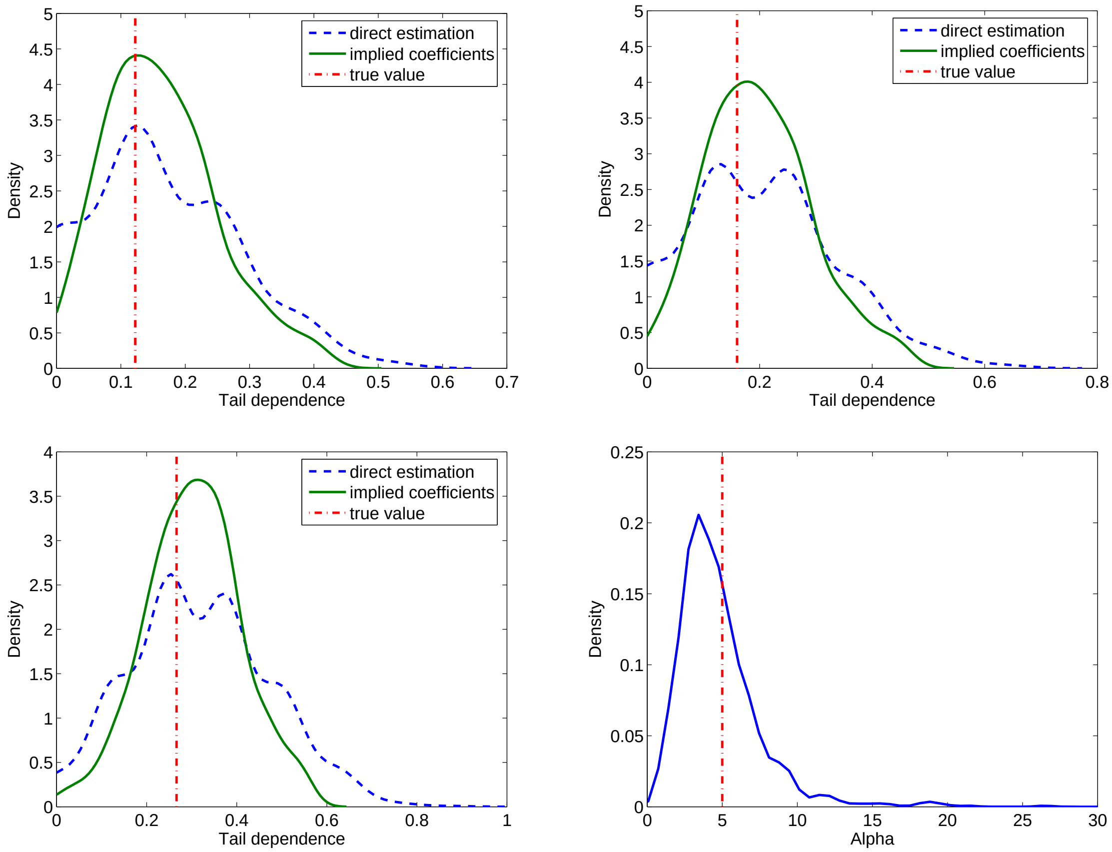

Figure 2: Upper row and bottom row, left (λ_{12}, λ_{13}, λ_{23}): The histograms of the direct estimates for tail dependence by (2.6), compared to the implied (by the estimated α and Kendall's tau) estimates for tail dependence and to the true values for Example 4.1 with sample size $n = 1000$. The implied estimates are more centered around the true values, and have less empirical variance. Bottom row, right: the histogram of the estimates for the tail index α of the copula compared to the true value $\alpha^* = \nu = 5$ for Example 4.1 with sample size $n = 1000$. The histogram is not symmetric around the true value 5. On the other hand, most of its mass is in the region [2,8], which includes the true value 5, and the empirical mean is 5.0003.

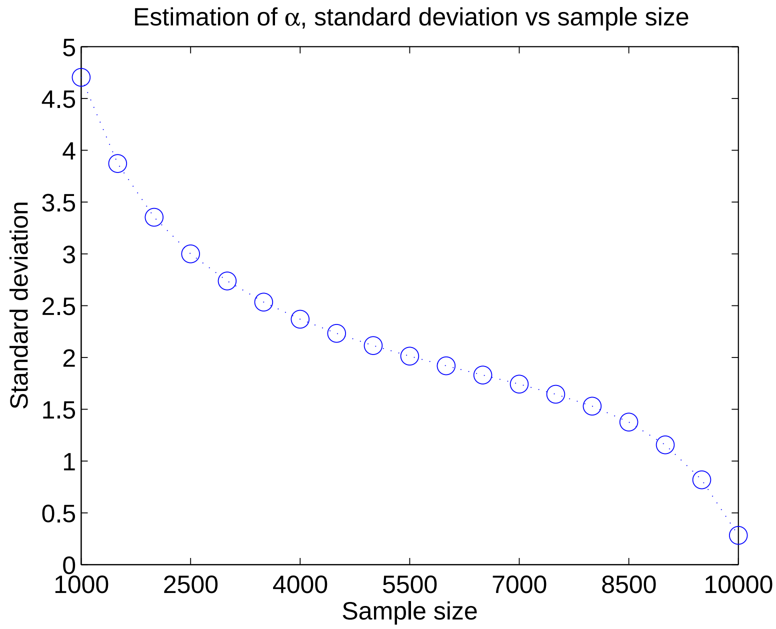

Figure 3: The empirical standard deviation of the estimator (3.4) for the tail index α of the copula in Example 4.1, as a function of the sample size.

Note that (2.6) uses only the extreme observations, which further increases the variance of the estimators (see Frahm et al. [13] for a detailed discussion).

In Figure 2 we present the results on the tail index α estimated by (3.4). Most of the realizations are in the region [2,8] which includes the true value 5. The empirical mean of the estimate is equal to the true value. Furthermore, taking the estimated α and the estimated Kendall's tau matrix, we compute also the implied tail dependence coefficients as in step (5) of Algorithm 3.3. We observe that in this way the estimates are improved as compared to the direct estimates using (2.6). However, the histogram of the estimated α is not centered around its mean, it is very skewed and there are cases when the estimate $\widehat{\alpha}^n$ takes very large values compared to the true one. This makes the asymptotic confidence bounds derived through (3.5) mainly of theoretical interest, at least for smaller samples.

In order to assess the accuracy of the method when applied to larger samples, we increase gradually the sample size to $n = 10000$ and at each step apply Algorithm 3.3. In Figure 3 we observe that the empirical standard deviation of the estimator (3.4) of the tail index α is quite satisfactory at sample size $n = 10000$. $\square$

In the next example we use simulated data to examine the robustness of the estimation when the copula is not elliptical.

Example 4.2. We consider the following model: The marginals of the random vector (X_1, X_2) are standard normal $N(0, 1)$. The copula of (X_1, X_2) is the 2-dimensional Clayton

	mean estimate	true value	m.s.e.	std
τ	0.3336	1/3	0.0003	0.0196
λ	0.4765	0.5	0.0235	0.1549
α	1.6482	1	9.5512	1.6306

Table 2: Application of Algorithm 3.3 to a sample of $n = 1000$ i.i.d. vectors with Clayton copula and standard normal margins as in Example 4.2. The estimates for Kendall's tau are accurate (low empirical standard deviation (std) and mean square error (m.s.e)). The estimates for the tail dependence have high empirical variance. This results in high variance in the estimation of the tail index α. The 'true value' of the copula parameter α is chosen such that the elliptical copula has the same tail dependence as the Clayton copula.

copula

$$C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

with $\theta = 1$.

This copula is not elliptical, and in particular it is not radially symmetric, see Figure 4. It has a tail dependence coefficient $\lambda = 2^{-1/\theta} = 0.5$ and Kendall's tau $\tau = \frac{\theta}{\theta+2} = \frac{1}{3}$. However, all elliptical copulas with Kendall's tau $\tau = \frac{1}{3}$ and tail index $\alpha = 1$ have the same tail dependence coefficient. In Figure 4 we compare $C(p, p)$ with $C_t(p, p)$, a 2-dimensional t-copula with $\nu = \alpha = 1$ degrees of freedom and Kendall's tau $\tau = \frac{1}{3}$. We focus on the small values of p (in the region $[0.0001, 0.025]$). We observe that $C(p, p)$ and $C_t(p, p)$ are practically indistinguishable. Therefore, even if we assume a wrong elliptical copula model, we would obtain comparatively similar results with respect to joint extreme event probabilities, provided that the parameters of the elliptical copula are selected appropriately.

Our goal is to assess the performance of Algorithm 3.3 when applied to data, which is coming from a non-elliptical copula model. For this reason we simulate $n = 1000$ i.i.d copies of (X_1, X_2). We apply Algorithm 3.3 to estimate Kendall's tau, tail dependence and tail index α. We repeat the simulation 1000 times. In Table 2 we summarize the results.

In Figure 5 we present the results on Kendall's tau estimated as in (2.4). We observe that the estimation is accurate (low empirical variance, histogram symmetric arround the true value). We present also the results on the tail dependence coeffient estimated as in (2.6). As in Example 4.1, we observe that the estimation has a rather high empirical variance. Furthermore, the estimator (3.4) of the tail index α inherits the errors from (2.6). The histogram of the estimates of α is not centered around its mean, it is skewed and there are cases when the estimate $\widehat{\alpha}^n$ takes very large values. On the other hand, its mode is exactly equal to 1, and most of the mass is in the region [0,5]. Hence the estimation method seems robust with respect to joint extreme event probabilities. $\square$

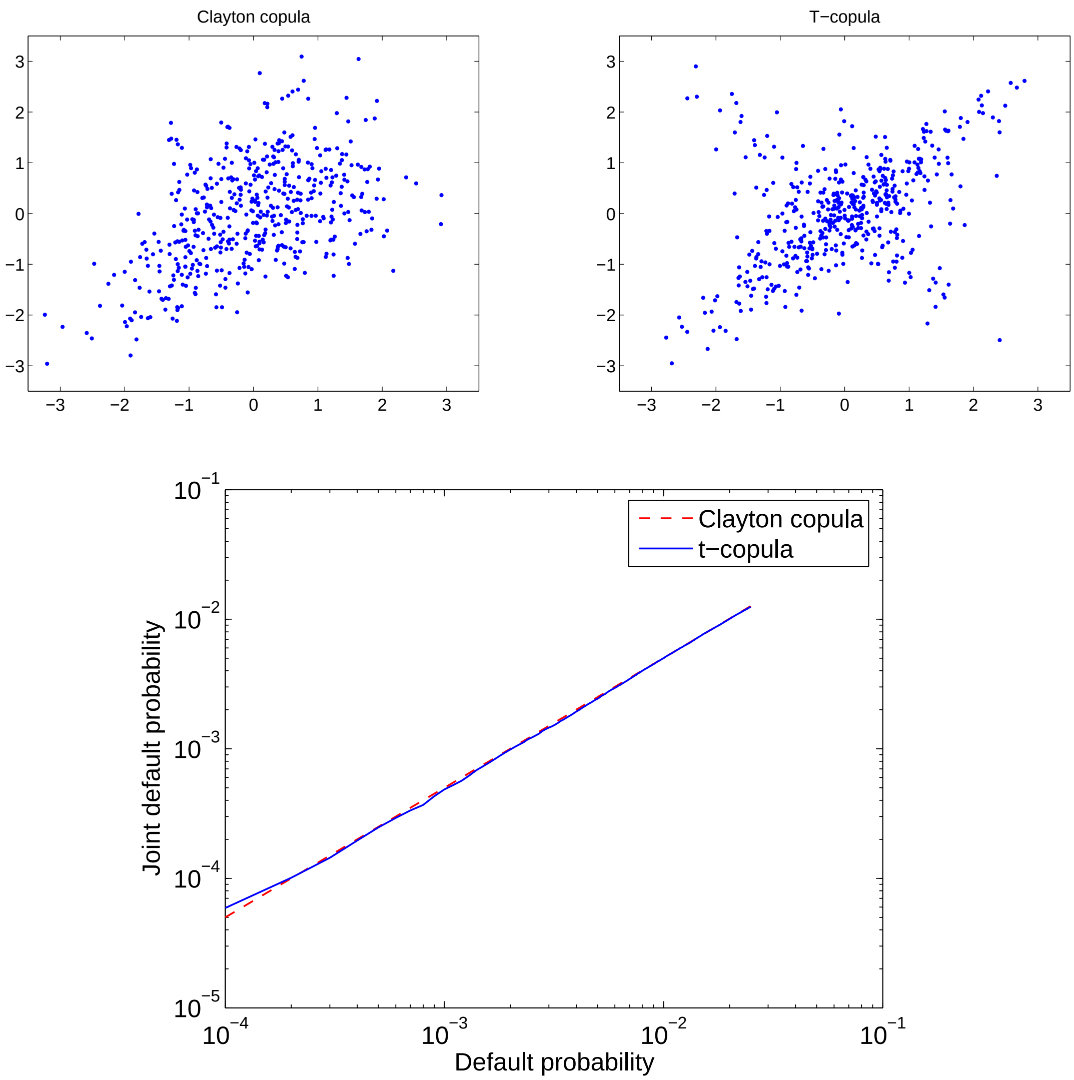

Figure 4: Upper row, left figure: 1000 realizations of the random vector (X_1, X_2) from Example 4.2. Upper row, right figure: 1000 realizations of the random vector (Y_1, Y_2) with standard normal $N(0,1)$ marginals and 2-dimensional t-copula with $\nu = \alpha = 1$ degrees of freedom and Kendall's tau $\tau = \frac{1}{3}$. Both vectors have the same marginals, same Kendall's tau and same tail dependence coefficients. The plots are quite different in the center, i.e. the two copulas are different close to the mean values. Bottom row: $C(p,p) = P(X_1 < \Phi^{-1}(p), X_2 < \Phi^{-1}(p))$ compared to $C_t(p,p) = P(Y_1 < \Phi^{-1}(p), Y_2 < \Phi^{-1}(p))$. The two joint probabilities are practically indistinguishable. This is particularly important in view of the credit risk model in Section 6, where $C(p,p)$ is interpreted as joint default probabiliy of two credits, see formula (6.3).

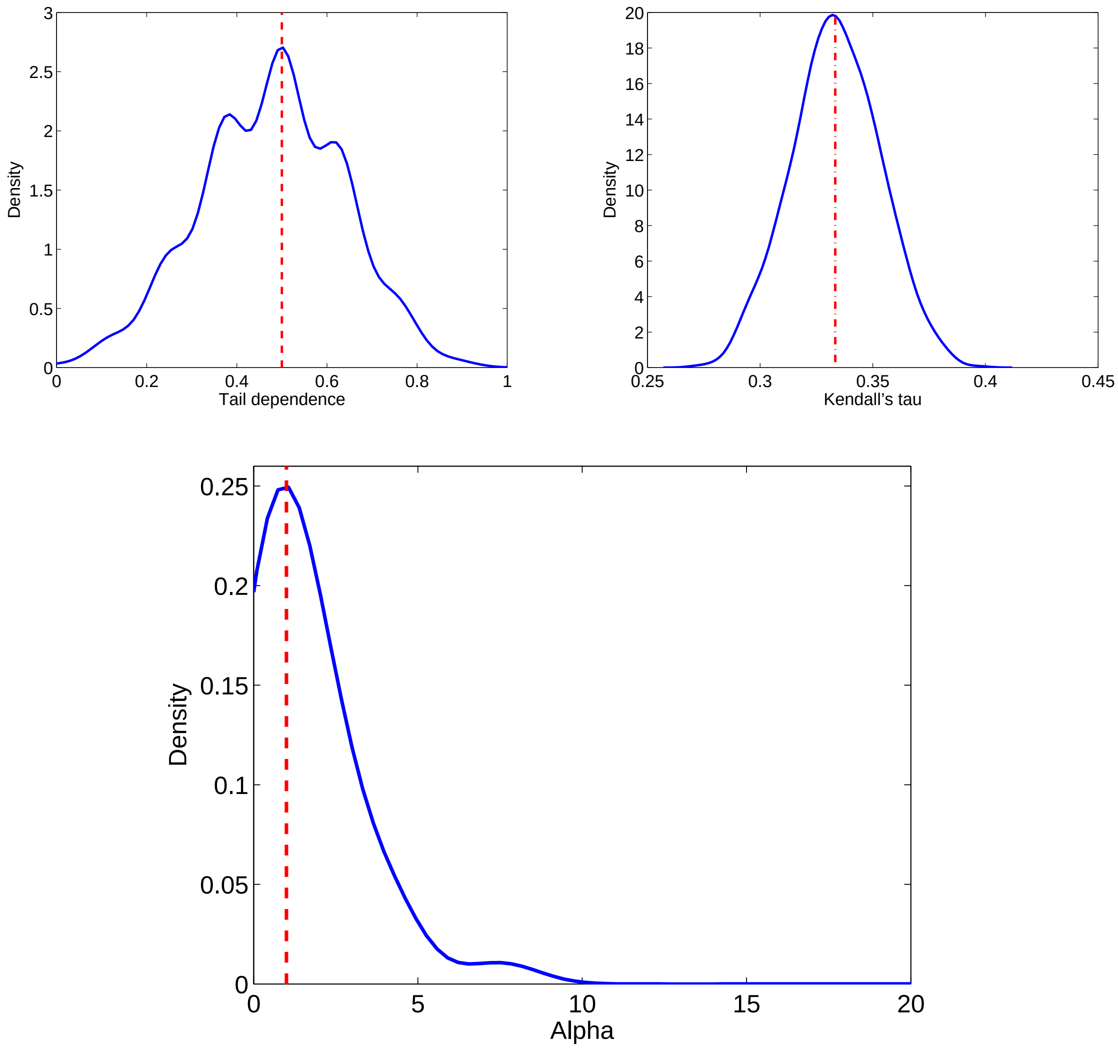

Figure 5: Upper row, left figure: The histogram of the tail dependence estimates for a sample of $n = 1000$ copies of the random vector (X_1, X_2) from Example 4.2. The estimate has high empirical variance. Upper row, right figure: The histogram of Kendall's tau estimates for the same sample. The estimate has low empirical variance. Bottom row: The histogram of the estimated tail index α of the (incorreclty assumed) elliptical copula. The mode is exacly equal to 1, which is the value for which the elliptical copula is closest to the true copula with respect to joint extremes, see Figure 4. Most of the mass is in the region [0,8], however, the histogram is not symmetric.

5 Alternative tail dependence estimation

The examples in the previous section show that the main problem in the proposed method comes from the high variance in the estimation of the tail dependence coefficients by (2.6). This is expected to happen and is due to the small number of observations on which the estimation is based (only the extreme observations are taken into account). In this section we suggest an alternative method for the estimation of the tail dependence coefficients. We prove consistency of the proposed estimator. Note that Proposition 3.2 works for any consistent estimates of the tail dependence coefficients, hence step (3) of Algorithm 3.3 can be applied with any consistent tail dependence estimator.

It is sufficient to consider a 2-dimensional random vector (X_1, X_2) with arbitrary continuous marginals and elliptical copula, equal to the copula of the elliptical random vector $Y = RAU$, where the spectral r.v. R has tail index α. Due to Proposition 2.4, this means that X_1 and X_2 are tail dependent with coefficient which we denote by λ^*.

The tail dependence estimator (2.6) is a simple empirical estimator based on the extreme observations, i.e. denoting

$$U_j^{(k)} = F_j^E(X_j^{(k)}),\ j = 1, 2,\ k = 1, \ldots, n\,, \tag{5.1}$$

where F_j^E is the empirical d.f. of X_j,

$$\widehat{\lambda}^{n,u} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{u}1_{\{U_1^{(k)}<u,\, U_2^{(k)}<u\}}\,,$$

where u is some small threshold. We consider the extremes in a different set and use a weighted empirical estimator with weights proportional to the distance to the diagonal $U_1 = U_2$. To this end, we transform further (U_1^k, U_2^k), $k = 1, \ldots, n$, into polar coordinates

$$U_1^{(k)} = Q^{(k)}\sin\phi^{(k)},\ U_2^{(k)} = Q^{(k)}\cos\phi^{(k)},\ k = 1, \ldots, n, \tag{5.2}$$

where the r.v.s $(Q^{(k)}, \phi^{(k)})$ satisfy $0 \le Q^{(k)} \le 1$ and $0 \le \phi^{(k)} \le \frac{\pi}{2}$. Then we select a small $0 < r < 1$ and we suggest the following estimator

$$\widetilde{\lambda}^{n,r} = \frac{1}{n}\sum_{k=1}^{n}\frac{\sqrt{2}}{r}1_{\{Q^{(k)}<r\}}\sin(2\phi^{(k)}). \tag{5.3}$$

In the following proposition we prove consistency of estimator (5.3).

Proposition 5.1. *For $n \in \mathbb{N}$ let $X^{(k)} = (X_1^{(k)}, X_2^{(k)})$, $k = 1, \ldots, n$, be a sequence of i.i.d random vectors with elliptical copula C and arbitrary continuous marginals F_1, F_2. Let also $(Q^{(k)}, \phi^{(k)})$, $k = 1, \ldots, n$, be the r.v.s in (5.2). Let $s = s(n)$ be a sequence of positive constants such that $s(n) \to \infty$ and $\frac{s(n)}{n} \to 0$, $n \to \infty$. Then*

$$\widetilde{\lambda}^{n,\,s/n} \xrightarrow{P} \lambda^*,\ n \to \infty, \tag{5.4}$$

where λ^* is the tail dependence coefficient of (X_1, X_2).

Proof. Denote by

$$\overline{U}_j^{(k)} \quad = \quad F_j(X_j^{(k)}),\ j = 1,2,\ k = 1,\ldots,n\,,$$

where F_j is the true d.f. of Y_j, and the corresponding $(\overline{Q}^{(k)}, \overline{\phi}^{(k)})_{k=1,\ldots,n}$ as in (5.2) and note that this is an i.i.d sequence, since $(X_1^{(k)}, X_2^{(k)})_{k=1,\ldots,n}$ is i.i.d. Denote also by $\overline{\lambda}^{n,s/n}$ the corresponding r.v. as in (5.3). Since $F_j,\ j = 1,2$, is a monotone function, we have for the copula $C_{\overline{U}_1^{(k)},\overline{U}_2^{(k)}} = C,\ k = 1,\ldots,n$. Fix $k \in \{1,\ldots,n\}$ and denote $Q = \overline{Q}^{(k)}$, $\phi = \overline{\phi}^{(k)}$, $U_j = \overline{U}_j^{(k)}$, $j = 1,2$. We have

$$\begin{aligned} 1_{\{Q<r\}}\sin(2\phi) &= 1_{\{Q<r\}}2\sin\phi\cos\phi \\ &= 1_{\{Q<r\}}\frac{2(Q\sin\phi)(Q\cos\phi)}{Q^2} \\ &\stackrel{d}{=} 1_{\{\sqrt{U_1^2+U_2^2}<r\}}\frac{2U_1U_2}{U_1^2+U_2^2}\,. \end{aligned}$$

Therefore

$$E[1_{\{Q<r\}}\sin(2\phi)] \quad = \quad \int\int_{D=\{\sqrt{u_1^2+u_2^2}<r\}}\frac{2u_1u_2}{u_1^2+u_2^2}dC(u_1,u_2).$$

Changing variables by $z = \sqrt{u_1^2+u_2^2}$, $t = \sqrt{2u_1u_2}$ we obtain

$$E[1_{\{Q<r\}}\sin(2\phi)] = \int_0^r\int_0^r\frac{t^2}{z^2}dC\left(\frac{\sqrt{z^2+t^2}+\sqrt{z^2-t^2}}{2},\frac{\sqrt{z^2+t^2}-\sqrt{z^2-t^2}}{2}\right)$$

Therefore, applying L'Hopital's rule

$$\lim_{r\to 0}\frac{\sqrt{2}}{r}E[1_{\{R<r\}}\sin(2\phi)] = \lim_{r\to 0}\frac{\partial}{\partial u_1}C(u_1,u_2)_{|u_1=u_2=r} + \lim_{r\to 0}\frac{\partial}{\partial u_2}C(u_1,u_2)_{|u_1=u_2=r}\,. \quad (5.5)$$

However, by definition,

$$\lambda^* = \lim_{u\to 0}\frac{C(u,u)}{u} = \lim_{r\to 0}\frac{\partial}{\partial u_1}C(u_1,u_2)_{|u_1=u_2=r} + \lim_{r\to 0}\frac{\partial}{\partial u_2}C(u_1,u_2)_{|u_1=u_2=r}.$$

Then (5.4) follows from Chebishev's inequality. More precisely, we have for every $\epsilon > 0$,

$$P\left(\left|\widetilde{\lambda}^{n,s/n} - \lambda^*\right| > \epsilon\right) \quad \le \quad \frac{1}{\epsilon}\left(E\left|\widetilde{\lambda}^{n,s/n} - \overline{\lambda}^{n,s/n}\right| + E\left|\overline{\lambda}^{n,s/n} - E\overline{\lambda}^{n,s/n}\right| + E\left|E\overline{\lambda}^{n,s/n} - \lambda^*\right|\right)\,.$$

From (5.5) we have that $E\overline{\lambda}^{n,s/n} \to \lambda^*$, $n \to \infty$. By the SLLN we have $\overline{\lambda}^{n,s/n} \stackrel{\text{a.s.}}{\to} E\overline{\lambda}^{n,s/n}$, $n \to \infty$. Finally, since $F_j^E(X^{(k)}) \stackrel{\text{a.s.}}{\to} F_j(X^{(k)})$ we have also $(Q^k, \phi^k) \stackrel{\text{a.s.}}{\to} (\overline{Q}^k, \overline{\phi}^k)$, $k = 1,\ldots,n$, and hence $E\left|\widetilde{\lambda}^{n,s/n} - \overline{\lambda}^{n,s/n}\right| \to 0$, $n \to \infty$. This leads to the required result. $\square$

	mean estimate	true value	m.s.e	std
α	4.9488	5	5.8749	2.4293
λ_{12} direct	0.1560	0.1224	0.0151	0.1213
λ_{12} new	0.1440	0.1224	0.0090	0.1022
λ_{12} implied	0.1454	0.1224	0.0052	0.0771
λ_{13} direct	0.1955	0.1559	0.0192	0.1357
λ_{13} new	0.1808	0.1559	0.0108	0.1268
λ_{13} implied	0.1817	0.1559	0.0063	0.0991
λ_{23} direct	0.3145	0.2666	0.0259	0.1536
λ_{23} new	0.2919	0.2666	0.0166	0.1352
λ_{23} implied	0.2874	0.2666	0.0074	0.1238

Table 3: Estimation of the t-copula with different marginals model from Example 4.1, sample size $n = 1000$. We observe that the new method (5.3) for estimation of the tail dependence coefficients improves the empirical variance and mean square error (m.s.e.), as compared to the direct estimator (2.6). This results immediatelly through Algorithm 5.2 in improved estimates of the tail index α (see Table 1 for comparison). Then the implied tail dependence coefficients estimates (step (5) of the Algorithm 5.2) have also quite satisfactory empirical variance and m.s.e.

We consider a modification of Algorith 3.3, where instead of using (2.6) at step (3) we apply the new estimator (5.3).

Algorithm 5.2. *(1) Estimate Kendall's tau matrix by* $\widehat{\tau}^n = [\widehat{\tau_{i,j}}^n]_{i,j=1,\dots,d}$ *as in (2.4).*

(2) Estimate the correlation matrix by $[\widehat{\rho_{ij}}^n]_{i,j=1,\dots,d}$ *using (2.3) and Kendall's tau estimates* $\widehat{\tau}^n$*.*

(3) Estimate the lower tail dependence coefficients by $\widetilde{\Lambda}^n = [\widetilde{\lambda_{ij}}^n]_{i,j=1,\dots,d}$ *as in (5.3), i.e. using only the extreme observations.*

(4) Estimate the tail index of the spectral r.v. R *by* $\widehat{\alpha}^n$ *as in (3.4).*

(5) In order to quantify the extremal dependence implied by the estimated $\widehat{\tau}^n$ *and* α*, compute the implied tail dependence matrix* $L(\widehat{\alpha}^n, \widehat{\tau}^n)$*, where* $L \in \mathbb{R}^d \times \mathbb{R}^d$ *is defined in (3.3).*

Example 5.3. (continuation of Example 4.1) Using the simulated data from Example 4.1, we apply Algorithm 5.2. Recall that we consider samples of size $n = 1000$ i.i.d copies of the vector (X_1, X_2, X_3) with t-copula and various marginals. Since steps (1) and (2) of Algorithms 3.3 and 5.2 are the same, we consider the differences only with respect to the estimated tail dependence coefficients and tail index α.

In Table 3 we summarize the results, see also Figure 6. We observe that the new tail dependence estimator has smaller emprical variance than (2.6). This results immediatelly in improved estimation of the tail index α by (3.4). Furthermore, the histogram of the

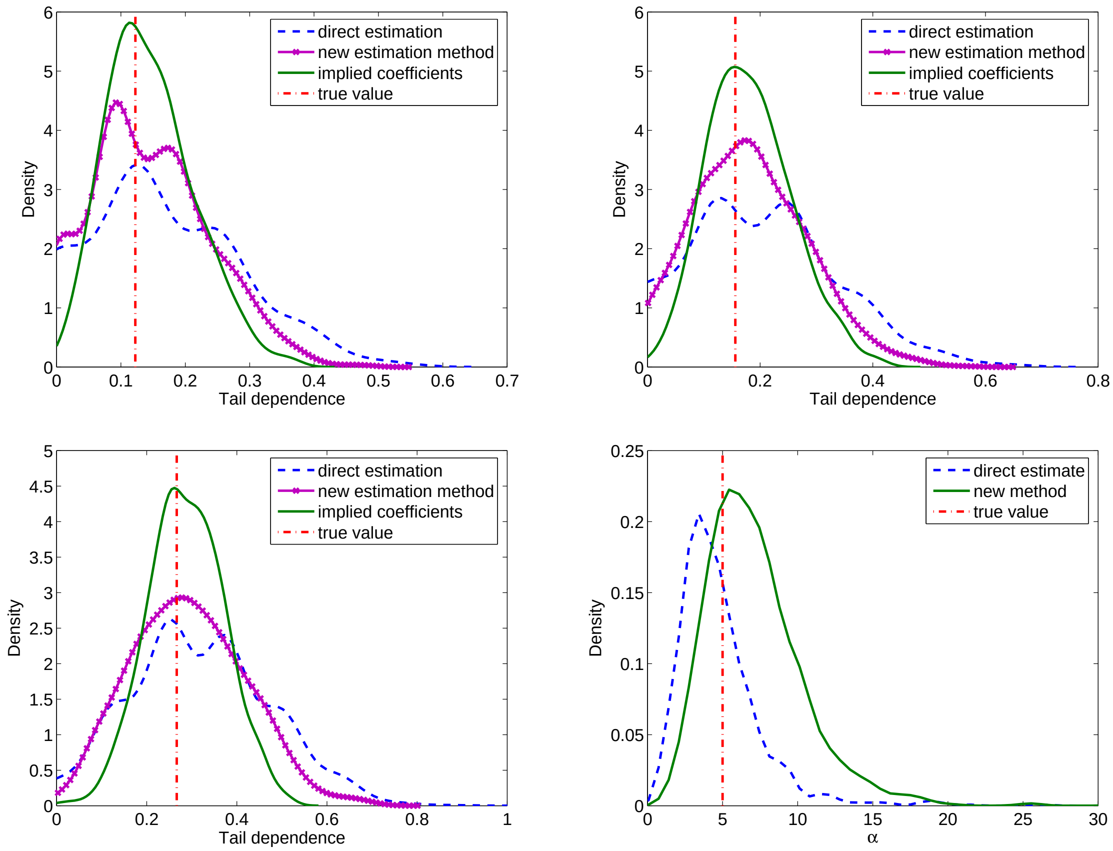

Figure 6: Upper row and bottom row, left (λ_{12}, λ_{13}, λ_{23}): The histograms of the direct estimates of tail dependence by (2.6), compared to the estimates by the new method (5.3) and to the implied (by the estimated Kendall's tau and tail index α by algorithm 5.2) estimates for tail dependence and to the true values. The direct estimates have the highest empirical variance. The new method (5.3) reduces the variance. The implied estimates improve further (5.3). Bottom row, right: the histograms of the estimates for the index α using the direct tail dependece estimates (2.6) as in Algorithm 3.3 and the new method (5.3) as in Algorithm 5.2. By the new method, the histogram is more centered arround the true value.

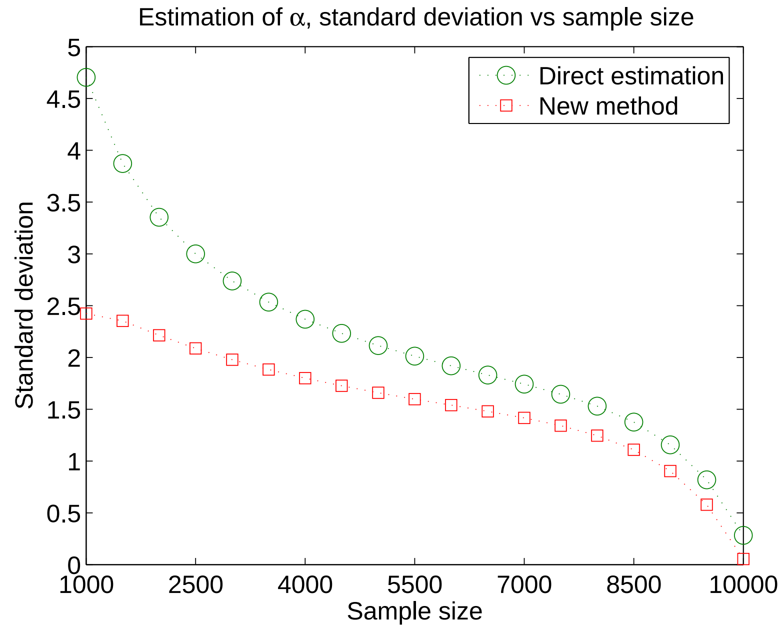

Figure 7: The empirical standard deviation of the estimator (3.4) for the tail index α, as a function of the sample size, using the direct tail dependence estimator (2.6) as in Algorithm 3.3 and the new method (5.3) as in Algorithm 5.2 for the model in Example 4.1. In all cases the new method provides lower empirical variance.

estimated α by Algorithm 5.2 is more centered arround the true value, i.e. applying (3.5) in order to obtain confidence bounds is now possible. Besides, the implied (by the estimated α and Kendall's tau matrix as in step (5) of Algorithm 5.2) tail dependence coefficients have also quite a low empirical variance.

In order to assess the accuracy of the method when applied to larger samples, we increase gradually the sample size to $n = 10000$ and at each step apply Algorithm 5.2 and compare it to Algorithm 3.3. In Figure 7 we observe that the new method reduces the empirical standard deviation of the estimator of the tail index α also for larger samples. $\square$

6 Application to credit risk

We consider a portfolio credit risk model in the spirit of CreditMetrics [19] and investigate the loss distribution over fixed time horizon T. The dependence structure in the portfolio is given through a set of underlying risk factors which we model by a general multivariate elliptical distribution with heavy-tailed marginals, introducing tail-dependence.

For $m \in \mathbb{N}$ let $X = (X_1, \dots, X_m)$ be a random vector with discrete marginals, all having the same range $\{1, 2, \dots, K\}$ – the unknown rating (the credit quality) of the credits at the time horizon T . The loss of a portfolio of m credit risks (loans, bonds or credit derivatives) is modelled by the r.v.

$$L = \sum_{j=1}^{m} e_j L_j, \tag{6.1}$$

where for $j = 1, \dots, m$:

- e_j is a known positive constant: the exposure;

- L_j is a real-valued r.v., defined on the probability space $(\Omega, \mathcal{F}, P(\cdot \mid X_j))$, where $P(\cdot \mid X_j)$ denotes the conditional probability measure: the loss given rating.

Most of the credit risk models used in practice fit within (6.1). For instance, when $K = 2$ (default and non-default rating) and $L_j = \mathrm{I}_{\{X_j=1\}}$, L is the loss of a credit portfolio under the so called 'actuarial valuation' (see Gordy [18], Section 1). With the actuarial valuation one takes care only of the default risk, and the uncertainty in the recovery of a credit in the event of default is ignored. An extension to random recovery rates has been considered by various authors, see for example Bluhm et al. [2], Section 1.1.3. A further extension to multiple ratings is necessary for the so called 'mark-to-market' valuation, see Gordy [18], Section 3, or CreditMetrics [19].

The complexity of model (6.1) is in the joint distribution of $X = (X_1, \dots, X_m)$. Usually the marginals of $(X_1, \dots, X_m)$ are calibrated to historical default and rating transition data, see Lando and Skodeberg [23] and Cantor and Hamilton [4] for some recent methods. We denote these probabilities by $P(X_j = k) = p_{j,k}$ and

$$P\left(X_j \leq s\right) = \sum_{k=1}^{s} p_{j,k} = p_j^s, \;\; s = 1, \dots, K, \; j = 1, \dots, m.$$

In order to model the dependence structure of $X = (X_1, \dots, X_m)$ we introduce the random vector $Y = (Y_1, \dots, Y_m)$ with continuous marginal distributions G_j and a copula C, i.e. the multivariate d.f. of Y is given by

$$G_Y(y_1, \dots, y_m) = C(G_1(y_1), \dots, G_m(y_m)). \tag{6.2}$$

The r.v. Y_j, $j = 1, \dots, m$, is interpreted as asset return (wealth return) of the company standing behind credit j in the portfolio.

Following the approach in CreditMetrics [19], we set for $j = 1, \dots, m$

$$X_j = k \iff G_j^{-1}(p_j^{k-1}) \;\; < \;\; Y_j \;\; \leq \;\; G_j^{-1}(p_j^k), \; k = 1, \dots, K, \tag{6.3}$$

where we interpret $G_j^{-1}(p_j^0) = -\infty$ and $G_j^{-1}(p_j^K) = \infty$.

Thus we reduce the calibration of the distribution of $X = (X_1, \dots, X_m)$ to the calibration of the marginal default and transition probabilities and the copula of $Y = (Y_1, \dots, Y_m)$

(see Frey and McNeil [14], Proposition 3.3). Assuming that the marginal parameters are given, we focus on the dependence structure of the asset returns $Y_1, \ldots, Y_m$.

We assume that $Y = (Y_1, \ldots, Y_m)$ has an elliptical copula. If we had a sample of i.i.d copies $Y^{(k)} = \left(Y_1^{(k)}, \ldots, Y_m^{(k)}\right)$, $k = 1, \ldots, n$, of asset returns, we could apply Algorithm 5.2 and estimate the parameters of the copula. Unfortunately, in practice such sample is not available. Instead, we assume that we observe only $S^{(k,s)} = (S_1^{(k,s)}, \ldots, S_m^{(k,s)}),\ k, s = 1, \ldots, n$, where

$$S_j^{(k,s)} = \operatorname{sign}[Y_j^{(k)} - Y_j^{(s)}]\ , j = 1, \ldots, m,\ k, s = 1, \ldots, n, \tag{6.4}$$

i.e. we observe only whether the asset returns in a given period are higher or lower than the returns in the other periods.

Additionally we have the problem of multidimensionality (m is large in the contemporary credit portfolios). We simplify the situation by assuming that the assets $Y_1, \ldots, Y_m$ follow a factor model:

$$Y_j = \sum_{l=1}^{d} \alpha_{j,l} W Z_l + \sigma_j W \epsilon_j,\ j = 1, \ldots, m, \tag{6.5}$$

where:

- $Z = (Z_1, \ldots, Z_d)$ is d-dimensional multivariate normal with standard normal $N(0,1)$ marginals and with correlation matrix Σ;

- W is a positive r.v., independent of Z;

- ϵ_j, $j = 1, \ldots, m$, are i.i.d $N(0,1)$, independent of W and Z;

- the loadings $\alpha_{j,l} \in \mathbb{R}$ and $\sigma_j > 0$, $j = 1, \ldots, m$, $l = 1, \ldots, d$, are normalized so that $\operatorname{var}[Y_j \,|\, W] = W$.

Given (6.5), $Y \in N_m(0, W\Sigma_Y)$ (multivariate normal variance mixture with mixing variable W or, otherwise, substituting $R = W\sqrt{\chi_m^2}$ in (2.1) we obtain a multivariate elliptical distribution. The marginal distributions of Y_j are all one-dimensional normal variance mixtures, i.e. $Y_j \stackrel{d}{=} W Z_0$, where W is defined as above and $Z_0 \in N(0,1)$, $Z_0 \perp W$. Note that the assumption on the marginal distributions in not restrictive, as we are interested only in the copula of $Y = (Y_1, \ldots, Y_m)$.

The part $\sum_{l=1}^{d} \alpha_{j,l} W Z_l$ of (6.5) is frequently referred to as the systematic part and $\sigma_j W \epsilon_j$ as the the specific part for credit j. Note that, unlike in other models, the specific parts in our case are no longer independent of the systematic part, nor between each other. They are uncorrelated, but depend through the r.v. W. We interpret W as a common shock affecting simultaneously all companies across countries and industries.

For the systematic part, we assume that

$$W Z_l = H_l(I_l),\ l = 1, \ldots, d\,, \tag{6.6}$$

where $H_l : \mathbb{R} \to \mathbb{R}$ are some continuous and strictly increasing functions and $I = (I_1, \dots, I_d)$ are observable macroeconomic factors. In practice, $I = (I_1, \dots, I_d)$ are taken to be the log-returns of regional / industry stock indices. A similar assumption is taken in CreditMetrics [19], see also Daul et al. [6]. Note that, again, we do not impose any further restrictions on the mapping functions H_l, $l = 1, \dots, d$, hence $I = (I_1, \dots, I_d)$ is a random vector with elliptical copula and arbitrary marginals.

Remark 6.1. The simplest special case of (6.5) is the one-factor Gaussian model, obtained when $W = 1$ a.s. and $d = 1$. This model is used extensively in regulatory capital allocation, see BIS [1]. It is calibrated for homogeneous portfolios ($\alpha_{1,1} = \alpha_{2,1} = \ldots = \alpha_{m,1}$) or for portfolios consisting of homogeneous groups by fitting the historical default rate only, see Bluhm et al. [2], Section 2.5.1, Gordy [18], or Frey and McNeil [15]. As we are interested in non-homogeneous portfolios and on measuring the diversification of such into countries / industries, we have $d > 1$ in (6.5) and these methods are not applicable. □

Remark 6.2. The popular in practice model CreditMetrics [19] can be obtained from (6.5) by setting $W = 1$ a.s. However, in this paper we are particularly interested in model (6.5), when W belongs to the class of distributions with regularly varying tail at infinity. In this case we have also that the r.v. $R = W\sqrt{\chi^2_m}$ in representation (2.1) of $Y = (Y_1, \dots, Y_m)$ is also regulary varying with the same index α, see Breiman [3]. By means of Proposition 2.4, only in this case Y_i and Y_j, $i \neq j$, exhibit tail dependence, i.e.

$$\lim_{p \to 0} \frac{P\left(Y_i < G_i^{-1}(p), Y_j < G_j^{-1}(p)\right)}{p} > 0\,.$$

Note that by (6.3) the probability of joint defaults is given by

$$P\left(X_i = 1, X_j = 1\right) = P\left(Y_i < G_i^{-1}(p_i^1), Y_j < G_j^{-1}(p_j^1)\right)$$

and, taking into account that usually the default probabilities p_i^1 and p_j^1 are small, the pairwise tail dependence of assets Y_i and Y_j results in an increased likelihood for simultaneous defaults in the credit portfolio, thus having an important impact on the credit loss distribution, in particular on its tail (see Frey and McNeil [14] for some numerical examples). □

The parameter space of model (6.5) with (6.6) consists of:
(a) The correlation matrix Σ of the common factors $WZ_1, \dots, WZ_d$, which, due to (6.6) is equal to the correlation matrix of the observable market indices $I = (I_1, \dots, I_d)$
(b) The d.f. of W. As seen in Remark 6.2, the only parameter in the d.f. of W with significant influence on the joint extremes of the asset returns $Y_1, \dots, Y_m$ is the tail index α. Due to (6.6) this tail index is equal to the corresponding parameter of the copula of

the observable market indices $I = (I_1, \ldots, I_d)$. From now on we consider the tail index α to be the only parameter in the d.f. of W, and we denote the d.f. of WZ_0, $Z_0 \in N(0,1)$, Z_0 independent of W, by $F_\alpha(x)$, $x \in \mathbb{R}$.
(c) The factor loadings $\alpha_{j,l}$ and σ_j, $l = 1, \ldots, d$, $j = 1, \ldots, m$ as in (6.5). Due to the assumption that $\mathrm{var}[Y_j \,|\, W] = W$ in (6.5), for $j = 1, \ldots, m$ we have the following relation between the loadings:

$$\sum_{l,p=1}^{d} \alpha_{j,l} \Sigma_{lp} \alpha_{j,p} = 1 - \sigma_j^2 \,, \tag{6.7}$$

where $\Sigma = [\Sigma_{lp}]_{l,p=1,\ldots,d}$ is the correlation matrix of $(WZ_1, \ldots, WZ_d)$, i.e. the parameters (a).

In order to estimate the parameters (a) and (b), it is sufficient to apply Algorithm 5.2 to the available data for the market indices $I = (I_1, \ldots, I_d)$. We suggest two principle approaches to estimate the factor loadings (c):

Method I: Estimate $\alpha_{j,l}$, $j = 1, \ldots, m$, $l = 1, \ldots, d$ and then use (6.7) to obtain σ_j, $j = 1, \ldots, m$.

Method II: Assume a special functional form of $\alpha_{j,l}$, $j = 1, \ldots, m$, $l = 1, \ldots, d$, namely

$$\alpha_{j,l} = \frac{\sqrt{1 - \sigma_j^2}}{\sqrt{\sum_{l,p=1}^{d} w_{jl} w_{jp} \Sigma_{lp}}} w_{jl} \,, \;\; j = 1, \ldots, m, \; l = 1, \ldots, d, \tag{6.8}$$

where $w_{j,l} > 0$, $j = 1, \ldots, m$, $l = 1, \ldots, d$, are known quantities. Then we are left only with the parameters σ_j, $j = 1, \ldots, m$, to specify. Note that (6.8) is consistent with (6.7). This method is similar to the approach in CreditMetrics [19], where $w_{j,l} > 0$, $j = 1, \ldots, m$, $l = 1, \ldots, d$, are called country / industry participations.

We need the following proposition, parts (1) and (2) are for Method I and parts (3) and (4) are for Method II. In part (1) we show that $\alpha_{j,l}$, $j = 1, \ldots, m$, $l = 1, \ldots, d$, satisfy a system of linear equations. In part (2) we suggest an estimate for the unknown coefficients of the system. In part (3) we show that σ_j, $j = 1, \ldots, m$, can be expresses as a function of Kendall's tau of the marginal asset return Y_j and a particular transformation of the observable market risk factors $I = (I_1, \ldots, I_d)$. Based on that, in part (4) we suggest an estimate for σ_j, $j = 1, \ldots, m$.

Proposition 6.3. *Let $Y = (Y_1, \ldots, Y_m)$, $S = (S_1, \ldots, S_m)$ and $I = (I_1, \ldots, I_d)$ satisfy (6.4), (6.5) and (6.6) and $\alpha_{j,l}, \sigma_j$, $j = 1, \ldots, m$, $l = 1, \ldots, d$ satisfy (6.7). Then, for $j = 1, \ldots, m$:*

(1) The random vector $(Y_j, \alpha_{j,1} W Z_1, \ldots, \alpha_{j,d} W Z_d, \sigma_j W \epsilon_j)$ is elliptical and we have

$$\sum_{l=1}^{d}\alpha_{j,p}\alpha_{j,l}\Sigma_{lp}=\sin\left(\frac{\pi}{2}\tau(Y_j,I_p)\right),\ p=1,\ldots,d.$$

(2) Denoting

$$\widehat{\tau_{j,p}}^{\,n}=\left(\begin{array}{c} n\\ 2\end{array}\right)^{-1}\sum_{k>s}S_j^{(k,s)}sign\left[I_p^{(k)}-I_p^{(s)}\right],\ p=1,\ldots,d\,,$$

where $I^{(k)}=(I_1^{(k)},\ldots,I_d^{(k)})$, $k=1,\ldots,n$ *are i.i.d copies of* I, *we have*

$$\widehat{\tau_{j,p}}^{\,n}\xrightarrow{P}\tau(Y_j,I_p),\ n\to\infty.$$

(3) If additionally (6.8) holds, then we have

$$\sqrt{1-\sigma_j^2}=\sin\left(\frac{\pi}{2}\tau(Y_j,\,A_j)\right), \tag{6.9}$$

where

$$A_j=\sum_{l=1}^{d}\gamma_{j,l}F_\alpha^{-1}(F_l(I_l))\,,$$

$$\gamma_{j,l}=\frac{w_{jl}}{\sqrt{\sum_{l,p=1}^{d}w_{jl}w_{jp}\Sigma_{lp}}}\,,$$

$F_l(x)$ *is the d.f. of* I_l, $l=1,\ldots,d$, *and* F_α *is the d.f. of* WZ_0, $Z_0\in N(0,1)$, *independent of* W, *where* W *is from (6.5).*

(4) Furthermore, denoting

$$\widehat{\tau_j}^{\,n}=\left(\begin{array}{c} n\\ 2\end{array}\right)^{-1}\sum_{k>s}S_j^{(k,s)}sign\left[\widehat{A}_j^{(k)}(n)-\widehat{A}_j^{(s)}(n)\right],$$

where

$$\widehat{A}_j^{(k)}(n)=\sum_{l=1}^{d}\gamma_{j,l}F_\alpha^{-1}(F_l^E(I_l^{(k)}))\,,\ k=1,\ldots,n,$$

$F_l^E(x)$ *is the empirical d.f. of* I_l, $l=1,\ldots,d$, *we have*

$$\widehat{\tau_j}^{\,n}\xrightarrow{P}\tau\left(Y_j,\,A_j\right)\ ,n\to\infty. \tag{6.10}$$

Proof. Fix $j\in\{1,\ldots,m\}$. By (6.5), the random vector $(Y_j,\alpha_{j,1}WZ_1,\ldots,\alpha_{j,d}WZ_d,\sigma_jW\epsilon_j)$ can be obtained by a linear transformation of the elliptical random vector $(\alpha_{j,1}WZ_1,\ldots,\alpha_{j,d}WZ_d,\sigma_jW\epsilon_j)$. For the correlation we have $\rho(Y_j,\alpha_{j,p}WZ_p)=\sum_{l=1}^{d}\alpha_{j,p}\alpha_{j,l}\Sigma_{lp}$, and, applying (2.3) we obtain

$$\sum_{l=1}^{d}\alpha_{j,p}\alpha_{j,l}\Sigma_{lp}=\sin(\frac{\pi}{2}\tau(Y_j,\alpha_{j,p}WZ_p)),\ p=1,\ldots,d\,.$$

As Kendall's tau is invariant under strictly increasing marginal transformations, we get by means of (6.6) $\tau(Y_j, \alpha_{j,p} W Z_p) = \tau(Y_j, I_p)$, i.e. (1).

(2) follows directly from (1) and (2.4).

To prove (3), we note that the random vector $(Y_j, \sum_{l=1}^d \alpha_{j,l} W Z_l) = (Y_j, \sum_{l=1}^d \frac{\gamma_{j,l}}{\sqrt{1-\sigma_j^2}} W Z_l)$ has elliptical distribution and correlation $\rho(Y_j, \sum_{l=1}^d \alpha_{j,l} W Z_l) = \sqrt{1-\sigma_j^2}$. By means of (2.3),

$$\sqrt{1-\sigma_j^2} = \sin\left(\frac{\pi}{2}\tau\left(Y_j, \sum_{l=1}^d \frac{\gamma_{j,l}}{\sqrt{1-\sigma_j^2}} W Z_l\right)\right).$$

As Kendall's tau is invariant under strictly increasing transformation of the marginals, we have

$$\tau\left(Y_j, \sum_{l=1}^d \frac{\gamma_{j,l}}{\sqrt{1-\sigma_j^2}} W Z_l\right) = \tau\left(Y_j, \sum_{l=1}^d \gamma_{j,l} W Z_l\right).$$

By means of Theorem 3.1. in Embrechts et al. [9], as WZ_l is comonotone with I_l, $l = 1, \dots, d$, we have

$$(WZ_1, \dots, WZ_d) \stackrel{d}{=} (F_\alpha^{-1}(F_1(I_1)), \dots, F_\alpha^{-1}(F_d(I_d))),$$

hence we obtain (6.9).

To prove (6.10) note that for $k, s = 1, \dots, n$

$$\begin{aligned} E\left[S_j^{(k,s)} \operatorname{sign}\left[\widehat{A}_j^{(k)}(n) - \widehat{A}_j^{(s)}(n)\right]\right] &= P((Y_j^{(k)} - Y_j^{(s)})(\widehat{A}_j^{(k)}(n) - \widehat{A}_j^{(s)}(n)) > 0) \\ &\quad - P((Y_j^{(k)} - Y_j^{(s)})(\widehat{A}_j^{(k)}(n) - \widehat{A}_j^{(s)}(n)) < 0). \end{aligned}$$

By the continuous mapping theorem we have for any $k = 1, \dots, n$

$$\widehat{A}_j^{(k)}(n) \xrightarrow{\text{a.s.}} A_j^{(k)}, \; n \to \infty,$$

therefore

$$\lim_{n\to\infty} E\left[S_j^{(k,s)} \operatorname{sign}\left[(\widehat{A}_j^{(k)}(n) - \widehat{A}_j^{(s)}(n))\right]\right] = \tau\left(Y_j, A_j\right).$$

Then (6.10) follows directly from Chebishev's inequality. □

In the final example we consider a real data sample, consisting of monthly log-returns of 8 German stock indices. In view of (6.6), this data represents a sample of i.i.d. copies of the market index vector $I = (I_1, \dots, I_8)$ as in (6.6). We apply Algorithm 5.2 to estimate the parameters of copula of I. For more examples with high-dimensional market risk vectors, as well as for a simulation study of the impact of the heavy-tailed risk factors on credit portfolio risk measures like Value-at-Risk (VaR) see Schwarz [31].

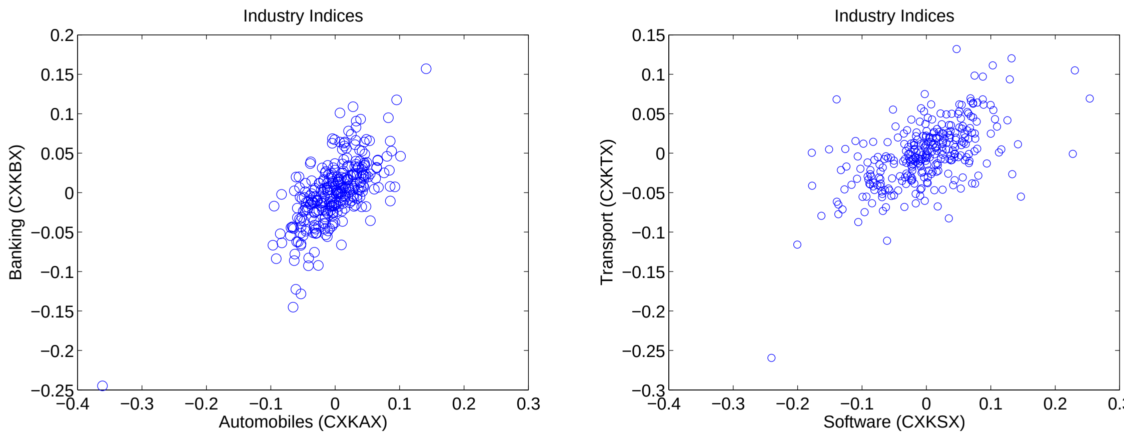

Figure 8: Log-returns of the indices Banking / Automobiles and Software / Transport. In both plots the point in the lower left quadrant represents the returns in the week of the 11/9'th terrorist attacks. However, even if we ignore this extreme point, we observe significant dependence in the extremely small values.

Example 6.4. Our data consists of weekly log-returns of the stock indices: I_1 Automobiles (CXKAX), I_2 Banking (CXKBX), I_3 Chemicals (CXKCX), I_4 Construction (CXKOX), I_5 Insurance (CXPIX), I_6 Media (CXKDX), I_7 Software (CXKSX) and I_8 Transport (CXKTX). By standard time series analysis we conclude that the hypothesis that the data are i.i.d. cannot be rejected at sufficiently high confidence level. Plotting the bivariate marginals (see Figure 8) we detect that there is a significant dependence in the extremes. In order to quantify this dependence we apply the Algorithm 5.2.

In Table 4 we present results on the estimation of Kendall's tau matrix and the correlation matrix. The estimated correlations are positive, and the hypothesis for zero or negative correlation can be rejected with high significance. In Table 5 we present results on the estimation of the tail dependence coefficients. We obtain positive tail dependence estimates. However, due to the small sample size ($n = 300$), we cannot reject the hypothesis for tail-independence at confidence levels higher than 90%. Finally, we estimate the tail index $\widehat{\alpha} = 4.05$ and by means of (3.5) we obtain also a 90% confidence interval $\alpha \in [2.98,\ 5.12]$. □

7 Conclusions

In this paper we estimate the dependence structure of a multivariate random vector regardless of its marginals. We use the class of elliptical copulas, which generalize the standard for the practice Gaussian copula, and provide flexible models for the joint extremes. Our results indicate that:

(1) The elliptical copula model retains some of the advantages of the Gaussian model. In particular, besides the correlation matrix determining the linear dependence, every

	CXKAX	CXKBX	CXKCX	CXKOX	CXPIX	CXKDX	CXKSX	CXKTX
I_1	τ_{ij}	0.47	0.44	0.26	0.36	0.18	0.34	0.44
	90%CI	[.39 .55]	[.34 .54]	[.11 .41]	[.24 .48]	[.04 .32]	[.22 .48]	[.34 .55]
	ρ_{ij}	0.67	0.64	0.40	0.54	0.28	0.52	0.64
	90%CI	[.57 .76]	[.51 .75]	[.18 .60]	[.37 .69]	[.06 .49]	[.34 .68]	[.50 .76]
I_2	0.47	τ_{ij}	0.46	0.27	0.51	0.23	0.38	0.48
	[.39 .55]	90%CI	[.36 .56]	[.11 .44]	[.41 .60]	[.10 .36]	[.26 .50]	[.39 .57]
	0.67	ρ_{ij}	0.66	0.42	0.72	0.35	0.56	0.68
	[.57 .76]	90%CI	[.53 .77]	[.17 .63]	[.60 .81]	[.15 .54]	[.39 .71]	[.57 .78]
I_3	0.44	0.46	τ_{ij}	0.24	0.43	0.17	0.28	0.47
	[.34 .54]	[.36 .56]	90%CI	[.08 .40]	[.32 .53]	[.03 .32]	[.12 .43]	[.38 .57]
	0.64	0.66	ρ_{ij}	0.37	0.62	0.27	0.42	0.69
	[.51 .75]	[.53 .77]	90%CI	[.13 .59]	[.49 .74]	[.05 .48]	[.19 .62]	[.56 .78]
I_4	0.26	0.27	0.24	τ_{ij}	0.28	0.17	0.17	0.30
	[.11 .41]	[.11 .44]	[.08 .40]	90%CI	[.13 .43]	[.02 .32]	[.00 .34]	[.14 .46]
	0.40	0.42	0.37	ρ_{ij}	0.43	0.27	0.26	0.46
	[.18 .60]	[.17 .63]	[.13 .59]	90%CI	[.20 .63]	[.03 .49]	[.00 .50]	[.22 .66]
I_5	0.36	0.51	0.43	0.28	τ_{ij}	0.21	0.35	0.42
	[.24 .48]	[.41 .60]	[.32 .53]	[.13 .43]	90%CI	[.08 .34]	[.22 .48]	[.29 .54]
	0.54	0.72	0.62	0.43	ρ_{ij}	0.33	0.52	0.61
	[.37 .69]	[.60 .81]	[.49 .74]	[.20 .63]	90%CI	[.13 .51]	[.34 .68]	[.45 .75]
I_6	0.18	0.23	0.17	0.17	0.21	τ_{ij}	0.31	0.27
	[.04 .32]	[.10 .36]	[.03 .32]	[.02 .32]	[.08 .34]	90%CI	[.17 .44]	[.15 .40]
	0.28	0.35	0.27	0.27	0.33	ρ_{ij}	0.46	0.42
	[.06 .49]	[.15 .54]	[.05 .48]	[.03 .49]	[.13 .51]	90%CI	[.27 .64]	[.23 .59]
I_7	0.34	0.38	0.28	0.17	0.35	0.31	τ_{ij}	0.40
	[.22 .48]	[.26 .50]	[.12 .43]	[.00 .34]	[.22 .48]	[.17 .44]	90%CI	[.28 0.53]
	0.52	0.56	0.42	0.26	0.52	0.46	ρ_{ij}	0.59
	[.34 .68]	[.39 .71]	[.19 .62]	[.00 .50]	[.34 .68]	[.27 .64]	90%CI	[.42 .74]
I_8	0.44	0.48	0.47	0.30	0.42	0.27	0.40	τ_{ij}
	[.34 .55]	[.39 .57]	[.38 .57]	[.14 .46]	[.29 .54]	[.15 .40]	[.28 .53]	90%CI
	0.64	0.68	0.69	0.46	0.61	0.42	0.59	ρ_{ij}
	[.50 .76]	[.57 .78]	[.56 .78]	[.22 .66]	[.45 .75]	[.23 .59]	[.42 .74]	90%CI

Table 4: Estimation of Kendall's tau and correlation for the German stock index data. In brackets are given the 90% confidence intervals, based on the empirical variance.

	CXKAX	CXKBX	CXKCX	CXKOX	CXPIX	CXKDX	CXKSX	CXKTX
I_1	λ_{ij}	.28	.16	.16	.13	.25	.27	.25
	90%UB	.64	.36	.35	.29	.57	.60	.56
	λ_{ij}^I	.36	.34	.20	.27	.15	.26	.34
I_2	.28	λ_{ij}	.16	.46	.42	.16	.16	.40
	.64	90%UB	.36	.99	.95	.36	.36	.90
	.36	λ_{ij}^I	.35	.20	.40	.18	.28	.37
I_3	.16	.16	λ_{ij}	.16	.13	.16	.25	.25
	.36	.36	90%UB	.36	.29	.36	.58	.58
	.34	.35	λ_{ij}^I	.19	.32	.15	.20	.37
I_4	.16	.46	.16	λ_{ij}	.32	.16	.16	.43
	.35	.99	.36	90%UB	.72	.35	.37	.98
	.20	.20	.19	λ_{ij}^I	.21	.15	.15	.23
I_5	.13	.42	.13	.32	λ_{ij}	.12	.21	.57
	.29	.95	.29	.72	90%UB	.29	.48	.99
	.27	.40	.32	.21	λ_{ij}^I	.17	.26	.31
I_6	.25	.16	.16	.16	.12	λ_{ij}	.31	.31
	.57	.36	.36	.35	.29	90%UB	.71	.72
	.15	.18	.15	.15	.17	λ_{ij}^I	.23	.20
I_7	.27	.16	.25	.16	.21	.31	λ_{ij}	.47
	.60	.36	.58	.37	.48	.71	90%UB	.99
	.26	.28	.20	.15	.26	.23	λ_{ij}^I	.30
I_8	.25	.40	.25	.43	.57	.31	.47	λ_{ij}
	.56	.90	.58	.98	.99	.72	.99	90%UB
	.34	.37	.37	.23	.31	.20	.30	λ_{ij}^I

Table 5: Estimated tail dependence matrix and implied tail dependence matrix (step (5) of Algorithm 5.2) for the German stock indices. In the second rows are given the 90% upper confidence bounds, based on the empirical variance.

elliptical copula has only one important additional parameter (the tail index α, which determines the dependence in the extremes). Our main results in Section 3 concern the estimation of these parameters.

(2) The proposed estimation method is based on the joint extremes in the data. On the one hand, this makes the method robust, see Example 4.2. On the other hand, as any statistical method based on rare events, it is subject to high variance. The way to decrease the variance is to consider all possible sources of infomation. The modification of our main method developed in Section 5 is a step in this direction.

(3) The relative simplicity of the elliptical model makes it applicable to high-dimensional cases which are typical in credit risk, see Section 6.

(4) Real data often exhibits greater dependence in the extremes than the implied dependence by the Gaussian copula, see Example 6.4. Since modelling the dependence structure in a credit portfolio by copulas with tail-dependence property instead by the standard Gaussian copula affects dramatically portfolio risk measures like VaR, accurate and robust in the extremes statistical methods are important for precise estimation of VaR.

Acknowledgement

I take pleasure in thanking my Ph.D. advisor Claudia Klüppelberg for helpful comments and careful proofreading. Christian Schwarz [31] applied the method to multiple examples in higher dimensions.

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