Subexponential distributions - large deviations with applications to insurance and queueing models

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Abstract

We present a fine large deviations theory for heavy-tailed distributions whose tails are heavier than $\exp(-\sqrt{t})$ and have finite second moment. Asymptotics for first passage times are derived. The results are applied to estimate the finite time ruin probabilities in insurance as well as the busy period in a GI/G/1 queueing model.

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1 Introduction

Subexponential distributions are a special class of heavy-tailed distributions which have been prominent in applied probability, whenever samples have to be modelled, where large values appear with non-negligible probability. Such pattern is often seen in insurance data, e.g. in fire, wind-storm or flood insurance, but also in queueing models - in particular in telecommunication data.

First order approximations to ruin probabilities and waiting time distributions have already been derived in the 1970/80s, this is by now folklore found in the relevant textbooks; e.g. Asmussen [2] and Embrechts, Klüppelberg and Mikosch [19]; see also the review papers by Goldie and Klüppelberg [21], Greiner, Jobmann and Klüppelberg [22] and Sigman [28].

The present paper concentrates on more recent research in the area, leading not only to higher order approximations for the ruin probability and the waiting time distribution in a queueing system, but in particular to approximations for the ruin probability in finite time, the queue length and the busy period.

Mathematically, fine large deviations asymptotics are called for. Classical large deviations theory uses a logarithmic approximation which is based on the existence of exponential moments. For subexponential distributions exponential moments do not exist. Consequently, large deviations theory as introduced by Nagaev [26] in 1977 has always been technically very demanding. Nevertheless, in this paper we try to make such large deviations concepts transparent and show some of their applications in the area of insurance and queueing.

The applications we have in mind can be embedded in a random walk setting. For an i.i.d. sequence $(Z_k)_{k\in\mathbb{N}}$ define the random walk

$$S_0 = 0\,, \quad S_n = \sum_{k=1}^{n} Z_k\,, \quad n \in \mathbb{N}\,. \tag{1.1}$$

Throughout we denote by F the distribution function (d.f.) of Z_1 and assume that $EZ_1 = \mu < 0$. Then the random walk $(S_n)_{n\in\mathbb{N}}$ drifts to $-\infty$ and

$$\widetilde{B} = \sum_{n=1}^{\infty} \frac{1}{n} P(S_n > 0) < \infty\,; \tag{1.2}$$

see e.g. Feller [20], Chapter XII.7, Theorem 2. Obviously, $p_n = P(S_n > 0)/n \to 0$ as $n \to \infty$, moreover, in the heavy-tailed case the sequence $(p_n)_{n\in\mathbb{N}}$ decreases like the tail of Z_1. A finer analysis leads to a large deviations result for $(S_n)_{n\geq 0}$ as well as to a result on first passage times of $(S_n)_{n\geq 0}$. These results are applied to derive the asymptotic behaviour of finite time ruin probabilities in insurance as well as that of the busy period in queueing.

The aim of this paper is to show the intrinsic similarity of the problems. Related results and proofs can be found in Baltrūnas [10] and in Baltrūnas, Daley and Klüppelberg [11].

2 Subexponential distributions

We start by recalling some definitions concerning subexponential d.f.s and subexponential sequences. Throughout this section, B denotes the d.f. of a nonnegative random variables (r.v.) X.

Definition 2.1. (a) B *is* subexponential ($B \in \mathcal{S}$) *if*

$$\lim_{t\to\infty} \frac{\overline{B^{2*}}(t)}{\overline{B}(t)} = 2. \tag{2.1}$$

(b) *If B has finite mean, it belongs to the class $\mathcal{S}^*$ if*

$$\lim_{t\to\infty} \int_0^t \frac{\overline{B}(t-u)}{\overline{B}(t)}\, \overline{B}(u)\, du = 2\int_0^\infty \overline{B}(u)\, du. \tag{2.2}$$

As shown in Klüppelberg [24], when $B \in \mathcal{S}^*$ it follows that $B \in \mathcal{S}$ and hence $B \in \mathcal{L}$, i.e.

$$\lim_{x\to\infty} \frac{\overline{B}(x+y)}{\overline{B}(x)} = 1 \quad \text{locally uniformly in } y \in \mathbb{R}. \tag{2.3}$$

A discrete analogue of $\mathcal{S}^*$ is the following class.

Definition 2.2. *The summable nonnegative sequence $(h_n)_{n\geq 0}$ is in the class $\mathcal{S}_D^*$ if both*

$$\lim_{n\to\infty} h_{n+1}/h_n = 1 \tag{2.4}$$

and the terms $h_n^{2\oplus} := \sum_{i=0}^n h_i h_{n-i}$, $n \in \mathbb{N}$, of its second convolution power satisfy

$$\lim_{n\to\infty} h_n^{2\oplus}/h_n = 2\sum_{i=0}^\infty h_i < \infty. \tag{2.5}$$

Lemma 2.3. (Baltrūnas et al. [11])
If the d.f. $B \in \mathcal{S}^*$, then the sequence $(\overline{B}(cn))_{n\in\mathbb{N}} \in \mathcal{S}_D^*$ for every $c > 0$.

The following result plays an important role in our investigations.

Proposition 2.4. (Chover, Ney and Wainger [14]).
Let the probability distribution $(\nu_n)_{n\geq 0}$ with generating function $\hat{\nu}(z) = \sum_{n=1}^\infty \nu_n z^n$, $|z| \leq 1$, satisfy the conditions

(i) $\lim_{n\to\infty} \nu_n^{2\oplus}/\nu_n = c$ *exists and is finite,*

(ii) $\lim_{n\to\infty} \nu_{n+1}/\nu_n = 1/R$ *for some* $1 \le R < \infty$, *and*

(iii) $d = \hat{\nu}(R)$ *is finite.*

Assume that the function $\Psi(w)$ *is analytic in a region containing the range of* $\hat{\nu}(z)$ *for* $|z| \le R$. *Then* $c = 2d$ *and there exists a sequence of real numbers* $\Psi_\nu \equiv ((\Psi_\nu)_n)_{n\ge 0}$ *satisfying*

$$\widehat{\Psi}_\nu(z) \equiv \sum_{n=0}^{\infty} (\Psi_\nu)_n z^n = \Psi\big(\hat{\nu}(z)\big), \qquad |z| \le R, \tag{2.6}$$

and such that

$$\lim_{n\to\infty} (\Psi_\nu)_n/\nu_n = \Psi'(d). \tag{2.7}$$

If in fact $\Psi(w) = \sum_{n=0}^{\infty} c_k w^k$ *for* $|w| \le 1$, *where* $\sum_{n=0}^{\infty} |c_k| < \infty$, *then*

$$\widehat{\Psi}_\nu(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} c_k \nu_n^{k\oplus} \right) z^n. \tag{2.8}$$

As a weak condition in the class $\mathcal{L}$ we require the existence of a density b. Then the *hazard function* $Q = -\log \overline{B}$ has a density $q = b/\overline{B}$, which is called the *hazard rate.* We define the *hazard ratio index*

$$r := \limsup_{t\to\infty} \frac{tq(t)}{Q(t)}. \tag{2.9}$$

The following result provides a simple sufficient condition for $B \in \mathcal{S}^*$. It is the extension of the result which was proved in Lemma 2.3 of Baltrunas [10] for eventually decreasing hazard rate q.

Proposition 2.5. *Suppose that*

(i) $r < 1$,

(ii) *for some* $\varepsilon > 0$ *such that* $r_\varepsilon = r + \varepsilon < 1$ *the function* $\overline{B}^{2-2^{r_\varepsilon}}$ *is integrable over* $\mathbb{R}^+$.

Then $B \in \mathcal{S}^*$.

Proof. For $0 < v < u < t/2$ write

$$\int_0^t \frac{\overline{B}(t-u)}{\overline{B}(t)} \overline{B}(u)\, du \;=\; 2\Big(\int_0^v + \int_v^{t/2} \Big) \frac{\overline{B}(t-u)}{\overline{B}(t)} \overline{B}(u)\, du \;=:\; 2(I + II)\,.$$

Then we estimate

$$II \;=\; \int_v^{t/2} \exp\big(Q(t) - Q(t-u) - Q(u)\big)du\,.$$

As shown in Baltrunas et al. [11] for all $t \geq v$ and $y \geq 1$, $Q(ty) \leq y^{r_\varepsilon} Q(t)$. From this we conclude

$$\begin{aligned}
Q(t) - Q(t-u) &\leq \left(1 - \left(1 - \frac{u}{t}\right)^{r_\varepsilon}\right) Q(t) \\
&\leq \left(1 - \left(1 - \frac{u}{t}\right)^{r_\varepsilon}\right) \left(\frac{t}{u}\right)^{r_\varepsilon} Q(u) \\
&= \left(t^{r_\varepsilon} - (t-u)^{r_\varepsilon}\right) u^{-r_\varepsilon} Q(u) \\
&\leq \left((2u)^{r_\varepsilon} - u^{r_\varepsilon}\right) u^{-r_\varepsilon} Q(u) \\
&= (2^{r_\varepsilon} - 1) Q(u)\,.
\end{aligned}$$

Hence,

$$II \;\leq\; \int_v^{t/2} \exp((2^{r_\varepsilon} - 2)Q(u))du < \infty\,.$$

The result follows now from dominated convergence and the fact that $B \in \mathcal{L}$. $\square$

3 Large deviations results

In this section we present a large deviations result for the random walk $(S_n)_{n\geq 0}$ whose increments have subexponential right tail. For the situation of nonnegative r.v.s $(X_k)_{k\in\mathbb{N}}$ with d.f. B as in the previous section $B \in \mathcal{S}$ is equivalent to

$$P(\sum_{k=1}^n X_k > t) \sim P(\max_{1\leq k\leq n} X_k > t)\,, \quad t \to \infty\,, \quad \forall n \in \mathbb{N}\,.$$

Thus, for any d.f. $B \in \mathcal{S}$ there exists a positive sequence $(t_n)_{n\in\mathbb{N}}$ such that uniformly in $t \in (t_n, \infty)$

$$P(\sum_{k=1}^n X_k > t) \sim P(\max_{1\leq k\leq n} X_k > t)\,, \quad n \to \infty\,.$$

In the investigation of precise large deviations for subexponential distributions the main problem becomes finding the sequences $(t_n)_{n\in\mathbb{N}}$. First results of this kind are due to Nagaev [26]; see also Pinelis [27] and Baltrūnas [9].

Coming back to the situation in the introduction, the increments $(Z_k)_{k\in\mathbb{N}}$ of the random walk $(S_n)_{n\in\mathbb{N}}$ have support $\mathbb{R}$, i.e. are not subexponential in the sense of Definition 2.1.

It does, however, suffice that the right tail of Z_1 is subexponential, in combination with certain conditions on the left tail behaviour, which we formulate now. Define

$$\begin{aligned}\alpha &= \sup\{k : \mathbf{E}[Z_1^k\,;\, Z_1 > 0] < \infty\}\,,\\ \beta &= \sup\{k : \mathbf{E}[|Z_1|^k\,;\, Z_1 < 0] < \infty\}\,.\end{aligned}$$

Denote for the hazard rate index $r < 1$ as defined in (2.9)

$$a(r) = \begin{cases} 2\,, & \text{if } r = 0,\\ 4/(1-r)\,, & \text{if } r \neq 0,\end{cases} \tag{3.1}$$

We require the following conditions:

Conditions C

(i) $r < 1/2$;

(ii) $\liminf_{t\to\infty} tq(t) > a(r)$;

(iii) $\beta > 2$.

Condition (i) is satisfied for all d.f.s whose right tail is heavier than a Weibull tail with exponent 1/2, i.e. $Q(t) = o(1)\sqrt{t}$ as $t \to \infty$. Condition (iii) requires the existence of a finite second moment of the negative part of the increment. Lemma 3.6 of Baltrunas et al. [11] and (ii) imply that $\alpha = \liminf_{t\to\infty} Q(t)/\log t > a(r)$. Hence, (ii) is a moment condition on the positive part of the increment and limits the pathological cases, which have been prominent in the subexponential area. In non-pathological cases (if e.g. $\lim_{t\to\infty} tq(t)$ exists), the case $r \neq 0$ corresponds to d.f.s with moments of all order, hence (ii) is satisfied, whereas such d.f.s with infinite moments correspond to $r = 0$ and then (ii) requires a finite second moment.

With this notation we can formulate the following result, which is an obvious extension of Theorem 4.1 of [11].

Theorem 3.1. (Large deviations property for r.v.s with subexponential tail).
Let $(Z_k)_{k\in\mathbb{N}}$ be i.i.d. r.v.s with d.f. F, hazard function Q and assume that the hazard rate q exists such that the hazard ratio index (2.9) *satisfies $r < 1$. Assume furthermore that conditions $\mathbf{C}$(ii) and (iii) hold. Then for any sequence $(t_n)_{n\in\mathbb{N}}$ satisfying*

$$\limsup_{n\to\infty} \sqrt{n}Q(t_n)/t_n < \infty\,, \tag{3.2}$$

the random walk $(S_n)_{n\in\mathbb{N}}$, satisfies

$$\lim_{n\to\infty} \sup_{t\geq t_n} \left| \frac{P(S_n - \mathbf{E}S_n > t)}{n\overline{F}(t)} - 1 \right| = 0. \tag{3.3}$$

4 Applications

4.1 Transient random walks

Define the first passage time $\widetilde{N}_0$ of the random walk $(S_n)_{n\in\mathbb{N}}$ to $(-\infty, 0]$ by

$$\widetilde{N}_0 = \inf\{n \geq 1 : S_n \leq 0\}\,.$$

Setting

$$m_n = \min_{1\leq k\leq n} S_k\,, \quad n \in \mathbb{N}\,,$$

we have

$$P(\widetilde{N}_0 > n) = P(m_n > 0)\,, \quad n \in \mathbb{N}\,.$$

The key to the next result is the Baxter-Spitzer identity in combination with Proposition 2.4 and Theorem 3.1. The Baxter-Spitzer identity for $\widetilde{N}_0$ is given by

$$\sum_{n=0}^{\infty} z^n P(\widetilde{N}_0 > n) = \exp\left(\sum_{n=1}^{\infty} z^n \frac{P(S_n > 0)}{n}\right)\,, \quad |z| < 1\,.$$

Since $P(S_n > 0) = P(S_n - \mu n > |\mu| n)$, we set $t_n = |\mu| n$. Then condition (3.2) holds for $(t_n)_{n\in\mathbb{N}}$ if $r < 1/2$.

The next result follows from Theorem 3.1.

Proposition 4.1. (Baltrūnas [10])
If conditions **C** *hold, then*

$$\frac{1}{n} P(S_n > 0) \sim P(Z_1 > |\mu| n)\,, \quad n \to \infty\,. \tag{4.1}$$

By Proposition 2.5, under conditions **C**, the right tail $FI_{[0,\infty)}$ of F belongs to $\mathcal{S}^*$. Hence by Lemma 2.3 the sequence $(P(S_n > 0)/n)_{n\in\mathbb{N}}$ belongs to $\mathcal{S}^*_D$. Setting $\Psi(w) = e^w$, Proposition 2.4 implies for $\widetilde{B}$ as defined in (1.2)

$$P(\widetilde{N}_0 > n) \sim e^{\widetilde{B}} \frac{P(S_n > 0)}{n}\,, \quad n \in \mathbb{N}\,. \tag{4.2}$$

From this we obtain the next result.

Proposition 4.2. (Baltrūnas et al. [11])
Assume that conditions **C** *hold. Then*

$$P(\widetilde{N}_0 > n) \sim e^{\widetilde{B}} P(Z_1 > |\mu| n)\,, \quad n \to \infty\,. \tag{4.3}$$

Define the first passage time $\widetilde{T}_x$ to $[x,\infty)$ by

$$\widetilde{T}_x = \inf\{n \in \mathbb{N} : S_n > x\}\,, \quad x \geq 0\,. \tag{4.4}$$

Setting

$$M_n = \max_{1\leq k\leq n} S_k\,, \quad n \in \mathbb{N}\,,$$

we have for all $x \geq 0$

$$P(\widetilde{T}_x > n) = P(M_n \leq x)\,, \quad n \geq 0\,. \tag{4.5}$$

Denote

$$L_0 = 0 \quad \text{and} \quad L_n = \min\{r \geq 0 : S_r = M_n\}\,, \quad n \in \mathbb{N}\,;$$

and set

$$V(x) = \sum_{n=0}^{\infty} P(L_n = n, 0 \leq S_n \leq x)\,, \quad x \geq 0\,,$$

the renewal function of the strict ladder heights of $(S_n)_{n\geq 0}$.

Lemma 4.3. (Baltrūnas [10])
If the sequence $(P(S_n > 0)/n)_{n\in\mathbb{N}}$ belongs to $\mathcal{S}_D^*$, then

$$\lim_{n\to\infty} \frac{P(\infty > \widetilde{T}_x > n)}{P(\infty > \widetilde{T}_0 > n)} = V(x) < \infty \tag{4.6}$$

for each continuity point $x \geq 0$ of V.

Lemma 4.3 shows that it suffices to investigate only the case $x = 0$. We have

$$\sum_{n=0}^{\infty} z^n P(\widetilde{T}_0 = n) = 1 - \exp\left(-\sum_{n=1}^{\infty} z^n \frac{P(S_n > 0)}{n}\right)\,, \quad |z| < 1\,.$$

Using the fact that $(P(S_n > 0)/n)_{n\in\mathbb{N}}$ belongs to $\mathcal{S}_D^*$, setting $\Psi(w) = 1 - e^{-w}$, Proposition 2.4 implies

$$P(\infty > \widetilde{T}_0 > n) \sim e^{-\widetilde{B}} \sum_{k=n+1}^{\infty} \frac{P(S_k > 0)}{k}\,, \quad n \to \infty\,.$$

Combining this with Proposition 4.1 yields the following result, which can be found in Baltrunas [10] for d.f.s with eventually decreasing hazard rates. A careful analysis shows, however, that this condition can be avoided.

Theorem 4.4. *Let $(Z_k)_{k\in\mathbb{N}}$ be i.i.d. r.v.s with d.f. F, hazard function Q and hazard rate q. Assume that conditions* **C** *hold. Set $Z_+ = Z_1 \vee 0$ and*

$$\overline{F}_1(t) = \frac{1}{E[Z_+]}\int_t^\infty \overline{F}(u)du\,, \quad t > 0\,.$$

Then

$$P(\infty > T_0 > n) \sim e^{-\widetilde{B}}\frac{E[Z_+]}{|\mu|}\overline{F}_1(|\mu|n) = \frac{e^{-\widetilde{B}}}{|\mu|}\int_{|\mu n|}^\infty \overline{F}(u)du\,, \quad n \to \infty\,.$$

Using different methods, similar results have been derived by Asmussen; see [1], [2] and [3].

The following applications are based on the discrete skeleton of a Lévy process, which is nothing else but a renewal reward process with positive rewards. Consequently we replace conditions **C** by two separate sets of conditions, one on the renewal process with i.i.d. inter-arrival times $(Y_k)_{k\in\mathbb{N}}$ with d.f. A, and another one on the (independent of $(Y_k)_{k\in\mathbb{N}}$) i.i.d. sequence $(X_k)_{k\in\mathbb{N}}$ of rewards with d.f. B.

Condition A *The inter-arrival time d.f. A is such that for every increasing function $g(n)$ satisfying $g(n)/n \to 0$ and $g(n)/\sqrt{n} \to \infty$ as $n \to \infty$, there is a positive constant c_A and an integer n_g such that for $n \geq n_g$,*

$$P\big(|S_n^Y - nEY| > g(n)\big) \;\leq\; \exp\big(-c_A g^2(n)/n\big). \tag{4.7}$$

Condition A is for instance satisfied when Y has any finite exponential moment.

Conditions B *The reward d.f. B is absolutely continuous with density b so that its hazard function $Q = -\log\overline{B}$ has a hazard rate $q = Q' = b/\overline{B}$ satisfying*

(i) $r := \limsup_{t\to\infty} tq(t)/Q(t) < 1/2$;

(ii) $\liminf_{t\to\infty} tq(t) > a(r)$ *given in (3.1).*

The conditions imply that X_1 has at least a finite second moment and that $B \in \mathcal{S}^*$.

The increments of the corresponding random walk are then

$$Z_k = X_k - Y_k\,, \quad k \in \mathbb{N}\,,$$

with d.f.

$$F(t) = \int_0^\infty \overline{B}(t+u)dA(u)\,, \quad t \geq 0\,.$$

Hence for $B \in \mathcal{L}$ we obtain by dominated convergence

$$\overline{F}(t) \sim \overline{B}(t)\,, \quad t \to \infty\,,$$

which implies in particular that $F \in \mathcal{S}$ or $F \in \mathcal{S}^*$, whenever B is.

4.2 Finite time ruin probabilities

We consider the *Sparre-Andersen model*, which is defined by the following quantities:

(i) The claim times constitute a renewal process, i.e. the interclaim times $(T_n)_{n\in\mathbb{N}}$ are i.i.d. r.v.s and we assume that they have finite second moment.

(ii) The claim sizes $(X_k)_{k\in\mathbb{N}}$ are i.i.d. r.v.s with common d.f. B with $EX_1 = m < \infty$. The claim sizes and interclaim times are independent.

(iii) We denote by $R_0 = x$ the initial risk reserve and by $c > 0$ the premium rate. We also assume that $\mu = EX_1 - cET_1 = m - c/\lambda < 0$.

Define the *risk process*

$$R(x,t) = x + ct - \sum_{k=1}^{N(t)} X_k\,, \quad t \geq 0\,, \tag{4.8}$$

where

$$N(t) = \sup\{k \geq 0 : \sum_{i=1}^{k} T_i \leq t\}\,.$$

Defining by R_n the level of the *risk process* immediately after the n-th payoff, the embedded random walk structure becomes visible. Note that

$$R_{n+1} = R_n + cT_{n+1} - X_{n+1}\,, \quad n \geq 0\,. \tag{4.9}$$

Then setting

$$Z_k = X_k - cT_k\,, \quad k \in \mathbb{N}\,,$$

$S_0 = 0$ and $S_n = x - R_n$, $n \in \mathbb{N}$, defines a random walk with mean $\mu < 0$.

Define in this discrete time setting the *ruin time*

$$\widetilde{T}_x = \inf\{n \in \mathbb{N} : R_n < 0\} = \inf\{n \in \mathbb{N} : S_n > x\}\,. \tag{4.10}$$

Then the *ruin probability before the n-th payoff* is given by

$$\Psi(x,n) = P(\widetilde{T}_x \leq n)\,,$$

and the *infinite time ruin probability* is

$$\Psi(x) = \Psi(x,\infty) = P(\widetilde{T}_x < \infty)\,.$$

The following result is an immediate consequence of Theorem 4.4 and the fact that by l'Hospital $\overline{F}_1(t) \sim \overline{B}_1(t)$ as $t \to \infty$.

Note that S^T and S^Y only differ by the factor c, so condition **A** is satisfied for $(Y_k)_{k\in\mathbb{N}}$, whenever it is satisfied for $(T_k)_{k\in\mathbb{N}}$.

Proposition 4.5. (Baltrūnas [10])
Under conditions **A** *and* **B**,

$$P(\infty > \widetilde{T}_x > n) \sim P(\infty > \widetilde{T}_0 > n)V(x) \sim e^{-\widetilde{B}}(m/|\mu|)\overline{B}_1(|\mu|n)V(x)\,, \quad n \to \infty$$

where $V(x)$ *is the renewal function of the strictly increasing ladder heights of* $(S_n)_{n\geq 0}$.

We are interested in the ruin time

$$\tau_x = \inf\{t \geq 0 : R(x,t) < 0\}\,.$$

Using the discrete skeleton above we obtain the following result. We give a proof of this result, since the proof in (Baltrūnas [10]), though correct, is somewhat obscure.

Proposition 4.6. *Under conditions* **A** *and* **B**,

$$P(\infty > \tau_x > t) \sim P(\infty > \tau_0 > t)V(x) \sim \sum_{n=0}^{\infty} \overline{B}_1(|\mu|n)P(N(t) = n)V(x)\,, \quad t \to \infty$$

where $V(x)$ *is the renewal function of the strictly increasing ladder heights of* $(S_n)_{n\geq 0}$.

Proof. We have

$$\mathbf{P}(\infty > \widetilde{T}_0 > N(t) + 1) \leq \mathbf{P}(\infty > \tau_0 > t) \leq \mathbf{P}(\infty > \widetilde{T}_0 > N(t)).$$

Define

$$Z_n = \frac{P(\infty > \widetilde{T}_0 > n)}{\overline{B}_1(|\mu|n)} \to e^{-\widetilde{B}}\frac{m}{|\mu|}\,, \quad n \to \infty\,.$$

Since $(N(t))_{t\geq 0}$ is ergodic with rate $\lambda > 0$, Anscombe's theorem (see e.g. Embrechts et al. [19], Lemma 2.5.8) applies giving

$$Z_{N(t)} = \frac{P(\infty > \widetilde{T}_0 > N(t))}{\overline{B}_1(|\mu|N(t))} \to e^{-\widetilde{B}}\frac{m}{|\mu|}\,, \quad t \to \infty\,.$$

Hence,

$$P(\infty > \widetilde{T}_0 > N(t)) \sim e^{-\widetilde{B}}\frac{m}{|\mu|}\overline{B}_1(|\mu|N(t))\,, \quad t \to \infty\,.$$

Next, we use the fact that the integrated tail distribution B_1 of X_1 is independent of $(N(t))_{t\geq 0}$. We obtain by conditioning

$$\overline{B}_1(|\mu|N(t)) = \sum_{n=0}^{\infty} \overline{B}_1(|\mu|n)P(N(t) = n)\,.$$

Since $B \in \mathcal{L}$, we have that for t large enough

$$\mathbf{P}(\infty > \tau_0 > t) \sim \sum_{n=0}^{\infty} \overline{B}_1(|\mu|n)P(N(t) = n)\,.$$

The left-hand asymptotic equivalence follows from a similar argument and Proposition 4.5.

□

Theorem 4.7. (Baltrūnas [10])
Assume that conditions **A** *and* **B** *are satisfied and that the hazard rate* q_1 *of* B_1 *satisfies* $q_1(t) = O(1)q(t)$*, then*

$$P(\infty > \tau_0 > t) \sim e^{-\widetilde{B}} \frac{m}{|\mu|} \overline{B}_1(|\mu|(\lambda/c)t) = e^{-\widetilde{B}} \frac{m}{|\mu|} \overline{B}_1\left((1 - \frac{EX_1}{cET_1})t \right) , \quad t \to \infty .$$

This section was based on Baltrūnas [10]. Further relevant work in this area can be found in particular in Asmussen's work, also with various collaborators, see [2, 3, 6]. Klüppelberg, Kyprianou and Maller [25] extend the results of [6] to the infinite mean setting within the context of Lévy processes.

4.3 The busy period

In this section we study the busy period $\widetilde{T}$ in a stable GI/G/1 queue under condition **A** on the inter-arrival times represented by a r.v. Y and conditions **B** on the service times represented by the r.v. X and we assume that $EX = \rho EY$ for some $\rho \in (0, 1)$.

Denote the number of customers served in a busy period by

$$\widetilde{N} = \widetilde{N}_0 = \inf\{n \in \mathbb{N} : S_n \leq 0\} ,$$

Proposition 4.1 with (4.2) yields

$$P(S_n > 0) \sim nP(Z_1 > |\mu|n) \sim nP(X_1 > |\mu|n) , \quad n \to \infty .$$

Moreover, the busy period satisfies

$$\widetilde{T} = S^X_{\widetilde{N}} . \tag{4.11}$$

Furthermore, by definition $\widetilde{T}$ is the first index such that $S^X_n \leq S^Y_n$ giving

$$S^Y_{\widetilde{N}-1} < S^X_{\widetilde{N}} = \widetilde{T} \leq S^Y_{\widetilde{N}} .$$

Thus

$$P(\widetilde{T} \geq S^Y_n) = P(\widetilde{N} \geq n + 1) , \quad n \in \mathbb{N} . \tag{4.12}$$

Putting all this together gives the following result.

Theorem 4.8. (Baltrūnas et al. [11])
Under conditions **B**,

$$P(\widetilde{T} \geq S^Y_n) = P(\widetilde{N} \geq n + 1) \sim e^{\widetilde{B}} \overline{B}(|\mu|n) , \quad n \to \infty . \tag{4.13}$$

Now define

$$N(t) = \sup\{k \geq 0 : \sum_{i=1}^{k} Y_i \leq t\}, \quad t \geq 0.$$

Then

$$S^Y_{N(t)} \leq t < S^Y_{N(t)+1}, \quad t \geq 0, \text{ a.s. }. \tag{4.14}$$

Define

$$Z_n = \frac{P(\widetilde{T} \geq S^Y_n)}{P(X_1 > |\mu| n)} \to e^{\widetilde{B}}, \quad n \to \infty.$$

By ergodicity of $(N(t))_{t\geq 0}$ we can apply again Anscombe's theorem and conclude

$$Z_{N(t)} = \frac{P(\widetilde{T} \geq S^Y_{N(t)})}{P(X_1 > |\mu| N(t))} \to e^{\widetilde{B}}, \quad t \to \infty.$$

Since $B \in \mathcal{L}$, we obtain by (4.14)

$$P(\widetilde{T} > t) \sim e^{\widetilde{B}} P(X_1 > |\mu| N(t)) = e^{\widetilde{B}} \sum_{n=0}^{\infty} \overline{B}(|\mu| n) P(N(t) = n), \quad t \to \infty,$$

where the last identity comes from the independence of X_1 and $N(t)$.

Estimating the final sum on the r.h.s. making use of conditions **A** and **B** yields the final result.

Theorem 4.9. *Assume that conditions* **A** *and* **B** *hold and $EX = \rho EY$ for some $\rho \in (0,1)$, then*

$$P(\widetilde{T} > t) \sim e^{\widetilde{B}} P(X > (1-\rho)t), \quad t \to \infty.$$

This section was based on Baltrūnas et al. [11]. Jelenković and Momčilović solved the problem for subexponential service time distributions with log-concave tails and all moments finite. The busy period asymptotics in the realm of regular variation has been solved by de Meyer and Teugels [18], Boxma [12, 13] and Zwart [29]. Daley and collaborators have studied related problems in $GI/G/k$ queues including $k = \infty$, see [15, 16, 17]. Further relevant work in this area can be found in particular in Asmussen's work, also with various collaborators, see [1, 4, 5, 7, 8].

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