Growth estimates for sine-type-functions and applications to Riesz bases of exponentials

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Abstract

We present explicit estimates for the growth of sine-type-functions as well as for the derivatives at their zero sets, thus obtaining explicit constants in a result of Levin. The estimates are then used to derive explicit lower bounds for exponential Riesz bases, as they arise in Avdonin's Theorem on 1/4 in the mean or in a Theorem of Bogmér, Horváth, Joó and Seip. An application is discussed, where knowledge of explicit lower bounds of exponential Riesz bases is desirable.

1 Introduction

Consider the function $\sin(\pi\cdot)$. From the triangle inequality, it follows that

$$|\sin \pi z| \leq e^{\pi|\Im z|} \quad \forall\, z \in \mathbb{C}$$

and

$$\frac{1-e^{-2\pi}}{2} e^{\pi|\Im z|} \leq |\sin \pi z| \quad \forall\, z \in \mathbb{C} : |\Im z| \geq 1.$$

Furthermore, the integers are the zero set of $\sin(\pi\cdot)$, and $(\frac{1}{\sqrt{2\pi}} e^{in(\cdot)})_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(-\pi,\pi)$. Motivated by this, Levin asked for functions having similar properties to the sine function, such that their zero sets give rise to more general bases of exponentials. In [12], he invented sine-type-functions. We have the following

Definition 1.1 (a) An entire function F is of *exponential type at most* σ ($\sigma > 0$), if for any $\varepsilon > 0$ there exists an $A_\varepsilon > 0$ such that

$$|F(z)| \;\leq\; A_\varepsilon \cdot e^{(\sigma+\varepsilon)|z|} \quad \forall\, z \,\in\, \mathbb{C}.$$

If there is $\sigma > 0$, such that F is of exponential type at most σ, then F is called *of exponential type*.

(b) Let $\sigma > 0$. An entire function F of exponential type is called σ-*sine-type-function*, if there are positive constants C_1, C_2 and τ such that

$$C_1 \cdot e^{\sigma|y|} \;\leq\; |F(x+iy)| \;\leq C_2 \cdot e^{\sigma|y|} \quad \forall\; x,\, y \,\in\, \mathbb{R} \,:\, |y| \,\geq\, \tau. \tag{1}$$

We shall say that F has *growth constants* $(C_1,\, C_2,\, \tau)$.

A large class of sine-type-functions was established by Sedleckiǐ [19], see also Young [24, Ch. 4, Sec. 5, Problem 3].

In [12], Levin proved that $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ will be a Schauder basis for $L^2(-\sigma,\sigma)$ if $(\lambda_n)_{n\in\mathbb{Z}}$ is the zero set of a σ-sine-type-function and if it is separated. In [5], Golovin showed that $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ will then even be a Riesz basis for $L^2(-\sigma,\sigma)$. Here we have the following definitions:

Definition 1.2 A sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers is called *separated*, if there is $\delta > 0$ such that

$$|\lambda_n - \lambda_m| \geq \delta \;\; \forall\, n, m \in \mathbb{Z} : n \neq m.$$

The constant δ is called a *separation constant*.

Definition 1.3 Let $\sigma > 0$ and $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers. Then $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ is a *Riesz basis* for $L^2(-\sigma,\sigma)$, if $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ is complete in $L^2(-\sigma,\sigma)$ and if there are positive constants A and B such that

$$A \sum |c_k|^2 \leq \int_{-\sigma}^{\sigma} \left|\sum c_k\, e^{i\lambda_k x}\right|^2 dx \leq B \sum |c_k|^2 \tag{2}$$

holds for all finite sequences (c_k) of complex scalars. The constants A and B are called *lower* and *upper bounds* of the Riesz basis.

Riesz bases of exponentials allow representations of functions as infinite series. They also give rise to irregular sampling series. For the corresponding truncation error, explicit estimates for the bounds of the Riesz basis are needed. This will be discussed in Section 3.

To obtain the result of Levin and Golovin mentioned above, Levin [12, 13] showed that a sine-type-function has the following properties. The proofs can be found in Levin [12], [13], [14, Lectures 6, 17, 22].

Theorem 1.4 *Let F be a σ-sine-type-function with zero sequence $(\lambda_n)_{n\in\mathbb{Z}}$, counted according to multiplicity, and ordered such that*

$$\Re\lambda_n \leq \Re\lambda_{n+1} \ \forall \, n \in \mathbb{Z}. \tag{3}$$

Then holds:

(a)

$$\exists \, M > 0 : |\lambda_{n+1} - \lambda_n| \leq M \ \ \forall \, n \in \mathbb{Z}. \tag{4}$$

(b) For any positive number r, there exists some positive integer S_r such that for every $t \in \mathbb{R}$,

$$\text{card}\,\{n \in \mathbb{Z} : -r \leq \Re\lambda_n - t < r\} \ \leq \ S_r. \tag{5}$$

(c) If we put

$$G_{\lambda_n}(z) := \begin{cases} 1 - z/\lambda_n, & \text{if } \lambda_n \neq 0 \\ z, & \text{if } \lambda_n = 0, \end{cases}$$

then there is a constant $C_F \neq 0$ such that

$$F(z) = C_F \cdot \lim_{R\to\infty} \prod_{n\in\mathbb{Z}:|\lambda_n|\leq R} G_{\lambda_n}(z) \ \ \forall \, z \in \mathbb{C}. \tag{6}$$

The product converges locally uniformly on $\mathbb{C}$.

(d)

$$\exists \, C_2' > 0 : |F(z)| \leq C_2' \cdot e^{\sigma|\Im z|} \ \ \forall \, z \in \mathbb{C}.$$

(e) For any $\varepsilon > 0$ there exists $m_\varepsilon > 0$ such that

$$|F(z)| \ \geq \ m_\varepsilon \cdot e^{\sigma|\Im z|} \quad \forall \, z \in \mathbb{C} : \text{dist}\,(z, \{\lambda_n : n \in \mathbb{Z}\}) \geq \varepsilon. \tag{7}$$

(f) If $(\lambda_n)_{n\in\mathbb{Z}}$ is separated, then there exist positive constants C_3 and C_4 such that

$$C_3 \ \leq \ |F'(\lambda_n)| \ \leq \ C_4 \quad \forall \, n \in \mathbb{Z}. \tag{8}$$

Properties (a) and (b) refer to the sequence $(\lambda_n)_{n\in\mathbb{Z}}$, and are usually easy to check. We also remark that by the Hadamard Factorization Theorem, the zero-set of any sine-type-function must be countable infinite. Furthermore, the exponential type σ is related with the numbers S_r via the following Lemma by Horváth and Joó [6, Lemma 2]:

Lemma 1.5 *Let F be a σ-sine-type function with zero sequence $(\lambda_n)_{n\in\mathbb{Z}}$. Then it holds*

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{r}\max_{t\in\mathbb{R}}\operatorname{card}\{n\in\mathbb{Z}: -r\le\Re\lambda_n-t<r\} &= \\ \lim_{r\to\infty}\frac{1}{r}\min_{t\in\mathbb{R}}\operatorname{card}\{n\in\mathbb{Z}: -r\le\Re\lambda_n-t<r\} &= \frac{\sigma}{\pi}.\end{aligned}$$

In particular, if S_r denotes the best constant occuring in (5), then

$$\inf_{r>0}\frac{S_r}{2r}=\lim_{r\to\infty}\frac{S_r}{2r}=\frac{\sigma}{\pi}.$$

Suppose that F is a σ-sine-type-function which has growth constants (C_1, C_2, τ). Then it is easy to show that possible choices for C_2' and C_4 are

$$\begin{aligned}C_2' &:= C_2\cdot e^{\sigma\tau},\\ C_4 &:= \frac{2}{\delta}\cdot C_2\cdot e^{\sigma\tau+\sigma\delta/2}\end{aligned}$$

(using the maximum principle for C_2' and Levin's proof for C_4). However, Levin's proof for the existence of the sequence $(m_\varepsilon)_{\varepsilon>0}$ was indirect. He thus obtained no explicit values for m_ε, in terms of the growth constants, of σ and of constants describing the sequence $(\lambda_n)_{n\in\mathbb{Z}}$, e.g. a separation constant or the constants M and S appearing in (a) and (b). Since he derived the existence of C_3 using $m_{\delta/2}$ (where δ is the separation constant), he neither did obtain an explicit expression for C_3. In this paper, we shall obtain explicit estimates for m_ε and for C_3. This will be done in the next section, where we also give examples to discuss the goodness of these estimates.

In [11, Theorem 2], Katsnel'son proved a generalization of Levin and Golovin's Theorem, which was further generalized by Avdonin [1, Theorem 2]. The latter proved, that if the deviation of $(\lambda_n)_{n\in\mathbb{Z}}$ from the zero set of a sine-type-function is less than "1/4 in the mean", then $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2(-\pi,\pi)$. However, while it is easy to give an explicit expression for an upper bound, Avdonin's proof for the lower bound rested on property (f) of Theorem 1.4. Thus, its existence was proved indirectly and he did not obtain an explicit expression for the lower bound, in terms of the restricting data. Using our results on sine-type-functions from Section 2 enables us to give an explicit lower bound for Avdonin's Theorem on 1/4 in the mean. This will be done in Section 3, along with applications for Riesz bases of exponentials and their bounds. We shall also give explicit lower bounds for a Theorem of Bogmér–Horváth–Joó–Seip [2, Theorem 3], [20, Theorem 2.3] and for a Theorem of Duffin–Schaeffer [4, Theorem I]. For special cases, explicit estimates have already been given in [15], [21]. Here, we shall obtain lower bounds for the Theorems in full generality. Lower bounds for the Theorem of Duffin–Schaeffer in full generality (i.e., for complex sequences) have also been obtained by Voß [22, 23].

The motivation for our studies is that for practical applications of bases of exponentials, knowledge of the occuring bounds is essential (cf. Section 3). To obtain such bounds, explicit estimates for sine-type-functions turn out to be a good tool.

It should be noted that our results are aimed to provide first estimates for the occuring theorems. While the derived estimates for the sine-type-function seem to be quite good, our values for Avdonin's theorem are too small to be valuable in practice so that further research would be desirable.

To conclude this section we want to mention the characterization of exponential Riesz bases by Pavlov [18, Theorem 1]:

Theorem 1.6 *Let* $(\lambda_n)_{n\in\mathbb{Z}}$ *be a sequence of complex numbers satisfying*

$$0 < c_1 \le \Im\lambda_n \le c_2 \quad \forall\ n \in \mathbb{Z}$$

with two positive constants c_1 *and* c_2*. Let* $a > 0$*. Then* $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ *forms a Riesz basis for* $L^2(0,a)$ *if and only if* $(\lambda_n)_{n\in\mathbb{Z}}$ *is separated, if*

$$\begin{aligned} \lim_{r\to\infty} \frac{\text{card}\,\{n\in\mathbb{Z} : |\lambda_n| \le r, \Re\lambda_n \ge 0\}}{r} &= \\ \lim_{r\to\infty} \frac{\text{card}\,\{n\in\mathbb{Z} : |\lambda_n| \le r, \Re\lambda_n < 0\}}{r} &= \frac{a}{2\pi}, \end{aligned}$$

$\lim_{r\to\infty}\sum_{|\lambda_n|\le r}\lambda_n^{-1}$ *exists, and the function* $W:\mathbb{R}\to]0,\infty[$*, defined by*

$$W(x) := \left|\lim_{r\to\infty}\prod_{|\lambda_n|\le r}\left(1-\frac{x}{\lambda_n}\right)\right|^2$$

fulfills the (A_2)*-Muckenhoupt-condition, i.e. there exists a constant* C *such that for any bounded interval* I *in* $\mathbb{R}$ *we have*

$$\frac{1}{|I|}\int_I W(x)\,dx \cdot \frac{1}{|I|}\int_I W^{-1}(x)\,dx \;\leq\; C.$$

We note that by a Theorem of Hunt, Muckenhoupt and Wheeden [9], the (A_2)-Muckenhoupt-condition is equivalent to the Helson-Szegö condition. Using this condition, Hruščev [7] could derive Avdonin's Theorem from Pavlov's characterization. A good account of all of this is given in Hruščev, Nikol'skiĭ and Pavlov [8, Sections III.1/2]. For another approach to Pavlov's Theorem, which covers also L^p-spaces, we refer to Lyubarskii and Seip [17]. We mention that explicit constants for Pavlov's characterization have been given in [16, Section 3.4]. However, it is not clear how estimates for Avdonin's Theorem could be obtained using this approach.

It should be noted that, while Pavlov's Theorem gives a full characterization of exponential Riesz bases, it may be hard to apply. In contrast, the conditions of Avdonin are usually easy to check. This is the reason why we concentrate on Avdonin's Theorem and want to derive explicit bounds for this.

The results of this paper are part of the author's doctoral thesis [16, Ch. 2, 3].

2 Explicit growth estimates

The following Theorem gives explicit values for the numbers m_ε, appearing in part (e) of Theorem 1.4.

Theorem 2.1 *Let* $\sigma > 0$ *and let* F *be a* σ*-sine-type-function with growth constants* (C_1, C_2, τ) *and with zero sequence* $(\lambda_n)_{n\in\mathbb{Z}}$*, ordered according to* (3)*. Suppose* (5) *holds for* $r > 0$ *and* $S_r \in \mathbb{N}$*. Define*

$$\tau' := \sup_{k\in\mathbb{Z}} \Im\lambda_k - \inf_{k\in\mathbb{Z}} \Im\lambda_k.$$

Then for any $\varepsilon > 0$ *and any* $z \in \mathbb{C}$ *such that* $\operatorname{dist}(z, \{\lambda_k : k \in \mathbb{Z}\}) = \varepsilon$*, it holds*

$$\begin{aligned} |F(z)| \;&\geq\; C_1 e^{-\sqrt{\tau^2+2\tau\min(2\tau,\tau'+\varepsilon)}\pi S_r/(2r)+\sigma\tau} \\ &\cdot\; \left(1+\frac{\tau^2+2\tau\min(2\tau,\tau'+\varepsilon)}{\varepsilon^2}\right)^{-S_r/2} e^{\sigma|\Im z|}. \end{aligned} \tag{9}$$

In particular, if m_ε is defined by

$$m_\varepsilon := C_1 e^{-\sqrt{5}\pi\tau S_r/(2r)+\sigma\tau}\left(1+\frac{5\tau^2}{\varepsilon^2}\right)^{-S_r/2}, \tag{10}$$

then (7) *holds for any* $\varepsilon > 0$ *with* m_ε *defined as above.*

Proof. Let $z \in \mathbb{C}$ such that $|\Im z| \le \tau$ and $\operatorname{dist}(z, \{\lambda_k : k \in \mathbb{Z}\}) = \varepsilon > 0$. Put

$$\tilde{\tau} := \begin{cases} \tau, & \text{if } \Im z \ge 0, \\ -\tau, & \text{else.} \end{cases}$$

Then $|\Im(z + i\tilde{\tau})| \ge \tau$. Since

$$\left|\frac{G_{\lambda_k}(z+i\tilde{\tau})}{G_{\lambda_k}(z)}\right| = \left|\frac{\lambda_k - z - i\tilde{\tau}}{\lambda_k - z}\right| = \left|1 - \frac{i\tilde{\tau}}{\lambda_k - z}\right|,$$

we obtain from (6)

$$\left|\frac{F(z+i\tilde{\tau})}{F(z)}\right| = \lim_{R\to\infty} \prod_{k\in\mathbb{Z}:|\lambda_k|\le R} \left|1 - \frac{i\tilde{\tau}}{\lambda_k - z}\right|. \tag{11}$$

Furthermore, it holds

$$\left|1 - \frac{i\tilde{\tau}}{\lambda_k - z}\right|^2 = 1 + \frac{\tau^2 - 2\tilde{\tau}\Im(\lambda_k - z)}{|\lambda_k - z|^2} =: 1 + a_k. \tag{12}$$

Thus (11) gives

$$\left|\frac{F(z+i\tilde{\tau})}{F(z)}\right|^2 = \lim_{R\to\infty} \prod_{k\in\mathbb{Z}:|\lambda_k|\le R} (1 + a_k). \tag{13}$$

For $j \in \mathbb{Z}$ define

$$K_j := K_j(z) := \{k \in \mathbb{Z} : r(2j-1) \le \Re\lambda_k - \Re z < r(2j+1)\}.$$

By (5),

$$|K_j| \le S_r \quad \forall\, j \in \mathbb{Z}. \tag{14}$$

Furthermore, we have $|\lambda_k - z| \ge r(2|j|-1)$ for $k \in K_j$. Thus, for $k \in K_j$ and $|j| \ge 1$,

$$|a_k| \le \frac{\tau^2 + 2\tau \min(2\tau, \tau' + \varepsilon)}{r^2(2|j|-1)^2}. \tag{15}$$

Recalling $\cos \pi w = \prod_{j=1}^{\infty}(1-4w^2(2j-1)^{-2})$ for $w \in \mathbb{C}$, we obtain, using (14) and (15),

$$\begin{aligned}\prod_{|j|\geq j_1}\prod_{k\in K_j}(1+a_k) &\leq \left(\prod_{j=1}^{\infty}\left(1+\frac{\tau^2+2\tau\min(2\tau,\tau'+\varepsilon)}{r^2(2j-1)^2}\right)\right)^{2S_r} \\ &= \left(\cosh \pi\sqrt{\tau^2+2\tau\min(2\tau,\tau'+\varepsilon)}/(2r)\right)^{2S_r} \\ &\leq e^{\pi\sqrt{\tau^2+2\tau\min(2\tau,\tau'+\varepsilon)}S_r/r}. \qquad (16)\end{aligned}$$

Since

$$\prod_{k\in K_0}(1+a_k) \leq \left(1+\frac{\tau^2+2\tau\min(2\tau,\tau'+\varepsilon)}{\varepsilon^2}\right)^{S_r}, \qquad (17)$$

it follows from (13) that

$$|F(z)| \geq |F(z+i\tilde{\tau})|\left(\prod_{j\in\mathbb{Z}}\prod_{k\in K_j}(1+a_k)\right)^{-1/2}.$$

Using (16) and (1) implies (9) for $|\Im z| \leq \tau$. Since $\pi S_r/(2r) \geq \sigma$ by Lemma 1.5, (1) implies (9) for $|\Im z| \geq \tau$, too. That (7) holds with m_ε as defined by (10) now follows immediately. □

From Lemma 1.5 we know that $\lim_{r\to\infty}\frac{\pi S_r}{2r} = \sigma$. Furthermore, if τ' and ε are small compared to τ, then $\sqrt{\tau^2+2\tau\min(2\tau,\tau'+\varepsilon)} \approx \tau+\tau'+\varepsilon$. Thus, if additionally $\frac{\pi S_r}{2r}$ is close to σ, the exponential term in (9) is close to $e^{-\sigma(\tau'+\varepsilon)}$. However, the larger r the larger S_r will be, so that the non-exponential term in (9) will grow with increasing r. Thus we expect that formula (9) will yield in particular good estimates when already for small r, we have $\frac{\pi S_r}{2r} \approx \sigma$. We illustrate that by a few examples.

Example 2.2 a) Let $F(z) = (\sin \pi z)^n$. Then $\sigma = n\pi$, $S_{1/2} = n$, $\frac{\pi S_{1/2}}{2\cdot 1/2} = \pi n$, and $\tau' = 0$. We are mainly interested in the behavior of (9) for small ε. It is clear that the true asymptotic behavior of $|F(z)|$, as $\varepsilon = \operatorname{dist}(z,\mathbb{Z}) \to 0$, is given by $\pi^n\varepsilon^n$. On the other hand, (9) gives that $|F(z)|$ is asymptotically greater or equal than $C_1\tau^{-1}\varepsilon^n$. For $\tau = 1$, we have $C_1 = \left(\frac{1-e^{-2\pi}}{2}\right)^n$, and the

estimate appears quite good. Also note that in (16) we used for convenience the estimate $\cosh x \le e^x$. For large x, however, we would have $\cosh x \approx e^x/2$, so that with this estimate we would obtain an additional improving factor which is a bit less than $2^{S_{1/2}} = 2^n$. Comparing $(1-e^{-2\pi})^n$ and π^n now is still better.
b) Let $F(z) = (\sin \pi z) z^{n-1} \prod_{k=1}^{n-1}(z-m-k)^{-1}$ for some $n, m \in \mathbb{N}$, $n \ll m$. Then $|F(z)|$ behaves approximately as $m^{-(n-1)}\pi\varepsilon^n$, as $\varepsilon = |z| \to 0$. Now F is a π-sine-type-function, where for $\tau = 1$ we have a constant C_1 which is about $C_1 \approx m^{-(n-1)}/2$. Using (9) with $r = 1/2$ and $S_{1/2} = n$ gives that the asymptotic behavior of $|F(z)|$ as $\varepsilon = |z| \to 0$ is given by $C_1 e^{-(n-1)\tau\pi}\tau^{-n}\varepsilon^n \approx \frac{1}{2} m^{-(n-1)} e^{-(n-1)\pi}\varepsilon^n$ for $\tau = 1$. Thus we see that in that case (9) gives estimates which differ by a factor of about $\frac{1}{2\pi}e^{-(n-1)\pi}$.
c) Let $F(z) = (\sin \pi z)^n (\sin \pi(z - i/2))^n$. Then $S_{1/2} = 2n$, $\frac{\pi S_{1/2}}{2\cdot 1/2} = 2\pi n = \sigma$, and $\tau' = 1/2$. Then $|F(z)|$ behaves asymptotically as $\left(\frac{e^{\pi/2}-e^{-\pi/2}}{2}\right)^n \pi^n \varepsilon^n$, as $\varepsilon = \operatorname{dist}(z, \mathbb{Z} \cup (\frac{i}{2} + \mathbb{Z})) \to 0$. However, (9) gives only that $|F(z)|$ is asymptotically greater or equal than $C_1 e^{-(\sqrt{2}-1)\tau 2\pi n}(\sqrt{2}\tau)^{-2n}\varepsilon^{2n}$, where $C_1 = \left(\frac{1-e^{-2\pi}}{2}\frac{1-e^{-\pi}}{2}e^{-\pi/2}\right)^n$ and $\tau = 1$. The worst shortcoming of this result is of course that it behaves like a constant times ε^{2n} as $\varepsilon \to 0$, whereas the true behavior is like a constant times ε^n. This is because the derivation of (9) is done for quite general F, which does not make use of special structures of certain sine-type-functions.

We can now obtain explicit estimates for the derivate of a sine-type-function at its zero set, as they appear in part (f) of Theorem 1.4.

Theorem 2.3 *Let $\sigma > 0$ and let F be a σ-sine-type-function with growth constants (C_1, C_2, τ). Let $(\lambda_n)_{n\in\mathbb{Z}}$ be its zero sequence, ordered according to* (3) *and separated with separation constant δ. Let* (5) *be fulfilled for some $r > 0$ and $S_r \in \mathbb{N}$. Put*

$$\tau' := \sup_{k\in\mathbb{Z}} \Im\lambda_k - \inf_{k\in\mathbb{Z}} \Im\lambda_k, \tag{18}$$

$$C_3 := C_1 e^{-\sqrt{\tau^2+2\tau\tau'}\pi S_r/(2r)+\sigma\tau}(\tau^2+2\tau\tau')^{-1/2}\left(1+\frac{\tau^2+2\tau\tau'}{\delta^2}\right)^{-(S_r-1)/2} \tag{19}$$

$$C_4 := \frac{2}{\delta}C_2 e^{\sigma\tau+\sigma\delta/2}. \tag{20}$$

Then (8) *holds with* C_3, C_4 *defined as above. If* $\tau' = 0$, *then* $r = \delta/2$ *and* $S_{\delta/2} = 1$ *can be chosen and* (19) *can be replaced by*

$$C_3 := C_1 e^{(\sigma - \pi/\delta)\tau}\tau^{-1}. \tag{21}$$

Proof. Noting that $(\lambda_n)_{n\in\mathbb{Z}}$ is separated by δ, it follows from (12) and (17) in the proof of Theorem 2.3, that (with the notations used there),

$$\lim_{\varepsilon\to 0} \varepsilon^2 \prod_{k\in K_0} (1 + a_k) \leq \left(1 + \frac{\tau^2 + 2\tau\tau'}{\delta^2}\right)^{-(S_r - 1)} (\tau^2 + 2\tau\tau').$$

Then it follows with the same proof as the one of Theorem 2.3, that for $n \in \mathbb{Z}$, it holds

$$|F'(\lambda_n)| = \lim_{z\to\lambda_n} \left|\frac{F(z)}{z - \lambda_n}\right| \geq C_3,$$

with C_3 as defined by (19). That $S_{\delta/2} = 1$ for $\tau' = 0$ and that this implies (21) is clear. To obtain (20), note that by the maximum principle,

$$|F(z)| \leq C_2 e^{\sigma \max(\tau, |\Im z|)} \ \forall\, z \in \mathbb{C}.$$

The claim then follows using the maximum principle again, noting

$$|F'(\lambda_n)| \ \leq \ \max_{|z-\lambda_n| = \delta/2} \left|\frac{F(z)}{z - \lambda_n}\right|. \square$$

Example 2.4 a) Let $F(z) = \sin(\pi z/\delta)$. Then $\sigma = \pi/\delta$, the zeroes $(\lambda_n)_{n\in\mathbb{Z}}$ of F are separated by $\delta > 0$, and for $\tau = 1$ we can choose $C_1 = (1 - e^{-2\pi/\delta})/2$. Thus, (21) gives $|F'(\lambda_n)| \geq C_1\tau^{-1} = (1 - e^{-2\pi/\delta})/2$. This is quite a good estimate for the true value $|F'(\lambda_n)| = \frac{\pi}{\delta}$. Since

$$\lim_{\delta\to\infty} \frac{C_1}{\pi/\delta} = \lim_{\delta\to\infty} \frac{1 - e^{-2\pi/\delta}}{2\pi/\delta} = 1,$$

the estimate is optimal in the limit $\delta \to \infty$.
b) Let $F(z) = \frac{z-\delta}{z-m}\sin \pi z$ for some large integer m and $0 < \delta \ll 1$. Then $\inf_{n\in\mathbb{Z}} |F'(\lambda_n)| \approx |F'(0)| = \frac{\pi\delta}{m}$. The function F is a π-sine-type-function. For $\tau = 1$ we can choose approximately $C_1 \approx \frac{1}{2m}$. Thus (21) gives

$$\inf_{n\in\mathbb{Z}} |F'(\lambda_n)| \geq C_1 e^{\pi(1-1/\delta)} \approx \frac{1}{2m} e^{\pi(1-1/\delta)},$$

which can be quite a bad estimate in comparison with the true value $\frac{\pi\delta}{m}$. Here it is better to use (19) with $r = 1$ and $S_1 = 2$ to obtain

$$\inf_{n\in\mathbb{Z}} |F'(\lambda_n)| \geq C_1 e^{-\pi\tau}\tau^{-1}\left(1+\frac{\tau^2}{\delta^2}\right)^{-1/2} \approx \frac{\delta}{2m}e^{-\pi\tau}\tau^{-2} = \frac{\delta}{2m}e^{-\pi}.$$

In that case, (19) gives an estimate which approximately differs from the true value only by a factor of $\frac{e^{-\pi}}{2\pi}$.
c) For $F(z) = (\sin \pi z/\delta)(\sin \pi(z-i\delta)/\delta)$, it follows $\inf_{n\in\mathbb{Z}} |F'(\lambda_n)| \geq \frac{\pi}{\delta}\frac{e^{\pi}-e^{-\pi}}{2}$. On the other hand, (19) gives for $\tau = 1$, $r = \delta/2$ and $S_r = 2$, that

$$\inf_{n\in\mathbb{Z}} |F'(\lambda_n)| \geq C_1 e^{-(\sqrt{2}-1)2\pi/\delta}(\sqrt{2})^{-1}\left(1+\frac{2}{\delta^2}\right)^{-1/2}.$$

For $\delta \ll 1$, we have $C_1 \approx \left(\frac{1-e^{-2\pi}}{2}\right)^2$ for $\tau = 1$, and thus we have approximately

$$\inf_{n\in\mathbb{Z}} |F'(\lambda_n)| \geq \left(\frac{1-e^{-2\pi}}{2}\right)^2 \frac{\delta}{2}e^{-(\sqrt{2}-1)2\pi/\delta},$$

which is quite a bad estimate in comparison with the true value. The reason is that for $\tau' \neq 0$, already the estimates in Theorem 2.1 were quite bad.

Remark 2.5 Originally, Levin [12] defined σ-sine-type-functions F as entire functions of exponential type, for which constants $C_1, C_2 > 0$ and $H > h > 0$ exist, such that all zeros of F lie in the strip $|\Im z| \leq h$,

$$C_1 e^{\sigma H} \leq |F(x+iH)| \leq C_2 e^{\sigma H} \quad \forall\ x \in \mathbb{R}$$

and

$$\limsup_{y\to+\infty} \frac{\log|F(iy)|}{y} + \limsup_{y\to-\infty} \frac{\log|F(iy)|}{|y|} = 2\sigma.$$

This definition was used by Avdonin [1], too. Putting

$$\begin{aligned} a &:= \limsup_{y\to+\infty} \frac{\log|F(iy)|}{y} \quad \text{and} \\ G(z) &:= F(z)\cdot e^{i(a-\sigma)z}\cdot e^{(a-\sigma)H} \quad \forall\ z \in \mathbb{C}, \end{aligned}$$

one can show (using a result of Levin and the method applied in the proof of Theorem 2.1), that G is a σ-sine-type-function with growth constants $(C_1 b, C_2 e^{2\sigma H}, H)$ in accordance with Definition 1.1, where b is defined by

$$b := e^{-3\pi^2 H^2 S} \cdot \left(1 + \frac{12H^2}{(H-h)^2}\right)^{-S(\sqrt{6}H+1)}$$

(and S fulfills (5)). The zero sets of F and G are the same. Hence it is no restriction to work with the definition of sine-type-functions given by 1.1.

3 Applications to Riesz bases of exponentials

Suppose that $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers such that $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\sigma,\sigma)$ with bounds A and B. Then, there exists a unique biorthogonal system $(h_n)_{n\in\mathbb{Z}}$ to $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ in $L^2(-\sigma,\sigma)$. It is complete in this space, and any $f \in L^2(-\sigma,\sigma)$ can be written as

$$\begin{aligned} f &= \sum_{n\in\mathbb{Z}} < f, e^{i\lambda_n(\cdot)} > h_n \qquad (22)\\ &= \sum_{n\in\mathbb{Z}} < f, h_n > e^{i\lambda_n(\cdot)}, \qquad (23)\end{aligned}$$

where $< f, g > := \int_{-\sigma}^{\sigma} f(x)\overline{g(x)}\,dx$ denotes the inner product in $L^2(-\sigma,\sigma)$ (cf., e.g., Young [24, Ch. 1, Sec. 7]). Define, for $n \in \mathbb{N}$ and $f \in L^2(-\sigma,\sigma)$,

$$\begin{aligned} s_n^{(1)}(f) &:= \sum_{|k|\le n} < f, e^{i\lambda_k(\cdot)} > h_k,\\ s_n^{(2)}(f) &:= \sum_{|k|\le n} < f, h_k > e^{i\lambda_k(\cdot)}.\end{aligned}$$

Then, the *n-th truncation errors* satisfy

$$\|s_n^{(1)}(f) - f\| \le \frac{1}{\sqrt{A}} \left(\sum_{|k|>n} |< f, e^{i\lambda_k(\cdot)} >|^2\right)^{1/2}, \qquad (24)$$

$$\|s_n^{(2)}(f) - f\| \le \sqrt{B} \left(\sum_{|k|>n} |< f, h_k >|^2\right)^{1/2} \qquad (25)$$

(cf. [15, Proposition 2] for (24); (25) follows by duality). Thus, Riesz bases of exponentials allow representations of functions as nonharmonic Fourier series, and for the corresponding truncation errors estimates for the occuring bounds are needed.

Equation (22) also gives rise to an irregular sampling series. For, let PW_σ denote the Paley-Wiener space, consisting of all entire functions of exponential type at most σ, whose restriction to $\mathbb{R}$ belong to $L^2(\mathbb{R})$. Endowed with the $L^2(\mathbb{R})$-norm, PW_σ becomes a Hilbert space. Since the Fourier Laplace transform $\mathcal{F}$, defined by

$$(\mathcal{F}f)(z) := \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} e^{-izt} f(t)\, dt \quad (f \in L^2(-\sigma,\sigma), z \in \mathbb{C}),$$

is an isometry from $L^2(-\sigma,\sigma)$ onto PW_σ by the Paley-Wiener theorem, and since

$$< f, e^{i\lambda_n(\cdot)} >= \sqrt{2\pi}\, (\mathcal{F}f)(\overline{\lambda_n}),$$

(22) is equivalent to

$$F = \sum_{n\in\mathbb{Z}} F(\overline{\lambda_n})\, \sqrt{2\pi}\, \mathcal{F}h_n \ \ \forall\ F \in PW_\sigma. \tag{26}$$

Convergence holds in the PW_σ-norm as well as uniformly on any horizontal strip of finite width (cf. Young [24, Ch. 2, Pt. 2, Sec. 5]). Since (26) recovers any function $F \in PW_\sigma$ from its samples $F(\overline{\lambda_n})$ and since the sample points $(\overline{\lambda_n})_{n\in\mathbb{Z}}$ are (usually) spaced non-equidistantly, (26) is an irregular sampling formula. The corresponding trunction error satisfies

$$\|F - \sum_{|k|\le n} F(\overline{\lambda_k})\, \sqrt{2\pi}\, \mathcal{F}h_n\|_{PW_\sigma} \le \sqrt{\frac{2\pi}{A}} \left(\sum_{|k|>n} |F(\overline{\lambda_k})|^2 \right)^{1/2}. \tag{27}$$

A broad class of exponential Riesz bases is given by Avdonin's Theorem on 1/4 in the mean, considering perturbations of zero sets of sine-type-functions. For its formulation, we need the following

Definition 3.1 A sequence $(\Lambda_j)_{j\in\mathbb{Z}}$ of non-empty, disjoint subintervals of $\mathbb{R}$ is called an *A-partition* of $\mathbb{R}$, if

$$\begin{aligned} \bigcup_{j\in\mathbb{Z}} \Lambda_j &= \mathbb{R}, \\ \sup \Lambda_j &= \inf \Lambda_{j+1} \quad \forall\ j \in \mathbb{Z} \end{aligned}$$

and if the lengths $|\Lambda_j|$ of the intervals are uniformly bounded by some constant l. We shall say, the A-partition is *bounded by* l.

Theorem 3.2 (Avdonin's Theorem on 1/4 in the mean [1, Theorem 2]) . *Let $\sigma > 0$ and $(\lambda_k)_{k\in\mathbb{Z}}$ be the zero set of a σ-sine-type-function, ordered according to* (3)*. Let $(\delta_k)_{k\in\mathbb{Z}}$ be a bounded sequence of complex numbers, such that $(\lambda_k+\delta_k)_{k\in\mathbb{Z}}$ is separated. For an A-partition $(\Lambda_j)_{j\in\mathbb{Z}}$ of $\mathbb{R}$ define the sets K_j ($j \in \mathbb{Z}$) by*

$$K_j := \{k \in \mathbb{Z} : \Re\lambda_k \in \Lambda_j\}.$$

Suppose there is $d \in [0, 1/4[$, such that

$$\left| \sum_{k\in K_j} \Re\delta_k \right| \le d \cdot |\Lambda_j| \quad \forall \; j \in \mathbb{Z}. \tag{28}$$

Then $(e^{i(\lambda_k+\delta_k)\cdot})_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\sigma, \sigma)$.

Note that for $d = 0$ and $\Im\lambda_k = 0$, Avdonin's Theorem reduces to the Theorem of Levin and Golovin. On the other hand, for $\sigma = \pi$, $\Lambda_j := [j - 1/2, j+1/2[$ and $\lambda_k := k$ (i.e. consider the π-sine-type-function $\sin\pi(\cdot)$), (28) reduces to

$$|\Re\delta_k| \le d \;\; \forall \; k \in \mathbb{Z},$$

and we have Kadec's 1/4-theorem [10, Theorem 2].

We have seen that explicit estimates for the occuring bounds of the Riesz bases are important for the truncation errors in (24), (25) and (27). The following Lemma, a version of a Theorem of Plancherel and Pólya, shows that an explicit upper bound for sequences of exponentials is usually easy to obtain. We have

Lemma 3.3 *Let $\sigma > 0$ and $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers, with imaginary parts bounded by some constant τ. Suppose furthermore that $(\lambda_n)_{n\in\mathbb{Z}}$ is separated, with separation constant $\delta > 0$. Then, the right hand side inequality in* (2) *is fulfilled with*

$$B := B(\delta, \tau, \sigma) := \frac{2}{\sigma}(e^{2\sigma(\tau+1)} - 1)\left(\frac{2}{\delta} + 1\right)^2.$$

The proof follows from [15, Lemma 1] and a Theorem of Boas (cf. Young [24, Ch. 4, Theorem 3]).

While it is easy to obtain an explicit upper bound for exponential sequences, Avdonin's proof for the existence of the lower bound was indirect. For, in his proof he constructed a certain 2σ-sine-type-function and then applied property (f) of Theorem 1.4. To obtain an explicit estimate for the lower bound, an explicit estimate for C_3 in (8) would have been needed. Having now obtained such an explicit expression in Theorem 2.3, we can also obtain an explicit lower bound for Avdonin's Theorem on 1/4 in the mean. The proof follows by explicating Avdonin's proof (and that of Katsnel'son [11], which Avdonin's proof is based on) and applying Theorem 2.3. We omit the long and technical details, which are carried out in [16, Ch. 3].

Theorem 3.4 (An explicit lower bound for Avdonin's Theorem). *Let the assumptions of Theorem 3.2 be valid. Let the σ-sine-type-function have growth constants (C_1, C_2, τ), and let its zero set $(\lambda_k)_{k\in\mathbb{Z}}$ fulfill (4) and (5) with some constants M and $S = S_1$ (for $r = 1$). Let $(\delta_k)_{k\in\mathbb{Z}}$ be bounded by some $L \geq 1$, and let $\delta > 0$ be a separation constant for $(\lambda_k + \delta_k)_{k\in\mathbb{Z}}$. Suppose the A-partition $(\Lambda_j)_{j\in\mathbb{Z}}$ is bounded by l. Define*

$$N := 36\tau + 20L + 25M + 12 + l + \frac{64S(L+2M)(L+2M+1)}{1-4d}$$

and

$$\delta' := \min\left\{\delta, \frac{M}{(NS)^3} \cdot 2^{-NS-1}\right\}.$$

Then, an explicit lower bound for the Riesz basis $(e^{i(\lambda_k+\delta_k)\cdot})_{k\in\mathbb{Z}}$ in $L^2(-\sigma,\sigma)$ is given by

$$\begin{aligned} A_{Avd} &= A_{Avd}(L, \delta, M, S, d, l, C_1/C_2, \tau) \\ &:= \left(\frac{C_1}{C_2}\right)^{44} \cdot (\delta')^{73N^2 S} \cdot (8N)^{-234(2N)^{2NS}}. \end{aligned}$$

Remark 3.5 A possible choice for S in Theorem 3.4 is

$$S := \frac{4(2+2L+\delta)(2\tau+2L+\delta)}{\pi\delta^2}.$$

Thus, the lower bound in the Theorem depends only on $L, \delta, M, d, l, C_1/C_2$ and τ.

An important special case of Theorem 3.4 is the following. For the case of real sequences, an explicit lower bound for it has already been given in [15, Theorem 1].

Corollary 3.6 *Let $(\delta_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers, bounded by $L \geq 1$. Suppose $(k+\delta_k)_{k\in\mathbb{Z}}$ is separated, with separation constant $\delta > 0$. Suppose there is $d \in [0, 1/4[$ and a natural number T such that*

$$\left| \sum_{k=jT+1}^{(j+1)T} \Re\delta_k \right| \leq T \cdot d \quad \forall\ j \in \mathbb{Z}.$$

Define

$$N := 73 + 20L + T + \frac{192(L+2)(L+3)}{1-4d}$$

and

$$\delta' := \min\left\{\delta, \frac{1}{(3N)^3} \cdot 2^{-3N-1}\right\}.$$

Then, $(e^{i(k+\delta_k)\cdot})_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$ with lower bound

$$A = A(L,\delta,d,T) := 2^{-88} \cdot (\delta')^{219N^2} \cdot (8N)^{-234(2N)^{6N}}.$$

Proof. Set $\Phi(z) := \sin \pi z$ and $\Lambda_j := [jT + 1/2, (j+1)T + 1/2[$. Then Φ is a π-sine-type-function with growth constants $(1/4, 1, 1)$ and zero set $\mathbb{Z}$. $(\Lambda_j)_{j\in\mathbb{Z}}$ is an A-Partition of $\mathbb{R}$ satisfying $|\Lambda_j| = T \quad \forall\ j \in \mathbb{Z}$ and $\{k \in \mathbb{Z} : k \in \Lambda_j\} = \{jT+1, \ldots, (j+1)T\}$. Putting $M := 1, S := 3, l := T$ und $\sigma := \pi$, the claim follows from Theorem 3.4. □

Using Corollary 3.6, we can obtain an explicit lower bound in a Theorem of Bogmér–Horváth–Joó and Seip [2, Theorem 3], [20, Theorem 2.3]:

Theorem 3.7 (An explicit lower bound in the Theorem of Bogmér–Horváth–Joó and Seip). *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of complex numbers, with separation constant $\delta > 0$. Let $\sigma > 0$, $L \geq 0$, $\gamma > \sigma/\pi$, and suppose*

$$\left|\lambda_n - \frac{n}{\gamma}\right| \leq L \ \ \forall\, n \in \mathbb{Z}.$$

Then there exists a subsequence $(\lambda_{n_k})_{k\in\mathbb{Z}}$ *of* $(\lambda_n)_{n\in\mathbb{Z}}$ *such that* $(e^{i\lambda_{n_k}(\cdot)})_{k\in\mathbb{Z}}$ *is a Riesz basis for* $L^2(-\sigma,\sigma)$ *with lower bound*

$$A_{BHJS}(L,\delta,\sigma,\gamma) := \frac{\sigma}{\pi}(\delta')^{1,3\cdot 10^9 M^6} \cdot M^{-M^{1,3\cdot 10^5 M^3}},$$

where

$$M := \max\left\{4, \left\lceil \frac{3+2\gamma L}{\pi\gamma/\sigma - 1}\right\rceil\right\} + 1, \quad \delta' := \min\left\{\frac{\sigma\delta}{\pi}, 2^{-7070M^3}\right\}.$$

Bogmér–Horváth–Joó and Seip obtained the existence of the subsequence mentioned in Theorem 3.7 using the special case of Avdonin's Theorem considered in Corollary 3.6 (without its explicit lower bound). Using the explicit lower bound in Corollary 3.6 and explicating the proof of Bogmér–Horváth–Joó and Seip then gives the lower bound in the Theorem above. The details are carried out in [16, Satz 3.27]. For the case of real sequences, this has already been done in [15, Remark 3].

Remark 3.8 Theorem 3.7 also gives rise to explicit lower bounds in a Theorem of Duffin and Schaeffer [4, Theorem I]. The latter states that if the assumptions of Theorem 3.7 are valid, then $(e^{i\lambda_n(\cdot)})_{n\in\mathbb{Z}}$ is a *frame* for $L^2(-\sigma,\sigma)$, i.e. there exist positive constants A, B such that

$$A\int_{-\sigma}^{\sigma} |f(x)|^2\,dx \le \sum_{n\in\mathbb{Z}} \left|\int_{-\sigma}^{\sigma} f(x)e^{-i\overline{\lambda_n}x}\,dx\right|^2 \le B\int_{-\sigma}^{\sigma} |f(x)|^2\,dx$$

holds for all $f \in L^2(-\sigma,\sigma)$. While an explicit value for B is given by Lemma 3.3, Duffin and Schaeffer's proof for the existence of A was indirect. However, using the fact that every Riesz basis is a frame with the same constants, it is now easy to see that $A_{\mathrm{BHJS}}(L,\delta,\sigma,\gamma)$ is a possible choice for A. A better estimate for A was found by Voß in [22, Korollar 5.1.3], [23, Corollary 4.1.2]. The considerations of Voß also cover generalizations to L^p-spaces and include derivatives. For the case of Duffin-Schaeffer's Theorem with real sequences, explicit lower bounds have already been given in Lindner [15, Theorem 2] and Voß [21, Corollary to Theorem 1].

It should be noted that the lower bounds obtained in Theorems 3.4, 3.7 and Corollary 3.6 are too small to be useful in practice. For special cases of Avdonin's Theorem, such as Kadec's 1/4-Theorem or the Levin-Golovin

Theorem, better estimates for the lower bounds are known, cf., e.g., Christensen and Lindner [3, Theorem 1.3 and the example in Section 2].

Acknowledgements. The author would like to thank his thesis advisor D. Kölzow for introducing him to the subject, for asking for explicit lower bounds for the Theorems considered in Section 3, and for his steady support. Thanks also to B. Bittner for discussions to O. Christensen for his support, and to an anonymous refeeree for valueable suggestions which helped to improve the results of the paper.

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