Technische Universität München
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Constructing isospectral manifolds

Bachelor Thesis
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I hereby declare that I have written this Bachelor Thesis on my own and have used none other than the stated sources and aids.

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Abstract

Two closed Riemannian manifolds are called isospectral, if the Laplace operators on them have the same set of eigenvalues counted with multiplicities. Riemannian manifolds which are isometric are trivially isospectral and up to date there are only two methods to systematically construct isospectral, but not isometric, manifolds — the Sunada method and the construction via effective torus actions.

In this thesis we investigate the question whether isospectral metrics constructed via effective torus actions descend along Riemannian covering maps. This is used to construct continuous families of isospectral, not isometric metrics on certain lens spaces of dimension at least seven, on the spaces $\mathbb{RP}^{2m-1\geq 5} \times \mathbb{S}^1$ and on the Lie group $\mathrm{SO}(3) \times \mathbb{S}^1$.

Contents

1 Introduction

The *Laplace operator* Δ is a fundamental differential operator in physics, e.g. it is used in the modeling of wave propagation, heat flow and fluid mechanics. It is a second order differential operator acting on functions $f \in C^2(\mathbb{R}^n)$ as

$$\Delta f := -(\operatorname{div} \circ \operatorname{grad}) f.$$

So if we want to define the Laplace operator on a manifold, we have to define the gradient and divergence operators first. Since the gradient of a scalar field is a vector field which points in the direction of the highest rate of increase of the scalar field and whose magnitude is the greatest rate of change, and since the divergence of a vector field measures the change of volume density the flow of the vector field generates, we need a method to assign magnitudes to tangential vectors and a method to measure volume, i.e. we need a *Riemannian metric* on our manifold.

Definition 1.1 (Riemannian manifold)**.** Let M be a smooth manifold and let $g \in \Gamma(M, T^2M)$ be a 2-tensor field on M, which is *symmetric* (i.e., $g(X,Y) = g(Y,X)$) and *positive-definite* (i.e., $g(X,X) > 0$ if $X \neq 0$). Then g is called a *Riemannian metric* on M and the pair (M, g) is called a *Riemannian manifold*.

Now given a Riemannian manifold, we define the *gradient* of a function $f \in C^1(M)$ as the unique vector field $\operatorname{grad} f$ determined by the fact, that for all vectors $X \in TM$ we have

$$df(X) = \langle \operatorname{grad} f, X \rangle,$$

and we define the *divergence* of a vector field $X \in \Gamma(M, TM)$ as the unique scalar field $\operatorname{div} X$ determined by the equation

$$(\operatorname{div} X) dM = d(i_X dM),$$

where dM is the Riemannian volume form and i_X the interior multiplication with X: for any k-form ω, $i_X\omega$ is the $(k-1)$-form defined by $i_X\omega(Y_1, \dots, Y_{k-1}) := \omega(X, Y_1, \dots, Y_{k-1})$.

Using these two operators, we now define the Laplace operator on Riemannian manifolds as in the Euclidean case as

$$\Delta := -(\operatorname{div} \circ \operatorname{grad}) : C^\infty(M, g) \to C^\infty(M, g).$$

From Stoke's Theorem we can conclude two very nice properties of the Laplace operator:

Lemma 1.2 (Properties of the Laplace operator)**.** *Let (M, g) be a closed, oriented Riemannian manifold. Then the following holds true:*

- *The Laplace operator is self-adjoint, i.e. we have $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$ for any two functions $f, g \in C^\infty(M)$.*
- *The Laplace operator is non-negative, i.e. we have $\langle \Delta f, f\rangle \geq 0$ for any function $f \in C^\infty(M)$.*

Because of these properties we are able to prove a *spectral theorem* for the Laplace operator, which leads to the area of *spectral geometry.* We will give an idea of the proof in the next section and the whole proof can be found, e.g., in [Ros97].

We denote by $L^2(M, g)$ the completion of $C^\infty(M)$ with respect to the norm $\|f\| := \int_M f^2 \, dM$ and by *multiplicity* of an eigenvalue we mean the dimension of the corresponding eigenspace.

Theorem 1.3 (Spectral theorem)**.** *Let (M, g) be a connected, closed and oriented Riemannian manifold. Then there exists an orthonormal basis of $L^2(M, g)$ consisting of smooth eigenfunctions of the Laplace operator. All eigenvalues are non-negative, have finite multiplicity and accumulate only at infinity.*

The condition of orientability can easily be dropped if we use a Riemannian density instead of a volume form for integration, but if the manifold has boundary we have to impose boundary conditions, because without them the Laplace operator would no longer be self-adjoint, because we get an additional integral over the boundary from Stoke's Theorem.

The *Dirichlet boundary condition* requires the functions to vanish on the boundary and the *Neumann boundary condition* requires the derivative of the functions with respect to the outer normal field to vanish. With either of this conditions the spectral theorem for the Laplace operator holds true in case the manifold has boundary (for a proof see, e.g., [Sak92]).

After computing with local coordinates, the Laplace operator is given by

$$\begin{aligned}\Delta f &= -\frac{1}{\sqrt{\det g}} \partial_j (g^{ij} \sqrt{\det g} \; \partial_i f) \\ &= -g^{ij} \partial_j \partial_i f + (\text{lower order terms}),\end{aligned}$$

which shows that not only is the Laplace operator determined by the Riemannian metric but the Laplace operator also determines the metric (by evaluating Δ on a function which is locally given by $x^i x^j$, we can recover g^{ij} and hence g_{ij}). Thus we can investigate the relation between the spectral theory of the Laplace operator and the geometry of (M, g).

1.1 Heat kernel and the spectral theorem

We have seen that knowing the effect of the Laplace operator on smooth functions is equivalent to knowing the Riemannian metric. But in physical experiments we can often only estimate the eigenvalues of the Laplace operator, so we are led to the question: How much information is stored in its spectrum.

Definition 1.4 (Spectrum of a manifold)**.** The *spectrum* of a closed, connected Riemannian manifold is the spectrum of eigenvalues of the Laplace operator, counted with multiplicities.

The *Dirichlet spectrum* (resp. *Neumann spectrum*) of a compact, connected Riemannian manifold with boundary is the spectrum of eigenvalues, counted with multiplicities, corresponding to eigenfunctions of the Laplace operator, which satisfy the Dirichlet boundary condition (resp. the Neumann boundary condition).

The main tool for extracting information about the geometry of a Riemannian manifold from its spectrum is the *heat kernel*, which is defined by the following two properties:

Definition 1.5 (Heat kernel)**.** A function $e(t,x,y) \in C^\infty(\mathbb{R}^+ \times M \times M)$ is called the *heat kernel* if it satisfies the following two conditions:

$$\begin{aligned}(\partial_t + \Delta_x)e(t,x,y) &= 0\\ \lim_{t\to 0}\langle e(t,x,y), f(y)\rangle_y &= f(x).\end{aligned}$$

The name comes from the fact that the heat kernel is a *fundamental solution* to the *heat equation*

$$\begin{aligned}(\partial_t + \Delta_x)f(t,x) &= 0 \text{ and}\\ f(0,x) &= f(x),\end{aligned}$$

which models the heat flow on a manifold with initial distribution $f(x)$.

The term *fundamental solutions* means that we can easily get a solution to the heat equation for *any* initial distribution: Just let

$$f(t,x) := \langle e(t,x,y), f(y)\rangle_y(x). \tag{1.1}$$

It is easy to verify that this function solves the heat equation using the two defining properties of the heat kernel from Definition 1.5.

For a proof of the heat kernel being a fundamental solution and its construction, see, e.g., [Ros97].

There we can also find proofs to the two fundamental properties of it: It is symmetric in the space variables, i.e. $e(t,x,y) = e(t,y,x)$, and for a compact manifold M, it is also unique.

Motivated by the Equation (1.1), which we use to define solutions to the heat flow for arbitrary initial values $f(x)$, we define the *heat operator*:

Definition 1.6 (Heat operator)**.** For any $t > 0$ the heat operator

$$e^{-t\Delta} : L^2(M,g) \to L^2(M,g)$$

is defined to be

$$(e^{-t\Delta} f)(x) := \int_M e(t,x,y)f(y)\ dM.$$

This operator enjoys some wonderful properties, which are summarized in the following lemma and for which a proof can be found again in [Ros97].

Lemma 1.7. *For any $t > 0$ the heat operator $e^{-t\Delta} : L^2(M,g) \to L^2(M,g)$ is a continuous and compact operator (compact means, that it maps bounded sets to relatively compact ones). Moreover, the heat operator is self-adjoint, positive and has the semigroup property $e^{-t\Delta}e^{-s\Delta} = e^{-(t+s)\Delta}$.*

Because of the spectral theorem for self-adjoint compact operators on Hilbert spaces and the Lemma 1.7, $L^2(M,g)$ has an orthonormal basis consisting of eigenfunctions for the heat operator $e^{-t\Delta}$ with all eigenvalues $\gamma_i(t)$ positive, accumulating only at zero and having finite multiplicity. From this we can deduce easily the spectral theorem for the Laplace operator by showing that the eigenvalues λ_i of the Laplace operator satisfy the equation $\gamma_i(t) = e^{-\lambda_i t}$. For a elaborate proof, refer to [Ros97].

1.2 Asymptotics of the heat kernel

In [Min53] it was shown that for closed, connected Riemannian manifolds for any positive integer N and all $t > 0$ there is the asymptotic expansion

$$\operatorname{trace} e(t,x,x) = (4\pi t)^{-\frac{n}{2}} \left(1 + \sum_{i=1}^{N} t^i k_i(x)\right) + \mathcal{O}(t^{N-\frac{n}{2}+1}) \text{ as } t \downarrow 0, \quad (1.2)$$

where, for all i, k_i is a C^∞-function on M.

The functions k_i are polynomials in the components of the curvature tensor and its covariant derivatives and k_1 and k_2 were explicitly calculated in [MS67] and are given by

$$k_1 = \frac{1}{6} scal \text{ and } k_2 = \frac{1}{72} scal^2 - \frac{1}{180}|Ric|^2 + \frac{1}{180}|Riem|^2 + \frac{1}{30}\Delta scal, \quad (1.3)$$

where *scal* is the scalar curvature, *Ric* the Ricci tensor and *Riem* the Riemannian curvature tensor.

Now let $\{\lambda_i\}$ be the eigenvalues of the closed, connected Riemannian manifold (M,g), repeated as many times as its multiplicity indicates, and let $\{f_i\}$ denote the corresponding orthonormal eigenfunctions. In [Min53] it was shown that the series

$$\sum_i \exp(-\lambda_i t) \cdot f_i(x) \cdot f_i(y) \quad (1.4)$$

converges compactly on $\mathbb{R}^+ \times M^2$ to the heat kernel $e(t,x,y)$ and together with equation (1.2) we get (for any positive integer N and all $t > 0$)

$$\sum_i \exp(-\lambda_i t) = (4\pi t)^{-\frac{n}{2}} \left(\operatorname{vol} M + \sum_{i=1}^{N} t^i a_i\right) + \mathcal{O}(t^{N-\frac{n}{2}+1}) \text{ as } t \downarrow 0, \quad (1.5)$$

where the coefficients a_i for all i are given by $a_i = \int_M k_i(x)\, dx$.

Combining equation (1.5) with (1.3) we get the following corollary.

Corollary 1.8. *We can hear, i.e. recover from the spectrum, the dimension, the volume and the total scalar curvature of a closed, connected Riemannian manifold.*

1.3 Isospectral manifolds

Besides the asymptotic expansions of functions connected to the spectrum of the Laplace operator, e.g. the heat kernel, there are no other known ways to recover information from the spectrum. But another possibility to find out how much of the geometry is stored in the spectrum is by comparing isospectral manifolds.

Definition 1.9 (Isospectral manifolds)**.** Two closed, connected Riemannian manifolds are called *isospectral* if they have the same spectrum (including multiplicities).

Two compact, connected Riemannian manifolds with boundary are called *Dirichlet isospectral* (resp. *Neumann isospectral*), if the have the same Dirichlet spectrum, resp. the same Neumann spectrum.

From the above discussion we know already that two manifolds, which are isospectral, must have the same dimension, volume and total scalar curvature. But if one of them has some additional curvature properties, then in [Tan73] it was shown that the other one must have them too — at least in low dimensions.

Lemma 1.10. *Let (M, g) and (M', g') be isospectral, closed, connected Riemannian manifolds.*

1. *If their dimension is between two and five, then M is of constant sectional curvature sec_0 if and only if M' is so.*
2. *For dimension six we have that*
   - *M is conformally flat and has constant scalar curvature $scal_0$ if and only if M' is so and that*
   - *M is of constant sectional curvature $sec_0 > 0$ if and only if M' is so.*

From Lemma 1.10 it follows now that the round metric on the sphere is, in dimension up to six, completely characterized by its spectrum.

Theorem 1.11. *Let (M, g) be a closed, connected Riemannian manifold of dimension up to six, which is isospectral to the round sphere.*

Then (M, g) is isometric to the round sphere.

Remark 1.12. It is not known whether Theorem 1.11 is true in dimensions higher than six. In fact, the proof of Lemma 1.10 uses an explicit comparison of the coefficients in the asymptotic expansion of the heat kernel and because these terms become complicated rather quickly, there is little hope that one can raise the dimension with such a proof.

2 Sunada construction method

We have seen that although the complete information about the geometry of a Riemannian manifolds is stored in the Laplace operator, it is very hard to retrieve it from the spectrum — in fact, almost all methods used to recover geometric information form the spectrum use the coefficients (1.3) in the asymptotic expansion of the heat kernel, but they become complicated very quickly. But, despite these problems of extracting information from the spectrum, it was thought for a long time that it contains a wealth of information — just not accessible by the methods developed to that time — and that if two manifolds were isospectral they should share many geometric properties. This perception was strengthened by theorems like 1.11 and 1.10 and the fact that there was no known way to construct isospectral, not isometric manifolds in a systematic way.

The first example of isospectral, not isometric manifolds was given in 1964 by John Milnor in [Mil64]: a pair of flat tori in dimension 16; which was the first proof of the fact that the spectrum of the Laplace operator does not determine the isometry class of a Riemannian manifold. But this construction of a pair of isospectral manifolds could not be generalised, because it used in an essential way properties of flat tori.

The breakthrough came in 1985 when Toshikazu Sunada established the first general method for a systematic construction of isospectral but not isometric manifolds (see [Sun85]). This also led (via a generalization to orbifolds by Pierre Bérard in [B́92] and [B́93]) to the famous first examples of bounded plane domains with the same Dirichlet (and Neumann) spectrum found in 1991 by Carolyn Gordon, David Webb and Scott Wolpert (see [GWW92]) answering Mark Kac's question of 1966 "Can one hear the shape of a drum?" (see [Kac66]) negatively. However, these domains do not have a smooth boundary and the answer to Mark Kac's question is in the smoothly bounded case still open.

2.1 Main theorem

We now present the construction method, now called *Sunada method*, (as it is shown in [Bro88]), which reduces the whole problem to finite group theory at once.

Let M and N be closed, connected Riemannian manifolds, $M \xrightarrow{\pi} N$ be a normal Riemannian covering with finite covering group G and $e_M(t,x,y)$ and $e_N(t,x,y)$ the corresponding heat kernels.

Proposition 2.1. *We have the formula*

$$e_N(t,x,y) = \sum_{g\in G} e_M(t,\tilde{x},g\tilde{y}), \tag{2.1}$$

where $\tilde{x}$ and $\tilde{y}$ are chosen such that $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$.

Proof. We have to verify its two defining properties (see Definition 1.5).

We first have

$$(\partial_t + \Delta_x)e_N(t,x,y) = \sum_{g\in G}(\partial_t + \Delta_{\tilde{x}})e_M(t,\tilde{x},g\tilde{y}) = 0$$

and for a function $f \in L^2(N)$

$$\begin{aligned}
\lim_{t\to 0} \langle e_N(t,x,y), f(y)\rangle_y &= \lim_{t\to 0} \int_N e_N(t,x,y)\cdot f(y)\ dy \\
&= \lim_{t\to 0} \int_N \sum_{g\in G} e_M(t,\tilde{x},g\tilde{y})\cdot f(y)\ dy \\
&= \lim_{t\to 0} \sum_{g\in G} \int_N e_M(t,\tilde{x},g\tilde{y})\cdot f(y)\ dy \\
&= \lim_{t\to 0} \int_M e_M(t,\tilde{x},\tilde{y})\cdot (\pi^* f)(\tilde{y})\ d\tilde{y} \\
&= \lim_{t\to 0} \langle e_M(t,\tilde{x},\tilde{y}), (\pi^* f)(\tilde{y})\rangle_{\tilde{y}} \\
&= (\pi^* f)(\tilde{x}) = f(x).
\end{aligned}$$

□

Lemma 2.2. *We have*

$$(\operatorname{trace} e_N)(t) = \sum_{[g]\subset G} \frac{\operatorname{card}[g]}{\operatorname{card} G} \int_M e_M(t,\tilde{x},g\tilde{x})\ d\tilde{x}, \tag{2.2}$$

where $[g]$ denotes the conjugacy class of g in G.

Proof. Because of equation (2.1) we have

$$(\operatorname{trace} e_N)(t) = \int_N e_N(t, x, x)\ dx = \frac{1}{\operatorname{card} G} \sum_{g \in G} \int_M e_M(t, \tilde{x}, g\tilde{x})\ d\tilde{x}.$$

Now we have to check that the term on the right-hand side is unchanged under conjugation.

We have for any isometry h of M

$$e_M(t, h\tilde{x}, h\tilde{y}) = e_M(t, \tilde{x}, \tilde{y}),$$

therefore

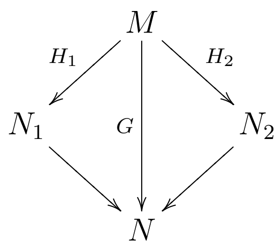
$$\begin{aligned} \int_M e_M(\tilde{x}, hgh^{-1}\tilde{x})\ d\tilde{x} &= \int_M e_M(h^{-1}\tilde{x}, gh^{-1}\tilde{x})\ d\tilde{x} \\ &= \int_M e_M(t, \tilde{x}, g\tilde{x})\ d\tilde{x} \text{ (changing variables)}, \end{aligned} \tag{2.3}$$

and the claim results.

□

Now we are able to prove the construction theorem of Sunada.

Theorem 2.3. *Let M, N, N_1 and N_2 be closed Riemannian manifolds, such that we have the following diagram of finite Riemannian coverings.*



Suppose furthermore that the three labeled coverings are normal and that we have for all $g \in G$

$$\operatorname{card}([g] \cap H_1) = \operatorname{card}([g] \cap H_2), \tag{2.4}$$

where $[g]$ denotes the conjugacy class of g in G. Then N_1 is isospectral to N_2.

If also H_1 is not conjugate to H_2 and M has no extra isometries not in G, then N_1 is not isometric to N_2.

Proof. Using equation (2.2) on the covering $M \xrightarrow{\pi_1} N_1$ and using the conjugacy invariance (2.3), we get

$$(\operatorname{trace} e_{N_1})(t) = \sum_{[g]\subset G} \frac{\operatorname{card}([g] \cap H_1)}{\operatorname{card} H_1} \int_M e_M(t, \tilde{x}, g\tilde{x})\ d\tilde{x}.$$

Now on the right-hand side of the above equation, the integral depends only on G, so if for all $g \in G$ we have $\operatorname{card}([g] \cap H_1) = \operatorname{card}([g] \cap H_2)$, then N_1 and N_2 will have an identical trace of the heat kernel and therefore will be isospectral by use of the following Lemma 2.4.

Now if H_1 is conjugate to H_2 in G (or an even larger group of isometries of M), then N_1 will be isometric to N_2. But if the metric on N is such that M has no extra isometries not in G and N_1 and N_2 are isometric, then H_1 must be conjugate to H_2 in G. □

Lemma 2.4. *The trace of the heat kernel determines all of the eigenvalues, counted with multiplicities.*

Proof. Let $\{\lambda_i\}$ be the set of all eigenvalues, counted with multiplicities, and ordered by size, i.e. $0 \le \lambda_0 \le \lambda_1 \le \dots$. Using that the series (1.4) converges compactly to the heat kernel (this was shown in [Min53]), we get for the trace of the heat kernel

$$(\operatorname{trace} e)(t) = \sum_i \exp(-\lambda_i t).$$

Now assume that $\lambda_0, \lambda_1, \dots, \lambda_k$ have been found. Then λ_{k+1} is the largest value λ, such that

$$\lim_{t\to\infty} \frac{(\operatorname{trace} e)(t) - \sum_{i=0}^{k} \exp(-\lambda_i t)}{\exp(-\lambda t)}$$

is finite. □

Now we can construct manifolds which are isospectral but not isometric (even not homeomorphic, if H_1 and H_2 are not isomorphic as abstract groups), if we have found examples of groups fulfilling (2.4), because every finite group arises as the fundamental group of a compact, smooth 4-manifold.

Examples of groups fulfilling (2.4) will be given in Section 2.2.

2.2 Examples

The examples of groups fulfilling (2.4) — called *almost conjugate* — are all taken from the book [Bus92].

Symmetric group $\mathfrak{S}_6$

Let $\mathfrak{S}_6$ be the group of all permutation of the set $\{1, \ldots, 6\}$. Denote by (a, b) the cyclic permutation which interchanges a with b and leaves the remaining elements fixed. For $(a, b), (c, d) \in \mathfrak{S}_6$ we let $(a, b)(c, d)$ denote the product "(c, d) followed by (a, b)". The following subgroups H_1, H_2 of $\mathfrak{S}_6$ are almost conjugate but not conjugate.

$$H_1 := \{\mathrm{id}, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\},$$
$$H_2 := \{\mathrm{id}, (1, 2)(3, 4), (1, 2)(5, 6), (3, 4)(5, 6)\}.$$

The index of both H_1 and H_2 in $\mathfrak{S}_6$ is 180.

Proof. Every element in $(H_1 \cup H_2) - \{\mathrm{id}\}$ acts in the same way: it changes a string $(ab\ cd\ ef)$ into a string $(ba\ dc\ ef)$. Since $\mathfrak{S}_6$ permutes all possible strings, all elements in $(H_1 \cup H_2) - \{\mathrm{id}\}$ belong to the same conjugacy class. Hence, H_1 and H_2 are almost conjugate. Now H_1 acts with two fixed points and H_2 acts without fixed points, which shows that H_1 and H_2 are not conjugate in $\mathfrak{S}_6$. □

Semidirect product $\mathbb{Z}_8^* \ltimes \mathbb{Z}_8$

Let $\mathbb{Z}_8$ be the additive group with eight elements and let $\mathbb{Z}_8^*$ be the multiplicative group of integers modulo eight. The semidirect product $\mathbb{Z}_8^* \ltimes \mathbb{Z}_8$ is the group with underlying set $\mathbb{Z}_8^* \times \mathbb{Z}_8$ and group operation defined by

$$(a, b) \circ (c, d) := (a \cdot c, a \cdot d + b) \pmod 8.$$

$\mathbb{Z}_8^* \ltimes \mathbb{Z}_8$ has the following isomorphic abelian subgroups of index eight.

$$H_1 := \{(1, 0), (3, 0), (5, 0), (7, 0)\},$$
$$H_2 := \{(1, 0), (3, 4), (5, 4), (7, 0)\}.$$

They are almost conjugate but not conjugate.

Proof. Writing

$$\sigma = (a, b) = (1, b) \circ (a, 0) \text{ and } \sigma^{-1} = (a, 0) \circ (1, -b)$$

we find

$$\sigma \circ (m,0) \circ \sigma^{-1} = (1,b) \circ (m,0) \circ (1,-b)$$
$$\equiv_8 (m, b(1-m)) .$$

From this we see, that H_1 and H_2 are not conjugate. To see that H_1 and H_2 are almost conjugate we use the fact that $\mathbb{Z}_8^*$ is an abelian factor. It is easily checked that elements in H_1, resp. H_2, are paiswise not conjugate and that for each element in H_1 there exists a conjugate in H_2. □

Groups with prime exponent — Heisenberg groups

Let $p \geq 3$ be a prime number and let H_1 and H_2 be finite groups with the same cardinality n. Assume that both groups have exponent p (i.e. $x^p = 1$ for all $x \in H_1$ and $x \in H_2$). Interpret H_1 and H_2 as subgroups of the symmetric group $\mathfrak{S}_n$, where H_1 and similarly H_2 is identified with the set $\{1, \ldots, n\}$ in some fixed way, and where the permutation corresponding to $g \in H_i, i = 1, 2$, is obtained via its action on H_i by left multiplication.

Under this hypothesis, H_1 and H_2 are almost conjugate in $\mathfrak{S}_n$.

Proof. We have to show that

$$\mathrm{card}([h] \cap H_1) = \mathrm{card}([h] \cap H_2)$$

for all $h \in H_1 \cup H_2$. For $h = \mathrm{id}$ there is nothing to prove. If $h \neq \mathrm{id}$ then h is a permutation of $\{1, \ldots, n\}$ which has no fixed point. Since h has prime order p, there exists a partition of $\{1, \ldots, n\}$ into pairwise disjoint subsets of cardinality p, such that h operates by cyclic permutation on each of this subsets. This proves that all $h \in H_1 \cup H_2 - \{\mathrm{id}\}$ belong to the same conjugacy class in $\mathfrak{S}_n$, and therefore $\mathrm{card}([h] \cap H_1) = \mathrm{card}([h] \cap H_2)$ for all $h \in \mathfrak{S}_n$. □

Obviously, H_1 and H_2 are conjugate in $\mathfrak{S}_n$ if and only if H_1 is isomorphic to H_2.

Examples of not isomorphic groups which satisfy the above hypotheses are, for instance, given by the Heisenberg groups: let $k, p \in \mathbb{N}, k \geq 3, p \geq k, p$ prime. Define

$$H_1 := \{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : * \in \mathbb{Z}_p \}$$

to be the group of all upper triangular $(k \times k)$-matrices with coefficients from $\mathbb{Z}_p$ and all diagonal elements equal to 1. Each such matrix G may be written in the form

$$G = A + I,$$

where I is the identity matrix and A is an upper triangular matrix with zero diagonal. Since A and I commute, the binomial formula yields

$$G^p = (A + I)^p = \sum_{i=0}^{p} \binom{p}{i} A^i.$$

In $\mathbb{Z}_p$ we have $\binom{p}{i} = 0$ for $i = 1, \ldots, p-1$. As the matrix A is k-nilpotent and $p \geq k$, we have $A^p = 0$. Hence, $G^p = I$. It follows that H_1 has exponent p and order p to the power $\frac{1}{2}k(k-1)$. The same holds for

$$H_2 := \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p, \qquad (\frac{1}{2}k(k-1) \text{ factors}).$$

As H_2 is abelian and H_1 is not, these groups are not isomorphic.

3 Construction via effective torus actions

The Sunada method is a very effective one but the manifolds constructed by it are always locally isometric, because they arise from a common Riemannian covering. The first example of a pair of isospectral, not locally isometric manifolds was discovered by Zoltán Szabó (see [Sza99]); these were manifolds with boundary, diffeomorphic to the product of an eight-dimensional ball and a three-dimensional torus, arising as domains in quotients of certain harmonic manifolds (a Riemannian manifold is said to be harmonic, if the volume density function $\omega_p = \sqrt{|\det(g_{ij})|}$ of M around p is a function of the geodesic distance $\mathrm{dist}(p, \cdot)$ alone). Motivated by Szabó's examples, Carolyn Gordon constructed, in [Gor94], the first pairs of isospectral manifolds without boundary. Her isospectrality proof revealed another general principle which does not necessarily imply the local isometry of the constructed manifolds.

Theorem 3.1 ([Gor94]). *Let $(G/\Gamma, g)$ and $(G'/\Gamma', g')$ be compact Riemannian nilmanifolds (i.e., G is a simply-connected and nilpotent Lie-group, Γ a [possibly trivial] discrete subgroup and the lift of g to G is left-invariant).*

The center $Z(\Gamma)$ of Γ is a lattice in $Z(G)$. Denote with $Z(\Gamma)^*$ the dual lattice to $Z(\Gamma)$ and define an equivalence relation on $Z(\Gamma)^*$ by $\lambda \sim \mu$ if and only if $\mathrm{kern}(\lambda) = \mathrm{kern}(\mu)$. Denote the set of equivalence by $[Z(\Gamma)^*]$.

Let $\lambda \in Z(\Gamma)^*$ and define $G_\lambda := G/\mathrm{kern}(\lambda)$. Let $\pi_\lambda : G \to G_\lambda$ be the projection, let $\Gamma_\lambda := \pi_\lambda(\Gamma)$ and let g_λ denote the induced left-invariant Riemannian metric on G_λ. The quotient $(G_\lambda/\Gamma_\lambda, g_\lambda)$ depends only on the equivalence class of λ.

Assume now that there exists a one-to-one correspondence $[\lambda] \to [\lambda']$ between $[Z(\Gamma)^*]$ and $[Z(\Gamma')^*]$ such that $\mathrm{spec}(G_\lambda/\Gamma_\lambda, g_\lambda) = \mathrm{spec}(G'_\lambda/\Gamma'_\lambda, g'_\lambda)$ for every $[\lambda] \in [Z(\Gamma)^*]$.

Then $\mathrm{spec}(G/\Gamma, g) = \mathrm{spec}(G'/\Gamma', g')$.

This method was further developed in a series of papers by various authors and led then, e.g., to the following reinterpretation (from [Sch00]):

If a torus acts on two Riemannian manifolds freely and isometrically with totally geodesic fibers, and if the quotients of the manifolds by any subtorus of codimension at most one are isospectral when endowed with the submersion metric, then the original two manifolds are isospectral.

But since no sphere is a principal $T^{\geq 2}$-bundle, the construction of isospectral metrics on spheres was beyond reach up to the year 2001. Then Carolyn Gordon reformulated her theorem in [Gor01], which now does not need a free action of the torus on the whole manifold, but instead, just on the principal orbits (and also changed the condition of totally geodesic fibers to the condition that the projected mean curvature vector fields of the submersions are mapped onto each other), and constructed isospectral metrics on spheres of dimension at least eight. In [Sch01] Dorothee Schüth reformulated again the theorem, now instead assuming the condition on the projected mean curvature vector fields, she assumed only that the volume elements are pulled back onto one another — see below for the precise statement. While these two conditions actually are equivalent in this context, the volume preserving condition is easier to check in application and therefore Dorothee Schüth was able to lower the dimension to five for pairs of isospectral metrics and to seven for continuous families of isospectral metrics.

Now assume we have two Riemannian manifolds, which were shown to be isospectral by use of the torus action method and both admit Riemannian coverings with isomorphic covering groups. We will now develop general criteria, such that the covered spaces are again isospectral and using this we will construct isospectral metrics e.g. on certain lens spaces.

3.1 Isospectrality

We begin by stating the construction method for isospectral manifolds via torus actions as in [Sch01].

Notation 3.2. By a *torus*, we always mean a nontrivial, compact, connected and abelian Lie group. If a torus T acts smoothly and effectively by isometries on a compact connected Riemannian manifold (M, g) then we denote by $\hat{M}$ the union of those orbits on which T acts freely. Note that $\hat{M}$ is an open and dense submanifold of M. By g^T we denote the unique Riemannian metric on the quotient manifold $\hat{M}/T$ such that the canonical projection $\pi : (\hat{M}, g) \to (\hat{M}/T, g^T)$ is a Riemannian submersion.

Theorem 3.3 ([Sch01])**.** *Let T be a torus which acts effectively on two compact, connected Riemannian manifolds (M, g) and (N, h), with or without boundary, by isometries.*

For each subtorus $W \subset T$ of codimension one, assume that there exists a T-equivariant diffeomorphism $F_W : M \to N$ which satisfies

$$F_W^* dN = dM$$

and induces an isometry

$$\hat{F}_W : (\hat{M}/W, g^W) \to (\hat{N}/W, h^W).$$

Then (M, g) and (N, h) are isospectral; if the manifolds have boundary then they are Dirichlet and Neumann isospectral.

Now let G be a finite group, which acts freely on (M, g) and on (N, h) by isometries (therefore, $M \overset{\pi_M}{\to} M/G$ and $N \overset{\pi_N}{\to} N/G$ are finite Riemannian coverings).

If we want now to use Theorem 3.3 to show that $(M/G, g_G)$ and $(N/G, h_G)$ are isospectral, we have to construct a new torus, which acts effectively on them. Therefore, we need first to pass to the quotient groups $T/(T \cap G)_M$ and $T/(T \cap G)_N$ (where $(T \cap G)_M$ means that we identify T and G with the corresponding subgroups of the isometry group of M and then identify the intersection again with a subgroup of T) and second, that these two quotient groups are equal.

Definition 3.4. We call a group G, which acts freely and by isometries on M and N, *compatible with* T, if G is T-equivariant, $T/(T \cap G)_M = T/(T \cap G)_N$ and $(T \cap G)$ is normal in G.

We first prove Proposition 3.5, which guarantees that we still have a torus which acts effectively and by isometries on the covered spaces.

Proposition 3.5. *Let (M, g) be a connected Riemannian manifold, with or without boundary, let T be a torus which acts effectively on (M, g) by isometries and let G be a finite group, which is T-equivariant and acts freely and by isometries on (M, g). Then the torus*

$$T_G := T/(T \cap G)$$

acts effectively and by isometries on $(M/G, g_G)$.

Proof. Because G is a finite group acting freely and by isometries on (M, g), we have a finite Riemannian covering $M \stackrel{\pi_M}{\to} M/G$, and because G is T-equivariant, the action of T on (M, g) descends to a well-defined action on $(M/G, g_G)$. Clearly, the action of the quotient $T_G = T/(T \cap G)$ is also well-defined on $(M/G, g_G)$, and because we have a Riemannian covering and both G and T act by isometries, T_G acts also by isometries on $(M/G, g_G)$.

It remains to show that T_G acts effectively. This would be the case if we could prove that if there exists a $t \in T$, such that for all $x \in M$ exists a $g_x \in G$ with $t(x) = g_x(x)$, then g_x is independent of x; because then t would be the trivial element in T_G.

We show this by an open-closed argument. Therefore, let $t \in T$ be as above, let x_0 be a fixed element in M and define $\mathcal{X} := \{x \in M : g_x = g_{x_0}\}$. If we show that $\mathcal{X}$ is both open and closed, it follows by the connectedness of M that we have $\mathcal{X} = M$ (the other case, i.e. $\mathcal{X} = \emptyset$, can not occur, because we have $x_0 \in \mathcal{X}$).

- $\mathcal{X}$ is closed: This is clear by the continuity of t and g_{x_0}, i.e. if we have $(x_n) \to x$ and $g_{x_n} = g_{x_0} \forall n$, then we get

  $$g_x(x) = t(x) = \lim t(x_n) = \lim g_{x_n}(x_n) = \lim g_{x_0}(x_n) = g_{x_0}(x).$$

  Because G acts freely, this is sufficient to conclude that $g_x = g_{x_0}$.

- $\mathcal{X}$ is open: Let $x \in M$, denote by $[x] \in M/G$ its G-orbit and let

  $$d := \min\{\text{dist}(y, z) : y, z \in [x], y \neq z\}.$$

  Then we have $d > 0$, because G is finite. Because both g_{x_0} and t are isometries, we have

  $$g_{x_0}(B_{d/2}(x)) = B_{d/2}(g_{x_0}(x)) = B_{d/2}(t(x)) = t(B_{d/2}(x)) \tag{3.1}$$

  and also for all $h, i \in G$ with $h(x) \neq i(x)$ we have

  $$h(B_{d/2}(x)) \cap i(B_{d/2}(x)) = \emptyset, \tag{3.2}$$

because of the definition of d and because both h and i are isometries. Now let $y \in B_{d/2}(x)$. Using (3.1) we get

$$g_y(y) = t(y) \in g_{x_0}(B_{d/2}(x)),$$

i.e.

$$g_y(B_{d/2}(x)) \cap g_{x_0}(B_{d/2}(x)) \neq \emptyset.$$

Together with (3.2) it follows that $g_y(x) = g_{x_0}(x)$, i.e. $g_y = g_{x_0}$, because G acts freely.

□

The following Lemma 3.6 now gives a satisfying answer to the question whether the covered spaces are again isospectral, which is one of the main results of this thesis. To the knowledge of the author, this lemma is new.

Lemma 3.6. *Let (M, g), (N, h) and the torus T be as in Theorem 3.3. Let the finite group G act freely and by isometries on both (M, g) and (N, h) and let G be compatible with T. Assume further that all diffeomorphisms F_W from Theorem 3.3 are G-equivariant.*

Then $(M/G, g_G)$ and $(N/G, h_G)$ are isospectral; if they have boundary then they are Dirichlet and Neumann isospectral.

Proof. We prove this theorem by showing that $(M/G, g_G)$ and $(N/G, h_G)$ satisfy the hypothesis of Theorem 3.3.

Define the new torus

$$T_G := T/(T \cap G).$$

This quotient is well-defined, because G is compatible with T (i.e., because $T/(T \cap G)_M = T/(T \cap G)_N$), and by Proposition 3.5 the torus T_G acts effectively and by isometries on both $(M/G, g_G)$ and $(N/G, h_G)$.

Now let $W_G \subset T_G$ be a subtorus of codimension one. Then there exists a subtorus $W \subset T$ of codimension one, such that $W/(T \cap G) = W_G$ (we can take the preimage of W_G under the quotient map and then choose the connected component containing the identity element). The T-equivariant diffeomorphism $F_W : M \to N$ descends to a well-defined T_G-equivariant diffeomorphism $F_{W_G} : M/G \to N/G$, because F_W is G-equivariant, and it clearly satisfies

$$F_{W_G}^* d(N/G) = d(M/G).$$

It remains to show that F_{W_G} induces an isometry

$$\hat{F}_{W_G} : (\widehat{M/G})/W_G \to (\widehat{N/G})/W_G,$$

but this follows from

$$(\widehat{M/G})/W_G = (\hat{M}/G)/W_G = (\hat{M}/W)/G_W,$$

where $G_W := G/(W \cap G)$ (and analogously with N). □

Now we have a convenient way of deciding whether isospectral metrics descend down the covered manifolds — the compatibility of the group G with the torus T is a natural requirement and the G-equivariance of the diffeomorphisms F_W will be easy to check in our applications.

3.2 Non-isometry

It now remains to develop criteria such that our constructed metrics are not isometric, because isometric metrics are always isospectral. We will again first state the corresponding result from [Sch01] concerning the metrics on the covering spaces and then give criteria such that this non-isometry will keep holding on the covered spaces.

We first recall the notation from 1.5 of [Sch01].

Notation 3.7. We fix a torus T with Lie algebra $\mathfrak{z} = T_eT$, a compact and connected Riemannian manifold (M, g_0), with or without boundary, and a smooth and effective action of T on (M, g_0) by isometries.

1. For $Z \in \mathfrak{z}$ we denote by $Z^\star$ the vector field $x \mapsto \frac{d}{dt}|_{t=0} \exp(tZ)x$ on M. For each $x \in M$ and each subspace $\mathfrak{w}$ of $\mathfrak{z}$ we let $\mathfrak{w}_x := \{Z^\star_x : Z \in \mathfrak{w}\}$.
2. We call a smooth $\mathfrak{z}$-valued 1-form on M *admissible* if it is T-invariant and horizontal (i.e. vanishes on the vertical spaces $\mathfrak{z}_x$).
3. For any admissible 1-form λ on M we denote by g_λ the Riemannian metric on M given by
$$g_\lambda(X, Y) := g_0(X + \lambda(X)^\star, Y + \lambda(Y)^\star).$$
4. We say that a diffeomorphism $F : M \to M$ is *T-preserving* is conjugation by F preserves $T \subset \mathrm{Diffeo}(M)$. In that case, we denote by Ψ_F the automorphism of $\mathfrak{z}$ induced by conjugation by F.
5. We denote by $\mathrm{Aut}^T_{g_0}(M)$ the group of all T-preserving diffeomorphisms F of M which preserve the g_0 norm of vectors tangent to the T-orbits and induce an isometry of $(\hat{M}/T, g_0^T)$. We denote the corresponding group of induced isometries by $\overline{\mathrm{Aut}}^T_{g_0}(M) \subset \mathrm{Isom}(\hat{M}/T, g_0^T)$.
6. We define $\mathcal{D} := \{\Psi_F : F \in \mathrm{Aut}^T_{g_0}(M)\} \subset \mathrm{Aut}(\mathfrak{z})$.

There are two sufficient criteria, which the $\mathfrak{z}$-valued 1-forms λ and μ have to fulfill in order for the metrics g_λ and g_μ to be not isometric.

Proposition 3.8 ([Sch01]). *Let λ, μ be admissible, $\mathfrak{z}$-valued 1-forms on M, such that the associated curvature forms Ω_λ and Ω_μ on $\hat{M}/T$ satisfies the following conditions:*

1. *No non-trivial 1-parameter group in $\overline{\mathrm{Aut}}^T_{g_0}(M)$ preserves Ω_μ; and*

2. $\Omega_\lambda \notin \mathcal{D} \circ \overline{\mathrm{Aut}}^T_{g_0}(M)^* \Omega_\mu$.

Then (M, g_λ) and (M, g_μ) are not isometric.

Given a group G as in Lemma 3.6, we want the induced metrics on the quotient manifolds to also be not isometric. To verify this, we would like to show that the conditions of Proposition 3.8 are satisfied, but for this we need the $\mathfrak{z}$-valued 1-forms λ and μ to descend to the quotient manifolds, i.e. that they are invariant under the action of G.

Lemma 3.9. *Let (M, g_λ) and (M, g_μ) satisfy the conditions of Proposition 3.8, let the group G be compatible with the torus T and assume furthermore that λ and μ are invariant under the action of G.*

Then $(M/G, g_{\lambda_G})$ and $(M/G, g_{\mu_G})$ are not isometric.

Proof. By assumption, λ and μ descend to $\mathfrak{z}$-valued 1-forms λ_G and μ_G on M/G and it is clear that they remain admissible.

From 2.1(vi) in [Sch01] we get $\Omega_\lambda = \Omega_0 + d\bar{\lambda}$, so Ω_λ is also invariant under the action of G and therefore descends to the associated form Ω_{λ_G} on $\hat{M}/T/G_T = \widehat{M/G}/T_G$ and analogously with Ω_μ.

We now show that Ω_{λ_G} and Ω_{μ_G} satisfy the conditions from Proposition 3.8.

1. Let $\overline{F^G_t} \in \overline{\mathrm{Aut}}^{T_G}_{g_0}(M/G) \subset \mathrm{Isom}(\widehat{M/G}/T_G, g_0^{T_G})$ be an one-parameter group preserving Ω_{μ_G}.

   Then every $\overline{F^G_t}$ is induced by an $\overline{F_t} \in \overline{\mathrm{Aut}}^T_{g_0}(M) \subset \mathrm{Isom}(\hat{M}/T, g_0^T)$, which we can choose, such that they again form a one-parameter group. Then $\overline{F_t}$ preserves Ω_λ, therefore it must be trivial, so also $\overline{F^G_t}$.

2. Assume that
$$\Omega_{\lambda_G} \in \mathcal{D}_G \circ \overline{\mathrm{Aut}}^{T_G}_{g_0}(M/G)^* \Omega_{\mu_G},$$
   i.e. there exists an $\overline{F^G} \in \overline{\mathrm{Aut}}^{T_G}_{g_0}(M/G)$ and a $D \in \mathcal{D}_G$, such that
$$\Omega_{\lambda_G} = D \circ \overline{F^G}^* \Omega_{\mu_G}.$$
   But then we get
$$\begin{aligned}\Omega_\lambda &= \pi_G^* \Omega_{\lambda_G} \\ &= \pi_G^*(D \circ \overline{F^G}^* \Omega_{\mu_G}) \\ &= D \circ \pi_G^*(\overline{F^G}^* \Omega_{\mu_G}) \\ &= D \circ \bar{F}^*(\pi_G^* \Omega_{\mu_G}) \\ &= D \circ \bar{F}^* \Omega_\mu,\end{aligned}$$

which contradicts the assumption $\Omega_\lambda \notin \mathcal{D} \circ \overline{\mathrm{Aut}}^T_{g_0}(M)^*\Omega_\mu$, because we have $\bar{F} \in \overline{\mathrm{Aut}}^T_{g_0}(M)$ and $\mathcal{D}_G \subset \mathcal{D}$.

□

Again as in Lemma 3.6, the assumed conditions of Lemma 3.9 on the action of the group G are natural and will also be easy to check in our applications.

3.3 Isospectral metrics on $\mathbb{RP}^{2m+1\geq 5}$ and on certain lens spaces of dimension ≥ 5

We show that the continuous families of isospectral, not isometric metrics on spheres of odd dimension at least five, constructed by Dorothee Schüth in [Sch01], induce continuous families of isospectral, not isometric metrics on real projective spaces and on certain lens spaces.

Such metrics on real projective spaces can already be found in the literature, but only in dimensions at least nine: In [Sch03] Dorothee Schüth constructed examples of such metrics on real Grassmann manifolds $Gr_{k,n}$ of k-planes in $\mathbb{R}^{n\geq 9}, 1 \leq k \leq n-1$, via a general method using principal connections. Also in the unpublished Diploma Thesis [RÖ6] of Ralf Rückriemen he mentions that the metrics on spheres constructed by Dorothee Schüth in [Sch01] and by Carolyn Gordon in [Gor01] descend down to isospectral, not isometric metrics on the real projective spaces — though he proves this by an explicit computation and not via a general lemma as we do.

But to the knowledge of the author, the continuous families of isospectral, not isometric metrics on lens spaces constructed here are new — though pairs of such metrics on lens spaces, which are not even homotopy equivalent, were already constructed by Akira Ikeda in [Ike80].

We first give a definition of lens spaces.

Definition 3.10 (Lens spaces)**.** Let ω be a primitive p-th root of unity and let $q_1, \ldots, q_{m+1}$ be integers coprime to p.

Consider on the $2m+1$-dimensional sphere $\mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}$ the $\mathbb{Z}_p$-action generated by multiplication with the matrix

$$W := \begin{pmatrix} \omega^{q_1} & & \\ & \ddots & \\ & & \omega^{q_{m+1}} \end{pmatrix} \in \mathrm{U}(m+1).$$

The orbit space $L(p; q_1, \ldots, q_{m+1}) := \mathbb{S}^{2m+1}/\mathbb{Z}_p$ is called *lens space*.

In order to evoke Lemma 3.6 we have to make sure that the above defined $\mathbb{Z}_p$-action commutes with all diffeomorphisms F_W. The definition of these maps is given in Proposition 3.2.5 of [Sch01]:

$$F_W := (A_Z, \mathrm{id}) \in \mathrm{SO}(2m+2),$$

where $A_Z \in \mathrm{SU}(m) \subset \mathrm{SO}(2m)$.

But since maps in $\mathrm{U}(m)$, which commute with an arbitrary $A_Z \in \mathrm{SU}(m)$, are of the form $\{\lambda \cdot \mathrm{id} : \lambda \in S^1 \subset \mathbb{C}\}$, this $\mathbb{Z}_p$-actions have to satisfy $q_1 = \ldots = q_m$.

Now recall that the action of the torus

$$T := \mathbb{R}^2/(2\pi\mathbb{Z} \times 2\pi\mathbb{Z})$$

on $\mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1} = \mathbb{C}^m \oplus \mathbb{C}$ is given by

$$(p, q) \mapsto (e^{ia}p, e^{ib}q)$$

for all $a, b \in \mathbb{R}, p \in \mathbb{C}^m$ and $q \in \mathbb{C}$.

We see that both the $\mathbb{Z}_2$ action given by the antipodal map and the $\mathbb{Z}_p$ action given above are part of the torus action itself, so all needed conditions from the Lemmas 3.6 and 3.9 are automatically satisfied (i.e. the T-equivariance of the actions and the $\mathbb{Z}_2/\mathbb{Z}_p$-equivariance of the $\mathfrak{z}$-valued one-forms) and we have proved the following theorem.

Theorem 3.11. *There exist continuous families of isospectral, pairwise not isometric metrics on the real projective spaces of dimension $2m + 1 \geq 7$ and there also exists a pair of isospectral, not isometric metrics on $\mathbb{RP}^5$.*

There exist continuous families of isospectral, pairwise not isometric metrics on the lens spaces $L(p; q_1, \ldots, q_{m+1})$ with $q_1 = \ldots = q_m$ of dimension $2m + 1 \geq 7$ and there also exist pairs of isospectral, not isometric metrics on such five-dimensional lens spaces.

3.4 Isospectral metrics on $\mathbb{RP}^{n\geq 8}$ and on certain lens spaces of dimension ≥ 9

The examples from Section 3.3 were all trivial because the actions of the groups G were completely part of the action of the torus. We now discuss the examples of Carolyn Gordon (which were first constructed in [Gor01]), i.e. continuous families of isospectral, pairwise not isometric metrics on $\mathbb{S}^{n\geq 8}$, as they are presented by Dorothee Schüth in [Sch01] — she reformulated the examples in such a way that the construction now uses her method and not Carolyn Gordon's. We will see that neither the $\mathbb{Z}_2$-action of the antipodal map nor the $\mathbb{Z}_p$-actions used to construct the lens spaces are part of the torus actions, so we will have to check all conditions from the Lemmas 3.6 and 3.9 (contrary to Section 3.3 where we did not have to check this because it followed automatically from the fact that the torus actions satisfy it all).

Let $m \geq 5$, let $\mathbb{S}^{m+3} \subset \mathbb{R}^m \oplus \mathbb{C} \oplus \mathbb{C}$, let the action of the torus $T := \mathbb{R}^2/(2\pi\mathbb{Z} \times 2\pi\mathbb{Z})$ be induced by its canonical action on the $\mathbb{C} \oplus \mathbb{C}$ component (generated by multiplication with i on each summand) and denote the standard basis of its Lie Algebra $\mathfrak{z} \cong \mathbb{R}^2$ by $\{Z_1, Z_2\}$.

For each linear map $j : \mathbb{R}^2 \to \mathfrak{so}(m)$ we define a $\mathfrak{z}$-valued one-form on $\mathbb{R}^m \oplus \mathbb{C} \oplus \mathbb{C} \cong \mathbb{R}^m \oplus \mathbb{R}^4$ (and hence on $\mathbb{S}^{m+3}$) by

$$\lambda^k_{(p,q)}(X, U) := \langle j_{Z_k} p, X \rangle$$

for $k = 1, 2$ and all $(X, U) \in T_p\mathbb{R}^m \oplus T_q\mathbb{R}^4$.

From the proof of Theorem 3.3 in [Gor01] we see that the action of the diffeomorphisms F_W is given by $(A_Z, \mathrm{id}) \in \mathrm{SO}(m + 4)$ for some map $A_Z \in \mathrm{O}(m)$.

We define now the $\mathbb{Z}_2$ action to be the action of the antipodal map and for $m = 2r + 1$ the $\mathbb{Z}_p$ action to be the action generated by multiplication with the matrix

$$W := \begin{pmatrix} \omega^{q_1} & & & & \\ & \ddots & & & \\ & & \omega^{q_1} & & \\ & & & \omega^{q_2} & \\ & & & & \omega^{q_3} \end{pmatrix} \in \mathrm{U}(r + 1),$$

where ω is a primitive p-th root of unity and q_1, q_2, q_3 are integers coprime to p. The reason we have to choose the q's in the upper left block of the matrix W all equal is the same as in Section 3.3: We need the $\mathbb{Z}_p$-action to commute with the action of all F_W's.

It is easy to verify that all conditions form the Lemmas 3.6 and 3.9 are satisfied, so we get:

Theorem 3.12. *There exist continuous families of isospectral, pairwise not isometric metrics on the real projective spaces of dimension* $n \geq 8$ *and there also exist continuous families of isospectral, pairwise not isometric metrics on the lens spaces* $L(p; q_1, \ldots, q_{r+1})$ *with* $q_1 = \ldots = q_{r-1}$ *of dimension* $2r+1 \geq 9$.

Note that in these examples for the lens spaces, we have one more free parameter than in Theorem 3.11, i.e. here we are free to choose q_1, q_r and q_{r+1} independent from each other (and the rest of the q_i' is determined by the condition $q_1 = \ldots = q_{r-1}$), whereas in Theorem 3.11 only q_1 and q_{m+1} can be chosen.

3.5 Isospectral metrics on $\mathbb{RP}^{2m-1\geq 5} \times \mathbb{S}^1$ and $\mathrm{SO}(3) \times \mathbb{S}^1$

Dorothee Schüth also has seen that a little modification of the main construction of isospectral metrics on spheres in [Sch01] can yield continuous isospectral families on $\mathbb{S}^{2m-1\geq 5} \times \mathbb{S}^1$ and pairs of isospectral metrics on $\mathbb{S}^3 \times \mathbb{S}^1$.

We use these to give another application of our Lemma 3.6.

The little modification of the construction from Section 3.3 is as following: Instead of considering the unit sphere $\mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1} = \mathbb{C}^m \oplus \mathbb{C}$ we consider the manifold $\mathbb{S}^{2m-1} \times \mathbb{S}^1 \subset \mathbb{C}^m \oplus \mathbb{C}$.

The antipodal map on the first factor, i.e. on $\mathbb{S}^{2m-1}$, is again part of the torus action, therefore all conditions from the Lemmas 3.6 and 3.9 are again automatically satisfied.

For $m = 2$ we get a pair of isospectral, not isometric metrics on $\mathbb{RP}^3 \times \mathbb{S}^1$. Because $\mathbb{RP}^3$ is diffeomorphic to SO(3) we can use the induced diffeomorphism $\mathbb{RP}^3 \times \mathbb{S}^1 \xrightarrow{\cong} \mathrm{SO}(3) \times \mathbb{S}^1$ to get isospectral, not isometric metrics on the Lie group $\mathrm{SO}(3) \times \mathbb{S}^1$ (note that the two isospectral metrics on $\mathbb{RP}^3 \times \mathbb{S}^1$ are *not* product metrics).

Theorem 3.13. *There exist continuous families of isospectral, pairwise not isometric metrics on* $\mathbb{RP}^{2m-1\geq 5} \times \mathbb{S}^1$ *and there also exist a pair of isospectral, not isometric metrics on the Lie group* $\mathrm{SO}(3) \times \mathbb{S}^1$.

Again, to the knowledge of the author, there are no other examples of isospectral, not isometric metrics on $\mathrm{SO}(3) \times \mathbb{S}^1$.

References

[B9́2] Pierre Bérard, *Transplantation et isospectralité I*, Math. Ann. **292** (1992), no. 3, 547–559.

[B9́3] ______, *Transplantation et isospectralité II*, J. London Math. Soc. **48** (1993), 565–576.

[Bro88] Robert Brooks, *Constructing Isospectral Manifolds*, American Mathematical Monthly **95** (1988), no. 9, 823–839.

[Bus92] Peter Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Progress in Mathematics, vol. 106, Birkhäuser, 1992.

[Gor94] Carolyn S. Gordon, *Isospectral Riemannian manifolds which are not locally isometric, II*, Contemp. Math. **173** (1994), 121–132.

[Gor01] ______, *Isospectral deformations of metrics on spheres*, Invent. Math. **145** (2001), no. 2, 317–331.

[GWW92] Carolyn S. Gordon, David Webb, and Scott Wolpert, *Isospectral plane domains ans surfaces via Riemannian orbifolds*, Invent. Math. **110** (1992), 1–22.

[Ike80] Akira Ikeda, *On lens spaces which are isospectral but not isometric*, Ann. Sci. École Norm. Sup. **13** (1980), 303–315.

[Kac66] Mark Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly **73** (1966), no. 2, 1–23.

[Mil64] John W. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci. USA **51** (1964), 542.

[Min53] Subbaramiah Minakshisundaram, *Eigenfunctions on Riemannian manifolds*, Journal of the Indian Math. Soc. **17** (1953), 158–165.

[MS67] Henry P. McKean, Jr. and Isadore M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967), 43–69.

[R0̈6] Ralf Rückriemen, *Isospectral metrics on projective spaces*, Diploma Thesis, Humboldt-Universität zu Berlin, 2006.

[Ros97] Steven Rosenberg, *The Laplacian on a Riemannian Manifold*, London Mathematical Society Student Texts, vol. 31, Cambridge University Press, 1997.

[Sak92] Takashi Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, 1992.

[Sch00] Dorothee Schüth, *Isospectral manifolds with different local geometries*, Habilitation, Rheinische Friedrich-Wilhelms-Universität Bonn, 2000.

[Sch01] ———, *Isospectral metrics on five-dimensional sheres*, J. Differential Geometry **58** (2001), 87–111.

[Sch03] ———, *Constructing isospectral metrics via principal connections*, Geometric Analysis and Nonlinear Partial Differential Equations (Stefan Hildebrandt and Hermann Karcher, eds.), Springer, 2003, pp. 69–79.

[Sun85] Toshikazu Sunada, *Riemannian coverings and isospectral manifolds*, Annals of Mathematics **121** (1985), 169–186.

[Sza99] Zoltán I. Szabó, *Locally non-isometric yet super isospectral spaces*, Geom. Funct. Anal. **9** (1999), no. 1, 185–214.

[Tan73] Shukichi Tanno, *Eigenvalues of the Laplacian of Riemannian manifolds*, Tôhoku Math. Journ. **25** (1973), 391–403.