Functional relationships between price and volatility jumps and its consequences for discretely observed data

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Abstract

Many prominent continuous-time stochastic volatility models exhibit certain functional relationships between price jumps and volatility jumps. We show that stochastic volatility models like the Ornstein-Uhlenbeck and other continous-time CARMA models as well as continous-time GARCH and EGARCH models all exhibit such functional relations. We investigate the asymptotic behaviour of certain functionals of price and volatility processes for discrete observations of the price process on a grid, which are relevant for estimation and testing problems.

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1 Introduction

Models for financial data involving a stochastic volatility and allowing for sample path discontinuities in the volatility as well as in the underlying asset price (stock, index, exchange rate) have become more and more popular in recent years. Since Merton's paper [24] there is an increasing number of publications showing empirical evidence of jumps in the asset prices: from the distribution of log-returns viewpoint as in [1, 10, 11], or from a non-parametric statistical viewpoint as in [4, 6, 14] using multipower variations, or with other methods as in [3, 22]; see also the references in these papers.

Naturally, if the asset price process has jumps, one might suppose that the volatility process also exhibits jumps, which is supported by empirical evidence; cf. [18]. Starting from the assumption of jumps present in the price process as well as in the volatility process, it is important to investigate in a next step a possible relation between price jumps and volatility jumps. In a first statistical analysis [15] we have investigated, whether for such common jumps some functions of the jump sizes are correlated.

However, many specific models which feature jumps in price and volatility exhibit much more than correlated price and volatility jumps, they impose a functional relationship between these jumps, when they occur at the same times. For example the following one-factor continuous-time models, sometimes including a leverage term, exhibit such functional relationships:

(1) the Ornstein-Uhlenbeck or BNS (Barndorff-Nielsen Shephard) model [5];
(2) the CARMA models [7, 25];
(3) the COGARCH model [13, 19]; and
(4) the ECOGARCH model [12].

All such models fit into a semimartingale framework, and the aim of this paper is to investigate the possibility of testing whether, in this general framework, a functional relationship occurs between the jumps of price and volatility.

One problem, which immediately arises is then the structure of the data: the price is observed at discrete times only, typically at times $i\Delta_n$ for all integers i, up to the final horizon T, and where Δ_n is a time lag, which is small. However, the price jumps are not really observed, although "big" jumps are reasonably well known. For the volatility, the problem is much worse, because it is never observed *per se*: we need a preliminary estimation, which unfortunately is not really accurate in view of the typically available data. So, price jumps can only be detected, when they are big enough, and even more so for volatility jumps. To tackle the problem of local volatility estimation various methods have been proposed in different settings; see [2, 18] and references therein.

On top of these "discretization" errors, we have another problem here: on the one hand, checking for a given functional relationship can be done as soon as there are at least two (joint) jumps, and this usually occurs in relatively short periods of time, if there are jumps at all.

Below, we propose a method for testing a (relatively) arbitrary functional relationship between price and volatility jumps, for those which occur simultaneously. As mentioned above, it has been shown in [18] that such common jumps are likely to happen. We then apply this method to the four different stochastic volatility models above. These models allow for joint upwards and (perhaps also) downwards jumps in the prices and the volatilities. This is done in the natural discrete observation scheme mentioned above, in spite of the estimation problems already mentioned.

The paper is organized as follows: In Section 2 we set the stage by formulating the (log) price and stochastic volatility models in its full generality of a bivariate Itô semimartingale. We also formulate the functional relationships we want to investigate in detail in this paper. We introduce the discretized version of the process and recall the spot volatility estimator. Then we present the relevant statistics and prove a central limit theorem. In Section 3 we present the four model classes and their functional relationships between jump sizes in price and volatility. We also explain the forms of the corresponding test statistics and the rejection mechanism. Finally, the test itself is presented in Section 4.

2 The model framework

Our (non-parametric) stochastic volatility model is a "joint" model for (log) price X and volatility σ. All processes are 1-dimensional here, and we set $c_t = \sigma_t^2$, and the model is really for the pair (X, c). We have a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, on which

$$
\begin{aligned}
X_t &= X_0 + \int_0^t b_s ds + \int_0^t \sigma_s \, dW_s \\
&\quad + \int_0^t \int_E (\delta(s,z) 1_{\{|\delta(t,z)|\leq 1\}})(\mu - \nu)(ds, dz) + \int_0^t \int_E (\delta(s,z) 1_{\{|\delta(t,z)|> 1\}})\mu(ds, dz) \\
c_t &= c_0 + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s + \int_0^t \widetilde{\sigma}'_s dW'_s \\
&\quad + \int_0^t \int_E (\widetilde{\delta}(s,z) 1_{\{|\widetilde{\delta}(t,z)|\leq 1\}})(\mu - \nu)(ds, dz) + \int_0^t \int_E (\widetilde{\delta}(s,z) 1_{\{|\widetilde{\delta}(t,z)|> 1\}})\mu(ds, dz)
\end{aligned}
$$

Here, W and W' are two independent standard Brownian motions, $\mu = \mu(\omega, dt, dz)$ is a Poisson random measure on $\mathbb{R}_+ \times E$, where $(E, \mathcal{E})$ is an auxiliary Polish space, and the intensity measure of μ is $\nu(dt, dx) = dt \otimes \lambda(dz)$ for some σ-finite measure λ on $(E, \mathcal{E})$ (there is a lot of freedom for choosing μ and E and λ). The "coefficients" of the model are the two drifts b_t and $\widetilde{b}_t$, the two processes $\widetilde{\sigma}_t$ and $\widetilde{\sigma}'_t$, and the two functions δ and $\widetilde{\delta}$ on $\Omega \times \mathbb{R}_+ \times E$.

Equivalently, the pair (X, c) is an Itô semimartingale, whose characteristics (B, C, ν) have the form:

$$
\left.
\begin{array}{l}
B_t^1 = \int_0^t b_s ds, \qquad B_t^2 = \int_0^t \widetilde{b}_s ds \\
C_t^{11} = \int_0^t c_s ds, \quad C_t^{12} = C_t^{21} = \int_0^t \sigma_s \, \widetilde{\sigma}_s \, ds, \quad C_t^{22} = \int_0^t (\widetilde{\sigma}_s^2 + \widetilde{\sigma}_s'^2) ds \\
\nu(\omega; dt, dx, dy) = dt \, F_t(\omega; dx, dy), \quad \text{where } \ F_t(\omega; .) \ \text{ is the image} \\
\qquad \text{of } \lambda \text{ by } z \mapsto (\delta(\omega, t, z), \widetilde{\delta}(\omega, t, z)), \text{ restricted to } \mathbb{R}^2 \backslash \{0\}.
\end{array}
\right\} \tag{2.1}
$$

This formulation is general enough to accommodate a large variety of situations:

- Cases without or with leverage for the "continuous part", according to whether $\widetilde{\sigma}_t$ is identically 0 or not.

- The case where X and σ do *not* jump together, if F_t is supported by the two axes $\mathbb{R}\times\{0\}$ and $\{0\}\times\mathbb{R}$.

- The case where X and σ have common jumps, when F_t puts some positive mass outside of the two axes, including the case when they *always* jump together, when F_t does not charge the two axes.

2.1 Functional relationships

The pair of processes (X,c) is of the form (2.1), where the processes b_t, $\widetilde{b}_t$, $\widetilde{\sigma}_t$ and $\widetilde{\sigma}'_t$ should be progressively measurable, and the functions δ and $\widetilde{\delta}$ should be predictable. All these terms are "not too big", so that all integrals make sense, and they should also be such that $c_t \geq 0$ identically. We refer to [17] for all unexplained, but classical, notions or notation.

In particular, we may have a functional relationship between the jumps of X and c, at least when jumps of X belong to some subset A of $\mathbb{R}\backslash\{0\}$ (with possibly $A=\mathbb{R}\backslash\{0\}$); this can be expressed as

$$\phi(c_{t-},c_t) \;=\; \gamma\psi(X_{t-},X_t) \quad \text{if } \Delta X_t\in A \tag{2.2}$$

for some given (known) functions ψ and ϕ and a (usually unknown) parameter $\gamma\in\mathbb{R}$. An equivalent formulation of this property is that the restriction of F_t to the set $A\times\mathbb{R}$ is supported by the "curve" $\{(x,y):\phi(c_{t-},c_{t-}+y)=\gamma\psi(X_{t-},X_{t-}+x)\}$.

The specific assumption which we need is as follows, where r is a given number inside $[0,2)$:

Assumption (A-r) a) The processes b_t, $\widetilde{b}_t$, $\widetilde{\sigma}_t$ and $\widetilde{\sigma}'_t$ are locally bounded.

b) There exist a locally bounded process Γ_t and a (non-random) function $\chi\geq 0$ such that $|\delta(\omega,t,z)|\leq\Gamma_t(\omega)\chi(z)$ and $|\widetilde{\delta}(\omega,t,z)|\leq\Gamma_t(\omega)\chi(z)$ and $\int_E(\chi(z)^r\wedge 1)\,\lambda(dz)<\infty$.

c) The processes c_t and c_{t-} do not vanish. $\square$

The (finite) time horizon T here is fixed. We want to investigate (2.2), in restriction to the time interval $[0,T]$, and its consequences for a realistic modeling. In the examples of Section 3, but also in most cases one can think of, the relation (2.2) should hold for all jumps of X, or for all positive jumps, or for all negative ones, thus taking leverage between price and volatility jumps into account. Since very small jumps of X cannot be realistically inferred, these considerations lead us to take for A one of the following three sets:

$$A=(-\infty,-\varepsilon), \quad \text{or} \quad A=(\varepsilon,\infty), \quad \text{or} \quad A=(-\infty,-\varepsilon)\cup(\varepsilon,\infty) \tag{2.3}$$

for some given $\varepsilon>0$. We make the following additional assumption:

$$X \text{ has a.s. no jump with size } \Delta X_t \text{ in the boundary } \partial A \text{ of } A. \tag{2.4}$$

This is quite weak: typically the first marginal $F_t(dx,\mathbb{R})$ of the Lévy measures has a density, implying that (2.4) is satisfied when A has the form (2.3), and otherwise it is satisfied for all $\varepsilon > 0$ except countably many values.

As for the functions ψ and ϕ occurring in (2.2), and although weaker assumptions are possible for parts of what follows, we will assume throughout the following (with obvious notation for the derivatives or partial derivatives):

$$\begin{aligned}&\psi \text{ and } \phi \text{ are } C^2 \text{ functions, with } \psi(x,y) > 0 \text{ if } x \neq y,\\ &\qquad\qquad \text{and } |\phi_1'(x,y)| + |\phi_2'(x,y)| > 0 \text{ for all } x,y \in \mathbb{R}.\end{aligned} \tag{2.5}$$

Obviously (2.2) is meaningless if no jump of X before time T has size in A, and it is automatically satisfied if there is a single such jump, say at time S, because it then holds with $\gamma = \phi(c_{S-}, c_S)/\psi(X_{S-}, X_S)$. Therefore, we define

$$\Omega_T^A \;=\; \{\omega \in \Omega \;:\; \sum_{s\leq T} 1_{\{\Delta X_s(\omega)\in A\}} \geq 2\} \tag{2.6}$$

as the set on which there are at least two jumps of X within $[0,T]$ with size in A. Furthermore, define

$$\begin{aligned}\Omega_T^{(A,\gamma)} \;=\; &\{\omega \in \Omega_T^A:\ \text{there is some } \gamma(\omega)\in\mathbb{R} \text{ such that, if } s\in(0,T] \text{ and}\\ &\ \Delta X_s(\omega) \in A, \text{ then }\ \phi(\sigma_{s-}(\omega),\sigma_s(\omega)) = \gamma(\omega)\psi(X_{s-}(\omega),X_s(\omega))\}.\end{aligned} \tag{2.7}$$

Being in $\Omega_T^{(A,\gamma)}$ seems more general than having (2.2), because it allows for $\gamma = \gamma(\omega)$ to be random; however in practice this is immaterial because, upon observing a single path of (X,σ), it is of course impossible to decide whether γ is random or not.

2.2 The discretized process

For any process Y we write

$$\Delta_i^n Y \;=\; Y_{i\Delta_n} - Y_{(i-1)\Delta_n}, \tag{2.8}$$

and refer to [16] for background reading. We choose two numbers $\varpi, \rho \in (0,1/2)$ and two sequences $u_n > 0$ and $k_n \geq 1$ such that for some constant $C > 1$,

$$\frac{\Delta_n^{\varpi}}{C} \;\leq\; u_n \;\leq\; C\Delta_n^{\varpi}, \qquad \frac{1}{C\Delta_n^{\rho}} \;\leq\; k_n \;\leq\; \frac{C}{\Delta_n^{\rho}}. \tag{2.9}$$

The next variables serve as "local estimators" of the squared volatility $c_t = \sigma_t^2$, for a window of size $k_n\Delta_n$ starting at time $i\Delta_n$ (cf.[16], Eq. (9.3.1)):

$$\widehat{c}_i^n \;=\; \frac{1}{k_n\Delta_n}\sum_{j=1}^{k_n} |\Delta_{i+j}^n X|^2\, 1_{\{|\Delta_{i+j}^n X|\leq u_n\}}. \tag{2.10}$$

It is always nonnegative, and even a.s. positive under (A-r)-(a), which implies $\Delta_i^n X \neq 0$ a.s. for all i,n.

The functions ϕ and ψ satisfying (2.5) and A satisfying (2.3) are given, and we assume (2.4). For any two integers $p, q \geq 0$ we set

$$U(p,q)_T = \sum_{s \leq T} 1_A(\Delta X_s)\, \psi(X_{s-}, X_s)^p\, \phi(c_{s-}, c_s)^q, \tag{2.11}$$

$$\Phi_T = U(1,1)_T^2 - U(2,0)_T\, U(0,2)_T, \tag{2.12}$$

Being in $\Omega_T^{(A,\gamma)}$ amounts to be in Ω_T^A, plus having $U(0,2)_T - 2\gamma\, U(1,1)_T + \gamma^2\, U(2,0)_T = 0$ for some $\gamma = \gamma(\omega)$. Since $\gamma \mapsto U(0,2)_T - 2\gamma\, U(1,1)_T + \gamma^2 U(2,0)_T$ is quadratic and always nonnegative, we thus have

$$\Omega_T^{(A,\gamma)} = \Omega_T^A \cap \{\Phi_T = 0\}. \tag{2.13}$$

This leads us to construct a sequence of statistics which converges to Φ_T. For this, we set

$$U^n(p,q)_T = \sum_{i=k_n+1}^{[T/\Delta_n]-k_n} 1_A(\Delta_i^n X)\, \psi(X_{(i-1)\Delta_n}, X_{i\Delta_n})^p\, \phi(\widehat{c}^n_{i-k_n-1}, \widehat{c}^n_i)^q \tag{2.14}$$

and

$$\Phi_T^n = U^n(1,1)_T^2 - U^n(2,0)_T\, U^n(0,2)_T. \tag{2.15}$$

Observe that, by the Cauchy-Schwarz inequality, we necessarily have $\Phi_T \leq 0$ and $\Phi_T^n \leq 0$.

2.3 Some limit theorems

We give the asymptotic behavior of Φ_T^n. The properties (2.5) are in force below, without special mention. We have consistency (that is, $\Phi_T^n \stackrel{\mathbb{P}}{\longrightarrow} \Phi_T$, see Theorem 2.1 below), but also present a Central Limit Theorem, and this necessitates some preparation.

We consider an auxiliary probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, on which we have a double sequence $(V_m^-, V_m^+)_{m\geq 1}$ of i.i.d. $\mathcal{N}(0,1)$ variables. We denote by $S_1, S_2, \ldots$ the successive times at which a jump of X with size within A occurs, so this sequence increases to infinity. Then we define an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$ by setting

$$\widetilde{\Omega} = \Omega \times \Omega', \quad \widetilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \quad \widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'. \tag{2.16}$$

Any variable or process defined on Ω or Ω' is extended to $\widetilde{\Omega}$ in the usual way, without change of notation.

We then recall the notion of *stable convergence in law*. This kind of convergence is slightly stronger than convergence in law. For details and extensions see [16], Section 2.2.1. A formal definition is as follows: let $(V_n)_{n\geq 1}$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and V be a variable defined on the extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ as defined in (2.16). Then V_n converges to V stably in law ($V_n \stackrel{\mathcal{L}-(s)}{\longrightarrow} V$) if $\mathbb{E}[Y f(V_n)] \to \widetilde{\mathbb{E}}[Y f(V)]$ as $n \to \infty$ for all bounded $\mathcal{F}$-measurable random variables Y and all bounded Lipschitz functions f. In contrast with the usual convergence in law, saying that V_n converges stably in law to V "in restriction to a set B" is meaningful, and means that $\mathbb{E}[Y f(V_n)] \to \widetilde{\mathbb{E}}[Y f(V)]$ for all Y, f as above, with further $Y = 0$ outside B.

We have (ψ, ϕ) satisfying (2.5), and A of the form (2.3) with $\varepsilon > 0$ satisfying (2.4). We denote by $(T_m)_{m\geq 1}$ the successive jump times of the counting process $N'_t = \mu((0,t] \times \{z : \chi(z) > \varepsilon/2\})$ (a Poisson process with parameter $\lambda(\{z : \chi(z) > \varepsilon/2\})$, recall that the function χ is given in Assumption (A-r)(b)). We introduce the notation:

$$\left.\begin{array}{ll} \multicolumn{2}{l}{i(n,m) \text{ is the unique (random) integer with } (i(n,m)-1)\Delta_n < T_m \leq i(n,m)\Delta_n} \\ \widehat{c}^n(m-) = \widehat{c}^n_{i(n,m)-k_n-1}, & \widehat{c}^n(m+) = \widehat{c}^n_{i(n,m)} \\ \kappa^n_{m-} = \sqrt{k_n}\,\big(\widehat{c}^n(m-) - c_{T_m-}\big), & \kappa^n_{m+} = \sqrt{k_n}\,\big(\widehat{c}^n(m+) - c_{T_m}\big) \\ \kappa'^n_{m-} = \frac{1}{\sqrt{\Delta_n}}\big(X_{T_m-} - X_{(i(n,m)-1)\Delta_n}\big), & \kappa'^n_{m+} = \frac{1}{\sqrt{\Delta_n}}\big(X_{i(n,m)\Delta_n} - X_{T_m}\big) \\ \kappa'^n_m = \kappa'^n_{m-} + \kappa'^n_{m+}. & \end{array}\right\} \tag{2.17}$$

Then we first have:

$$\text{each sequence } (\kappa'^n_{m\pm})_{m\geq 1} \text{ is bounded in probability} \tag{2.18}$$

Moreover, under (A-r) and if ρ in (2.9) satisfies

$$\rho \;\leq\; \frac{2-r}{r}\,, \tag{2.19}$$

then from Theorem 13.3.3 for $\beta = 0$ of [16] we know

$$\big(\kappa^n_{m-}, \kappa^n_{m+}\big)_{m\geq 1} \;\stackrel{\mathcal{L}-(s)}{\longrightarrow}\; \big(c_{T_m-}\sqrt{2}\,V^-_m, c_{T_m}\sqrt{2}\,V^+_m\big)_{m\geq 1}\,. \tag{2.20}$$

We also set

$$N_t \;=\; \sum_{m\geq 1} 1_{\{S_m\leq t\}} \;=\; \sum_{t\leq t} 1_{\{\Delta X_t\in A\}} \tag{2.21}$$

for the number of jumps of X with size in A, up to time t. Recalling that A satisfies (2.3), by Assumption (A-r)(b) we have $N_t \leq N'_t$, and in fact every S_m equals some $T_{m'}$ (with usually a random index m').

Theorem 2.1. *Assume (A-r) for some $r \in [0,2)$ and (2.4) and (2.5). Then for Φ_T as in* (2.12) *we have as $n \to \infty$*

$$\Phi^n_T \stackrel{\mathbb{P}}{\longrightarrow} \Phi_T.$$

If further ρ in (2.9) satisfies (2.19), we have the following stable convergence in law, as $n \to \infty$:

$$k_n\Phi^n_T \;\stackrel{\mathcal{L}-(s)}{\longrightarrow}\; \overline{\Phi}_T, \quad \text{in restriction to the set } \Omega^{(A,\gamma)}_T,$$

where

$$\begin{aligned} \overline{\Phi}_T \;=\; & 2\Big(\textstyle\sum_{m=1}^{N_T} \psi(X_{S_m-}, X_{S_m})\big(c_{S_m-}\phi'_1(c_{S_m-}, c_{S_m})V^-_m + c_{S_m}\phi'_2(c_{S_m-}, c_{S_m})V^+_m\big)\Big)^2 \\ & -2\Big(\textstyle\sum_{m=1}^{N_T} \psi(X_{S_m-}, X_{S_m})^2\Big)\Big(\sum_{m=1}^{N_T} \big(c_{S_m-}\,\phi'_1(c_{S_m-}, c_{S_m})\,V^-_m + c_{S_m}\,\phi'_2(c_{S_m-}, c_{S_m})\,V^+_m\big)^2\Big). \end{aligned} \tag{2.22}$$

Moreover, $\overline{\Phi}_T < 0$ a.s. on Ω^ε_T and $\overline{\Phi}_T = 0$ a.s. on the complement $(\Omega^\varepsilon_T)^c$.

Proof. 1) In view of (2.13) and (2.15), for $\Phi_T^n \xrightarrow{\mathbb{P}} \Phi_T$ it is enough to prove that for all integers $p,q\geq 0$ we have

$$U^n(p,q)_T \ \stackrel{\mathbb{P}}{\longrightarrow} \ U(p,q)_T. \tag{2.23}$$

From (A-r) and the definition of the stopping times T_m, which imply that every s with $|\Delta X_s|>\varepsilon/2$ is equal to some (unique) T_m, we see that for all $n\geq n_T(\omega)$ for some finite variable n_T we have (we use the fact that A is open and no jump of X lies on its boundary ∂A):

$$\begin{array}{l} \bullet\ T_1>2k_n\Delta_n, \quad T_{N'_T}<T-2k_n\Delta_n, \quad 1\leq m\leq N'_T \ \ \Rightarrow \ \ T_m-T_{m-1}>3k_n\Delta_n \\ \bullet\ i\in\{1,\dots,[T/\Delta_n]\}\backslash\{i(n,1),\dots,i(n,N'_T)\} \ \ \Rightarrow \ \ |\Delta_i^n X|\leq\varepsilon \\ \bullet\ 1\leq m\leq N'_T \ \ \Rightarrow \ \ \left\{\begin{array}{ll} \Delta^n_{i(n,m)}X\in A & \text{if } \ \Delta X_{T_m}\in A \\ \Delta^n_{i(n,m)}X\notin A & \text{if } \ \Delta X_{T_m}\notin A. \end{array}\right. \end{array} \tag{2.24}$$

Hence if $n\geq n_T$ all summands in (2.14) are 0 except those corresponding to $i=i(n,m)$, and those $i(n,m)$ are all distinct. Since (2.18) and (2.20) imply $\Delta^n_{i(n,m)}X \xrightarrow{\mathbb{P}} \Delta X_{T_m}$ and $(\widehat{c}^n(m-),\widehat{c}^n(m+)) \xrightarrow{\mathbb{P}} (c_{T_m-},c_{T_m})$, we readily deduce (2.23) because ϕ and ψ are continuous.

2) Next, we introduce the simplifying notation

$$v_n=1/\sqrt{k_n}, \qquad \eta_m \ = \ (c_{T_m-},c_{T_m}), \qquad \eta'_m \ = \ (X_{T_m-},X_{T_m}).$$

Using (2.24) again, we see that

$$\begin{array}{c} n\geq n_T \quad \Longrightarrow \qquad U^n(p,q)_T \ = \ \sum_{m=1}^{N'_T}\zeta(p,q)^n_m \qquad \text{where} \\ \zeta(p,q)^n_m=1_A(\Delta X_{T_m})\,\psi(X_{T_m-}-\sqrt{\Delta_n}\,\kappa'^n_{m-},X_{T_m}+\sqrt{\Delta_n}\,\kappa'^n_{m+})^p \\ \qquad\qquad\qquad \phi(c_{T_m-}+v_n\kappa^n_{m-},c_{T_m}+v_n\kappa^n_{m+})^q \end{array} \tag{2.25}$$

As seen before, we have

$$\zeta(p,q)^n_m \ \stackrel{\mathbb{P}}{\longrightarrow} \ \zeta(p,q)_m \ := \ 1_A(\Delta X_{T_m})\,\psi(\eta'_m)^p\,\phi(\eta_m)^q. \tag{2.26}$$

3) We use the notation $Y_n=\mathrm{o}_P(a_n)$ for a sequence a_n of positive numbers if $Y_n/a_n \xrightarrow{\mathbb{P}} 0$. We recall once more that the variables $\sqrt{\Delta_n}\,\kappa'^n_{m\pm}$ and $v_n\kappa^n_{m\pm}$ tend to 0 in probability. Then, with obvious notation for the derivatives, and since $\sqrt{\Delta_n}=\mathrm{o}(v_n)$ by (2.9), we deduce from (2.25) that

$$\begin{array}{rcl} \zeta(p,q)^n_m &=& \zeta(p,q)_m+1_A(\Delta X_{T_m})\Big(p\sqrt{\Delta_n}\psi(\eta'_m)^{p-1}\phi(\eta_m)^q\big(\psi'_2(\eta'_m)\kappa'^n_{m+}-\psi'_1(\eta'_m)\kappa'^n_{m-}\big) \\ && +qv_n\psi(\eta'_m)^p\phi(\eta_m)^{q-1}\big(\phi'_1(\eta_m)\kappa^n_{m-}+\phi'_2(\eta_m)\kappa^n_{m+}\big) \\ && +\frac{q(q-1)}{2}\,v_n^2\,\psi(\eta'_m)^p\phi(\eta_m)^{q-2}\big(\phi'_1(\eta_m)\kappa^n_{m-}+\phi'_2(\eta_m)\kappa^n_{m+}\big)^2 \\ && +\frac{q}{2}\,v_n^2\,\psi(\eta'_m)^p\phi(\eta_m)^{q-1}\big(\phi''_{11}(\eta_m)(\kappa^n_{m-})^2 \\ && \quad +2\phi''_{12}(\eta_m)\kappa^n_{m-}\kappa^n_{m+}+\phi''_{22}(\eta_m)(\kappa^n_{m+})^2\big)\Big)+\mathrm{o}_P(v_n^2). \end{array}$$

At this stage we set

$$Y^j_n(p,q)_T=\begin{cases} \sum_{m=1}^{N'_T} 1_A(\Delta X_{T_m})\psi(\eta'_m)^p\,\phi(\eta_m)^q\big(\psi'_2(\eta'_m)\kappa'^n_{m+}-\psi'_1(\eta'_m)\kappa'^n_{m-}\big) & \text{if } j=1\\ \sum_{m=1}^{N'_T} 1_A(\Delta X_{T_m})\,\psi(\eta'_m)^p\,\phi(\eta_m)^q\big(\phi'_1(\eta_m)\,\kappa^n_{m-}+\phi'_2(\eta_m)\,\kappa^n_{m+}\big) & \text{if } j=2\\ \frac12\sum_{m=1}^{N'_T} 1_A(\Delta X_{T_m})\psi(\eta'_m)^p\,\phi(\eta_m)^q\big(\phi'_1(\eta_m)\,\kappa^n_{m-}+\phi'_2(\eta_m)\,\kappa^n_{m+}\big)^2 & \text{if } j=3\\ \frac12\sum_{m=1}^{N'_T} 1_A(\Delta X_{T_m})\,\psi(\eta'_m)^p\,\phi(\eta_m)^q\big(\phi''_{11}(\eta_m)(\kappa^n_{m-})^2 & \\ \qquad\qquad +\phi''_{22}(\eta_m)(\kappa^n_{m+})^2+2\phi''_{12}(\eta_m)\,\kappa^n_{m-}\,\kappa^n_{m+}\big) & \text{if } j=4. \end{cases}$$

We use (2.25) to deduce that, for all $n\geq n_0$:

$$\begin{aligned} U^n(p,q)_T &= U(p,q)_T+p\sqrt{\Delta_n}\,Y^1_n(p-1,q)_T+v_n\,qY^2_n(p,q-1)_T\\ &\qquad +v_n^2\,q\big((q-1)Y^3_n(p,q-2)_T+Y^4_n(p,q-1)\big)_T+\mathrm{o}_P(v_n^2). \end{aligned}\tag{2.27}$$

4) Now we turn to proving the stable convergence in law of Φ^n_T. For simplicity we write $U=U(2,0)_T$ and $U'=U(0,2)_T$ and $U''=U(1,1)_T$. Observe that on the set $\Omega_T^{(A,\gamma)}$ there is a (possibly random) number γ such that $U'=\gamma U''=\gamma^2U$, and also $Y^j_n(p,q)_T=\gamma^qY^j_n(p+q,0)_T$ for all $j=1,2,3,4$ and all n. Therefore, if $Y^j_n(p)_T=Y^j_n(p,0)_T$, we deduce from (2.27):

$$\begin{aligned} \Phi^n_T &= (U'')^2+2U''\gamma\sqrt{\Delta_n}\,Y^1_n(1)_T+2U''v_nY^2_n(1)_T+v_n^2(Y^2_n(1)_T)^2+2U''v_n^2Y^4_n(1)_T\\ &\quad -UU'-2\sqrt{\Delta_n}\,U'Y^1_n(2)_T-2v_n\gamma UY^2_n(1)-2v_n^2U\big(Y^3_n(0)+\gamma Y^4_n(1)\big)\\ &= \Phi_T+v_n^2\big((Y^2_n(1))^2-2UY^3_n(0)\big)+\mathrm{o}_P(v_n^2). \end{aligned}\tag{2.28}$$

Applying (2.20), we obtain that in restriction to the set $\Omega_T^{(A,\gamma)}$ the variables $v_n^{-2}\,\Phi^n_T=k_n\Phi^n_T$ converge stably in law to

$$\begin{aligned} &2\Big(\sum_{m=1}^{N'_T} 1_A(\Delta X_{T_m})\,\psi(X_{T_m-},X_{T_m})\big(c_{T_m-}\phi'_1(c_{T_m-},c_{T_m})V^-_m+c_{T_m}\phi'_2(c_{T_m-},c_{T_m})V^+_m\big)\Big)^2\\ &-2\,\Big(\sum_{m=1}^{N'_T} 1_A(\Delta X_{T_m})\,\psi(X_{T_m-},X_{T_m})^2\Big)\\ &\quad\Big(\sum_{m=1}^{N'_T} 1_A(\Delta X_{T_m})\,\big(c_{T_m-}\,\phi'_1(c_{T_qm-},c_{T_m})\,V^-_m+c_{T_m}\,\phi'_2(c_{T_m-},c_{T_m})\,V^+_m\big)^2\Big). \end{aligned}\tag{2.29}$$

Now, the variables $V^\pm_m$ are independent of $\mathcal{F}$, the $\mathcal{F}$-conditional distribution of the right hand side of (2.22) is the same as the $\mathcal{F}$-conditional distribution of (2.29). Therefore, $k_n\Phi^n_T\stackrel{\mathcal{L}-(s)}{\longrightarrow}\overline{\Phi}_T$ in restriction to $\Omega_T^{(A,\gamma)}$, with $\overline{\Phi}_T$ defined by (2.22).

5) That $\overline{\Phi}_T=0$ when $N_T=0$ or $N_T=1$ is obvious, and that $\overline{\Phi}_T\leq 0$ everywhere follows from the Cauchy-Schwarz inequality. Finally, let us introduce the simplifying notation

$$a^-_m=c_{S_m-}\,\phi'_1(c_{S_m-},c_{S_m}),\quad a^+_m=c_{S_m}\,\phi'_2(c_{S_m-},c_{S_m}),\quad a_m=\psi(X_{S_m-},X_{S_m}).$$

Then (2.22) rewrites as

$$\overline{\Phi}_T=2\Big(\sum_{m=1}^{N_T}a_m(a^-_mV^-_m+a^+_mV^+_m)\Big)^2-2\Big(\sum_{m=1}^{N_T}a_m^2\Big)\Big(\sum_{m=1}^{N_T}(a^-_mV^-_m+a^+_mV^+_m)^2\Big).$$

Since $a_m \neq 0$ for all m, if $\overline{\Phi}_T = 0$, we must have

$$\frac{a_m^- V_m^- + a_m^+ V_m^+}{a_m} = \frac{a_1^- V_1^- + a_1^+ V_1^+}{a_1}$$

for all $m = 2, 3, \ldots, N_T$. Since $|a_m^-| + |a_m^+| > 0$ for all m, whereas all V_m^+ and V_m^- are independent and with a density, conditionally on $(a_p, a_p^-, a_p^+)_{p \geq 1}$, this property is almost never true on the set $\{N_T \geq 2\} = \Omega_T^A$. Therefore, $\overline{\Phi}_T < 0$ almost surely on $\Omega_T^{A)}$, and the proof is complete. □

Note that, since $\rho < \frac{1}{2}$, (2.19) is always true when $r \leq \frac{4}{3}$. Otherwise, and since the number r is typically unknown, we need to know beforehand an upper bound on it: the closer r is to 2, the more close to a Brownian motion the compensated sum of jumps is (apart from the "big" jumps, of course). So in practice we should say that although r is unknown, we "know" (or, pretend) that it is smaller than some prescribed value $r_0 < 2$. Then we choose $\rho = \frac{2-r_0}{r_0}$ if $r_0 > \frac{4}{3}$, for example.

The limit $\overline{\Phi}_T$ in (2.22) depends on the values X_{S_m}, X_{S_m-}, c_{S_m} and c_{S_m-}, which are not observed. However, we can "approximate" this limit in the same way as for the consistency above in the following sense. We endow the auxiliary space $(\Omega', \mathcal{F}', \mathbb{P}')$ with another double sequence $(\widetilde{V}_m^-, \widetilde{V}_m^+)_{m \geq 1}$ of variables, with the same description as the sequence (V_m^-, V_m^-); if one wants to, one can take the same two double sequences because only their laws do matter:

Theorem 2.2. *Assume that the assumptions of Theorem 2.1 hold and define*

$$\begin{aligned} \overline{\Phi}_T^n \; = \; & 2\Big(\textstyle\sum_{i=k_n+1}^{[T/\Delta_n]-k_n} 1_A(\Delta_i^n X)\, \psi(X_{(i-1)\Delta_n}, X_{i\Delta_n}) \\ & \quad \big(\widehat{c}_{i-k_n-1}^n\, \phi_1'(\widehat{c}_{i-k_n-1}^n, \widehat{c}_i^n)\, \widetilde{V}_i^- + \widehat{c}_i^n\, \phi_2'(\widehat{c}_{i-k_n-1}^n, \widehat{c}_i^n)\, \widetilde{V}_i^+\big)\Big)^2 \\ & -2\,\Big(\textstyle\sum_{i=k_n+1}^{[T/\Delta_n]-k_n} 1_A(\Delta_i^n X)\, \psi(X_{(i-1)\Delta_n}, X_{i\Delta_n})^2\Big)\Big(\textstyle\sum_{i=k_n+1}^{[T/\Delta_n]-k_n} 1_A(\Delta_i^n X) \\ & \quad \big(\widehat{c}_{i-k_n-1}^n\, \phi_1'(\widehat{c}_{i-k_n-1}^n, \widehat{c}_i^n)\, \widetilde{V}_i^- + \widehat{c}_i^n\, \phi_2'(\widehat{c}_{i-k_n-1}^n, \widehat{c}_i^n)\, \widetilde{V}_i^+\big)^2\Big). \end{aligned} \tag{2.30}$$

Then $\overline{\Phi}_T^n$ are non-positive variables. Moreover, on the whole space Ω, and not only on $\Omega_T^{(A,\gamma)}$,

$$\overline{\Phi}_T^n \stackrel{\mathcal{L}-(s)}{\longrightarrow} \Phi_T.$$

Proof. The property $\overline{\Phi}_T^n \leq 0$ again follows from the Cauchy-Schwarz inequality. The rest of the proof is the same as for Theorem 2.1, upon observing that if (2.24) holds, the sequence $(\widetilde{V}_{i(n,m)}^-, \widetilde{V}_{i(n,m)}^+)_{m \geq 1}$ is independent of $\mathcal{F}$ and has the same distribution as the sequence $(V_m^-, V_m^+)_{m \geq 1}$. □

3 Examples

We shall investigate four model classes for the volatility as listed in the introduction: the Lévy-driven Ornstein-Uhlenbeck model (also called BNS model in finance) [5], more general CARMA models [7], with the CARMA(2,1) model as specific example (cf. [7, 25]), the COGARCH(1,1) model [19, 21] and its higher order versions [7, 8] and ECOGARCH models [12]. For readable review papers on continuous-time volatility models we refer to [20, 23]. Lévy processes are the natural driving processes in all continuous-time jump models.

In most models below the classical Gaussian semimartingale set-up for the price process has been extended by a leverage term to allow for the possibility of an upwards jump in the price, whenever there is a jump in the volatility (cf. [5], Eq. (8), [7], Eq. (1.2), [13], Eq. (5.1)). In all these models all jumps of X and σ^2 are common jumps, since they are caused by the same driving Lévy process L.

Example 3.1. [Ornstein-Uhlenbeck or BNS model; cf. [5]]
The BNS model suggests a continuous-time semimartingale as pricing model, and an Ornstein-Uhlenbeck or CAR(1) model driven by a subordinator L as stochastic volatility:

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \rho L_t \\ \sigma_t^2 &= \sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dL_t, \end{aligned}$$

where $\lambda > 0$, and $\rho \leq 0$. Here all price jumps are negative and coincide with a positive jump in the volatility.

Example 3.2. [CARMA(p,q) model; cf. [7, 25]]
Motivated to obtain a more flexible autocorrelation function for the stochastic volatility process, [7, 25] suggest a generalisation of the CAR(1) model to a general CARMA(p,q) model. Price model can be the same as in Example 3.1 and volatility models are in the CARMA framework given as

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \rho L_t \\ \sigma_t^2 &= Y_t \end{aligned}$$

where $\rho \leq 0$, L_t is a subordinator and Y_t is a CARMA(p,q) process driven by L_t.

Since volatility models have to be positive, not all CARMA models (even when driven by a subordinator) can be chosen as volatility models. For certain sets of parameters a subordinator-driven CARMA(2,1) model is non-negative, and has been suggested as volatility model in [25].

The resulting equation for the squared volatility is

$$\sigma_t^2 = \sigma_0^2 + \int_0^t g(t-u) dL(u)$$

and the kernel $g(u)$ can be given explicitly. The CARMA(2,1) process has for negative and different roots ρ_1, ρ_2 of $a(\cdot)$ the kernel

$$g(h) = \frac{1 + b_1\rho_1}{\rho_1 - \rho_2} e^{\rho_1 h} + \frac{1 + b_1\rho_2}{\rho_2 - \rho_1} e^{\rho_2 h}, \quad h \geq 0.$$

It is nonnegative if and only if $0 \le b_1 \le \max\{-1/\rho_1, -1/\rho_2\}$. Hence for this set of parameters the CARMA(2,1) model is a possible volatility model, which generalizes the CAR(1) model of Example 3.1 in allowing for more flexible autocorrelation functions. Again all price jumps are negative and coincide with a positive jump in the volatility.

Example 3.3. [COGARCH(p,q) process; cf. [13, 19]]
The COGARCH(1,1) model has been suggested as a continuous-time version of the GARCH(1,1) model by replacing the discrete noise variables by the jumps of a Lévy process L. Introducing a possible leverage term results in the following model, and we also include a drift in the price process for completeness:

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s ds + \int_0^t \sigma_{s-} dL_s + \rho \sum_{s \le t\,,\, \Delta L_s < 0} \sigma_{s-}^2 (\Delta L_s)^2 \\ \sigma^2 t &= \sigma_0^2 + \int_0^t \Big(\beta - \eta \sigma_s^2\Big) ds + \varphi \sum_{s \le t} \sigma_{s-}^2 (\Delta L_s)^2, \end{aligned}$$

where $\beta > 0$, $\eta > 0$, $\varphi \ge 0$, and $\rho \ge 0$. Extensions to higher order COGARCH models have been suggested in [8], but this does not change the relation of the jumps in price and volatility as given in Proposition 3.5 below. Here the price can have positive and negative jumps, but negative jumps have a higher impact than positive ones by the leverage term.

Example 3.4. [ECOGARCH processes; cf. [12]]
Another model, which involves a CARMA model, but takes care of the necessary non-negativity of the volatility, models $\log \sigma^2$ by a CARMA model:

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s ds + \int_0^t \sigma_{s-} dL_s \\ \sigma_t^2 &= e^{Y_t}, \end{aligned}$$

where Y_t is a CARMA(q,p-1) process with mean μ. This model has a leverage effect in-built.

In COGARCH and ECOGARCH models it is immediate by the definition that the models allow for common jumps in price and volatility.

Proposition 3.5. *With these definitions we obtain the following functional relationships:*
• *In all linear models of Examples 3.1 and 3.2, (2.2) holds with $A = \mathbb{R}\backslash\{0\}$ and*

$$\psi_{OU}(x,y) = \phi_{OU}(x,y) = y - x \tag{3.1}$$

(in these models, the jumps of X are always negative and those of σ^2 positive).
• *In all* COGARCH *models, (2.2) holds with $A = \mathbb{R}\backslash\{0\}$ and*

$$\psi_{COG}(x,y) = (y-x)^2, \qquad \phi_{COG}(x,y) = y - x, \tag{3.2}$$

and with $A = (0,\infty)$ and $A = (-\infty, 0)$, with two different values of the constant γ on those two sets.
• *for the* ECOGARCH *models, (2.2) holds with*

$$\psi_{ECOG}(x,y) = y - x, \qquad \phi_{ECOG}(x,y) = \sqrt{x}\,(\log y - \log x), \tag{3.3}$$

and with $A=(0,\infty)$ and $A=(-\infty,0)$, with two different values of the constant γ on those two sets.

4 Test for a functional relationship

As mentioned before, we want to test whereas the observed outcome lies in the set $\Omega_T^{(A,\gamma)}$ of (2.7), for a *given* pair of functions (ϕ,ψ). The "natural" alternative is then $\Omega_T^A\backslash\Omega_T^{(A,\gamma)}$. This is not the complementary set of $\Omega_T^{(A,\gamma)}$, since outside Ω_T^A there is "always" a functional relationship, which is in fact meaningless for the model.

We recall that testing a null hypothesis "we are in a subset Ω_0" of Ω, against the alternative "we are in a subset Ω_1", with of course $\Omega_0\cap\Omega_1=\emptyset$, amounts to finding a critical (rejection) region $C_n\subset\Omega$ at stage n. The *asymptotic size* for this sequence C_n of critical regions is the number:

$$\alpha \;=\; \sup\Big\{\limsup_{n\to\infty}\ \mathbb{P}(C_n\mid A):\ A\in\mathcal{F},\ A\subset\Omega_0,\ \mathbb{P}(A)>0\Big\}. \tag{4.1}$$

Moreover we say that the sequence C_n is *consistent for the alternative* Ω_1 if

$$\mathbb{P}((C_n)^c\cap\Omega_1)\;\to\;0 \tag{4.2}$$

(where $(C_n)^c$ is the complement of C_n in Ω). The latter also amounts to say that $\mathbb{P}(C_n\mid A)\to 1$ as $n\to\infty$ for all $A\subset\Omega_1$ having $\mathbb{P}(A)>0$.

The previous limiting results allow us to construct a test with a given asymptotic level $\alpha\in(0,1)$ for the null hypothesis $\Omega_T^{(A,\gamma)}$. The procedure is as follows, at any given stage n. We single out two cases:

- (a) We have $\Delta_i^n X\notin A$ for all $i\le[T/\Delta_n]$ except perhaps one value of i: then, at stage n, it "looks like" there is 0 or 1 jump of X with size in A, and we cannot perform the test at all (asymptotically, as $\Delta_n\to 0$, this cannot occur if we are inside Ω_T^A).
- (b) We have $\Delta_i^n X\in A$ for at least two values of i smaller than $[T/\Delta_n]$. Then we proceed.

We simulate M copies of the variables (V_i^-,V_i^+) and plug them into (2.30), always with the same observed values of the increments $\Delta_i^n X$. This gives a sequence $(\overline{\Phi}_{T,j}^n: j=1,\dots,M)$ of variables which, conditionally on the partially observed path, is i.i.d. with the same (conditional) law as $\overline{\Phi}_T^n$ in (2.30). In particular, the simulated values $\overline{\Phi}_{T,j}^n$ are all distinct (almost surely). We then consider the empirical α-quantile

$$\widehat{A}_{n,M}(\alpha)=\overline{\Phi}_{n,([\alpha M])},\quad \begin{array}{l}\text{where } \overline{\Phi}_{T,(1)}^n<\overline{\Phi}_{T,(2)}^n<\cdots<\overline{\Phi}_{T,(M)}^n \text{ is the increasing}\\ \text{re-ordering of the sequence } (\overline{\Phi}_{T,j})_{1\le j\le M}^n.\end{array} \tag{4.3}$$

Then $\widehat{A}_{n,M}(\alpha)$ converges to the (random, $\mathcal{F}$-measurable) number $A_n(\alpha)$ satisfying, as $M\to\infty$,

$$\widetilde{\mathbb{P}}(\overline{\Phi}_T^n<A_n(\alpha)\mid\mathcal{F})\;=\;\alpha, \tag{4.4}$$

Doing this with a number of simulations $M = M_n$, we can take the critical region

$$C_n = \Big\{\Phi_T^n < \frac{\widehat{A}_{n,M_n}(\alpha)}{k_n}\Big\}. \tag{4.5}$$

Theorem 4.1. *Assume (A-r) for some $r \in [0,2)$ and (2.5) and (2.4). Then, and as soon as $M_n \to \infty$, the critical regions defined by (4.5) have the asymptotic level α for testing the null hypothesis $\Omega_T^{(A,\gamma)}$, and are consistent for the (partial) alternative $\Omega_T^A \backslash \Omega_T^{(A,\gamma)}$.*

Proof. First, $A_n(\alpha)$ converges in probability to the α-quantile $A(\alpha)$ of $\overline{\Phi}$ knowing $\mathcal{F}$, and this is true under the null, and under the alternative as well. So, since by Theorem 2.1 $k_n\Phi_T^n$ converges stably in law to $\overline{\Phi}_T$ under the null and to $-\infty$ under the alternative (because $\Phi_n \xrightarrow{\mathbb{P}} \Phi < 0$ on $\Omega_T^A \backslash \Omega_T^{(A,\gamma)}$)), the proof is the same as for Theorem 4.2 of [18] for example. $\square$

5 Conclusion

A simulation study has shown that the test procedure works reasonably well. However, a real data analysis manifested that such a simple functional relationship cannot be found in the data. For the COGARCH model the statistical analysis of [15], cf. Table 2, gave a partly positive answer, where a test for non-correlation between common jumps in the price and the squared volatility was not rejected even on a 2% level in case of the SPDR S&P 500 ETF (SPY), an exchange traded fund that tracks the S&P 500 index. The possibility to test for a finite number of factors or regimes γ is currently investigated.

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