Web supplement: Derivatives and Fisher information of bivariate copulas

Ulf Schepsmeier
and
Jakob Stöber

Lehrstuhl für Mathematische Statistik,
Technische Universität München,
Parkring 13, 85748 Garching-Hochbrück, Germany,
contact email: schepsmeier@ma.tum.de,
phone: +49 89 289 17425.

1 Introduction

This manuscript contains additional material to the paper "Derivatives and Fisher Information" of Schepsmeier and Stöber (2012) and considers partial derivatives corresponding to several parametric copula families. In Chapter 2 and Chapter 3 we list the partial derivatives of considered elliptical and Archimedean copulas, respectively. Especially the Gauss and Student's t-copula are treated in detail. For these two families as well as for the (Archimedean) Clayton copula the partial derivatives of the density function and the conditional distribution function of first and second order are derived. For the remaining Archimedean copulas, i.e. Gumbel, Frank and Joe, only the partial derivatives of first order are considered. The results for Student's t-copula are based on Dakovic and Czado (2011), where we corrected some flaws.

To improve the standalone readability of this manuscript, definitions and notation from the main paper are repeated where necessary.

2 Derivatives corresponding to elliptical copulas

The following listed derivatives of the elliptical copula families, Gauss and Student's t, are the prosecution of Chapter 3 in the main article by Schepsmeier and Stöber (2012). For better readability we also note the definitions of the copulas, although they are already noted in the manuscript.

2.1 Gaussian copula

The cumulative distribution function (cdf) of the Gaussian copula family is given by

$$C(u_1, u_2; \rho) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \rho),$$

where $\Phi_2(\cdot, \cdot, \rho)$ is the cdf of two standard normally distributed random variables with correlation $\rho \in (-1, 1)$, Φ is the cdf of $N(0, 1)$ (standard normal) and Φ^{-1} (the quantile function) is its functional inverse. The density is

$$c(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} \exp\left\{ -\frac{\rho^2(x_1^2 + x_2^2) - 2\rho x_1 x_2}{2(1-\rho^2)} \right\},$$

where $x_1 = \Phi^{-1}(u_1)$ and $x_2 = \Phi^{-1}(u_2)$.
The cdf of the first variable U_1 given the second is

$$h(u_1, u_2; \rho) = \frac{\partial}{\partial u_2} C(u_1, u_2; \rho) = \Phi_2\left(\frac{\Phi^{-1}(u_1) - \rho\Phi^{-1}(u_2)}{\sqrt{1-\rho^2}} \right).$$

Derivatives of the density function

Taking the first derivative of the density $c(u_1, u_2; \rho)$ with respect to the correlation parameter ρ we obtain

$$\frac{\partial c}{\partial \rho} = -\frac{(\rho^3 - x_1 x_2 \rho^2 + \rho x_2^2 + \rho x_1^2 - \rho - x_1 x_2) \exp\left\{ \frac{1}{2} \frac{\rho(\rho x_1^2 + \rho x_2^2 - 2x_1 x_2)}{(\rho-1)(\rho+1)} \right\}}{(1-\rho^2)^{\frac{5}{2}}}.$$

The derivative with respect to u_1 is

$$\frac{\partial c}{\partial u_1} = c(u_1, u_2; \rho) \left(-\frac{(2\rho^2 x_1 \frac{\partial x_1}{\partial u_1} - 2\rho x_2 \frac{\partial x_1}{\partial u_1})}{2(1-\rho^2)} \right),$$

where

$$\frac{\partial x_i}{\partial u_i} = \frac{\sqrt{2\pi}}{exp\{-\Phi^{-1}(u_i)^2/2\}} \quad i = 1, 2.$$

Further, $\partial_2 c(u_1, u_2) = \partial_1 c(u_2, u_1)$.

Derivatives of the h-function

The first derivative of the conditional cdf $h(u_1, u_2; \rho)$ with respect to the copula parameter ρ is

$$\frac{\partial h}{\partial \rho} = \phi\left(\frac{\Phi^{-1}(u_1) - \rho\Phi^{-1}(u_2)}{\sqrt{1-\rho^2}} \right) \cdot \frac{-\Phi^{-1}(u_2)\sqrt{1-\rho^2} - [\Phi^{-1}(u_1) - \rho\Phi^{-1}(u_2)]\frac{-\rho}{\sqrt{1-\rho^2}}}{1-\rho^2}.$$

The derivative with respect to u_1 is c and the derivative with respect to u_2 is

$$\frac{\partial h}{\partial u_2} = \phi\left(\frac{\Phi^{-1}(u_1) - \rho\Phi^{-1}(u_2)}{\sqrt{1-\rho^2}}\right) \cdot \frac{-\rho}{\sqrt{1-\rho^2}} \cdot \frac{\partial x_2}{\partial u_2}.$$

Second derivatives of the density function

The second derivative of $c(u_1, u_2; \rho)$ w.r.t. the correlation parameter ρ is

$$\begin{aligned}
\frac{\partial^2 c}{\partial^2 \rho} = \frac{1}{(1-\rho^2)^5}\Bigg(&-(1-\rho^2)^{\frac{5}{2}}\Bigg(\Bigg[3\rho^2 - \frac{\partial x_1}{\partial \rho}x_2\rho^2 - x_1\frac{\partial x_2}{\partial \rho}\rho^2 - 2x_1x_2\rho + x_2^2 + 2\rho\frac{\partial x_2}{\partial \rho}x_2 \\
&+ x_1^2 + 2\rho\frac{\partial x_1}{\partial \rho}x_1 - 1 - \frac{\partial x_1}{\partial \rho}x_2 - x_1\frac{\partial x_2}{\partial \rho}\Bigg] \exp\left\{\frac{1}{2}\frac{\rho(\rho x_1^2 + \rho x_2^2 - 2x_1x_2)}{(\rho-1)(\rho+1)}\right\} \\
&+ \left(\rho^3 - x_1x_2\rho^2 + \rho x_2^2 + \rho x_1^2 - \rho - x_1x_2\right)\exp\left\{\frac{1}{2}\frac{\rho(\rho x_1^2 + \rho x_2^2 - 2x_1x_2)}{(\rho-1)(\rho+1)}\right\} \\
&\cdot \frac{1}{2}\Bigg(\Bigg[(\rho x_1^2 + \rho x_2^2 - 2x_1x_2) + \rho\left(x_1^2 + 2\rho x_1\frac{\partial x_1}{\partial \rho} + x_2^2 + 2\rho x_2\frac{\partial x_2}{\partial \rho} - 2\frac{\partial x_1}{\partial \rho}x_2 - 2\frac{\partial x_2}{\partial \rho}x_1\right)\Bigg] \\
&\cdot (\rho-1)(\rho+1) - \left(\rho(\rho x_1^2 + \rho x_2^2 - 2x_1x_2)\right)[(\rho-1)+(\rho+1)]\Bigg)\Bigg/(\rho-1)^2(\rho+1)^2\Bigg) \\
&- \left(\rho^3 - x_1x_2\rho^2 + \rho x_2^2 + \rho x_1^2 - \rho - x_1x_2\right)\exp\left\{\frac{1}{2}\frac{\rho(\rho x_1^2 + \rho x_2^2 - 2x_1x_2)}{(\rho-1)(\rho+1)}\right\}(1-\rho^2)^{\frac{3}{2}}5\rho\Bigg).
\end{aligned}$$

The next two derivatives involve the first copula argument u_1. We differentiate twice with respect to u_1 and in the second case we differentiate with respect to u_1 and the correlation parameter ρ. Thus, for the second derivative w.r.t. u_1 we obtain

$$\frac{\partial^2 c}{\partial^2 u_1} = \frac{\partial c}{\partial u_1}\left(-\frac{2\rho^2 x_1\frac{\partial x_1}{\partial u_1} - 2\rho x_2\frac{\partial x_1}{\partial u_1}}{2(1-\rho^2)}\right) + c(u_1,u_2)\left(-\frac{2\rho^2\left(\left(\frac{\partial x_1}{\partial u_1}\right)^2 + x_1\frac{\partial^2 x_1}{\partial^2 u_1}\right) - 2\rho x_2\frac{\partial^2 x_1}{\partial^2 u_1}}{2(1-\rho^2)}\right),$$

the partial derivative w.r.t. ρ and u_1 is

$$\begin{aligned}
\frac{\partial^2 c}{\partial \rho u_1} = &-\frac{1}{(1-\rho^2)^{\frac{5}{2}}}\left(-\frac{\partial x_1}{\partial u_1}x_2\rho^2 + 2\rho x_1\frac{\partial x_1}{\partial u_1} - \frac{\partial x_1}{\partial u_1}\right)\exp\left\{\frac{1}{2}\frac{\rho(\rho x_1^2 + \rho x_2^2 - 2x_1x_2)}{(\rho-1)(\rho+1)}\right\} \\
&+ \frac{\partial c}{\partial \rho}\frac{\rho}{2(\rho-1)(\rho+1)}\left(2\rho x_1\frac{\partial x_1}{\partial u_1} - 2\frac{\partial x_1}{\partial u_1}x_2\right).
\end{aligned}$$

Finally, we have the partial derivative of c with respect to u_1, u_2:

$$\frac{\partial c}{\partial u_1 \partial u_2} = c(u_1,u_2;\rho)\left[\left(\frac{(2\rho^2 x_2\frac{\partial x_2}{\partial u_2} - 2\rho x_1\frac{\partial x_2}{\partial u_2})}{2(1-\rho^2)}\right)\left(\frac{(2\rho^2 x_1\frac{\partial x_1}{\partial u_1} - 2\rho x_2\frac{\partial x_1}{\partial u_1})}{2(1-\rho^2)}\right) + \frac{\rho\frac{\partial x_1}{\partial u_1}\frac{\partial x_2}{\partial u_2}}{1-\rho^2}\right].$$

Second derivatives of the h-function

The second derivatives of h only need to be determined with respect to the second argument u_2 and the copula parameter ρ, since $\frac{\partial h}{\partial u_2} = c$. For the second derivative w.r.t. ρ we obtain

$$\frac{\partial^2 h}{\partial^2 \rho} = \frac{\partial}{\partial t}\phi(t)\bigg|_{t=\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right)} \left(\frac{-x_2\sqrt{1-\rho^2} + (x_1-\rho x_2)\frac{\rho}{\sqrt{1-\rho^2}}}{1-\rho^2}\right)^2$$
$$+ \phi\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right)\left(\frac{(x_1-\rho x_2)(\sqrt{1-\rho^2}+\frac{\rho^2}{\sqrt{1-\rho^2}}) + \left(x_1\sqrt{1-\rho^2} - (x_1-\rho x_2)\frac{\rho}{\sqrt{1-\rho^2}}\right)}{(1-\rho^2)^2}\right),$$

where

$$\phi(t) = \frac{1}{\sqrt{2\pi}}\exp\{-t^2/2\}$$

and

$$\frac{\partial}{\partial t}\phi(t) = \frac{1}{\sqrt{2\pi}}\exp\{-t^2/2\}(-t) = -t\phi(t).$$

Thus

$$\frac{\partial}{\partial t}\phi(t)\bigg|_{t=\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right)} = -\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right)\phi\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right).$$

Secondly, we have the second derivative w.r.t. u_2.

$$\frac{\partial^2 h}{\partial^2 u_2} = \frac{\partial}{\partial t}\phi(t)\bigg|_{t=\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right)} \frac{\rho^2}{1-\rho^2}\left(\frac{\partial x_2}{\partial u_2}\right)^2 - \Phi\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right)\frac{\rho}{\sqrt{1-\rho^2}}\frac{\partial^2 x_2}{\partial^2 u_2},$$

where

$$\frac{\partial^2 x_2}{\partial^2 u_2} = \frac{\sqrt{2\pi}}{exp\{-\Phi^{-1}(u_i)^2/2\}}\Phi^{-1}(u_i)\frac{\partial x_i}{\partial u_i} = \left(\frac{\partial x_i}{\partial u_i}\right)^2\Phi^{-1}(u_i) \quad i = 1, 2,$$

and finally the partial derivative w.r.t ρ and u_2

$$\frac{\partial^2 h}{\partial\rho\partial u_2} = \frac{\partial}{\partial t}\phi(t)\bigg|_{t=\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right)} \left(\frac{-x_2\sqrt{1-\rho^2} + (x_1-\rho x_2)\frac{\rho}{\sqrt{1-\rho^2}}}{1-\rho^2}\right)\frac{-\rho}{\sqrt{1-\rho^2}}\frac{\partial x_2}{\partial u_2}$$
$$+ \phi\left(\frac{x_1-\rho x_2}{\sqrt{1-\rho^2}}\right)\frac{-\sqrt{1-\rho^2}+\frac{\rho^2}{\sqrt{1-\rho^2}}}{1-\rho^2}\frac{\partial x_2}{\partial u_2}.$$

2.2 Student's t-copula

Derivatives of the density function

The log-density of Student's t-copula is given by

$$
\begin{aligned}
l(u_1,u_2;\rho,\nu) &= \ln c(u_1,u_2;\rho,\nu)\\
&= -\ln(2) + \frac{\nu+1}{2}\ln(1-\rho^2) - 2\ln\left(\Gamma\left(\frac{\nu+1}{2}\right)\right) + 2\ln\left(\Gamma\left(\frac{\nu}{2}\right)\right)\\
&- \frac{\nu-2}{2}\ln(\nu) + \frac{\nu+1}{2}\left[\ln\left(\nu+x_1^2\right)+\ln\left(\nu+x_2^2\right)\right] - \frac{\nu+2}{2}\ln(M(\nu,\rho,x_1,x_2)),
\end{aligned}
$$

where

$$
\begin{aligned}
M(\nu,\rho,x_1,x_2) &:= \nu(1-\rho^2)+x_1^2+x_2^2-2\rho x_1x_2,\\
x_i &:= t_\nu^{-1}(u_i), \quad u_i\in(0,1), \quad i=1,2.
\end{aligned}
$$

We follow the approach by Dakovic and Czado (2011) to write the cdf $t_\nu(x_i)$ of the univariate t-distribution in terms of the regularized beta function I:

$$
t_\nu(x_i) = \begin{cases} 1-\frac{1}{2}I_{\frac{\nu}{\nu+x_i^2}}\left(\frac{\nu}{2},\frac{1}{2}\right), & x_i\geq 0,\\ \frac{1}{2}I_{\frac{\nu}{\nu+x_i^2}}\left(\frac{\nu}{2},\frac{1}{2}\right), & x_i<0,\end{cases}
$$

where

$$
I_{\frac{\nu}{\nu+x_i^2}}\left(\frac{\nu}{2},\frac{1}{2}\right) = \frac{\int_0^{\frac{\nu}{\nu+x_i^2}} t^{\frac{\nu}{2}-1}(1-t)^{-\frac{1}{2}}dt}{B\left(\frac{\nu}{2},\frac{1}{2}\right)} \quad i=1,2,
$$

(Abramowitz and Stegun, 1992, p. 948). Let us denote the density of the univariate t-distribution with degrees of freedom ν by $dt(x_i;\nu)$. Then, this implies for the partial derivative $\frac{\partial}{\partial\nu}t_\nu^{-1}(u_i)$, which we denote as $\frac{\partial x_i}{\partial \nu}$, that

$$
\frac{\partial x_i}{\partial\nu} = -\frac{\left(\frac{\partial}{\partial\nu}t_\nu\right)(x_i)}{dt(x_i;\nu)}, \quad i=1,2. \tag{1}
$$

Expanding the expression for the cdf $t_\nu(x_i)$ in Equation (1) we obtain

$$
\begin{aligned}
\frac{\partial x_i}{\partial\nu} &= \frac{1}{2dt(x_i;\nu)}\frac{1}{B\left(\frac{\nu}{2},\frac{1}{2}\right)}\left[\left(\frac{1}{x_i^2+\nu}\right)^{\frac{\nu+1}{2}}\nu^{\frac{\nu}{2}-1}x_i + \frac{1}{2}\int_0^{\frac{\nu}{\nu+x_i^2}} t^{\frac{\nu}{2}-1}(1-t)^{-\frac{1}{2}}\ln(t)dt\right]\\
&- \frac{1}{4dt(x_i;\nu)}I_{\frac{\nu}{\nu+x_i^2}}\left(\frac{\nu}{2},\frac{1}{2}\right)\left[\Psi\left(\frac{\nu}{2}\right)-\Psi\left(\frac{\nu+1}{2}\right)\right]\\
&= \frac{1}{dt(x_1;\nu)}\left[\frac{1}{2}\partial_1 I_{\frac{\nu}{\nu+x_i^2}}\left(\frac{\nu}{2},\frac{1}{2}\right)+\left(\frac{1}{x_i^2+\nu}\right)^{\frac{\nu+1}{2}}\nu^{\frac{\nu}{2}-1}x_i\right], \quad u_i\geq\frac{1}{2}, \ i=1,2.
\end{aligned}
$$

Here, $\partial_1 I_{\frac{\nu}{\nu+x_i^2}}\left(\frac{\nu}{2},\frac{1}{2}\right)$ is the partial derivative of the regularized beta function with respect to the first argument in brackets, i.e. $\frac{d}{dx}I_{\frac{\nu}{\nu+x_i^2}}(t,\frac{1}{2})$ evaluated at t.

The partial derivative of l with respect to ρ is

$$\frac{\partial l}{\partial \rho} = -(\nu+1)\frac{\rho}{1-\rho^2} + (\nu+2)\frac{\nu\rho + x_1 x_2}{M(\nu,\rho,x_1,x_2)},$$

which implies for the copula density c that

$$\frac{\partial c}{\partial \rho} = \frac{\partial l}{\partial \rho} \cdot c(u_1,u_2).$$

Similarly, the derivative with respect to the degrees of freedom is

$$\begin{aligned}
\frac{\partial l}{\partial \nu} &= -\Psi\left(\frac{\nu+1}{2}\right) + \Psi\left(\frac{\nu}{2}\right) + \frac{1}{2}\ln(1-\rho^2) - \frac{\nu-2}{2\nu} - \frac{1}{2}\ln(\nu) \\
&+ \frac{\nu+1}{2}\left[\frac{1+2x_1\frac{\partial x_1}{\partial \nu}}{\nu+x_1^2} + \frac{1+2x_2\frac{\partial x_2}{\partial \nu}}{\nu+x_2^2}\right] + \frac{1}{2}\ln\left((\nu+x_1^2)(\nu+x_2^2)\right) \\
&- \frac{\nu+2}{2}\frac{1-\rho^2+2x_1\frac{\partial x_1}{\partial \nu} + 2x_2\frac{\partial x_2}{\partial \nu} - 2\rho\left(x_1\frac{\partial x_2}{\partial \nu} + x_2\frac{\partial x_1}{\partial \nu}\right)}{M(\nu,\rho,x_1,x_2)} - \frac{1}{2}\ln(M(\nu,\rho,x_1,x_2)),
\end{aligned}$$

where $\Psi(\cdot)$ denotes the digamma function.
The partial derivative of the Student's t-copula density with respect to u_1 is

$$\frac{\partial c}{\partial u_1} = -\frac{c(u_1,u_2)}{dt(x_1;\nu)}\left(\frac{(\nu+2)(x_1-\rho x_2)}{\nu(1-\rho^2)+x_1^2+x_2^2-2\rho x_1 x_2} + \frac{\partial}{\partial u_1}dt(x_1;\nu)\right),$$

where

$$\frac{\partial}{\partial u_i}dt(x_i(u_i);\nu) = \frac{dt'(x_i;\nu)}{dt(x_i;\nu)} = -\frac{x_i\frac{\nu+1}{\nu}}{1+\frac{x_i^2}{\nu}}.$$

Derivatives of the h-function with respect to ρ, ν and u_2

The partial derivatives with respect to ρ, ν and u_2, respectively, are given as follows:

$$\frac{\partial h}{\partial \rho} = dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}};\nu+1\right)\left[\frac{-x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}} - \frac{1}{2}\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(-2\rho)\frac{\nu+x_2^2}{\nu+1}\right],$$

$$\begin{aligned}
\frac{\partial h}{\partial \nu} = &\left(\frac{\partial}{\partial \nu}t_{\nu+1}\right)\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}}\right) + dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}};\nu+1\right) \\
&\cdot\left(\frac{\frac{\partial x_1}{\partial \nu} - \rho\frac{\partial x_2}{\partial \nu}}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}} - \frac{1}{2}\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(1-\rho^2)\left[\frac{1+2x_2\frac{\partial x_2}{\partial \nu}}{\nu+1} - \frac{\nu+x_2^2}{(\nu+1)^2}\right]\right),
\end{aligned}$$

$$\frac{\partial h}{\partial u_2} = dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}};\nu+1\right)\frac{1}{dt(x_2;\nu)}\left[\frac{-\rho}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}} - \frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{\nu+1}x_2\right].$$

These can be evaluated using our previous result for $\frac{\partial x_i}{\partial \nu}$, since

$$\left(\frac{\partial}{\partial \nu}t_\nu\right)(x_i) = -\frac{1}{2}\frac{\partial}{\partial \nu}I_{\frac{\nu}{\nu+x_i^2}}\left(\frac{\nu}{2},\frac{1}{2}\right) = -\frac{\partial x_i}{\partial \nu}dt(x_i;\nu).$$

Second derivatives of the density function

We compute the second partial derivatives of the copula density c. Where this leads to simpler expressions, we state the corresponding derivatives of the log density l from which (for ρ) are related as follows:

$$\frac{\partial^2 c}{\partial^2\rho} = \frac{\partial}{\partial\rho}\left(\frac{\partial c}{\partial \rho}\right) = \frac{\partial}{\partial\rho}\left(\frac{\partial l}{\partial\rho}\cdot c\right) = \frac{\partial^2 l}{\partial^2\rho}\cdot c + \frac{\partial l}{\partial\rho}\frac{\partial c}{\partial\rho} = \frac{\partial^2 l}{\partial^2\rho}\cdot c + \frac{\frac{\partial c}{\partial\rho}}{c}\frac{\partial c}{\partial\rho},$$

The second derivatives of the log-likelihood function l with respect to ρ and ν are:

$$\frac{\partial^2 l}{\partial^2\rho} = -(\nu+1)\frac{1+\rho^2}{(1-\rho^2)^2} + \frac{(\nu+2)\nu}{M(\nu,\rho,x_1,x_2)} + 2(\nu+2)\frac{(\nu\rho+x_1x_2)^2}{M(\nu,\rho,x_1,x_2)^2}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial^2\nu} &= -\frac{1}{2}\Psi_1\left(\frac{\nu+1}{2}\right) + \frac{1}{2}\Psi_1\left(\frac{\nu}{2}\right) - \frac{1}{\nu^2} - \frac{1}{2\nu} + \frac{1}{2}\left(\frac{1+2x_1\frac{\partial x_1}{\partial\nu}}{\nu+x_1^2} + \frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+x_2^2}\right)\\
&+ \frac{\nu+1}{2}\left(\frac{2\left(\frac{\partial x_1}{\partial\nu}\right)^2 + 2x_1\frac{\partial^2 x_1}{\partial^2\nu}}{\nu+x_1^2} - \frac{1+2x_1\frac{\partial x_1}{\partial\nu}}{(\nu+x_1^2)^2}\left(1+2x_1\frac{\partial x_1}{\partial\nu}\right)\right.\\
&+ \left.\frac{2\left(\frac{\partial x_2}{\partial\nu}\right)^2 + 2x_2\frac{\partial^2 x_2}{\partial^2\nu}}{\nu+x_2^2} - \frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{(\nu+x_2^2)^2}\left(1+2x_2\frac{\partial x_2}{\partial\nu}\right)\right)\\
&+ \frac{1}{2}\frac{1+2x_1\frac{\partial x_1}{\partial\nu}}{\nu+x_1^2} + \frac{1}{2}\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+x_2^2} - \frac{\frac{\partial M}{\partial\nu}}{M(\nu,\rho,x_1,x_2)} - \left(\frac{\nu}{2}+1\right)\frac{M(\nu,\rho,x_1,x_2)\frac{\partial^2 M}{\partial^2\nu} - \left(\frac{\partial M}{\partial\nu}\right)^2}{M(\nu,\rho,x_1,x_2)^2}.\end{aligned}$$

Here,

$$\frac{\partial M}{\partial\nu} = (1-\rho^2) + 2x_1\frac{\partial x_1}{\partial\nu} + 2x_2\frac{\partial x_2}{\partial\nu} - 2\rho\left(x_2\frac{\partial x_1}{\partial\nu} + x_1\frac{\partial x_2}{\partial\nu}\right),$$

$$\frac{\partial^2 M}{\partial^2\nu} = 2\left(\frac{\partial x_1}{\partial\nu}\right)^2 + 2x_1\frac{\partial^2 x_1}{\partial^2\nu} + 2\left(\frac{\partial x_2}{\partial\nu}\right)^2 + 2x_2\frac{\partial^2 x_2}{\partial^2\nu} - 4\rho\frac{\partial x_1}{\partial\nu}\frac{\partial x_2}{\partial\nu} - 2\rho\left(x_2\frac{\partial^2 x_1}{\partial^2\nu} + x_1\frac{\partial^2 x_2}{\partial^2\nu}\right),$$

and

$$\begin{aligned}\frac{\partial^2 x_i}{\partial^2\nu} &= -\frac{1}{dt(x_i;\nu)}\left(\left(\frac{\partial^2}{\partial^2\nu}t_\nu\right)(x_i) + \left(\frac{\partial}{\partial\nu}dt\right)(x_i;\nu)\frac{\partial x_i}{\partial\nu}\right)\\
&+ \frac{\left(\frac{\partial}{\partial\nu}t_\nu\right)(x_i)}{dt(x_i;\nu)^2}\left[\left(\frac{\partial}{\partial\nu}dt\right)(x_i;\nu) + dt'(x_i;\nu)\frac{\partial x_i}{\partial\nu}\right].\end{aligned}$$

$\Psi_1(\cdot)$ is the trigamma function, the second derivative of the logarithm of the gamma function ($\Psi_1(z) = \frac{d^2}{d^2 z}\ln\Gamma(z)$). This expression for $\frac{\partial^2 x_i}{\partial^2\nu}$ involves the second partial derivative of the cdf $t_\nu(x_i)$, as well as the partial derivatives of $dt(x_i,\nu)$ with respect to ν (denoted $\left(\frac{\partial}{\partial\nu}dt\right)$) and x_i (denoted dt'), respectively:

$$
\begin{aligned}
\frac{\partial^2}{\partial^2\nu}t_\nu(x_i) = &-\frac{1}{2}\left[-\frac{1}{2}\frac{1}{B\left(\frac{\nu}{2},\frac{1}{2}\right)}\left[\left(\frac{1}{x_i^2+\nu}\right)^{\frac{\nu+1}{2}}\nu^{\frac{\nu}{2}-1}x_i+\frac{1}{2}\int_0^{\frac{\nu}{\nu+x_i^2}}t^{\frac{\nu}{2}-1}(1-t)^{-\frac{1}{2}}\ln(t)dt\right]\right]\\
&\cdot\left[\Psi\left(\frac{\nu}{2}\right)-\Psi\left(\frac{\nu}{2}+\frac{1}{2}\right)\right]\\
&-\frac{1}{2}\left(\frac{\partial}{\partial\nu}t_\nu\right)(x_i)\left[\Psi\left(\frac{\nu}{2}\right)-\Psi\left(\frac{\nu}{2}+\frac{1}{2}\right)\right]+\frac{1}{8}I_{\frac{\nu}{\nu+x_i^2}}\left(\frac{\nu}{2},\frac{1}{2}\right)\left[\Psi_1\left(\frac{\nu}{2}\right)-\Psi_1\left(\frac{\nu+1}{2}\right)\right]\\
&-\frac{1}{2}\frac{1}{B\left(\frac{\nu}{2},\frac{1}{2}\right)}\left[\left(\left(\frac{1}{x_i^2+\nu}\right)^{\frac{\nu+1}{2}}\nu^{\frac{\nu}{2}-1}x_i\right)\right.\\
&\cdot\left[-\frac{1}{x_i^2+\nu}\frac{\nu+1}{2}+\left(\frac{\nu}{2}-1\right)\frac{1}{\nu}-\frac{\ln(x_i^2+\nu)}{2}+\frac{\ln(\nu)}{2}\right]\\
&+\frac{1}{2}\left(\frac{\nu}{x_i^2+\nu}\right)^{\frac{\nu}{2}-1}\left(\frac{x_i^2}{\nu+x_i^2}\right)^{-\frac{1}{2}}\ln\left(\frac{\nu}{\nu+x_i^2}\right)\left(\frac{x_i^2}{(\nu+x_i^2)^2}\right)\\
&\left.+\frac{1}{4}\int_0^{\frac{\nu}{\nu+x_i^2}}t^{\frac{\nu}{2}-1}(1-t)^{-\frac{1}{2}}\ln(t)^2dt\right],
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial}{\partial\nu}dt\right)(x_i;\nu) = &\frac{\Psi\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)-\Gamma\left(\frac{\nu+1}{2}\right)\left[\Psi\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)+\Gamma\left(\frac{\nu}{2}\right)\nu^{-1}\right]}{2\Gamma\left(\frac{\nu}{2}\right)^2\sqrt{\pi\nu}}\cdot\left(1+\frac{x_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}\\
&+\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\cdot\left(\left(1+\frac{x_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}\left(-\frac{1}{2}\ln\left(1+\frac{x_i^2}{\nu}\right)-\frac{\left(-\frac{\nu+1}{2}\right)x_i^2}{\nu^2\left(1+\frac{x_i^2}{\nu}\right)}\right)\right),
\end{aligned}
$$

$$
dt'(x_i;\nu) = -\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\frac{\nu+1}{\nu}x_i\left(1+\frac{x_i^2}{\nu}\right)^{-\frac{\nu+3}{2}}.
$$

Taking the partial derivatives of l with respect to both parameters yields

$$
\begin{aligned}
\frac{\partial^2 l}{\partial\rho\partial\nu} = &-\frac{\rho}{1-\rho^2}+(\nu+2)\left[\frac{\rho+\left(x_1\frac{\partial x_2}{\partial\nu}+x_2\frac{\partial x_1}{\partial\nu}\right)}{M(\nu,\rho,x_1,x_2)}+\frac{1}{2}\frac{\partial M}{\partial\nu}\frac{\frac{\partial M}{\partial\rho}}{M(\nu,\rho,x_1,x_2)^2}\right]\\
&-\frac{1}{2}\frac{\frac{\partial M}{\partial\rho}}{M(\nu,\rho,x_1,x_2)},
\end{aligned}
$$

where

$$
\frac{\partial M}{\partial\rho} = -2(\nu\rho+x_1x_2).
$$

Further, the derivative of the log-density with respect to ρ and u_1 is

$$
\frac{\partial^2 l}{\partial\rho\partial u_1} = \frac{\nu+2}{M(\nu,\rho,x_1,x_2)dt(x_1;\nu)}\left(x_2-2\frac{\nu\rho+x_1x_2}{M(\nu,\rho,x_1,x_2)}(x_1-\rho x_2)\right).
$$

The derivatives of the copula density with respect to ν and u_1 can be determined similarly:

$$\frac{\partial^2 c}{\partial\nu\partial u_1} = \left[-\frac{\frac{\partial c}{\partial \nu}}{dt(x_1;\nu)} + \frac{c(u_1,u_2)}{dt(x_1;\nu)^2} \cdot \left(\left(\frac{\partial}{\partial \nu} dt \right)(x_1;\nu) + dt'(x_1;\nu)\frac{\partial x_1}{\partial \nu} \right) \right]$$
$$\cdot \left(\frac{(\nu+2)(x_1-\rho x_2)}{M(\nu,\rho,x_1,x_2)} + \frac{\partial}{\partial u_1} dt(x_1;\nu) \right)$$
$$- \frac{c(u_1,u_2)}{dt(x_1;\nu)} \cdot \left(\frac{x_1-\rho x_2}{M(\nu,\rho,x_1,x_2)} - \frac{(\nu+2)(x_1-\rho x_2)}{M(\nu,\rho,x_1,x_2)^2}\frac{\partial M}{\partial \nu} + \frac{(\nu+2)\left(\frac{\partial x_1}{\partial \nu} - \rho\frac{\partial x_2}{\partial \nu}\right)}{M(\nu,\rho,x_1,x_2)} \right.$$
$$\left. + \left[-\frac{\partial x_1}{\partial \nu}\frac{\frac{\nu+1}{\nu}}{1+\frac{x_1^2}{\nu}} + \frac{x_1}{\nu^2+\nu x_1^2} + \frac{x_1\frac{\nu+1}{\nu}}{\left(1+\frac{x_1^2}{\nu}\right)^2}\left(\frac{2x_1\frac{\partial x_1}{\partial \nu}}{\nu} - \frac{x_1^2}{\nu^2} \right) \right] \right).$$

For the second partial derivative with respect to the argument u_1 we obtain

$$\frac{\partial^2 c}{\partial^2 u_1} = \left(\frac{-\frac{\partial c}{\partial u_1}}{dt(x_1;\nu)} + \frac{\frac{\partial}{\partial u_1}dt(x_1;\nu)}{dt(x_1;\nu)^2} \cdot c(u_1,u_2) \right) \left(\frac{(\nu+2)(x_1-\rho x_2)}{M(\nu,\rho,x_1,x_2)} + \frac{\partial}{\partial u_1} dt(x_1;\nu) \right)$$
$$- \frac{c(u_1,u_2)}{dt(u_1;\nu)} \left(\frac{1}{dt(x_1;\nu)} \left[\frac{\nu+2}{M(\nu,\rho,x_1,x_2)} - \frac{2(\nu+2)(x_1-\rho x_2)^2}{M(\nu,\rho,x_1,x_2)^2} \right] + \frac{\partial^2}{\partial^2 u_1} dt(x_1;\nu) \right),$$

where

$$\frac{\partial^2}{\partial^2 u_1} dt(x_1;\nu) = \frac{1}{dt(x_1;\nu)} \left[-\frac{\nu+1}{\nu+x_1^2} + \frac{2x_1^2\frac{\nu+1}{\nu^2}}{\left(1+\frac{x_1^2}{\nu}\right)^2} \right].$$

Finally, the second partial derivative with respect to u_1 and u_2 is

$$\frac{\partial^2 c}{\partial u_1 \partial u_2} = \frac{c(u_1,u_2)}{dt(x_1;\nu)dt(x_2;\nu)} \left[\left(\frac{(\nu+2)(x_1-\rho x_2)}{M(\nu,\rho,x_1,x_2)} + \frac{\partial}{\partial u_1} dt(x_1;\nu) \right) \right.$$
$$\left. \cdot \left(\frac{(\nu+2)(x_2-\rho x_1)}{M(\nu,\rho,x_1,x_2)} + \frac{\partial}{\partial u_2} dt(x_2;\nu) \right) + \frac{(\nu+2)\rho}{M(\nu,\rho,x_1,x_2)} + 2\frac{(\nu+2)(x_1-\rho x_2)(x_2-\rho x_1)}{M(\nu,\rho,x_1,x_2)^2} \right].$$

Second derivatives of the h-function

In this section, we obtain the second partial derivatives of the conditional distribution function (h-function). Taking derivatives with respect to the association parameter ρ we get

$$\frac{\partial^2 h}{\partial^2 \rho} = dt'\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)\left[\frac{-x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}+\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\rho\frac{\nu+x_2^2}{\nu+1}\right]^2$$

$$+\,dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)\left[\left(\frac{x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}-\frac{3}{2}\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{5}{2}}}\rho\frac{\nu+x_2^2}{\nu+1}\right)\right.$$

$$\left.\cdot\,(-2\rho)\frac{\nu+x_2^2}{\nu+1}+\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{\nu+x_2^2}{\nu+1}\right].$$

Using similar notation and techniques as for the derivatives of the log-likelihood function we obtain

$$\frac{\partial^2 h}{\partial^2 \nu} = \left(\frac{\partial^2}{\partial^2 \nu}t_{\nu+1}\right)\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}\right)+2\left(\frac{\partial}{\partial \nu}dt\right)\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)$$

$$\cdot\left(\frac{\frac{\partial x_1}{\partial\nu}-\rho\frac{\partial x_2}{\partial\nu}}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}-\frac{1}{2}\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(1-\rho^2)\left[\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right]\right)$$

$$+\,dt'\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)$$

$$\cdot\left(\frac{\frac{\partial x_1}{\partial\nu}-\rho\frac{\partial x_2}{\partial\nu}}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}-\frac{1}{2}\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(1-\rho^2)\left[\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right]\right)^2$$

$$+ dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)$$

$$\cdot\left(\left(-\frac{\frac{\partial x_1}{\partial\nu}-\rho\frac{\partial x_2}{\partial\nu}}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}+\frac{3}{4}\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{5}{2}}}(1-\rho^2)\left[\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right]\right)\right.$$

$$\cdot(1-\rho^2)\left[\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right]+\frac{\frac{\partial^2 x_1}{\partial^2\nu}-\rho\frac{\partial^2 x_2}{\partial^2\nu}}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}-\frac{1}{2}\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(1-\rho^2)$$

$$\left.\cdot\left[\frac{2\left(\frac{\partial x_2}{\partial\nu}\right)^2+2x_2\frac{\partial^2 x_2}{\partial^2\nu}}{\nu+1}-2\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{(\nu+1)^2}+2\frac{\nu+x_2^2}{(\nu+1)^3}\right]\right).$$

Also the second partial derivatives involving u_2 follow similarly:

$$\frac{\partial^2 h}{\partial^2 u_2}=\left(\frac{dt'\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)}{dt(x_2;\nu)^2}\cdot\left[\frac{-\rho}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}-\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{\nu+1}x_2\right]\right.$$

$$\left.-\,dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)\cdot\frac{dt'(x_2;\nu)}{dt(x_2;\nu)^3}\right)\left[\frac{-\rho}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}-\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{\nu+1}x_2\right]$$

$$+\,dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)\cdot\left(\left(\left[\frac{1}{2}\frac{\rho}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}+\frac{3}{2}\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{5}{2}}}\frac{1-\rho^2}{\nu+1}x_2\right]\right.\right.$$

$$\left.\cdot\left(\frac{1-\rho^2}{\nu+1}\frac{2x_2}{dt(x_2;\nu)}\right)-\frac{1-\rho^2}{\nu+1}\frac{1}{dt(x_2;\nu)}\frac{\left((x_1-\rho x_2)-\rho x_2\right)}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\right)\frac{1}{dt(x_2;\nu)},$$

$$\frac{\partial^2 h}{\partial\rho\partial u_2} = \frac{dt'\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}};\nu+1\right)}{dt(x_2;\nu)}\cdot\left[\frac{-x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}}-\frac{1}{2}\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(-2\rho)\frac{\nu+x_2^2}{\nu+1}\right]$$

$$\cdot\left[\frac{-\rho}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}}-\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{\nu+1}x_2\right]$$

$$+\frac{dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}};\nu+1\right)}{dt(x_2;\nu)}\cdot\left[\frac{-1}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}}+\frac{x_2^2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{\nu+1}\right.$$

$$+\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{2\rho x_2}{\nu+1}+\left(\frac{3}{2}\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{5}{2}}}\frac{1-\rho^2}{1-\nu}x_2+\frac{\rho}{2\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\right)$$

$$\left.\cdot\left(\frac{-2\rho(\nu+x_2^2)}{\nu+1}\right)\right].$$

Finally, we obtain the partial derivatives with respect to ν, and ρ and u_2, respectively.

$$\frac{\partial^2 h}{\partial\rho\partial\nu} = \left[\left(\frac{\partial}{\partial\nu}dt\right)\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}};\nu+1\right)+dt'\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}};\nu+1\right)\right.$$

$$\left.\cdot\left(\frac{\frac{\partial x_1}{\partial\nu}-\rho\frac{\partial x_2}{\partial\nu}}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}}-\frac{1}{2}\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(1-\rho^2)\left[\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right]\right)\right]$$

$$\cdot\left[\frac{-x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}}+\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\rho\frac{\nu+x_2^2}{\nu+1}\right]$$

$$+\,dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)\cdot\left[\left(\frac{1}{2}\frac{x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}-\frac{3}{2}\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{5}{2}}}\rho\frac{\nu+x_2^2}{\nu+1}\right)\right.$$

$$\cdot\,(1-\rho^2)\left[\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right]$$

$$-\frac{\frac{\partial x_2}{\partial\nu}}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}+\frac{\frac{\partial x_1}{\partial\nu}-\rho\frac{\partial x_2}{\partial\nu}}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\rho\frac{\nu+x_2^2}{\nu+1}$$

$$\left.+\,\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\rho\frac{(\nu+1)\left(1+2x_2\frac{\partial x_2}{\partial\nu}\right)-(\nu+x_2^2)}{(\nu+1)^2}\right]$$

$$\frac{\partial^2 h}{\partial\nu\partial u_2}=\left(\frac{\left(\frac{\partial}{\partial\nu}dt\right)\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)}{dt(x_2;\nu)}+\frac{dt'\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)}{dt(x_2;\nu)}\right.$$

$$\cdot\left(\frac{\frac{\partial x_1}{\partial\nu}-\rho\frac{\partial x_2}{\partial\nu}}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}-\frac{1}{2}\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(1-\rho^2)\left[\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right]\right)$$

$$\left.-\,\frac{dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}};\nu+1\right)}{dt(x_2;\nu)^2}\left(\left(\frac{\partial}{\partial\nu}dt\right)(x_2;\nu)+dt'(x_2;\nu)\frac{\partial x_2}{\partial\nu}\right)\right)$$

$$\cdot\left(\frac{-\rho}{\sqrt{\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}}}-\frac{x_1-\rho x_2}{\left(\frac{\left(\nu+x_2^2\right)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{\nu+1}x_2\right)$$

$$+\frac{dt\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}}};\nu+1\right)}{dt(x_2;\nu)}\cdot\left[\frac{1}{2}\frac{\rho}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}(1-\rho^2)\left(\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right)\right.$$

$$-\frac{\frac{\partial x_1}{\partial\nu}-\rho\frac{\partial x_2}{\partial\nu}}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{\nu+1}x_2-\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{\nu+1}\frac{\partial x_2}{\partial\nu}$$

$$\left.+\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{3}{2}}}\frac{1-\rho^2}{(\nu+1)^2}x_2+\frac{3}{2}\frac{x_1-\rho x_2}{\left(\frac{(\nu+x_2^2)(1-\rho^2)}{\nu+1}\right)^{\frac{5}{2}}}\frac{(1-\rho^2)^2}{\nu+1}x_2\left(\frac{1+2x_2\frac{\partial x_2}{\partial\nu}}{\nu+1}-\frac{\nu+x_2^2}{(\nu+1)^2}\right)\right]$$

3 Derivatives corresponding to Archimedean copulas

In the class of Archimedean copulas we consider the Clayton, Gumbel, Frank and Joe copula, mentioned in Chapter 3.2. For the Clayton copula we will exemplarily list all first and second partial derivatives of the density function as well as of the conditional distribution function (h-function), while in the other case we restrict ourself to the first partial derivatives. The remaining partial derivatives not listed here can easily be computed by a computer algebra system such as Mathematica[1] or Maple[2].

3.1 Clayton copula

The Clayton copula, or sometimes referred to as MTCJ copula, is defined as

$$C(u_1,u_2;\theta)=(u_1^{-\theta}+u_2^{-\theta}-1)^{-\frac{1}{\theta}}=A(u_1,u_2,\theta)^{-\frac{1}{\theta}},$$

with $A(u_1,u_2,\theta):=u_1^{-\theta}+u_2^{-\theta}-1$. The density of the Clayton copula is

$$\begin{aligned}c(u_1,u_2;\theta)&=(1+\theta)(u_1\cdot u_2)^{-1-\theta}(u_1^{-\theta}+u_2^{-\theta}-1)^{-\frac{1}{\theta}-2}\\&=\frac{(1+\theta)(u_1\cdot u_2)^{-1-\theta}}{A(u_1,u_2,\theta)^{\frac{1}{\theta}+2}},\end{aligned}$$

where $0<\theta<\infty$ controls the degree of dependence. Using the same notation, the h-function is given by

$$h(u_1,u_2;\theta)=u_2^{-\theta-1}\cdot A(u_1,u_2,\theta)^{-1-\frac{1}{\theta}}.$$

[1]Wolfram Research, Inc., Mathematica, Version 8.0, Champaign, IL (2012).
[2]Maple 13.0. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.

Derivatives of the density function

The partial derivative of the density c with respect to the parameter θ is

$$\begin{aligned}\frac{\partial c}{\partial \theta} &= (u_1u_2)^{-\theta-1}(u_1^{-\theta}+u_2^{-\theta}-1)^{-2-\frac{1}{\theta}} - (1+\theta)(u_1u_2)^{-\theta-1}\ln(u_1u_2)(u_1^{-\theta}+u_2^{-\theta}-1)^{-2-\frac{1}{\theta}} \\ &+ (1+\theta)(u_1u_2)^{-\theta-1}(u_1^{-\theta}+u_2^{-\theta}-1)^{-2-\frac{1}{\theta}} \\ &\left(\frac{\ln(u_1^{-\theta}+u_2^{-\theta}-1)}{\theta^2} + \frac{(-2-\frac{1}{\theta})(-u_1^{-\theta}\ln(u_1)-u_2^{-\theta}\ln(u_2))}{u_1^{-\theta}+u_2^{-\theta}-1}\right) \\ &= -c(u_1,u_2)\left(\ln(u_1u_2) - \left(\frac{\ln(A(u_1,u_2,\theta))}{\theta^2} + \frac{(-2-\frac{1}{\theta})(-u_1^{-\theta}\ln(u_1)-u_2^{-\theta}\ln(u_2))}{A(u_1,u_2,\theta)}\right)\right),\end{aligned}$$

and the derivative with respect to u_1 is

$$\begin{aligned}\frac{\partial c}{\partial u_1} &= (1+\theta)\,(u_1u_2)^{-\theta-1}\,(-\theta-1)\left({u_1}^{-\theta}+{u_2}^{-\theta}-1\right)^{-2-\theta^{-1}}{u_1}^{-1} - (1+\theta)\,(u_1u_2)^{-\theta-1} \\ &\left({u_1}^{-\theta}+{u_2}^{-\theta}-1\right)^{-2-\theta^{-1}}\left(-2-\theta^{-1}\right){u_1}^{-\theta}\theta{u_1}^{-1}\left({u_1}^{-\theta}+{u_2}^{-\theta}-1\right)^{-1} \\ &= -\frac{c(u_1,u_2)\cdot(\theta+1)}{u_1} + \frac{c(u_1,u_2)\cdot\left(2+\frac{1}{\theta}\right)\theta}{u_1^{\theta+1}\cdot A(u_1,u_2,\theta)}.\end{aligned}$$

Derivatives of the h-function

We calculate the derivatives of the h-function with respect to the copula parameter θ and u_2, respectively:

$$\begin{aligned}\frac{\partial h}{\partial \theta} &= -u_2^{-\theta-2}(u_1^{-\theta}+u_2^{-\theta}-1)^{-\frac{\theta+1}{\theta}}\theta - u_2^{-\theta-2}(u_1^{-\theta}+u_2^{-\theta}-1)^{-\frac{\theta+1}{\theta}} \\ &+ u_2^{-2\theta-2}(u_1^{-\theta}+v^{-\theta}-1)^{-\frac{2\theta+1}{\theta}}\theta + u_2^{-2\theta-2}(u_1^{-\theta}+u_2^{-\theta}-1)^{-\frac{2\theta+1}{\theta}} \\ &= \left(-\frac{h}{u_2} + u_2^{-2\theta-2}A(u_1,u_2,\theta)^{-\frac{2\theta+1}{\theta}}\right)(\theta+1), \\ \frac{\partial h}{\partial u_2} &= (-\theta-1)u_2^{-\theta-2}A(u_1,u_2,\theta)^{-1-\frac{1}{\theta}} + (-1-\frac{1}{\theta})A(u_1,u_2,\theta)^{-2-\frac{1}{\theta}}\frac{\partial A}{\partial u_2},\end{aligned}$$

where

$$\frac{\partial A}{\partial u_1} = -\theta u_1^{-\theta-1}.$$

Second derivatives of the density function

For the following, we use the partial derivative of $A(u_1,u_2,\theta)$ to shorten the notation.

$$\begin{aligned}\frac{\partial A}{\partial \theta} &= -u_1^{-\theta}\ln(u_1) - u_2^{-\theta}\ln(u_2), & \frac{\partial^2 A}{\partial^2 \theta} &= -u_1^{-\theta}\ln(u_1)^2 - u_2^{-\theta}\ln(u_2)^2, \\ \frac{\partial A}{\partial u_1} &= -\theta u_1^{-\theta-1}, & \frac{\partial^2 A}{\partial^2 u_1} &= \theta(\theta+1)u_1^{-\theta-2}\end{aligned}$$

The second partial derivative of the Clayton copula density with respect to θ is

$$\frac{\partial^2 c}{\partial^2\theta} = \frac{\partial c}{\partial\theta}\cdot\left(-\ln(u_2) + \frac{\ln(A(u_1,u_2,\theta))}{\theta^2} + \frac{\left(-2-\frac{1}{\theta}\right)\frac{\partial A}{\partial\theta}}{A(u_1,u_2,\theta)}\right)$$
$$+ c(u_1,u_2)\cdot\left(\frac{\frac{\frac{\partial A}{\partial\theta}}{A(u_1,u_2,\theta)}\theta^2 - 2\ln(A(u_1,u_2,\theta))\theta}{\theta^4}\right.$$
$$\left. + \frac{\left(\frac{1}{\theta^2}\frac{\partial A}{\partial\theta} + \left(-2-\frac{1}{\theta}\right)\frac{\partial^2 A}{\partial^2\theta}\right)\cdot A(u_1,u_2,\theta) - \left(-2-\frac{1}{\theta}\right)\left(\frac{\partial A}{\partial\theta}\right)^2}{A(u_1,u_2,\theta)^2}\right).$$

Further, the partial derivative with respect to u_1 is

$$\frac{\partial^2 c}{\partial^2 u_1} = -\frac{\frac{\partial c}{\partial u_1}(\theta+1)u_1 - (\theta+1)c(u_1,u_2)}{u_1^2}$$
$$+ \frac{\left(2+\frac{1}{\theta}\right)\left(\frac{\partial c}{\partial u_1}\frac{\partial A}{\partial u_1} + c(u_1,u_2)\frac{\partial^2 A}{\partial^2 u_1}\right) - c(u_1,u_2)\left(2+\frac{1}{\theta}\right)\left(\frac{\partial^2 A}{\partial^2 u_1}\right)^2}{A(u_1,u_2,\theta)^2},$$

and finally, we obtain the derivatives with respect to u_1, and θ and u_2, respectively.

$$\frac{\partial^2 c}{\partial u_1\partial\theta} = -\frac{\frac{\partial c}{\partial\theta}(\theta+1) + c(u_1,u_2)}{u_1} + \frac{\left(u_1^{\theta+1}A(u_1,u_2,\theta)\right)\left[\frac{\partial c}{\partial\theta}(2\theta+1) + 2c(u_1,u_2)\right]}{u_1^{2\theta+2}A(u_1,u_2,\theta)^2}$$
$$- \frac{c(u_1,u_2)(2\theta+1)\left[u_1^{\theta+1}\ln(u_1)A(u_1,u_2,\theta) + u_1^{\theta+1}\frac{\partial A}{\partial\theta}\right]}{u_1^{2\theta+2}A(u_1,u_2,\theta)^2},$$
$$\frac{\partial^2 c}{\partial u_1\partial u_2} = -\frac{\frac{\partial c}{\partial u_2}(\theta+1)}{u_1} + \frac{\frac{\partial c}{\partial u_2}(2\theta+1)}{u_1^{\theta+1}A(u_1,u_2,\theta)} - \frac{c(u_1,u_2)(2\theta+1)}{u_1^{2\theta+2}A(u_1,u_2,\theta)^2}\cdot\frac{\partial A}{\partial u_2}.$$

Second derivatives of the h-function

The second partial derivatives of the conditional distribution function with respect to θ and u_2 are given as follows,

$$\frac{\partial^2 h}{\partial^2\theta} = \frac{\partial h}{\partial\theta}\cdot\left(-\ln(u_2) + \frac{\ln(A(u_1,u_2,\theta))}{\theta^2} + \frac{\left(-1-\frac{1}{\theta}\right)\frac{\partial A}{\partial\theta}}{A(u_1,u_2,\theta)}\right)$$
$$+ h(u_1,u_2)\cdot\left(\frac{\frac{\frac{\partial A}{\partial\theta}}{A(u_1,u_2,\theta)}\theta^2 - 2\ln(A(u_1,u_2,\theta))\theta}{\theta^4}\right.$$
$$\left. + \frac{\left(\frac{1}{\theta^2}\frac{\partial A}{\partial\theta} + \left(-1-\frac{1}{\theta}\right)\frac{\partial^2 A}{\partial^2\theta}\right)\cdot A(u_1,u_2,\theta) - \left(-1-\frac{1}{\theta}\right)\left(\frac{\partial A}{\partial\theta}\right)^2}{A(u_1,u_2,\theta)^2}\right),$$

$$
\begin{aligned}
\frac{\partial^2 h}{\partial^2 u_2} &= (-\theta-1)(-\theta-2)u_2^{-\theta-3}A(u_1,u_2,\theta)^{-1-\frac{1}{\theta}} \\
&\quad + (-\theta-1)u_2^{-\theta-1}\left(-1-\frac{1}{\theta}\right)A(u_1,u_2,\theta)^{-2-\frac{1}{\theta}}\frac{\partial A}{\partial u_2} \\
&\quad + \left(-1-\frac{1}{\theta}\right)\left(-2-\frac{1}{\theta}\right)A(u_1,u_2,\theta)^{-3-\frac{1}{\theta}}\left(\frac{\partial A}{\partial u_2}\right)^2 \\
&\quad + \left(-1-\frac{1}{\theta}\right)A(u_1,u_2,\theta)^{-2-\frac{1}{\theta}}\frac{\partial^2 A}{\partial^2 u_2}, \\
\frac{\partial^2 h}{\partial\theta\partial u_2} &= (\theta+1)\left(-\frac{\frac{\partial h}{\partial u_2}u_2 - h(u_1,u_2)}{u_2^2} + (-2\theta-2)u_2^{-2\theta-3}A(u_1,u_2,\theta)^{-2-\frac{1}{\theta}}\right. \\
&\quad \left. + u_2^{-2\theta-2}\left(-2-\frac{1}{\theta}\right)A(u_1,u_2,\theta)^{-3-\frac{1}{\theta}}\frac{\partial A}{\partial u_2}\right).
\end{aligned}
$$

3.2 Gumbel copula

The Gumbel copula is given by

$$C(u_1,u_2;\theta) = \exp[-\{(-\ln(u_1))^\theta + (-\ln(u_2))^\theta\}^{\frac{1}{\theta}}] = \exp[-(t_1+t_2)^{\frac{1}{\theta}}],$$

where $t_i := (-\ln(u_i))^\theta, i = 1,2$ and $\theta \geq 1$. The h-function and the density are as follows:

$$h(u_1,u_2;\theta) = -\frac{\mathrm{e}^{-(t_1+t_2)^{\frac{1}{\theta}}}(t_1+t_2)^{\frac{1}{\theta}-1}t_2}{u_2\ln(u_2)},$$

$$
\begin{aligned}
c(u_1,u_2;\theta) &= C(u_1,u_2;\theta)(u_1u_2)^{-1}\{(-\ln(u_1))^\theta + (-\ln(u_2))^\theta\}^{-2+\frac{2}{\theta}}(\ln(u_1)\ln(u_2))^{\theta-1} \\
&\quad \times \{1+(\theta-1)((-\ln(u_1))^\theta + (-\ln(u_2))^\theta)^{-\frac{1}{\theta}}\} \\
&= C(u_1,u_2;\theta)\frac{1}{u_1u_2}(t_1+t_2)^{-2+\frac{2}{\theta}}(\ln(u_1)\ln(u_2))^{\theta-1} \times \{1+(\theta-1)(t_1+t_2)^{-\frac{1}{\theta}}\}.
\end{aligned}
$$

The derivative of the density c with respect to the copula parameter θ is

$$
\begin{aligned}
\frac{\partial c}{\partial\theta} &= c(u_1,u_2)\left[-(t_1+t_2)^{\frac{1}{\theta}}\left(-\frac{\ln(t_1+t_2)}{\theta^2} + \frac{t_1\ln(-\ln(u_1)) + t_2\ln(-\ln(u_2))}{\theta\,(t_1+t_2)}\right)\right. \\
&+ \left.\left(-2\frac{\ln(t_1+t_2)}{\theta^2} + \left(-2+\frac{2}{\theta}\right)\frac{t_1\ln(-\ln(u_1)) + t_2\ln(-\ln(u_2))}{t_1+t_2}\right) + \ln(\ln(u_1)\ln(u_2))\right] \\
&+ C(u_1,u_2)\,(t_1+t_2)^{-2+\frac{2}{\theta}}\frac{(\ln(u_1)\ln(u_2))^{\theta-1}}{u_1u_2} \\
&\left((t_1+t_2)^{-\frac{1}{\theta}} + (\theta-1)\,(t_1+t_2)^{-\frac{1}{\theta}}\left(\frac{\ln(t_1+t_2)}{\theta^2} - \frac{t_1\ln(-\ln(u_1)) + t_2\ln(-\ln(u_2))}{\theta\,(t_1+t_2)}\right)\right).
\end{aligned}
$$

Similarly, the derivative of c with respect to u_1 is obtained as

$$\frac{\partial c}{\partial u_1} = c(u_1,u_2)\left[-(t_1+t_2)^{\frac{1}{\theta}-1}\frac{t_1}{u_1\ln(u_1)} - \frac{1}{u_1} + \frac{\left(-2+\frac{2}{\theta}\right)t_1\theta}{u_1} + \frac{\theta-1}{u_1\ln(u_1)}\right]$$
$$-C(u_1,u_2)\,(t_1+t_2)^{-2+\frac{2}{\theta}}\,\frac{(\ln(u_1)\ln(u_2))^{\theta-1}}{u_1u_2}\,(\theta-1)\,(t_1+t_2)^{-\frac{1}{\theta}-1}\,\frac{t_1}{u_1\ln(u_1)}.$$

Finally, the partial derivatives of the h-function w.r.t. θ and u_2 are:

$$\frac{\partial h}{\partial \theta} = h(u_1,u_2)\left[(t_1+t_2)^{\theta^{-1}}\left(-\frac{\ln(t_1+t_2)}{\theta^2} + \frac{t_1\ln(-\ln(u_1)) + t_2\ln(-\ln(u_2))}{\theta\,(t_1+t_2)}\right)\right.$$
$$\left. -\left(-\frac{\ln(t_1+t_2)}{\theta^2} + \frac{\left(\theta^{-1}-1\right)\left(t_1\ln(-\ln(u_1)) + t_2\ln(-\ln(u_2))\right)}{t_1+t_2}\right) - \ln(-\ln(u_2))\right],$$

$$\frac{\partial h}{\partial u_2} = h(u_1,u_2)\left[\frac{(t_1+t_2)^{\frac{1}{\theta}-2} - \theta + 1}{u_2\ln(u_2)} - \frac{(\frac{1}{\theta}-1)t_2\theta}{u_2\ln(u_2)(t_1+t_2)} + \frac{1}{u_2}\right].$$

3.3 Frank copula

The Frank copula is defined as

$$C(u_1,u_2;\theta) = -\frac{1}{\theta}\ln\left(\frac{1}{1-e^{-\theta}}[(1-e^{-\theta}) - (1-e^{-\theta u_1})(1-e^{-\theta u_2})]\right),$$

and has the following density function:

$$c(u_1,u_2;\theta) = \theta(1-e^{-\theta})e^{-\theta(u_1+u_2)}[(1-e^{-\theta}) - (1-e^{-\theta u_1})(1-e^{-\theta u_2})]^{-2},$$

where $\theta \in [-\infty,\infty]\backslash\{0\}$. The corresponding h-function is

$$h(u_1,u_2;\theta) = -\frac{\mathrm{e}^{\theta}\left(\mathrm{e}^{\theta\,u_1}-1\right)}{\mathrm{e}^{\theta\,u_2+\theta\,u_1} - \mathrm{e}^{\theta\,u_2+\theta} - \mathrm{e}^{\theta\,u_1+\theta} + \mathrm{e}^{\theta}}.$$

The partial derivative of the density function c with respect to the copula parameter θ is given by

$$\frac{\partial c}{\partial \theta} = c(u_1,u_2)\cdot\left(\frac{1}{\theta} + e^{-\theta}\left(1-e^{-\theta}\right)^{-1} - (u_1+u_2)\right.$$
$$\left. -2\left[\left(1-e^{-\theta}\right) - t_1t_2\right]^{-1}\left(e^{-\theta} - \frac{\partial t_1}{\partial \theta}t_2 - t_1\frac{\partial t_2}{\partial \theta}\right)\right).$$

Here, we use $t_i := (1-e^{-\theta u_i}), i = 1,2$ which implies

$$\frac{\partial t_i}{\partial \theta} = u_i e^{-\theta u_i}, \quad i = 1,2.$$

The derivative with respect to u_1 is

$$\frac{\partial c}{\partial u_1} = c(u_1, u_2) \cdot \left(-\theta + 2t_2 \frac{\partial t_1}{\partial u_1} \left[(1 - e^{-\theta}) - t_1 t_2 \right]^{-1} \right),$$

where

$$\frac{\partial t_i}{\partial u_i} = \theta e^{-\theta u_i}, \quad i = 1, 2.$$

Similarly, we obtain the partial derivatives of the h-function

$$\begin{aligned} \frac{\partial h}{\partial \theta} &= h(u_1, u_2; \theta) \left(1 + \frac{u_1 \mathrm{e}^{\theta u_1}}{(\mathrm{e}^{\theta u_1} - 1)} \right. \\ &\quad \left. - \frac{\left((u_2 + u_1)\, \mathrm{e}^{\theta\, u_2 + \theta\, u_1} - (u_2 + 1)\, \mathrm{e}^{\theta\, u_2 + \theta} - (u_1 + 1)\, \mathrm{e}^{\theta\, u_1 + \theta} + \mathrm{e}^{\theta} \right)}{(\mathrm{e}^{\theta\, u_2 + \theta\, u_1} - \mathrm{e}^{\theta\, u_2 + \theta} - \mathrm{e}^{\theta\, u_1 + \theta} + \mathrm{e}^{\theta})} \right), \\ \frac{\partial h}{\partial u_2} &= -h(u_1, u_2; \theta) \frac{\left(\theta\, \mathrm{e}^{\theta\, u_2 + \theta\, u_1} - \theta\, \mathrm{e}^{\theta\, u_2 + \theta} \right)}{(\mathrm{e}^{\theta\, u_2 + \theta\, u_1} - \mathrm{e}^{\theta\, u_2 + \theta} - \mathrm{e}^{\theta\, u_1 + \theta} + \mathrm{e}^{\theta})}. \end{aligned}$$

3.4 Joe copula

The Joe copula is defined for $\theta \geq 1$ as

$$\begin{aligned} C(u_1, u_2; \theta) &= 1 - \left((1 - u_1)^{\theta} + (1 - u_2)^{\theta} - (1 - u_1)^{\theta} (1 - u_2)^{\theta} \right)^{\frac{1}{\theta}} \\ &= 1 - (t_1 + t_2 - t_1 t_2)^{\frac{1}{\theta}}, \end{aligned}$$

where $t_i := (1 - u_i)^{\theta} \quad i = 1, 2$. Its density and h-function are

$$\begin{aligned} c(u_1, u_2; \theta) &= \left((1 - u_1)^{\theta} + (1 - u_2)^{\theta} - (1 - u_1)^{\theta} (1 - u_2)^{\theta} \right)^{\frac{1}{\theta} - 2} \cdot (1 - u_1)^{\theta - 1} (1 - u_2)^{\theta - 1} \\ &\quad \cdot \left[\theta - 1 + (1 - u_1)^{\theta} + (1 - u_2)^{\theta} - (1 - u_1)^{\theta} (1 - u_2)^{\theta} \right] \\ &= (t_1 + t_2 - t_1 t_2)^{\frac{1}{\theta} - 2} (\theta - 1 + t_1 + t_2 - t_1 t_2)(1 - u_1)^{\theta - 1} (1 - u_2)^{\theta - 1}, \\ h(u_1, u_2; \theta) &= \left((1 - u_1)^{\theta} + (1 - u_2)^{\theta} - ((1 - u_1)(1 - u_2))^{\theta} \right)^{\frac{1 - \theta}{\theta}} (1 - u_2)^{\theta - 1} \left(1 - (1 - u_1)^{\theta} \right) \\ &= (t_1 + t_2 - t_1 t_2)^{\frac{1}{\theta} - 1} (1 - u_2)^{\theta - 1} (1 - u_1)^{\theta - 1}. \end{aligned}$$

Using the partial derivative of $t_i(u_i, \theta)$,

$$\frac{\partial t_i}{\partial \theta} = t_i \ln(1 - u_i) \quad i = 1, 2,$$

the derivative of the density with respect to θ can be determined as

$$\begin{aligned} \frac{\partial c}{\partial \theta} &= c(u_1, u_2) \left(-\frac{\ln(t_1 + t_2 - t_1 t_2)}{\theta^2} \right. \\ &\quad \left. + \frac{\left(\frac{1}{\theta} - 2 \right) \left(\frac{\partial t_1}{\partial \theta} + \frac{\partial t_2}{\partial \theta} - \frac{\partial t_1}{\partial \theta} t_2 - t_1 \frac{\partial t_2}{\partial \theta} \right)}{t_1 + t_2 - t_1 t_2} + \ln(1 - u_1) + \ln(1 - u_2) \right) \\ &\quad + (t_1 + t_2 - t_1 t_2)^{\frac{1}{\theta} - 2} (1 - u_1)^{\theta - 1} (1 - u_2)^{\theta - 1} \left(1 + \frac{\partial t_1}{\partial \theta} + \frac{\partial t_2}{\partial \theta} - \frac{\partial t_1}{\partial \theta} t_2 - t_1 \frac{\partial t_2}{\partial \theta} \right). \end{aligned}$$

The derivative with respect to the first copula argument u_1 is

$$\frac{\partial c}{\partial u_1} = c(u_1, u_2) \left(\frac{\frac{1}{\theta} - 2}{t_1 + t_2 - t_1 t_2} \left(-\frac{t_1 \theta}{1 - u_1} + \frac{t_1 \theta \, t_2}{1 - u_1} \right) - \frac{(\theta - 1)}{1 - u_1} \right)$$
$$+ \left(t_1 + t_2 - t_1 t_2\right)^{\frac{1}{\theta} - 2} \left(1 - u_1\right)^{\theta - 1} \left(1 - u_2\right)^{\theta - 1} \left(-\frac{t_1 \theta}{1 - u_1} + \frac{t_1 \theta \, t_2}{1 - u_1} \right).$$

Finally, we determine the derivatives of the h-function with respect to the copula parameter and u_2, respectively.

$$\frac{\partial h}{\partial \theta} = h(u_1, u_2) \left(-\frac{\ln\left(t_1 + t_2 - t_1 t_2\right)}{\theta^2} + \frac{\left(\frac{1}{\theta} - 1\right)\left(\frac{\partial t_1}{\partial \theta} + \frac{\partial t_2}{\partial \theta} - \frac{\partial t_1}{\partial \theta} t_2 - t_1 \frac{\partial t_2}{\partial \theta}\right)}{t_1 + t_2 - t_1 t_2} + \ln\left(1 - u_2\right) \right)$$
$$- \left(t_1 + t_2 - t_1 t_2\right)^{\frac{1}{\theta} - 1} \left(1 - u_2\right)^{\theta - 1} \frac{\partial t_1}{\partial \theta},$$
$$\frac{\partial h}{\partial u_2} = h(u1, u_2) \left(\frac{\frac{1}{\theta} - 1}{t_1 + t_2 - t_1 t_2} \left(-\frac{t_2 \theta}{1 - u_2} + \frac{t_1 t_2 \theta}{1 - u_2} \right) - \frac{(\theta - 1)}{1 - u_2} \right).$$

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