Conditional characteristic functions of Molchan-Golosov fractional Lévy processes with application to credit risk

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Abstract

Molchan-Golosov fractional Lévy processes (MG-fLps) are introduced by a multivariate componentwise Molchan-Golosov transformation based on an n-dimensional driving Lévy process. Using results of fractional calculus and infinitely divisible distributions we are able to calculate the conditional characteristic function of integrals driven by MG-fLps. This leads to important prediction results including the case of multivariate fractional Brownian motion, fractional subordinators or general fractional stochastic differential equations. Examples are the fractional Lévy Ornstein-Uhlenbeck or Cox-Ingersoll-Ross model. As an application we present a fractional credit model with a long range dependent hazard rate and calculate bond prices.

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1 Introduction

In many financial or technical applications it is very useful to know the conditional characteristic function of certain stochastic processes, because it is closely related to prediction: given the past evolution of a stochastic process, we are interested in its distribution at some time in the future. Consider the following example from the field of credit risk: given a short rate process $r = (r(v))_{v\in[0,T]}$ and a hazard rate process $\lambda = (\lambda(v))_{v\in[0,T]}$ with some market maturity date $T > 0$, the price of a defaultable zero coupon bond at time s with bond maturity t, $0 \le s \le t \le T$, is given by

$$B(s,t) = \mathbf{1}_{\{\tau>s\}} E^{\mathbb{Q}}\left[e^{-\int_s^t (r(v)+\lambda(v))dv} \middle| (r(v), \lambda(v)), v \in [0,s] \right], \tag{1.1}$$

where the random time τ describes the default time of the bond and $\mathbb{Q}$ is some risk-neutral measure. To calculate this price it is useful to know the conditional characteristic function of the bivariate process (r, λ).

Standard credit models describe (r, λ) by an affine Markov process (see e.g. Duffie [11] and Duffie, Filipovic and Schachermayer [12]). Due to the Markov property, (1.1) can be calculated and takes a nice form which does only depend on the random variable $(r(s), \lambda(s))$; i.e. today's level of the process (r, λ). The past evolution of the path does not play any role.

However, the ongoing financial crisis, which had its origin in the US credit market, showed that in real markets this assumption may be violated. In fact the work of Henry and Zaffaroni [20] suggests that in many macroeconomic variables like interest and hazard rates, domestic gross products, supply and demand rates or volatilities, there is strong empirical evidence against the Markov property. In particular these processes show signs of long range dependence in their increments. This supports the use of fractional processes like fractional Brownian motion (fBm) or fractional Lévy processes for modeling the dynamics of (r, λ) in (1.1). However, the prediction problem gets a lot more complicated, since all past information will enter the price. The key is now that these fractional processes can be described by kernel representations of affine Markov processes and thus certain structures remain.

This idea has been present since Biagini, Fuschini and Klüppelberg [7], where the authors focused however on contagion effects. Biagini, Fink and Klüppelberg [6] introduced a fractional Brownian credit market. Interest rate models driven by fBm were considered in Fink, Klüppelberg and Zähle [17] and also in Ohashi [28]. However all these approaches do not leave the Brownian framework.

In this paper we introduce the class of (multivariate) Molchan-Golosov fractional Lévy processes (MG-fLps), including fractional Brownian motion and fractional subordinators (as defined in Bender and Marquardt [5]) by a Molchan-Golosov transformation based on a general finite second moment Lévy process. This idea as been proposed by Tikanmäki and Mishura [37], who however considered only the univariate case and used centered driving Lévy processes without Brownian parts, basically excluding fBm and fractional subordinators. We then calculate the conditional characteristic functions of MG-fLp-related processes using fractional calculus and results on infinitely divisible distributions. Important examples like fractional Lévy Ornstein-Uhlenbeck processes (fLOUps) or Cox-Ingersoll-Ross processes (fLCIRps) are considered. The calculation of bond prices as in (1.1) is then straightforward.

Our paper is organized as follows. Section 2 will state preliminaries about Lévy processes, including integration and distributional results. In section 3 we will introduce MG-fLps by an integral representation with respect to a multivariate Lévy process. Second order structure, path properties and integration concepts are considered. In section 4 we will present our main results on conditional characteristic functions of MG-fLp-related processes, including important examples like fLOUps or fLCIRps. As an application, section 5 will consider fractional credit models described by Vasicek dynamics and calculate bond prices. An example using a two-dimensional fractional Poisson subordinator for (r, λ) is presented.

We always assume a complete probability space $(\Omega, \mathcal{F}, P)$. Denote by $L^2(\Omega)$ the space of square integrable random variables. For a family of random variables $(X(i))_{i\in I}$, I some index set, let $\sigma\overline{\{X(i), i \in I\}}$ denote the completion of the generated σ-algebra. For $A \subseteq \mathbb{R}$ and $n \in \mathbb{N}$, the spaces of integrable and square integrable functions $f : A \to \mathbb{R}^n$ are denoted by $L^1(A, \mathbb{R}^n)$ and $L^2(A, \mathbb{R}^n)$. In the case $n = 1$ we shall just write $L^1(A)$ and $L^2(A)$. Furthermore $\|\cdot\|$ is the L^2-norm and $\langle\cdot,\cdot\rangle$ the corresponding Euclidian scalar product. $\mathbb{R}_+$ ($\mathbb{R}_-$) are the positive (negative) real half lines. For a matrix A, $A^\top$ shall be the adjoint. The gamma function shall be denoted by Γ. Denote by $\mathbb{R}^+$ the right halfline including zero.

2 Preliminaries

In the literature there are many possible approaches to define fractional Lévy processes and a readable overview of two main concepts can be found in Tikanmäki and Mishura [37]. Both ways are mainly based on the idea of integrating memory into a Lévy process by choosing an appropriate kernel function. This can either be done by integration over the whole real line like in Marquardt [25] or on a compact interval which has been done by Tikanmäki and Mishura [37]. However these are not the only ways to obtain a generalization of fractional Brownian motion. For example, another possibility is to extend the harmonizable representation of fBm as has been done by Benassi, Cohen and Istas [2].

In this work we will choose the approach of Tikanmäki and Mishura [37] and generalize it to the multivariate case. Also we will introduce a broader class of processes by using general finite second moment Lévy processes. To avoid confusion with the other concepts we will call our class of processes *Molchan-Golosov fractional Lévy processes (MG-fLps).*

Throughout the whole paper we will always consider a given multivariate Lévy process $\mathbf{L} = (\mathbf{L}(t))_{t\in[0,T]} = (L^1(t), \ldots, L^n(t))^\top_{t\in[0,T]}$, for $n \in \mathbb{N}$ and $T > 0$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions of right-continuity and completeness. The filtration (after possible augmentation) is assumed to be generated by $\mathbf{L}$ (cf. Theorem 2.1.9 of Applebaum [1]).

Then $\mathbf{L}$ can be described in terms of the characteristic triple (γ, Σ, ν) by its characteristic function $E[\exp\{i\langle u, \mathbf{L}(t)\rangle\}] = \exp\{t\psi(u)\}$, $t \in [0,T]$, with

$$\psi(u) \;=\; i\langle \gamma, u\rangle - \frac{1}{2}u^\top \Sigma u + \int_{\mathbb{R}^n} (\exp\{i\langle u, x\rangle\} - 1 - i\langle u, x\rangle \mathbf{1}_{\{\|x\|<1\}})\nu(dx), \quad u \in \mathbb{R}^n.$$

Here we have $\gamma \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ is symmetric and positive semidefinite, and the measure ν satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} (\|x\|^2 \wedge 1)\nu(dx) < \infty.$$

Since we consider only finite second moment Lévy processes we also have that

$$\int_{\mathbb{R}^n} \|x\|^2 \nu(dx) < \infty.$$

Integration with respect to Lévy processes shall be understood in the usual $L^2(\Omega)$-sense (e.g. see Rajput and Rosinski [31] or Sato [36]). The following theorem will be crucial when predicting MG-fLps. It is the multivariate version of Theorem 2.7 of Rajput and Rosinski [31] and can be obtained using Proposition 2.17 of Sato [36].

Theorem 2.1. *For $f \in L^2([0,T], \mathbb{R}^{n\times n})$ the integral $\int_0^T f(t)d\boldsymbol{L}(t)$ exists as an $L^2(\Omega)$-limit of approximating step functions. Moreover, we have for $u \in \mathbb{R}^n$*

$$E\Big[\exp\Big\{i\Big\langle u, \int_0^T f(t)d\boldsymbol{L}(t)\Big\rangle\Big\}\Big] = \exp\Big\{\int_0^T \psi(f(t)^\top u)dt\Big\}.$$

Before stating the definition of MG-fLps we need a few more concepts: Define the fractional Riemann-Liouville integral with finite time horizon for $d > 0$,

$$(I^d_{T-}f)(s) = \frac{1}{\Gamma(d)}\int_s^T f(r)(r-s)^{d-1}dr, \quad 0 \le s \le T. \tag{2.1}$$

For $f \in L^2(\mathbb{R})$ this always exists. We shall also need the fractional derivative with finite time horizon for $d \in (0,1)$,

$$(D^d_{T-}g)(u) = \frac{1}{\Gamma(1-d)}\Big(\frac{g(u)}{(T-u)^d} + d\int_u^T \frac{g(u)-g(s)}{(s-u)^{d+1}}ds\Big), \quad 0 < u < T. \tag{2.2}$$

The question of the existence of the fractional derivative is more sophisticated, for more details we refer to Zähle [39]. For simplicity we shall only take fractional derivatives for suitable functions g such that it always exists.

We shall write $I^{-d}_{T-} = D^d_{T-}$. For $d = 0$ we will set $I^d_{T-} = D^d_{T-} = id$.

We will also make use of the following spaces, introduced by Pipiras and Taqqu [29, 30]:

$$\widetilde{\Lambda}^d_T := \begin{cases} f:[0,T]\to\mathbb{R} \;\Big|\; \int_0^T [s^{-d}I^d_{T-}((\cdot)^d f(\cdot))(s)]^2 ds < \infty\Big\}, & d \in (0,\frac{1}{2}), \\ f:[0,T]\to\mathbb{R} \;\Big|\; \exists \phi_f \in L^2[0,T] : f(s) = s^{-d}I^{-d}_{T-}((\cdot)^d\phi_f(\cdot))(s)\Big\}, & d \in (-\frac{1}{2},0). \end{cases}$$

Because of our notation, for $d = 0$ both sets are equal to $L^2[0,T]$. In the light of Lemma 4.3 of Bender and Elliott [3] we shall adjust these spaces such that they are closed with respect to multiplication with an indicator function. Therefore define for $d \in (-\frac{1}{2}, \frac{1}{2})$

$$\Lambda_T^d := \Big\{ f : [0,T] \to \mathbb{R} \Big| \forall [s,t] \subseteq [0,T] : f \mathbf{1}_{[s,t]} \in \widetilde{\Lambda}_T^d \Big\}.$$

For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, define

$$\Lambda_T^d := \Big\{ f : [0,T] \to \mathbb{R}^{n \times n} \Big| f_{ij} \in \Lambda_T^{d(j)}, 1 \le i, j \le n \Big\},$$

where f_{ij} denotes the ij-th component of f.

3 Multivariate Molchan-Golosov fractional Lévy processes

As already mentioned in section 2, we will generalize the concept introduced by Tikanmäki and Mishura [37] to the multivariate case. Therefore, our processes will be defined by a compactly supported Molchan-Golosov transformation. However, in contrast to [37] we will also allow for a Brownian part which results in a fractional Brownian motion, and a non-central driving Lévy process, possibly leading to a fractional subordinator introduced by Bender and Marquardt [5]. For $f \in \Lambda_T^d$ and $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, define for $s \in [0,T]$ the convolution operator

$$z^d(f,s) := \begin{pmatrix} c_{d(1)} s^{-d(1)} I_{T-}^{d(1)}((\cdot)^{d(1)} f_{11}(\cdot))(s) & \dots & c_{d(n)} s^{-d(n)} I_{T-}^{d(n)}((\cdot)^{d(n)} f_{1n}(\cdot))(s) \\ \vdots & \ddots & \vdots \\ c_{d(1)} s^{-d(1)} I_{T-}^{d(1)}((\cdot)^{d(1)} f_{n1}(\cdot))(s) & \dots & c_{d(n)} s^{-d(n)} I_{T-}^{d(n)}((\cdot)^{d(n)} f_{nn}(\cdot))(s) \end{pmatrix}$$

where for $1 \le j \le n$

$$c_{d(j)} = \Big(\frac{(2d(j)+1)\Gamma(d(j)+1)\Gamma(1-d(j))}{\Gamma(1-2d(j))} \Big)^{\frac{1}{2}}.$$

The very general definition of MG-fLps follows.

Definition 3.1. *For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, we define the kernel function $z^d(\hat{\mathbf{1}}_{[0,\cdot)}, \cdot) : [0,T] \times [0,T] \to \mathbb{R}^{n \times n}$ using for $s, t \in [0,T]$*

$$\hat{\mathbf{1}}_{[0,t]}(s) := \begin{pmatrix} \mathbf{1}_{[0,t]}(s) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{1}_{[0,t]}(s) \end{pmatrix}.$$

Then a Molchan-Golosov fractional Lévy process (MG-fLp)

$$\boldsymbol{L}^d = (\boldsymbol{L}^d(t))_{t \in [0,T]} = (L^{d(1)}(t), \dots, L^{d(n)}(t))_{t \in [0,T]}^\top$$

is defined by

$$\boldsymbol{L}^d(t) = \int_0^t z^d(\hat{\mathbf{1}}_{[0,t]}, s) d\boldsymbol{L}(s), \quad t \in [0,T]. \tag{3.1}$$

Remark 3.2. The integral in (3.1) can be considered in the $L^2(\Omega)$- or in the pathwise Riemann-Stieltjes-sense. The first assertion is clear by Rajput and Rosinski [31] and the fact that the kernel is square-integrable. The second one follows because as a finite second moment Lévy process, $\mathbf{L}$ is of bounded p-variation for all $p > 2$ (cf. Monroe [27], Theorem 2, based on the Blumenthal-Getoor-index introduced in Blumenthal and Getoor [9]) and $z^d(t,s)$ is on $(0,t)$ componentwise of bounded variation in the variable s, cf. Young [38].

Using the Lévy-Itô decomposition we obtain

$$\begin{aligned}\mathbf{L}(t) &= E[\mathbf{L}(t)] + (\mathbf{L}(t) - E[\mathbf{L}(t)]) \\ &= E[\mathbf{L}(1)] \cdot t + \mathbf{B}(t) + \mathbf{S}(t), \quad t \in [0,T],\end{aligned}$$

where $\mathbf{B}$ is a n-dimensional Brownian motion with $\mathbf{B}(1) \sim N(0,\Sigma)$ and $\mathbf{S}$ is a zero-mean Lévy process without Brownian part. Furthermore $\mathbf{B}$ and $\mathbf{S}$ are independent. This leads to the MG-fLp decomposition

$$\mathbf{L}^d(t) = \int_0^t z^d(\hat{\mathbf{1}}_{[0,t]}, s) ds \cdot E[\mathbf{L}(1)] + \mathbf{B}^d(t) + \mathbf{S}^d(t), \quad t \in [0,T], \tag{3.2}$$

where the first integral is understood componentwise. $\mathbf{B}^d$ and $\mathbf{S}^d$ are defined as in (3.1), i.e. are a multivariate fBm and a zero-mean MG-fLp.

Example 3.3. Let $n = 1$ in Definition 3.1.

(i) Choosing as driving Lévy process a standard Brownian motion, we get a classical fBm on $[0,T]$, cf. Samorodnitsky and Taqqu [34].

(ii) Taking a strictly increasing subordinator as driving Lévy process leads to a fractional subordinator in the sense of Example 1 of Bender and Marquardt [5]. In particular, the resulting MG-fLp is a.s. increasing.

The next results follow by standard properties of Lévy processes, see e.g. Rajput and Rosinski [31], Marcus and Rosinski [24] or Sato [35]. A brief look at the autocovariance of a MG-fLp leads to the following proposition.

Proposition 3.4. *For $s,t \in [0,T]$ we have for the mean-value and autocovariance function*

(i) $E[\boldsymbol{L}^d(t)] = \int_0^t z^d(\hat{\mathbf{1}}_{[0,t]}, s) ds \cdot E[\boldsymbol{L}(1)]$.

(ii) $Cov[\boldsymbol{L}^d(t), \boldsymbol{L}^d(s)]$
$= \frac{1}{2}\left(c_{d(i),d(j)} Cov[L^i(1), L^j(1)](t^{d(i)+d(j)+1} + s^{d(i)+d(j)+1} - |t-s|^{d(i)+d(j)+1})\right)_{1 \le i,j \le n}$
where

$$c_{d(i),d(j)} = \frac{\sqrt{\Gamma(2d(i)+2)\sin(\pi(d(i)+\frac{1}{2}))}\sqrt{\Gamma(2d(j)+2)\sin(\pi(d(j)+\frac{1}{2}))}}{\Gamma(d(i)+d(j)+2)\sin(\pi(d(i)+d(j)+1)/2)}.$$

Proof. The first part is clear from the decomposition (3.2). The second one follows by

$$\begin{aligned} Cov[\mathbf{L}^d(t),\mathbf{L}^d(s)] &= \int_0^T z^d(\hat{\mathbf{1}}_{[0,t]},u)Cov[\mathbf{L}(t),\mathbf{L}(s)](z^d(\hat{\mathbf{1}}_{[0,s]},u))^\top du \\ &= \left(Cov[L^i(t),L^j(s)]\int_0^T z^d_{ii}(\hat{\mathbf{1}}_{[0,t]},u)z^d_{jj}(\hat{\mathbf{1}}_{[0,s]},u)du\right)_{1\le i,j\le n}. \end{aligned}$$

The calculation of the last integrals is similar to (2.17) of Elliott and von der Hoek [14]. □

Remark 3.5. Proposition 3.4 (*ii*) shows that MG-fLps have the same second-order structure as fBm (up to a constant) and therefore it is clear that for all $1 \le i \le n$ the increments of the univariate process $L^{d(i)}$ exhibit a similar memory structure as in the case of fBm.

The next lemma summarizes main properties of MG-fLps and is the multivariate extension of Proposition 3.7 of Tikanmäki and Mishura [37].

Lemma 3.6. *We have for $d = (d(1),\dots,d(n))^\top$:*

(i) A MG-fLp without Gaussian component has a.s. continuous paths if and only if $d \in (0,\frac{1}{2})^n$.

(ii) A MG-fLp without Gaussian component has a.s. Hölder continuous paths of any order $\alpha < \min[d]$ if and only if $d \in (0,\frac{1}{2})^n$.

(iii) If $d(i) \in (-\frac{1}{2},0)$ for some $1 \le i \le n$, then the MG-fLp has discontinuous and unbounded sample paths with positive probability.

Before coming to conditional characteristic functions of MG-fLps we need to define integration. This will be done by the usual $L^2(\Omega)$- approach, as e.g. in Pipiras and Taqqu [29, 30], Marquardt [25] or Tikanmäki and Mishura [37].

Consider simple functions of the form

$$f(\cdot) = \sum_{k=1}^m a_k \mathbf{1}_{[t_k,t_{k+1})}(\cdot),$$

where $m \in \mathbb{N}$, $m \ge 1$, $0 \le t_1 \le \dots \le t_m \le T$ and $a_k \in \mathbb{R}^{n\times n}$ for $1 \le k \le m$. Then we define

$$\int_0^T f(s)d\mathbf{L}^d(s) := \sum_{k=1}^m a_k(\mathbf{L}^d(t_{k+1}) - \mathbf{L}^d(t_k)) = \sum_{k=1}^m \begin{pmatrix} \sum_{j=1}^n (a_k)_{1j}(L^{d(j)}(t_{k+1}) - L^{d(j)}(t_k)) \\ \vdots \\ \sum_{j=1}^n (a_k)_{nj}(L^{d(j)}(t_{k+1}) - L^{d(j)}(t_k)) \end{pmatrix}.$$

A simple calculation leads to

$$\int_0^T f(s)d\mathbf{L}^d(s) = \int_0^T z^d(f,s)d\mathbf{L}(s).$$

Now we obtain from the definition of Λ^d_T and Theorem 2.1:

Theorem 3.7. *For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in \Lambda_T^d$. Then the integral $\int_0^T f(s) d\boldsymbol{L}^d(s)$ exists as a (componentwise) $L^2(\Omega)$-limit of approximating step functions in Λ_T^d (also componentwise). Furthermore, we have the identity*

$$\int_0^T f(s) d\boldsymbol{L}^d(s) = \int_0^T z^d(f, s) d\boldsymbol{L}(s),$$

which holds (componentwise) in $L^2(\Omega)$.

Further we find some distributional results on MG-fLp driven integrals. The proof uses Theorem 2.1, Theorem 3.7 and the fact that $z^d(t, s)$ is Hermitian and symmetric for $s, t \in [0, T]$.

Theorem 3.8. *For $d = (d(1), \dots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in \Lambda_T^d$. Then we have for all $u \in \mathbb{R}^n$*

$$\begin{aligned} & E\Big[\exp\Big\{i\Big\langle u, \int_0^T f(s) d\boldsymbol{L}^d(s)\Big\rangle\Big\}\Big] = \exp\Big\{\int_0^T \psi(z^d(f, s)^\top u) ds\Big\} \\ = \quad & \exp\Big\{i \int_0^T \langle z^d(f, s)\gamma, u\rangle ds - \frac{1}{2}\int_0^T u^\top z^d(f, s) \Sigma z^d(f, s)^\top u ds \\ & + \int_0^T \int_{\mathbb{R}^n} \Big(e^{i\langle u, z^d(f,s)x\rangle} - 1 - i\langle u, z^d(f, s)x\rangle \mathbf{1}_{\{\|z^d(f,s)x\|<1\}}\Big)\nu(dx) ds\Big\}. \end{aligned}$$

The characteristic function of a MG-fLp follows directly from Theorem 3.9:

Example 3.9. For each fixed $t \in [0, T]$ the random vector $\mathbf{L}^d(t)$ is infinitely divisible and its characteristic function is for $u \in \mathbb{R}^n$ given by

$$\begin{aligned} & E[\exp\{i\langle u, \mathbf{L}^d(t)\rangle\}] = \exp\Big\{\int_0^T \psi(z^d(t, s) u) ds\Big\} \\ = \quad & \exp\Big\{i \int_0^T \langle z^d(t, s)\gamma, u\rangle ds - \frac{1}{2}\int_0^T u^\top z^d(t, s) \Sigma z^d(t, s) u ds \\ & + \int_0^T \int_{\mathbb{R}^n} \Big(e^{i\langle u, z^d(t,s)x\rangle} - 1 - i\langle u, z^d(t, s)x\rangle \mathbf{1}_{\{\|z^d(t,s)x\|<1\}}\Big)\nu(dx) ds\Big\} \end{aligned}$$

Remark 3.10. When $d \in (0, \frac{1}{2})^n$, it is also possible to define pathwise integration with respect to MG-fLps using Hölder continuity like in Buchmann and Klüppelberg [10] or a p-variation approach like in Fink and Klüppelberg [16]. Another possible approach would be via a Skorohod-type integral using the S-transform as suggested by Bender and Marquardt [4].

4 Results on conditional characteristic functions

In this section we will state and prove our main theorems about the conditional characteristic functions of MG-fLp driven integrals and related processes. Define for $d \in (-\frac{1}{2}, \frac{1}{2})$ and suitable $f : [0, T] \to \mathbb{R}^{n \times n}$ the deconvolution operator

$$z_\star^d(f, s) := \begin{pmatrix} c_{-d(1)}^{-1} s^{d(1)} I_{T-}^{d(1)}((\cdot)^{-d(1)} f_{11}(\cdot))(s) & \dots & c_{-d(n)}^{-1} s^{d(n)} I_{T-}^{d(n)}((\cdot)^{-d(n)} f_{1n}(\cdot))(s) \\ \vdots & \ddots & \vdots \\ c_{-d(1)}^{-1} s^{d(1)} I_{T-}^{d(1)}((\cdot)^{-d(1)} f_{n1}(\cdot))(s) & \dots & c_{-d(n)}^{-1} s^{d(n)} I_{T-}^{d(n)}((\cdot)^{-d(n)} f_{nn}(\cdot))(s) \end{pmatrix}.$$

First we need a technical lemma, which will be crucial to derive the prediction formula and ensure that all appearing deconvolution operators exist.

Lemma 4.1. *For $d = (d(1), \ldots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in \Lambda_T^d$. Then the following assertions hold for all $0 \leq s \leq t \leq T$:*

(i) For all $1 \leq i, j \leq n$ we have that $z_{ij}^d(f\mathbf{1}_{[s,t]}, \cdot) \in L^2([0,T]) \subset L^1([0,T])$.

(ii) The function

$$[0, s] \to \mathbb{R}^{n\times n}, \quad v \mapsto z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(f\mathbf{1}_{[s,t]}, \cdot), v)$$

exists componentwise and belongs to $\widetilde{\Lambda}_s^d$.

(iii) For all $v \in [0, T]$,

$$z^d(z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(f\mathbf{1}_{[s,t]}, \cdot), \cdot), v) = \mathbf{1}_{[0,s]}(v) z^d(f\mathbf{1}_{[s,t]}, v).$$

Proof. Since $f \in \Lambda_T^d$ it follows that $f\mathbf{1}_{[s,t]} \in \Lambda_T^d$. By definition of Λ_T^d we therefore get that $z_{ij}^d(f\mathbf{1}_{[s,t]}, \cdot) \in L^2([0,T]) \subset L^1([0,T])$, which leads to (i). The existence of the function in assertion (ii) can be obtained by using a similar approximation argument as in the proof of Lemma 1 of Duncan [13]. The second statement in (ii) and (iii) follows by definition of $\widetilde{\Lambda}_s^d$ and by applying Theorem 2.5 of Samko, Kilbas and Marichev [33] componentwise. □

Remark 4.2. Clearly $\mathbf{L}^d$ is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ which can now be directly seen by the deconvolution formula (cf. Jost [22])

$$\mathbf{L}(t) = \int_0^t z_\star^{-d}(\hat{\mathbf{1}}_{[0,t]}, s) d\mathbf{L}^d(s), \quad t \in [0, T].$$

Further we want to remark that this is not the case for fractional Lévy processes as defined in Marquardt [25]. However in contrast to the processes in [25], MG-fLps do not have stationary increments in general, cf. Proposition 3.11 of Tikanmäki and Mishura [37].

Theorem 4.3. *For $d = (d(1), \ldots, d(n))^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in \Lambda_T^d$ and*

$$\mathcal{F}_t = \overline{\sigma\Big\{ \int_0^s f(v) d\mathbf{L}^d(v), s \in [0,t] \Big\}}, \quad t \in [0, T].$$

Then we have for all $0 \leq s \leq t \leq T$ and $u \in \mathbb{R}^n$

$$\begin{aligned} & E\Big[\exp\Big\{ i\Big\langle u, \int_0^t f(v) d\boldsymbol{L}^d(v) \Big\rangle \Big| \mathcal{F}_s \Big\}\Big] \\ = \ & \exp\Big\{ i\Big\langle u, \int_0^s f(v) d\boldsymbol{L}^d(v) + \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]} z^d(f\mathbf{1}_{[s,t]}, \cdot), v) d\boldsymbol{L}^d(v) \Big\rangle + \int_s^t \psi(z^d(f\mathbf{1}_{[s,t]}, v)^\top u) dv \Big\}. \end{aligned}$$

Proof. Recall that a MG-fLp is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the corresponding Lévy process. Since the random variable $\int_0^t f(v)d\mathbf{L}^d(v)$ is $\mathcal{F}_s$-measurable, it is enough to consider the conditional characteristic function of $\int_s^t f(v)d\mathbf{L}^d(v)$. Applying Theorem 3.7 we switch from the MG-fLp to the corresponding Lévy process and obtain

$$\int_s^t f(v)d\mathbf{L}^d(v) \quad = \quad \int_0^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v) = \int_0^s z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v) + \int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v).$$

The first summand on the righthand side is again $\mathcal{F}_s$-measurable and therefore we obtain

$$\begin{aligned} & E\Big[\exp\Big\{i\Big\langle u,\int_0^t f(v)d\mathbf{L}^d(v)\Big\rangle\Big|\mathcal{F}_s\Big\}\Big] \\ = \;& \exp\Big\{i\Big\langle u,\int_0^s f(v)d\mathbf{L}^d(v)+\int_0^s z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big\} \\ & \times E\Big[\exp\Big\{i\Big\langle u,\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big|\mathcal{F}_s\Big\}\Big]. \end{aligned}$$

However due to independent increments of Lévy processes we have

$$E\Big[\exp\Big\{i\Big\langle u,\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big|\mathcal{F}_s\Big\}\Big] = E\Big[\exp\Big\{i\Big\langle u,\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big\}\Big]$$

and applying Theorem 2.1 leads to

$$E\Big[\exp\Big\{i\Big\langle u,\int_s^T z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v)\Big\rangle\Big\}\Big] = \exp\Big\{\int_s^t \psi(z^d(f\mathbf{1}_{[s,t]},v)^\top u)dv\Big\},$$

where we have used the fact that $z^d(f\mathbf{1}_{[s,t]},v)=0$ if $v\in[t,T]$. Since we want the prediction formula in terms of the MG-fLp and not the driving Lévy process, we invoke Lemma 4.1 (ii) and apply again Theorem 3.7 to obtain

$$\int_0^s z^d(f\mathbf{1}_{[s,t]},v)d\mathbf{L}(v) = \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(f\mathbf{1}_{[s,t]},\cdot),v)d\mathbf{L}^d(v).$$

Putting everything together we get the assertion. □

Remark 4.4. Every $f\in\Lambda_T^d$, with $f_{ij}(u)\neq 0$ for all $u\in[0,T]$ and $1\le i,j\le n$, satisfies the conditions of Theorem 4.3. This follows since an MG-fLp is adapted to the natural filtration of the driving Lévy process, cf. Remark 3.2 of Tikanmäki and Mishura [37].

Example 4.5. [Univariate fBm] Choose in Theorem 4.3 $n=1$, $f=\mathbf{1}_{[0,t]}$, $0\le t\le T$, and take as driving Lévy process a standard Brownian motion, i.e. $\mathbf{L}=B$. Then $\mathbf{L}^d=B^d$ is an univariate fBm. Using Theorem 3.1 and Remark 3.2 of Fink, Klüppelberg and Zähle [17], the conditional characteristic function is for $0\le s\le t\le T$ given by

$$\begin{aligned} & E[\exp\{iuB^d(t)|\mathcal{F}_s\}] \\ = \;& \exp\Big\{iu\big[B^d(s)+\int_0^s \Psi^d(s,t,v)dB^d(v)\big]-\frac{u^2}{2}\big[\|\mathbf{1}_{[s,t]}\|_{d,T}^2-\|\Psi^d(s,t,\cdot)\mathbf{1}_{[0,s]}\|_{d,T}^2\big]\Big\}, \end{aligned}$$

where

$$\Psi^d(s,t,v) = v^{-d}(I_{s-}^{-d}(I_{t-}^{d}(\cdot)^d \mathbf{1}_{[s,t]}(\cdot)))(v), \quad v \in (0,s),$$

and $\|\cdot\|_{d,T}^2$ as defined in [17], section 2. This matches the result of Theorem 4.3 since $I_{t-}^{-d}\mathbf{1}_{[0,s]}(\cdot)f(\cdot) = I_{s-}^{-d}f(\cdot)$ in the situation above. Furthermore we have

$$\|\mathbf{1}_{[s,t]}\|_{d,T}^2 - \|\Psi^d(s,t,\cdot)\mathbf{1}_{[0,s]}\|_{d,T}^2 = \|z^d(\mathbf{1}_{[s,t]},\cdot)\mathbf{1}_{[s,t]}(\cdot)\|^2.$$

Example 4.6. [Univariate Gamma MG-fLp] In Theorem 4.3 choose $n=1$, $f = \mathbf{1}_{[0,t]}$, $0 \le t \le T$, and take as driving Lévy process a univariate Gamma process $G = (G(t))_{t\in[0,T]}$. Its distribution is then characterized by

$$\psi(u) = -\gamma \log\big(1 - \frac{u}{\lambda}\big)\mathbf{1}_{[0,\lambda)}(u),$$

with $\gamma, \lambda > 0$. By Theorem 4.3 we obtain for the Gamma MG-fLp $G^d = (G^d(t))_{t\in[0,T]}$ and $0 \le s \le t \le T$

$$\begin{aligned}
& E[\exp\{iuG^d(t)|\mathcal{F}_s\}] \\
= \;& \exp\Big\{iu\big[G^d(s) + \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(\mathbf{1}_{[s,t]},\cdot),v)dG^d(v)\big]\Big\} \\
& \times \exp\Big\{\gamma\log(\lambda)[t-s] - \gamma\int_s^t \log(\lambda - z^d(\mathbf{1}_{[s,t]},v)u)dv\Big\}.
\end{aligned}$$

Example 4.7. [Bivariate Poisson MG-fLp] Choose in Theorem 4.3 $n=2$, $f = \mathbf{1}_{[0,t]}$, $0 \le t \le T$, and take as driving Lévy process a bivariate Poisson process, i.e. here take independent Poisson processes Z_i on $[0,T]$ with intensities $\eta_i \ge 0$ for $i = 1,2,3$ and define

$$\mathbf{L} := (Z_1 + Z_2, Z_2 + Z_3)^\top.$$

The distribution of this bivariate Lévy process is then characterized by

$$\psi(u) = \int_{\mathbb{R}^2}(\exp\{i\langle u,x\rangle - 1 - i\langle u,x\rangle\mathbf{1}_{\{\|x\|<1\}})\nu(dx) = \int_{\mathbb{R}^2}(\exp\{i\langle u,x\rangle\} - 1)\nu(dx), \quad u \in \mathbb{R}^2,$$

with $\nu(dx) = \eta_1\delta_{\{1\}\times\{0\}}(dx) + \eta_2\delta_{\{1\}\times\{1\}}(dx) + \eta_3\delta_{\{0\}\times\{1\}}(dx)$. Theorem 4.3 leads now for $0 \le s \le t \le T$ to

$$\begin{aligned}
& E[\exp\{i\langle u,\mathbf{L}^d(t)\rangle|\mathcal{F}_s\}] \\
= \;& \exp\Big\{i\Big\langle u, \mathbf{L}^d(s) + \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(\mathbf{1}_{[s,t]},\cdot),v)d\mathbf{L}^d(v)\Big\rangle\Big\} \\
& \times \exp\Big\{\eta_1\int_s^t \big(\exp\big(i\sum_{j=1}^2 z_{1j}^d(\mathbf{1}_{[s,t)},v)u_j\big) - 1\big)dv\Big\} \\
& \times \exp\Big\{\eta_2\int_s^t \big(\exp\big(\sum_{k=1}^2\sum_{j=1}^2 z_{kj}^d(\mathbf{1}_{[s,t)},v)u_j\big) - 1\big)dv\Big\} \\
& \times \exp\Big\{\eta_3\int_s^t \big(\exp\big(i\sum_{j=1}^2 z_{2j}^d(\mathbf{1}_{[s,t)},v)u_j\big) - 1\big)dv\Big\}
\end{aligned}$$

In a next step we will consider MG-fLp-driven Ornstein-Uhlenbeck processes, starting with the definition. Similar processes were considered by Marquardt [26] and Klüppelberg and Matsui [23]. However, in contrast to our work, they define their underlying fractional Lévy processes by an integral representation over the whole real line like in Marquardt [25]. Postponing the usual question of existence and uniqueness until Proposition 4.8 we define:

Definition 4.8. *For $d = (d(1), \ldots, d(n))^\top \in (0, \frac{1}{2})^n$, take $\sigma \in \Lambda_T^d$, $k : [0,T] \to \mathbb{R}^n$ and $a : [0,T] \to \mathbb{R}^{n\times n}$, k, a componentwise locally integrable such that $e^{-\int_\cdot^t a(v)dv}\sigma(\cdot) \in \Lambda_T^d$ for all $t \in [0,T]$. Then the (unique) solution to the stochastic differential equation (sde)*

$$d\mathfrak{L}^d(t) = (k(t) - a(t)\mathfrak{L}^d(t))dt + \sigma(t)d\mathbf{L}^d(t), \quad t \in [0,T], \quad \mathfrak{L}^d(0) \in \mathbb{R}^n.$$

is called a fractional Lévy Ornstein-Uhlenbeck process (fLOUp).

The next proposition ensures the existence of a solution. Its uniqueness follows by a simple application of Gronwall's Lemma (e.g. Theorem 3.1 of Ikeda and Watanabe [21]) similar to the classical Brownian case.

Proposition 4.9. *In the situation of Definition 4.8 we have for $t \in [0,T]$*

$$\mathfrak{L}^d(t) = e^{-\int_0^t a(s)ds}\mathfrak{L}^d(0) + \int_0^t e^{-\int_s^t a(v)dv}k(s)ds + \int_0^t e^{-\int_s^t a(v)dv}\sigma(s)d\mathbf{L}^d(s),$$

where the matrix exponential is defined as usual.

Proof. Define $\mathfrak{L}^d$ as above and calculate for $0 \le s \le t \le T$

$$\begin{aligned}
& -\int_s^t a(z)\mathfrak{L}^d(z)dz = -\int_s^t a(z)e^{-\int_0^z a(v)dv}dz\mathfrak{L}^d(0) - \int_s^t a(z)\int_0^z e^{-\int_w^z a(v)dv}k(w)dwdz \\
& -\int_s^t a(z)\int_0^z e^{-\int_w^z a(v)dv}\sigma(w)d\mathbf{L}^d(w)dz \\
= \; & \left[e^{-\int_0^t a(v)dv} - e^{-\int_0^s a(v)dv}\right]\mathfrak{L}^d(0) \\
& -\int_s^t a(z)\int_0^s e^{-\int_w^z a(v)dv}k(w)dwdz - \int_s^t a(z)\int_s^z e^{-\int_w^z a(v)dv}k(w)dwdz \\
& -\int_s^t a(z)\int_0^s e^{-\int_w^z a(v)dv}\sigma(w)d\mathbf{L}^d(w)dz - \int_s^t a(z)\int_s^z e^{-\int_w^z a(v)dv}\sigma(w)d\mathbf{L}^d(w)dz \\
= \; & \left[e^{-\int_0^t a(v)dv} - e^{-\int_0^s a(v)dv}\right]\mathfrak{L}^d(0) \\
& -\int_0^s \int_s^t a(z)e^{-\int_w^z a(v)dv}k(w)dzdw - \int_s^t \int_w^t a(z)e^{-\int_w^z a(v)dv}k(w)dzdw \\
& -\int_0^s \int_s^t a(z)e^{-\int_w^z a(v)dv}\sigma(w)dzd\mathbf{L}^d(w) - \int_s^t \int_w^t a(z)e^{-\int_w^z a(v)dv}\sigma(w)dzd\mathbf{L}^d(w) \\
= \; & \left[e^{-\int_0^t a(v)dv} - e^{-\int_0^s a(v)dv}\right]\mathfrak{L}^d(0) \\
& +\int_0^s \left[e^{-\int_w^t a(v)dv} - e^{-\int_w^s a(v)dv}\right]k(w)dw + \int_s^t \left[e^{-\int_w^t a(v)dv} - I\right]k(w)dw \\
& +\int_0^s \left[e^{-\int_w^t a(v)dv} + e^{-\int_w^s a(v)dv}\right]\sigma(w)d\mathbf{L}^d(w) + \int_s^t \left[e^{-\int_w^t a(v)dv} - I\right]\sigma(w)d\mathbf{L}^d(w) \\
= \; & \mathfrak{L}^d(t) - \mathfrak{L}^d(s) - \int_s^t \sigma(w)d\mathbf{L}^d(w) - \int_s^t k(w)dw.
\end{aligned}$$

□

Remark 4.10. Using Hölder continuity of the MG-fLp paths (Lemma 3.6 (*ii*)), we can also define integration via a pathwise approach, cf. Young [38]. If the pathwise and the L^2-integral both exist, they have to be equal. The next lemma is based on this fact.

Lemma 4.11. *In addition to the assumptions of Definition 4.8, let the matrix $\sigma(t)$ be non-singular for every $0 \le t \le T$ and $d = (d(1), \ldots, d(n))^\top \in (0, \frac{1}{2})^n$. Furthermore assume σ_{ij} and $(\sigma)^{-1}_{ij}$ are of bounded $p(j)$-variation for some $0 < p(j) < 1/(1 - d(j))$ (cf. Young [38]) for all $1 \le i, j \le n$. Then we have*

$$d\mathbf{L}^d(t) = \Big(-\sigma(t)^{-1}k(t) + \sigma(t)^{-1}a(t)\mathfrak{L}^d(t)\Big)\,dt + \sigma(t)^{-1}d\mathfrak{L}^d(t), \quad 0 \le t \le T.$$

Proof. The proof is analogous to the proof of Proposition 4.9. □

The prediction result follows. The proof is a combination of Theorem 4.3 and Lemma 4.11.

Theorem 4.12 (fLOUp)**.** *For $d = (d(1), \ldots, d(n))^\top \in (0, \frac{1}{2})^n$ take $\sigma \in \Lambda^d_T$, $k : [0, T] \to \mathbb{R}^n$ and $a : [0, T] \to \mathbb{R}^{n\times n}$, k, a componentwise locally integrable such that $e^{-\int_\cdot^t a(v)dv}\sigma(\cdot) \in \Lambda^d_T$ for all $t \in [0, T]$. Let further $\sigma(t)$ be non-singular for every $0 \le t \le T$ and assume σ_{ij} and $(\sigma)^{-1}_{ij}$ are of bounded $p(j)$-variation for some $0 < p(j) < 1/(1 - d(j))$ (cf. Young [38]) for all $1 \le i, j \le n$. Then we have for $0 \le s \le t \le T$ and $u \in \mathbb{R}^n$*

$$\begin{aligned}
& E[\exp\{i\langle u, \mathfrak{L}^d(t)\rangle | \mathcal{F}_s\}] \\
= \;& \exp\Big\{i\Big\langle u, e^{-\int_s^t a(v)dv}\mathfrak{L}^d(s) + \int_s^t e^{-\int_w^t a(v)dv}k(w)dw\Big\rangle\Big\} \\
& \times \exp\Big\{i\Big\langle u, \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(h\mathbf{1}_{[s,t]}, \cdot), v)d\boldsymbol{L}^d(v)\Big\rangle + \int_s^t \psi(z^d(h\mathbf{1}_{[s,t]}, v)^\top u)dv\Big\} \\
= \;& \exp\Big\{i\Big\langle u, e^{-\int_s^t a(v)dv}\mathfrak{L}^d(s) + \int_s^t e^{-\int_w^t a(v)dv}k(w)dw\Big\rangle\Big\} \\
& \times \exp\Big\{-i\Big\langle u, \int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(h\mathbf{1}_{[s,t]}, \cdot), v)\sigma(v)^{-1}k(v)dv\Big\rangle\Big\} \\
& \times \exp\Big\{\int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(h\mathbf{1}_{[s,t]}, \cdot), v)\sigma(v)^{-1}a(v)\mathfrak{L}^d(v)dv\Big\} \\
& \times \exp\Big\{\int_0^s z_\star^{-d}(\mathbf{1}_{[0,s]}z^d(h\mathbf{1}_{[s,t]}, \cdot), v)\sigma(v)d\mathfrak{L}^d(v) + \int_s^t \psi(z^d(h\mathbf{1}_{[s,t]}, v)^\top u)dv\Big\},
\end{aligned} \tag{4.1}$$

where $h(\cdot) = e^{-\int_\cdot^t a(v)dv}\sigma(\cdot)$.

Remark 4.13. [General sdes] Using the MG-fLp decomposition (3.2) and Proposition 3.9 of Tikanmäki and Mishura [37] we see that the i-th component of a MG-fLp is of zero-quadratic variation if $d(i) \in (0, \frac{1}{2})$. Therefore for $d = (d(1), \ldots, d(n))^\top \in (0, \frac{1}{2})^n$, general sdes driven by MG-fLps can be considered using for example the theory of Zähle [40], section 5. However as already mentioned in Fink and Klüppelberg [16], this does not cover CIR type processes. Using the zero-quadratic variation property and the integration concept of Russo and Vallois [32], a similar theory as in [16] can be proven (at least up to stationary solutions, where we have to face Remark 4.2). Therefore it will be interesting to calculate the prediction of transforms of MG-fLp driven integrals, which can be achieved by Fourier methods (cf. Theorem 3.7 of Fink, Klüppelberg and Zähle [17] for the univariate fBm case). An example is the next theorem which can be proven similar to Buchmann and Klüppelberg [10] using the idea of Theorem 3.7 and Example 3.8 of [17].

Theorem 4.14 (fLCIRp)**.** *For $d \in (0, \frac{1}{2})$ and $a > 0$ let $X = (X(t))_{t\in[0,T]}$ be the solution to the fLOUp sde*

$$dX(t) = -\frac{a}{2}X(t)dt + dL^d(t), \quad t \in [0,T], \quad X(0) \in \mathbb{R}.$$

Assume further that $E[\exp\{X(t)\}] < \infty$ for $t \in [0,T]$. Take $f(x) = \operatorname{sign}(x)x^2\frac{\sigma^2}{4}$ for $x \in \mathbb{R}$ and $\sigma > 0$. Define the process $Z = (Z(t))_{t\in[0,T]}$ by $Z(t) = f(X(t))$. Then for $0 \le s \le t \le T$

$$Z(t) - Z(s) = -a\int_s^t Z(v)dv + \sigma\int_s^t \sqrt{|Z(v)|}d^{\mathrm{RV}}L^d(v)$$

holds with $Z(0) = f(X(0))$. Here the integral $\int_s^t \sqrt{|Z(v)|}d^{\mathrm{RV}}L^d(v)$ is the forward integral of Definition 1 of Russo and Vallois [32]. Furthermore for $u \in \mathbb{R}$ we have

$$E\Big[e^{iuZ(t)}\Big|\mathcal{F}_s\Big] \;=\; \int_{\mathbb{R}}\Big(E\Big[e^{(i\xi+1)X(t)}\Big|\mathcal{F}_s\Big]g_+(\xi,u) + E\Big[e^{(i\xi-1)X(t)}\Big|\mathcal{F}_s\Big]g_-(\xi,u)\Big)\,d\xi$$

with $g_\star(\xi,u) = (2\pi)^{-1}\int_{\mathbb{R}_\star} e^{-(i\xi\star1)x+iuf(x)}dx$, $\star \in \{+,-\}$, where $E[e^{(i\xi+1)X(t)}|\mathcal{F}_s]$ is given by the analytic extension of (4.1) *to $\mathbb{C}$.*

5 Fractional credit models

We shall work in the framework of the most reduced-form credit risk models in the literature. Given a finite time horizon $T^\star > 0$, a credit market shall be described by the bivariate process $(r,H) = (r(t),H(t))_{0\le t\le T^\star}$ on a given probability space $(\Omega,\mathcal{F},\mathcal{Q})$ endowed with the filtration $(\mathcal{F}_t)_{0\le t\le T^\star}$, which represents the market information and satisfies the usual conditions of completeness and right continuity. The process r models the *short rate* and H the *default indicator*, i.e.

$$H(t) = \mathbf{1}_{\{\tau\le t\}}, \quad 0 \le t \le T^\star,$$

for a given $(\mathcal{F}_t)_{0\le t\le T^\star}$-stopping time τ, the *default time.* Denote by $(\mathcal{H}_t)_{0\le t\le T^\star}$ the filtration generated by H.

Assumption 5.1 (Market structure; cf. Frey and Backhaus [19], Ass. 3.1)**.**

(i) We assume that there is a subfiltration $(\mathcal{G}_t)_{0\le t\le T^\star}$ of $(\mathcal{F}_t)_{0\le t\le T^\star}$ with

$$\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t\,, \quad 0 \le t \le T^\star,$$

r *is* $(\mathcal{G}_t)_{0\le t\le T^\star}$*-progressive, and that there exists a positive* $(\mathcal{G}_t)_{0\le t\le T^\star}$*-progressive process* $\lambda = (\lambda_t)_{0\le t\le T^\star}$*, called the* default rate*, describing the intensity of* H *(cf. Corollary 5.1.5 of Bielecki and Rutkowski [8]) with* $\int_0^t \lambda(s)ds < \infty$ *a.s. for all* $0 \le t \le T^\star$*. Furthermore assume that*

$$P(\tau > t \mid \mathcal{G}_t) = E[1 - H(t) \mid \mathcal{G}_t] = \exp\left\{-\int_0^t \lambda(s)ds\right\}. \tag{5.1}$$

Setting $\mathcal{G}_\infty := \bigvee_{0\le t\le T^\star} \mathcal{G}_t$*, assume that for all bounded* $\mathcal{G}_\infty$*-measurable random variables* η*,*

$$E[\eta \mid \mathcal{F}_t] = E[\eta \mid \mathcal{G}_t] \tag{5.2}$$

holds.

(ii) $\mathcal{Q}$ *is a risk neutral pricing measure, such that the price of any* $\mathcal{F}_T$*-measurable claim* $X \in L^1(\Omega)$ *with maturity* $0 \le T \le T^\star$ *at time* $0 \le t \le T$ *is given by* $\mathcal{V}(t,T) = E[X \mid \mathcal{F}_t]$ *for* $0 \le t \le T$*.*

In the framework above, the default history $(\mathcal{F}_t)_{0\le t\le T^\star}$ is the investor information at time t, meaning that the investor knows the short rate r, the default rate λ and the default indicator process H at time t. Using Lemma 13.2 of Filipovic [15] we see that the price of a defaultable zero coupon bond is for $0 \le t \le T \le T^\star$ given by

$$\overline{B}(t,T) \quad = \quad E\Big[\mathbf{1}_{\{\tau>T\}} e^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\Big] = \mathbf{1}_{\{\tau>t\}} E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big].$$

Therefore it is sufficient to specify the dynamics of the bivariate process (r,λ), forgetting H. We propose a fractional Vasicek model:

Assumption 5.2 (Fractional Vasicek model)**.** *For* $d = (d(1),\dots,d(n))^\top \in (0,\frac{1}{2})^n$*,* $n \in \mathbb{N}$*, take* $\sigma \in \Lambda^d_{T^\star}$*,* $k : [0,T^\star] \to \mathbb{R}^n$ *and* $a : [0,T^\star] \to \mathbb{R}^{n\times n}$*,* k, a *componentwise locally integrable such that* $e^{-\int_\cdot^t a(v)dv}\sigma(\cdot) \in \Lambda^d_{T^\star}$ *for all* $t \in [0,T^\star]$*. Consider the corresponding Ornstein-Uhlenbeck sde*

$$d\mathfrak{L}^d(t) = (k(t) - a(t)\mathfrak{L}^d(t))dt + \sigma(t)d\mathbf{L}^d(t), \quad t \in [0,T^\star], \quad \mathfrak{L}^d(0) \in \mathbb{R}^{n\times n}.$$

Assume further as in Lemma 4.11 that $\sigma(t)$ *is non-singular for every* $0 \le t \le T^\star$ *and* σ_{ij} *and* $(\sigma)^{-1}_{ij}$ *are of bounded* $p(j)$*-variation for some* $0 < p(j) < 1/(1-d(j))$ *(cf. Young [38]) for all* $1 \le i,j \le n$*. Then define for fixed weights* $\theta, \phi \in (\mathbb{R}_+)^n$*, where* $\theta \ne 0$*,*

$$r(t) = \langle \theta, \mathfrak{L}^d(t)\rangle \quad \text{and} \quad \lambda(t) = \langle \phi, \mathfrak{L}^d(t)\rangle, \quad t \in [0,T^\star]. \tag{5.3}$$

Therefore we have $\mathcal{G}_t = \sigma\overline{\{\mathbf{L}^d(s), s \in [0,t]\}}$ *for all* $t \in [0,T^\star]$*.*

The following theorem considers the price of a defaultable zero coupon bond in the fractional Vasicek credit market 5.2.

Theorem 5.3. *Let* $0 \le t \le T \le T^\star$*. In the model of Assumption 5.2, set* $D(t,T) := \int_t^T e^{-\int_t^s a(v)dv} ds$ *where the integral is taken componentwise. Assume further that* $D(\cdot,T)\sigma(\cdot) \in \Lambda^d_T$ *and*

$$E\Big[\exp\Big\{-\Big\langle \theta + \phi, \int_t^T D(w,T)\sigma(w)d\mathbf{L}^d(w)\Big\rangle\Big\}\Big] < \infty. \tag{5.4}$$

Then we have

$$
\begin{aligned}
\overline{B}(t,T) &= \mathbf{1}_{\{\tau>t\}} E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big] \\
&= \mathbf{1}_{\{\tau>t\}} \exp\Big\{-\Big\langle \theta+\phi, D(t,T)\mathfrak{L}^d(t) + \int_t^T D(v,T)k(v)dv\Big\rangle\Big\} \\
&\quad \times \exp\Big\{-\Big\langle \theta+\phi, \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(h\mathbf{1}_{[t,T]},\cdot),v)d\boldsymbol{L}^d(v)\Big\rangle\Big\} \\
&\quad \times \exp\Big\{\int_t^T \psi(z^d(h\mathbf{1}_{[t,T]},v)^\top i(\theta+\phi))dv\Big\},
\end{aligned}
$$

with $h(\cdot) = D(\cdot,T)\sigma(\cdot)$.

Proof. By Assumption 5.2 the stochastic integrals also exist in the pathwise sense. By the proof of Proposition 4.9 and Fubini's Theorem (pathwise) we get

$$
\begin{aligned}
&\int_t^T (r(s)+\lambda(s))ds = (\theta+\phi)^\top \int_t^T \mathfrak{L}^d(s)ds \\
=\; & (\theta+\phi)^\top \int_t^T \Big[e^{-\int_t^s a(v)dv}\mathfrak{L}^d(t) + \int_t^s e^{-\int_w^s a(v)dv}k(w)dw \\
&+ \int_t^s e^{-\int_w^s a(v)dv}\sigma(w)d\mathbf{L}^d(w)\Big]ds \\
=\; & (\theta+\phi)^\top\Big[D(t,T)\mathfrak{L}^d(t) + \int_t^T D(v,T)k(v)dv + \int_t^T D(w,T)\sigma(w)d\mathbf{L}^d(w)\Big].
\end{aligned}
$$

We calculate

$$
\begin{aligned}
& E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big] \\
=\; & \exp\Big\{-\Big\langle \theta+\phi, \Big[D(t,T)\mathfrak{L}^d(t) + \int_t^T D(v,T)k(w)dw\Big\rangle\Big\} \\
& \times E\Big[\exp\Big\{-\Big\langle \theta+\phi, \int_t^T D(w,T)\sigma(w)d\mathbf{L}^d(w)\Big\rangle\Big\}\Big|\mathcal{G}_t\Big].
\end{aligned}
$$

By assumption, the conditional expectation is a.s. smaller than infinity and therefore we can invoke the prediction result of Theorem 4.3 (by extending it to $u \in \mathbb{C}$) to achieve

$$
\begin{aligned}
& E\Big[\exp\Big\{-\Big\langle \theta+\phi, \int_t^T D(w,T)\sigma(w)d\mathbf{L}^d(w)\Big\rangle\Big\}\Big|\mathcal{G}_t\Big] \\
=\; & \exp\Big\{-\Big\langle \theta+\phi, \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(h\mathbf{1}_{[t,T]},\cdot),v)d\mathbf{L}^d(v)\Big\rangle\Big\} \\
& \times \exp\Big\{\int_t^T \psi(z^d(h\mathbf{1}_{[t,T]},v)^\top i(\theta+\phi))dv\Big\},
\end{aligned}
$$

with $h(\cdot) = D(\cdot,T)\sigma(\cdot)$. Putting everything together we obtain the assertion. □

Remark 5.4. Condition (5.4) is met if we assume the components of $\mathbf{L}^d$ to be fractional subordinators and $\sigma(\cdot)$ to be componentwise positive. These assumptions are economically justified since interest rates should be positive in most cases. We refer to Theorem 3.3 of Rajput and Rosinski [31] for more general conditions.

Remark 5.5. The Gaussian case was already considered in Fink, Klüppelberg and Zäehle [17] and Biagini, Fink and Klüppelberg [6] using different, partly more direct approaches. It has the serious drawback that processes like the short rate could be negative. However in practice the Gaussian models are fast to implement and very tractable. Of course it is always possible to scale and shift a Gaussian Vasicek model such that the probability of becoming negative is arbitrarily small. However with the theory in this paper we can propose models with non-negative short and default rates where bond prices are comparably easy to calculate. On the other hand we want to mention that when considering credit derivatives like *credit default swaps*, negative interest and hazard rates might be suitable to model market anomalies like inverted CDS curves, cf. Fink and Scherr [18].

The remainder of the paper is dedicated to the consideration of a specific example using the bivariate Poisson MG-fLp of Example 4.7. The components of this process are fractional subordinators in the sense of Example 3.3 and have therefore the advantage of delivering non-negative short and hazard rates.

Example 5.6. [Fractional Poisson market] Assume $d(1), d(2) > 0$. Take the bivariate Poisson MG-fLp of Example 4.7 as driving process in the fractional market 5.2. Further, for simplicity, take $k(\cdot) = (k_1, k_2)^\top$, with $k_1, k_2 > 0$,

$$a(\cdot) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \sigma(\cdot) = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

for $a_1, a_2, \sigma_1, \sigma_2 > 0$ and $\theta = (1,0)^\top$, $\phi = (0,1)^\top$. Therefore

$$\begin{aligned} dr(t) &= (k_1 - a_1 r(t))dt + \sigma_1 dL^{d(1)}(t), \quad r(0) = r_0 \in \mathbb{R}, \\ d\lambda(t) &= (k_2 - a_2 \lambda(t))dt + \sigma_2 dL^{d(2)}(t), \quad \lambda(0) = \lambda_0 \in \mathbb{R}, \end{aligned}$$

where $L^{d(1)}$ and $L^{d(2)}$ are dependent.

We apply now Theorem 5.3. Condition (5.4) is met since the process in the exponential is non-positive. Further we have

$$D(t,T) = \int_t^T e^{-\int_t^s a(v)dv} ds = \begin{pmatrix} \int_t^T e^{-a_1(s-t)} ds & 0 \\ 0 & \int_t^T e^{-a_2(s-t)} ds \end{pmatrix} =: \begin{pmatrix} D_1(t,T) & 0 \\ 0 & D_2(t,T) \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} & -\Big\langle \theta + \phi, D(t,T)\mathfrak{L}^d(t) + \int_t^T D(v,T)k(v)dv \Big\rangle \\ = \ & -D_1(t,T)r(t) - D_2(t,T)\lambda(t) - k_1 \int_t^T D_1(v,T)dv - k_2 \int_t^T D_2(v,T)dv \end{aligned}$$

and

$$
\begin{aligned}
& -\Big\langle \theta+\phi, \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(h(\cdot)\mathbf{1}_{[t,T]},\cdot),v)\, d\mathbf{L}^d(v)\Big\rangle \\
= \;& -\sigma_1 \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_1(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)\, dL^{d(1)}(v) - \sigma_2 \int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_2(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)\, dL^{d(2)}(v) \\
= \;& -\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_1(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)(-k_1 dv + a_1 r(v)dv + dr(v)) \\
& -\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_2(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)(-k_2 dv + a_2 \lambda(v)dv + d\lambda(v))
\end{aligned}
$$

where we used Lemma 4.11 in the last line. Applying the characterization of the Lévy measure from Example 4.7, we calculate

$$
\begin{aligned}
\psi(z^d(h\mathbf{1}_{[t,T]},v)^\top i(\theta+\phi)) &= \psi\left(\begin{pmatrix} \sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v) & 0 \\ 0 & \sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v) \end{pmatrix} i(\theta+\phi)\right) \\
&= \psi\left(\begin{pmatrix} i\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v) \\ i\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v) \end{pmatrix}\right) \\
&= \eta_1(\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1) \\
&\quad + \eta_2(\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v) - \sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1) \\
&\quad + \eta_3(\exp(-\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1)
\end{aligned}
$$

Putting everything together and applying Theorem 5.3 leads for all $0 \le t \le T \le T^\star$ to the bond prices

$$
\begin{aligned}
& \overline{B}(t,T) \\
= \;& \mathbf{1}_{\{\tau>t\}} E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big] \\
= \;& \mathbf{1}_{\{\tau>t\}} \exp\Big\{-D_1(t,T)r(t) - D_2(t,T)\lambda(t) - k_1\int_t^T D_1(v,T)dv - k_2\int_t^T D_2(v,T)dv\Big\} \\
& \times \exp\Big\{-\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_1(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)(-k_1 dv + a_1 r(v)dv + dr(v))\Big\} \\
& \times \exp\Big\{-\int_0^t z_\star^{-d}(\mathbf{1}_{[0,t]} z^d(D_2(\cdot,T)\mathbf{1}_{[t,T]},\cdot),v)(-k_2 dv + a_2 \lambda(v)dv + d\lambda(v))\Big\} \\
& \times \exp\Big\{\eta_1 \int_t^T (\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1)dv\Big\} \\
& \times \exp\Big\{\eta_2 \int_t^T (\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[t,T]},v) - \sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1)dv\Big\} \\
& \times \exp\Big\{\eta_3 \int_t^T (\exp(-\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[t,T]},v)) - 1)dv\Big\}.
\end{aligned}
$$

Remark 5.7. If we want to choose $d = (0,0)^\top$ in the context of Example 5.6, short and hazard rate are driven by Lévy processes which are a subclass of affine Markov processes. Therefore the bond prices can be calculated to

$$
\begin{aligned}
B(t,T) &= \exp\Big\{ - D_1(0,T)r(0) - D_2(0,T)\lambda(0) - k_1 \int_0^T D_1(v,T)dv - k_2 \int_0^T D_2(v,T)dv \Big\} \\
&\quad \times \exp\Big\{ \eta_1 \int_t^T (\exp(-\sigma_1 D_1(v,T)) - 1)dv + \eta_3 \int_t^T (\exp(-\sigma_2 D_2(v,T)) - 1)dv \Big\} \\
&\quad \times \exp\Big\{ \eta_2 \int_t^T (\exp(-\sigma_1 D_1(v,T) - \sigma_2 D_2(v,T)) - 1)dv \Big\},
\end{aligned}
$$

which represents the affine structure, see e.g. Duffie [11] and Duffie, Filipovic and Schachermayer [12]. However, if $d \neq (0,0)^\top$, the past paths of short/default rate matter and will enter the prices.

To compare prices we consider the case $t = 0$.

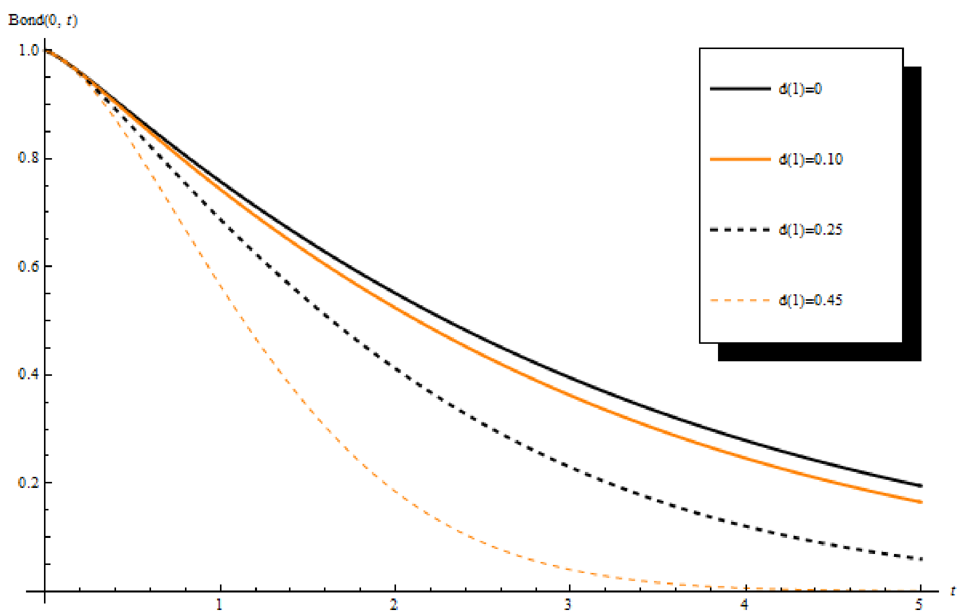

Figure 1: Bond prices $B(0,t)$ in the fractional Poisson market 5.6 for varying $d(1)$ and maturity t, using $(\eta_1, \eta_2, \eta_3)^\top = (2,1,2)^\top$, $(r(0), \lambda(0))^\top = (0.1, 0.05)^\top$, $k_1 = 0.5$, $k_2 = 1$, $a_1 = 4$, $a_2 = 8$, $\sigma_1 = 2$, $\sigma_2 = 1$ and $d(2) = 0.25$. Recall that $(d(1), d(2))^\top = (0,0)^\top$ corresponds to the Lévy Vasicek model of Remark 5.7. Prices decrease with $d(1)$ as a consequence of the long range dependence, which is very surprising, cf. Remark 5.9.

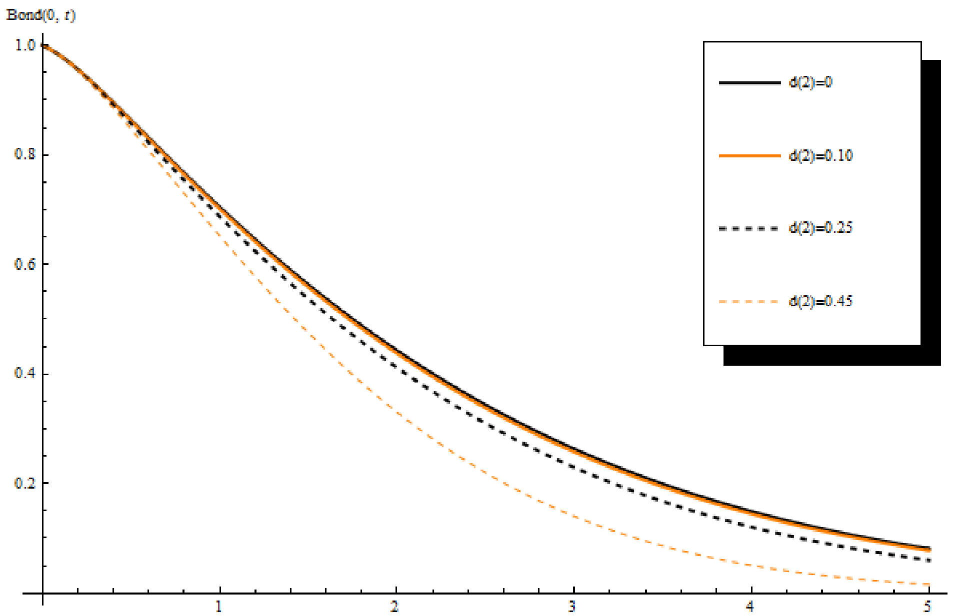

Figure 2: Bond prices $B(0,t)$ in the fractional Poisson market 5.6 for varying $d(2)$ and maturity t, using $(\eta_1,\eta_2,\eta_3)^\top = (2,1,2)^\top$, $(r(0),\lambda(0))^\top = (0.1, 0.05)^\top$, $k_1 = 0.5$, $k_2 = 1$, $a_1 = 4$, $a_2 = 8$, $\sigma_1 = 2$, $\sigma_2 = 1$ and $d(1) = 0.25$. Recall that $(d(1), d(2))^\top = (0,0)^\top$ corresponds to the Lévy Vasicek model of Remark 5.7. Prices decrease with $d(1)$ and $d(2)$ as a consequence of the long range dependence, cf. Remark 5.9.

Example 5.8. In the situation of Example 5.6 we have

$$\begin{aligned}
B(0,T) &= \exp\Big\{ - D_1(0,T)r(0) - D_2(0,T)\lambda(0) - k_1 \int_0^T D_1(v,T)dv - k_2 \int_0^T D_2(v,T)dv \Big\} \\
&\quad \times \exp\Big\{ \eta_1 \int_0^T (\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[0,T]}, v)) - 1)dv \Big\} \\
&\quad \times \exp\Big\{ \eta_2 \int_0^T (\exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[0,T]}, v) - \sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[0,T]}, v)) - 1)dv \Big\} \\
&\quad \times \exp\Big\{ \eta_3 \int_0^T (\exp(-\sigma_2 z^{d(2)}(D_2(\cdot,T)\mathbf{1}_{[0,T]}, v)) - 1)dv \Big\}.
\end{aligned}$$

Because of the singularities in the operator z^d classical numerical methods have to be used with care. We will choose a similar discretization scheme, as in the fBm case, cf. Fink, Klüppelberg and Zähle [17]. This will be explained for the first occuring fractional integration:

For $d(1) \in (0, \frac{1}{2})$ and $0 \le t \le T \le T^\star$ we have

$$\begin{aligned}
&\int_0^T \exp(-\sigma_1 z^{d(1)}(D_1(\cdot,T)\mathbf{1}_{[0,T]}, v))dv \\
&= \int_0^T \exp\Big(- \sigma_1 c_{d(1)} v^{-d(1)} \int_v^{T^\star} \frac{r^{d(1)} D(r,T)\mathbf{1}_{[0,T]}(r)}{(r-v)^{1-d(1)}} dr \Big) dv.
\end{aligned}$$

First we decompose the outer integral for $m \in \mathbb{N}$ and $0 = v_0 \leq v_1 \leq \cdots \leq v_m = T$

$$= \sum_{i=0}^{n-1} \int_{v_i}^{v_{i+1}} \exp\Big(- \sigma_1 c_{d(1)} v^{-d(1)} \int_v^T \frac{r^{d(1)} D(r,T) \mathbf{1}_{[0,T]}(r)}{(r-v)^{1-d(1)}} dr \Big) dv.$$

Now by Remark 3.3.13 we have for sufficiently small intervals $[v_i, v_{i+1}]$, subpartitions $v_i = w_0^i \leq w_1^i \leq \cdots \leq w_{m_i}^i = v_{i+1}$ for some $m_i \in \mathbb{N}$, $i = 0, \ldots, m-1$, and $v \in [v_i, v_{i+1}]$

$$\begin{aligned}
&\int_v^T \frac{r^{d(1)} D(r,T) \mathbf{1}_{[0,T]}(r)}{(r-v)^{1-d(1)}} dr \\
\approx\ & \frac{1}{2d(1)} \sum_{j=0}^{m_i - 1} [(w_{j+1}^i - v_i)^{d(1)} - (w_j^i - v_i)^{d(1)}][(w_j^i)^{d(1)} D(w_j^i, T) + (w_{j+1}^i)^{d(1)} D(w_{j+1}^i, T)] \\
=:\ & A(v_i)
\end{aligned}$$

Putting everything together we get

$$\begin{aligned}
&\int_0^T \exp(-\sigma_1 z^{d(1)} (D_1(\cdot, T) \mathbf{1}_{[0,T]}, v)) dv \\
=\ & \sum_{i=0}^{n-1} \int_{v_i}^{v_{i+1}} \exp\Big(- \sigma_1 c_{d(1)} v^{-d(1)} \int_v^T \frac{r^{d(1)} D(r,T) \mathbf{1}_{[0,T]}(r)}{(r-v)^{1-d(1)}} dr \Big) dv \\
\approx\ & \sum_{i=0}^{n-1} \int_{v_i}^{v_{i+1}} \exp\Big(- \sigma_1 c_{d(1)} A(v_i) v^{-d(1)} \Big) dv \approx \sum_{i=0}^{n-1} [v_{i+1} - v_i] \exp\Big(- \sigma_1 c_{d(1)} A(v_i) [v_{i+1} - v_i]/2 \Big).
\end{aligned}$$

Choosing now $v_i = 0.01i$, $i = 0, \ldots, 100t$, and $w_j^i = 0.01(i+j)$, $j = 0, \ldots, 100t - i$, we obtain

$$A(v_i) \quad = \quad \sum_{j=0}^{100t-i-1} [(j+1)^\kappa - j^\kappa][(i+j)^\kappa D(0.01(i+j), t) + (i+j+1)^\kappa D(0.01(i+j+1), t)]$$

and

$$\int_0^T \exp(-\sigma_1 z^{d(1)} (D_1(\cdot, T) \mathbf{1}_{[0,T]}, v)) dv \approx 0.01 \sum_{i=0}^{n-1} \exp\Big(- 0.005 \sigma_1 c_{d(1)} A(v_i) \Big). \qquad (5.5)$$

Remark 5.9. At first sight it is surprising that in the case of a fractional Poisson market prices decrease with $d(1)$ and $d(2)$, since in the Gaussian case the contrary is the case (cf. Fink, Klüppelberg and Zähle [17]). The reason behind this is the following: in a fractional Poisson market, short and default rate increase with the shocks of the driving Poisson subordinators and decrease between these exponentially. An increase in $d(1)$ and $d(2)$ means an increase in the (positive) correlation between these shocks. Therefore short and default rate are more likely to go up, which leads the bond price to drop.

In the Gaussian case, the driving processes are not increasing and an increase in $d(1)$ and $d(2)$ does no longer affect short and default rate as above, since there is now also a (positive) correlation between decreases of the driving processes.

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