A TAYLOR-LIKE EXPANSION OF A COMMUTATOR WITH A FUNCTION OF SELF-ADJOINT, PAIRWISE COMMUTING OPERATORS

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Abstract

Let A be a ν-vector of self-adjoint, pairwise commuting operators and B a bounded operator of class $C^{n_0}(A)$. We prove a Taylor-like expansion of the commutator $[B, f(A)]$ for a large class of functions $f\colon \mathsf{R}^\nu \to \mathsf{R}$, generalising the one-dimensional result where A is just a self-adjoint operator. This is done using almost analytic extensions and the higher-dimensional Helffer-Sjöstrand formula.

1. Introduction

It is well-known that if A is a self-adjoint operator, B is a bounded operator of class $C^{n_0}(A)$ in the sense of [1] and f satisfies $|f^{(n)}(x)| \leq C_n \langle x \rangle^{s-n}$ for all n, then for $0 \leq t_1 \leq n_0$, $0 \leq t_2 \leq 1$ with $s + t_1 + t_2 < n_0$,

$$[B, f(A)] = \sum_{k=1}^{n_0-1} \frac{1}{k!} f^{(k)}(A) \operatorname{ad}_A^k(B) + R_{n_0}(A, B)$$

where $\operatorname{ad}_A^k(B)$ is the k'th iterated commutator, $R_{n_0}(A,B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}; \mathcal{H}_A^{t_1})$ and $\mathcal{H}_A^t$ is defined as $\mathcal{D}(\langle A \rangle^t)$ equipped with the graph-norm $\|v\|_t = \|\langle A \rangle^t v\|$ for $t \geq 0$ and $\mathcal{H}_A^{-t}$ is the dual space of $\mathcal{H}_A^t$. This follows relatively easily from using the (one-dimensional) Helffer-Sjöstrand formula

$$f(A) = \frac{1}{\pi} \int_{\mathsf{C}} \bar{\partial} \tilde{f}(z)(A - z)^{-1} dz, \tag{1}$$

where $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\tilde{f}$ is an almost analytic extension of f, and the identity

$$\begin{aligned}[B, f(A)] = &\sum_{k=1}^{n_0-1} \frac{1}{k!} \frac{k!}{\pi} \int_{\mathsf{C}} \bar{\partial} \tilde{f}(z)(-1)^k (A - z)^{-k-1} dz \operatorname{ad}_A^k(B) \\ &+ \frac{(-1)^{n_0}}{\pi} \int_{\mathsf{C}} \bar{\partial} \tilde{f}(z)(A - z)^{-n_0} \operatorname{ad}_A^{n_0}(B)(A - z)^{-1} dz\end{aligned}$$

when $\frac{k!}{\pi} \int_{\mathsf{C}} \bar{\partial} \tilde{f}(z)(-1)^k (A - z)^{-k-1} dz$ is recognised as $f^{(k)}(A)$ using (1). Such commutator expansions where first proved in [7]. See e.g. [4] for details. Due to the higher complexity of the general Helffer-Sjöstrand formula, these calculations do not lead directly to the generalised result where A is a vector of self-adjoint, pairwise commuting operators. However, we will follow the same idea.

The theorem may be viewed as an abstract analogue of pseudo-differential calculus. The one-dimensional version is an often used result, see e.g. [2] and [4]. Apart from the obvious interest in generalising the result to higher dimensions, our improvement has proven useful in the treatment of models in quantum field theory, see [6]. In particular, a lemma in [6] whose proof depends on our result, extends the results of [5] to a larger class of models.

2. The setting and result

In the following, $A = (A_1, \dots, A_\nu)$ is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space $\mathcal{H}$, and $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on $\mathcal{H}$. We shall use the notion of B being of class $C^{n_0}(A)$ introduced in [1]. For notational convenience, we adobt the following convention: If $0 \le j \le \nu$, then δ_j denotes the multi-index $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the j'th entry.

DEFINITION 1. Let $n_0 \in \mathsf{N} \cup \{\infty\}$. Assume that the multi-commutator form defined iteratively by $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^\alpha(B) = [\mathrm{ad}_A^{\alpha-\delta_j}(B), A_j]$ as a form on $\mathcal{D}(A_j)$, where $\alpha \ge \delta_j$ is a multi-index and $1 \le j \le \nu$, can be represented by a bounded operator also denoted by $\mathrm{ad}_A^\alpha(B)$, for all multi-indices α, $|\alpha| < n_0 + 1$. Then B is said to be of class $C^{n_0}(A)$ and we write $B \in C^{n_0}(A)$.

REMARK 2. The definition of $\mathrm{ad}_A^\alpha(B)$ does not depend on the order of the iteration since the A_j are pairwise commuting. We call $|\alpha|$ the *degree* of $\mathrm{ad}_A^\alpha(B)$.

In the following, $\mathcal{H}_A^s := D(\langle A\rangle^s)$ for $s \ge 0$ will be used to denote the scale of spaces associated to A. For negative s, we define $\mathcal{H}_A^s := (\mathcal{H}_A^{-s})^*$.

THEOREM 3. *Assume that $B \in C^{n_0}(A)$ for some $n_0 \ge n + 1 \ge 1$, $0 \le t_1 \le n + 1$, $0 \le t_2 \le 1$ and that $\{f_\lambda\}_{\lambda \in I}$ satisfies*

$$\forall \alpha\, \exists C_\alpha \colon |\partial^\alpha f_\lambda(x)| \le C_\alpha \langle x\rangle^{s-|\alpha|}$$

uniformly in λ for some $s \in \mathsf{R}$ such that $t_1 + t_2 + s < n + 1$. Then

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha f_\lambda(A)\ \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A, B)$$

as an identity on $\mathcal{D}(\langle A\rangle^s)$, where $R_{\lambda,n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})$ and there exist a constant C independent of A, B and λ such that

$$\|R_{\lambda,n}(A, B)\|_{\mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})} \le C \sum_{|\alpha|=n+1} \|\mathrm{ad}_A^\alpha(B)\|.$$

REMARK 4. A similar statement holds with the $\mathrm{ad}_A^\alpha(B)$ and $\partial^\alpha f_\lambda(A)$ interchanged at the cost of a sign correction given by $(-1)^{|\alpha|-1}$, and the corresponding remainder term $R'_{\lambda,n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_1}, \mathcal{H}_A^{t_2})$. This can be seen either by proving it analogously or by taking the adjoint equation and replacing B by $-B$.

REMARK 5. If $k \le t_1$ and $n_0 \ge n + 1 + k$, then $R_{\lambda,n}(A, B)$ can be replaced by $R^k_{\lambda,n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2+k}, \mathcal{H}_A^{t_1-k})$. This can be seen by commuting $|A - z|^{-2}$ and $\mathrm{ad}_A^\alpha(B)$ in the terms of the remainder, see page 6.

3. The Proof

Let $z \in \mathsf{C}^\nu$, $\mathrm{Im}\, z \ne 0$, $1 \le \ell \le \nu$ and $g, g_\ell \colon \mathsf{R}^\nu \to \mathsf{C}$ be given as $g(t) = |t - z|^{-2}$ and $g_\ell(t) = t_\ell - \bar{z}_\ell$. Write for $2\beta \le \alpha$

$$T_\alpha^\beta(t, z) := \tfrac{(-2)^{|\alpha-\beta|}|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t - \mathrm{Re}\, z)^{\alpha-2\beta}|t - z|^{-2|\alpha-\beta|}.$$

LEMMA 6. *Let g be as above and α be any multi-index. Then*

$$\partial^\alpha g(t) = \sum_{2\beta \le \alpha} \alpha! T_\alpha^\beta(t, z)|t - z|^{-2}.$$

PROOF. For brevity, we will write α^i or β^i for $\alpha + \delta_i$ or $\beta + \delta_i$, respectively. The formula is obviously true for $\alpha = 0$. Now assume that we have proven the formula for $|\alpha| \le k$. Let $|\alpha| = k$ and $0 \le i \le \nu$ be arbitrary. It suffices to prove the formula for α^i. One easily verifies using the chain rule that

$$(\partial^{\delta_i} g^n)(t) = -2n(t_i - \mathrm{Re}\, z_i)|t - z|^{-2n-2}. \tag{2}$$

Now by the induction hypothesis, we see that

$$\partial^{\alpha+\delta_i} g(t) = \partial_t^{\delta_i} \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}|t-z|^{-2|\alpha-\beta|-2}$$

$$= \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(\partial_t^{\delta_i}(t-\operatorname{Re} z)^{\alpha-2\beta})|t-z|^{-2|\alpha-\beta|-2} \tag{3}$$

$$+ \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}(\partial_t^{\delta_i}|t-z|^{-2|\alpha-\beta|-2}). \tag{4}$$

For the sake of clarity, we will now consider each sum independently.

$$(3) = \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(\alpha_i-2\beta_i)(t-\operatorname{Re} z)^{\alpha-2\beta-\delta_i}|t-z|^{-2|\alpha-\beta|-2}$$

$$= \sum_{\substack{2\beta\le\alpha\\2\beta_i<\alpha_i}} 2(\beta_i+1)\tfrac{(-2)^{|\alpha^i-\beta^i|}\alpha!|\alpha^i-\beta^i|!}{2^{|\beta^i|}\beta^i!(\alpha^i-2\beta^i)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta^i}|t-z|^{-2|\alpha^i-\beta^i|-2}$$

$$= \sum_{2\beta\le\alpha+\delta_i} 2\beta_i \tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \tag{5}$$

Using (2), we see that (4) equals

$$\sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}(-2)(|\alpha-\beta|+1)(t_i-\operatorname{Re} z_i)|t-z|^{-2|\alpha-\beta|-4}$$

$$= \sum_{2\beta\le\alpha}(\alpha_i+1-2\beta_i)\tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}$$

$$= \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha^i-\beta|}\alpha^i!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2} \tag{6}$$

$$- \sum_{2\beta\le\alpha} 2\beta_i \tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \tag{7}$$

Now (7) cancels (5) except for possible terms with $2\beta = \alpha+\delta_i$:

$$(5)+(7) = \sum_{2\beta=\alpha+\delta_i} \tfrac{(-2)^{|\alpha^i-\beta|}\alpha^i!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \tag{8}$$

Adding (6) and (8) finishes the induction.

LEMMA 7. *Let $B\in C^{n_0}(A)$ for some $n_0\ge 1$ and let $n\in \mathsf{N}_0$ and α_0 be a multi-index satisfying $|\alpha_0|+n+1\le n_0$. Then*

$$[\operatorname{ad}_A^{\alpha_0}(B), g(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha g(A)\operatorname{ad}_A^{\alpha_0+\alpha}(B) + R_n^g(A, \operatorname{ad}_A^{\alpha_0}(B)), \tag{9}$$

where

$$R_n^g(A,\operatorname{ad}_A^{\alpha_0}(B))$$

$$= \sum_{|\alpha|=n-1}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)|A-z|^{-2} \tag{10}$$

$$+ \sum_{|\alpha|=n}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i}(A,z)(A_i-\bar z_i)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)|A-z|^{-2} \tag{11}$$

$$+ \sum_{|\alpha|=n}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)(A_i-z_i)|A-z|^{-2}. \tag{12}$$

PROOF. The proof goes by induction. One may check by inspection of the following identity that the statement is true for $n = 0$.

$$\begin{aligned}[\operatorname{ad}_A^{\alpha_0}(B), |A-z|^{-2}] = &-\sum_{i=1}^{\nu} |A-z|^{-2}(A_i - \bar{z}_i)\operatorname{ad}_A^{\alpha_0+\delta_i}(B)|A-z|^{-2} \\ &-\sum_{i=1}^{\nu} |A-z|^{-2}\operatorname{ad}_A^{\alpha_0+\delta_i}(B)(A_i - z_i)|A-z|^{-2}.\end{aligned} \tag{13}$$

Now assume that we have proven the formula for $k \le n$, $|\alpha_0| + n + 2 \le n_0$. We will now show that this implies that the formula holds for $k = n + 1$. We begin by noting two useful identities.

$$T_\alpha^\beta(t,z)|t-z|^{-2} = -\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(t,z), \quad \forall j. \tag{14}$$

$$(\beta_i+1)T_{\alpha+2\delta_i}^{\beta+\delta_i}(t,z)2(t_i - \operatorname{Re} z_i) = (\alpha_i + 1 - 2\beta_i)T_{\alpha+\delta_i}^{\beta}(t,z). \tag{15}$$

Now using (13) and (14) we see that

$$(10) = \sum_{|\alpha|=n-1}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)|A-z|^{-2}\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B) \tag{16}$$

$$\begin{aligned}&+\sum_{|\alpha|=n-1}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}\tfrac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T_{\alpha+2\delta_i+2\delta_j}^{\beta+\delta_i+\delta_j}(A,z) \\ &\quad\times (A_j - \bar{z}_j)\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B)|A-z|^{-2}\end{aligned} \tag{17}$$

$$\begin{aligned}&+\sum_{|\alpha|=n-1}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}\tfrac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T_{\alpha+2\delta_i+2\delta_j}^{\beta+\delta_i+\delta_j}(A,z) \\ &\quad\times \operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B)(A_j - z_j)|A-z|^{-2},\end{aligned} \tag{18}$$

and by reordering and reindexing the sum in (16), (17) and (18), we get

$$(16) = \sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1\\ \alpha_i\ge 2}}\sum_{\substack{2\beta\le\alpha\\ \beta_i\ge 1}} \tfrac{\beta_i}{|\alpha-\beta|} T_\alpha^\beta(A,z)|A-z|^{-2}\operatorname{ad}_A^{\alpha_0+\alpha}(B), \tag{19}$$

and (17) equals

$$\begin{aligned}&\sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1\\ \alpha_i\ge 2}}\sum_{\substack{2\beta\le\alpha\\ \beta_i\ge 1}}\sum_{j=1}^{\nu} \tfrac{\beta_i}{|\alpha-\beta|}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \\ &\quad\times (A_j - \bar{z}_j)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}\end{aligned} \tag{20}$$

and similarly for (18) with $(A_j - \bar{z}_j)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)$ replaced by $\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j - z_j)$. Note that we may relax the extra conditions on α and β in the above statements, as a term with $\beta_i = 0$ contributes nothing.

Instead of continuing in the same fashion with (11) and (12), we note using (15) that

$$(11) + (12) = \sum_{|\alpha|=n}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)|A-z|^{-2} \tag{21}$$

$$+\sum_{|\alpha|=n}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu} \tfrac{\alpha_i+1-2\beta_i}{|\alpha+\delta_i-\beta|} T_{\alpha+\delta_i}^{\beta}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)|A-z|^{-2}, \tag{22}$$

so we may focus our attention on (22):

$$(22) = \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta \leq \alpha \\ 2\beta_i < \alpha_i}} \tfrac{\alpha_i - 2\beta_i}{|\alpha-\beta|} T_\alpha^\beta(A,z)|A-z|^{-2} \operatorname{ad}_A^{\alpha_0+\alpha}(B) \tag{23}$$

$$+ \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta \leq \alpha \\ 2\beta_i < \alpha_i}} \sum_{j=1}^{\nu} \tfrac{\alpha_i - 2\beta_i}{|\alpha-\beta|} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \times (A_j - \bar{z}_j) \operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}. \tag{24}$$

$$+ \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta \leq \alpha \\ 2\beta_i < \alpha_i}} \sum_{j=1}^{\nu} \tfrac{\alpha_i - 2\beta_i}{|\alpha-\beta|} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \times \operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j - z_j)|A-z|^{-2} \tag{25}$$

We note again that the additional conditions on α and β are superfluous.

We may now recollect the terms. First we see using Lemma 6:

$$\sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha g(A) \operatorname{ad}_A^{\alpha_0+\alpha}(B) + (19) + (23) = \sum_{|\alpha|=1}^{n+1} \frac{1}{\alpha!} \partial^\alpha g(A) \operatorname{ad}_A^{\alpha_0+\alpha}(B), \tag{26}$$

then

$$(20) + (24) = \sum_{|\alpha|=n+1} \sum_{2\beta \leq \alpha} \sum_{j=1}^{\nu} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z)(A_j - \bar{z}_j) \operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}, \tag{27}$$

and

$$(18) + (25) = \sum_{|\alpha|=n+1} \sum_{2\beta \leq \alpha} \sum_{j=1}^{\nu} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j - z_j)|A-z|^{-2}, \tag{28}$$

so adding up, we have proved that (9) equals the sum of (26), (21), (27) and (28) as stated.

The following lemma plays the same role for g_ℓ as Lemma 7 plays for g, but contrary to Lemma 7, the proof is trivial.

LEMMA 8. *For $n \in \mathsf{N}_0$ and $|\alpha_0| + 1 \leq n_0$*

$$[\operatorname{ad}_A^{\alpha_0}(B), g_\ell(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha g_\ell(A) \operatorname{ad}_A^{\alpha_0+\alpha}(B) + R_n^{g_\ell}(A, \operatorname{ad}_A^{\alpha_0}(B)),$$

where $R_n^{g_\ell}(A, \operatorname{ad}_A^{\alpha_0}(B)) = 0$ for $n \geq 1$, $R_0^{g_\ell}(A, \operatorname{ad}_A^{\alpha_0}(B)) = \operatorname{ad}_A^{\alpha_0+\delta_\ell}(B)$.

The following lemma also follows by induction.

LEMMA 9. *Assume $B \in C^{n_0}(A)$ and that $h_i \in C^\infty(\mathsf{R}^\nu)$, $1 \leq i \leq k$, satisfies*

$$[\operatorname{ad}_A^{\alpha_0}(B), h_i(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha h_i(A) \operatorname{ad}_A^{\alpha_0+\alpha}(B) + R_n^{h_i}(A, \operatorname{ad}_A^{\alpha_0}(B)),$$

where $R_n^{h_i}(A, \operatorname{ad}_A^{\alpha_0}(B))$ is bounded for all $n + |\alpha_0| \le n_0$ and $\partial^\alpha h_i(A)$ is bounded for all $0 \le |\alpha| + 1 \le n_0$. Then

$$\Big[B, \prod_{i=1}^k h_i(A)\Big] = \sum_{|\alpha|=1}^n \frac{1}{\alpha!}\partial^\alpha\Big(\prod_{i=1}^k h_i\Big)(A)\operatorname{ad}_A^\alpha(B) + \sum_{j=1}^k \sum_{|\alpha|=0}^n \frac{1}{\alpha!}\partial^\alpha\Big(\prod_{i=1}^{j-1} h_i\Big)(A) R_{n-|\alpha|}^{h_j}(A, \operatorname{ad}_A^\alpha(B)) \prod_{i=j+1}^k h_i(A).$$

Let $n+1 \le n_0$. If we put $k = \nu+1$, $h_i = g$ for $i \ne \nu$, $h_\nu = g_\ell$ and apply Lemma 7, 8 and 9 we see that

$$[B, |A-z|^{-2\nu}(A_\ell - \bar z_\ell)] = \sum_{|\alpha|=1}^n \frac{1}{\alpha!}\partial^\alpha\big(|\cdot - z|^{-2\nu}(\cdot_\ell - \bar z_\ell)\big)(A)\operatorname{ad}_A^\alpha(B) + R_{\ell,n}(A,B), \tag{29}$$

where

$$R_{\ell,n}(A,B) = \sum_{j=1}^{\nu-1}\sum_{|\alpha|=0}^n \frac{1}{\alpha!}\partial^\alpha(g^{j-1})(A) R^g_{n-|\alpha|}(A, \operatorname{ad}_A^\alpha(B))|A-z|^{-2(\nu-j)}(A_\ell - \bar z_\ell) \tag{30}$$

$$+ \sum_{|\alpha|=n} \frac{1}{\alpha!}\partial^\alpha(g^{\nu-1})(A)\operatorname{ad}_A^{\alpha+\delta_\ell}(B)|A-z|^{-2} \tag{31}$$

$$+ \sum_{|\alpha|=0}^n \frac{1}{\alpha!}\partial^\alpha(g^{\nu-1}g_\ell)(A) R^g_{n-|\alpha|}(A, \operatorname{ad}_A^\alpha(B)) \tag{32}$$

In the following, we will refer to the terms of $R_{\ell,n}(A,B)$ as the remainder terms. Let $0 \le t_1 \le n+1$ and $0 \le t_2 \le 1$. By Hadamard's three-line lemma and using (10–12), (30–32), Lemma 6 and the identity

$$\partial^\alpha\Big(\prod_{i=1}^j f_i\Big) = \sum_{\sum \alpha_i = \alpha} \frac{\alpha!}{\prod_{i=1}^j \alpha_i!}\prod_{i=1}^j \partial^{\alpha_i} f_i,$$

we may inspect that each remainder term (with $R_{\ell,n}(A,B)$ replaced by the remainder term) and hence $R_{\ell,n}(A,B)$ satisfies the inequality

$$\|\langle A\rangle^{t_1} R_{\ell,n}(A,B)\langle A\rangle^{t_2}\| \le C\langle z\rangle^{t_1+t_2}|\operatorname{Im} z|^{-n-2\nu}. \tag{33}$$

We will now use the functional calculus of almost analytic extensions. See [3] for details. In the following, we write $\bar\partial = (\bar\partial_1, \dots, \bar\partial_\nu)$ where $\bar\partial_j = \frac{1}{2}(\partial_{u_j} + i\partial_{v_j})$ and u_j and v_j are real and imaginary parts of $z_j \in \mathsf{C}$, $z = (z_1, \dots, z_n) \in \mathsf{C}^\nu$. The following proposition is inspired by [8, Chap. X.2] and [4].

PROPOSITION 10. *Let $s \in \mathsf{R}$ and $\{f_\lambda\}_{\lambda\in I} \subset C^\infty(\mathsf{R}^\nu)$ satisfy*

$$\forall\alpha\ \exists C_\alpha\colon |\partial^\alpha f_\lambda(x)| \le C_\alpha\langle x\rangle^{s-|\alpha|}.$$

There exists a family of almost analytic extensions $\{\tilde f_\lambda\}_{\lambda\in I} \subset C^\infty(\mathsf{C}^\nu)$ satisfying

(i) $\operatorname{supp}(\tilde f_\lambda) \subset \{u + iv \mid u \in \operatorname{supp}(f_\lambda), |v| \le C\langle u\rangle\}$.
(ii) $\forall \ell \ge 0\ \exists C_\ell\colon |\bar\partial \tilde f_\lambda(z)| \le C_\ell\langle z\rangle^{s-\ell-1}|\operatorname{Im} z|^\ell$.

PROOF. We define a mapping $C^\infty(\mathsf{R}^\nu) \ni f \mapsto \tilde f \in C^\infty(\mathsf{C}^\nu)$ in the following way. Choose a function $\kappa \in C_0^\infty(\mathsf{R})$ which equals 1 in a neighbourhood of 0 and put $\lambda_0 = C_0$,

$\lambda_k = \max\{\max_{|\alpha|=k} C_\alpha, \lambda_{k-1}+1\}$ for $k \geq 1$. Writing $z = u + iv \in \mathsf{R}^\nu \oplus i\mathsf{R}^\nu$, we now define

$$\tilde{f}(z) = \sum_\alpha \frac{\partial^\alpha f(u)}{\alpha!}(iv)^\alpha \prod_{j=1}^{\nu} \kappa\Big(\frac{\lambda_{|\alpha|} v_j}{\langle u\rangle}\Big).$$

One can now check that the properties hold.

REMARK 11. Note that if we define for a $\chi \in C_0^\infty(\mathsf{R}^\nu;[0,1])$ with $\chi(0)=1$ a sequence of functions by $f_{k,\lambda}(x) = \chi(\frac{x}{k}) f_\lambda(x)$, then

$$[B, f_\lambda(A)] = \lim_{k\to\infty}[B, f_{k,\lambda}(A)]$$

as a form identity on $\mathcal{D}(\langle A\rangle^s)$ and we have the dominated pointwise convergence

$$\bar\partial \tilde f_{k,\lambda}(x) \to \bar\partial \tilde f_\lambda(x) \text{ for } k \to \infty.$$

Let $\{f_\lambda\}_{\lambda\in I}$ satisfy the assumption of Proposition 10 with $s < 0$. Then the almost analytic extensions provide a functional calculus via the formula

$$f_\lambda(A) = C_\nu \sum_{\ell=1}^{\nu} \int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)(A_\ell - \bar z_\ell)|A-z|^{-2\nu} dz, \tag{34}$$

where C_ν is a positive constant (see [3, formula (8.18)] for details). Note that the integrals are absolutely convergent by Proposition 10(ii).

Multiplying $\langle A\rangle^{t_1} R_{\ell,n}(A,B)\langle A\rangle^{t_2}$ with $\bar\partial \tilde f_\lambda(z)$, we get from Proposition 10(ii) and (33) that

$$\|\langle A\rangle^{t_1}\bar\partial \tilde f_\lambda(z) R_{\ell,n}(A,B)\langle A\rangle^{t_2}\| \leq C\langle z\rangle^{t_1+t_2+s-n-1-2\nu}. \tag{35}$$

Hence, if $t_1+t_2+s < n+1$, $\langle A\rangle^{t_1}\bar\partial \tilde f_\lambda(z) R_{\ell,n}(A,B)\langle A\rangle^{t_2}$ is integrable over C^ν. Using (29), (34) and (35), we see that

$$\begin{aligned}
[B, f_\lambda(A)] &= C_\nu \sum_{\ell=1}^{\nu}\int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)[B,(A_\ell-\bar z_\ell)|A-z|^{-2\nu}]\,dz \\
&= C_\nu \sum_{\ell=1}^{\nu}\int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z) \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha\big(|\cdot - z|^{-2\nu}(\cdot_\ell - \bar z_\ell)\big)(A)\,dz\ \mathrm{ad}_A^\alpha(B) \\
&\quad + C_\nu \sum_{\ell=1}^{\nu}\int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z) R_{\ell,n}(A,B)\,dz.
\end{aligned} \tag{36}$$

We denote (36) by $R_{\lambda,n}(A,B)$. Note that

$$\begin{aligned}
C_\nu &\sum_{\ell=1}^{\nu}\int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)\frac{1}{\alpha!}\partial_t^\alpha\big(|t-z|^{-2\nu}(t_\ell-\bar z_\ell)\big)\,dz \\
&= \frac{C_\nu}{\alpha!}\partial_t^\alpha \sum_{\ell=1}^{\nu}\int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)|t-z|^{-2\nu}(t_\ell - \bar z_\ell)\,dz = \frac{1}{\alpha!}\partial^\alpha f_\lambda(t),
\end{aligned}$$

which implies

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha f_\lambda(A)\ \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A,B).$$

We have now proved Theorem 3 in the case $s < 0$. For the general case, we use Remark 11 to see that $[B, f_\lambda(A)] = \lim_{k\to\infty}[B, f_{k,\lambda}(A)]$ and clearly, $f_{k,\lambda}$ satisfies the assumption of Proposition 10 with the same s, so the estimate corresponding to (35) is now uniform in k and λ. The pointwise convergence and Lebesgue's theorem on dominated convergence now finishes the argument.

ACKNOWLEDGEMENTS. The author would like to thank J. S. Møller for suggestions, fruitful discussions and ultimately for proposing this problem. Part of this work was done while participating in the Summer School on Current Topics in Mathematical Phyics at the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna.

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