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Defaultable term structure models: macroeconomic impact and valuation of complex credit- and inflation-linked derivatives

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Abstract

This thesis is concerned with the pricing of credit- and inflation-linked products within a defaultable term structure framework that incorporates macroeconomic and firm-specific factors. In particular, we introduce a general pricing framework from which several models are derived differing in the assumptions regarding the number of economic factors, observability and correlation of these factors. For this family of models, we study the determinants of non-defaultable and defaultable bond prices by directly including observable as well as unobservable macroeconomic factors into the different set-ups.

Based on the general version of the defaultable term structure model, we determine prices for credit default swaps in closed form and further deduce exact dynamics of credit default swap spreads. Approximating these exact dynamics enables us to present closed-form solutions for complex credit derivatives like credit default swaptions and constant maturity credit default swaps. We use a full simulation approach to test the pricing formulas for these credit derivatives and to compare our results to literature.

Further, we apply a variant of our general term structure framework to the pricing of inflation-linked assets. We use a framework that decomposes the short rate into a real short rate and an inflation short rate. Starting with standard inflation-linked derivatives like zero-coupon inflation-linked swaps and year-on-year inflation-linked swaps, we extend our framework to the pricing of complex hybrid inflation-linked derivatives incorporating interest rate, equity or credit components. We derive closed-form solutions for inflation-linked equity options and credit default swaps. Also, we present a feasible approximation for pricing hybrid inflation-linked derivatives in closed form enabling a fast and accurate pricing for such complex derivatives.

Zusammenfassung

Diese Dissertation befasst sich mit der Bewertung von kreditrisikobehafteten und inflationsindexierten Produkten innerhalb eines ausfallbehafteten Zinsstrukturmodells, das sowohl makroökonomische als auch firmenspezifische Faktoren integriert. Ausgehend von einem allgemeinen Bewertungsansatz werden mehrere Modelle abgeleitet, welche sich in den Annahmen bezüglich der Anzahl ökonomischer Faktoren und deren Beobachtbarkeit und Korrelation unterscheiden. Für diese verschiedenen Ansätze werden anhand der Integration von beobachtbaren und unbeobachtbaren makroökonomischen Faktoren potentielle Treiber risikoloser und ausfallbehafteter Bondpreise analysiert.

Basierend auf der allgemeinen Version des ausfallbehafteten Zinsstrukturmodells werden Preise für Credit Default Swaps in geschlossener Form bestimmt und des Weiteren exakte Dynamiken der Credit Default Swap Spreads abgeleitet. Das Approximieren dieser exakten Dynamiken erlaubt nun die Bewertung von komplexen Kreditderivaten wie Credit Default Swaptions und Constant Maturity Credit Default Swaps in geschlossener Form. Abschließend werden diese Ergebnisse gegen eine simulationsbasierte Bewertung getestet und mit der bestehenden Literatur verglichen.

Eine Variante des allgemeinen Bewertungsmodells wird zudem verwendet, um inflationsindexierte Produkte zu bewerten. Dieser Ansatz zerlegt die Shortrate in eine reale Shortrate und eine Inflations-Shortrate. Ausgehend von Standard-Inflationsderivaten wie Zero-Coupon- und Year-on-Year Inflation-Linked Swaps wird die Bewertung auf komplexe, hybride, inflationsindexierte Derivate ausgeweitet. Diese hybriden Derivate beinhalten zusätzliche Zins-, Equity- und Kreditkomponenten. Es werden geschlossene Bewertungsformeln für inflationsindexierte Equity Optionen und Credit Default Swaps hergeleitet. Des Weiteren wird eine Approximation für die Bewertung von hybriden, inflationsindexierten Derivaten in geschlossener Form vorgestellt, welche eine schnelle und akkurate Bewertung für komplexe Derivate erlaubt.

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Contents

Chapter 1

Introduction

1.1 Motivation

The recent financial crisis turned the spotlight to credit risk pricing. The deterioration in prices and ratings of complex credit derivatives left the community wondering if the models in use are capable of pricing highly structured products, and whether default and its determinants are captured correctly. The majority of losses and provisions which occured during the crisis were not due to actual losses caused by default but corrections of prices with respect to counterparty risk. So far, the assumptions for pricing derivatives have been that there is no counterparty risk inherent especially for interbank transactions resulting in risk-free values. But since the bail-out of AIG one of the biggest player in so-called credit default swaps, which are a type of insurance against the default of a certain reference asset, the focus of traders, financial engineers and regulators lies in adjusting derivatives' prices with respect to counterparty risk (CVA).
There are two main approaches to credit risk pricing, structural and reduced-form models. While the former tries to model default directly by assuming its occurrence when the firm's value crosses a certain threshold (i.e. outstanding debt), the latter focuses on modelling the default probability instead. Although the rational behind structural models is easy to understand, they fail in exactly specifying default. Contrarily, reduced-form models assume the default event of being exogenously given. For these models default is not explainable by any observable data and comes totally unexpected. In order to overcome the shortcomings of both approaches, a third class of models have arosen. Hybrid models combine characteristics of both approaches therefore linking default probabilities to macroeconomic or microeconomic data.
The literature on determinants of sovereign und bond spreads is extensive.

Yet, the discussion is still going on about which economic factors are driving the spreads, how spreads and determinants are related and how to uncover the relationships respectively find the determinants. A popular approach for specifying determinants is to use regression analysis for spreads and a set of candidate determinants. However, the results of these studies do not link the economic risk dynamics to asset prices. The true relationship of the spread and its driving factors remains unexplained. Therefore, more recent approaches use economic risk factors in no-arbitrage term structure models directly linking the determinants to prices and emphasizing the growing interest in hybrid credit models. All approaches have in common that although the choice of factors to be included in the test varied substantially, only a portion of credit spread changes could be explained. The majority of variation, however, appeared to be driven by a common factor that is still unexplained.

1.2 Objectives and Structure

The main objective of this thesis is to study hybrid credit risk models with respect to their ability in explaining credit spreads and their usage for pricing complex derivatives. It is our aim to further develop and promote hybrid credit risk models because of their linkage to economic factors, which we believe crucial for pricing and forecasting credit risk especially for risk management purposes like stress testing, future exposure and counterparty risk. Concerning the pricing of derivatives we want to improve the usage of our proposed defaultable term structure model by proposing closed-form solutions that could help to reduce the computational burden of risk management applications.

The remainder of this thesis is organized as follows: In Chapter 2 we introduce and familiarize the reader with the basic concepts in (financial) mathematics that are used throughout this thesis. Chapter 3 outlines the origins and building blocks of the main credit risk pricing models and embeds our defaultable term structure framework into these approaches.

In Section 4.1 of Chapter 4 we introduce the general version of our defaultable term structure model and derive pricing formulas for non-defaultable zero-coupon bonds in Theorem 4.2 as well as for defaultable zero-coupon bonds in Theorem 4.3. From this general set-up we deduce several models differing in the assumptions regarding the number of economic factors, observability and correlation. For example, the extended Schmid-Zagst model of Section 4.2 was first introduced by Antes, Ilg, Schmid & Zagst (2008) and incorp-

orates an observable macroeconomic factor in its term structure, whereas the real and inflation short-rate model of Section 4.4, for which a variant of it was first published by Hagedorn, Meyer & Zagst (2007), makes use of a second unobservable macroeconomic factor. Based on these models we test in Sections 4.7 and 4.8 a set of macroeconomic factors with respect to their impact on sovereign and bond spreads. We use factors that either represent a single driving factor or are a composition of several factors representing the current or future state of the economy. Our choice of factors is based on their recurrent appearance in literature. Among our set of factors are widely accepted factors like the gross domestic product, that was used in several studies by e.g. Bonfim (2009), Glen (2005), Hilscher & Nosbusch (2010) and Rowland (2005), the consumer price index, that was used by Ang & Piazzesi (2003) and Cantor & Packer (1996) in addition to some of the previously mentioned studies, and the industrial production, that was analyzed by Figlewski, Frydman & Liang (2012), Krishnan, Ritchken & Thomson (2005) and Krishnan, Ritchken & Thomson (2010). In addition to those well-known macroeconomic factors, we study the composite indices of leading and coincident indicators which are an aggregate of macroeconomic factors and give indications concerning the state of the economy. These indices are published by The Conference Board (see TCB (2001)) and appeared e.g. in the work of Huang & Kong (2003). In Sections 4.7 and 4.8 we describe in detail the calibration as well as the analysis of the obtained results.
Based on the defaultable term structure model of Chapter 4, we determine in Chapter 5 prices for credit default swaps in closed form also after controlling for counterparty risk. The results for credit default swaps of Theorems 5.13, 5.15 and 5.18 extend the work of Schmid (2002) and Antes, El Moufatich, Schmid & Zagst (2009) to our general framework introduced in Section 4.1 of Chapter 4 with respect to different assumptions concerning the recovery payments. Then, in Section 5.4.3 we further extend these results by incorporating counterparty risk based on the work of Jarrow & Yu (2001) who used so-called primary and secondary firms in order to model default dependencies. In Section 5.4.1 we deduce from the closed-form solutions of Theorems 5.13, 5.15 and 5.18 dynamics of credit default swap spreads in a consistent way while keeping the link to economic factors. After approximating the exact dynamics in Section 5.4.2 by lognormal and shifted-lognormal dynamics, we present closed-form solutions based on these approximations for credit default swaptions in Theorems 5.33 and 5.34, and for constant maturity credit default swaps in Theorems 5.38, 5.39 and 5.40. In addition, we show in Section 5.5.1 how to incorporate the new quoting mechanism for credit default swaps, i.e. a constant cds spread (cf. Markit (2009a) and Markit (2009b)), into the pricing of credit default swaptions and we outline in Theorem 5.35

how to price a credit default swaption if the option maturity does not coincide with the start of the credit default swap. We use a full simulation approach to test the pricing formulas for those credit derivatives and to compare our results to literature, e.g. Krekel & Wenzel (2006) and Brigo & Mercurio (2006).

In Chapter 6 we outline the pricing of inflation-linked derivatives within our term structure model. This chapter extends the work of Hagedorn et al. (2007) to pricing hybrid inflation-linked derivatives. Starting with standard derivatives like zero-coupon inflation swaps we extend our pricing framework to hybrid products combining inflation with interest rates in Theorem 6.6 according to the work of Dodgson & Kainth (2006). We test the approximated semi-analytical solution of Theorem 6.6 against the pricing by means of simulation. Further, we introduce in Theorem 6.7 derivatives combining the characteristics of inflation and equity analogously to Hammarlid (2010), and in Theorem 6.9 we extend our inflation set-up to credit derivatives and make use of results obtained in Chapter 5 in order to price an inflation-indexed credit default swap introduced by Avogaro (2006). Finally, Chapter 7 concludes.

Chapter 2

Mathematical Fundamentals

This chapter is meant to introduce and familiarize the reader with the mathematical fundamentals and notations which will be used in this thesis. The first section deals with point processes and intensities while the next section outlines the basics of stochastic differential equations. Section 2.3 introduces the concepts of financial markets and Section 2.4 presents the Kalman filtering technique which we will use later on as suggested in Schmid (2002). Mainly this chapter is based on Zagst (2002) but the usage of other sources will be explicitly stated at the appropriate places.

2.1 Point Processes and Intensities

The concept of point processes is an important source for credit risk modelling. Therefore, we start with these processes and further introduce intensities of point processes. A main class of credit risk models, the so-called reduced-form models (cf. Chapter 3), make use of intensities.

In the following we assume a filtered probability space $(\Omega, \mathcal{F}, Q, \mathbb{F})$, i.e. a sigma-algebra $\mathcal{F}$ on the non-empty sample space Ω which is further equipped with a probability measure Q and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

Definition 2.1 ***(Point Processes)***
Let $(T_n)_{n\in\mathbb{N}}$ be a monotonously increasing series of random variables with values in $[0,\infty]$ and $T(0) = 0$. If it holds for $T_n < \infty$: $T_n(\omega) < T_{n+1}(\omega)$, $\forall \omega \in \Omega$, then $N(t)$ defined as

$$N(t) := \sum_{n\geq 1} 1_{\{t\geq T_n\}}$$

is called the $(T_n)_{n\in\mathbb{N}_0}$*-associated point process.*
Further, $N(t)$ *is non-explosive if it holds* $\sup_{n\in\mathbb{N}} T_n = \infty$, $Q-a.s.$.

Definition 2.2 ***(Stopping Time)***
Let τ *be a random variable in* $\mathbb{R}^+ \cup \{\infty\}$ *with* $\{\tau \le t\} \in \mathcal{F}_t$ *for any* $t \ge 0$, *then* τ *is a stopping time with respect to the filtration* $\mathbb{F}$.

Lemma 2.3
A point process N *is adapted if and only if the associated series* $(T_n)_{n\in\mathbb{N}}$ *is a series of stopping times.*

Proof:
see Protter (1990), Theorem I.22. □

Definition 2.4 ***(Intensity)***
Let N *be a non-explosive, adapted point process and* c *a non-negative, progressively measurable process, such that it holds for all* $t \ge 0$

$$\int_0^t c(s)ds < \infty \, Q - a.s..$$

If it further holds for all non-negative, predictable processes C

$$\mathbb{E}_Q\left[\int_0^\infty C(s)dN(s)\right] = \mathbb{E}_Q\left[\int_0^\infty C(s)\cdot c(s)ds\right],$$

then N *is said to admit the intensity* c.

Theorem 2.5 ***(Martingale Characterization of Intensity)***

(i) Assume $N(t)$ *admits the intensity* c, M *is given as* $M(t) := N(t) - \int_0^t c(s)ds$, *and* C *is a predictable process with* $\mathbb{E}_Q\left[\int_0^t |C(s)|c(s)ds\right] < \infty$, $t \ge 0$, *then* $\int_0^t C(s)dM(s)$ *is a martingale.*

(ii) If it additionally holds $\mathbb{E}_Q\left[\int_0^t c(s)ds\right] < \infty$, $t \ge 0$, *then* M *is a martingale.*

(iii) Let $N(t)$ *be a non-explosive, adapted,* $(T_n)_{n\in\mathbb{N}_0}$ *associated point process and let* $N(t\wedge T_n) - \int_0^{t\wedge T_n} c(s)ds$ *be a martingale* $\forall n \in \mathbb{N}_0$. *Then* $c(t)$ *is the intensity of* $N(t)$.

Proof:
see Brémaud (1981), pages 27-28. □

If we now assume the point process $N(t)$ to be represented by the indicator function $1_{\{t\geq\tau\}}$ for a stopping time τ and that N admits a right-continuous intensity c with $\mathbb{E}_Q\left[\sup_{0\leq s\leq t} c(s)\right] < \infty$, $\forall t \geq 0$. Then according to Theorem 2.5 (ii) $M(t)$ is a martingale and it holds for $\epsilon > 0$:

$$\begin{aligned}
Q\left(t < \tau \leq t+\epsilon\middle|\,\mathcal{F}_t\right) &= \mathbb{E}_Q\left[1_{\{t<\tau\leq t+\epsilon\}}\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}_Q\left[N(t+\epsilon) - N(t)\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}_Q\left[M(t+\epsilon) - M(t)\middle|\,\mathcal{F}_t\right] + \mathbb{E}_Q\left[\int_t^{t+\epsilon} c(s)ds\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}_Q\left[\int_t^{t+\epsilon} c(s)ds\middle|\,\mathcal{F}_t\right] .
\end{aligned}$$

Additionally, it holds (see e.g. Schmid (2002))

$$c(t) = \lim_{\epsilon\to 0} \frac{Q\left(t < \tau \leq t+\epsilon\middle|\,\mathcal{F}_t\right)}{\epsilon} .$$

In credit risk models the stopping time τ is defined as the time of default of a reference entity, e.g. the time when a company is unable to meet its financial obligations. With this in mind, the intensity c, which is often also referred to as hazard rate, can be interpreted as the arrival rate of default within the next infinitesimal time period $[t, t+\epsilon]$ given all available information at time t.

2.2 Itô Processes and Stochastic Differential Equations

An important tool in financial mathematics are Itô processes for describing the performance of prices. In this section we introduce those processes and further important applications of stochastic analysis. If not stated otherwise we consult Zagst (2002). For further reading we also recommend Øksendal (1998) and Karatzas & Shreve (1991).

Definition 2.6 ***(Itô Process)***
Let W be an m-dimensional Brownian motion. A stochastic process is called an Itô process if for all $t \geq 0$

$$X_t = X_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s),$$

with X_0 being $\mathcal{F}_0$-measurable and μ and $\sigma = (\sigma_1, ..., \sigma_m)$ (m-dimensional) progressively measurable stochastic processes with

$$\int_0^t |\mu(s)|ds < \infty$$

and

$$\int_0^t \sigma_j^2(s)ds < \infty$$

Q-a.s. $\forall\, t \geq 0$, $j = 1, \dots, m$.
An n-dimensional Itô process is given by an n-dimensional vector $X = (X_1, \dots, X_n)'$, $n \in \mathbb{N}$, whose elements are an Itô process.

The Itô process is often denoted in another way via a so-called stochastic differential equation (SDE):

$$\begin{aligned} dX(t) &= \mu(t)dt + \sigma(t)dW(t) \\ &= \mu(t)dt + \sum_{j=1}^{m} \sigma_j(t)dW_j(t). \end{aligned}$$

Since financial derivatives are often constructed as a function of an Itô process it is helpful to know how this new process looks like and under which conditions it will be an Itô process again. The following lemma states the necessary conditions for a one-dimensional Itô process but can be extended for higher dimension (see e.g. Zagst (2002), page 29).

Theorem 2.7 ***(Itô's Lemma)***
Let $X = (X(t))_{t \geq 0}$ be an Itô process with

$$dX(t) = \mu(t)dt + \sum_{j=1}^{m} \sigma_j(t)dW_j(t)$$

and $G : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ be twice continuously differentiable in the first variable and once continuously differentiable in the second. Then it holds for all $t \in [0, \infty)$

$$\begin{aligned} dG(X(t), t) &= [G_t(X(t), t) + G_x(X(t), t)\mu(t) + \frac{G_{xx}(X(t), t)}{2}||\sigma(t)||^2]dt \\ &\quad + G_x(X(t), t)\sigma(t)dW(t). \end{aligned}$$

Proof:
See Korn & Korn (1999), page 48-50. □

Now, we define a strong solution of a given SDE and give conditions for the existence and uniqueness of such a strong solution.

Definition 2.8 ***(Strong Solution)***
Let $\mu : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^{n\times m}$, $n, m \in \mathbb{N}$, be measurable with respect to the corresponding Borel σ-algebras. If there exists an n-dimensional Itô-process X on the filtered probability space $(\Omega, \mathcal{F}, Q, \mathbb{F})$ such that

$$X(t) = x + \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dW(s) \quad Q\text{-a.s.}, \; X(0) = x,$$

with $x \in \mathbb{R}^n$, then X is called a strong solution of the SDE

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t), \; \forall\, t \geq 0, X(0) = x\,.$$

Theorem 2.9 ***(Existence and Uniqueness)***
Let the functions μ and σ of the previously stated SDE be continuous such that for all $t > 0$, $x, y \in \mathbb{R}^n$ and a constant $K > 0$ the following conditions hold [I]*:*

1. $||\mu(x,t)-\mu(y,t)||+||\sigma(x,t)-\sigma(y,t)|| \leq K\cdot||x-y||$ *(Lipschitz condition)*
2. $||\mu(x,t)||^2 + ||\sigma(x,t)||^2 \leq K^2(1 + ||x||^2)$ *(growth condition).*

Then there exists a unique, continuous strong solution X of the SDE and a constant C which depends only on K and $T > 0$ such that it holds:

$$\mathbb{E}_Q[||X(t)||^2] \leq C(1 + ||x||^2)e^{C\cdot t} \; \forall t \in [0, T].$$

Furthermore it holds that

$$\mathbb{E}_Q[\sup_{0\leq t\leq T} ||X(t)||^2] < \infty.$$

Proof:
See Korn & Korn (1999), page 127-133. □

In this thesis, we will work with linear stochastic differential equations that are defined in the following. Further, we present the unique strong solution of this special class of SDEs.

[I] $||x||, x \in \mathbb{R}^{n\times m}$, denotes the Euclidean norm with $||x|| := \sqrt{\sum_{i=1}^n \sum_{j=1}^m x_{ij}^2}$.

Definition 2.10 ***(Linear Stochastic Differential Equation)***
Consider the matrices $H \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{n\times m}$ and a continuous function $J : [0,\infty) \to \mathbb{R}^n$, then

$$dX(t) = [HX(t) + J(t)]dt + VdW(t)$$

with initial condition $X(0) = x$ is called a linear stochastic differential equation.

Theorem 2.11 ***(Solution of Linear Stochastic Differential Equation)***
The previously introduced linear SDE has a unique strong solution of the form

$$X(t) = e^{Ht}x + \int_0^t e^{H(t-s)}J(s)ds + \int_0^t e^{H(t-s)}VdW(s).$$

Moreover, $X(t)$ follows a normal distribution for $t > 0$ with

$$\mathbb{E}_Q[X(t)] = e^{Ht}x + \int_0^t e^{H(t-s)}J(s)ds$$

and

$$Cov_Q[X(t)] = \int_0^t e^{Hs}VV'e^{H's}ds.$$

Proof:
see Karatzas & Shreve (1991), page 354-355. □

In the following, we outline an important link between partial differential equations (PDE) and stochastic analysis, the so-called Feynman-Kac representation. Given certain assumptions, this representation allows us to interpret the solution of a PDE as the expectation of a function of a diffusion process where the drift and coefficient are represented in terms of the PDE coefficients. First, we define the PDE for which the Feynman-Kac representation holds.

Definition 2.12 ***(Cauchy Problem)***
Let the differential operator $\mathcal{D}$ be defined by

$$(\mathcal{D}v)(x,t) := v_t(x,t) + \sum_{i=1}^{n} \mu_i(x,t)v_{x_i}(x,t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x,t)v_{x_i x_j}(x,t)$$

with $v : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ being twice continuously differentiable in x and once continuously differentiable in t, and with functions $\mu : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ and $a : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times n}$. Additionally, let $r : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a continuous function and $T > 0$ be arbitrary but fixed. Then the Cauchy problem is the problem of finding a function $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ which is continuously differentiable in t, twice continuously differentiable in x and solves the following partial differential equation, the so-called backward Kolmogorov equation,

$$(\mathcal{D}v)(x,t) = r(x,t)v(x,t), \quad \forall (x,t) \in \mathbb{R}^n \times [0,T]$$

and terminal condition $v(x,T) = D(x)$ for all $x \in \mathbb{R}^n$.

Theorem 2.13 ***(Uniqueness of Solutions for the Cauchy Problem)***
If $(a_{ij}(x,t))_{1\le i,j\le n}$ is positive semi-definite and it holds for a constant $K > 0$

$$|a_{ij}(x,t)| \le K,$$

$$|\mu_i(x,t)| \le K\left(1 + \|x\|\right),$$
$$-r(x,t) \le K\left(1 + \|x\|^2\right),$$

then there exists at most one solution v of the Cauchy problem satisfying

$$|v(x,t)| \le K_1 e^{K_2\|x\|^2}$$

for positive constants K_1, K_2.

Proof:
see Friedman (1975), page 139-140. □

We now present the Feynman-Kac representation for linear stochastic differential equations which will be used later in this thesis. More general applications of Feynman-Kac can be found in Friedman (1975), e.g. Theorem 4.6, page 142 and Theorem 5.3, page 148.

Theorem 2.14 ***(Feynman-Kac Representation)***
Assume $T \ge 0$, $X(t)$ being the solution of the linear stochastic differential equation (see Definition 2.10) and VV' being positive definite. Furthermore, let $f, r : \mathbb{R}^n \to \mathbb{R}$, $f(x) := F'x + d$, $r(x) := G'x + c$ be affine linear functions, $F, G \in \mathbb{R}^n$, $c, d \in \mathbb{R}$, $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$,

$$v(x,t) := \mathbb{E}_Q^{t,x}\left[e^{-\int_t^T r(X(l))dl} f(X(T))\right]$$

and [II] *the differential operator* $\mathcal{D}$ *be defined as in Definition 2.12 with* $\mu(x,t) := Hx + J(t)$, $a_{ij}(x,t) := \sum_{k=1}^{m} V_{ik}V_{jk} = (VV')_{ij}$. *Then it holds that*

$$v(X^{0,\tilde{x}}(t),t) = \mathbb{E}_Q^{0,\tilde{x}}\left[e^{-\int_t^T r(X(l))dl} f(X(T))|\mathcal{F}_t\right]$$

and $v(x,t)$ *is the unique solution of the Cauchy problem and fulfills the growth condition*

$$|v(x,t)| \leq K_1 e^{K_2\|x\|^2}$$

for positive constants K_1, K_2.

Proof:
see Antes (2004), page 36-37. □

Hence, the unique solution of the Cauchy problem is given by this expected value as a function depending on the initial parameters (x,t) of the SDE. In general, the reverse is not true. But if it is possible to determine the expected value and to show that this expected value solves the Cauchy problem then it is the unique solution.
In order to solve the PDE that it is obtained by means of the Feynman-Kac representation, the next theorem will be used within this thesis.

Theorem 2.15 ***(Linear Differential Equation)***
Consider the inhomogeneous linear differential equation

$$y'(x) = a(x)y(x) + b(x)$$

with continuous functions a *and* b, $b \neq 0$. *Then, the solution of this differential equation is*

$$y(x) = e^{A(x)}\left(\int_{x_0}^{x} b(t)e^{-A(t)}dt + C\right) ,$$

with $C \in \mathbb{R}$ *and* $A' = a$.

Proof:
see Walter (1986), §2. □

[II] The superscript in $\mathbb{E}_Q^{t,x}$ indicates that $X(t) = x$.

2.3 Financial Markets

In order to get a consistent framework we present below the most important building blocks for financial markets. We start with introducing a general model for financial markets. Throughout this section we consult Zagst (2002). Other textbooks regarding introductions of financial markets are Brigo & Mercurio (2006) with an emphasis on interest-rate markets, Musiela & Rutkowski (1997) and Bingham & Kiesel (2004).

Definition 2.16 ***(Financial Market)***
The primary financial market $\mathcal{M}(Q)$ on the filtered probability space $(\Omega, \mathcal{F}, Q, \mathbb{F})$ with the filtration $\mathbb{F}(W)$, $\mathcal{F} = \mathcal{F}_T(W)$, consists of $n+1$ primary traded assets whose prices are non-negative Itô processes on $[0,T]$:

$$dP_i(t) = \mu_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW_j(t)\,, \quad i = 0, \ldots, n\,,$$

with an m-dimensional Brownian motion W and progressively measurable stochastic processes μ_i and σ_{ij}. Furthermore, these processes satisfy the conditions

$$\int_0^T |\mu_i(s)|ds < \infty \;\; Q-a.s.$$

and

$$\mathbb{E}_Q\left[\int_0^T \sigma_{ij}^2(s)ds\right] < \infty \;\; \forall\, j = 1, \ldots, m\,.$$

For pricing purposes we want to rewrite the primary traded assets with respect to another unit price (numéraire).

Definition 2.17 ***(Numéraire)***
A price process $(X(t))_{t\in[0,T]}$ that satisfies

$$X(t) > 0 \;\; \forall\, t \in [0,T]$$

is a numéraire in the financial market $\mathcal{M}(Q)$.

In the following, we want to use P_0 as numéraire and hence define it as the riskless cash account by taking a stochastic process r which satisfies the above condition such that

$$dP_0(t) = r(t) \cdot P_0(t)dt\,,\; P_0(0) = 1.$$

Hence, the discounted prices of the primary traded assets are

$$\widetilde{P}_i(t) := P_0^{-1}(t) \cdot P_i(t),\ t \in [0,T],\ i = 0, \ldots, n\,,$$

with

$$\widetilde{P}_0(t) = 1\,,$$

$$d\widetilde{P}_i(t) = \widetilde{\mu}_i(t)dt + \sum_{j=1}^{m} \widetilde{\sigma}_{ij}(t)dW_j(t)\,,$$

$$\widetilde{\mu}_i(t) = (\mu_i(t) - r(t) \cdot P_i(t)) \cdot P_0^{-1}(t)\,,$$

and

$$\widetilde{\sigma}_{ij}(t) = \sigma_{ij}(t) \cdot P_0^{-1}(t)$$

for all $i = 1, \ldots, n$, $j = 1, \ldots, m$, $t \in [0,T]$.

In order to simplify the calculation of prices, respectively expected values, we need to find a measure under which the discounted price processes are martingales.

Definition 2.18 ***(Equivalent Martingale Measure)***
A probability measure $\widetilde{Q}$ on the measure space $(\Omega, \mathcal{F})$ is called an equivalent martingale measure to Q if:

(i) $\widetilde{Q}$ is equivalent to Q, i.e. $\widetilde{Q}$ and Q have the same null sets.

(ii) The discounted price process $\widetilde{P} = (\widetilde{P}_1(t), \ldots, \widetilde{P}_n(t))_{t\in[0,T]}$ is an n-dimensional $\widetilde{Q}$-martingale, i.e.

$$\widetilde{P}(t) = \mathbb{E}_{\widetilde{Q}}\left[\widetilde{P}(s)\middle|\,\mathcal{F}_t\right]\,, s > t$$

and

$$\mathbb{E}_{\widetilde{Q}}\left[\int_0^T ||\sigma_{\widetilde{P}}(s)||^2 ds\middle|\,\mathcal{F}_t\right] < \infty.$$

The set of equivalent martingale measures to Q is denoted by $\mathbb{M}(Q)$.

The next theorem describes how such an equivalent martingale measure $\widetilde{Q}$ can be constructed. As a result we get an arbitrage-free financial market.

Theorem 2.19 ***(Discounted Market Characterization)***
Suppose there exists an m-dimensional progressively measurable stochastic process γ *such that the no-arbitrage condition*

$$\mu_i(t) - \sigma_i(t) \cdot \gamma(t) = r(t) \cdot P_i(t) \;\; \lambda \otimes Q - a.s. \text{ on } [0,T],\; i = 1, \ldots, n\,,$$

with $\sigma_i := (\sigma_{i1}, \ldots, \sigma_{im})$*, and the Novikov condition*

$$\mathbb{E}_Q\left[e^{\frac{1}{2}\int_0^T ||\gamma(s)||^2 ds}\right] < \infty$$

are fulfilled.
Furthermore, let the probability measure $\widetilde{Q}$ *on* $(\Omega, \mathcal{F})$ *be defined as*

$$\widetilde{Q}(A) = Q_{L(\gamma,T)}(A) = \mathbb{E}_Q\left[1_A \cdot L(\gamma,T)\right] \;\forall A \in \mathcal{F}$$

with

$$L(\gamma,T) := e^{-\int_0^T \gamma(s)' dW(s) - \frac{1}{2}\int_0^T ||\gamma(s)||^2 ds}.$$

Then the stochastic process $\widetilde{W} = \left(\widetilde{W}(t)\right)_{t\in[0,T]}$ *defined by*

$$d\widetilde{W}(t) := \gamma(t)dt + dW(t) \text{ on } [0,T]$$

is a $\widetilde{Q}$*-Brownian motion and the price processes have the following representation in terms of* $\widetilde{W}$*:*

$$\begin{aligned} d\widetilde{P}_0(t) &= 0, \\ d\widetilde{P}_i(t) &= \widetilde{\sigma}_i(t)d\widetilde{W}(t),\; \widetilde{\sigma}_i := (\widetilde{\sigma}_{i1}, \ldots, \widetilde{\sigma}_{im})\,,\; i = 1, \ldots, n\,, \\ dP_i(t) &= r(t) \cdot P_i(t)dt + \sigma_i(t)d\widetilde{W}(t),\; i = 1, \ldots, n. \end{aligned}$$

If additionally the martingale condition

$$\mathbb{E}_{\widetilde{Q}}\left[\int_0^T \widetilde{\sigma}_{ij}^2(t)dt\right] < \infty \;\; \forall\, i = 1, \ldots, n\,,\; j = 1, \ldots, m\,,$$

holds, then $\widetilde{Q}$ *is an equivalent martingale measure with* L *being the Radon-Nikodym derivative of* $\widetilde{Q}$ *with respect to* Q*.*

Proof:
See Zagst (2002), pages 59f. □

Having found an equivalent martingale measure $\widetilde{Q}$ we wonder about the prices of financial products like e.g. derivatives with primary traded assets as underlyings.

Definition 2.20 ***(Contingent Claim)***
A random variable $D(T)$ on $(\Omega, \mathcal{F})$ whose discounted value up to time t $P_0(t) \cdot \widetilde{D}(T)$ is lower bounded for all $t \in [0,T]$, is named a European contingent claim with maturity T.

Definition 2.21 ***(Contingent Claim Prices)***
Under $\widetilde{Q} \in \mathbb{M}(Q)$ the expected-value process of a European contingent claim D is given by

$$V_D^{\widetilde{Q}}(t) := P_0(t) \cdot \mathbb{E}_{\widetilde{Q}}\left[\widetilde{D}(T)|\mathcal{F}_t\right], \; t \in [0,T].$$

If this process $V_D^{\widetilde{Q}}(t)$ is unique in $\mathbb{M}(Q)$, it is called the price of the contingent claim D, $V_D(t)$.

If our financial market $\mathcal{M}(Q)$ is complete, the prices of European contingent claims are unique. We call a financial market complete if all contingent claims $D(T)$ can be replicated by an admissible trading strategy [III].

A powerful tool for pricing financial derivatives is the change of numéraire where the martingale property of the newly discounted price process is preserved under the changed probability measure.

Theorem 2.22 ***(Change of Numéraire)***
Let $X = (X(t))_{t\in[0,T]}$ be a non-dividend-paying numéraire in $\mathcal{M}(Q)$ and $\widetilde{Q} \in \mathbb{M}(Q)$. If the discounted numéraire process $\widetilde{X} = (\widetilde{X}(t))_{t\in[0,T]}$ with $\widetilde{X}(t) := P_0^{-1}(t) \cdot X(t)$, $t \in [0,T]$, is a $\widetilde{Q}$-martingale, then there exists a probability measure Q^X on $(\Omega, \mathcal{F})$, defined by its Radon-Nikodym derivative $L(T)$ with respect to $\widetilde{Q}$,

$$L(t) = \left.\frac{dQ^X}{d\widetilde{Q}}\right|_{\mathcal{F}_t} = \frac{X(t)}{X(0) \cdot P_0(t)}, \; t \in [0,T],$$

and

$$dL(t) = -L(t)\gamma(t)d\widetilde{W}(t),$$

such that the discounted primary traded asset prices $\widetilde{P}_i^X$, $i = 1, \ldots, n$, are Q^X-martingales. Furthermore, the expected-value process of a contingent

[III] An admissible trading strategy is a self-financing trading strategy with (discounted) price processes which are $\lambda \otimes Q$-a.s. bounded below.

claim $D = D(T)$ with maturity T under $\widetilde{Q}$ and numéraire P_0 coincides with the expected-value process of D under Q^X and numéraire X, i.e.

$$P_0(t) \cdot \mathbb{E}_{\widetilde{Q}} \left[\widetilde{D}(T) \middle| \mathcal{F}_t \right] = X(t) \cdot \mathbb{E}_{Q^X} \left[\widetilde{D}^X(T) \middle| \mathcal{F}_t \right]$$

for all $t \in [0, T]$.

Proof:
See Zagst (2002), pages 87f. □

A popular application of the above financial market is the famous Black-Scholes Model (see Black & Scholes (1973)) of which we present a generalized version (see e.g. Zagst (2002)). Within the terms of this model the financial market is free of arbitrage as well as complete, i.e. the price process of a European contingent claim is unique.

Theorem 2.23 ***(Generalized Black-Scholes)***
Suppose that $m = n = 1$ and that the primary traded assets with prices P_0 and P_1 are given by

$$\begin{aligned} dP_0(t) &= r(t) \cdot P_0(t)dt \,, P_0(0) = 1 \,, \\ dP_1(t) &= \mu(t) \cdot P_1(t)dt + \sigma(t) \cdot P_1(t)dW(t) \,,\ P_1(0) > 0 \,, \end{aligned}$$

with $\sigma > 0$ such that the no-arbitrage, the Novikov and the martingale conditions of Theorem 2.19 are satisfied. Then this financial market is free of arbitrage, and the price process of any European contingent claim D=D(T) with maturity T is given by

$$V_D(t) = P_0(t) \cdot \mathbb{E}_{\widetilde{Q}} \left[\widetilde{D}(T) | \mathcal{F}_t \right] = \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^T r(s)ds} \cdot D(T) | \mathcal{F}_t \right]$$

for $t \in [0, T]$, $\widetilde{Q} \in \mathbb{M}(Q)$.

Proof:
See Zagst (2002), pages 77-78. □

An important and well known result of this theorem are the formulas for European options. Here we present the call option price within the generalized Black-Scholes framework.

Theorem 2.24 ***(Generalized Black-Scholes Call Option Price)***
Let the assumptions of Theorem 2.23 be satisfied and let r and σ be deterministic. Then the price at time $t \in [0, T]$ of a European call option with strike X and terminal payoff $D(T) = \max\{P_1(T) - X, 0\}$ is given by

$$Call^{BS}(t, T, X) = P_1(t) \cdot \mathcal{N}(d_1) - e^{-\int_t^T r(s)ds} \cdot X \cdot \mathcal{N}(d_2)$$

with

$$d_1 := \frac{\ln\left(\frac{P_1(t)}{X}\right) + \int_t^T r(s)ds + \frac{1}{2}\sigma_Y^2}{\sigma_Y}\ ,\ d_2 := d_1 - \sigma_Y\ ,$$

and

$$\sigma_Y = \sigma_Y(t,T) := \sqrt{\int_t^T \sigma^2(s)ds}\ .$$

$\mathcal{N}$ denotes the standard normal cumulative distribution function.

Proof:
See Zagst (2002), pages 79-80. □

An extension of the Black-Scholes formula is the so-called Black formula (see Black (1976)) for futures prices. Since we make use of Black's formula in the following chapters we present it here too.
Let $F(t,T)$ be defined as

$$F(t,T) := e^{\int_t^T r(s)ds} \cdot P_1(t)\, t \in [0,T].$$

Theorem 2.25 ***(Generalized Black Price)***
Let the assumptions of Theorem 2.23 be satisfied and let r and σ be deterministic. Then the price at time $t \in [0,T]$ of a European call option written on a financial instrument with price process $(F(t,T))_{t\in[0,T]}$ and terminal payoff $D(T) = \max\{F(T,T) - X, 0\}$ is given by

$$Call^{Black}(t,T,X) = e^{-\int_t^T r(s)ds} \cdot \left(F(t,T)\cdot\mathcal{N}(d_1) - X\cdot\mathcal{N}(d_2)\right)$$

with

$$d_1 := \frac{\ln\left(\frac{F(t,T)}{X}\right) + \frac{1}{2}\sigma_Y^2}{\sigma_Y}\ ,\ d_2 := d_1 - \sigma_Y\ ,$$

and

$$\sigma_Y = \sigma_Y(t,T) := \sqrt{\int_t^T \sigma^2(s)ds}\ .$$

$\mathcal{N}$ denotes the standard normal cumulative distribution function.

Proof:
See Zagst (2002), pages 81-87. □

Interest-Rate Markets

Interest-rate markets are a special case of the introduced financial markets where in general the set of primary traded assets consists of zero-coupon bonds with different maturities. A zero-coupon bond is a financial contract which pays its holder a nominal N (:=1) at the end of the maturity T. Its price at time t is given by

$$P(t,T) = Ne^{-R(t,T)\cdot(T-t)}$$

where $R(t,T)$ denotes the continuous zero or spot rate, i.e. the interest rate which is guaranteed for the time period $[t,T]$.
Describing an interest market completely is a challenge since there are infinitely many zero-coupon bonds with different maturities on the market. Therefore an approach is to concentrate on a single interest rate instead of trying to model all possible rates $R(t,T)$ and to describe the whole term structure $T \rightarrow R(t,T)$ by means of this special rate. There are two rates which are commonly used, namely the short rate and the forward short rate.

Definition 2.26 ***(Short Rate and Forward Short Rate)***
The short rate $r(t)$ at time t is the interest rate for an infinitesimal time period. It is defined as

$$r(t) := R(t,t) := -\lim_{\Delta t \to 0} \frac{\ln P(t,t+\Delta t)}{\Delta t} = -\frac{\partial}{\partial T} \ln P(t,T)|_{T=t}.$$

The forward short rate $f(t,T)$ at time t is the interest rate for an infinitesimal time period at time T but derived at time t. It is defined as

$$\begin{aligned} f(t,T) &:= R(t,T,T) := -\lim_{\Delta t \to 0} \frac{\ln P(t,T+\Delta t) - \ln P(t,T)}{\Delta t} \\ &= -\frac{\partial}{\partial T} \ln P(t,T), \end{aligned}$$

where $R(t,T_1,T_2)$ denotes the forward zero rate given by

$$R(t,T_1,T_2) := -\frac{\ln P(t,T_2) - \ln P(t,T_1)}{T_2 - T_1},$$

i.e. the interest rate for the time period $[T_1,T_2]$ derived at time t.

We now define our primary interest-rate market $\mathcal{M}^{IRM}(Q)$ on the complete probability space $(\Omega,\mathcal{F},Q)$ with filtration $\mathbb{F}(W)$. The market is supposed to be frictionless and trading is allowed continuously up to a fixed time T^*. The numéraire of our interest-rate market is the so-called cash account P_0 with

$$P_0(t) = e^{\int_{t_0}^t r(s)ds} \, , t_0 \leq t \leq T \leq T^* \, .$$

The SDE of the cash account is

$$dP_0(t) = r(t)P_0(t)dt$$

with $P_0(0) = 1$ and r being a progressively measurable process with

$$\int_{t_0}^{T^*} |r(s)|ds < \infty \;\; Q\text{-a.s.}.$$

The primary traded assets, which are driven by an m-dimensional Brownian motion $W = (W_1(t), \ldots, W_m(t))_{t \in [t_0, T^*]}$ with $t_0 \in [0, T^*]$, consist of zero-coupon bonds with prices $P(t,T)$, $t \leq T$. Those prices are described by non-negative Itô processes as in Definition 2.16 with

$$dP(t,T) = \mu_P(t,T)dt + \sum_{j=1}^{m} \sigma_{Pj}(t,T)dW_j(t),$$

where μ_P and σ_{Pj}, $j = 1, \ldots, m$ are progressively measurable stochastic processes such that it holds for all $T \in [t_0, T^*]$:

$$\int_{t_0}^{T} |\mu_P(s,T)|ds < \infty \qquad Q\text{-a.s.}$$

and

$$\mathbb{E}_Q \left[\int_{t_0}^{T} \sigma_{Pj}^2(s,T)ds \right] < \infty, \qquad \forall j = 1, \ldots, m.$$

So far, the only differences between the general financial market $\mathcal{M}(Q)$ and the interest-rate market $\mathcal{M}^{IRM}(Q)$ are the number of primary assets, which is not limited anymore to n, and the time horizon which was changed to $[t_0, T^*]$ instead of [0,T].
$\mathcal{M}^{IRM}(Q)$ is defined to be arbitrage-free if any finite interest-rate market $\mathcal{M}^{IRM}(Q, \mathcal{T}_n)$, which is based on a finite number of zero-coupon bonds with maturities $T \in \mathcal{T}_n := \{T_1, \ldots, T_n\} \subset [t_0, T^*]$, is free of arbitrage.
The definition of an equivalent martingale measure has to be slightly extended compared to Definition 2.18 in order to fit into the new framework.

Definition 2.27 ***(Equivalent Martingale Measure in $\mathcal{M}^{IRM}(Q)$)***
A probability measure $\tilde{Q}$ on $(\Omega, \mathcal{F})$ is called an equivalent martingale measure with respect to Q if

1. *$\tilde{Q}$ is equivalent to Q,*

2. *The discounted price process $(\tilde{P}(t,T))_{t\in[t_0,T]}$ is a $\tilde{Q}$-martingale for all $T \in [t_0, T^*]$.*

The conditions under which the existence of an equivalent martingale measure is guaranteed are similar to Theorem 2.19. We just have to make sure that the time horizon is changed to $[t_0, T^*]$, especially for the integrals in the Novikov and martingale conditions. Additionally, the martingale condition and the no-arbitrage condition have to be fulfilled for all $t_0 \leq t \leq T \leq T^*$ (see Zagst (2002), page 103ff). The completeness of our primary interest-rate market is linked to the completeness of a finite interest rate market since $\mathcal{M}^{IRM}(Q)$ is said to be complete if any contingent claim $D(T_D)$, $T_D \in [t_0, T^*]$, is attainable in a finite interest-rate market $\mathcal{M}^{IRM}(Q, \mathcal{T}_n)$. Thus, if there exists an equivalent martingale measure for $\mathcal{M}^{IRM}(Q)$ and if this interest-rate market is complete then the expected-value process of the contingent claim D is unique. For more general conditions about pricing contingent claims see Zagst (2002), page 107f.

2.4 Kalman Filter

In this section we present the Kalman filter which will be used later on for calibration purposes. The main application of the Kalman filter technique, which was introduced by Kalman (1960), is the modelling and estimation of unobservable processes. Furthermore, if there are any parameters within the set-up of the model which are to be estimated, this can also be done by means of the Kalman filter and a maximum likelihood estimation. In this section we refer to Harvey (1989). Other textbooks covering this topic are e.g. Øksendal (1998) who devotes a chapter for the linear filtering problem, especially the Kalman-Bucy filter. He also cites references for non-linear cases. Greg Welch and Gary Bishop of the University of North Carolina provide on their webpage[IV] an extensive overview of books, articles, tutorials and research related to the Kalman filter.

State Space Model

The state space model describes the development of the unobservable process and its linkage to given data. The dynamics of the process, i.e. its evolution from one point in time to another, are given by the transition equation whereas the measurement equation determines the relation of this

[IV] http://www.cs.unc.edu/∼welch/kalman/

process to measurable information. We consider a linear state space model for $t = 1, \dots, T$

$$\begin{aligned} Y_t &= Z_t \alpha_t + d_t + \epsilon_t && \text{(measurement equation)},\\ \alpha_t &= T_t \alpha_{t-1} + c_t + \eta_t && \text{(transition equation)}, \end{aligned}$$

with

Y_t	$N \times 1$ vector with observable information at time t,
α_t	$m \times 1$ state vector at time t,
$c_t \in \mathbb{R}^m$	constant term of transition equation at time t,
$d_t \in \mathbb{R}^N$	constant term of measurement equation at time t,
$Z_t \in \mathbb{R}^{N \times m}$	coefficient matrix of state vector for measurement equation,
$T_t \in \mathbb{R}^{m \times m}$	coefficient matrix of state vector for transition equation,
$\epsilon_t \sim \mathcal{N}_N(0, H_t)$	disturbance term of measurement equation,
$\eta_t \sim \mathcal{N}_m(0, Q_t)$	disturbance term of transition equation.

Furthermore, it must hold that ϵ_t and η_t are sequences of independent random vectors with $\mathbb{E}(\epsilon_t \eta_s') = 0$ for all $s, t = 1, \dots, T$. Additionally, the initial state α_0 has to be independent of ϵ_t and η_t with α_0 being normally distributed, i.e. $\alpha_0 \sim \mathcal{N}_m(a_0, P_0)$ for $a_0 \in \mathbb{R}^m$ and $P_0 \in \mathbb{R}^{m \times m}$.

Based on this state space model, we now present the Kalman filter algorithm which will be used in order to get an estimate of α_t with respect to all available information up to time t.

Algorithm

- *Initialize a_0 and P_0.*
- *For $t = 1, \dots, T$ evaluate*
  - *the prediction equation*

$$\begin{aligned} a_{t|t-1} &= T_t a_{t-1} + c_t\\ P_{t|t-1} &= T_t P_{t-1} T_t' + Q_t, \end{aligned}$$

  - *and the update equation*

$$\begin{aligned} a_t &= a_{t|t-1} + P_{t|t-1} Z_t' F_t^{-1} (y_t - Z_t a_{t|t-1} - d_t)\\ P_t &= P_{t|t-1} - P_{t|t-1} Z_t' F_t^{-1} Z_t P_{t|t-1}\\ &\textit{with } F_t = Z_t P_{t|t-1} Z_t' + H_t. \end{aligned}$$

In order to check if the model is well-specified Harvey (1989), e.g. page 256, suggests to test the standardized innovations

$$\tilde{\nu}_t := \frac{y_t - Z_t a_{t|t-1} - d_t}{\sqrt{f_t}}$$

with f_t being the corresponding element on the diagonal of F_t since these residuals should be independent and standard normally distributed. He proposes testing e.g. for serial correlation, for heteroscedasticity and for normality.

Theorem 2.28 ***(Properties of the Kalman Filter)***
It holds that

$$\begin{pmatrix} \alpha_t \\ Y_t \end{pmatrix} | Y_{t-1} = y_{t-1}, \dots, Y_1 = y_1$$

$$\sim \mathcal{N}_{m+N}\left(\begin{pmatrix} a_{t|t-1} \\ Z_t a_{t|t-1} + d_t \end{pmatrix}, \begin{pmatrix} P_{t|t-1} & P_{t|t-1} Z_t' \\ Z_t P_{t|t-1} & Z_t P_{t|t-1} Z_t' + H_t \end{pmatrix} \right)$$

and

$$\alpha_t | Y_t = y_t, \dots, Y_1 = y_1 \sim \mathcal{N}_m(a_t, P_t)$$

for $t = 1, \dots, T$.
Moreover, the minimum mean square estimate of α_t *for available data* $y_1, \dots, y_t$ *is given by* a_t.

Proof:
see Harvey (1989), page 109-110. □

With the help of this theorem we are now able to estimate any unknown parameters of the state space model. If the disturbance terms and the initial state α_0 are normally distributed, then by Theorem 2.28 it follows that $\mathbb{E}[\alpha_t | y_{t-1}, \dots, y_1] = a_{t|t-1}$ and $Cov[\alpha_t | y_{t-1}, \dots, y_1] = P_{t|t-1}$. Hence, if we condition the measurement equation with respect to $t-1$ we obtain a normal distribution with

$$\mathbb{E}_{t-1}[y_t] = \tilde{y}_{t|t-1} = Z_t a_{t|t-1} + d_t$$

and covariance matrix F_t. Since we are dealing with a normal distribution, the log-likelihood sums up to

$$\log(L(y_1, \dots, y_T, \Theta)) = -\frac{NT}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \log|F_t| - \frac{1}{2}\sum_{t=1}^{T} v_t^T F_t^{-1} v_t,$$

with L denoting the likelihood function, Θ the vector of unknown parameters, N the length of Y_t and $v_t := y_t - \tilde{y}_{t|t-1}$ for $t = 1, \ldots, T$. This is also called prediction error decomposition as v_t can be seen as a prediction error. For further information about maximum likelihood estimation and the prediction error decomposition refer to Harvey (1989), Chapter 3.4, page 125-147.

Within this thesis we use the software package S-PLUS finmetrics for any computations regarding the Kalman filter.

Chapter 3

Pricing Credit Risk

This chapter outlines the main approaches of credit risk modelling: structural models and reduced-form models. The former approach tries to model default by directly using the assets of the firm, whereas the latter approach does not concentrate on modelling the firm's asset process. Here, the default event is typically given exogenously and default happens completely unexpected.
Also, there exists a third approach where so-called hybrid models use characteristics of both the structural and the reduced-form models. These models assume a linkage between the hazard rate of default and the value of the firm's assets. The models presented in this thesis belong to this class of credit risk models since they do not try to specify exactly the firm's assets but incorporate market data as well as firm-specific information.

3.1 Structural Models

Characteristic of this approach is the attempt to model the evolution of the firm's assets in order to deduce the value of corporate debt and to price credit risk. The most utilized credit event is the firm's default. Therefore, the attention is directed to a lower barrier which represents the default threshold. If the firm's assets reach this boundary for the first time, the default will be triggered and the firm will go bankrupt. This mechanism can be seen as a savety covenant whose goal is to protect bondholders against stockholders.
Structural models have their intellectual roots in the work of Merton (1974). His approach to corporate debt assumes a constant rate of interest r and several standard conditions like e.g. unrestricted borrowing and lending, no taxes and transaction costs, and continuously trading in time. The firm is

assumed to have one liability with a terminal payoff L and default may only happen at the debt's maturity T. The firm's value process is modelled as a geometric Brownian motion

$$dV(t) = V(t) \cdot ((r - k)dt + \sigma dW(t))$$

with constants σ and k where the latter represents the payout ratio in case it is positive otherwise the capital inflow. The price process X of the defaultable claim is given at time T as:

$$X = L \cdot 1_{\{V(T) \geq L\}} + V(T) \cdot 1_{\{V(T) < L\}} = L - \max(L - V(T), 0).$$

Hence, the payoff of a defaultable zero-coupon bond can be interpreted as the payoff of a default-free zero-coupon bond with face value L less the payoff of a European put option which is written on the assets V of the firm with strike price L and exercise date T. Therefore, the value of the firm's debt at time t is the difference of a zero-coupon bond with face value L and the price of a European put option at t. The value of this European put option can be written in closed form with the help of the Black-Scholes formula (see Theorem 2.24). And since the assets of the firm are the sum of the firm's debt and equity, we get the value of the equity as the price of a European call option also written on the firm's assets by means of the put-call parity for European options.

First-passage-time models are an extension to the Merton model allowing default to happen before and at the debt's maturity. The time of default is specified as the first-passage time of the firm's assets relative to a barrier, which can be random and either exogenously or endogenously given. Black & Cox (1976) extend Merton's framework by letting default happen if the firm's assets are below some triggering level at maturity or if they cross a time-dependent level before maturity. Kim, Ramaswamy & Sundaresan (1993) and Longstaff & Schwartz (1995) incorporate stochastic interest rates into the model by assuming either a Cox-Ingersoll-Ross process or a Vasicek process.
An advantage of structural models is that default is modelled endogenously by means of the firm's assets and therefore allows for the usage of market information. But a major drawback of the above introduced structural models is the fact that short-term credit spreads are close to zero due to the asset value being modelled as a continuous process. In order to circumvent this shortcoming, Zhou (2001) adds a jump process to the dynamics of the assets.

3.2 Reduced-Form Models

The reduced-form approach is motivated by the difficulty of exactly specifying default, i.e. it is often impossible to find variables such as the firm's assets on whose particular constellation default depends with certainity. Default often happens without meeting all the defined requirements or it fails to happen although all requirements are met. Therefore, the idea is not to focus on the exact definition of the default event and the modelling of the firm's value, but to work with the evolution of the probability of default at any point in time instead. In order to model the default event as a total surprise, the default time (τ) is set as a non-predictable stopping time (see Section 2.1). Then, default is described as the first jump of a special point process (see also Section 2.1), i.e. a Poisson process (see e.g Brigo & Mercurio (2006), Appendix C). The Poisson process can have either constant, deterministic or stochastic (Cox process) intensities. For example, if we assume the intensity c to be a positive, stochastic, adapted and right-continuous process with $\Lambda(T) := \int_0^T c(s)ds$ being strictly increasing and denoting its cumulated intensity or hazard function. Then, for Poisson processes the jump time τ can be transformed according to its cumulated intensity Λ:

$$\Lambda(\tau) =: \zeta \Rightarrow \tau = \Lambda^{-1}(\zeta)$$

with ζ being a standard exponential random variable (see McNeil, Frey & Embrechts (2005), Lemma 9.13). Therefore, using the cumulated distribution of an exponential random variable, we can determine the probability of the jump being after time t, also called the survival probability up to time t:

$$Q(\tau > t) = Q(\Lambda(\tau) > \Lambda(t)) = Q(\zeta > \Lambda(t)) = \mathbb{E}_Q\left[e^{-\int_0^t c(s)ds}\right].$$

The variable ζ is independent of all other variables, hence being an external source of randomness. With these assumptions, monitoring basic market observables gives not a complete information with respect to default since the exogenous component is independent of the default-free market data.

Jarrow & Turnbull (1992) introduce the reduced-form approach by assuming a constant intensity and a pre-defined payoff at default. The work of Lando (Lando (1994), Lando (1997), and Lando (1998)) extends this framework using stochastic intensities (Cox processes).

Advantages of reduced-form models are their positive credit spreads even for short maturities as opposed to structural models and the fact that they are completely data-driven, i.e. their parameters can be fitted easily to market data. However, a shortcoming of this type of models is the fact that the intensity process is specified exogenously. Hence, there exists no linkage between default and any drivers of default, therefore making default completely unexpected.

3.3 Hybrid Models

Hybrid models try to circumvent the drawbacks of structural and reduced-form models (i.e. short-term credit spreads of zero, intensities that are specified completely exogenously) and therefore combine characteristics of structural and reduced-form models. By doing this, they provide a linkage between the likelihood of default and data that is supposed to drive or indicate default.

Starting with a structural framework, Duffie & Lando (2001) assume that the bondholders only receive incomplete information about the firm's value. They show that this set-up is consistent with a reduced-form approach since it admits an intensity and short-term credit spreads greater than zero.

Another way to build hybrid models is to start with reduced-form models and relate the probability of default to observable or unobservable factors. Cathcart & El-Jahel (1998) assume default to be driven by a signaling process, whereas Bakshi, Madan & Zhang (2006) incorporate an unobservable macroeconomic factor as well as an observable firm-specific factor for which they use e.g. stock prices.

The models presented in this thesis are also hybrid models and are based on the work of Schmid & Zagst (2000). Schmid & Zagst (2000) assume credit spreads to be driven by an unobservable uncertainty index that aggregates all available information concerning the quality of a firm. This model is further extended with an additional observable macroeconomic factor influencing interest rates as well as credit spreads by Antes et al. (2008).

Chapter 4

A Generalized Five Factor Model

Within this chapter we present a hybrid model which links macroeconomic and firm-specific information to the performance of interest rates and credit spreads. Our framework is mainly based on the work of Schmid & Zagst (2000) who introduced a defaultable term structure model which is driven by an additional factor comprising an aggregation of market and/or firm-specific data. This model is built by three factors, namley the short rate r, the so-called uncertainty index u and the short-rate spread s. The short rate r was first modelled as a mean-reverting Hull-White or square-root process, both with a time-dependent mean-reversion level. The short-rate spread s which is meant to be the difference between the spreads of defaultable and non-defaultable bonds for an infinitesimal maturity follows a square-root process and is influenced by the uncertainty index u. This uncertainty index is to be understood as an aggregation of all available information regarding the creditworthiness of the firm and/or relevant macroeconomic data. Higher values of this index u indicate a deterioration in the obligor's state and lead to increasing credit spreads. As before, this index is also described by a square-root process. Cathcart & El-Jahel (1998) were the first to introduce a process similar to the uncertainty index u. The so-called signaling process explicitely drives the default in their framework. Kalemanova & Schmid (2002) tested the three factor model of Schmid and Zagst on German and Italian government bonds and obtained good approximations of the given term structures. The choice of square-root processes prevents the short rate and the short-rate spread to take on negative values which is a desirable characteristic of this framework since e.g. credit spreads should be thought of as a compensation for bearing credit risk and thus should be non-negative. Unfortunately, these square-root processes complicate the estimation proced-

ure considerably. Therefore Roth & Zagst (2004) simplified the three factor Schmid-Zagst model by replacing the square-root processes by Vasicek processes. Although this change leads to possible negative values for the short rate and the short-rate spread, the authors showed that neglecting the positivity constraint does not influence the pricing quality compared to the preceding model.
There are many articles in literature which analyze the impact of macroeconomic factors on interest rates as well as credit spreads. Additionally, the dependence of credit spreads on factors stemming from firm-specific information is examined. E.g. Ang & Piazzesi (2003) analyzed the effect of macro variables on non-defaultable bond prices and on the dynamics of the yield curve using inflation and economic growth factors. They found that the forecasting performance is improved by incorporating macroeconomic factors which are also found to be able to explain a great portion of the variation in bond yields. Krishnan et al. (2005) showed that firm-specific and market variables are important in explaining credit spread levels and changes for banking and non-banking firms. A similar study was done by Avramov, Jostova & Philipov (2007) who found that more than 50 % of the variation of credit spread changes can be explained by a combination of common and firm-specific fundamentals.
Hence, a further enhancement of the Schmid-Zagst model was developed by Antes et al. (2008) who incorporated an additional macroeconomic factor in both the short rate and the short-rate spread. Since literature indicates that there is more than just one explanatory macroeconomic variable we devote this chapter to work out a framework which incorporates two factors representing economic data in the short rate as well as the short-rate spread.

This chapter is organized as follows. In Section 4.1 we set up our general framework which is used to derive the various types of models which will be presented in the following five sections. Section 4.7 is devoted to the data and the estimation procedure. Afterwards, a comparison of the calibration results is presented in Section 4.8.

4.1 The Set-Up

We assume a frictionless market where trading takes place continuously and where investors act as price takers. Additionally there are no transaction costs, no taxes and no informational asymmetries. All random variables and stochastic processes will be defined on a probability space $(\Omega, \mathcal{G}, Q)$ which describes the uncertainty in the financial market. Furthermore, we assume

this probability space to be equipped with three filtrations $\mathbb{H}$, $\mathbb{F}$, and $\mathbb{G}$ which fulfill the assumptions of completeness and right-continuity. $\mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T^*}$ is the filtration generated by the process H with $H(t) = \mathbf{1}_{\{T^d\leq t\}}$ for a default time T^d and a fixed terminal time horizon T^*. This default time is a non-negative random variable on the probability space with $Q\left(T^d = 0\right) = 0$ and $Q\left(T^d > t\right) > 0$ for every $t \in (0, T^*]$. $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T^*}$ is supposed to be the filtration which is generated by the multi-dimensional Brownian motion $W(t)$ with $\mathcal{F}_0$ being trivial , whereas $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T^*}$ is to be the enlarged filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, namely $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for every t. Additionally, there exist on the probability space two $\mathbb{F}$-adapted processes, the short rate process $r(t)$ and the short spread process $s(t)$.
In the following we will assume that under the martingale measure $\widetilde{Q}$ $\mathbb{F}$ has the martingale invariance property with respect to $\mathbb{G}$, meaning any $\mathbb{F}-$martingale follows also a $\mathbb{G}-$martingale (see Bielecki & Rutkowski (2004), page 167). This assumption is equivalent to the fact that for any $t \in (0, T^*]$ and any $\widetilde{Q}-$integrable $\mathcal{F}_{T^*}-$measurable random variable X with $\widetilde{Q}$ being a martingale measure it holds that $\mathbb{E}_{\widetilde{Q}}\left[X|\ \mathcal{G}_t\right] = \mathbb{E}_{\widetilde{Q}}\left[X|\ \mathcal{F}_t\right]$ (see Bielecki & Rutkowski (2004), page 242).

The introduced interest-rate market contains four different types of traded assets. As numéraire serves the non-defaultable cash account

$$P_0(t) = e^{\int_0^t r(l)dl},$$

which is an investment of value one for an infinitesimal short maturity with successive reinvestment up to time t.
Furthermore, we can invest into non-defaultable zero-coupon bonds and defaultable zero-coupon bonds with maturities $T \in [0, T^*]$.

Definition 4.1 ***(Defaultable Zero-Coupon Bond)***
A zero-coupon bond with face value 1 *and maturity* T *which pays* 1 *at maturity, if there has been no default before time* T*, and the recovery rate* $z\left(T^d\right)$ *at default* T^d*, if* $T^d \leq T$*, is called a defaultable zero-coupon bond with price* $P^d(t,T)$*.*

The recovery rate is to be understood as a fraction of the market value of the bond just before the default $P^d_-(T^d, T)$. Additionally it is assumed that $z(t)$ is a $\mathcal{F}_t$-adapted, continuous process with $z(t) \in [0, 1)$ for all t.
The fourth traded asset is the defaultable money-market account defined by

$$P_0^d(t) = \left(1 + \int_0^t (z(l) - 1)dH(l)\right) e^{\int_0^t r(l)+s(l)L(l)dl},$$

with $L(t) = \mathbf{1}_{\{T^d>t\}}$ being the survival indicator. This defaultable account is defined analogously to the non-defaultable case, i.e. it is an investment of value one in a defaultable zero-coupon bond of infinitesimal short maturity with subsequent reinvestment in case of no default.
The prices of the financial instruments can be determined under the martingale measure $\widetilde{Q}$ as the conditional present value of all future payoffs. Hence, the price of the non-defaultable zero-coupon bond is given by

$$P(t,T) = \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl} \middle| \mathcal{F}_t \right].$$

The price of a defaultable zero-coupon bond is determined for $t < \min(T^d, T)$ by the expected value of the recovery payment in case of a default between $[t, T]$ and the payment at the maturity T if there is no default:

$$\mathbf{1}_{\{T^d>t\}} \cdot P^d(t,T) = \mathbb{E}_{\widetilde{Q}}\left[\int_t^T e^{-\int_t^u r(l)dl} z(u) P_-^d(u,T) dH(u) + e^{-\int_t^T r(l)dl} L(T) \middle| \mathcal{G}_t \right].$$

Analogously to e.g. Schmid (2004) and Antes (2004) it can be shown that by means of some technical conditions with respect to r and s the price of a defaultable zero-coupon bond is determined by

$$P^d(t,T) = \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T (r(l)+s(l))dl} \middle| \mathcal{F}_t \right]$$

for $t < \min(T^d, T)$.

Having generally introduced our financial market, we now present in detail the processes which are crucial for our five factor framework.
For a fixed terminal time horizon T^*, let the following stochastic differential equations be satisfied for $0 \leq t \leq T^*$:
The short rate r which is driven by two macroeconomic factors (w_1 and w_2) is described by a three-factor Hull-White process.

$$\begin{aligned} dr(t) &= (\theta_r(t) + b_{rw_1} w_1(t) + b_{rw_2} w_2(t) - a_r r(t))\, dt \\ &\quad + \sigma_r \sqrt{1 - \rho_{rw_1}^2 - \rho_{rw_2}^2} dW_r(t) + \sigma_r \rho_{rw_1} dW_{w_1}(t) + \sigma_r \rho_{rw_2} dW_{w_2}(t). \end{aligned}$$

The macroeconomic factors w_1 and w_2 are given by correlated Vasicek processes and can be chosen to be observable or unobservable.

$$\begin{aligned} dw_1(t) &= (\theta_{w_1} - a_{w_1} w_1(t))\, dt + \sigma_{w_1} dW_{w_1}(t), \\ dw_2(t) &= (\theta_{w_2} - a_{w_2} w_2(t))\, dt + \sigma_{w_2} \rho_{w_1 w_2} dW_{w_1}(t) + \sigma_{w_2} \sqrt{1 - \rho_{w_1 w_2}^2} dW_{w_2}(t). \end{aligned}$$

The uncertainty index u summarizes all available information concerning the creditworthiness of a company. This index is assumed to be unobservable and is described by a Vasicek process.

$$du(t) \quad = \quad (\theta_u - a_u u(t))\, dt + \sigma_u dW_u(t).$$

The short-rate spread s represents the difference between the spreads of defaultable and non-defaultable bonds and is also given by a Vasicek process. This process is affected by the firm-specific uncertainty index u as well as the macroeconomic factors w_1 and w_2.

$$\begin{aligned} ds(t) \quad = \quad & (\theta_s + b_{su}u(t) - b_{sw_1}w_1(t) - b_{sw_2}w_2(t) - a_s s(t))\, dt \\ & + \sigma_s\sqrt{1-\rho_{su}^2-\rho_{sw_1}^2-\rho_{sw_2}^2}\, dW_s(t) + \sigma_s\rho_{su}dW_u(t) \\ & + \sigma_s\rho_{sw_1}dW_{w_1}(t) + \sigma_s\rho_{sw_2}dW_{w_2}(t), \end{aligned}$$

For the constants it holds

$$a_r, a_{w_1}, a_{w_2}, a_u, a_s > 0\,,$$

$$\sigma_r, \sigma_{w_1}, \sigma_{w_2}, \sigma_u, \sigma_s > 0\,,$$

$$\theta_{w_1}, \theta_{w_2}, \theta_u, \theta_s \geq 0\,,$$

$$b_{rw_1}, b_{rw_2}, b_{su}, b_{sw_1}, b_{sw_2} \in \mathbb{R}\,,$$

$$\rho_{w_1w_2}, \rho_{rw_1}, \rho_{rw_2}, \rho_{su}, \rho_{sw_1}, \rho_{sw_2} \in [-1, 1]\,,$$

and θ_r is a continuous deterministic function.
Furthermore, $W := (W_r, W_{w_1}, W_{w_2}, W_u, W_s)'$ is a five-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{G}, Q, \mathbb{G})$.
Then, the above system of five stochastic differential equations has a unique solution for any given vector of initial values ($r(0)$, $w_1(0)$, $w_2(0)$, $u(0)$, $s(0)$)' $\in \mathbb{R}^5$ (see Theorem 2.11).
Suppose there exists a progressively measurable process $\gamma(t) = (\gamma_r(t), \gamma_{w_1}(t), \gamma_{w_2}(t), \gamma_u(t), \gamma_s(t))'$ with

$$\frac{d\widetilde{Q}_t}{dQ_t} = e^{-\int_0^t \gamma(l)dW(l) - \frac{1}{2}\int_0^t \|\gamma(l)\|^2 dl},$$

where $\widetilde{Q}_t$ and Q_t are the restrictions of $\widetilde{Q}$ and Q on $\mathcal{G}_t$.
Additionally, let γ satisfy the Novikov condition

$$\mathbb{E}_Q\left[e^{\frac{1}{2}\int_0^{T^*} \|\gamma(l)\|^2 dl}\right] < \infty$$

and let the following equations be true for real constants λ_r, λ_{w_1}, λ_{w_2}, λ_u, λ_s:[I]

$$\begin{aligned}
\gamma_r(t) &= \lambda_r\sigma_r r(t) - \delta\lambda_r\sigma_r w_2(t) \\
&\quad + \frac{1}{\sqrt{1-\rho_{rw_1}^2-\rho_{rw_2}^2}}\left(\rho_{rw_2}\frac{\rho_{w_1w_2}}{\sqrt{1-\rho_{w_1w_2}^2}} - \rho_{rw_1}\right)\gamma_{w_1}(t) \\
&\quad - (1-\delta)\frac{\rho_{rw_2}}{\sqrt{1-\rho_{rw_1}^2-\rho_{rw_2}^2}}\lambda_{w_2}\sigma_{w_2}w_2(t) \quad \text{with } \delta\in\{0,1\}, \\
\gamma_{w_1}(t) &= \lambda_{w_1}\sigma_{w_1}w_1(t), \\
\gamma_{w_2}(t) &= \lambda_{w_2}\sigma_{w_2}w_2(t) - \frac{\rho_{w_1w_2}}{\sqrt{1-\rho_{w_1w_2}^2}}\gamma_{w_1}(t), \\
\gamma_u(t) &= \lambda_u\sigma_u u(t), \\
\gamma_s(t) &= \lambda_s\sigma_s s(t) - \frac{\rho_{su}\gamma_u(t)+\rho_{sw_1}\gamma_{w_1}(t)+\rho_{sw_2}\gamma_{w_2}(t)}{\sqrt{1-\rho_{su}^2-\rho_{sw_1}^2-\rho_{sw_2}^2}}.
\end{aligned}$$

According to Theorem 2.19, the process

$$\widetilde{W}(t) := W(t) + \int_0^t \gamma(l)dl$$

is now a $\widetilde{Q}$-Brownian motion. Therefore, under the measure $\widetilde{Q}$ the stochastic differential equations can be written as:

$$\begin{aligned}
dr(t) &= \Big(\theta_r(t) + b_{rw_1}w_1(t) + \hat{b}_{rw_2}w_2(t) - \hat{a}_r r(t)\Big)\,dt \\
&\quad + \sigma_r\sqrt{1-\rho_{rw_1}^2-\rho_{rw_2}^2}\,d\widetilde{W}_r(t) + \sigma_r\rho_{rw_1}d\widetilde{W}_{w_1}(t) + \sigma_r\rho_{rw_2}d\widetilde{W}_{w_2}(t), \\
dw_1(t) &= (\theta_{w_1} - \hat{a}_{w_1}w_1(t))\,dt + \sigma_{w_1}d\widetilde{W}_{w_1}(t), \\
dw_2(t) &= (\theta_{w_2} - \hat{a}_{w_2}w_2(t))\,dt + \sigma_{w_2}\rho_{w_1w_2}d\widetilde{W}_{w_1}(t) + \sigma_{w_2}\sqrt{1-\rho_{w_1w_2}^2}\,d\widetilde{W}_{w_2}(t), \\
du(t) &= (\theta_u - \hat{a}_u u(t))\,dt + \sigma_u d\widetilde{W}_u(t), \\
ds(t) &= (\theta_s + b_{su}u(t) - b_{sw_1}w_1(t) - b_{sw_2}w_2(t) - \hat{a}_s s(t))\,dt \\
&\quad + \sigma_s\sqrt{1-\rho_{su}^2-\rho_{sw_1}^2-\rho_{sw_2}^2}\,d\widetilde{W}_s(t) + \sigma_s\rho_{su}d\widetilde{W}_u(t) \\
&\quad + \sigma_s\rho_{sw_1}d\widetilde{W}_{w_1}(t) + \sigma_s\rho_{sw_2}d\widetilde{W}_{w_2}(t),
\end{aligned}$$

with [II] $\hat{a}_r = a_r + \lambda_r\sigma_r^2\sqrt{1-\rho_{rw_1}^2-\rho_{rw_2}^2}$, $\hat{a}_{w_2} = a_{w_2} + \lambda_{w_2}\sigma_{w_2}^2\sqrt{1-\rho_{w_1w_2}^2}$, $\hat{a}_s = a_s + \lambda_s\sigma_s^2\sqrt{1-\rho_{su}^2-\rho_{sw_1}^2-\rho_{sw_2}^2}$, $\hat{a}_i = a_i + \lambda_i\sigma_i^2$, $i = w_1, u$, and

[I] This approach is adapted to Schmid (2002), page 54.

[II] Throughout this work we assume $\hat{a}_r$, $\hat{a}_{w_1}$, $\hat{a}_{w_2}$, $\hat{a}_u$, $\hat{a}_s$ to be positive in order to preserve the mean-reverting quality of the processes under the measure $\widetilde{Q}$.

$\hat{b}_{rw_2} = b_{rw_2} + \delta\sigma_r(\lambda_r\sigma_r\sqrt{1-\rho_{rw_1}^2-\rho_{rw_2}^2} - \lambda_{w_2}\sigma_{w_2}\rho_{rw_2}),\ \delta \in \{0,1\}.$

Within this framework the price of a non-defaultable zero-coupon bond has an affine term structure given in the next theorem.

Theorem 4.2 ***(Price of a Non-Defaultable Zero-Coupon Bond)***
The price of a non-defaultable zero-coupon bond is given by

$$P(t,T) = \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl}|\mathcal{F}_t\right] = P(t,T,r(t),w_1(t),w_2(t)),$$

with

$$P(t,T,r,w_1,w_2) = e^{A(t,T)-B(t,T)r-E_1(t,T)w_1-E_2(t,T)w_2}$$

and

$$\begin{aligned}
B(t,T) &= \frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right),\\
E_1(t,T) &= b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),\\
E_2(t,T) &= \hat{b}_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right),\\
A(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2(B(l,T))^2 + \frac{1}{2}\sigma_{w_1}^2(E_1(l,T))^2 + \frac{1}{2}\sigma_{w_2}^2(E_2(l,T))^2\\
&\quad + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1(l,T)E_2(l,T) + \sigma_r\sigma_{w_1}\rho_{rw_1}B(l,T)E_1(l,T)\\
&\quad + \sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)B(l,T)E_2(l,T)\\
&\quad - \theta_r(l)B(l,T) - \theta_{w_1}E_1(l,T) - \theta_{w_2}E_2(l,T)dl.
\end{aligned}$$

Proof:
According to Feynman-Kac (see Theorem 2.14) the following differential equation must hold:

$$\begin{aligned}
rP &= P_t\\
&\quad + \left(\theta_r(t) + b_{rw_1}w_1 + \hat{b}_{rw_2}w_2 - \hat{a}_r r\right)P_r\\
&\quad + \left(\theta_{w_1} - \hat{a}_{w_1}w_1\right)P_{w_1}\\
&\quad + \left(\theta_{w_2} - \hat{a}_{w_2}w_2\right)P_{w_2}\\
&\quad + \frac{1}{2}\Big(\sigma_r^2P_{rr} + \sigma_{w_1}^2P_{w_1w_1} + \sigma_{w_2}^2P_{w_2w_2} + 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}P_{w_1w_2}\\
&\quad + 2\sigma_r\sigma_{w_1}\rho_{rw_1}P_{rw_1} + 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)P_{rw_2}\Big).
\end{aligned}$$

Using the affine term structure, we derive the partial derivatives of P:[III]

$$
\begin{aligned}
&P_t = (A_t - B_t r - (E_1)_t w_1 - (E_2)_t w_2) \cdot P\,, \\
&P_r = -B \cdot P\,, \qquad P_{rr} = B^2 \cdot P\,, \\
&P_{w_1} = -E_1 \cdot P\,, \quad P_{w_1 w_1} = (E_1)^2 \cdot P\,, \\
&P_{w_2} = -E_2 \cdot P\,, \quad P_{w_2 w_2} = (E_2)^2 \cdot P\,, \\
&P_{rw_1} = BE_1 \cdot P\,, \quad P_{w_1 w_2} = E_1 E_2 \cdot P\,, \\
&P_{rw_2} = BE_2 \cdot P\,.
\end{aligned}
$$

Substituting these terms and dividing by $P > 0$, we arrive at:

$$
\begin{aligned}
r \;=\;& A_t - B_t r - (E_1)_t w_1 - (E_2)_t w_2 \\
&+ \Big(\theta_r(t) + b_{rw_1} w_1 + \hat{b}_{rw_2} w_2 - \hat{a}_r r\Big)\,(-B) \\
&+ (\theta_{w_1} - \hat{a}_{w_1} w_1)\,(-E_1) \\
&+ (\theta_{w_2} - \hat{a}_{w_2} w_2)\,(-E_2) \\
&+ \frac{1}{2}\Big(\sigma_r^2 B^2 + \sigma_{w_1}^2 (E_1)^2 + \sigma_{w_2}^2 (E_2)^2 + 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1 w_2} E_1 E_2 \\
&+ 2\sigma_r \sigma_{w_1}\rho_{rw_1} BE_1 + 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1 w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1 w_2}^2}\right) BE_2\Big).
\end{aligned}
$$

Regrouping the terms, the equation takes on the form:

$$
\begin{aligned}
0 \;=\;& r\,(\hat{a}_r B - 1 - B_t) \\
&+ w_1\,(\hat{a}_{w_1} E_1 - b_{rw_1} B - (E_1)_t) \\
&+ w_2\,\Big(\hat{a}_{w_2} E_2 - \hat{b}_{rw_2} B - (E_2)_t\Big) \\
&+ A_t - \theta_r(t) B - \theta_{w_1} E_1 - \theta_{w_2} E_2 \\
&+ \frac{1}{2}\Big(\sigma_r^2 B^2 + \sigma_{w_1}^2 (E_1)^2 + \sigma_{w_2}^2 (E_2)^2 + 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1 w_2} E_1 E_2 \\
&+ 2\sigma_r \sigma_{w_1}\rho_{rw_1} BE_1 + 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1 w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1 w_2}^2}\right) BE_2\Big).
\end{aligned}
$$

[III] Throughout this thesis, we denote with P_x, $x \in \{t, r, w_1, w_2, s, u\}$ the partial derivative of the function P with respect to x. The same logic holds for functions like $A(t,T)$ and $B(t,T)$.

We obtain a system of linear differential equations for A, B, E_1, and E_2 by comparing the coefficients:

$$
\begin{aligned}
B_t &= \hat{a}_r B - 1 \\
(E_1)_t &= \hat{a}_{w_1} E_1 - b_{rw_1} B \\
(E_2)_t &= \hat{a}_{w_2} E_2 - \hat{b}_{rw_2} B \\
-A_t &= \frac{1}{2}\Big(\sigma_r^2 B^2 + \sigma_{w_1}^2 (E_1)^2 + \sigma_{w_2}^2 (E_2)^2 + 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2} E_1 E_2 \\
&\quad + 2\sigma_r\sigma_{w_1}\rho_{rw_1} B E_1 + 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) B E_2 \Big) \\
&\quad - \theta_r(t) B - \theta_{w_1} E_1 - \theta_{w_2} E_2.
\end{aligned}
$$

Since the condition $P(T,T) = 1$ must be fulfilled for all $r, w_1, w_2 \in \mathbb{R}$ it holds $A(T,T) = B(T,T) = E_1(T,T) = E_2(T,T) = 0$. By means of the transformation $\tau = T - t$ and the given terminal conditions, the differential equations result in (cf. Theorem 2.15):

$$
\begin{aligned}
B(t,T) &= e^{-\hat{a}_r(T-t)} \int_0^{T-t} e^{\hat{a}_r l} dl = e^{-\hat{a}_r(T-t)} \frac{1}{\hat{a}_r}\left(e^{\hat{a}_r(T-t)} - 1\right) \\
&= \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(T-t)}\right), \\
E_1(t,T) &= e^{-\hat{a}_{w_1}(T-t)} \int_0^{T-t} e^{\hat{a}_{w_1} l} b_{rw_1} B(0,l) dl \\
&= b_{rw_1} \frac{1}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1} - \hat{a}_r} \right), \\
E_2(t,T) &= e^{-\hat{a}_{w_2}(T-t)} \int_0^{T-t} e^{\hat{a}_{w_2} l} \hat{b}_{rw_2} B(0,l) dl \\
&= \hat{b}_{rw_2} \frac{1}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2} - \hat{a}_r} \right), \\
A(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2 (B(l,T))^2 + \frac{1}{2}\sigma_{w_1}^2 (E_1(l,T))^2 + \frac{1}{2}\sigma_{w_2}^2 (E_2(l,T))^2 \\
&\quad + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2} E_1(l,T) E_2(l,T) + \sigma_r\sigma_{w_1}\rho_{rw_1} B(l,T) E_1(l,T) \\
&\quad + \sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) B(l,T) E_2(l,T) \\
&\quad - \theta_r(l) B(l,T) - \theta_{w_1} E_1(l,T) - \theta_{w_2} E_2(l,T) dl.
\end{aligned}
$$

□

In Appendix A we show how the deterministic function θ_r can be derived.

Analogously to the non-defaultable case, the price of a defaultable zero-coupon bond also exhibits an affine term structure.

Theorem 4.3 ***(Price of a Defaultable Zero-Coupon Bond)***
For t <min(T^d, T) the price of a defaultable zero-coupon bond is given by

$$P^d(t,T) = \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T (r(l)+s(l))dl} | \mathcal{F}_t\right] = P^d(t,T,r(t),w_1(t),w_2(t),s(t),u(t)),$$

with

$$P^d(t,T,r,w_1,w_2,s,u) = e^{A^d(t,T)-B^d(t,T)r-C^d(t,T)s-D^d(t,T)u-E_1^d(t,T)w_1-E_2^d(t,T)w_2}$$

and

$$\begin{aligned}
B^d(t,T) &= B(t,T) = \frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right), \\
C^d(t,T) &= \frac{1}{\hat{a}_s}\left(1-e^{-\hat{a}_s(T-t)}\right),
\end{aligned}$$

$$\begin{aligned}
D^d(t,T) &= b_{su}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_u(T-t)}}{\hat{a}_u} + \frac{e^{-\hat{a}_u(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_u-\hat{a}_s}\right), \\
E_1^d(t,T) &= -b_{sw_1}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_1}-\hat{a}_s}\right) \\
&\quad + b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right), \\
E_2^d(t,T) &= -b_{sw_2}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_2}-\hat{a}_s}\right) \\
&\quad + \hat{b}_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right), \\
A^d(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2(B^d(l,T))^2 + \frac{1}{2}\sigma_s^2(C^d(l,T))^2 + \frac{1}{2}\sigma_u^2(D^d(l,T))^2 \\
&\quad + \frac{1}{2}\sigma_{w_1}^2(E_1^d(l,T))^2 + \frac{1}{2}\sigma_{w_2}^2(E_2^d(l,T))^2 \\
&\quad + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^d(l,T)E_2^d(l,T) + \sigma_r\sigma_{w_1}\rho_{rw_1}B^d(l,T)E_1^d(l,T) \\
&\quad + \sigma_s\sigma_u\rho_{su}C^d(l,T)D^d(l,T) + \sigma_s\sigma_{w_1}\rho_{sw_1}C^d(l,T)E_1^d(l,T) \\
&\quad + \sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)B^d(l,T)E_2^d(l,T) \\
&\quad + \sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)C^d(l,T)E_2^d(l,T) \\
&\quad + \sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})B^d(l,T)C^d(l,T) - \theta_r(l)B^d(l,T) \\
&\quad - \theta_s C^d(l,T) - \theta_u D^d(l,T) - \theta_{w_1}E_1^d(l,T) - \theta_{w_2}E_2^d(l,T)dl.
\end{aligned}$$

Proof:
According to Feynman-Kac (see Theorem 2.14) the following differential equation must hold:

$$\begin{aligned}
(r+s)P^d &= P_t^d \\
&+ \left(\theta_r(t) + b_{rw_1}w_1 + \hat{b}_{rw_2}w_2 - \hat{a}_r r\right) P_r^d \\
&+ \left(\theta_{w_1} - \hat{a}_{w_1}w_1\right) P_{w_1}^d \\
&+ \left(\theta_{w_2} - \hat{a}_{w_2}w_2\right) P_{w_2}^d \\
&+ \left(\theta_u - \hat{a}_u u\right) P_u^d \\
&+ \left(\theta_s + b_{su}u - b_{sw_1}w_1 - b_{sw_2}w_2 - \hat{a}_s s\right) P_s^d \\
&+ \frac{1}{2}\Big(\sigma_r^2 P_{rr}^d + \sigma_s^2 P_{ss}^d + \sigma_u^2 P_{uu}^d + \sigma_{w_1}^2 P_{w_1w_1}^d + \sigma_{w_2}^2 P_{w_2w_2}^d \\
&+ 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}P_{w_1w_2}^d + 2\sigma_r\sigma_{w_1}\rho_{rw_1}P_{rw_1}^d \\
&+ 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) P_{rw_2}^d + 2\sigma_s\sigma_u\rho_{su}P_{su}^d \\
&+ 2\sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})P_{sr}^d + 2\sigma_s\sigma_{w_1}\rho_{sw_1}P_{sw_1}^d \\
&+ 2\sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) P_{sw_2}^d\Big).
\end{aligned}$$

Using the affine term structure, we get the following partial derivatives:

$$\begin{array}{lll}
P_t^d = (A_t^d - B_t^d r - (E_1^d)_t w_1 - (E_2^d)_t w_2 - C_t^d s - D_t^d u)\cdot P^d\,, & & \\
P_r^d = -B^d\cdot P^d\,, & P_{w_1w_1}^d = (E_1^d)^2\cdot P^d\,, & P_{w_1w_2}^d = E_1^d E_2^d\cdot P^d\,, \\
P_{w_1}^d = -E_1^d\cdot P^d\,, & P_{w_2w_2}^d = (E_2^d)^2\cdot P^d\,, & P_{sr}^d = B^d C^d\cdot P^d\,, \\
P_{w_2}^d = -E_2^d\cdot P^d\,, & P_{ss}^d = (C^d)^2\cdot P^d\,, & P_{sw_1}^d = C^d E_1^d\cdot P^d\,, \\
P_s^d = -C^d\cdot P^d\,, & P_{uu}^d = (D^d)^2\cdot P^d\,, & P_{sw_2}^d = C^d E_2^d\cdot P^d\,, \\
P_u^d = -D^d\cdot P^d\,, & P_{rw_1}^d = B^d E_1^d\cdot P^d\,, & P_{su}^d = C^d D^d\cdot P^d\,, \\
P_{rr}^d = (B^d)^2\cdot P^d\,, & P_{rw_2}^d = B^d E_2^d\cdot P^d. &
\end{array}$$

Substituting these terms and dividing by $P^d > 0$, we arrive at:

$$\begin{aligned}
r+s &= A_t^d - B_t^d r - (E_1^d)_t w_1 - (E_2^d)_t w_2 - C_t^d s - D_t^d u \\
&+ \left(\theta_r(t) + b_{rw_1}w_1 + \hat{b}_{rw_2}w_2 - \hat{a}_r r\right)(-B^d) \\
&+ \left(\theta_{w_1} - \hat{a}_{w_1}w_1\right)(-E_1^d) \\
&+ \left(\theta_{w_2} - \hat{a}_{w_2}w_2\right)(-E_2^d) \\
&+ \left(\theta_u - \hat{a}_u u\right)(-D^d)
\end{aligned}$$

$$
\begin{aligned}
&+ (\theta_s + b_{su}u - b_{sw_1}w_1 - b_{sw_2}w_2 - \hat{a}_s s)\left(-C^d\right) \\
&+ \frac{1}{2}\Big(\sigma_r^2 (B^d)^2 + \sigma_s^2 (C^d)^2 + \sigma_u^2 (D^d)^2 + \sigma_{w_1}^2 (E_1^d)^2 + \sigma_{w_2}^2 (E_2^d)^2 \\
&+ 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^d E_2^d + 2\sigma_r\sigma_{w_1}\rho_{rw_1}B^d E_1^d \\
&+ 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) B^d E_2^d + 2\sigma_s\sigma_u\rho_{su}C^d D^d \\
&+ 2\sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})B^d C^d + 2\sigma_s\sigma_{w_1}\rho_{sw_1}C^d E_1^d \\
&+ 2\sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) C^d E_2^d\Big).
\end{aligned}
$$

Regrouping the terms, the equation takes on the form:

$$
\begin{aligned}
0 \;=\;& r\left(\hat{a}_r B^d - 1 - B_t^d\right) \\
&+ w_1\left(\hat{a}_{w_1}E_1^d - b_{rw_1}B^d + b_{sw_1}C^d - (E_1^d)_t\right) \\
&+ w_2\left(\hat{a}_{w_2}E_2^d - \hat{b}_{rw_2}B^d + b_{sw_2}C^d - (E_2^d)_t\right) \\
&+ u\left(\hat{a}_u D^d - b_{su}C^d - D_t^d\right) \\
&+ s\left(\hat{a}_s C^d - 1 - C_t^d\right) \\
&+ A_t^d - \theta_r(t)B^d - \theta_s C^d - \theta_u D^d - \theta_{w_1}E_1^d - \theta_{w_2}E_2^d \\
&+ \frac{1}{2}\Big(\sigma_r^2 (B^d)^2 + \sigma_s^2 (C^d)^2 + \sigma_u^2 (D^d)^2 + \sigma_{w_1}^2 (E_1^d)^2 + \sigma_{w_2}^2 (E_2^d)^2 \\
&+ 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^d E_2^d + 2\sigma_r\sigma_{w_1}\rho_{rw_1}B^d E_1^d \\
&+ 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) B^d E_2^d + 2\sigma_s\sigma_u\rho_{su}C^d D^d \\
&+ 2\sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})B^d C^d + 2\sigma_s\sigma_{w_1}\rho_{sw_1}C^d E_1^d \\
&+ 2\sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) C^d E_2^d\Big).
\end{aligned}
$$

We obtain a system of linear differential equations for $A^d, B^d, C^d, D^d, E_1^d$, and E_2^d by comparing the coefficients:

$$
\begin{aligned}
B_t^d &= \hat{a}_r B^d - 1 \\
C_t^d &= \hat{a}_s C^d - 1 \\
D_t^d &= \hat{a}_u D^d - b_{su}C^d \\
(E_1^d)_t &= \hat{a}_{w_1}E_1^d - b_{rw_1}B^d + b_{sw_1}C^d \\
(E_2^d)_t &= \hat{a}_{w_2}E_2^d - \hat{b}_{rw_2}B^d + b_{sw_2}C^d
\end{aligned}
$$

$$\begin{aligned}
-A_t^d &= \frac{1}{2}\Big(\sigma_r^2(B^d)^2 + \sigma_s^2(C^d)^2 + \sigma_u^2(D^d)^2 + \sigma_{w_1}^2(E_1^d)^2 + \sigma_{w_2}^2(E_2^d)^2 \\
&\quad + 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^dE_2^d + 2\sigma_r\sigma_{w_1}\rho_{rw_1}B^dE_1^d \\
&\quad + 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)B^dE_2^d + 2\sigma_s\sigma_u\rho_{su}C^dD^d \\
&\quad + 2\sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})B^dC^d + 2\sigma_s\sigma_{w_1}\rho_{sw_1}C^dE_1^d \\
&\quad + 2\sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)C^dE_2^d\Big) \\
&\quad - \theta_r(t)B^d - \theta_sC^d - \theta_uD^d - \theta_{w_1}E_1^d - \theta_{w_2}E_2^d.
\end{aligned}$$

Since the condition $P^d(T,T) = 1$ must be fulfilled for all $r, w_1, w_2, s, u \in \mathbb{R}$ it holds that $A^d(T,T) = B^d(T,T) = C^d(T,T) = D^d(T,T) = E_1^d(T,T) = E_2^d(T,T) = 0$. By means of the transformation $\tau = T - t$ and the given terminal conditions, the differential equations result in (cf. Theorem 2.15):

$$\begin{aligned}
B^d(t,T) &= B(t,T) = \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(T-t)}\right), \\
C^d(t,T) &= \frac{1}{\hat{a}_s}\left(1 - e^{-\hat{a}_s(T-t)}\right), \\
D^d(t,T) &= e^{-\hat{a}_u(T-t)}\int_0^{T-t} e^{\hat{a}_u l} b_{su} C^d(0,l)\,dl \\
&= b_{su}\frac{1}{\hat{a}_s}\left(\frac{1 - e^{-\hat{a}_u(T-t)}}{\hat{a}_u} + \frac{e^{-\hat{a}_u(T-t)} - e^{-\hat{a}_s(T-t)}}{\hat{a}_u - \hat{a}_s}\right), \\
E_1^d(t,T) &= e^{-\hat{a}_{w_1}(T-t)}\int_0^{T-t} e^{\hat{a}_{w_1} l}\left(b_{rw_1}B^d(0,l) - b_{sw_1}C^d(0,l)\right)dl \\
&= -b_{sw_1}\frac{1}{\hat{a}_s}\left(\frac{1 - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_1} - \hat{a}_s}\right) \\
&\quad + b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1} - \hat{a}_r}\right), \\
E_2^d(t,T) &= e^{-\hat{a}_{w_2}(T-t)}\int_0^{T-t} e^{\hat{a}_{w_2} l}\left(\hat{b}_{rw_2}B^d(0,l) - b_{sw_2}C^d(0,l)\right)dl \\
&= -b_{sw_2}\frac{1}{\hat{a}_s}\left(\frac{1 - e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}(T-t)} - e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_2} - \hat{a}_s}\right) \\
&\quad + \hat{b}_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2} - \hat{a}_r}\right),
\end{aligned}$$

$$
\begin{aligned}
A^d(t,T) \;=\; & \int_t^T \frac{1}{2}\sigma_r^2(B^d(l,T))^2 + \frac{1}{2}\sigma_s^2(C^d(l,T))^2 + \frac{1}{2}\sigma_u^2(D^d(l,T))^2 \\
& + \frac{1}{2}\sigma_{w_1}^2(E_1^d(l,T))^2 + \frac{1}{2}\sigma_{w_2}^2(E_2^d(l,T))^2 \\
& + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^d(l,T)E_2^d(l,T) + \sigma_r\sigma_{w_1}\rho_{rw_1}B^d(l,T)E_1^d(l,T) \\
& + \sigma_s\sigma_u\rho_{su}C^d(l,T)D^d(l,T) + \sigma_s\sigma_{w_1}\rho_{sw_1}C^d(l,T)E_1^d(l,T) \\
& + \sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)B^d(l,T)E_2^d(l,T) \\
& + \sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)C^d(l,T)E_2^d(l,T) \\
& + \sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})B^d(l,T)C^d(l,T) - \theta_r(l)B^d(l,T) \\
& - \theta_s C^d(l,T) - \theta_u D^d(l,T) - \theta_{w_1}E_1^d(l,T) - \theta_{w_2}E_2^d(l,T)dl.
\end{aligned}
$$

□

In the following, we want to test this general framework by specifying and comparing various types of models stemming from this setting.

4.2 The Extended Schmid-Zagst Model

This model is an extension of the three factor model introduced by Schmid & Zagst (2000) where additionally to the short rate r, the short-rate spread s and the uncertainty index u a macroeconomic factor w_1 is incorporated. This factor which acts as an indicator of the economy's state influences both the short rate r and the short-rate spread s. We assume the macroeconomic factor to be positively related to interest rates (i.e. $b_{rw_1} > 0$ with a positive sign in the drift of r) and oppositely to credit spreads (i.e. $b_{sw_1} > 0$ with a negative sign in the drift of s). That is, increasing values of w_1 indicate a healthy economy which is often accompanied by increasing interest rates and decreasing credit spreads. Therefore, the extended model of Schmid and Zagst is a special case of our generalized framework and is derived by setting $\rho_{rw_1} = \rho_{rw_2} = \rho_{sw_1} = \rho_{sw_2} = \rho_{su} = 0$, $\delta = 0$ and by eliminating the second macroeconomic factor w_2 as well as all coefficients with respect to w_2, e.g. $\rho_{w_1w_2}$, b_{rw_2} and b_{sw_2}.
This approach is therefore based on the following stochastic differential equations.

Model 4.4 *Let* $W := (W_r, W_{w_1}, W_u, W_s)'$ *be a four-dimensional Brownian motion on the filtered probability space* $(\Omega, \mathcal{G}, Q, \mathbb{G})$*, then the extended model* *of Schmid and Zagst (SZ4) is given by*

$$\begin{aligned}
dr(t) &= (\theta_r(t) + b_{rw_1} w_1(t) - a_r r(t))\, dt + \sigma_r dW_r(t) \\
dw_1(t) &= (\theta_{w_1} - a_{w_1} w_1(t))\, dt + \sigma_{w_1} dW_{w_1}(t), \\
du(t) &= (\theta_u - a_u u(t))\, dt + \sigma_u dW_u(t), \\
ds(t) &= (\theta_s + b_{su} u(t) - b_{sw_1} w_1(t) - a_s s(t))\, dt + \sigma_s dW_s(t),
\end{aligned}$$

with $a_r, a_{w_1}, a_u, a_s, \sigma_r, \sigma_{w_1}, \sigma_u, \sigma_s, b_{rw_1}, b_{su}, b_{sw_1} > 0$, $\theta_{w_1}, \theta_u, \theta_s \geq 0$ *and* θ_r *being a continuous deterministic function.*

The prices of non-defaultable and defaultable bonds within this approach are as follows:

Lemma 4.5 *In the extended model of Schmid and Zagst (SZ4), the price of a non-defaultable zero-coupon bond is given by*

$$P(t, T, r, w_1) = e^{A(t,T) - B(t,T) r - E_1(t,T) w_1}$$

with

$$\begin{aligned}
B(t,T) &= \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r (T-t)}\right), \\
E_1(t,T) &= b_{rw_1} \frac{1}{\hat{a}_r} \left(\frac{1 - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_r (T-t)}}{\hat{a}_{w_1} - \hat{a}_r} \right), \\
A(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2 (B(l,T))^2 + \frac{1}{2}\sigma_{w_1}^2 (E_1(l,T))^2 - \theta_r(l) B(l,T) \\
&\quad - \theta_{w_1} E_1(l,T)\, dl.
\end{aligned}$$

For a defaultable zero-coupon bond the price is determined by

$$P^d(t, T, r, w_1, s, u) = e^{A^d(t,T) - B^d(t,T) r - C^d(t,T) s - D^d(t,T) u - E_1^d(t,T) w_1}$$

with

$$
\begin{aligned}
B^d(t,T) &= B(t,T) = \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(T-t)}\right), \\
C^d(t,T) &= \frac{1}{\hat{a}_s}\left(1 - e^{-\hat{a}_s(T-t)}\right), \\
D^d(t,T) &= b_{su}\frac{1}{\hat{a}_s}\left(\frac{1 - e^{-\hat{a}_u(T-t)}}{\hat{a}_u} + \frac{e^{-\hat{a}_u(T-t)} - e^{-\hat{a}_s(T-t)}}{\hat{a}_u - \hat{a}_s}\right), \\
E_1^d(t,T) &= -\,b_{sw_1}\frac{1}{\hat{a}_s}\left(\frac{1 - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_1} - \hat{a}_s}\right) \\
&\quad + b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1} - \hat{a}_r}\right), \\
A^d(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2(B^d(l,T))^2 + \frac{1}{2}\sigma_s^2(C^d(l,T))^2 + \frac{1}{2}\sigma_u^2(D^d(l,T))^2 \\
&\quad + \frac{1}{2}\sigma_{w_1}^2(E_1^d(l,T))^2 - \theta_r(l)B^d(l,T) - \theta_s C^d(l,T) - \theta_u D^d(l,T) \\
&\quad - \theta_{w_1}E_1^d(l,T)dl.
\end{aligned}
$$

Schmid et al. (see Antes et al. (2008)) showed that the introduction of a macroeconomic factor improves the power of the Schmid and Zagst framework by comparing the extended version with its preceding three factor version. They obtained as a result that both the levels and the changes of credit spreads could be explained better by this additional factor.
Following the work of Schmid et al. we test five factor models against the extended model of Schmid and Zagst. Furthermore, since Schmid et al. used the growth rate of the nominal GDP as representative of w_1, we want to analyze the impact of other macroeconomic indicators, which are supposed to be good proxies of the economy, on the performance of the extended framework.

4.3 A Further Enhancement of the Schmid-Zagst Model - The Five Factor Approach

Since the introduction of a macroeconomic factor yields promising results in explaining credit spreads and pricing defaultable bonds (see Antes et al. (2008)), the performance could be further improved by a second macroeconomic factor. There can be found various articles in the literatur which analyze the impact of macroeconomic factors on credit spreads and which found that there is more than just one explanatory variable. E.g. Amatoa

& Luisi (2006) analyzed the impact of aggregate risk factors on corporate spreads. These risk factors comprised of macroeconomic data like consumer price index, industrial production and unemployment rates. The authors found several factors which exhibit strong effects on corporate spreads. Wu & Zhang (2008) used a dynamic factor model in order to identify three fundamental risk dimensions namely inflation, real output growth, and financial market volatility. For each risk dimension they summarized several time series and extracted a common factor capturing the systematic dynamics. Then, they linked the fundamental risk dimensions to US Treasury yields and corporate bond spreads.

The model is reached by enhancing the extended four factor model with an additional macroeconomic factor w_2 and by allowing this factor to be correlated with w_1. As before, we set $\rho_{rw_1} = \rho_{rw_2} = \rho_{sw_1} = \rho_{sw_2} = \rho_{su} = 0$ and $\delta = 0$, i.e. $\hat{b}_{rw_2} = b_{rw_2}$.

Model 4.6 *Let $W := (W_r, W_{w_1}, W_{w_2}, W_u, W_s)'$ be a five-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{G}, Q, \mathbb{G})$, then the five factor version of the model of Schmid and Zagst (SZ5) is given by*

$$\begin{aligned}
dr(t) &= (\theta_r(t) + b_{rw_1} w_1(t) + b_{rw_2} w_2(t) - a_r r(t))\, dt + \sigma_r dW_r(t) \\
dw_1(t) &= (\theta_{w_1} - a_{w_1} w_1(t))\, dt + \sigma_{w_1} dW_{w_1}(t), \\
dw_2(t) &= (\theta_{w_2} - a_{w_2} w_2(t))\, dt + \sigma_{w_2} \rho_{w_1 w_2} dW_{w_1}(t) + \sigma_{w_2} \sqrt{1 - \rho_{w_1 w_2}^2} dW_{w_2}(t), \\
du(t) &= (\theta_u - a_u u(t))\, dt + \sigma_u dW_u(t), \\
ds(t) &= (\theta_s + b_{su} u(t) - b_{sw_1} w_1(t) - b_{sw_2} w_2(t) - a_s s(t))\, dt + \sigma_s dW_s(t),
\end{aligned}$$

with $a_r, a_{w_1}, a_{w_2}, a_u, a_s, \sigma_r, \sigma_{w_1}, \sigma_{w_2}, \sigma_u, \sigma_s, b_{rw_1}, b_{su}, b_{sw_1} > 0$, $b_{rw_2}, b_{sw_2} \in \mathbb{R}$, $\theta_{w_1}, \theta_{w_2}, \theta_u, \theta_s \geq 0$, $\rho_{w_1 w_2} \in [-1, 1]$ and θ_r being a continuous deterministic function.

Here, we skip the restrictions regarding the influence of w_2 on the short rate and the short-rate spread, i.e. $b_{rw_2}, b_{sw_2} \in \mathbb{R}$, since there are macroeconomic factors, e.g. inflation, whose impact is not known for sure.
Prices for zero-coupon bonds also exist within this framework and can be derived from the general case, see Theorem 4.2 and Theorem 4.3.

Lemma 4.7 *In the five factor version of the Schmid-Zagst model (SZ5), the price of a non-defaultable zero-coupon bond is given by*

$$P(t, T, r, w_1, w_2) = e^{A(t,T) - B(t,T) r - E_1(t,T) w_1 - E_2(t,T) w_2}$$

with

$$\begin{aligned}
B(t,T) &= \frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right),\\
E_1(t,T) &= b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),\\
E_2(t,T) &= b_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right),\\
A(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2(B(l,T))^2+\frac{1}{2}\sigma_{w_1}^2(E_1(l,T))^2+\frac{1}{2}\sigma_{w_2}^2(E_2(l,T))^2\\
&\quad+\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1(l,T)E_2(l,T)-\theta_r(l)B(l,T)\\
&\quad-\theta_{w_1}E_1(l,T)-\theta_{w_2}E_2(l,T)\,dl.
\end{aligned}$$

For a defaultable zero-coupon bond the price is determined by

$$P^d(t,T,r,w_1,w_2,s,u)=e^{A^d(t,T)-B^d(t,T)r-C^d(t,T)s-D^d(t,T)u-E_1^d(t,T)w_1-E_2^d(t,T)w_2}$$

with

$$\begin{aligned}
B^d(t,T) &= B(t,T)=\frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right),\\
C^d(t,T) &= \frac{1}{\hat{a}_s}\left(1-e^{-\hat{a}_s(T-t)}\right),
\end{aligned}$$

$$\begin{aligned}
D^d(t,T) &= b_{su}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_u(T-t)}}{\hat{a}_u}+\frac{e^{-\hat{a}_u(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_u-\hat{a}_s}\right),\\
E_1^d(t,T) &= -b_{sw_1}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_1}-\hat{a}_s}\right)\\
&\quad+b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),\\
E_2^d(t,T) &= -b_{sw_2}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_2}-\hat{a}_s}\right)\\
&\quad+b_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right),\\
A^d(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2(B^d(l,T))^2+\frac{1}{2}\sigma_s^2(C^d(l,T))^2+\frac{1}{2}\sigma_u^2(D^d(l,T))^2\\
&\quad+\frac{1}{2}\sigma_{w_1}^2(E_1^d(l,T))^2+\frac{1}{2}\sigma_{w_2}^2(E_2^d(l,T))^2\\
&\quad+\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^d(l,T)E_2^d(l,T)-\theta_r(l)B^d(l,T)\\
&\quad-\theta_sC^d(l,T)-\theta_uD^d(l,T)-\theta_{w_1}E_1^d(l,T)-\theta_{w_2}E_2^d(l,T)dl.
\end{aligned}$$

As a special case of this approach, we follow the work of Merz (2007) where w_2 is supposed to be an unobservable factor. Therefore, w_2 takes on the role as an unobservable macroeconomic index which is orthogonal to w_1 and which could be seen as an aggregation of influences on the economy. This means that in Model 4.6 $\rho_{w_1 w_2}$ is set to zero as well as b_{sw_2}. The last condition is necessary since otherwise there would be two unobservable terms in the drift of the short-rate spread which could be hard to discriminate. Hence, for this special case, the pricing formula of a defaultable zero-coupon bond $P^d(t,T)$ simplifies to the one given in Lemma 4.5.

4.4 The Real and Inflation Short-Rate Model

The real and inflation short-rate model was first introduced by Hagedorn et al. (2007)(see also Hagedorn (2005) and Meyer (2005)). It decomposes the short rate r into the real short rate r_R and the inflation short rate r_I:

$$r(t) = r_R(t) + r_I(t)$$

where the real short rate evolves according to the SDE

$$dr_R(t) = (\theta_R(t) + b_{Rw_1} w_1(t) - a_R r_R(t))\, dt + \sigma_R dW_R(t),$$

with positive constants b_{Rw_1}, a_R, σ_R and a continuous, deterministic function $\theta_R(t)$. Furthermore, the inflation short rate follows the SDE

$$dr_I(t) = (\theta_I - a_I r_I(t))\, dt + \sigma_I \rho_{w_1 I} dW_{w_1}(t) + \sigma_I \sqrt{1 - \rho_{w_1 I}^2} dW_I(t),$$

with positive constants a_I, σ_I, a non-negative constant θ_I and independent Brownian motions W_R, W_I and W_{w_1}. In contrast to Hagedorn et al. (2007), where the constant $\rho_{w_1 I}$ was set to zero, we allow $\rho_{w_1 I}$ to be within $[-1, 1]$. As in the models before, w_1 is a macroeconomic factor represented here by the growth rate of the real GDP and satisfies the SDE

$$dw_1(t) = (\theta_{w_1} - a_{w_1} w_1(t))\, dt + \sigma_{w_1} dW_{w_1}(t),$$

with positive constants a_{w_1}, σ_{w_1} and a non-negative constant θ_{w_1}.
This approach also fits in our general framework of a five factor model. If we let the inflation short rate r_I be represented by the second macroeconomic factor w_2, and if we take the process r as the sum of real short rate r_R and inflation short rate r_I, respectively w_2, with

$$\theta_r(t) := \theta_R(t) + \theta_I\,, a_r := a_R\,, b_{rw_1} := b_{Rw_1}\,, W_r := W_R\,,$$

$$b_{rw_2} := a_R - a_I \ (\hat{b}_{rw_2} := \hat{a}_R - \hat{a}_I \text{ with } \delta = 1),$$
$$\sigma_r := \sigma_R \text{ where } \sigma_R \text{ equals the term } \sigma_{\tilde{r}}\sqrt{1-\rho^2_{\tilde{r}w_1}-\rho^2_{\tilde{r}w_2}} \text{ of Section 4.1,}$$

$$\rho_{\tilde{r}w_1} = \frac{\sigma_{w_2}}{\sigma_{\tilde{r}}}\rho_{w_1w_2}\,, \rho_{\tilde{r}w_2} = \frac{\sigma_{w_2}}{\sigma_{\tilde{r}}}\sqrt{1-\rho^2_{w_1w_2}}\,, \rho_{sw_1} = \rho_{sw_2} = \rho_{su} = 0\,,$$

and $\lambda_r := \frac{\lambda_{\tilde{r}}}{\sqrt{1-\rho^2_{\tilde{r}w_1}-\rho^2_{\tilde{r}w_2}}}$, we end up with the following model. [IV] Model 4.8 extends the non-defaultable set-up of Hagedorn et al. (2007) by introducing a firm-specific uncertainty index u and the short-rate spread s.

Model 4.8 *Let $W := (W_r, W_{w_1}, W_{w_2}, W_u, W_s)'$ be a five-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{G}, Q, \mathbb{G})$, then the real and inflation short-rate model (INF) is given by*

$$\begin{aligned}
dr(t) &= (\theta_r(t) + b_{rw_1}w_1(t) + b_{rw_2}w_2(t) - a_r r(t))\,dt + \sigma_r dW_r(t) \\
&\quad + \sigma_{w_2}\rho_{w_1w_2}dW_{w_1}(t) + \sigma_{w_2}\sqrt{1-\rho^2_{w_1w_2}}dW_{w_2}(t) \\
dw_1(t) &= (\theta_{w_1} - a_{w_1}w_1(t))\,dt + \sigma_{w_1}dW_{w_1}(t), \\
dw_2(t) &= (\theta_{w_2} - a_{w_2}w_2(t))\,dt + \sigma_{w_2}\rho_{w_1w_2}dW_{w_1}(t) + \sigma_{w_2}\sqrt{1-\rho^2_{w_1w_2}}dW_{w_2}(t), \\
du(t) &= (\theta_u - a_u u(t))\,dt + \sigma_u dW_u(t), \\
ds(t) &= (\theta_s + b_{su}u(t) - b_{sw_1}w_1(t) - b_{sw_2}w_2(t) - a_s s(t))\,dt + \sigma_s dW_s(t),
\end{aligned}$$

with $a_r, a_{w_1}, a_{w_2}, a_u, a_s, \sigma_r, \sigma_{w_1}, \sigma_{w_2}, \sigma_u, \sigma_s, b_{rw_1}, b_{su}, b_{sw_1} > 0$, $b_{rw_2}, b_{sw_2} \in \mathbb{R}$, $\theta_{w_1}, \theta_{w_2}, \theta_u, \theta_s \geq 0$, $\rho_{w_1w_2} \in [-1,1]$ and θ_r being a continuous deterministic function.

The pricing formulas of zero-coupon bonds for this set-up are a special case of Theorem 4.2 and Theorem 4.3.

Lemma 4.9 *In the real and inflation short-rate model (INF) the price of a non-defaultable zero-coupon bond is given by*

$$\begin{aligned}
P(t,T,r,w_1,w_2) &= e^{A(t,T)-B(t,T)r-E_1(t,T)w_1-E_2(t,T)w_2} \\
&= e^{A(t,T)-B(t,T)r_R-E_1(t,T)w_1-(E_2(t,T)+B(t,T))w_2} \\
&= P(t,T,r_R,w_1,w_2)
\end{aligned}$$

with

$$B(t,T) = \frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right),$$

[IV] We use here the notation $\tilde{r}$ in order to indicate the theoretical framework of Section 4.1.

$$
\begin{aligned}
E_1(t,T) &= b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),\\
E_2(t,T) &= \hat{b}_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right),\\
A(t,T) &= \int_t^T \frac{1}{2}\left(\sigma_r^2+\sigma_{w_2}^2\right)(B(l,T))^2+\frac{1}{2}\sigma_{w_1}^2(E_1(l,T))^2+\frac{1}{2}\sigma_{w_2}^2(E_2(l,T))^2\\
&\quad+\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1(l,T)\left(B(l,T)+E_2(l,T)\right)+\sigma_{w_2}^2B(l,T)E_2(l,T)\\
&\quad-\theta_r(l)B(l,T)-\theta_{w_1}E_1(l,T)-\theta_{w_2}E_2(l,T)\,dl.
\end{aligned}
$$

For a defaultable zero-coupon bond the price is determined by

$$
\begin{aligned}
P^d(t,T,r,w_1,w_2,s,u) &= e^{A^d(t,T)-B^d(t,T)r-C^d(t,T)s-D^d(t,T)u-E_1^d(t,T)w_1-E_2^d(t,T)w_2}\\
&= e^{A^d(t,T)-B^d(t,T)r_R-C^d(t,T)s-D^d(t,T)u-E_1^d(t,T)w_1}\\
&\quad\cdot e^{-\left(E_2^d(t,T)+B^d(t,T)\right)w_2}\\
&= P^d(t,T,r_R,w_1,w_2,s,u)
\end{aligned}
$$

with

$$
\begin{aligned}
B^d(t,T) &= B(t,T)=\frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right),\\
C^d(t,T) &= \frac{1}{\hat{a}_s}\left(1-e^{-\hat{a}_s(T-t)}\right),\\
D^d(t,T) &= b_{su}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_u(T-t)}}{\hat{a}_u}+\frac{e^{-\hat{a}_u(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_u-\hat{a}_s}\right),\\
E_1^d(t,T) &= -b_{sw_1}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_1}-\hat{a}_s}\right)\\
&\quad+b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),\\
E_2^d(t,T) &= -b_{sw_2}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_2}-\hat{a}_s}\right)\\
&\quad+\hat{b}_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right),\\
A^d(t,T) &= \int_t^T \frac{1}{2}\left(\sigma_r^2+\sigma_{w_2}^2\right)(B^d(l,T))^2+\frac{1}{2}\sigma_s^2(C^d(l,T))^2+\frac{1}{2}\sigma_u^2(D^d(l,T))^2\\
&\quad+\frac{1}{2}\sigma_{w_1}^2(E_1^d(l,T))^2+\frac{1}{2}\sigma_{w_2}^2(E_2^d(l,T))^2+\sigma_{w_2}^2B^d(l,T)E_2^d(l,T)\\
&\quad+\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^d(l,T)\left(B^d(l,T)+E_2^d(l,T)\right)-\theta_r(l)B^d(l,T)\\
&\quad-\theta_sC^d(l,T)-\theta_uD^d(l,T)-\theta_{w_1}E_1^d(l,T)-\theta_{w_2}E_2^d(l,T)dl.
\end{aligned}
$$

We consider this framework for two cases. First, we set $\rho_{w_1 w_2} = 0$ following the work of Hagedorn et al. (2007). Then, as a second step we allow $\rho_{w_1 w_2}$ to be non-zero.

4.5 A Simplified Version of the General Set-Up - The Correlated Five Factor Approach

This model is a combination of all previously introduced models and additionally is closely related to the general framework introduced in the first section of this chapter. It assumes both the short rate r and the short-rate spread s to be dependent on an observable macroeconomic factor and an unobservable factor aggregating information inherent in the market. Furthermore, both SDEs are driven by the Brownian motions of all factors represented in the drift term as it is done in the short-rate model with real and inflation short rates (see Model 4.8). We obtain this model from the general framework by assuming w_2 to be the unobservable factor of the short rate as described in Section 4.3 and by setting $\rho_{w_1 w_2} = 0$, $\delta = 0$ ($\hat{b}_{rw_2} = b_{rw_2}$), $b_{sw_2} = 0$, and $\rho_{sw_2} = 0$. The last assumptions are due to the factor u already being unobservable.

Model 4.10 *Let $W := (W_r, W_{w_1}, W_{w_2}, W_u, W_s)'$ be a five-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{G}, Q, \mathbb{G})$, then the correlated five factor model (5corr) is given by*

$$\begin{aligned}
dr(t) &= (\theta_r(t) + b_{rw_1} w_1(t) + b_{rw_2} w_2(t) - a_r r(t))\, dt \\
&\quad + \sigma_r \sqrt{1 - \rho_{rw_1}^2 - \rho_{rw_2}^2} dW_r(t) + \sigma_r \rho_{rw_1} dW_{w_1}(t) + \sigma_r \rho_{rw_2} dW_{w_2}(t), \\
dw_1(t) &= (\theta_{w_1} - a_{w_1} w_1(t))\, dt + \sigma_{w_1} dW_{w_1}(t), \\
dw_2(t) &= (\theta_{w_2} - a_{w_2} w_2(t))\, dt + \sigma_{w_2} dW_{w_2}(t), \\
du(t) &= (\theta_u - a_u u(t))\, dt + \sigma_u dW_u(t), \\
ds(t) &= (\theta_s + b_{su} u(t) - b_{sw_1} w_1(t) - a_s s(t))\, dt \\
&\quad + \sigma_s \sqrt{1 - \rho_{su}^2 - \rho_{sw_1}^2} dW_s(t) + \sigma_s \rho_{su} dW_u(t) + \sigma_s \rho_{sw_1} dW_{w_1}(t),
\end{aligned}$$

with $a_r, a_{w_1}, a_{w_2}, a_u, a_s, \sigma_r, \sigma_{w_1}, \sigma_{w_2}, \sigma_u, \sigma_s, b_{rw_1}, b_{rw_2}, b_{su}, b_{sw_1} > 0$, $\theta_{w_1}, \theta_{w_2}, \theta_u, \theta_s \geq 0$, $\rho_{rw_1}, \rho_{rw_2}, \rho_{su}, \rho_{sw_1} \in [-1, 1]$ and θ_r being a continuous deterministic function.

The pricing formulas for non-defaultable and defaultable bonds are similar to the ones presented in Section 4.1.

Lemma 4.11 *In the correlated five factor model (5corr) the price of a non-defaultable zero-coupon bond is given by*

$$P(t,T,r,w_1,w_2) \;=\; e^{A(t,T)-B(t,T)r-E_1(t,T)w_1-E_2(t,T)w_2}$$

with

$$\begin{aligned}
B(t,T) &= \frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right),\\
E_1(t,T) &= b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),\\
E_2(t,T) &= b_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right),\\
A(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2(B(l,T))^2+\frac{1}{2}\sigma_{w_1}^2(E_1(l,T))^2+\frac{1}{2}\sigma_{w_2}^2(E_2(l,T))^2\\
&\quad +\sigma_r\sigma_{w_1}\rho_{rw_1}B(l,T)E_1(l,T)\\
&\quad +\sigma_r\sigma_{w_2}\rho_{rw_2}B(l,T)E_2(l,T)\\
&\quad -\theta_r(l)B(l,T)-\theta_{w_1}E_1(l,T)-\theta_{w_2}E_2(l,T)dl.
\end{aligned}$$

For a defaultable zero-coupon bond the price is determined by

$$P^d(t,T,r,w_1,w_2,s,u) \;=\; e^{A^d(t,T)-B^d(t,T)r-C^d(t,T)s-D^d(t,T)u-E_1^d(t,T)w_1-E_2^d(t,T)w_2}$$

with

$$\begin{aligned}
B^d(t,T) &= B(t,T)=\frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right),\\
C^d(t,T) &= \frac{1}{\hat{a}_s}\left(1-e^{-\hat{a}_s(T-t)}\right),\\
D^d(t,T) &= b_{su}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_u(T-t)}}{\hat{a}_u}+\frac{e^{-\hat{a}_u(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_u-\hat{a}_s}\right),\\
E_1^d(t,T) &= -b_{sw_1}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_1}-\hat{a}_s}\right)\\
&\quad +b_{rw_1}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),
\end{aligned}$$

$$
\begin{aligned}
E_2^d(t,T) &= E_2(t,T) = b_{rw_2}\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right), \\
A^d(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2(B^d(l,T))^2 + \frac{1}{2}\sigma_s^2(C^d(l,T))^2 + \frac{1}{2}\sigma_u^2(D^d(l,T))^2 \\
&\quad + \frac{1}{2}\sigma_{w_1}^2(E_1^d(l,T))^2 + \frac{1}{2}\sigma_{w_2}^2(E_2^d(l,T))^2 \\
&\quad + \sigma_r\sigma_{w_1}\rho_{rw_1}B^d(l,T)E_1^d(l,T) + \sigma_r\sigma_{w_2}\rho_{rw_2}B^d(l,T)E_2^d(l,T) \\
&\quad + \sigma_s\sigma_u\rho_{su}C^d(l,T)D^d(l,T) + \sigma_s\sigma_{w_1}\rho_{sw_1}C^d(l,T)E_1^d(l,T) \\
&\quad + \sigma_r\sigma_s\rho_{rw_1}\rho_{sw_1}B^d(l,T)C^d(l,T) - \theta_r(l)B^d(l,T) \\
&\quad - \theta_sC^d(l,T) - \theta_uD^d(l,T) - \theta_{w_1}E_1^d(l,T) - \theta_{w_2}E_2^d(l,T)dl.
\end{aligned}
$$

4.6 Summary of Models

Table 4.1 and Table 4.2 outline all used models and parameter settings within this thesis. In the following, the extended model of Schmid and Zagst will be abbreviated by SZ4, the enhancement of this model by SZ5. Both frameworks will be further differentiated by the used macroeconomic factors, e.g. gross domestic product (GDP) or inflation (CPI), or by the unobservability of the newly introduced factor, i.e. SZ5 versus SZ5u. Also within the real and inflation model there will be two settings, namely one assuming uncorrelated macroeconomic factors (INF) and another abandoning this assumption (INFcorr). Furthermore, the correlated five factor model, which is a simplified version of the general set-up, will also be presented within these tables by the label 5corr.

Altogether, the newly introduced models can be summarized as follows:
The correlated five factor model (5corr, see Model 4.10, page 50) is the most general set-up because it assumes two macroeconomic factors (w_1 observable, w_2 non-observable) driving the short rate r. For the short-rate spread s it allows one observable macroeconomic factor (w_1) and one unobservable firm-specific factor (u). Additionally, the dynamics of the short rate r and the short-rate spread s make use of the Brownian shocks of the macroeconomic and firm-specific factors.
Omitting the Brownian shocks of the macroeconomic and firm-specific factors in the dynamics of the short rate r and the short-rate spread s leads us to the five factor approach of the Schmid-Zagst framework (see Model 4.6, page 45). Again, we incorporate two macroeconomic factors (w_1 and w_2). In the setting of SZ5 we assume both macroeconomic factors to be observable and both entering the drift of the short rate r and the short-rate spread s. The

setting SZ5u works with an observable (w_1) and an unobservable macroeconomic factor (w_2) in the short rate r and an observable macroeconomic (w_1) and an unobservable firm-specific factor (u) in the short-rate spread s.
Further, relaxing the assumption of a second macroeconomic factor driving the short rate r leads us to the four factor version of the Schmid-Zagst framework (SZ4, see Model 4.4, page 42). Here, only one observable macroeconomic factor (w_1) enters the short rate r and the short-rate spread s.
Finally, the real and inflation short-rate model with its two settings (INF and INFcorr, see Model 4.8, page 48) also assumes two macroeconomic factors (w_1 and r_I), where the second is thought to be unobservable and stems from an additional set of market data (inflation linked bonds) on which the model is calibrated. INFcorr assumes the second macroeconomic factor, the so-called inflation short rate r_I, to be also driven from the Brownian shocks of the observable macroeconomic factor (w_1). The variant called INF does not incorporate these shocks. Like 5corr, the real and inflation short-rate model makes use of several Brownian shocks driving the short rate r. But unlike 5corr, it does not incorporate any additional Brownian shocks in the short-rate spread s. Therefore, INF and INFcorr are similar to the SZ5 setup.

The next sections are dedicated to the calibration and the comparison of the above introduced models. First, we want to analyze which observable macroeconomic factor is the best input. For this purpose, we use the most simple of the above models namely the extended model of Schmid and Zagst (SZ4) which contains only one observable macroeconomic factor (w_1). Secondly, we further analyze if the introduction of a second macroeconomic factor improves the pricing ability of the models. Especially, we study if the observability of the second macroeconomic factor and correlated Brownian shocks have an impact on the pricing.

$$dr(t) = (\theta_r(t) + b_{rw_1}w_1(t) + b_{rw_2}w_2(t) - a_r r(t))\,dt + \sigma_r\sqrt{1-\rho^2_{rw_1}-\rho^2_{rw_2}}dW_r(t) + \sigma_r\rho_{rw_1}dW_{w_1}(t) + \sigma_r\rho_{rw_2}dW_{w_2}(t)$$

	$\theta_r(t)$	b_{rw_1}	b_{rw_2}	a_r	σ_r	ρ_{rw_1}	ρ_{rw_2}	
SZ4	$\in\mathbb{R}$	>0 $(\in\mathbb{R})$	$:=0$	>0	>0	$:=0$	$:=0$	one m.f.
SZ5	$\in\mathbb{R}$	>0	$\in\mathbb{R}$	>0	>0	$:=0$	$:=0$	two m.f.s
SZ5u	$\in\mathbb{R}$	>0	$:=1$	>0	>0	$:=0$	$:=0$	2nd m.f. unobs.
INF	$:=\theta_R(t)+\theta_I$ $\in\mathbb{R}$	$:=b_{Rw_1}$ >0	$:=a_R-a_I$ $\in\mathbb{R}$	$:=a_R$ >0	$:=\frac{\sigma_R}{\sqrt{1-\rho^2_{rw_1}-\rho^2_{rw_2}}}$ >0	$:=\frac{\sigma_{w_2}}{\sigma_r}\rho_{w_1w_2}$ $=0$	$:=\frac{\sigma_{w_2}}{\sigma_r}\sqrt{1-\rho^2_{w_1w_2}}$ >0	2nd m.f. unobs.
INFcorr	$:=\theta_R(t)+\theta_I$ $\in\mathbb{R}$	$:=b_{Rw_1}$ >0	$:=a_R-a_I$ $\in\mathbb{R}$	$:=a_R$ >0	$:=\frac{\sigma_R}{\sqrt{1-\rho^2_{rw_1}-\rho^2_{rw_2}}}$ >0	$:=\frac{\sigma_{w_2}}{\sigma_r}\rho_{w_1w_2}$ $\in[-1,1]$	$:=\frac{\sigma_{w_2}}{\sigma_r}\sqrt{1-\rho^2_{w_1w_2}}$ >0	2nd m.f. unobs.
5corr	$\in\mathbb{R}$	>0	$:=1$	>0	>0	$\in[-1,1]$	$\in[-1,1]$	2nd m.f. unobs.

$$dw_1(t) = (\theta_{w_1} - a_{w_1}w_1(t))\,dt + \sigma_{w_1}dW_{w_1}(t)$$

	θ_{w_1}	a_{w_1}	σ_{w_1}	
SZ4	≥ 0	>0	>0	m.f. given by GDPn, GDPr, CPI, IP, Prod, CILI, CICI
SZ5	≥ 0	>0	>0	m.f. given by GDPr
SZ5u	≥ 0	>0	>0	m.f. given by GDPr
INF	≥ 0	>0	>0	m.f. given by GDPr
INFcorr	≥ 0	>0	>0	m.f. given by GDPr
5corr	≥ 0	>0	>0	m.f. given by GDPr

Table 4.1: Assumptions regarding the parameters for the different models which are derived from the general framework in Section 4.1. Macroeconomic factor is abbreviated by m.f. and unobservable by unobs.

$$dw_2(t) = (\theta_{w_2} - a_{w_2} w_2(t))\,dt + \sigma_{w_2}\rho_{w_1 w_2} dW_{w_1}(t) + \sigma_{w_2}\sqrt{1-\rho^2_{w_1 w_2}}\,dW_{w_2}(t)$$

	θ_{w_2}	a_{w_2}	σ_{w_2}	$\rho_{w_1 w_2}$	
SZ4	$:= 0$	$:= 0$	$:= 0$	$:= 0$	no 2nd m.f.
SZ5	≥ 0	> 0	> 0	$\in [-1,1]$	m.f. given by CPI
SZ5u	≥ 0	> 0	> 0	$:= 0$	m.f. unobs.
INF	$:= \theta_I \geq 0$	$:= a_I > 0$	$:= \sigma_I > 0$	$:= 0$	m.f. unobs.
INFcorr	$:= \theta_I \geq 0$	$:= a_I > 0$	$:= \sigma_I > 0$	$\in [-1,1]$	m.f. unobs.
5corr	≥ 0	> 0	> 0	$:= 0$	m.f. unobs.

$$du(t) = (\theta_u - a_u u(t))\,dt + \sigma_u dW_u(t)$$

	θ_u	a_u	σ_u	
	≥ 0	> 0	> 0	unobs. in all models

$$ds(t) = (\theta_s + b_{su}u(t) - b_{sw_1}w_1(t) - b_{sw_2}w_2(t) - a_s s(t))\,dt + \sigma_s\rho_{su}dW_u(t)$$
$$+\sigma_s\sqrt{1-\rho^2_{su}-\rho^2_{sw_1}-\rho^2_{sw_2}}\,dW_s(t) + \sigma_s\rho_{sw_1}dW_{w_1}(t) + \sigma_s\rho_{sw_2}dW_{w_2}(t)$$

	θ_s	b_{su}	b_{sw_1}	b_{sw_2}	a_s	σ_s	ρ_{su}	ρ_{sw_1}	ρ_{sw_2}	
SZ4	≥ 0	$:= 1$	$> 0\,(\in \mathbb{R})$	$:= 0$	> 0	> 0	$:= 0$	$:= 0$	$:= 0$	one m.f.
SZ5	≥ 0	$:= 1$	> 0	$\in \mathbb{R}$	> 0	> 0	$:= 0$	$:= 0$	$:= 0$	two m.f.s
SZ5u	≥ 0	$:= 1$	> 0	$:= 0$	> 0	> 0	$:= 0$	$:= 0$	$:= 0$	w/o 2nd m.f.
INF	≥ 0	$:= 1$	> 0	$\in \mathbb{R}$	> 0	> 0	$:= 0$	$:= 0$	$:= 0$	two m.f.s
INFcorr	≥ 0	$:= 1$	> 0	$\in \mathbb{R}$	> 0	> 0	$:= 0$	$:= 0$	$:= 0$	two m.f.s
5corr	≥ 0	$:= 1$	> 0	$:= 0$	> 0	> 0	$\in [-1,1]$	$\in [-1,1]$	$:= 0$	w/o 2nd m.f.

Table 4.2: Assumptions regarding the parameters for the different models which are derived from the general framework in Section 4.1. Macroeconomic factor is abbreviated by m.f. and unobservable by unobs.

4.7 Calibrating the Models to Market Data

In this section we calibrate the above mentioned models on given US data for an insample period from January 1 1999 to December 27 2002. The calibration is done in several steps. First, the parameters of the observable macroeconomic factors are estimated. Then, in the second step we calibrate the short-rate models on non-defaultable zero rates. The last step consists of the estimation of the parameters for the short-rate spreads by means of defaultable zero rates. For all estimations we use the software package S-PLUS finmetrics whereas the optimization is mainly based on a combination of Downhill Simplex and Simulated Annealing Algorithm described in Press, Teukolsky, Vetterling & Flannery (1992).

Estimating the parameters of the observable macroeconomic factor
Since we use observable data for the macroeconomic factor w_1, respectively w_2 in the SZ5 framework, and since the SDEs of these factors do not depend on any unobservable processes, we use a maximum likelihood estimation procedure to determine those parameters.

The solution (see Theorem 2.11) of w_1's SDE is

$$\begin{aligned} w_1(t_{k+1}) &= e^{-a_{w_1}\Delta t_{k+1}} w_1(t_k) + \int_{t_k}^{t_{k+1}} e^{-a_{w_1}(t_{k+1}-l)}\theta_{w_1} dl \\ &\quad + \int_{t_k}^{t_{k+1}} e^{-a_{w_1}(t_{k+1}-l)}\sigma_{w_1} dW_{w_1}(l) \end{aligned}$$

with $\Delta t_{k+1} := t_{k+1} - t_k$. Thus, w_1 conditioned on a previous realisation follows a normal distribution

$$\begin{aligned} & w_1(t_{k+1})|w_1(t_k) \\ &\sim \mathcal{N}\left(e^{-a_{w_1}\Delta t_{k+1}} w_1(t_k) + \frac{\theta_{w_1}}{a_{w_1}}(1 - e^{-a_{w_1}\Delta t_{k+1}}), \frac{\sigma_{w_1}^2}{2a_{w_1}}(1 - e^{-2a_{w_1}\Delta t_{k+1}})\right), \end{aligned}$$

and the likelihood function L is given by

$$L(\theta_{w_1}, a_{w_1}, \sigma_{w_1}) = \prod_{i=1}^{m} f_{w_1(t_i)|w_1(t_{i-1})},$$

where $f_{w_1(t_i)|w_1(t_{i-1})}$ denotes the conditional density of $w_1(t_i)$ given $w_1(t_{i-1})$ and m is the length of the time series w_1.

Within the SZ4 framework (see Model 4.4) we test seven different macroeconomic factors as representatives for w_1. All of those factors are commonly supposed to have good predicting power with respect to the state of the economy.

(i) Nominal Gross Domestic Product (GDPn)
The gross domestic product is a measure of total production and total consumption of goods and services within the United States. Hence, it gives the most comprehensive picture of the power of the U.S. economy. Its value is published quarterly with a delay of one quarter, i.e. it is finally known at the end of the following quarter. Therefore we take into account a lag of 6 month in our calibration procedure.
The growth rate of GDP is used e.g. by Bonfim (2009) who analyzes empirically the determinants of corporate credit default taking into account firm-specific and macroeconomic information. The obtained results suggest that the GDP growth rate belongs to the most important ones within the group of all considered variables.
Furthermore, the growth rate of GDP is incorporated in several studies analyzing the impact of macroeconomic variables on credit risk and sovereign ratings: Glen (2005) finds with the help of GDP growth rates a strong link between macroeconomic conditions and the ability to service debt. Hilscher & Nosbusch (2010) investigate the impact of macroeconomic fundamentals on sovereign credit spreads with the GDP being the main input in form of its growth, its volatility and several ratios, e.g. debt/GDP.

(ii) Real Gross Domestic Product (GDPr)
The real gross domestic product is adjusted for price changes in order to measure the GDP, respectively the production within the United States, regardless of changes in the purchasing power. The publication follows the same schedule as for the nominal GDP. Thus, we also consider a time lag of 2 quarters. Both the nominal and the real GDP are published by the U.S. Department of Commerce: Bureau of Economic Analysis[V].
As mentioned above, the growth rate of GDP is often used in empirical studies regarding credit risk and ratings. Some explicitly state the real GDP as input variable. E.g. Rowland (2005) incorporates the real GDP growth rate in his study of determinants of ratings, creditworthiness and spreads for emerging market sovereign debt. One of his

[V] http://www.bea.gov

findings is that the real GDP growth rate seems to have a significant impact on spreads.

(iii) Consumer Price Index (CPI)
The consumer price index measures the development of the average price of goods and services consumed by households. Thus, its percentage change indicates inflation. Since around 80 % of the U.S. population lives in urban areas, the CPI-U is the most popular representative of the CPI. A further differentiation within the CPI is its value apart from prices of energy and food. Since those products lead to a high volatility within the CPI time series and often overlap long-term trends, we use the index called: "Consumer Price Index for All Urban Consumers: All Items Less Food & Energy" which is published by the U.S. Department of Labor: Bureau of Labor Statistics[VI]. The CPI is published on a monthly basis with a delay of two and a half months. Therefore, we incorporate the CPI with a lag of three months in our calibration.
Some of the above works also make use of the CPI next to the GDP, e.g. Glen (2005) and Rowland (2005). Additionally, Ang & Piazzesi (2003) incorporate the CPI in their analysis of macro variables and their effect on bond prices and on the dynamics of the yield curve. Cantor & Packer (1996) study determinants of sovereign ratings and find that inflation belongs to the group of factors which seem to play an important role.

(iv) Industrial Production (IP)
This production index measures real output. Since the majority of variation in the national output of the U.S. is due to the industry sector, this index indicates structural developments in the U.S. economy. It is released by the Board of Governors of the Federal Reserve System[VII] with a monthly frequency and a time delay of 1 and a half months including revisions for the previous 3 months. Hence, we allow for a time lag of 3 months which is in line with the above indices whose final publications are also preceded by preliminary reports.
The growth rate in industrial production is used by Figlewski et al. (2012) who analyze reduced-form models by allowing the hazard rate to depend on firm-specific factors and macroeconomic conditions. They include the industrial production as a factor that indicates the direction where the economy is moving to. They also claim that the growth rate

[VI] http://www.bls.gov
[VII] http://www.federalreserve.gov

in industrial production might be a better measure than the growth rate in real GDP since the latter covers all economic activity including sectors which may be unrelated to corporate credit conditions.
Also, Ang & Piazzesi (2003) include the growth rate of industrial production in their study of a term structure model with inflation and economic growth factors. Here, IP is assumed to capture real activity. Furthermore, Krishnan et al. (2005) and Krishnan et al. (2010) use the growth rate in industrial production for their analysis of changes in credit spreads and their study in predicting future firm-level credit spreads.

(v) Productivity (Prod)
The most often used measure of productivity within the United States is the so-called labor productivity which determines the output per hour of all persons. Its importance stems from the fact that labor costs are easily identified and account for the majority of the output's value. The most comprehensive measure of productivity is that of the business sector whose output covers about 80 % of the GDP. Hence, its growth is strongly correlated with the growth of the GDP. The publication of the labor productivity is every quarter by the U.S. Department of Labor: Bureau of Labor Statistics. Like the GDP this index is released at the end of the following quarter which leads to a lag of 6 months in our analysis.

(vi) Composite Index Of Leading Indicators (CILI)
The composite index of leading indicators (CILI) is an aggregate of ten economic releases which all show patterns that are related to the business cycle, e.g average weekly hours worked in the manufacturing industries as a predictor for changes in unemployment, manufacturer's new orders for consumer goods/materials indicating future revenues, S&P 500 and interest rate spread reflecting investors' expectations about the economy and changes in the yield curve (see TCB (2001)). This index of leading indicators tries to cover the overall state of the macroeconomy and to reveal common turning point patterns within the series of indicators in order to judge the future state of the economy, i.e. the next six to nine months. By aggregating several economic indicators it gives a summary of the economy and additionally decreases the impact of volatility given by a single indicator. The composite index of leading indicators is published by The Conference Board [VIII] [IX] at the end of

[VIII] http://www.conference-board.org and http://tcb-indicators.org
[IX] We got the time series from Reuters using its RIC aUSCLEAD/A.

every month with a lag of one month. Since several indicators within the composite index are also published with a timing lag and are hence represented by projected data, the composite index allows for revisions of the most recent months. Consistent to the above data, we use a three month publication lag.
Huang & Kong (2003) examine determinants of corporate bond credit spreads by using explanatory variables which capture different aspects of credit risk, e.g. default rates, interest rates, equity market factors and macroeconomic indicators. Their main findings are that also variables like the Conference Board's composite indices of indicators, which have not been used before in the literature, have significant explaining power for credit spread changes.

(vii) Composite Index Of Coincident Indicators (CICI)
The composite index of coincident indicators (CICI) is also published by The Conference Board[X]. The main purpose of this index is to describe the current state of the economy. It is composed by four individual indicators which are said to be in-step with the current economic cycle. These indicators are employees on non-agricultural payrolls reflecting actual changes in hiring and firing, personal income less transfer payments measuring the general health of the economy, index of industrial production which historically captured most of movements in total industrial output, and manufacturing and trade sales measuring real total spending. Like CILI, the composite index of coincident indicators is published on a monthly basis including available data as well as estimates. Hence, this index will also be revised in the following months when the actual data of the underlying indicators are finally published. In order to justify the publishing delay of almost 2 months and the revisions we allow for a lag of 3 months.

The model is calibrated on weekly data, therefore we need to break down the given macroeconomic data with a frequency of 1 respectively 3 months to a weekly time series. This is done by means of the interpolation used for inflation-linked bonds[XI]:

$$\omega(t_m) := \omega_{m-3} + \frac{t_m - 1}{d(m)}(\omega_{m-2} - \omega_{m-3}),$$

where $d(m)$ indicates the number of days in the corresponding month m, respectiveley quarter, t_m the actual date where we want to get a value of ω with

[X] We got the time series from Reuters using its RIC aUSCOINDIF/A.

[XI] For further information refer to e.g. Agence France Trésor (http://www.aft.gouv.fr/)

$0 \leq t_m \leq d(m)$ and ω_m the published index value for the month (quarter) m. The lag of one month (quarter) inherent in the above interpolation is due to the fact that the index value ω_m for the month (quarter) m will be valid for the whole month (quarter), i.e. starting from the first day and lasting until the last day of the month (quarter), but will be available at the earliest on some day in the following month (quarter) $m+1$. Since it is unlikely that the publishment is on the very first day of the following month (quarter) and therefore, it cannot be assumed that the index value ω_m of the month (quarter) before is already known on all days of the next month (quarter) $m+1$. If the lag between the end of the respective month (quarter) and the publishment of its index value is even longer, the interpolated index value is obtained by

$$\omega(t_m) := \omega_{m-\tilde{d}-2} + \frac{t_m - 1}{d(m)}(\omega_{m-\tilde{d}-1} - \omega_{m-\tilde{d}-2}),$$

with $\tilde{d}$ denoting the lag between the end of the period for which the index value is valid and its publishment.

Afterwards, we need to calculate growth rates for the respective macroeconomic data. As the original GDP time series is released quarterly we calculate every week the growth rate with respect to 3 months. For the CPI we determine annual growth rates because the index is published with an accuracy of just one decimal place and we observed that the values of the index do not change for several months. In order to prevent low growth rates and a fluctuation around zero we again use annual growth rates for industrial production and the composite indices since changes for months as well as quarters are negligible. For the growth rate in productivity we proceed as with the GDP and determine growth rates with respect to a quarter.

Although the whole model is calibrated on weekly data, we only use monthly data for calibrating the macroeconomic factors. This is done in order to avoid a possibly high autocorrelation within the interpolated data set. The results of the maximum likelihood estimation for the macroeconomic factors are given in Table 4.3. In all cases, the given data seem to fit into our model assumption since the mean reversion levels which are determined by $\frac{\theta_{w_1}}{a_{w_1}}$ are near the corresponding empirical means.

Estimating the parameters of the short rate r

In order to get the parameters of the short rate $(a_r, b_r, \sigma_r, \rho_{rw_1}, \rho_{rw_2}, \lambda_r, \lambda_{w_1}, \lambda_{w_2})$ we use the Kalman filtering method (see Section 2.4) which requires a state space form consisting of a measurement equation and a transition equation. The measurement equation is derived from the affine relationship between

	θ_{w_1}	a_{w_1}	σ_{w_1}	mean reversion level	empirical mean
GDPn	0.0168	1.387	0.0075	1.21%	1.17%
GDPr	0.0091	1.338	0.0084	0.69%	0.70%
CPI	0.0217	0.914	0.0033	2.38%	2.38%
IP	0.0049	0.311	0.0359	1.59%	1.53%
Prod	0.0131	1.800	0.0129	0.72%	0.78%
CILI	0.0035	0.1793	0.0200	1.94%	1.99%
CICI	0.0015	0.0846	0.0084	1.78%	1.57%

Table 4.3: Estimated parameters for the growth rates of different macroeconomic factors, namely the nominal gross domestic product (GDPn), the real gross domestic product (GDPr), the consumer price index (CPI), the industrial production (IP), the productivity (Prod) and the composite indices of leading (CILI) and coincident (CICI) indicators.

zero rates and the unobservable factor r:

$$R(t_k,T) = -\frac{1}{T-t_k}\ln P(t_k,T) = a_1(t_k,T) + b_1(t_k,T)r(t_k),$$

with $a_1(t_k,T) = -\frac{A(t_k,T)}{T-t_k} + \frac{E_1(t_k,T)}{T-t_k}w_1(t_k) + \frac{E_2(t_k,T)}{T-t_k}w_2(t_k)$ and $b_1(t_k,T) = \frac{B(t_k,T)}{T-t_k}$. Thus, we get for the measurement equation

$$\begin{pmatrix} R(t_k,t_k+\tau_1) \\ \vdots \\ R(t_k,t_k+\tau_N) \end{pmatrix} = \begin{pmatrix} a_1\left(t_k,t_k+\tau_1\right) \\ \vdots \\ a_1\left(t_k,t_k+\tau_N\right) \end{pmatrix} + \begin{pmatrix} b_1\left(0,\tau_1\right) \\ \vdots \\ b_1\left(0,\tau_N\right) \end{pmatrix} \cdot r(t_k) + \epsilon_k$$

where $\tau_1,...,\tau_N$ denote the maturities of the zero rates and ϵ_k represents the measurement error which we assume to be normally distributed with

$$\epsilon_k \sim \mathcal{N}_N\left(0, \begin{pmatrix} h_1^2 & 0 & \cdots & 0 \\ 0 & h_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & h_N^2 \end{pmatrix}\right).$$

The transition equation is derived from the solution of the SDE (see Theorem 2.11) of the short rate r which yields

$$\begin{aligned} r(t_{k+1}) &= e^{-a_r \Delta t_{k+1}} r(t_k) + \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)}(\theta_r(l) + b_{rw_1} w_1(l) + b_{rw_2} w_2(l))dl \\ &+ \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)} \sigma_r \sqrt{1-\rho_{rw_1}^2 - \rho_{rw_2}^2} dW_r(l) \\ &+ \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)} \sigma_r \rho_{rw_1} dW_{w_1}(l) + \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)} \sigma_r \rho_{rw_2} dW_{w_2}(l), \end{aligned}$$

Approximating $\theta_r(l), w_1(l)$ and $w_2(l)$ by $\theta_r(t_k), w_1(t_k), w_2(t_k)$, we obtain the transition equation

$$\begin{aligned} r(t_{k+1}) &= e^{-a_r \Delta t_{k+1}} r(t_k) + \int_0^{\Delta t_{k+1}} e^{-a_r l}(\theta_r(t_k) + b_{rw_1} w_1(t_k) + b_{rw_2} w_2(t_k))dl \\ &+ \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)} \sigma_r \rho_{rw_1} dW_{w_1}(l) + \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)} \sigma_r \rho_{rw_2} dW_{w_2}(l) \\ &+ \eta_{k+1}\,, \end{aligned}$$

where η_{k+1} is defined as $\int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)} \sigma_r dW_r(l)$ with

$$\eta_{k+1} \sim \mathcal{N}_1 \left(0, \int_0^{\Delta t_{k+1}} e^{-2a_r l} \sigma_r^2 dl \right) = \mathcal{N}_1 \left(0, \frac{\sigma_r^2}{2a_r} \left(1 - e^{-2a_r \Delta t_{k+1}}\right)\right).$$

The two stochastic integrals in the above equation are approximated with $\int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)} \sigma_r \rho_{rw_i} dW_{w_i}(l) \approx e^{-a_r(\Delta t_{k+1})} \sigma_r \rho_{rw_i} \Delta W_{w_i}(t_{k+1})$, $i = 1, 2$, where $\Delta W_{w_i}(t_{k+1}) := W_{w_i}(t_{k+1}) - W_{w_i}(t_k)$ is obtained by inserting the time series of w_i, $i = 1, 2$, into the solution of its SDE (see page 56).
This procedure is used if we have observable time series for w_1 and w_2. If one of these processes is unobservable the above equations have to be rewritten. We do not perform this in detail but refer to the next passage where we explain the procedure for two unobservable processes, namely s and u.
The vector of parameters for the short rates of the models SZ4, SZ5, SZ5u and 5corr, which only differ by the number of macroeconomic factors, by their correlation and by their observability, are determined by the same procedure. The data is given by weekly par yields of US Treasury Strips of maturities (τ) 3 months, 6 months, 1 year, 2, 3, 4, 5, 7, 10, 20 and 25 years which are collected from Bloomberg[XII]. We transform these par yields in continuous

[XII]The Bloomberg tickers for the US Treasury Strips are: C0793M, C0796M, C0791Y, C0792Y, C0793Y, C0794Y, C0795Y, C0797Y, C07910Y, C07920Y, C07925Y.

	GDPn	GDPr	CPI	IP	Prod	CILI	CICI
a_r	0.6042	0.5922	0.3144	0.6188	0.4707	0.3258	0.3290
b_{rw_1}	0.0355	0.0532	0.0416	0.0263	0.0971	0.0527	0.0042
σ_r	0.0095	0.0092	0.0101	0.0117	0.0124	0.0099	0.0113
λ_r	-2204.3	-1344.0	-1082.6	-1559.1	-471.0	-575.5	-561.4
λ_{w_1}	-13605.2	-3475.3	-4023.9	-66.4	-538.3	-54.3	-359.6
h_1	0.0103	0.0137	0.0130	0.0075	0.0122	0.0101	0.0106
h_2	0.0124	0.0094	0.0106	0.0105	0.0097	0.0119	0.0118
h_3	0.0089	0.0063	0.0094	0.0035	0.0068	0.0066	0.0080
h_4	0.0048	0.0021	0.0027	0.0019	0.0024	0.0026	0.0027
h_5	0.0021	0.0023	0.0017	0.0053	0.0007	0.0006	0.0003
h_6	0.0019	0.0047	0.0008	0.0048	0.0011	0.0019	0.0019
h_7	0.0025	0.0054	0.0023	0.0044	0.0035	0.0028	0.0028
h_8	0.0049	0.0058	0.0046	0.0062	0.0063	0.0045	0.0040
h_9	0.0063	0.0092	0.0050	0.0091	0.0062	0.0064	0.0050
h_{10}	0.0032	0.0028	0.0029	0.0071	0.0054	0.0037	0.0039
h_{11}	0.0133	0.0113	0.0127	0.0090	0.0110	0.0098	0.0111
$\hat{a}_r$	0.4070	0.4775	0.2030	0.4066	0.3980	0.2698	0.2570
$\hat{a}_{w_1}$	0.6195	1.0810	0.8710	0.2254	1.7100	0.1577	0.0593
mean reversion	3.18%	3.84%	3.24%	3.07%	4.08%	4.07%	3.84%

Table 4.4: Estimated parameters for the short rate r within the SZ4 framework for different macroeconomic factors w_1. The mean reversion of r, which is given in the last row, compares to an empirical mean of 3.91% of the zero rates with a maturity of 3 months.

zero rates and use them as input for $R(t_k, t_k + \tau)$ of the measurement equation.
As previously mentioned, it is not always clear in which way interest rates are influenced by macroeconomic factors. Therefore, we relax the restrictions of non-negativity regarding b_{rw_1} for the factor CPI in the SZ4 model (see Model 4.4). The results of the estimation (see Table 4.4) propose that all factors' influence is in the same direction. Table 4.4 shows the estimated parameters of the SZ4 framework for different representatives of the factor w_1. Based on these estimates we can calculate the mean reversion of the short rate r by $\frac{median(\theta_r)+b_{rw_1}\frac{\theta_{w_1}}{a_{w_1}}}{a_r}$ and compare it with the empirical mean of 3.91% of the observed 3-months zero rates. The fit of the different four factor models (SZ4)

is promising since the calculated mean reversion levels are near the empirical mean especially for GDPr, Prod, CILI, and CICI. In addition, the scales of the estimated volatilities of the measurement errors $h_1, \ldots, h_{11}$ indicate that all seven versions of the SZ4 framework have a similar ability of explaining non-defaultable zero rates.
For the five factor versions of the SZ and the correlated framework (i.e. SZ5, SZ5u, 5corr) we use the real gross domestic product (GDPr) as representative of w_1. The second factor is represented by the consumer price index (CPI) in the case of SZ5 (see Model 4.6) and it is chosen to be unobservable in the case of SZ5u (see Model 4.6) and 5corr (see Model 4.10). We assume for the estimation of the parameters of SZ5 that $b_{rw_2} \in \mathbb{R}$ since the interaction between the two given factors is not known for sure. Within the framework of SZ5u and 5corr we define b_{rw_2} to be 1 and $\rho_{w_1 w_2}$ to be 0 in order to prevent problems of identification. Table 4.5 presents the estimated parameters. We do not give the parameters of w_1 and of w_2 if they are assumed to be observable, since we already estimated these parameters in the first step (see Table 4.3). Compared to the results of the four factor model (SZ4) e.g. with input GDPr or the composite indices (see Table 4.4) we can conclude that the five factor versions (SZ5, SZ5u, 5corr) yield promising results especially if we look at the volatilities of the measurement errors $h_1, \ldots, h_{11}$ which are on average smaller than the ones of the SZ4 frameworks with the exception of the long-term maturities, i.e. h_{10} and h_{11}. This indicates that the additional factor is able to explain an extra portion of the variation. The second macroeconomic factor (w_2) also changes the influence of the GDPr. If we incorporate the CPI as an additional factor, the value of b_{rw_1} increases indicating a bigger impact of the GDPr. But since the CPI affects the short rate in the opposite direction ($b_{rw_2} < 0$) the increase in b_{rw_1} is mostly due to the interaction between GDPr and CPI. In the case of an unobservable factor w_2, where we assume the same direction of influence as for the GDPr, the impact of the GDPr decreases in the case of the SZ5u framework as b_{rw_1} is smaller than in the SZ4 framework. Here, the unobservable factor seems to be able to better explain the variation and therefore reduces the influence of the GDPr. In the case of the 5corr framework, where it is assumed that the short rate is correlated with both the macroeconomic factor and the unobservable factor, b_{rw_1} increases compared to the SZ4 framework in response to the newly incorporated shocks of the factors w_1 and w_2. If we calculate the mean reversion level of the short rate r by $\frac{median(\theta_r)+b_{rw_1}\frac{\theta_{w_1}}{a_{w_1}}+b_{rw_2}\frac{\theta_{w_2}}{a_{w_2}}}{a_r}$ and compare it to the mean of the given 3-months zero rates we see that all three models obtain a good fit. Especially the models with an additional unobservable factor approach easily the empirical mean.

The parameters of the short rate within the real and inflation short-rate framework (INF and INFcorr, see Model 4.8) are estimated in a different way than the parameters of the other models. Since it is assumed that the short rate r is the sum of the real short rate r_R and the inflation short rate r_I the parameters of these processes are estimated independently but both with the help of a Kalman filter and a state space model similar to the above mentioned. We use the same data as Hagedorn et al. (2007) who first introduced this interest rate model based on real and inflation short rates. The real zero rates are generated with the help of inflation-linked bonds [XIII]. In the U.S. these bonds come in the most common structure of capital-indexed bonds like in Sweden, the United Kingdom, France and Canada. Capital-indexed bonds pay a real coupon as the nominal of these bonds is indexed by a capital multiplier which is given as the ratio of an inflation-indexed process at time t and at a certain reference day t_{base}. Furthermore, a deflation floor is built in which prevents the capital multiplier of being smaller than 1. The indexing is based on the CPI-U with the linear interpolation technique introduced in the previous passage. The real rates are determined by assuming a Nelson-Siegel structure and by approximating at weekly measurement points the market prices of US Treasury Inflation Protection Securities (TIPS). The input data for estimating the parameters of the inflation short rate is determined by substracting the real rates from the nominal rates which we derived from US Treasury Strips. As the real rates are calculated by assuming the Nelson-Siegel framework, the nominal rates are also smoothed by Nelson-Siegel curves in order to avoid any systematic errors. For further information about the derivation of the real rates see Hagedorn et al. (2007). Table 4.6 gives the estimated parameters of the real short rate r_R and the inflation short rate r_I which is assumed to be uncorrelated with w_1 (GDPr) in the framework INF ($\rho_{w_1 w_2} := 0$) and correlated with w_1 in the framework INFcorr ($\rho_{w_1 w_2} \in \mathbb{R}$). We can observe that the influence of the real gross domestic product (measured by b_{rw_1}) is bigger for the real rates than the nominal ones (see Table 4.4). Both versions of the inflation short rate r_I do have a good fit since the volatilities of the measurement errors $h_1, \ldots, h_{11}$ are of similar scale. As in the case of the five factor model of Schmid-Zagst, SZ5 (see Model 4.6), the correlation between the factors w_1 and w_2 takes on a negative value. The additional impact of the inflation rate for the nominal short rate is given by the parameter b_{rw_2} which is determined under the real-world measure by $a_R - a_{w_2}$ and results in -0.1753 (INF), i.e. an opposite impact of w_1 and w_2, respectively 0.3111 (INFcorr), also indicating together

[XIII] For further information about inflation-linked bonds see Deacon, Derry & Mirfendereski (2004).

	SZ5u	SZ5	5corr
a_r	0.6918	0.4841	1.5860
b_{rw_1}	0.0160	0.3214	0.2410
b_{rw_2}	1	-0.1527	1
θ_{w_2}	0.0001	-	0.0154
a_{w_2}	0.0533	-	0.2758
σ_{w_2}	0.0081	-	0.0096
σ_r	0.0091	0.0117	0.0130
λ_r	-2118.1	-591.9	-6037.8
λ_{w_1}	-4549.4	-13551.8	-10620.1
λ_{w_2}	-44.4	-84996.8	-2215.0
$\rho_{w_1w_2}$	0	-0.1477	0
ρ_{rw_1}	0	0	0.5469
ρ_{rw_2}	0	0	-0.2122
h_1	0.0038	0.0010	0.0050
h_2	0.0032	0.0039	0.0012
h_3	0.0012	0.0003	0.0013
h_4	0.0011	0.0029	0.0019
h_5	0.0009	0.0054	0.0014
h_6	0.0004	0.0040	0.0008
h_7	0.0003	0.0059	0.0004
h_8	0.0010	0.0072	0.0017
h_9	0.0021	0.0073	0.0028
h_{10}	0.0067	0.0124	0.0055
h_{11}	0.0207	0.0253	0.0171
$\hat{a}_r$	0.5152	0.4028	0.7639
$\hat{a}_{w_1}$	1.0052	0.3694	0.5764
$\hat{a}_{w_2}$	0.0503	0.0130	0.0716
mean reversion	3.92%	4.70%	3.90%

Table 4.5: Estimated parameters for the short rate r within the frameworks SZ5u, SZ5 and 5corr where the first macroeconomic factor w_1 is given by the GDPr and the second factor w_2 is chosen to be unobservable for SZ5u and 5corr, respectively is represented by CPI for SZ5. The mean reversion of r, which is given in the last row, compares to an empirical mean of 3.91% of the zero rates with a maturity of 3 months.

with the negative correlation a tendency of opposite impact . Hence, if we allow for correlated factors w_1 and w_2 the sign of the correlation is the same as in the framework SZ5, but the direction of the impact of the factor w_2 changes since the sign of b_{rw_2} is the same as for b_{rw_1} as opposed to $b_{rw_1} > 0$ and $b_{rw_2} < 0$ for SZ5. But otherwise, if $\rho_{w_1 w_2}$ is to be 0, b_{rw_2} takes on the same sign as in the case of SZ5 namely opposite to the impact of the GDPr, though $\rho_{w_1 w_2} < 0$ in SZ5 . This result - in addition with the results for SZ5 - emphasizes the previously stated fact that the influence of certain factors on the short rate respectively interest rates is an open question. The mean reversion level of the short rate in the real and inflation short-rate model is determined by $\frac{median(\theta_R)+\theta_{w_2}+b_{Rw_1}\frac{\theta_{w_1}}{a_{w_1}}+(a_R-a_{w_2})\frac{\theta_{w_2}}{a_{w_2}}}{a_R}$ resulting in 3.30% (INF) and 3.32% (INFcorr). Compared to the empirical mean of the 3-months zero rates (3.91%), the real and inflation short-rate framework (INF and INFcorr) fits the data as good as the other discussed frameworks.

Estimating the parameters of the short-rate spread s and the uncertainty index u

The parameters $(a_s, \sigma_s, \lambda_s, \theta_s, b_{su}, b_{sw_1}, b_{sw_2}, a_u, \sigma_u, \lambda_u, \theta_u, \rho_{su}, \rho_{sw_1}, \rho_{sw_2})$ of the processes s and u are also estimated by means of the Kalman filter. Here, we obtain the measurement equation by substracting non-defaultable zero rates $R(t_k, T) = -\frac{1}{T-t_k}\ln P(t_k, T)$ from defaultable zero rates $R^d(t_k, T) = -\frac{1}{T-t_k}\ln P^d(t_k, T)$ in order to obtain the spread $S(t_k, T)$:

$$\begin{aligned} S(t_k, T) &= R^d(t_k, T) - R(t_k, T) \\ &= \frac{C^d(t_k, T)}{T - t_k} s(t_k) + \frac{D^d(t_k, T)}{T - t_k} u(t_k) - \frac{A^d(t_k, T) - A(t_k, T)}{T - t_k} \\ &\quad + \frac{E_1^d(t_k, T) - E_1(t_k, T)}{T - t_k} w_1(t_k) + \frac{E_2^d(t_k, T) - E_2(t_k, T)}{T - t_k} w_2(t_k). \end{aligned}$$

If we define $c^d(t_k, T) = \frac{C^d(t_k, T)}{T-t_k}$, $d^d(t_k, T) = \frac{D^d(t_k, T)}{T-t_k}$, $a^d(t_k, T)$ as the sum of all terms independent of s and u, and

$$X(t_k) := \begin{pmatrix} s(t_k) \\ u(t_k) \end{pmatrix},$$

the measurement equation yields

$$\begin{pmatrix} S(t_k, t_k + \tau_1) \\ \vdots \\ S(t_k, t_k + \tau_N) \end{pmatrix} = \begin{pmatrix} a^d\,(t_k, t_k + \tau_1) \\ \vdots \\ a^d\,(t_k, t_k + \tau_N) \end{pmatrix} + \begin{pmatrix} c^d\,(0, \tau_1) & d^d\,(0, \tau_1) \\ \vdots & \vdots \\ c^d\,(0, \tau_N) & d^d\,(0, \tau_N) \end{pmatrix} \cdot X(t_k) + \epsilon_k$$

	r_R		r_I (INF)	r_I (INFcorr)
a_R	0.4654	a_{w_2}	0.6407	0.1542
b_{Rw_1}	0.1554	θ_{w_2}	0.0105	0.0026
σ_R	0.0084	σ_{w_2}	0.0145	0.0089
λ_R	-4426.3	$\rho_{w_1 w_2}$	0	-0.3514
λ_{w_1}	-15362.2	λ_{w_2}	-659.0	-228.3
h_1	0.0014	h_1	0.0026	0.0076
h_2	0.0013	h_2	0.0021	0.0061
h_3	0.0010	h_3	0.0012	0.0041
h_4	0.0004	h_4	8e-11	0.0017
h_5	8e-10	h_5	0.0008	0.0002
h_6	0.0004	h_6	0.0014	0.0019
h_7	0.0007	h_7	0.0020	0.0032
h_8	0.0012	h_8	0.0029	0.0051
h_9	0.0018	h_9	0.0040	0.0061
h_{10}	0.0018	h_{10}	0.0046	0.0054
h_{11}	0.0029	h_{11}	0.0040	0.0056
$\hat{a}_{w_1}$	0.2392	$\hat{a}_{w_2}$	0.5032	0.1373
$\hat{a}_R$	0.1501			

Table 4.6: Estimated parameters for the real short rate r_R within the real and inflation short-rate model (INF and INFcorr) where the first macroeconomic factor w_1 is given by the GDPr and the second factor, the so-called inflation short rate r_I, is filtered by means of inflation-linked bonds.

where $\tau_1, ..., \tau_N$ denote the maturities of the spreads and ϵ_k represents the measurement error which we assume to be normally distributed with

$$\epsilon_k \sim \mathcal{N}_N\left(0, \begin{pmatrix} g_1^2 & 0 & \cdots & 0 \\ 0 & g_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_N^2 \end{pmatrix}\right).$$

In order to get the transition equation we define the matrices H, J, V and W with

$$H := \begin{pmatrix} -a_s & b_{su} \\ 0 & -a_u \end{pmatrix},$$

$$J(t_k) := \begin{pmatrix} \theta_s - b_{sw_1} w_1(t_k) - b_{sw_2} w_2(t_k) + \sigma_s \rho_{sw_1} \Delta W_{w_1}(t_{k+1}) + \sigma_s \rho_{sw_2} \Delta W_{w_2}(t_{k+1}) \\ \theta_u \end{pmatrix},$$

$$V := \begin{pmatrix} \sigma_s\sqrt{1-\rho_{su}^2+\rho_{sw_1}^2+\rho_{sw_2}^2} & \sigma_s\rho_{su} \\ 0 & \sigma_u \end{pmatrix}, W := (W_s, W_u)' \ ,$$

where dW_{w_1}, respectively dW_{w_2}, is approximated by $\Delta W_{w_1}(t_{k+1}) := W_{w_1}(t_{k+1}) - W_{w_1}(t_k)$. Thus, the SDEs of s and u can be written as

$$dX(t_k) = HX(t_k) + J(t_k)dt + VdW(t_k).$$

Using Theorem 2.11 the solution of dX is

$$X_{k+1} = e^{H\Delta t_{k+1}}X_k + \int_{t_k}^{t_{k+1}} e^{H(t_{k+1}-l)}J(l)dl + \int_{t_k}^{t_{k+1}} e^{H(t_{k+1}-l)}VdW(l).$$

If we approximate $J(l)$ by $J(t_k)$ the transition equation can be written as

$$X(t_{k+1}) = e^{H\Delta t_{k+1}}X(t_k) + \int_0^{\Delta t_{k+1}} e^{Hl}J(t_k)dl + \eta_{k+1},$$

with $\eta_{k+1} := \int_{t_k}^{t_{k+1}} e^{H(t_{k+1}-l)}VdW(l)$ following a normal distribution

$$\eta_{k+1} \sim \mathcal{N}_2\left(0, \int_0^{\Delta t_{k+1}} e^{Hl}VV'e^{H'l}dl\right).$$

The procedure to determine the parameters for the processes s and u is the same throughout all discussed models since the only differences are the use of a second factor w_2 and the correlated shocks in the SDE of the short-rate spread. The data is given by weekly par yields of US Industrials rated BBB1 and A2 whose maturities are 3 months, 6 months, 1 year, 2, 3, 4, 5, 7, 10, 20 and 25 years. As before this data stems from Bloomberg[XIV]. Again, we transform these par yields in continuous zero rates, substract the non-defaultable zero rates in order to derive the credit spreads and smooth the resulting rates by means of Nelson-Siegel curves. We use these credit spreads as input for $S(t_k, t_k+\tau)$ of the measurement equation.
Throughout all models we assume b_{su} to be 1 as the process u is already unobservable. In the case of the five factor models with an unobservable second macroeconomic factor w_2, SZ5u (see Model 4.6) and 5corr (see Model 4.10), we additionally assume $b_{sw_2} = 0$ since there is already an unobservable factor u in the short-rate spread. Including a second unobservable factor would have led to problems regarding the differentiation of the two unobservable factors. This assumption gives the short-rate spread of the SZ5u framework

[XIV] The Bloomberg tickers are for the rating A2: C0063M, C0066M, C0061Y,... and C0083M, C0086M, C0081Y,... for the rating BBB1.

the same form as the short-rate spreads within the four factor framework (SZ4, see Model 4.4) and a similar appearance to the SZ4 framework in case of the 5corr framework. Hence, we expect similar results for SZ5u, 5corr and SZ4 frameworks with respect to the short-rate spread s.
Figlewski et al. (2012) indicate in their study of corporate defaults that inflation is understood to be an important macroeconomic factor but its effect is unclear, since, unlike the common perception, high inflation could also decrease default risk by reducing the value of required debt service payments. Hence, as in the case of the short rate r we do not restrict the parameter b_{sw_1} in the case of the SZ4 model with CPI to non-negative values, but as before the results indicate that the impact of the CPI is of the same direction as for the other macroeconomic factors. Table 4.7 and Table 4.8 give the results of the four factor models (SZ4) where we can observe that the models exhibit a similar fitting ability with respect to the mean reversion levels. The mean reversion levels are determined with $\frac{\theta_s - b_{sw_1}\frac{\theta_{w_1}}{a_{w_1}} + b_{su}\frac{\theta_u}{a_u}}{a_s}$ and are close to the empirical means of the given 3-months credit spreads, which yield 0.83% for the rating A2 and 1.08% for the rating BBB1. Additionally, the volatilities of the measurement errors $g_1, \ldots, g_{11}$ are of the same scale indicating that all macroeconomic factors have a similar ability to explain the variation.
Table 4.9 and Table 4.10 present the estimated parameters for the five factor models (SZ5, SZ5u, 5corr, INF, INFcorr). Like before, we do not restrict the parameter b_{sw_2} and find that in case of the SZ5 framework where the factor w_2 is given by the CPI, its impact is opposite to the impact of the first factor w_1 (GDPr). Whereas for the two versions of the real and inflation short-rate framework (INF, INFcorr) the impact of the inflation short rate is of the same direction as for the GDPr regardless if we account for correlation between w_1 and w_2. The only difference seems to be the fact that there is a shift in the impact of the two factors meaning that b_{sw_1} decreases and b_{sw_2} increases if we allow $\rho_{w_1 w_2}$ to be non-zero. The different impact of inflation is in line with the findings of Figlewski et al. (2012) who show that incorporating several macroeconomic factors can lead to unexpected results in explaining credit risk. Although the 5corr model is a five factor framework, its short-rate spread is built similarly to the factor s of the four factor versions (SZ4) since it makes no use of the factor w_2. Therefore, we expect the parameters of the 5corr model for credit spreads to behave similarly to the parameters of the four factor version (SZ4) with GDPr input. Especially, since the only difference of these two versions are the correlated shocks in the SDE of the short-rate spread s (cf. Model 4.4 vs. Model 4.10). As expected, the impact of the macroeconomic factor w_1 measured by the parameter b_{sw_1} is of the same scale for both models and rating classes. Furthermore, the

volatilities of the measurement errors $g_1, \ldots, g_{11}$ also behave like the ones of the other four factor models (SZ4), i.e. there seem to be difficulties for very short and long maturities. But the incorporation of correlated shocks leads to smaller volatilities for mid-term maturities than it is the case for the SZ4 framework. As expected above, the SZ5u framework also exhibits the same behaviour with respect to parameters and volatilities of measurement errors as the models of the SZ4 framework. Compared with the four factor versions of SZ4 (see Table 4.7 and Table 4.8) the five factor models' mean reversion levels $\frac{\theta_s - b_{sw_1}\frac{\theta_{w_1}}{a_{w_1}} - b_{sw_2}\frac{\theta_{w_2}}{a_{w_2}} + b_{su}\frac{\theta_u}{a_u}}{a_s}$ come as close to the empirical means and the volatilities of the measurement errors (especially for SZ5, INF and INFcorr) are on average smaller as the ones for the four factor model. Thus, the incorporation of a fifth factor does also improve the fitting ability for credit spreads given this factor appears both in the short rate r and the short-rate spread s. Whereas the usage of correlated shocks within the credit spread framework does not seem to improve the fitting ability as opposed to an additional factor.

	GDPn		GDPr		CPI		IP	
	A2	BBB1	A2	BBB1	A2	BBB1	A2	BBB1
a_s	1.2514	0.5835	1.7272	0.7287	1.1386	0.6143	0.7883	0.3368
b_{sw_1}	0.0694	0.0976	0.0549	0.1266	0.0650	0.1608	0.0285	0.0093
σ_s	0.0012	0.0025	0.0092	0.0055	0.0044	0.0051	0.0042	0.0043
θ_s	0.0006	0.0004	0.0116	0.0075	0.0050	0.0039	0.0042	0.0029
λ_s	-37608.5	-7492.3	-13296.3	-11908.4	-7793.7	-5079.6	-1121.3	-8268.9
θ_u	0.0059	0.0016	0.0004	0.0005	0.0006	0.0013	0.0028	0.0005
a_u	0.5514	0.2287	0.1160	0.4612	0.0970	0.1953	1.0004	0.5993
σ_u	0.0044	0.0056	0.0038	0.0066	0.0031	0.0060	0.0096	0.0074
λ_u	-11538.0	-4550.6	-5251.9	-8501.3	-7001.3	-3477.9	-8200.1	-2834.4
g_1	0.0056	0.0009	0.0004	0.0007	0.0063	0.0013	0.0014	0.0007
g_2	0.0076	0.0023	0.0006	0.0004	0.0005	0.0002	0.0002	0.0003
g_3	0.0006	0.0007	0.0001	0.0001	0.0008	0.0003	0.0005	0.0004
g_4	0.0004	0.0005	0.0002	0.0003	0.0006	0.0011	0.0002	0.0004
g_5	0.0005	0.0006	0.0002	0.0002	0.0003	0.0049	7e-5	0.0038
g_6	0.0019	0.0002	0.0002	5e-5	0.0026	0.0095	0.0011	0.0003
g_7	0.0030	8e-5	0.0008	0.0003	0.0011	0.0007	0.0064	0.0001
g_8	0.0014	0.0006	0.0011	0.0009	0.0030	0.0004	0.0099	0.0029
g_9	0.0015	0.0070	0.0036	0.0131	0.0086	0.0005	0.0021	0.0014
g_{10}	0.0031	0.0067	0.0032	0.0058	0.0057	0.0030	0.0048	0.0085
g_{11}	0.0093	0.0098	0.0096	0.0046	0.0121	0.0130	0.0065	0.0113
$\hat{a}_s$	1.2012	0.5380	0.5914	0.3711	0.9852	0.4844	0.7689	0.1810
$\hat{a}_u$	0.3327	0.0846	0.0414	0.0946	0.0299	0.0708	0.2495	0.4459
mean reversion	0.83%	1.08%	0.83%	1.05%	0.82%	1.06%	0.83%	1.07%

Table 4.7: Estimated parameters for the short-rate spread s and the uncertainty index u within the SZ4 framework for different macroeconomic factors w_1. The mean reversion of s, which is given in the last row, compares to an empirical mean of 0.83% (A2) and 1.08% (BBB1) of the credit spreads with a maturity of 3 months.

	Prod		CILI		CICI	
	A2	BBB1	A2	BBB1	A2	BBB1
a_s	1.9363	1.2145	0.6851	0.6544	0.5167	0.3484
b_{sw_1}	0.1541	0.1327	0.0168	0.0142	0.2534	0.0684
σ_s	0.0056	0.0063	0.0074	0.0061	0.0039	0.0047
θ_s	0.0163	0.0129	0.0017	0.0019	0.0003	0.0004
λ_s	-6142.2	-2961.2	-8776.9	-8554.3	-13735.9	-1907.7
θ_u	0.0023	0.0009	0.0003	0.0003	0.0007	0.0004
a_u	1.9219	0.7769	0.0639	0.0499	0.0839	0.0954
σ_u	0.0125	0.0070	0.0019	0.0018	0.0021	0.0027
λ_u	-11620.9	-15598.6	-5026.1	-5687.4	-11403.2	-9697.5
g_1	0.0008	0.0011	0.0028	0.0019	0.0015	0.0016
g_2	0.0005	0.0010	0.0014	0.0019	0.0028	0.0021
g_3	0.0007	0.0002	0.0013	0.0008	0.0017	0.0014
g_4	0.0005	0.0011	0.0005	0.0007	0.0005	0.0005
g_5	0.0005	0.0005	0.0001	3e-5	0.0004	0.0002
g_6	0.0001	0.0002	2e-5	0.0001	0.0005	0.0001
g_7	0.0002	8e-5	0.0002	4e-5	0.0001	2e-5
g_8	0.0011	0.0004	0.0006	0.0006	0.0004	0.0013
g_9	0.0056	0.0019	0.0008	0.0010	0.0004	0.0012
g_{10}	0.0041	0.0027	0.0021	0.0054	0.0014	0.0016
g_{11}	0.0079	0.0031	0.0028	0.0050	0.0019	0.0047
$\hat{a}_s$	1.7451	1.0955	0.2053	0.3331	0.3050	0.3055
$\hat{a}_u$	0.0933	0.0120	0.0459	0.0323	0.0327	0.0226
mean reversion	0.85%	1.08%	0.80%	1.09%	0.78%	1.09%

Table 4.8: Estimated parameters for the short-rate spread s and the uncertainty index u within the SZ4 framework for different macroeconomic factors w_1. The mean reversion of s, which is given in the last row, compares to an empirical mean of 0.83% (A2) and 1.08% (BBB1) of the credit spreads with a maturity of 3 months.

	SZ5u		SZ5		5corr		INF		INFcorr	
	A2	BBB1	A2	BBB1	A2	BBB1	A2	BBB1	A2	BBB1
a_s	0.6774	0.5963	1.7385	1.3715	0.3742	0.3430	1.2941	1.2482	1.1084	0.9528
b_{sw_1}	0.0524	0.0114	0.0131	0.0473	0.0882	0.1343	0.1295	0.0483	0.0157	0.0340
b_{sw_2}	0	0	-0.0028	-0.0052	0	0	0.0729	0.0611	0.2205	0.1801
σ_s	0.0086	0.0102	0.0097	0.0083	0.0094	0.0099	0.0077	0.0056	0.0073	0.0067
θ_s	0.0030	0.0011	0.0113	0.0123	0.0029	0.0039	0.0035	0.0035	0.0015	0.0010
λ_s	-2575.5	-1910.9	-13131.8	-12954.9	-3151.3	-3353.6	-18367.2	-29956.5	-11967.6	-13148.1
θ_u	0.0006	0.0005	0.0003	0.0002	7e-5	0.0001	0.0010	0.0005	0.0005	0.0006
a_u	0.1969	0.0872	0.0971	0.0802	0.0958	0.1619	0.1124	0.0449	0.0446	0.0502
σ_u	0.0042	0.0028	0.0038	0.0037	0.0033	0.0028	0.0043	0.0037	0.0035	0.0035
λ_u	-5206.4	-5124.7	-4721.4	-4821.1	-4756.5	-3991.4	-3003.0	-1751.1	-2366.2	-1521.7
ρ_{su}	0	0	0	0	0.1401	0.1737	0	0	0	0
ρ_{sw_1}	0	0	0	0	-0.0591	-0.0842	0	0	0	0

Table 4.9: Estimated parameters for the short-rate spread s and the uncertainty index u within the five factor frameworks SZ5u, SZ5, 5corr, INF and INFcorr.

	SZ5u		SZ5		5corr		INF		INFcorr	
	A2	BBB1	A2	BBB1	A2	BBB1	A2	BBB1	A2	BBB1
g_1	0.0005	0.0010	0.0011	0.0015	0.0022	0.0016	0.0012	0.0014	0.0010	0.0013
g_2	0.0006	0.0013	0.0008	0.0012	0.0012	0.0039	0.0009	0.0006	0.0006	0.0006
g_3	0.0005	0.0010	0.0004	0.0003	0.0010	0.0006	0.0004	0.0003	0.0003	0.0003
g_4	0.0009	0.0002	3e-5	9e-5	0.0005	0.0002	9e-5	9e-5	2e-5	8e-5
g_5	0.0009	0.0003	6e-5	9e-5	0.0001	5e-5	0.0002	7e-5	0.0001	2e-5
g_6	0.0003	0.0007	5e-6	8e-5	2e-5	8e-6	8e-5	8e-5	9e-5	8e-6
g_7	0.0005	0.0011	0.0001	0.0001	4e-5	5e-5	8e-6	8e-6	0.0001	5e-5
g_8	0.0010	0.0004	0.0004	0.0004	0.0002	0.0012	0.0003	0.0003	0.0003	0.0003
g_9	0.0027	0.0005	0.0010	0.0013	0.0008	0.0006	0.0007	0.0005	0.0007	0.0006
g_{10}	0.0048	0.0020	0.0025	0.0018	0.0026	0.0029	0.0015	0.0016	0.0015	0.0014
g_{11}	0.0034	0.0046	0.0024	0.0032	0.0051	0.0042	0.0029	0.0026	0.0022	0.0020
$\hat{a}_s$	0.4859	0.3976	0.5024	0.4849	0.0966	0.0188	0.2191	0.3060	0.4641	0.3632
$\hat{a}_u$	0.1055	0.0464	0.0291	0.0124	0.0430	0.1300	0.0560	0.0211	0.0163	0.0313
mean reversion	0.87%	1.08%	0.81%	1.08%	0.83%	1.07%	0.81%	1.09%	0.83%	1.09%

Table 4.10: Estimated parameters for the short-rate spread s and the uncertainty index u within the five factor frameworks SZ5u, SZ5, 5corr, INF and INFcorr. The mean reversion of s, which is given in the last row, compares to an empirical mean of 0.83% (A2) and 1.08% (BBB1) of the credit spreads with a maturity of 3 months.

4.8 Comparing the Models

Having calibrated the different models with the help of market data, we now want to find out which model yields the best results in explaining the data. First of all, we compare the average pricing errors of the different frameworks by calculating the absolute deviations for the insample period from January 1 1999 to December 27 2002 and for the out-of-sample period from January 3 2003 to December 31 2004[XV]. In order to identify any structural differences we do this for every single maturity as well as for the average over all maturities. Then, we will apply a linear regression to the market prices for determining whether they are explained well by model prices. Afterwards, we analyze by means of the Akaike Information Criterion if the fitting ability is just due to an increase of factors. Finally, we conclude by testing the standardized innovations with respect to the requirements of the state space model.

Absolute Deviations: Pricing errors between US Treasury Strips and non-defaultable model prices
Table B.2 in the Appendix illustrates the pricing errors between the zero rates of US Treasury Strips and the ones determined by the various models. The introduction of a second macroeconomic factor w_2 improves the pricing power as long as this second factor is chosen to be unobservable as in the SZ5u and the 5corr frameworks. This is in line with Antes et al. (2008) who found that the short-rate model of Bakshi et al. (2006) outperforms the extended Schmid-Zagst model (SZ4) since Bakshi et al. (2006) assume their macroeconomic factor to be unobservable. Due to the fact that all short-rate models within this thesis incorporate at least one observable macroeconomic factor into the short-rate, we omit a comparison with the model of Bakshi and concentrate instead on models that are based on macroeconomic input. For most of the maturities between 3 months and 10 years the SZ5u or the 5corr version are the best models with respect to pricing error, often far better than the others. Only for long-term maturities, i.e. 20 and 25 years, these models exhibit the same problems as can be observed for the majority of the models. The SZ5 framework which consists of two observable processes w_1 and w_2 driven by the real gross domestic product and the consumer price index yields promising results for short-term maturities which are additionally to the maturities at the long end often problematic to fit. The pricing of SZ5 shows like the ones of SZ5u and 5corr better results for short maturities

[XV] We use this time period in order to match the calibration period and results of Hagedorn et al. (2007).

indicating that a second factor - even if it is observable - helps to explain the market data. But for mid-term and long-term maturities this effect fades and the SZ5 framework tends to reach worse results than the rest. The two versions of the five factor model built on a real and an inflation short-rate (INF and INFcorr), where the second factor w_2, the so-called inflation short rate, is also unobservable and filtered with the help of inflation-linked bonds, take on positions between the other five factor models. They do not reach the low pricing errors as SZ5u and 5corr do, but most of the time they range among the best models. Astonishingly, these two models yield the lowest pricing error for the maturity of 25 years. Within the extended model of Schmid and Zagst (SZ4) we tested seven different economic factors GDPn, GDPr, CPI, IP, Prod, CILI and CICI. By means of the absolute deviations there is no factor which can be singled out as the best one. The pricing errors are in the same range for all seven economic factors. For short-term maturities there are the industrial production (IP) and the real gross domestic product (GDPr) followed by the composite indices of leading (CILI) and coincident indicators (CICI) which perform best. But it changes for mid-term and long-term maturities where the consumer price index (CPI), the nominal gross domestic product (GDPn) and the composite index of leading indicators (CILI) take over the leading position among the four factor models.
For the out-of-sample period we get a similar picture as the SZ5u as well as the 5corr framework yield good results for short-term and mid-term maturities. But for longer maturities both versions are outperformed by almost all other models. The same holds for the other five factor models even for almost all maturities. This is in line with the common expectation that the out-of-sample performance gets worse if additional factors are incorporated, especially if they are observable. Within the four factor models (SZ4) it is again difficult to determine the best macroeconomic factor since the ranking within the seven factors changes often. But next to the industrial production (IP) for short maturities, the productivity (Prod) for mid-range maturities, and the gross domestic product (GDP) for long maturities, it is the composite index of leading indicators (CILI) that obtains one of the best results across all maturities.
With the help of Table B.1 showing the absolute deviations averaged over all maturities and over the maturities between one and ten years, we can conclude that the incorporation of an additional factor w_2 can help to improve the pricing power for non-defaultable bonds if this factor is chosen to be unobservable as in the frameworks SZ5u and 5corr, or if this factor is filtered with the help of inflation-linked bond data. Choosing w_2 to be observable like it is done in SZ5 does not have any additional impact as it can be seen in Table B.2 where its absolute deviations are one of the worst. The ranking

of the four factor models indicates CILI to be the best for the insample and out-of-sample.

Absolute Deviations: Pricing errors between US corporate credit spreads and defaultable model prices
Tables B.3 and B.4 in the Appendix contain the average pricing errors for the credit spreads rated A2 and BBB1. Here, we emphasize that the model 5corr is actually a five factor model but as in the case of SZ5u the short-rate spread is not influenced by the fifth factor. The only difference to the models of SZ4 are the correlated diffusion terms which appear in the SDE of the short-rate spread s. The diffusion term of the first macroeconomic factor links the short-rate spread of 5corr to the short-rate spreads of the other five factor models since the values of ΔW_{w_1} are exogenously given by the data of factor w_1 and the previously estimated parameters of its SDE (see Section 4.7). Therefore, we expect 5corr to behave similarly to both the four factor and the five factor models regarding the pricing of defaultable bonds. The insample pricing errors of the rating A2 (see Table B.3) indicate that the five factor models perform best as opposed to the four factor versions with 5corr marking the transition between both frameworks: For short-term maturities (3M, 6M, 1Y) almost all four factor versions (SZ4) yield better results than the five factor models with 5corr even being the worst of all. For the maturities from 2 years to 25 years the five factor models including 5corr take on the leading position. Only for long-term maturities (20Y, 25Y) 5corr gets in line with the four factor models at the end of the ranking. Except for short maturities the composite indices (CILI, CICI) tend to be the best four factor versions. As predicted, the performance of SZ5u is similar to the four factor version with GDPr since both are based on the same macroeconomic factor and the same set-up for the short-rate spread s. If we take a look at the averages over all maturities for rating A2 in Table B.1, we see the dominating role of the five factor frameworks over the four factor ones whose best insample representatives seems to be the composite indices CILI and CICI. Within the five factor framework the models which are based on the inflation short rate tend to be the best. For the credit spreads rated BBB1 (see Table B.4) we get a similar picture since the five factor versions are among the best ones for the maturities from 2 years to 25 years with 5corr being the exception as before for long-term maturities. Especially the model based on the incorporation of an inflation short rate takes on the majority of the top positions. As in the case of the A2 rated credit spreads, almost all four factor versions outperform the models with an additional factor for short-term maturities. The means over all maturities, respectively the ones from 1 to 10 years, also indicate the dominance of the five factor frameworks and

the composite indices being the best four factor ones (see Table B.1). The out-of-sample pricing errors for credit spreads rated A2 and BBB1 show that on average the totality of the five factor versions performs better than the four factor ones but there are exceptions like CPI, CICI and CILI for rating A2 and Prod, CICI and CILI for rating BBB1 (see Table B.1). Considering the absolute deviations for every single maturity shows that the five factor models are always under the best models for maturities longer than one year. But as before there are four factor versions like CILI, CICI, Prod, CPI and GDPr as well as the special five factor cases 5corr and SZ5u, whose credit spread set-up is similar - respectively equal - to the four factor case, which also yield top positions for several maturities (see Tables B.3 and B.4).
Altogether, the results for credit spreads confirm that the introduction of an additional factor improves the pricing power of our framework. An alternative to an additional factor in the short-rate spread s seems to be the incorporation of the diffusion terms of all factors influencing the short-rate spread, as it is done in 5corr. But since the differences in the pricing errors are small, it is difficult to determine the best model within the generalized framework as well as within the SZ4 framework.

Absolute Deviations: Summary
Taking also into account the results for non-defaultable interest rates, we can conclude that incorporating an additional factor yields promising results. But since it is difficult to choose the best macroeconomic factor we propose to use one observable macroeconomic factor w_1 which would probably be represented best by the real gross domestic product (GDPr) or the composite index of leading indicators (CILI). Our choice is based on the promising results obtained by all models that are based on the former factor (SZ4 with GDPr, 5corr, SZ5u, INF, INFcorr) and the fact that the four factor version (SZ4) using the composite index always yielded good results as opposed to the other tested macroeconomic factors. For the second macroeconomic factor w_2 we suggest to use an unobservable macroeconomic index which only drives the short rate r, or to filter the inflation short rate which is supposed to influence the short rate r as well as the short-rate spread s from additional data provided by inflation-linked bonds.
Comparing the pricing errors for the three tested categories (US Treasury Strips, US Industrials rated A2 and BBB1) we can conclude that the riskier the category, the closer are the results of the different models (see Table B.1): For credit spreads rated BBB1 the average pricing errors are the smallest of the three categories and do not differ much between the models. But the average pricing errors become already bigger for credit spreads rated A2 indicating that there are influences which cannot be captured by one or two

macroeconomic factors. The pricing errors of the interest rates finally display the importance of the right choice for the macroeconomic factor as well as the model set-up since there are big differences between the insample and out-of-sample performances of the different factors.

Linear Regression: Market price versus model implied price
As a further quantitative measure we apply the linear regression model

$$P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon,\ \epsilon \sim N(0, \sigma_\epsilon^2)\ i.i.d$$

where P^{Market} denotes the market price and P^{Model} the corresponding model price of non-defaultable zero rates respectively defaultable credit spreads. For good working models we expect the regression parameters β_0 and β_1 to be near 0 and 1 and the confidence intervals to be more dense. Tables B.5 to B.16 contain the values of R^2 and the 95%-confidence intervals of the different models for all given maturities.

Linear Regression: Non-defaultable bond prices
For non-defaultable zero rates we observe the same ranking and findings within the five factor frameworks as before. The SZ5u and the 5corr models yield the best R^2 values among the five factor models but have some problems for the two longest maturities (20Y, 25Y) where the R^2 decreases and β_0 as well as β_1 take on values significantly different from 0 and 1, respectively. The SZ5 model with two observable macroeconomic processes reaches high R^2 and dense intervals for short-term maturities 3M, 6M and 1Y. But results for the following maturities reflect the problems of this model since the values of R^2 are decreasing and the confidence intervals widen considerably and depart of the expected values, especially for the longer maturities. The two versions of the real and inflation short-rate model (INF, INFcorr), which only differ by the assumptions about $\rho_{w_1 w_2}$, can be grouped in between the other five factor models regarding their results for non-defaultable zero rates. For the majority of the maturities they yield better results than SZ5 but perform worse than SZ5u and 5corr. As observed for the other models, R^2 and especially the confidence intervals for longer maturities worsen. Within the four factor models the results do not differ very much. These models exhibit the same problems as the five factor models for longer maturities, namely lower R^2, wider confidence intervals and β_i, $i = 0, 1$, which back away from their expected values. In comparison to the five factor models we see that all four factor models are outperformed by SZ5u and 5corr but obtain better results than SZ5 apart from the short-term maturities. A differentiation

between the four factor models and the real and inflation short-rate models (INF, INFcorr) is not possible because the results do not favour any versions.

Linear Regression: Defaultable bond prices

The regressions for credit spreads yield results which are very similar across all models. Here, we omit the model SZ5u since its short-rate spread is modelled according to the short-rate spread in the SZ4 framework. The fifth factor within SZ5u only appears in the short rate r leaving the short-rate spread as in SZ4. The real and inflation short-rate models (INF, INFcorr) obtain better results than the SZ5 framework for longer maturities. Similar to the non-defaultable case the confidence intervals widen for short-term and long-term maturities but in contrast to the non-defaultable results the confidence intervals stay close around the expected values of 0 and 1. However, this does not hold for the 5corr model where the confidence intervals of β_1 for long-term maturities are far away from 1 although they yield better values of R^2 as e.g. SZ5 and the four factor models. Additionally, the R^2 of 5corr for short-term maturities are the worst across all models emphasizing the above mentioned problems. As before, the results within the four factor models do not differ very much. There are maturities or even rating classes for which one four factor model outperforms the others but it is impossible to single out one model as the best. Compared to the five factor models we conclude that for mid-term maturities there is almost no difference. The four factor models yield slightly better results for short-term maturities but are outperformed by the five factor models except 5corr for long-term maturities. Here, the R^2 are lower and the confidence intervals are considerably wider than those of the five factor models. Especially the confidence intervals for β_1 are often placed far away from 1, e.g. for the four factor versions with GDPn, Prod and IP. The only exceptions at the long end are the composite indices CILI and CICI whose confidence intervalls and R^2 differ distinctively from the other four factor versions and perform similar or even better, e.g. CICI, than several five factor models.

Linear Regression: Summary

Summarizing these results we conclude that the incorporation of an additional factor does improve the short rate r if this factor is unobservable. In case of the short-rate spread s the results are too close across all presented models to give a distinct answer. It seems that a second observable factor as in SZ5 and additionally included diffusion terms as in 5corr do not improve the performance and do not solve the problems arising for short and long maturities; on the contrary, they tend to further intensify those effects (e.g. 5corr).

Akaike Information Criterion: Best fit versus minimum number of parameters
A model's ability to fit the data is generally increased by including an additional factor. In order to justify the additional factors we compare by means of the Akaike Information Criterion (AIC, see Akaike (1974)) the five factor models with the four factor models which are considered as a reduced form of the previous mentioned ones. The AIC which is defined as

$$AIC = 2k - 2\ln L$$

with k being the number of parameters and L the likelihood function, links the number of parameters with the fitting ability of a given model. Hence, its aim is to find a model which explains best the data using a minimum number of parameters at the same time. By applying the AIC we want to verify that the improvement of the model's performance does not only depend on the increase of factors.
Table 4.11 shows the loglikelihoods for the different models. The loglikelihoods of the five factor models exceed the loglikelihoods of all four factor models even after controlling for the additional factors. There is just one exception for the non-defaultable case, namely the model of Schmid and Zagst (SZ5) where two given macroeconomic factors are incorporated. It yields a loglikelihood which has the same order of magnitude as the loglikelihoods of the four factor models. Even the loglikelihoods for credit spreads (A2, BBB1) suggest that 5corr takes on a position between the four and five factor models. Although its short-rate spread s involves as many factors as the models of the SZ4 framework, its loglikelihood yields higher values which, however, do not reach the level of the other five factor models. The loglikelihoods of the other special case SZ5u illustrate the fact that its credit spreads have the same set-up as the SZ4 framework, i.e. for rating A2 and BBB1 SZ5u ranks between the four factor versions with input Prod and GDPr, although its loglikelihood for the non-defaultable case is the best of all models. The fitting ability of the composite indices of several macroeconomic factors, CILI and CICI, tend to be better than the ability of one single factor and come extremely close to the results of 5corr.
Summarizing the outcome of the Akaike Information Criterion we infer that the five factor models can be considered better models than the four factor versions, although a second observable macroeconomic factor as in SZ5 continues to be questionable.

	GDPn	GDPr	CPI	IP	Prod	CILI
TS	8919	8832	9119	8719	9001	9143
A2	11150	12991	11471	11678	12476	13278
BBB1	12655	13202	11955	12396	13270	13413
	CICI	SZ5u	SZ5	5corr	INF	INFcorr
TS	9146	10885	8745	10689	-	-
A2	13141	12672	13748	13330	13826	13896
BBB1	13408	13125	14054	13620	14555	14746

Table 4.11: Loglikelihoods of the zero rate and credit spread estimations for the different models. The real and inflation short-rate framework (INF, INFcorr) is excluded for the non-defaultable case since its short rate is estimated in two steps and therefore the comparison of its loglikelihoods is not appropriate. The row labeled ”TS” shows the results for the non-defaultable interest rates. The rows marked ”A2” and ”BBB1” contain the loglikelihoods for credit spreads rated A2 and BBB1.

Standardized Innovations: Independent, normally distributed random variables

Finally, the standardized innovations stemming from the Kalman filter have to be tested for being i.i.d. random variables (see e.g. Harvey (1989), Schmid (2002)). Additionally, the state space model requires the standardized innovations to be normally distributed with mean 0. Therefore, we apply the Jarque-Bera test (see Jarque & Bera (1987)) for normal distribution and the Ljung-Box test (see Ljung & Box (1978)) against autocorrelation (both provided by S-PLUS) in order to verify these assumptions. Furthermore, we use a test for homoscedasticity described by Harvey (1981) and a t-test for the hypothesis regarding a mean of zero. We apply these tests for every maturity and every model. The results are presented in the Appendix, Tables B.17 to B.28.

For the short-rate models[XVI] we cannot reject the hypotheses of homoscedasticity and of normal distribution for almost all maturities throughout all frameworks, whereas the hypothesis concerning the mean can only be accepted for the five factor frameworks. Unfortunately, we must reject the hypothesis of no autocorrelation for every framework.

[XVI]We do not include the real and inflation short-rate model (INF, INFcorr) in this analysis since the short rate within this framework is built of two independent processes whose parameters are estimated seperately. Thus, there are no standardized innovations for the short rate but for the real short rate and the inflation short rate. Meyer (2005) and Hagedorn (2005) analyzed those standardized innovations separately.

The tests of the standardized innovations received from credit spreads favour the real and inflation short-rate framework (INF, INFcorr) over SZ5 and 5corr since the hypotheses of homoscedasticity, of normal distribution and of a zero mean cannot be rejected for the majority of maturities. However, within the SZ5 and 5corr models we must often reject the hypotheses of no autocorrelation, of homoscedasticity and of zero mean. In contrast to the short-rate models, the hypothesis of no autocorrelation cannot always be rejected throughout all frameworks. Analogous to previous findings, the results of the 5corr model resemble those of the four factor models, e.g. CILI, CICI, GDPr, GDPn and CPI, as well as the results of the five factor models, e.g. SZ5. Depending on the macroeconomic factor, the four factor models yield different results. The performance of the composite indices CILI and CICI are similar to the performances of five factor models like INF and INFcorr. The Tables B.22 and B.23 in the Appendix display more entries, especially for the categories normal distribution (ND), homoscedasticity (HS) and mean of zero (M0), than the tables of other SZ4 versions indicating that these hypotheses cannot be rejected for more maturities. The hypotheses of no autocorrelation and of a mean of zero must still be rejected for many SZ4 cases but there are exceptions, e.g. the models based on the gross domestic product (GDPn, GDPr) and the composite indices (CILI, CICI). For the frameworks depending on the productivity (Prod) and the industrial production (IP) even the hypotheses of homoscedasticity and of normal distribution must be rejected for many maturities.

Altogether, we can conclude that the assumptions regarding the standardized innovations are fulfilled sufficiently throughout all models whereas the five factor frameworks tend to yield more stable results with respect to the short-rate models. If we additionally consider the tests based on credit spreads, the real and inflation short-rate models (INF, INFcorr) seem to reach the most satisfactory results within the five factor models. Within the four factor frameworks the composite indices CILI and CICI tend to work best.

Conclusion

The pricing errors as well as the various tests lead us to the conclusion that the incorporation of an additional factor does indeed improve the performance of our framework. We showed that the better ability to fit market prices is not only due to the increased number of parameters. Furthermore, an additional factor seems to stabilize the estimation procedure as suggested by the results of the tests of the standardized innovations.

Within our different five factor frameworks the models with a macroeconomic factor which is not observable (SZu, 5corr, INF, INFcorr) yield the most promising results. Especially the performance of the short rate r improves by this additional factor (see SZ5u and 5corr). However, the results for the short-rate spread s do not clearly favour five factor models over four factor ones, but they do suggest that introducing a second observable macroeconomic factor (SZ5) and correlated diffusion terms (5corr) do not improve the models' overall performance.
We tested several macroeconomic fundamentals in order to get a proper choice for the economic factor. The results indicate that all tested variables do a good job in explaining non-defaultable and defaultable zero rates but altogether the composite indices of leading (CILI) and coincident indicators (CICI) as well as the gross domestic product (GDPn, GDPr) seem to be plausible representatives for the state of an economy.
Dependent on the purpose, we recommend the usage of the SZ5u or 5corr frameworks if the main intension lies on the pricing of non-defaultable and defaultable interest-rate products. However, if the focus is mainly on the pricing of defaultable assets, the choice of the four factor framework SZ4 would reduce the complexity of the calculations and would still yield satisfying results when using one of the composite indices (CILI, CICI) or the domestic product (GDPn, GDPr). For pricing inflation-linked products, we suggest the usage of the real and inflation short-rate model (INF, INFcorr). Since the results of INF and INFcorr only differ slightly, we favour the set-up INF because of its reduced number of parameters and therefore its reduced complexity. Although, we found the real and inflation short-rate model to be one of the best, we do not recommend it for pricing purposes in general since the availability of inflation-linked bonds for the calibration procedure is limited. Due to these findings, we continue in the next chapter to develop the pricing formulas for our general set-up of Section 4.1 such that the results will hold for all models discussed in this chapter. In order to get a better insight into the proposed dynamics of Section 5.4.2 and pricing formulas of Sections 5.5 and 5.6, we will use the four factor model SZ4 because of its reduced complexity and its good performance. But all results obtained for the SZ4 framework can be derived analogously for all other frameworks within this thesis.

Our findings are in line with the literature. First, the fact that the perfect macroeconomic factor is not easy to get is indicated by Figlewski et al. (2012) who try to find "stylized facts" about the importance of specific macro-factors and their impact on credit risk. Though there is an increase in explanatory power by adding macroeconomic factors, it is difficult to single out factors which dominate alternative ones. Furthermore, their work shows that the estimated relationships are not stable over time and that the coefficients and the signs of the macroeconomic factors vary widely depending on the additionally included factors. This suggests that there is a considerable correlation among the factors and that their inherent information about credit risk overlap. Second, the closeness of the four and five factor models in pricing credit spreads can be explained by the work of Collin-Dufresne, Goldstein & Martin (2001). Using a regression analysis they conclude that only one quarter of the variation in credit spreads can be explained by economic factors as e.g. the return on S&P and the change in its implied volatility (VIX), whereas the remaining residuals are highly cross-correlated and are mostly driven by a common factor. Hence, they suggest that this common factor is unlikely a firm-specific but a systematic one. Therefore, they redo the regression by incorporating several proxies for this macro factor and find that this only adds limited extra explanatory power. They conclude that there is an aggregate factor which is common to all corporate bonds and which seems to be more important in explaining credit spread changes than firm-specific factors. So, our incorporation of one macroeconomic factor like GDP or especially one of the composite indices in addition to an unobservable factor as an aggregation of firm-specific and/or systematic information, is justified by their findings. Furthermore, the disappointing results of the SZ5 model, which uses two observable factors, and the promising results obtained by the INF and INFcorr frameworks, which are based on a second unobservable factor entering the short-rate spread, are along the line with the study of Collin-Dufresne et al. (2001).

However, two developments which took place in the markets during the first decade of the new millennium are strongly related to the topic of determinants of credit spread and need to be stressed: the recent financial crisis and the growing importance of credit default swaps (CDS). While the majority of studies about determinants of credit spreads rely on bond data before the crisis, newer studies are based on spreads of credit default swaps also incorporating post-crisis data sets. This leaves us wondering whether the previously found determinants still influence credit spreads and whether the results obtained for bond spreads are still valid for CDS spreads.
Apart from the fact that CDS spreads already come as a spread and do not need the specification of a benchmark risk-free curve as it is required for extracting the credit spread out of bond data, CDS spreads also appear to reflect changes in credit risk more efficiently (cf. studies analyzing the relationship between CDS spreads and rating changes, e.g. Hull, Predescu & White (2004), Di Cesare (2005)). Blanco, Brennan & Marsh (2005) claim

that CDS spreads reflect more quickly changes in the underlying's risk than bond spreads do. Further, Blanco et al. (2005) provide evidence that CDS as well as bond markets work quite well in the long run, but in the short run CDS spreads react more timely. They also study the determinants of changes in credit spreads and CDS spreads and conclude that analogous to the study of Collin-Dufresne et al. (2001) a large part of the variation of both spreads cannot be explained and furthermore, that the "first principal component explains a large and essentially identical proportion of the variation of the residuals".

Ericsson, Jacobs & Oviedo (2009) analyze by means of a linear regression approach the dependence of CDS premia and variables suggested by economic theory for the period of 1999 until 2002. Their findings are that the estimated coefficients are consistent with theory and the explanatory power is higher than in existing works on corporate bond spreads, also emphasized by a limited evidence for a residual common factor. This indicates that the variables suggested by economic theory are important for describing the pricing of such instruments. Further, they argue by the similarity of bond and CDS cashflows that occur until (bond coupons vs. CDS spreads) - respectively at (loss in bond vs. replacement of loss for CDS buyer) - default that the implied relationship between theoretical factors and spreads still hold.

Di Cesare & Guazzarotti (2010) study the effects of the financial crisis on determinants of CDS spread changes for the period from January 2002 until March 2009. They confirm that the factors identified by the literature have maintained their importance by showing that the models explain the changes almost the same way before and during the crisis. Further, they claim that the CDS spreads were moving increasingly together during the crisis, indicating the existence of a common factor that still remains unexplained.

Taking this altogether, we are confident that our findings will also hold for the recent time period and for CDS spreads, respectively credit derivatives based on CDS in general. Therefore, we will develop in the next chapter a consistent pricing framework for derivatives written on CDS by deriving dynamics of the CDS spread that reflect the dependence on firm-specific and macroeconomic risk factors.

Chapter 5

Pricing Credit Derivatives

So far, the bond market was assumed to be the best place to monitor the creditworthiness of a borrower. But in the last decade, the market for credit derivatives has grown substantially. Credit derivatives allow investors to buy or sell easily the credit risk of a certain reference entity without being exposed to its default risk, e.g. not owning the defaultable bond of a company on which the derivative is written. Therefore, credit derivatives are used for hedging against credit risk of a certain reference entity as well as for pure speculation, e.g. short-selling credit risk. The most popular credit derivative is the Credit Default Swap (CDS) which works like an insurance. The buyer of a CDS (the protection buyer) makes regular premium payments to the protection seller in order to be compensated if a certain credit event (default) occurs. Usually, credit events that trigger such protection payments are bankruptcy, failure to pay or restructuring.

Antes et al. (2009) determine closed-form solutions for credit default options and credit default swaps for the extended Schmid-Zagst model which we study in Chapter 4 (see Model 4.4). In their work, the four factors of the SZ4 model are calibrated to historical data of the period 2002-2008 and fitted to market prices of credit default swaps. Antes et al. (2009) show that the model performs well even during the crisis of 2007/2008 and is capable of displaying the latest market signals, e.g. an increase of credit risk.

In this chapter we will build on the work of Antes et al. (2009) and rewrite the pricing of credit default options and swaps for our general five factor framework that was introduced in Section 4.1 of Chapter 4 and from which all previously discussed models were derived (cf. Sections 4.2 - 4.5). Therefore, if not stated otherwise all results refer to the pricing framework of Section 4.1. Further, we extend it to more complex derivatives like Credit Default Swaptions (see Section 5.5) and Constant Maturity Credit Default Swaps (see Section 5.6) which both rely on the future spread of a CDS. Therefore,

we introduce in Section 5.4 so-called Forward Credit Default Swaps for different assumptions concerning the proctection payments. Further, we derive the dynamics of the FCDS spread by means of its closed-form solution while keeping the link to macroeconomic and firm-specific factors. Our procedure ensures that we are consistent with our defaultable term structure model of Chapter 4 in addition to being able to price analytically complex credit derivatives.
During the credit crisis, investors began to worry about the creditworthiness of their counterparties as well as the missing standardization of credit derivatives, making it impossible to compare contracts. Especially for CDS, the tailor-made contracts lead to spreads that are not comparable. Therefore, a new quoting mechanism was introduced in 2009 (see e.g. Markit (2009a)) which proposes to only use a constant set of spreads for pricing CDS and exchanging an upfront payment instead. We outline in Section 5.5.1 how the pricing of a Credit Default Swaption could be amended in order to account for this standardization. In addition, we show how to include the creditworthiness of the derivative's counterparty in the derivation of a CDS spread (see Subsection 5.4.3).
In the following three sections, we introduce the necessary building terms for valuing CDS. The definitions, theorems and propositions of these subsections are cited or inspired by the work of Schmid (2002) and Antes et al. (2009).

In the previous chapter the short-rate spread s was used to price defaultable zero-coupon bonds. In this chapter, we want to price derivatives whose underlying is such a defaultable zero-coupon bond or is related to one. But, since the recovery rate of such a derivative deviates from the recovery rate of the underlying, we need a short-rate spread s^{zero} which is of the same quality as the short-rate spread s but with a recovery of zero.

Definition 5.1
The zero-recovery short-rate spread s^{zero} is implicitely given by

$$(1 - z(t)) \cdot s^{zero}(t) = s(t) \; , 0 \leq t \leq T^* \; ,$$

where $s(t)$ is the short-rate spread process and $z(t)$ is the recovery-rate process with $0 \leq z(t) < 1$.

By introducing the zero-recovery short-rate spread, we are now able to price under a zero-recovery assumption. As opposed to reduced-form frameworks we do not rely on a non-negative intensity for pricing defaultable contingent claims. Instead our model as well as the models in e.g. Antes et al. (2009)

and Schmid (2004) are based on a defaultable money-market account, i.e.

$$P_0^d(t) = \left(1 + \int_0^t (z(l) - 1)dH(l)\right) e^{\int_0^t r(l)+s(l)L(l)dl},$$

where the short-rate spread s is not necessarily non-negative.

Proposition 5.2
Let Y be a $\mathcal{F}_T$−measurable random variable with $\mathbb{E}_{\widetilde{Q}}\left[|Y|^q\right] < \infty$ for some $q > 1$. Under the zero-recovery assumption, i.e. under the assumption that the contingent claim is knocked out at default of the reference credit asset, and with the stochastic processes specified for r, w_1, w_2, s, u, and s^{zero}, the price process, $V_{L,T}$:

$$V_{L,T}(t) = \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl}\; Y \cdot L(T)\right|\left.\mathcal{G}_t\right],\; 0 \leq t < T^d,$$

is given by

$$V_{L,T}(t) = L(t) \cdot V_T(t),$$

with $L(t) = \mathbf{1}_{\{T^d > t\}}$. The adapted continuous process V_T is defined by

$$V_T(t) = \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} Y \mid \mathcal{F}_t\right],\; 0 \leq t < T,$$

and $V_T(t) = 0$ for $t \geq T$. This equation has a unique solution in the space consisting of every semimartingale, J, such that $\mathbb{E}_{\widetilde{Q}}\left[\sup_t |J_t|^q\right] < \infty$ for some $q > 1$.

Proof:
See Proposition 6.4.1 in Schmid (2004), page 230, where $\mathcal{F}_t$ needs to be replaced by $\mathcal{G}_t$ in order to get the result under the enlarged filtration $\mathbb{G}$. Using the martingale invariance property (see page 31 in Chapter 4) yields the stated result. □

Remark:
The above Proposition holds for our general framework of Section 4.1 as well as for all models derived from it (cf. Sections 4.2 - 4.5), since in all set-ups the processes r and s are solutions of linear stochastic differential equations or are an affine linear function of such solutions (cf. the short-rate r in Section 4.4 where the short rate is the sum of the inflation short rate and the real short rate). Therefore, r and s fulfill the technical requirements needed for Proposition 5.2.

In the following, we assume the recovery rate z to be a known constant. With this assumption, we are able to state the dynamics for s^{zero} as well as the price of a defaultable zero-recovery zero-coupon bond.

Proposition 5.3
Let $z(t) = z$ be a constant for all $0 \leq t \leq T^*$, then the SDE for the zero-recovery short-rate spread under the equivalent martingale measure $\widetilde{Q}$ is given by

$$\begin{aligned} ds^{zero}(t) &= [\theta_{s^{zero}} + b_{s^{zero}u}u(t) - b_{s^{zero}w_1}w_1(t) - b_{s^{zero}w_2}w_2(t) - \hat{a}_s s^{zero}(t)]dt \\ &\quad + \sigma_{s^{zero}}\sqrt{1 - \rho_{su}^2 - \rho_{sw_1}^2 - \rho_{sw_2}^2}\,d\widetilde{W}_s(t) + \sigma_{s^{zero}}\rho_{su}d\widetilde{W}_u(t) \\ &\quad + \sigma_{s^{zero}}\rho_{sw_1}d\widetilde{W}_{w_1}(t) + \sigma_{s^{zero}}\rho_{sw_2}d\widetilde{W}_{w_2}(t)\,, \end{aligned}$$

with $\theta_{s^{zero}} = \frac{\theta_s}{1-z}$, $b_{s^{zero}u} = \frac{b_{su}}{1-z}$, $b_{s^{zero}w_1} = \frac{b_{sw_1}}{1-z}$, $b_{s^{zero}w_2} = \frac{b_{sw_2}}{1-z}$, $\sigma_{s^{zero}} = \frac{\sigma_s}{1-z}$. Furthermore, the price of a zero-recovery zero-coupon bond is given by

$$\begin{aligned} P^{d,zero}(t,T) &= e^{A^{d,zero}(t,T) - B^{d,zero}(t,T)r - C^{d,zero}(t,T)s^{zero} - D^{d,zero}(t,T)u} \\ &\quad \cdot e^{-E_1^{d,zero}(t,T)w_1 - E_2^{d,zero}(t,T)w_2} \end{aligned}$$

where the functions $A^{d,zero}(t,T)$, $B^{d,zero}(t,T)$, $C^{d,zero}(t,T)$, $D^{d,zero}(t,T)$, $E_1^{d,zero}(t,T)$ and $E_2^{d,zero}(t,T)$ have the same structure as in Theorem 4.3 with θ_s, b_{su}, b_{sw_1}, b_{sw_2} and σ_s substituted by $\theta_{s^{zero}}$, $b_{s^{zero}u}$, $b_{s^{zero}w_1}$, $b_{s^{zero}w_2}$ and $\sigma_{s^{zero}}$.

Proof:
Analogously to the proof of Theorem 4.3. □

5.1 Survival Probability

With s^{zero} being an approximation of the intensity, it holds

$$\begin{aligned} \widetilde{Q}(T^d > t|\mathcal{F}_t) &= \mathbb{E}_{\widetilde{Q}}\left[L(t)|\mathcal{F}_t\right] \\ &= e^{-\int_0^t s^{zero}(l)dl}. \end{aligned}$$

Hence the survival probability within our general framework is as follows.

Theorem 5.4 ***(Survival Probability)***
For $0 \leq t \leq T$ the survival probability up to time T conditioned on the information of time t is:

$$
\begin{aligned}
L(t) \cdot P^S(t,T) : &= \mathbb{E}_{\widetilde{Q}}\left[L(T)|\mathcal{G}_t\right] \\
&= L(t) \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T s^{zero}(l)dl}|\mathcal{F}_t\right] \\
&= L(t) \cdot e^{A^S(t,T)-C^S(t,T)s^{zero}-D^S(t,T)u-E_1^S(t,T)w_1-E_2^S(t,T)w_2} \\
&= L(t) \cdot P^S(t,T,s^{zero},u,w_1,w_2)
\end{aligned}
$$

with

$$
\begin{aligned}
C^S(t,T) &= \frac{1}{\hat{a}_s}\left(1-e^{-\hat{a}_s(T-t)}\right), \\
D^S(t,T) &= \frac{b_{s^{zero}u}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_u(T-t)}}{\hat{a}_u}+\frac{e^{-\hat{a}_u(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_u-\hat{a}_s}\right), \\
E_1^S(t,T) &= -\frac{b_{s^{zero}w_1}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_1}-\hat{a}_s}\right), \\
E_2^S(t,T) &= -\frac{b_{s^{zero}w_2}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_{w_2}-\hat{a}_s}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
A^S(t,T) = \int_t^T &\Big[\frac{1}{2}\sigma^2_{s^{zero}}(C^S(l,T))^2+\frac{1}{2}\sigma_u^2(D^S(l,T))^2+\frac{1}{2}\sigma^2_{w_1}(E_1^S(l,T))^2 \\
&+\frac{1}{2}\sigma^2_{w_2}(E_2^S(l,T))^2+\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^S(l,T)E_2^S(l,T) \\
&+\sigma_u\sigma_{s^{zero}}\rho_{su}C^S(l,T)D^S(l,T)+\sigma_{w_1}\sigma_{s^{zero}}\rho_{sw_1}C^S(l,T)E_1^S(l,T) \\
&+\sigma_{w_2}\sigma_{s^{zero}}\left(\rho_{sw_1}\rho_{w_1w_2}+\rho_{sw_2}\sqrt{1-\rho^2_{w_1w_2}}\right)C^S(l,T)E_2^S(l,T) \\
&-\theta_{s^{zero}}C^S(l,T)-\theta_u D^S(l,T)-\theta_{w_1}E_1^S(l,T)-\theta_{w_2}E_2^S(l,T)\Big]dl.
\end{aligned}
$$

The proof of this Theorem follows directly from Theorem 4.3 by setting all terms related to r equal to zero and replacing all terms indexed by s with equivalent terms indexed by s^{zero}.

5.2 Default Digital Put Option

Credit derivatives are mainly dependent on the time of default (T^d) and payments that are triggered by the default event. A simple credit derivative pays a fixed payoff that is due on the default of a reference entity.

Definition 5.5
A default digital put option pays a fixed payoff p at the time of default (T^d) of a reference credit asset.

Theorem 5.6
The time t price of the default digital put option is given by

$$\mathbb{E}_{\widetilde{Q}}\left[\int_t^T p \cdot e^{-\int_t^u r(l)dl} dH(u) \middle| \mathcal{G}_t\right] = L(t) \cdot p \cdot V_{T^d}^{ddp}(t,T)$$

with

$$\begin{aligned} V_{T^d}^{ddp}(t,T) &= \mathbb{E}_{\widetilde{Q}}\left[\int_t^T e^{-\int_t^u (r(l)+s^{zero}(l))dl} s^{zero}(u)\, du \middle| \mathcal{F}_t\right] \\ &= \int_t^T \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^u (r(l)+s^{zero}(l))dl} s^{zero}(u) \middle| \mathcal{F}_t\right] du. \end{aligned}$$

Proof:
See page 243 in Schmid (2004) where $\mathcal{F}_t$ needs to be replaced by $\mathcal{G}_t$ in order to get the result under the enlarged filtration $\mathbb{G}$. The stated result follows from using the martingale invariance property (see page 31 in Chapter 4). □

Remark:
Analogously to Proposition 5.2, the above result also holds for our general framework of Section 4.1 because of r and s^{zero} being solutions of linear stochastic differential equations. Therefore, the necessary technical conditions hold for applying Corollary 6.2.1 in Schmid (2004) and for interchanging expectation and integration.

In order to calculate the expected value we need the following theorem.

Theorem 5.7

$$\begin{aligned} v(r, s^{zero}, u, w_1, w_2, t, T) &:= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} s^{zero}(T) \middle| \mathcal{F}_t\right] \\ &= P^{d,zero}(t,T) \\ &\quad \cdot (F(t,T) + H(t,T)s^{zero}(t) + I(t,T)u(t) \\ &\quad + J_1(t,T)w_1(t) + J_2(t,T)w_2(t)) \end{aligned}$$

with

$$
\begin{aligned}
H(t,T) &= e^{-\hat{a}_s(T-t)}, \\
J_1(t,T) &= b_{s^{zero}w_1} \cdot \frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_s - \hat{a}_{w_1}}, \\
J_2(t,T) &= b_{s^{zero}w_2} \cdot \frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_s - \hat{a}_{w_2}}, \\
I(t,T) &= -b_{s^{zero}u} \cdot \frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_u(T-t)}}{\hat{a}_s - \hat{a}_u}, \\
F(t,T) &= -\frac{1}{2}\left(\left(\sigma_{s^{zero}}C^{d,zero}(t,T)\right)^2 + \left(\sigma_u D^{d,zero}(t,T)\right)^2\right) \\
&\quad +\theta_{w_1} \cdot (E_1^{d,zero}(t,T) - E_1(t,T)) + \theta_{w_2} \cdot (E_2^{d,zero}(t,T) - E_2(t,T)) \\
&\quad +\theta_u D^{d,zero}(t,T) + \theta_{s^{zero}}C^{d,zero}(t,T) \\
&\quad -\int_t^T \sigma_{w_1}^2 E_1^{d,zero}(l,T)J_1(l,T) + \sigma_{w_2}^2 E_2^{d,zero}(l,T)J_2(l,T)dl \\
&\quad -\int_t^T \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}(E_1^{d,zero}(l,T)J_2(l,T) + E_2^{d,zero}(l,T)J_1(l,T))dl \\
&\quad -\int_t^T \sigma_{w_1}\sigma_r\rho_{rw_1}B^{d,zero}(l,T)J_1(l,T)dl \\
&\quad -\int_t^T \sigma_{w_2}\sigma_r(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2})B^{d,zero}(l,T)J_2(l,T)dl \\
&\quad -\int_t^T \sigma_{s^{zero}}\sigma_u\rho_{su}(C^{d,zero}(l,T)I(l,T) + D^{d,zero}(l,T)H(l,T))dl \\
&\quad -\int_t^T \sigma_{s^{zero}}\sigma_{w_1}\rho_{sw_1}(E_1^{d,zero}(l,T)H(l,T) + C^{d,zero}(l,T)J_1(l,T))dl \\
&\quad -\int_t^T \sigma_{s^{zero}}\sigma_r(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})B^{d,zero}(l,T)H(l,T)dl \\
&\quad -\int_t^T \sigma_{w_2}\sigma_{s^{zero}}(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2})(C^{d,zero}(l,T)J_2(l,T) \\
&\quad +E_2^{d,zero}(l,T)H(l,T))dl.
\end{aligned}
$$

The proof of this Theorem can be found in Appendix C on page 261.

5.3 Default Put Option

Instead of just paying a fixed amount at default, a Default Put Option makes a payment at the time of default (T^d) that is linked to the value of the reference asset at T^d.

Definition 5.8
A default put on a zero-coupon bond pays at the time of default (T^d) of the bond a payoff which depends on the underlying's value at default.

Theorem 5.9
For $t \leq T \leq T^*$ the price of a default put with maturity T whose underlying reference asset is a zero-coupon bond maturing at T^* is given by

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[\int_t^T e^{-\int_t^l r(x)dx}(1 - Z(l))dH(l) \,\middle|\, \mathcal{G}_t\right] \\
&= L(t) \cdot V_{T^d}^{dp}(t, T, T^*) \\
&= L(t) \cdot (V_{T^d}^{ddp}(t, T) - P^d(t, T^*) + P^{d,*}(t, T, T^*))
\end{aligned}$$

where the payoff takes place at default (T^d) by replacement to the difference of par, i.e. the difference between the face value and the market value Z at default. $P^{d,*}(t, T, T^*)$ is derived in Proposition 5.10.

Proof:
The proof is given by Antes et al. (2009). For a better understanding, we will state the proof as well.
Let Z be the value of the zero-coupon bond upon default, then we obtain

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[\int_t^T e^{-\int_t^l r(x)dx}(1 - Z(l))dH(l) \,\middle|\, \mathcal{G}_t\right] \\
&= L(t)V_{T^d}^{ddp}(t, T) - \mathbb{E}_{\widetilde{Q}}\left[\int_t^{T^*} e^{-\int_t^l r(x)dx} Z(l)dH(l) \,\middle|\, \mathcal{G}_t\right] \\
&\quad + \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)dx} L(T) \int_T^{T^*} e^{-\int_T^l r(x)dx} Z(l)dH(l) \,\middle|\, \mathcal{G}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= L(t)\cdot(V^{ddp}_{T^d}(t,T) - P^d(t,T^*) + P^{d,zero}(t,T^*)) \\
&\quad + \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)dx} L(T)\mathbb{E}_{\widetilde{Q}}\left[\int_T^{T^*} e^{-\int_T^l r(x)dx} Z(l)dH(l)\,\middle|\, \mathcal{G}_T\right]\middle|\, \mathcal{G}_t\right] \\
&\quad + \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)dx} L(T)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_T^{T^*} r(x)dx} L(T^*)\,\middle|\, \mathcal{G}_T\right]\middle|\, \mathcal{G}_t\right] \\
&\quad - \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)dx} L(T)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_T^{T^*} r(x)dx} L(T^*)\,\middle|\, \mathcal{G}_T\right]\middle|\, \mathcal{G}_t\right] \\
&= L(t)\cdot(V^{ddp}_{T^d}(t,T) - P^d(t,T^*) + P^{d,zero}(t,T^*)) \\
&\quad + \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)dx} L(T)P^d(T,T^*)\,\middle|\, \mathcal{G}_t\right] \\
&\quad - \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)dx} L(T)P^{d,zero}(T,T^*)\,\middle|\, \mathcal{G}_t\right] \\
&\overset{Prop.5.2}{=} L(t)\cdot(V^{ddp}_{T^d}(t,T) - P^d(t,T^*) + P^{d,zero}(t,T^*)) \\
&\quad + L(t)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T (r(x)+s^{zero}(x))dx} P^d(T,T^*)\,\middle|\, \mathcal{F}_t\right] \\
&\quad - L(t)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T (r(x)+s^{zero}(x))dx} P^{d,zero}(T,T^*)\,\middle|\, \mathcal{F}_t\right] \\
&= L(t)\cdot(V^{ddp}_{T^d}(t,T) - P^d(t,T^*)) \\
&\quad + L(t)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T (r(x)+s^{zero}(x))dx} P^d(T,T^*)\,\middle|\, \mathcal{F}_t\right] \\
&= L(t)\cdot(V^{ddp}_{T^d}(t,T) - P^d(t,T^*) + P^{d,*}(t,T,T^*)) \\
&=: L(t)\cdot V^{dp}_{T^d}(t,T,T^*),
\end{aligned}
$$

with $P^{d,*}(t,T,T^*) := \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)+s^{zero}(x)dx} P^d(T,T^*)\,\middle|\, \mathcal{F}_t\right]$ which is given in the following proposition. □

Proposition 5.10
For $t < T^d$

$$
\begin{aligned}
P^{d,*}(t,T,T^*) &:= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)+s^{zero}(x)dx} P^d(T,T^*)\,\middle|\, \mathcal{F}_t\right] \\
&= P^{d,*}(t,T,T^*,r(t),s(t),u(t),w_1(t),w_2(t))
\end{aligned}
$$

is given by

$$
\begin{aligned}
P^{d,*}(t,T,T^*) &= e^{A^{d,*}(t,T,T^*)-B^{d,*}(t,T,T^*)r-C^{d,*}(t,T,T^*)s-D^{d,*}(t,T,T^*)u} \\
&\quad \cdot e^{-E_1^{d,*}(t,T,T^*)w_1-E_2^{d,*}(t,T,T^*)w_2}
\end{aligned}
$$

with

$$
\begin{aligned}
B^{d,*}(t,T,T^*) &= B(t,T^*), \\
C^{d,*}(t,T,T^*) &= e^{-\hat{a}_s(T-t)}C^d(T,T^*) + \frac{1}{1-z}C^d(t,T), \\
D^{d,*}(t,T,T^*) &= e^{-\hat{a}_u(T-t)}D^d(T,T^*) + \frac{1}{1-z}D^d(t,T) \\
&\quad -b_{su}C^d(T,T^*)\left(\frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_u(T-t)}}{\hat{a}_s - \hat{a}_u}\right),
\end{aligned}
$$

$$
\begin{aligned}
&E_1^{d,*}(t,T,T^*) \\
&= e^{-\hat{a}_{w_1}(T-t)}E_1^d(T,T^*) \\
&\quad + \frac{b_{rw_1}}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + e^{-\hat{a}_r(T^*-T)}\frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1} - \hat{a}_r}\right) \\
&\quad + b_{sw_1}C^d(T,T^*)\left(\frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_s - \hat{a}_{w_1}}\right) \\
&\quad + \frac{1}{1-z}(E_1^d(t,T) - E_1(t,T)),
\end{aligned}
$$

$$
\begin{aligned}
&E_2^{d,*}(t,T,T^*) \\
&= e^{-\hat{a}_{w_2}(T-t)}E_2^d(T,T^*) \\
&\quad + \frac{\hat{b}_{rw_2}}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_{w_2}} + e^{-\hat{a}_r(T^*-T)}\frac{e^{-\hat{a}_{w_2}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_2} - \hat{a}_r}\right) \\
&\quad + b_{sw_2}C^d(T,T^*)\left(\frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_{w_2}(T-t)}}{\hat{a}_s - \hat{a}_{w_2}}\right) \\
&\quad + \frac{1}{1-z}(E_2^d(t,T) - E_2(t,T)),
\end{aligned}
$$

and

$$\begin{aligned}
&A^{d,*}(t,T,T^*)\\
&= A^d(T,T^*) + \int_t^T \frac{1}{2}\Big(\sigma_r^2(B^{d,*}(l,T,T^*))^2 + \sigma_s^2(C^{d,*}(l,T,T^*))^2\\
&\quad + \sigma_u^2(D^{d,*}(l,T,T^*))^2 + \sigma_{w_1}^2(E_1^{d,*}(l,T,T^*))^2 + \sigma_{w_2}^2(E_2^{d,*}(l,T,T^*))^2\\
&\quad + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^{d,*}(l,T,T^*)E_2^{d,*}(l,T,T^*)\\
&\quad + \sigma_r\sigma_{w_1}\rho_{rw_1}B^{d,*}(l,T,T^*)E_1^{d,*}(l,T,T^*)\\
&\quad + \sigma_s\sigma_u\rho_{su}C^{d,*}(l,T,T^*)D^{d,*}(l,T,T^*)\\
&\quad + \sigma_s\sigma_{w_1}\rho_{sw_1}C^{d,*}(l,T,T^*)E_1^{d,*}(l,T,T^*)\\
&\quad + \sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)B^{d,*}(l,T,T^*)E_2^{d,*}(l,T,T^*)\\
&\quad + \sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)C^{d,*}(l,T,T^*)E_2^{d,*}(l,T,T^*)\\
&\quad + \sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})B^{d,*}(l,T,T^*)C^{d,*}(l,T,T^*)\\
&\quad - \theta_r(l)B^{d,*}(l,T,T^*) - \theta_sC^{d,*}(l,T,T^*) - \theta_uD^{d,*}(l,T,T^*)\\
&\quad - \theta_{w_1}E_1^{d,*}(l,T,T^*) - \theta_{w_2}E_2^{d,*}(l,T,T^*)dl.
\end{aligned}$$

The proof of this proposition is given in Appendix C on page 263.

5.4 Forward Credit Default Swap

Credit Default Swaps (CDS) are recently the most popular credit derivatives. The CDS market has grown fastly during the last years due to the characteristics of these derivatives. Although a CDS is a form of insurance, it is not dependent on the real exposure of the underlying asset. Hence, Credit Default Swaps allow to buy and sell protection without the need of holding the respective underlying asset. Therefore, CDS are not only used for hedging purposes but also for taking speculative positions.

Definition 5.11 ***(Credit Default Swap)***
A Credit Default Swap (CDS) is a contract where the protection buyer pays a regular spread $s(T_0, T_0, T_m)$ at times T_i, $i = 1 \dots m$ to the protection seller as long as the reference entity has not defaulted. At default ($T^d < T_m$) of the reference asset, the protection seller makes a replacement payment to the protection buyer.

Definition 5.12 ***(Forward Credit Default Swap)***
A Forward Credit Default Swap (FCDS) is a contract which is entered at time $t < T_0$ and which consists of a CDS starting in T_0 with payments of $s(t, T_0, T_m)$ at times T_i, $i = 1 \ldots m$. The contract expires without any payments if there is a default before T_0.

The value of a Credit Default Swap is mainly determined by the sort of payments that are made at the time of default T^d. Assuming that the protection payment is linked to the value of the reference entity by replacing its difference to par at default, the price of the CDS can be derived by means of the Default Put Option of Theorem 5.9.

Theorem 5.13
If the underlying reference asset is a zero-coupon bond with maturity T^*, then for $t < T_0 < T_m \leq T^*$ the spread $s(t, T_0, T_m)$ of the FCDS is

$$s(t, T_0, T_m) \quad = \quad \frac{V^{dp}_{T^d}(t, T_0, T_m, T^*)}{\sum_{i=1}^m P^{d,zero}(t, T_i)}$$

with $V^{dp}_{T^d}(t, T_0, T_m, T^*) = V^{ddp}_{T^d}(t, T_m) - V^{ddp}_{T^d}(t, T_0) + P^{d,*}(t, T_m, T^*) - P^{d,*}(t, T_0, T^*)$.

Proof:
The value of the premium leg of a swap starting at T_0 and ending at T_m is given by the discounted sum of swap spread payments:

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[s \sum_{i=1}^m e^{-\int_t^{T_i} r(l)dl} L(T_i) \,\middle|\, \mathcal{G}_t \right] \\
&= \quad s \sum_{i=1}^m \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^{T_i} r(l)dl} L(T_i) \,\middle|\, \mathcal{G}_t \right] \\
&\overset{Prop.5.2}{=} s \sum_{i=1}^m L(t) \cdot \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^{T_i} r(l) + s^{zero}(l)dl} \,\middle|\, \mathcal{F}_t \right] \\
&= \quad s \cdot L(t) \sum_{i=1}^m P^{d,zero}(t, T_i)
\end{aligned}$$

The value of the protection leg at time t equals the value of a default put (c.f. Theorem 5.9) starting at T_0 and maturing at T_m:[I]

[I]For Fubini's Theorem see Duffie (1996), page 282.

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0) \int_{T_0}^{T_m} e^{-\int_{T_0}^{l} r(x)dx}(1-Z(l))dH(l) \,\middle|\, \mathcal{G}_t\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0) \underbrace{\mathbb{E}_{\widetilde{Q}}\left[\int_{T_0}^{T_m} e^{-\int_{T_0}^{l} r(x)dx}(1-Z(l))dH(l) \,\middle|\, \mathcal{G}_{T_0}\right]}_{\overset{Th.5.9}{=} L(T_0)\cdot V_{T^d}^{dp}(T_0,T_m,T^*) \text{ with } V_{T^d}^{dp}(T_0,\cdot,\cdot)\ \mathcal{F}_{T_0}-measurable} \,\middle|\, \mathcal{G}_t \right] \\
&\overset{Prop.5.2}{=} L(t)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} V_{T^d}^{dp}(T_0,T_m,T^*) \,\middle|\, \mathcal{F}_t\right] \\
&\overset{Th.5.9}{=} L(t)\Big(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl}(V_{T^d}^{ddp}(T_0,T_m) - P^d(T_0,T^*)) \,\middle|\, \mathcal{F}_t\right] \\
&\quad +\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} P^{d,*}(T_0,T_m,T^*)) \,\middle|\, \mathcal{F}_t\right]\Big) \\
&= L(t)\Big(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} V_{T^d}^{ddp}(T_0,T_m) \,\middle|\, \mathcal{F}_t\right] - P^{d,*}(t,T_0,T^*) \\
&\quad + \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_{T_0}^{T_m} r(l)+s^{zero}(l)dl} P^d(T_m,T^*) \,\middle|\, \mathcal{F}_{T_0}\right] \,\middle|\, \mathcal{F}_t\right]\Big) \\
&= L(t)\Big(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} V_{T^d}^{ddp}(T_0,T_m) \,\middle|\, \mathcal{F}_t\right] - P^{d,*}(t,T_0,T^*) \\
&\quad + \underbrace{\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r(l)+s^{zero}(l)dl} P^d(T_m,T^*) \,\middle|\, \mathcal{F}_t\right]}_{=P^{d,*}(t,T_m,T^*)}\Big) \\
&= L(t)\Big(\mathbb{E}_{\widetilde{Q}}\Big[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} \int_{T_0}^{T_m} \mathbb{E}_{\widetilde{Q}}\Big[e^{-\int_{T_0}^{l} r(x)+s^{zero}(x)dx} s^{zero}(l) \,\Big|\, \mathcal{F}_{T_0}\Big] dl \,\Big|\, \mathcal{F}_t\Big] \\
&\quad - P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*)\Big) \\
&= L(t)\Big(\mathbb{E}_{\widetilde{Q}}\left[\int_{T_0}^{T_m} e^{-\int_t^{l} r(x)+s^{zero}(x)dx} s^{zero}(l)dl \,\middle|\, \mathcal{F}_t\right] \\
&\quad - P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*)\Big) \\
&\overset{Fubini}{=} L(t)\Big(\int_{T_0}^{T_m} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{l} r(x)+s^{zero}(x)dx} s^{zero}(l) \,\middle|\, \mathcal{F}_t\right] dl \\
&\quad -P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= L(t)\Big(\int_t^{T_m} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^l r(x)+s^{zero}(x)dx} s^{zero}(l) \,\Big|\, \mathcal{F}_t \right] dl \\
&\quad - \int_t^{T_0} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^l r(x)+s^{zero}(x)dx} s^{zero}(l) \,\Big|\, \mathcal{F}_t \right] dl \\
&\quad - P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*) \Big) \\
&= L(t)(V_{T^d}^{ddp}(t,T_m) - V_{T^d}^{ddp}(t,T_0) - P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*))
\end{aligned}
$$

$\Rightarrow$ The swap spread of a Forward Credit Default Swap is determined by equating the premium leg and the protection leg:

$$
\begin{aligned}
s(t,T_0,T_m) &= \frac{V_{T^d}^{ddp}(t,T_m) - V_{T^d}^{ddp}(t,T_0) + P^{d,*}(t,T_m,T^*) - P^{d,*}(t,T_0,T^*)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \\
&=: \frac{V_{T^d}^{dp}(t,T_0,T_m,T^*)}{\sum_{i=1}^m P^{d,zero}(t,T_i)}
\end{aligned}
$$

□

Proposition 5.14
The spread $s(T_0,T_0,T_m)$ at T_0 of a CDS contract starting in T_0 with the same characteristics as in Definition 5.12 is given by

$$
s(T_0,T_0,T_m) = \frac{V_{T^d}^{ddp}(T_0,T_m) + P^{d,*}(T_0,T_m,T^*) - P^d(T_0,T^*)}{\sum_{i=1}^m P^{d,zero}(T_0,T_i)} \quad .
$$

Proof:
For $t := T_0$ the spread $s(t,T_0,T_m)$ of Theorem 5.13 simplifies to the stated result with $V_{T^d}^{ddp}(T_0,T_0) = 0$ and $P^{d,*}(T_0,T_0,T^*) = P^d(T_0,T^*)$. □

Alternatively, the protection payment of a Credit Default Swap can be fixed in advance similar to a Default Digital Put Option of Theorem 5.6. For example, it is common to assume for quoting purposes a recovery rate Z of 40% in the CDS market.

Theorem 5.15
If the CDS pays at default a fraction of the face value, the swap spread of a Forward Credit Default Swap simplifies to

$$s(t,T_0,T_m) \;=\; \frac{V^{dp}_{ZT^d}(t,T_0,T_m)}{\sum_{i=1}^m P^{d,zero}(t,T_i)}$$

with $V^{dp}_{ZT^d}(t,T_0,T_m) = (1-Z)(V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0))$ *and recovery rate* Z.

Proof:
The premium leg is the same as in Theorem 5.13. The protection leg is calculated analogously to Theorem 5.13:

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0)\int_{T_0}^{T_m} e^{-\int_{T_0}^{l} r(x)dx}(1-Z)dH(l)\,\middle|\, \mathcal{G}_t\right]\\
&= (1-Z)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl}L(T_0)\quad \underbrace{\mathbb{E}_{\widetilde{Q}}\left[\int_{T_0}^{T_m} e^{-\int_{T_0}^{l} r(x)dx}dH(l)\,\middle|\,\mathcal{G}_{T_0}\right]}_{\overset{Th.5.6}{=} L(T_0)\cdot V^{ddp}_{T^d}(T_0,T_m)\ with\ V^{ddp}_{T^d}(T_0,\cdot)\ \mathcal{F}_{T_0}-meas.}\,\middle|\,\mathcal{G}_t\right]\\
&\overset{Prop.5.2}{=} L(t)\cdot(1-Z)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl}V^{ddp}_{T^d}(T_0,T_m)\,\middle|\,\mathcal{F}_t\right]\\
&= \ldots \text{ (see Theorem 5.13)}\\
&= L(t)\cdot(1-Z)(V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0))\\
&=: L(t)\cdot V^{dp}_{ZT^d}(t,T_0,T_m)\,.
\end{aligned}$$

By equating the two legs and solving for $s(t,T_0,T_m)$ we obtain the stated result. □

Proposition 5.16
The spread $s(T_0,T_0,T_m)$ *at* T_0 *of a CDS contract starting in* T_0 *with the same characteristics as in Theorem 5.15 is given by*

$$s(T_0,T_0,T_m) = \frac{(1-Z)V^{ddp}_{T^d}(T_0,T_m)}{\sum_{i=1}^m P^{d,zero}(T_0,T_i)}\quad .$$

Proof:
For $t := T_0$ the spread $s(t, T_0, T_m)$ of Theorem 5.15 simplifies to the stated result with $V_{T^d}^{ddp}(T_0, T_0) = 0$. □

In order to calculate the protection leg of the above CDS, a numerical integration has to be performed since the function $V_{T^d}^{ddp}$ cannot be calculated analytically. For an approximation, we use the so-called default bucketing (see Brigo & Chourdakis (2009)) where we divide the period $[T_0, T_m]$ in intervals $[\widetilde{T}_{j-1}, \widetilde{T}_j]$, $j = 1 \dots n$, $T_0 = \widetilde{T}_0 < \widetilde{T}_1 < \cdots < \widetilde{T}_n = T_m$ and delay the default payment until the end of the corresponding interval. If the length of such an interval is chosen adequately, the time gap between the times when the payment should be due (T^d) and when it is assumed to be made ($\widetilde{T}_j$) is almost neglectable.
For the approximated CDS rate, we need the following proposition:

Proposition 5.17
For $t < T^d$

$$\begin{aligned} P^*(t, \widetilde{T}_{j-1}, \widetilde{T}_j) &:= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_{j-1}} r(x)+s^{zero}(x)dx} P(\widetilde{T}_{j-1}, \widetilde{T}_j) \,\middle|\, \mathcal{F}_t \right] \\ &= P^*(t, \widetilde{T}_{j-1}, \widetilde{T}_j, r(t), s^{zero}(t), u(t), w_1(t), w_2(t)) \end{aligned}$$

is given by

$$\begin{aligned} P^*(t, \widetilde{T}_{j-1}, \widetilde{T}_j) &= e^{A^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - B^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) r - C^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) s^{zero} - D^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) u} \\ &\quad \cdot e^{-E_1^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) w_1 - E_2^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) w_2} \end{aligned}$$

with $A^*, B^*, C^*, D^*, E_1^*$, and E_2^* given in the proof in Appendix C.

The proof of the proposition is given in Appendix C on page 266.

The next theorem illustrates how the default bucketing leads to a simplified calculation of the FCDS spread since the terms of the Default Digital Put Option ($V_{T^d}^{ddp}$) vanish.

Theorem 5.18
If the protection is paid as a fraction of the face value as in Theorem 5.15 and the protection payments are only made at certain dates $\widetilde{T}_j, j = 1 \dots n$,

the swap spread of a Forward Credit Default Swap simplifies to

$$
\begin{aligned}
s(t,T_0,T_m) &= \frac{(1-Z)\sum_{j=1}^{n}\left(P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - P^{d,zero}(t,\widetilde{T}_j)\right)}{\sum_{i=1}^{m} P^{d,zero}(t,T_i)} \\
&= \frac{V^{dp}_{Z\widetilde{T}}(t,T_0,T_m)}{\sum_{i=1}^{m} P^{d,zero}(t,T_i)}
\end{aligned}
$$

with recovery rate Z.

Proof:
The premium leg is the same as in Theorem 5.13. The protection leg is given by:

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0) \sum_{j=1}^{n} e^{-\int_{T_0}^{\widetilde{T}_j} r(x)dx} (1-Z)\mathbf{1}_{\{\widetilde{T}_{j-1}<T^d<\widetilde{T}_j\}} \,\middle|\, \mathcal{G}_t \right] \\
&= (1-Z)\sum_{j=1}^{n} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L(\widetilde{T}_{j-1})(1-L(\widetilde{T}_j)) \,\middle|\, \mathcal{G}_t \right] \\
&= (1-Z)\sum_{j=1}^{n} \Bigg(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L(\widetilde{T}_{j-1}) \,\middle|\, \mathcal{G}_t \right] \\
&\quad - \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t \right] \Bigg) \\
&\overset{Prop.5.2}{=} L(t)(1-Z)\sum_{j=1}^{n} \Bigg(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_{j-1}} r(x)+s^{zero}(x)dx} e^{-\int_{\widetilde{T}_{j-1}}^{\widetilde{T}_j} r(x)dx} \,\middle|\, \mathcal{F}_t \right] \\
&\quad - \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)+s^{zero}(x)dx} \,\middle|\, \mathcal{F}_t \right] \Bigg) \\
&= L(t)(1-Z)\sum_{j=1}^{n} \Bigg(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_{j-1}} r(x)+s^{zero}(x)dx} P(\widetilde{T}_{j-1},\widetilde{T}_j) \,\middle|\, \mathcal{F}_t \right] \\
&\quad - P^{d,zero}(t,\widetilde{T}_j) \Bigg) \\
&= L(t)(1-Z)\sum_{j=1}^{n} \left(P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - P^{d,zero}(t,\widetilde{T}_j) \right) \\
&=: L(t) V^{dp}_{Z\widetilde{T}}(t,T_0,T_m)\,.
\end{aligned}
$$

By equating the two legs and solving for $s(t,T_0,T_m)$ we obtain the stated result. □

5.4.1 The Dynamics of the Forward Credit Default Swap Spread

So far, we derived semi-analytical solutions for credit derivatives within our proposed framework incorporating macroeconomic and firm-specific factors. However, there exist credit derivatives that rely on the evolution of a CDS spread over time or its value for a certain future time. Determining such a future CDS spread can be time consuming as well as CPU consuming if factors have to be simulated and numerical integrals have to be calculated for each scenario. A popular way to overcome this is to just assume dynamics for the CDS spread that can be easily handled, e.g. lead to closed-form solutions for certain derivatives, or to use models that work well for interest rate derivatives and adapt it to a credit risk framework (see e.g. Schoenbucher (2000) for a LIBOR market model inspired adaption). The purpose of this section is to derive FCDS spread dynamcis that are consistent with our framework and that keep the link to macroeconomic and firm-specific factors.
Since all formulas are based on the generalized five factor framework of Chapter 4 we introduce for the sake of convenience the following notation

$$dx(t) = \mu_x(t)dt + (\vec{\sigma}_x)' d\widetilde{W}(t)$$

with $x \in \{r, u, w_1, w_2, s, s^{zero}\}$ based on the SDE of pages 34 and 92, and $d\widetilde{W}(t) := (d\widetilde{W}_r(t), d\widetilde{W}_{w_1}(t), d\widetilde{W}_{w_2}(t), d\widetilde{W}_u(t), d\widetilde{W}_s(t))'$.

First, we derive the dynamcis for a FCDS that assumes a replacement to the difference of par for the protection payment.

Theorem 5.19
Under the equivalent martingale measure $\widetilde{Q}$ the dynamics of a Forward Credit Default Swap spread determined by Theorem 5.13 evolve according to the following stochastic differential equation:

$$\begin{aligned}
&ds(t, T_0, T_m) \\
&\quad = \mu_{fcds}(t, T_0, T_m, T^*)dt + \sigma^r_{fcds}(t, T_0, T_m, T^*)d\widetilde{W}_r(t) \\
&\quad + \sigma^s_{fcds}(t, T_0, T_m, T^*)d\widetilde{W}_s(t) + \sigma^u_{fcds}(t, T_0, T_m, T^*)d\widetilde{W}_u(t) \\
&\quad + \sigma^{w_1}_{fcds}(t, T_0, T_m, T^*)d\widetilde{W}_{w_1}(t) + \sigma^{w_2}_{fcds}(t, T_0, T_m, T^*)d\widetilde{W}_{w_2}(t) \quad .
\end{aligned}$$

The functions $\mu_{fcds}(t,T_0,T_m,T^*), \sigma^r_{fcds}(t,T_0,T_m,T^*), \sigma^s_{fcds}(t,T_0,T_m,T^*)$, $\sigma^u_{fcds}(t,T_0,T_m,T^*), \sigma^{w_1}_{fcds}(t,T_0,T_m,T^*)$ *and* $\sigma^{w_2}_{fcds}(t,T_0,T_m,T^*)$ *are defined in the proof.*

Proof:
According to Theorem 5.13 the dynamcis of the Forward Credit Default Swap spread are determined by [II]

$$\begin{aligned} & ds(t,T_0,T_m) \\ & = \left(\sum_{i=1}^m P^{d,zero}(t,T_i)\right)^{-1} \cdot d\left(V^{dp}_{T^d}(t,T_0,T_m,T^*)\right) \\ & + V^{dp}_{T^d}(t,T_0,T_m,T^*) \cdot\ d\left(\sum_{i=1}^m P^{d,zero}(t,T_i)\right)^{-1} \\ & + d\left\langle V^{dp}_{T^d}(t,T_0,T_m,T^*), \left(\sum_{i=1}^m P^{d,zero}(t,T_i)\right)^{-1}\right\rangle . \end{aligned}$$

We obtain the following results by splitting this formula in several building blocks.

(i)

$$\begin{aligned} & d\left(\sum_{i=1}^m P^{d,zero}(t,T_i)\right) \\ & = \sum_{i=1}^m d(P^{d,zero}(t,T_i)) \\ & = \sum_{i=1}^m P^{d,zero}_t(t,T_i)dt + P^{d,zero}(t,T_i)\Big[- B^{d,zero}(t,T_i)dr(t) \\ & - C^{d,zero}(t,T_i)ds^{zero}(t) - D^{d,zero}(t,T_i)du(t) - E^{d,zero}_1(t,T_i)dw_1(t) \\ & - E^{d,zero}_2(t,T_i)dw_2(t)\Big] + \Big[\end{aligned}$$

[II] $\langle X_1, X_2\rangle := \sum_{j=1}^m \int_0^t \sigma_{1j}(s)\cdot\sigma_{2j}(s)ds$ denotes the quadratic covariance of the processes X_1 and X_2 with $dX_i(t) = \mu_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW_j(t), i = 1, 2$.

$$\left.\begin{array}{l}
\frac{1}{2}P^{d,zero}(t,T_i)\Big[\sigma_r^2(B^{d,zero}(t,T_i))^2\\
+\sigma_{s^{zero}}^2(C^{d,zero}(t,T_i))^2+\sigma_u^2(D^{d,zero}(t,T_i))^2\\
+\sigma_{w_1}^2(E_1^{d,zero}(t,T_i))^2+\sigma_{w_2}^2(E_2^{d,zero}(t,T_i))^2\\
+2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^{d,zero}(t,T_i)E_2^{d,zero}(t,T_i)\\
+2\sigma_r\sigma_{w_1}\rho_{rw_1}B^{d,zero}(t,T_i)E_1^{d,zero}(t,T_i)\\
+2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2}+\rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)\\
\cdot B^{d,zero}(t,T_i)E_2^{d,zero}(t,T_i)\\
+2\sigma_{s^{zero}}\sigma_u\rho_{su}C^{d,zero}(t,T_i)D^{d,zero}(t,T_i)\\
+2\sigma_r\sigma_{s^{zero}}(\rho_{rw_1}\rho_{sw_1}+\rho_{rw_2}\rho_{sw_2})\\
\cdot B^{d,zero}(t,T_i)C^{d,zero}(t,T_i)\\
+2\sigma_{s^{zero}}\sigma_{w_1}\rho_{sw_1}C^{d,zero}(t,T_i)E_1^{d,zero}(t,T_i)\\
+2\sigma_{s^{zero}}\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2}+\rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)\\
\cdot C^{d,zero}(t,T_i)E_2^{d,zero}(t,T_i)\Big]
\end{array}\right\}=:P_{xx}^{d,zero}(t,T_i)$$

$$\Big]dt$$

$$\begin{aligned}
&=\Big[\sum_{i=1}^{m}\Big(P_t^{d,zero}(t,T_i)+P^{d,zero}(t,T_i)\Big[-B^{d,zero}(t,T_i)\mu_r(t)\\
&-C^{d,zero}(t,T_i)\mu_{s^{zero}}(t)-D^{d,zero}(t,T_i)\mu_u(t)-E_1^{d,zero}(t,T_i)\mu_{w_1}(t)\\
&-E_2^{d,zero}(t,T_i)\mu_{w_2}(t)\Big]+P_{xx}^{d,zero}(t,T_i)\Big)\Big]dt\\
&-\Big[\sum_{i=1}^{m}P^{d,zero}(t,T_i)B^{d,zero}(t,T_i)\Big](\vec{\sigma}_r)'d\widetilde{W}(t)\\
&-\Big[\sum_{i=1}^{m}P^{d,zero}(t,T_i)C^{d,zero}(t,T_i)\Big](\vec{\sigma}_{s^{zero}})'d\widetilde{W}(t)\\
&-\Big[\sum_{i=1}^{m}P^{d,zero}(t,T_i)D^{d,zero}(t,T_i)\Big](\vec{\sigma}_u)'d\widetilde{W}(t)\\
&-\Big[\sum_{i=1}^{m}P^{d,zero}(t,T_i)E_1^{d,zero}(t,T_i)\Big](\vec{\sigma}_{w_1})'d\widetilde{W}(t)\\
&-\Big[\sum_{i=1}^{m}P^{d,zero}(t,T_i)E_2^{d,zero}(t,T_i)\Big](\vec{\sigma}_{w_2})'d\widetilde{W}(t)\\
&=:\mu_{\sum_i P^{d,z}}(t)dt+\sigma_{\sum_i P^{d,z}}^{r}(t)d\widetilde{W}_r(t)+\sigma_{\sum_i P^{d,z}}^{s}(t)d\widetilde{W}_s(t)\\
&+\sigma_{\sum_i P^{d,z}}^{u}(t)d\widetilde{W}_u(t)+\sigma_{\sum_i P^{d,z}}^{w_1}(t)d\widetilde{W}_{w_1}(t)+\sigma_{\sum_i P^{d,z}}^{w_2}(t)d\widetilde{W}_{w_2}(t)\\
&=:\mu_{\sum_i P^{d,z}}(t)dt+(\vec{\sigma}_{\sum_i P^{d,z}}(t))'d\widetilde{W}(t)
\end{aligned}$$

with

$$\mu_{\sum_i P^{d,z}}(t) = (r(t) + s^{zero}(t)) \sum_{i=1}^{m} P^{d,zero}(t, T_i)$$

according to the differential equations which hold for the functions $A^{d,zero}$, $B^{d,zero}$, $C^{d,zero}$, $D^{d,zero}$, $E_1^{d,zero}$ and $E_2^{d,zero}$.

Using Itô (see Theorem 2.7) with

$$g(x,t) = \frac{1}{x} \Rightarrow g_t(x,t) = 0\,,\ g_x(x,t) = -\frac{1}{x^2}\,,\ g_{xx}(x,t) = \frac{2}{x^3}\,,$$

we determine the first building block:

$$\begin{aligned}
& d\left(\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right)^{-1} \\
&\quad = -\frac{1}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2} \cdot d\left(\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right) \\
&\quad + \frac{1}{2} \cdot \frac{2}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^3} \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t) dt \\
&\quad = \Bigg(-\frac{1}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2} \cdot \mu_{\sum_i P^{d,z}}(t) \\
&\quad + \frac{1}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^3} \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t) \Bigg) dt \\
&\quad - \frac{1}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2} \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' d\widetilde{W}(t) \\
&\quad =: \mu_{(\sum_i P^{d,z})^{-1}}(t) dt - \frac{1}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2} \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' d\widetilde{W}(t)
\end{aligned}$$

(ii)

$$\begin{aligned}
& d\left(P^{d,*}(t,T,T^*)\right) \\
&\quad = P_t^{d,*}(t,T,T^*) dt + P^{d,*}(t,T,T^*) \Big[-B^{d,*}(t,T,T^*) dr(t) \\
&\quad - C^{d,*}(t,T,T^*) ds(t) - D^{d,*}(t,T,T^*) du(t) - E_1^{d,*}(t,T,T^*) dw_1(t) \\
&\quad - E_2^{d,*}(t,T,T^*) dw_2(t) \Big] + \Big[
\end{aligned}$$

$$
\left.\begin{aligned}
&\tfrac{1}{2}P^{d,*}(t,T,T^*)\Big[\sigma_r^2(B^{d,*}(t,T,T^*))^2\\
&+\sigma_s^2(C^{d,*}(t,T,T^*))^2+\sigma_u^2(D^{d,*}(t,T,T^*))^2\\
&+\sigma_{w_1}^2(E_1^{d,*}(t,T,T^*))^2+\sigma_{w_2}^2(E_2^{d,*}(t,T,T^*))^2\\
&+2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^{d,*}(t,T,T^*)E_2^{d,*}(t,T,T^*)\\
&+2\sigma_r\sigma_{w_1}\rho_{rw_1}B^{d,*}(t,T,T^*)E_1^{d,*}(t,T,T^*)\\
&+2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2}+\rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)\\
&\cdot B^{d,*}(t,T,T^*)E_2^{d,*}(t,T,T^*)\\
&+2\sigma_s\sigma_u\rho_{su}C^{d,*}(t,T,T^*)D^{d,*}(t,T,T^*)\\
&+2\sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1}+\rho_{rw_2}\rho_{sw_2})\\
&\cdot B^{d,*}(t,T,T^*)C^{d,*}(t,T,T^*)\\
&+2\sigma_s\sigma_{w_1}\rho_{sw_1}C^{d,*}(t,T,T^*)E_1^{d,*}(t,T,T^*)\\
&+2\sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2}+\rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)\\
&\cdot C^{d,*}(t,T,T^*)E_2^{d,*}(t,T,T^*)\Big]
\end{aligned}\right\} =: P_{xx}^{d,*}(t,T,T^*)
$$

$$
\begin{aligned}
&\Big]dt\\
&=\Big[P_t^{d,*}(t,T,T^*)+P^{d,*}(t,T,T^*)\Big[-B^{d,*}(t,T,T^*)\mu_r(t)\\
&-C^{d,*}(t,T,T^*)\mu_s(t)-D^{d,*}(t,T,T^*)\mu_u(t)-E_1^{d,*}(t,T,T^*)\mu_{w_1}(t)\\
&-E_2^{d,*}(t,T,T^*)\mu_{w_2}(t)\Big]+P_{xx}^{d,*}(t,T,T^*)\Big]dt\\
&-P^{d,*}(t,T,T^*)\left[B^{d,*}(t,T,T^*)\,(\vec{\sigma}_r)'+C^{d,*}(t,T,T^*)\,(\vec{\sigma}_s)'\right.\\
&+D^{d,*}(t,T,T^*)\,(\vec{\sigma}_u)'+E_1^{d,*}(t,T,T^*)\,(\vec{\sigma}_{w_1})'\\
&\left.+E_2^{d,*}(t,T,T^*)\,(\vec{\sigma}_{w_2})'\right]d\widetilde{W}(t)\\
&=:\mu_{P^{d*}}(t,T,T^*)dt+\sigma_{P^{d*}}^r(t,T,T^*)d\widetilde{W}_r(t)\\
&+\sigma_{P^{d*}}^s(t,T,T^*)d\widetilde{W}_s(t)+\sigma_{P^{d*}}^u(t,T,T^*)d\widetilde{W}_u(t)\\
&+\sigma_{P^{d*}}^{w_1}(t,T,T^*)d\widetilde{W}_{w_1}(t)+\sigma_{P^{d*}}^{w_2}(t,T,T^*)d\widetilde{W}_{w_2}(t)\\
&=:\mu_{P^{d*}}(t,T,T^*)dt+(\vec{\sigma}_{P^{d*}}(t,T,T^*))'\,d\widetilde{W}(t)
\end{aligned}
$$

with

$$\mu_{P^{d*}}(t,T,T^*)=(r(t)+s^{zero}(t))P^{d,*}(t,T,T^*)$$

according to the differential equations which hold for the functions $A^{d,*}$, $B^{d,*}$, $C^{d,*}$, $D^{d,*}$, $E_1^{d,*}$ and $E_2^{d,*}$ (see the proof of Proposition 5.10).

(iii)

$$
\begin{aligned}
&d\left(V_{T^d}^{ddp}(t,T)\right)\\
&= d\Big(\int_t^T \Big(P^{d,zero}(t,x)(F(t,x) + H(t,x)s^{zero}(t) + I(t,x)u(t)\\
&+ J_1(t,x)w_1(t) + J_2(t,x)w_2(t))\Big)dx\Big)\\
&= d\Big(\int_t^T P^{d,zero}(t,x)F(t,x)dx\Big) + d\Big(s^{zero}(t)\int_t^T P^{d,zero}(t,x)H(t,x)dx\Big)\\
&+ d\Big(u(t)\int_t^T P^{d,zero}(t,x)I(t,x)dx\Big) + d\Big(w_1(t)\int_t^T P^{d,zero}(t,x)J_1(t,x)dx\Big)\\
&+ d\Big(w_2(t)\int_t^T P^{d,zero}(t,x)J_2(t,x)dx\Big)
\end{aligned}
$$

We show the calculation only for the second term since the other terms will be done analogously. First, we consider the parametric integral

$$
\begin{aligned}
\widetilde{H}(t,r,s^{zero},u,w_1,w_2) &:= \int_t^T h(t,x,r,s^{zero},u,w_1,w_2)dx\\
&:= \int_t^T P^{d,zero}(t,x)H(t,x)dx
\end{aligned}
$$

with the function h being continuous on $[0,T^*] \times [t,T] \times I_r \times I_s \times I_u \times I_{w_1} \times I_{w_2}$ for intervals I_y, $y = r, s^{zero}, u, w_1, w_2$ since $P^{d,zero}(t,x)$ and $H(t,x)$ are continuous on this domain. Therefore $\widetilde{H}(t,r,s^{zero},u,w_1,w_2)$ is continuous on $[0,T^*] \times I_r \times I_s \times I_u \times I_{w_1} \times I_{w_2}$ (see e.g. Walter (1990), page 241). Furthermore, h is continuously partially differentiable with respect to t, r, s^{zero}, u, w_1, and w_2, hence $\widetilde{H}(t,r,s^{zero},u,w_1,w_2)$ is continuously differentiable with respect to either t, r, s^{zero}, u, w_1, or w_2 (see Walter (1990)):

$$
\frac{\delta \widetilde{H}}{\delta y}(t,r,s^{zero},u,w_1,w_2) = \int_t^T \frac{\delta h}{\delta y}(t,x,r,s^{zero},u,w_1,w_2)dx
$$

$y = t, r, s^{zero}, u, w_1, w_2$. By means of the same arguments we can show that $\frac{\delta \widetilde{H}}{\delta y}(t,r,s^{zero},u,w_1,w_2)$ is also differentiable with respect to z, $z = t, r, s^{zero}, u, w_1, w_2$ (also see Walter (1990), page 242). Applying Itô to the parametric integral $\widetilde{H}(t,r,s^{zero},u,w_1,w_2)$ we obtain the dynamics of $\int_t^T P^{d,zero}(t,x)H(t,x)dx$:

$$\begin{aligned} & d\Big(\int_t^T P^{d,zero}(t,x)H(t,x)dx\Big) \\ = \;& d\widetilde{H} = \frac{\delta \widetilde{H}}{\delta t}dt + \sum_i \frac{\delta \widetilde{H}}{\delta y_i}dy_i + \frac{1}{2}\sum_{i,j}\frac{\delta^2 \widetilde{H}}{\delta y_i \delta y_j} d < y_i, y_j > \end{aligned}$$

with

$$\begin{aligned} \frac{\delta \widetilde{H}}{\delta t} &= \int_t^T \frac{\delta h}{\delta t}(t,x,r,s^{zero},u,w_1,w_2)dx - h(t,t,r,s^{zero},u,w_1,w_2) \\ \frac{\delta \widetilde{H}}{\delta y} &= \int_t^T \frac{\delta h}{\delta y}(t,x,r,s^{zero},u,w_1,w_2)dx, \;\; y = r, s^{zero}, u, w_1, w_2 \\ \frac{\delta^2 \widetilde{H}}{\delta y_i \delta y_j} &= \int_t^T \frac{\delta^2 h}{\delta y_i \delta y_j}(t,x,r,s^{zero},u,w_1,w_2)dx, \;\; y_i, y_j = r, s^{zero}, u, w_1, w_2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} & d\Big(\int_t^T P^{d,zero}(t,x)H(t,x)dx\Big) \\ = \;& \Big(\int_t^T P^{d,zero}(t,x)H_t(t,x) + H(t,x)P_t^{d,zero}(t,x)dx \\ & - \underbrace{P^{d,zero}(t,t)H(t,t)}_{=1}\Big)dt \\ & - \Big(\int_t^T H(t,x)P^{d,zero}(t,x)B^{d,zero}(t,x)dx\Big)dr(t) \\ & - \Big(\int_t^T H(t,x)P^{d,zero}(t,x)C^{d,zero}(t,x)dx\Big)ds^{zero}(t) \\ & - \Big(\int_t^T H(t,x)P^{d,zero}(t,x)D^{d,zero}(t,x)dx\Big)du(t) \\ & - \Big(\int_t^T H(t,x)P^{d,zero}(t,x)E_1^{d,zero}(t,x)dx\Big)dw_1(t) \\ & - \Big(\int_t^T H(t,x)P^{d,zero}(t,x)E_2^{d,zero}(t,x)dx\Big)dw_2(t) \\ & + \Big(\int_t^T H(t,x)P_{xx}^{d,zero}(t,x)dx\Big)dt \end{aligned}$$

$$
\begin{aligned}
&=: \mu_{\int PH}(t,T)dt + \sigma^{r}_{\int PH}(t,T)d\widetilde{W}_r(t) + \sigma^{s^{zero}}_{\int PH}(t,T)d\widetilde{W}_s(t) \\
&+ \sigma^{u}_{\int PH}(t,T)d\widetilde{W}_u(t) + \sigma^{w_1}_{\int PH}(t,T)d\widetilde{W}_{w_1}(t) + \sigma^{w_2}_{\int PH}(t,T)d\widetilde{W}_{w_2}(t) \\
&=: \mu_{\int PH}(t,T)dt + (\vec{\sigma}_{\int PH}(t,T))'d\widetilde{W}(t)
\end{aligned}
$$

Finally, using Itô we get as a result for the second term:

$$
\begin{aligned}
& d\Big(s^{zero}(t)\int_t^T P^{d,zero}(t,x)H(t,x)dx\Big) \\
&\quad = \int_t^T P^{d,zero}(t,x)H(t,x)dx \cdot d\Big(s^{zero}(t)\Big) \\
&\quad + s^{zero}(t)\cdot d\Big(\int_t^T P^{d,zero}(t,x)H(t,x)dx\Big) \\
&\quad + d\left\langle s^{zero}(t), \int_t^T P^{d,zero}(t,x)H(t,x)dx\right\rangle \\
&\quad = \int_t^T P^{d,zero}(t,x)H(t,x)dx \cdot \Big(\mu_{s^{zero}}(t)dt + (\vec{\sigma}_{s^{zero}})'d\widetilde{W}(t)\Big) \\
&\quad + s^{zero}(t)\cdot\Big(\mu_{\int PH}(t,T)dt + (\vec{\sigma}_{\int PH}(t,T))'d\widetilde{W}(t)\Big) \\
&\quad + \Big((\vec{\sigma}_{s^{zero}})'(\vec{\sigma}_{\int PH}(t,T))\Big)dt.
\end{aligned}
$$

Therefore by doing the analogous calculations for the other terms and summing up the resulting terms with respect to dt and $d\widetilde{W}$, we get the following result:

$$
\begin{aligned}
& d\left(V^{ddp}_{T^d}(t,T)\right) \\
&\quad =: \mu_{V^{ddp}}(t,T)dt + \sigma^{r}_{V^{ddp}}(t,T)d\widetilde{W}_r(t) + \sigma^{s^{zero}}_{V^{ddp}}(t,T)d\widetilde{W}_s(t) \\
&\quad + \sigma^{u}_{V^{ddp}}(t,T)d\widetilde{W}_u(t) + \sigma^{w_1}_{V^{ddp}}(t,T)d\widetilde{W}_{w_1}(t) + \sigma^{w_2}_{V^{ddp}}(t,T)d\widetilde{W}_{w_2}(t) \\
&\quad =: \mu_{V^{ddp}}(t,T)dt + (\vec{\sigma}_{V^{ddp}}(t,T))'d\widetilde{W}(t).
\end{aligned}
$$

(iv)

$$
\begin{aligned}
& d\left\langle V_{T^d}^{dp}(t,T_0,T_m,T^*), \left(\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right)^{-1}\right\rangle \\
& \overset{Th.5.13}{=} d\left\langle V_{T^d}^{ddp}(t,T_m) - V_{T^d}^{ddp}(t,T_0) + P^{d,*}(t,T_m,T^*) - P^{d,*}(t,T_0,T^*), \right. \\
& \qquad \left. \left(\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right)^{-1}\right\rangle \\
& \overset{(i)-(iii)}{=} \left[\left(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) + \vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)\right)' \right. \\
& \qquad \left. \cdot \frac{-1}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2}\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)\right] dt \\
& =: \quad \mu_{\langle\ \rangle}(t,T_0,T_m,T^*)dt
\end{aligned}
$$

By combining (i)-(iv) we get the dynamics of $s(t,T_0,T_m)$:

$$
\begin{aligned}
& ds(t,T_0,T_m) \\
& \quad = \left(\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right)^{-1} \cdot d\left(V_{T^d}^{dp}(t,T_0,T_m,T^*)\right) \\
& \quad + V_{T^d}^{dp}(t,T_0,T_m,T^*) \cdot\ d\left(\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right)^{-1} \\
& \quad + d\left\langle V_{T^d}^{dp}(t,T_0,T_m,T^*), \left(\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right)^{-1}\right\rangle \\
& \quad = \frac{1}{\sum_{i=1}^{m} P^{d,zero}(t,T_i)}\left(\left(\mu_{V^{ddp}}(t,T_m) - \mu_{V^{ddp}}(t,T_0) + \mu_{P^{d*}}(t,T_m,T^*)\right.\right. \\
& \quad \left. - \mu_{P^{d*}}(t,T_0,T^*)\right) dt \\
& \quad \left. + \left(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) + \vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)\right)' d\widetilde{W}(t)\right) \\
& \quad + V_{T^d}^{dp}(t,T_0,T_m,T^*)\left(\mu_{(\sum_i P^{d,z})^{-1}}(t)dt - \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2} d\widetilde{W}(t)\right) \\
& \quad + \mu_{\langle\ \rangle}(t,T_0,T_m,T^*)dt
\end{aligned}
$$

$$
\begin{aligned}
&=: \mu_{fcds}(t,T_0,T_m,T^*)dt + \sigma^r_{fcds}(t,T_0,T_m,T^*)d\widetilde{W}_r(t) \\
&+ \sigma^s_{fcds}(t,T_0,T_m,T^*)d\widetilde{W}_s(t) + \sigma^u_{fcds}(t,T_0,T_m,T^*)d\widetilde{W}_u(t) \\
&+ \sigma^{w_1}_{fcds}(t,T_0,T_m,T^*)d\widetilde{W}_{w_1}(t) + \sigma^{w_2}_{fcds}(t,T_0,T_m,T^*)d\widetilde{W}_{w_2}(t) \\
&=: \mu_{fcds}(t,T_0,T_m,T^*)dt + (\vec{\sigma}_{fcds}(t,T_0,T_m,T^*))' \, d\widetilde{W}(t).
\end{aligned}
$$

□

The determination of the FCDS dynamics in case of protection payments by replacement to par and its results enable us to also give the dynamics of Forward Credit Default Swaps that pay a fraction of the face value in case of default.

Theorem 5.20
If the recovery of the reference asset is paid as a fraction of the face value (see Theorem 5.15), the dynamics of a Forward Credit Default Swap spread under the equivalent martingale measure $\widetilde{Q}$ evolve according to the following stochastic differential equation:

$$
\begin{aligned}
&ds(t,T_0,T_m) \\
&\quad = \mu^Z_{fcds}(t,T_0,T_m)dt + \sigma^{rZ}_{fcds}(t,T_0,T_m)d\widetilde{W}_r(t) + \sigma^{sZ}_{fcds}(t,T_0,T_m)d\widetilde{W}_s(t) \\
&\quad + \sigma^{uZ}_{fcds}(t,T_0,T_m)d\widetilde{W}_u(t) + \sigma^{w_1 Z}_{fcds}(t,T_0,T_m)d\widetilde{W}_{w_1}(t) \\
&\quad + \sigma^{w_2 Z}_{fcds}(t,T_0,T_m)d\widetilde{W}_{w_2}(t) \; .
\end{aligned}
$$

The functions $\mu^Z_{fcds}(t,T_0,T_m), \sigma^{rZ}_{fcds}(t,T_0,T_m), \sigma^{sZ}_{fcds}(t,T_0,T_m), \sigma^{uZ}_{fcds}(t,T_0,T_m)$, $\sigma^{w_1 Z}_{fcds}(t,T_0,T_m)$, and $\sigma^{w_2 Z}_{fcds}(t,T_0,T_m)$ are defined in the proof.

Proof:
Since the value of the protection leg simplifies to $(1-Z)(V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0))$, the building block (ii) in the proof of Theorem 5.19 is not needed anymore and the building block (iv) is reduced by the parts of $P^{d,*}$ and mul-

tiplied by $(1 - Z)$. Therefore we get for the dynamics

$$
\begin{aligned}
&ds(t, T_0, T_m) \\
&\quad = \left(\sum_{i=1}^{m} P^{d,zero}(t, T_i)\right)^{-1} \cdot d\left(V^{dp}_{ZT^d}(t, T_0, T_m)\right) \\
&\quad + V^{dp}_{ZT^d}(t, T_0, T_m) \cdot\ d\left(\sum_{i=1}^{m} P^{d,zero}(t, T_i)\right)^{-1} \\
&\quad + d\left\langle V^{dp}_{ZT^d}(t, T_0, T_m), \left(\sum_{i=1}^{m} P^{d,zero}(t, T_i)\right)^{-1}\right\rangle \\
&\quad = \frac{(1 - Z)}{\sum_{i=1}^{m} P^{d,zero}(t, T_i)} \Big(\Big(\mu_{V^{ddp}}(t, T_m) - \mu_{V^{ddp}}(t, T_0)\Big)dt \\
&\quad + \Big(\vec{\sigma}_{V^{ddp}}(t, T_m) - \vec{\sigma}_{V^{ddp}}(t, T_0)\Big)' d\widetilde{W}(t)\Big) \\
&\quad + V^{dp}_{ZT^d}(t, T_0, T_m)\Big(\mu_{(\sum_i P^{d,z})^{-1}}(t)dt - \frac{\big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\big)'}{\left[\sum_{i=1}^{m} P^{d,zero}(t, T_i)\right]^2} d\widetilde{W}(t)\Big) \\
&\quad + \mu^{Z}_{\langle\ \rangle}(t, T_0, T_m)dt \\
&\quad =: \mu^{Z}_{fcds}(t, T_0, T_m)dt + \sigma^{rZ}_{fcds}(t, T_0, T_m)d\widetilde{W}_r(t) + \sigma^{sZ}_{fcds}(t, T_0, T_m)d\widetilde{W}_s(t) \\
&\quad + \sigma^{uZ}_{fcds}(t, T_0, T_m)d\widetilde{W}_u(t) + \sigma^{w_1 Z}_{fcds}(t, T_0, T_m)d\widetilde{W}_{w_1}(t) \\
&\quad + \sigma^{w_2 Z}_{fcds}(t, T_0, T_m)d\widetilde{W}_{w_2}(t) \\
&\quad =: \mu^{Z}_{fcds}(t, T_0, T_m)dt + \big(\vec{\sigma}^{Z}_{fcds}(t, T_0, T_m)\big)' d\widetilde{W}(t).
\end{aligned}
$$

□

For the FCDS spread where we use the so-called default bucketing in order to approximate the protection leg, the dynamcis further simplify as opposed to the above cases.

Theorem 5.21
If the recovery of the reference entity is paid as a fraction of the face value and the protection leg is approximated according to Theorem 5.18, the dynamics of a Forward Credit Default Swap spread under the equivalent martingale

measure $\widetilde{Q}$ *evolve according to the following stochastic differential equation:*

$$
\begin{aligned}
& ds(t, T_0, T_m) \\
& \quad = \mu_{fcds}^{Z\widetilde{T}}(t, T_0, T_m)dt + \sigma_{fcds}^{rZ\widetilde{T}}(t, T_0, T_m)d\widetilde{W}_r(t) + \sigma_{fcds}^{sZ\widetilde{T}}(t, T_0, T_m)d\widetilde{W}_s(t) \\
& \quad + \sigma_{fcds}^{uZ\widetilde{T}}(t, T_0, T_m)d\widetilde{W}_u(t) + \sigma_{fcds}^{w_1Z\widetilde{T}}(t, T_0, T_m)d\widetilde{W}_{w_1}(t) \\
& \quad + \sigma_{fcds}^{w_2Z\widetilde{T}}(t, T_0, T_m)d\widetilde{W}_{w_2}(t) \; .
\end{aligned}
$$

The functions $\mu_{fcds}^{Z\widetilde{T}}(t, T_0, T_m), \sigma_{fcds}^{rZ\widetilde{T}}(t, T_0, T_m), \sigma_{fcds}^{sZ\widetilde{T}}(t, T_0, T_m), \sigma_{fcds}^{uZ\widetilde{T}}(t, T_0, T_m),$ $\sigma_{fcds}^{w_1Z\widetilde{T}}(t, T_0, T_m)$*, and* $\sigma_{fcds}^{w_2Z\widetilde{T}}(t, T_0, T_m)$ *are defined in the proof.*

Proof:
The protection leg of the previous theorem is further simplified by the assumptions of the protection payments. Therefore, the dynamics result in:

$$
\begin{aligned}
& ds(t, T_0, T_m) \\
& \quad = \left(\sum_{i=1}^{m} P^{d,zero}(t, T_i) \right)^{-1} \cdot d \left((1 - Z) \sum_{j=1}^{n} (P^*(t, \widetilde{T}_{j-1}, \widetilde{T}_j) - P^{d,zero}(t, \widetilde{T}_j)) \right) \\
& \quad + (1 - Z) \sum_{j=1}^{n} (P^*(t, \widetilde{T}_{j-1}, \widetilde{T}_j) - P^{d,zero}(t, \widetilde{T}_j)) \cdot d \left(\sum_{i=1}^{m} P^{d,zero}(t, T_i) \right)^{-1} \\
& \quad + d \left\langle (1 - Z) \sum_{j=1}^{n} (P^*(t, \widetilde{T}_{j-1}, \widetilde{T}_j) - P^{d,zero}(t, \widetilde{T}_j)), \left(\sum_{i=1}^{m} P^{d,zero}(t, T_i) \right)^{-1} \right\rangle \\
& \quad = \frac{(1 - Z)}{\sum_{i=1}^{m} P^{d,zero}(t, T_i)} \Bigg(\sum_{j=1}^{n} \Big(\mu_{P^*}(t, \widetilde{T}_{j-1}, \widetilde{T}_j) - \mu_{P^{d,z}}(t, \widetilde{T}_j) \Big) dt \\
& \quad + \sum_{j=1}^{n} \Big(\vec{\sigma}_{P^*}(t, \widetilde{T}_{j-1}, \widetilde{T}_j) - \vec{\sigma}_{P^{d,z}}(t, \widetilde{T}_j) \Big)' d\widetilde{W}(t) \Bigg) \\
& \quad + (1 - Z) \sum_{j=1}^{n} (P^*(t, \widetilde{T}_{j-1}, \widetilde{T}_j) - P^{d,zero}(t, \widetilde{T}_j)) \Big(\mu_{(\sum_i P^{d,z})^{-1}}(t) dt
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2} \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' d\widetilde{W}(t)\Big) \\
&+ \left((1-Z)\sum_{j=1}^{n}\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)\right)' \\
&\cdot \frac{-1}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2}\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)dt \\
&=: \mu_{fcds}^{Z\widetilde{T}}(t,T_0,T_m)dt + \sigma_{fcds}^{rZ\widetilde{T}}(t,T_0,T_m)d\widetilde{W}_r(t) + \sigma_{fcds}^{sZ\widetilde{T}}(t,T_0,T_m)d\widetilde{W}_s(t) \\
&+ \sigma_{fcds}^{uZ\widetilde{T}}(t,T_0,T_m)d\widetilde{W}_u(t) + \sigma_{fcds}^{w_1Z\widetilde{T}}(t,T_0,T_m)d\widetilde{W}_{w_1}(t) \\
&+ \sigma_{fcds}^{w_2Z\widetilde{T}}(t,T_0,T_m)d\widetilde{W}_{w_2}(t) \\
&=: \mu_{fcds}^{Z\widetilde{T}}(t,T_0,T_m)dt + \left(\vec{\sigma}_{fcds}^{Z\widetilde{T}}(t,T_0,T_m)\right)' d\widetilde{W}(t)\ .
\end{aligned}
$$

The dynamics of $P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)$ are calculated similar to the dynamics of $P^{d,zero}$ (see pages 107ff, (i) with $m=1$ such that $\mu_{P^{d,z}}(t) = \mu_{\sum_i P^{d,z}}(t)$ and $\vec{\sigma}_{P^{d,z}}(t) = \vec{\sigma}_{\sum_i P^{d,z}}(t)$) and $P^{d,*}$ (see pages 109ff, (ii)) and evolve according to:

$$
\begin{aligned}
&dP^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) \\
&\quad = \Big[P_t^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) + P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\Big[-B^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\mu_r(t) \\
&\quad - C^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\mu_{s^{zero}}(t) - D^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\mu_u(t) - E_1^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\mu_{w_1}(t) \\
&\quad - E_2^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\mu_{w_2}(t)\Big] + P_{xx}^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\Big]dt \\
&\quad - P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\Big[B^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\left(\vec{\sigma}_r\right)' + C^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\left(\vec{\sigma}_{s^{zero}}\right)' \\
&\quad + D^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\left(\vec{\sigma}_u\right)' + E_1^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\left(\vec{\sigma}_{w_1}\right)' \\
&\quad + E_2^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\left(\vec{\sigma}_{w_2}\right)'\Big]d\widetilde{W}(t) \\
&\quad =: \mu_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)dt + \sigma_{P^*}^r(t,\widetilde{T}_{j-1},\widetilde{T}_j)d\widetilde{W}_r(t) + \sigma_{P^*}^{w_1}(t,\widetilde{T}_{j-1},\widetilde{T}_j)d\widetilde{W}_{w_1}(t) \\
&\quad + \sigma_{P^*}^{w_2}(t,\widetilde{T}_{j-1},\widetilde{T}_j)d\widetilde{W}_{w_2}(t) + \sigma_{P^*}^{u}(t,\widetilde{T}_{j-1},\widetilde{T}_j)d\widetilde{W}_u(t) \\
&\quad + \sigma_{P^*}^{s}(t,\widetilde{T}_{j-1},\widetilde{T}_j)d\widetilde{W}_s(t) \\
&\quad =: \mu_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)dt + \left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)\right)' d\widetilde{W}(t)
\end{aligned}
$$

where $\mu_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)$ equals $(r(t)+s^{zero}(t))P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)$ according to the differential equations in the proof of Proposition 5.17. □

The following lemma is an important input to further simplify the FCDS spread dynamics.

Lemma 5.22
In the dynamcis of $V_{T^d}^{ddp}$ (cf. (iii) in the proof of Theorem 5.19) the drift is

$$\mu_{V^{ddp}}(t,T) = (r(t) + s^{zero}(t))V_{T^d}^{ddp}(t,T) - s^{zero}(t) \ .$$

Proof:
This follows directly from Corollary 6.2.1 in Schmid (2004), page 193, with (cf. Theorem 5.6)

$$L(t)\, V_{T^d}^{ddp}(t,T) = \mathbb{E}_{\widetilde{Q}}\left[\int_t^T e^{-\int_t^u r(l)dl} dH(u) \middle| \mathcal{G}_t\right] \ .$$

Analogously to pages 242 - 243 in Schmid (2004), we obtain

$$\begin{aligned} d\left(L(t)V_{T^d}^{ddp}(t)\right) &= -dH(t) + r(t)L(t)V_{T^d}^{ddp}(t)dt + dm(t) \\ &= -s^{zero}(t)dt + (r(t) + s^{zero}(t))L(t)V_{T^d}^{ddp}(t)dt + d\tilde{m}(t) \ , \end{aligned}$$

for some martingales m and $\tilde{m}$ and $t \leq T$. □

With these results the dynamics of the Forward Credit Default Swap spread can be simplified for the three cases introduced above: namely a CDS where the default payment takes place at default by replacement to the difference to par (see Theorem 5.13 and Theorem 5.19), a CDS where the default payment is assumed to be a fraction of the face value (see Theorem 5.15 and Theorem 5.20), and a CDS which also pays a fraction of the face value in case of default but where the payment is assumed to take place at certain dates (see Theorem 5.18 and Theorem 5.21).

Proposition 5.23
In the case of Theorem 5.19 the dynamics can be written as

$$\begin{aligned} & ds(t,T_0,T_m) \\ &\quad = \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} s(t,T_0,T_m) + \mu_{\langle\ \rangle}(t,T_0,T_m,T^*)\right)dt \\ &\quad + \left(\left(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) + \vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)\right)'\right. \\ &\quad \left. - s(t,T_0,T_m)\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'\right) \frac{1}{\sum_{i=1}^m P^{d,zero}(t,T_i)} d\widetilde{W}(t) \ , \end{aligned}$$

with $\mu_{\langle\ \rangle}(t,T_0,T_m,T^*)$ given in the proof of Theorem 5.19, (iv).

Proof:
With the help of Theorem 5.13, of Lemma 5.22 and of the relations $\mu_{P^{d*}}(t,T,T^*) = (r(t)+s^{zero}(t))P^{d,*}(t,T,T^*)$ and $\mu_{\sum_i P^{d,z}}(t) = (r(t)+s^{zero}(t))\sum_{i=1}^m P^{d,zero}(t,T_i)$ (see (i) and (ii) in the proof of Theorem 5.19) , the dynamics of the Forward Credit Default Swap derived in Theorem 5.19 simplify to

$$
\begin{aligned}
&ds(t,T_0,T_m)\\
&\quad = \frac{\mu_{V^{ddp}}(t,T_m) - \mu_{V^{ddp}}(t,T_0) + \mu_{P^{d*}}(t,T_m,T^*) - \mu_{P^{d*}}(t,T_0,T^*)}{\sum_{i=1}^m P^{d,zero}(t,T_i)}dt\\
&\quad + V^{dp}_{T^d}(t,T_0,T_m)\mu_{(\sum_i P^{d,z})^{-1}}(t)dt\\
&\quad + \mu_{\langle\ \rangle}(t,T_0,T_m,T^*)dt\\
&\quad + \frac{\Big(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) + \vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)\Big)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)}d\widetilde{W}(t)\\
&\quad - V^{dp}_{T^d}(t,T_0,T_m)\frac{1}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2}\cdot\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' d\widetilde{W}(t)\\
&\quad = (r(t)+s^{zero}(t))s(t,T_0,T_m)dt - (r(t)+s^{zero}(t))s(t,T_0,T_m)dt\\
&\quad + \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'\vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2}s(t,T_0,T_m)dt + \mu_{\langle\ \rangle}(t,T_0,T_m,T^*)dt\\
&\quad + \frac{\Big(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) + \vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)\Big)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)}d\widetilde{W}(t)\\
&\quad - s(t,T_0,T_m)\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)}d\widetilde{W}(t)\\
&\quad = \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'\vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2}s(t,T_0,T_m) + \mu_{\langle\ \rangle}(t,T_0,T_m,T^*)\right)dt\\
&\quad + \Bigg(\Big(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) + \vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)\Big)'\\
&\quad - s(t,T_0,T_m)\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'\Bigg)\frac{1}{\sum_{i=1}^m P^{d,zero}(t,T_i)}d\widetilde{W}(t)\ .
\end{aligned}
$$

□

Proposition 5.24
If the recovery of the reference entity is paid as a fraction of the face value, the dynamics of Theorem 5.20 simplify to

$$
\begin{aligned}
&ds(t,T_0,T_m)\\
&= \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} s(t,T_0,T_m) + \mu^Z_{\langle\ \rangle}(t,T_0,T_m) \right) dt \\
&+ \frac{1}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \Bigg((1-Z)\Big(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) \Big)' \\
&\quad - s(t,T_0,T_m) \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \Bigg) d\widetilde{W}(t)
\end{aligned}
$$

with

$$
\begin{aligned}
&\mu^Z_{\langle\ \rangle}(t,T_0,T_m)\\
&\quad = (1-Z)\Big(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) \Big)' \\
&\quad\quad \cdot \frac{-1}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} \Big(\vec{\sigma}_{\sum_i P^{d,z}}(t) \Big) .
\end{aligned}
$$

Proof:
By means of Theorem 5.15, of Lemma 5.22 and the relation which holds for $\mu_{\sum_i P^{d,z}}$ (cf. (i) in the proof of Theorem 5.19), the dynamics can be written as follows:

$$
\begin{aligned}
&ds(t,T_0,T_m)\\
&\quad = \frac{(1-Z)\Big(\mu_{V^{ddp}}(t,T_m) - \mu_{V^{ddp}}(t,T_0) \Big)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} dt \\
&\quad + V^{dp}_{ZT^d}(t,T_0,T_m)\mu_{(\sum_i P^{d,z})^{-1}}(t)dt + \mu^Z_{\langle\ \rangle}(t,T_0,T_m)dt \\
&\quad + \frac{(1-Z)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \Big(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) \Big)' d\widetilde{W}(t) \\
&\quad - V^{dp}_{ZT^d}(t,T_0,T_m) \frac{1}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' d\widetilde{W}(t)
\end{aligned}
$$

$$
\begin{aligned}
&= (r(t) + s^{zero}(t))s(t, T_0, T_m)dt - (r(t) + s^{zero}(t))s(t, T_0, T_m)dt \\
&+ \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} s(t, T_0, T_m)dt + \mu^Z_{\langle\ \rangle}(t, T_0, T_m)dt \\
&+ \frac{(1 - Z)}{\sum_{i=1}^m P^{d,zero}(t, T_i)} \left(\vec{\sigma}_{V^{ddp}}(t, T_m) - \vec{\sigma}_{V^{ddp}}(t, T_0)\right)' d\widetilde{W}(t) \\
&- s(t, T_0, T_m) \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^m P^{d,zero}(t, T_i)} d\widetilde{W}(t) \\
&= \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} s(t, T_0, T_m) + \mu^Z_{\langle\ \rangle}(t, T_0, T_m) \right) dt \\
&+ \frac{1}{\sum_{i=1}^m P^{d,zero}(t, T_i)} \Bigg((1 - Z) \left(\vec{\sigma}_{V^{ddp}}(t, T_m) - \vec{\sigma}_{V^{ddp}}(t, T_0)\right)' \\
&- s(t, T_0, T_m) \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \Bigg) d\widetilde{W}(t)\ .
\end{aligned}
$$

$\mu^Z_{\langle\ \rangle}(t, T_0, T_m)$ is calculated according to (iv) in the proof of Theorem 5.19 with $V^{dp}_{T^d}(t, T_0, T_m, T^*)$ being replaced by $V^{dp}_{ZT^d}(t, T_0, T_m)$ of Theorem 5.15:

$$
\begin{aligned}
&\mu^Z_{\langle\ \rangle}(t, T_0, T_m) \\
&= (1 - Z) \left(\vec{\sigma}_{V^{ddp}}(t, T_m) - \vec{\sigma}_{V^{ddp}}(t, T_0)\right)' \cdot \frac{-1}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)\ .
\end{aligned}
$$

□

Proposition 5.25
If the recovery of the reference entity is paid as a fraction of the face value and the protection leg is approximated by a sum, the dynamics of Theorem 5.21 can also be written as

$$
\begin{aligned}
&ds(t, T_0, T_m) \\
&\quad = \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} s(t, T_0, T_m) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left((1-Z)\sum_{j=1}^{n}\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)-\vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)\right)' \\
&\cdot\frac{-1}{\left[\sum_{i=1}^{m}P^{d,zero}(t,T_i)\right]^2}\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)\Bigg)dt \\
&+\left(\frac{(1-Z)}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)}\sum_{j=1}^{n}\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)-\vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)'\right. \\
&\left.-\,s(t,T_0,T_m)\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)}\right)d\widetilde{W}(t)\ .
\end{aligned}
$$

Proof:
Using Theorem 5.18, Theorem 5.21 and the relation for $\mu_{P^{d,z}}$ as in the proofs before, the FCDS dynamics reduce to

$$
\begin{aligned}
&ds(t,T_0,T_m) \\
&\quad=\frac{(1-Z)}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)}\sum_{j=1}^{n}\left(\mu_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)-\mu_{P^{d,z}}(t,\widetilde{T}_j)\right)dt \\
&\quad+(1-Z)\sum_{j=1}^{n}(P^*(t,\widetilde{T}_{j-1}\widetilde{T}_j)-P^{d,zero}(t,\widetilde{T}_j))\mu_{(\sum_i P^{d,z})^{-1}}(t)dt \\
&\quad+\left((1-Z)\sum_{j=1}^{n}\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)-\vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)\right)' \\
&\quad\cdot\frac{-1}{\left[\sum_{i=1}^{m}P^{d,zero}(t,T_i)\right]^2}\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)dt \\
&\quad+\frac{(1-Z)}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)}\sum_{j=1}^{n}\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)-\vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)'d\widetilde{W}(t) \\
&\quad-\frac{(1-Z)\sum_{j=1}^{n}(P^*(t,\widetilde{T}_{j-1}\widetilde{T}_j)-P^{d,zero}(t,\widetilde{T}_j))}{\left[\sum_{i=1}^{m}P^{d,zero}(t,T_i)\right]^2}\cdot\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'d\widetilde{W}(t) \\
&\quad=(r(t)+s^{zero}(t))s(t,T_0,T_m)dt-(r(t)+s^{zero}(t))s(t,T_0,T_m)dt \\
&\quad+\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'\vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^{m}P^{d,zero}(t,T_i)\right]^2}s(t,T_0,T_m)dt
\end{aligned}
$$

$$
\begin{aligned}
&+\left((1-Z)\sum_{j=1}^{n}\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)-\vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)\right)'\\
&\cdot\frac{-1}{\left[\sum_{i=1}^{m}P^{d,zero}(t,T_i)\right]^2}\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)dt\\
&+\frac{(1-Z)}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)}\sum_{j=1}^{n}\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j)-\vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)'d\widetilde{W}(t)\\
&-s(t,T_0,T_m)\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)}d\widetilde{W}(t)
\end{aligned}
$$

□

5.4.2 Exact versus Approximated Dynamics of the Forward Credit Default Swap Spread

In order to get a better insight into the proposed dynamcis, we now leave the general framework of Section 4.1 that was used so far and take one of its special cases, the four factor framework of Schmid and Zagst (SZ4, cf. Model 4.4), for an example and present its dynamics of a Forward Credit Default Swap in a more detailed way. The proposed dynamics can then be used e.g. for pricing derivatives written on a CDS.
If we want to work with these dynamics we can choose between the following alternatives:

1. We can simulate the exact dynamics. Hence we need for every time step the values of the factors $r(t), s(t), u(t)$ and $w_1(t)$. Also we have to integrate numerically several intergrals. Therefore, it would be easier to just simulate the factors over time and calculate the CDS spread $s(t, T_0, T_m)$ at the specific point in time we need.

2. We approximate the exact dynamcis in order to get dynamics that can be handled much easier (e.g. for simulation) and can be further processed, i.e. yielding closed-form solutions for certain derivatives.

Assumption 5.26
In the SZ4 framework the exact dynamics of the FCDS spread presented in Proposition 5.23, Proposition 5.24, and Proposition 5.25 can be approximated

by lognormal dynamcis

$$
\begin{aligned}
&ds(t,T_0,T_m)\\
&= \Bigg(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} \\
&\quad + \Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y}) \Big) \\
&\quad \cdot \frac{1}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]} \Big(\vec{\sigma}_{\sum_i P^{d,z}}(t) \Big) \Bigg) s(t,T_0,T_m) dt \\
&\quad - \Bigg(\Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y}) \Big) \\
&\quad + \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \Bigg) s(t,T_0,T_m) d\widetilde{W}(t)
\end{aligned}
$$

with $\tilde{y} \in [T_0, T_m]$ *and frozen factors of* r, s^{zero}, u, w_1 *in* $\vec{\sigma}_{\sum_i P^{d,z}}$ *and* $P^{d,zero}$ *at time* t.

Approximation for Proposition 5.23: Within the SZ4 framework the terms $\vec{\sigma}_{P^{d*}} = (\sigma^r_{P^{d*}}, \sigma^s_{P^{d*}}, \sigma^u_{P^{d*}}, \sigma^{w_1}_{P^{d*}})'$ and $\vec{\sigma}_{V^{ddp}} = (\sigma^r_{V^{ddp}}, \sigma^s_{V^{ddp}}, \sigma^u_{V^{ddp}}, \sigma^{w_1}_{V^{ddp}})'$ of the above dynamics are as follows. The vector $\vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)$ consists of

$$
\begin{aligned}
&\sigma^r_{P^{d*}}(t,T_m,T^*) - \sigma^r_{P^{d*}}(t,T_0,T^*)\\
&\quad = -\sigma_r P^{d*}(t,T_m,T^*) B^{d*}(t,T_m,T^*) + \sigma_r P^{d*}(t,T_0,T^*) B^{d*}(t,T_0,T^*)\\
&\quad = -\sigma_r B^{d,zero}(t,T^*)(P^{d*}(t,T_m,T^*) - P^{d*}(t,T_0,T^*))\ ,\\
&\sigma^s_{P^{d*}}(t,T_m,T^*) - \sigma^s_{P^{d*}}(t,T_0,T^*)\\
&\quad = -\sigma_s P^{d*}(t,T_m,T^*) C^{d*}(t,T_m,T^*) + \sigma_s P^{d*}(t,T_0,T^*) C^{d*}(t,T_0,T^*)\ ,\\
&\sigma^u_{P^{d*}}(t,T_m,T^*) - \sigma^u_{P^{d*}}(t,T_0,T^*)\\
&\quad = -\sigma_u P^{d*}(t,T_m,T^*) D^{d*}(t,T_m,T^*) + \sigma_u P^{d*}(t,T_0,T^*) D^{d*}(t,T_0,T^*)\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sigma^{w_1}_{P^{d*}}(t,T_m,T^*) - \sigma^{w_1}_{P^{d*}}(t,T_0,T^*)\\
&\quad = -\sigma_{w_1} P^{d*}(t,T_m,T^*) E_1^{d*}(t,T_m,T^*) + \sigma_{w_1} P^{d*}(t,T_0,T^*) E_1^{d*}(t,T_0,T^*)\ .
\end{aligned}
$$

For the vector $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$ it holds that

$$\begin{aligned}
&\sigma^r_{V^{ddp}}(t,T_m) - \sigma^r_{V^{ddp}}(t,T_0)\\
&\quad = -\sigma_r \int_{T_0}^{T_m} B^{d,zero}(t,y) P^{d,zero}(t,y)\Big(F(t,y) + s^{zero}(t)H(t,y) + u(t)I(t,y)\\
&\quad + w_1(t)J_1(t,y)\Big)dy \ ,\\
&\sigma^s_{V^{ddp}}(t,T_m) - \sigma^s_{V^{ddp}}(t,T_0)\\
&\quad = -\sigma_{s^{zero}} \int_{T_0}^{T_m} C^{d,zero}(t,y) P^{d,zero}(t,y)\Big(F(t,y) + s^{zero}(t)H(t,y) + u(t)I(t,y)\\
&\quad + w_1(t)J_1(t,y)\Big)dy + \sigma_{s^{zero}} \int_{T_0}^{T_m} P^{d,zero}(t,y)H(t,y)dy \ ,\\
&\sigma^u_{V^{ddp}}(t,T_m) - \sigma^u_{V^{ddp}}(t,T_0)\\
&\quad = -\sigma_u \int_{T_0}^{T_m} D^{d,zero}(t,y) P^{d,zero}(t,y)\Big(F(t,y) + s^{zero}(t)H(t,y) + u(t)I(t,y)\\
&\quad + w_1(t)J_1(t,y)\Big)dy + \sigma_u \int_{T_0}^{T_m} P^{d,zero}(t,y)I(t,y)dy \ ,
\end{aligned}$$

and

$$\begin{aligned}
&\sigma^{w_1}_{V^{ddp}}(t,T_m) - \sigma^{w_1}_{V^{ddp}}(t,T_0)\\
&\quad = -\sigma_{w_1} \int_{T_0}^{T_m} E_1^{d,zero}(t,y) P^{d,zero}(t,y)\Big(F(t,y) + s^{zero}(t)H(t,y) + u(t)I(t,y)\\
&\quad + w_1(t)J_1(t,y)\Big)dy + \sigma_{w_1} \int_{T_0}^{T_m} P^{d,zero}(t,y)J_1(t,y)dy \ .
\end{aligned}$$

Therefore, the components of the vector $\vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)$ are approximated as follows with $\tilde{y} \in [T_0,T_m]$.

$$\begin{aligned}
&\sigma^s_{P^{d*}}(t,T_m,T^*) - \sigma^s_{P^{d*}}(t,T_0,T^*)\\
&\quad \approx -\sigma_{s^zero} C^{d,zero}(t,\tilde{y})(P^{d*}(t,T_m,T^*) - P^{d*}(t,T_0,T^*)) \ ,
\end{aligned}$$

since it holds (see Proposition 5.10 on page 97)

$$C^{d,*}(t,T,T^*) = \frac{1}{1-z} C^d(t,T) + e^{-\hat{a}_s(T-t)} C^d(T,T^*)$$

and we assume it to be approximately

$$\sigma_s C^{d,*}(t,T,T^*) \approx \frac{\sigma_s}{1-z} C^d(t,T) = \sigma_{s^{zero}} C^{d,zero}(t,T) \ ,$$

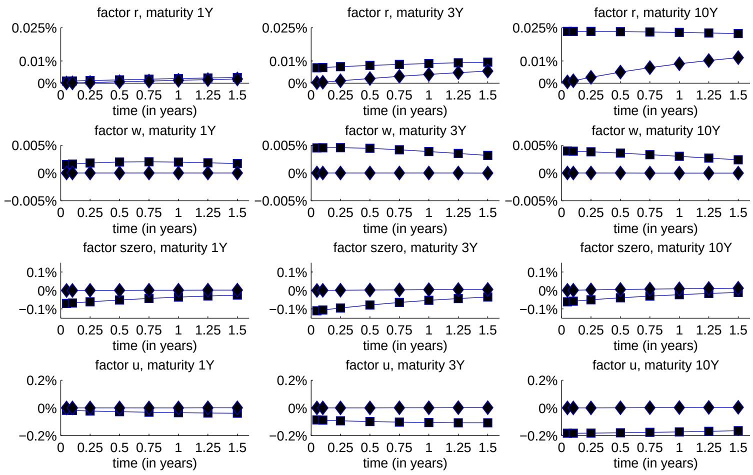

Figure 5.1: Each subplot of this figure gives the exact result (represented by squares) as well as the approximation (represented by diamonds) of $\vec{\sigma}_{P^{d*}}(t, T_m, T^*) - \vec{\sigma}_{P^{d*}}(t, T_0, T^*)$ (see page 126) for the factors r, w, s^{zero} and u in the SZ4 framework for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1, 3, 10\}$. The results are based on the parameters of GDPr for rating A2 (cf. Tables 4.4 and 4.7), $T_m = T_0$, $\tilde{y} = T_0$ and $z = 0.1$. The x-axis represents the forward starting time, i.e. $T_0 - t$.

especially for close T and T^*. Analogously, we assume

$$\begin{aligned}&\sigma^u_{P^{d*}}(t, T_m, T^*) - \sigma^u_{P^{d*}}(t, T_0, T^*)\\&\qquad \approx -\sigma_u D^{d,zero}(t, \tilde{y})(P^{d*}(t, T_m, T^*) - P^{d*}(t, T_0, T^*))\ ,\end{aligned}$$

with (cf. Proposition 5.10)

$$D^{d,*}(t, T, T^*) \approx \frac{1}{1-z} D^d(t, T) = D^{d,zero}(t, T)$$

since $\frac{b_{su}}{1-z} = b_{s^{zero}u}$, and

$$\begin{aligned}&\sigma^{w_1}_{P^{d*}}(t, T_m, T^*) - \sigma^{w_1}_{P^{d*}}(t, T_0, T^*)\\&\qquad \approx -\sigma_{w_1} E_1^{d,zero}(t, \tilde{y})(P^{d*}(t, T_m, T^*) - P^{d*}(t, T_0, T^*))\ ,\end{aligned}$$

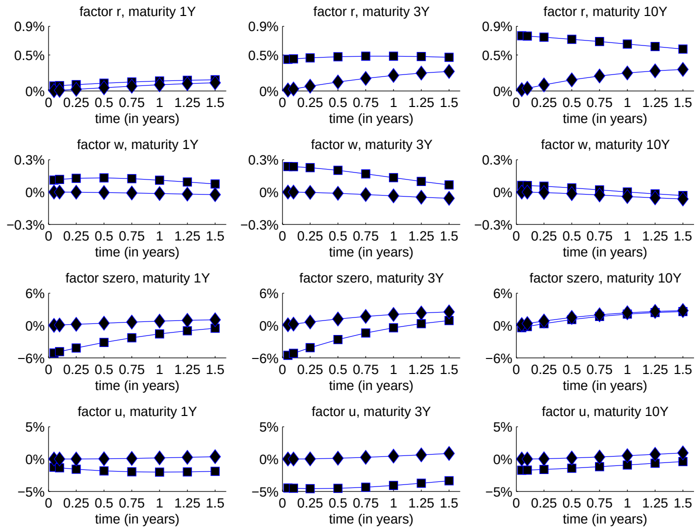

Figure 5.2: Each subplot of this figure gives the exact result (represented by squares) as well as the approximation (represented by diamonds) of $\vec{\sigma}_{P^{d*}}(t, T_m, T^*) - \vec{\sigma}_{P^{d*}}(t, T_0, T^*)$ (see page 126) for the factors r, w, s^{zero} and u in the SZ4 framework for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1, 3, 10\}$. The results are based on the parameters of GDPr for rating A2 (cf. Tables 4.4 and 4.7), $T_m = T_0$, $\tilde{y} = T_0$ and $z = 0.9$. The x-axis represents the forward starting time, i.e. $T_0 - t$.

with (cf. Proposition 5.10)

$$E_1^{d,*}(t, T, T^*)$$
$$\approx \frac{b_{rw_1}}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1} - \hat{a}_r}\right) + \frac{1}{1-z}\left(E_1^d(t, T) - E_1(t, T)\right)$$
$$= E_1^{d,zero}(t, T)$$

since $\frac{b_{sw_1}}{1-z} = b_{s^{zero}w_1}$.
These approximations will especially be satisfied if T_m - even better if T_0 also - is near T^* since $P^{d*}(t, T^*, T^*) = P^{d,zero}(t, T^*)$. Figure 5.1 and Figure 5.2 compare the exact results of $\vec{\sigma}_{P^{d*}}(t, T_m, T^*) - \vec{\sigma}_{P^{d*}}(t, T_0, T^*)$ with the approximations obtained above for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1, 3, 10\}$ and different values of z, $z = 0.1$ (see Figure 5.1) and $z = 0.9$ (see Figure 5.2). For $z = 0.1$ (see Figure 5.1) the approximated

values are close to the exact results of $\vec{\sigma}_{P^{d*}}(t, T_m, T^*) - \vec{\sigma}_{P^{d*}}(t, T_0, T^*)$ for all factors and maturities, especially for short maturities, i.e. $T_m - T_0 = 1$. However, for $z = 0.9$ (see Figure 5.2) the differences between the exact and approximated values increase especially for the factors s^{zero} and u. Here, the approximation seems to work better for longer maturities (10Y) for the factors w, u and s^{zero} indicating interdependencies between maturity and z.

Further, we approximate $\vec{\sigma}_{V^{ddp}}(t, T_m) - \vec{\sigma}_{V^{ddp}}(t, T_0)$ as follows where we assume certain integrals to be neglectable.

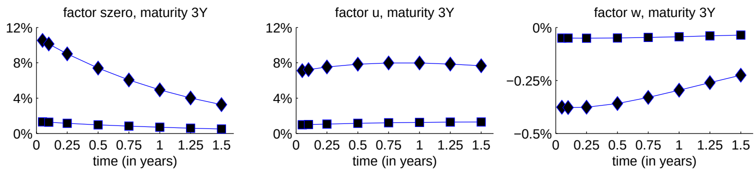

Figure 5.3: This figure contains the integrals (see page 129) $\sigma_{s^{zero}} \int_{T_0}^{T_m} P^{d,zero}(t, y) H(t, y) dy$, $\sigma_u \int_{T_0}^{T_m} P^{d,zero}(t, y) I(t, y) dy$ and $\sigma_{w_1} \int_{T_0}^{T_m} P^{d,zero}(t, y) J_1(t, y) dy$ for $z = 0.1$ (represented by squares) and $z = 0.9$ (represented by diamonds). The values are based on the parameters of GDPr for rating A2 (cf. Tables 4.4 and 4.7) and the maturity 3Y, i.e. $T_m - T_0 = 3$.

$$
\begin{aligned}
&\sigma^r_{V^{ddp}}(t, T_m) - \sigma^r_{V^{ddp}}(t, T_0) \\
&\quad \approx -\sigma_r B^{d,zero}(t, \tilde{y}) \int_{T_0}^{T_m} P^{d,zero}(t, y) \Big(F(t, y) + s^{zero}(t) H(t, y) \\
&\quad + u(t) I(t, y) + w_1(t) J_1(t, y) \Big) dy \\
&\quad = -\sigma_r B^{d,zero}(t, \tilde{y}) (V^{ddp}_{T^d}(t, T_m) - V^{ddp}_{T^d}(t, T_0)) \,,
\end{aligned}
$$

$$
\begin{aligned}
&\sigma^s_{V^{ddp}}(t, T_m) - \sigma^s_{V^{ddp}}(t, T_0) \\
&\quad \approx -\sigma_{s^{zero}} C^{d,zero}(t, \tilde{y}) (V^{ddp}_{T^d}(t, T_m) - V^{ddp}_{T^d}(t, T_0)) \\
&\quad + \sigma_{s^{zero}} \int_{T_0}^{T_m} P^{d,zero}(t, y) H(t, y) dy \\
&\quad \approx -\sigma_{s^{zero}} C^{d,zero}(t, \tilde{y}) (V^{ddp}_{T^d}(t, T_m) - V^{ddp}_{T^d}(t, T_0)) \,,
\end{aligned}
$$

$$\begin{aligned}
&\sigma^u_{V^{ddp}}(t,T_m) - \sigma^u_{V^{ddp}}(t,T_0) \\
&\quad \approx -\sigma_u D^{d,zero}(t,\tilde{y})(V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0)) \\
&\quad +\sigma_u \int_{T_0}^{T_m} P^{d,zero}(t,y) I(t,y) dy \\
&\quad \approx -\sigma_u D^{d,zero}(t,\tilde{y})(V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0)) \ ,
\end{aligned}$$

and

$$\begin{aligned}
&\sigma^{w_1}_{V^{ddp}}(t,T_m) - \sigma^{w_1}_{V^{ddp}}(t,T_0) \\
&\quad \approx -\sigma_{w_1} E^{d,zero}_1(t,\tilde{y})(V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0)) \\
&\quad +\sigma_{w_1} \int_{T_0}^{T_m} P^{d,zero}(t,y) J_1(t,y) dy \\
&\quad \approx -\sigma_{w_1} E^{d,zero}_1(t,\tilde{y})(V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0)) \ ,
\end{aligned}$$

with $\tilde{y} \in [T_0, T_m]$.
Figure 5.3 shows the values of the three integrals which we assume to be neglectable for a maturity of three years and different values of z, $z = 0.1$ and $z = 0.9$. As before, the approximation works well for $z = 0.1$ since the values of the integrals for all factors are close to zero. For $z = 0.9$ the values deviate from zero especially for the factors u and s^{zero}. Figure 5.4 and Figure 5.5 compare the exact results of $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$ with the approximations obtained above for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1, 3, 10\}$ and different values of z, $z = 0.1$ (see Figure 5.4) and $z = 0.9$ (see Figure 5.5). Similar to the results of the approximation for $\vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)$, the approximations are close to the exact results for $z = 0.1$ and the differences increase for $z = 0.9$, mainly for the factors u and s^{zero}.
Hence, by means of incorporating these approximations (see also page 114 for $\mu_{\langle\ \rangle}$) and by neglecting certain terms we obtain for the dynamics of the FCDS spread as given in Proposition 5.23

$$\begin{aligned}
&ds(t,T_0,T_m) \\
&\quad = \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} s(t,T_0,T_m) + \mu_{\langle\ \rangle}(t,T_0,T_m,T^*) \right) dt
\end{aligned}$$

$$+\Bigg(\Big(\vec{\sigma}_{V^{ddp}}(t,T_m)-\vec{\sigma}_{V^{ddp}}(t,T_0)+\vec{\sigma}_{P^{d*}}(t,T_m,T^*)-\vec{\sigma}_{P^{d*}}(t,T_0,T^*)\Big)'$$

$$-s(t,T_0,T_m)\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'\Bigg)\frac{1}{\sum_{i=1}^m P^{d,zero}(t,T_i)}d\widetilde{W}(t)$$

$$\overset{Th.5.13}{\approx}\Bigg(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'\vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2}$$

$$+\Big(\sigma_r B^{d,zero}(t,\tilde{y}),\sigma_{s^{zero}}C^{d,zero}(t,\tilde{y}),\sigma_u D^{d,zero}(t,\tilde{y}),\sigma_{w_1}E_1^{d,zero}(t,\tilde{y})\Big)$$

$$\cdot\frac{1}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]}\Big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\Big)\Bigg)s(t,T_0,T_m)dt$$

$$+\Bigg(-\Big(\sigma_r B^{d,zero}(t,\tilde{y}),\sigma_{s^{zero}}C^{d,zero}(t,\tilde{y}),\sigma_u D^{d,zero}(t,\tilde{y}),\sigma_{w_1}E_1^{d,zero}(t,\tilde{y})\Big)$$

$$-\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)}\Bigg)s(t,T_0,T_m)d\widetilde{W}(t)$$

Approximation for Proposition 5.24: Also, in case of SZ4 it holds that

$$\begin{aligned}
&\mu^Z_{\langle\ \rangle}(t,T_0,T_m)\\
&\quad=(1-Z)\Big(\sigma^r_{V^{ddp}}(t,T_m)-\sigma^r_{V^{ddp}}(t,T_0)\Big)\frac{\sigma_r\sum_{i=1}^m P^{d,zero}(t,T_i)B^{d,zero}(t,T_i)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2}\\
&\quad+(1-Z)\Big(\sigma^s_{V^{ddp}}(t,T_m)-\sigma^s_{V^{ddp}}(t,T_0)\Big)\frac{\sigma_{s^{zero}}\sum_{i=1}^m P^{d,zero}(t,T_i)C^{d,zero}(t,T_i)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2}\\
&\quad+(1-Z)\Big(\sigma^u_{V^{ddp}}(t,T_m)-\sigma^u_{V^{ddp}}(t,T_0)\Big)\frac{\sigma_u\sum_{i=1}^m P^{d,zero}(t,T_i)D^{d,zero}(t,T_i)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2}\\
&\quad+(1-Z)\Big(\sigma^{w_1}_{V^{ddp}}(t,T_m)-\sigma^{w_1}_{V^{ddp}}(t,T_0)\Big)\frac{\sigma_{w_1}\sum_{i=1}^m P^{d,zero}(t,T_i)E_1^{d,zero}(t,T_i)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2}
\end{aligned}$$

with the vector $\vec{\sigma}_{V^{ddp}}$ as before (see page 124ff).
Then, similar to the case discussed before (see also Proposition 5.24 for $\mu^Z_{\langle\ \rangle}$ and Figures 5.3 -5.5 for the approximations of $\vec{\sigma}_{V^{ddp}}(t,T_m)-\vec{\sigma}_{V^{ddp}}(t,T_0)$), the

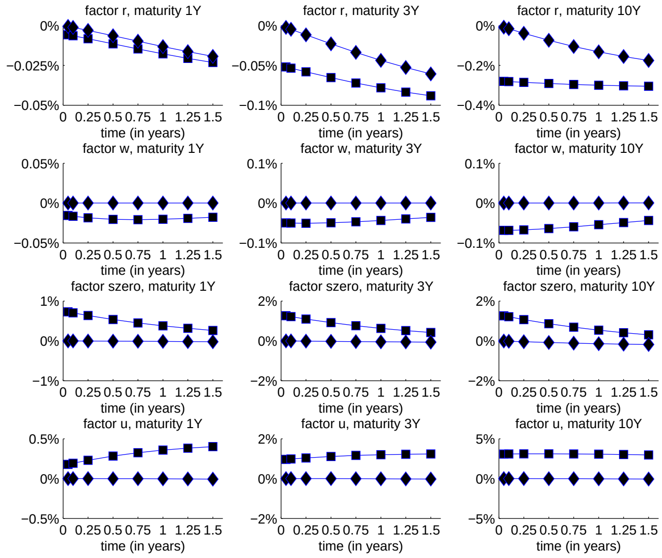

Figure 5.4: Each subplot of this figure gives the exact result (represented by squares) as well as the approximation (represented by diamonds) of $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$ (see page 129) for the factors r, w, s^{zero} and u in the SZ4 framework for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1, 3, 10\}$. The results are based on the parameters of GDPr for rating A2 (cf. Tables 4.4 and 4.7), $\tilde{y} = T_0$ and $z = 0.1$. The x-axis represents the forward starting time, i.e. $T_0 - t$.

approximated dynamics for Proposition 5.24 are

$$
\begin{aligned}
& ds(t,T_0,T_m) \\
&\quad = \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} s(t,T_0,T_m) + \mu^Z_{\langle\ \rangle}(t,T_0,T_m) \right) dt \\
&\quad + \frac{1}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \Bigg((1-Z)\left(\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) \right)' \\
&\quad - s(t,T_0,T_m) \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t) \right)' \Bigg) d\widetilde{W}(t) \\
&\quad \overset{Th.5.15}{\approx} \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y})\Big) \\
&\cdot \frac{1}{[\sum_{i=1}^m P^{d,zero}(t,T_i)]}\Big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\Big)\Big) s(t,T_0,T_m)dt \\
&+\Big(\Big(-\sigma_r B^{d,zero}(t,\tilde{y}), -\sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), -\sigma_u D^{d,zero}(t,\tilde{y}), \\
&-\sigma_{w_1} E_1^{d,zero}(t,\tilde{y})\Big) - \frac{\big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\big)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)}\Big) s(t,T_0,T_m) d\widetilde{W}(t)
\end{aligned}
$$

with $\tilde{y} \in [T_0, T_m]$.

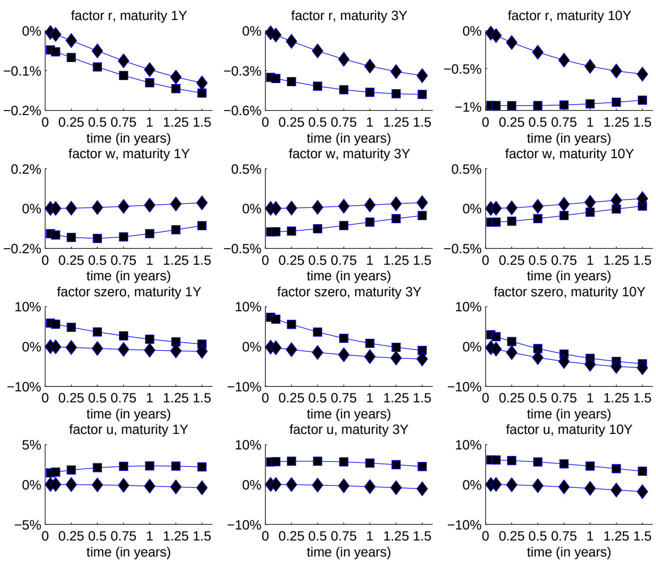

Figure 5.5: Each subplot of this figure gives the exact result (represented by squares) as well as the approximation (represented by diamonds) of $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$ (see page 129) for the factors r, w, s^{zero} and u in the SZ4 framework for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1, 3, 10\}$. The results are based on the parameters of GDPr for rating A2 (cf. Tables 4.4 and 4.7), $\tilde{y} = T_0$ and $z = 0.9$. The x-axis represents the forward starting time, i.e. $T_0 - t$.

Approximation for Proposition 5.25: Finally, for the third case we need the vector $\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)$ which consists of

$$\begin{aligned}
&\sigma^r_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \sigma^r_{P^{d,z}}(t,\widetilde{T}_j)\\
&\quad = -\sigma_r(B^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - B^{d,zero}(t,\widetilde{T}_j)P^{d,zero}(t,\widetilde{T}_j))\\
&\quad = -\sigma_r B^{d,zero}(t,\widetilde{T}_j)(P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - P^{d,zero}(t,\widetilde{T}_j))\ ,\\
&\sigma^s_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \sigma^s_{P^{d,z}}(t,\widetilde{T}_j)\\
&\quad = -\sigma_{s^{zero}}(C^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - C^{d,zero}(t,\widetilde{T}_j)P^{d,zero}(t,\widetilde{T}_j))\\
&\quad = -\sigma_{s^{zero}}(C^{d,zero}(t,\widetilde{T}_{j-1})P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - C^{d,zero}(t,\widetilde{T}_j)P^{d,zero}(t,\widetilde{T}_j))\ ,\\
&\sigma^u_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \sigma^u_{P^{d,z}}(t,\widetilde{T}_j)\\
&\quad = -\sigma_u(D^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - D^{d,zero}(t,\widetilde{T}_j)P^{d,zero}(t,\widetilde{T}_j))\\
&\quad = -\sigma_u(D^{d,zero}(t,\widetilde{T}_{j-1})P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - D^{d,zero}(t,\widetilde{T}_j)P^{d,zero}(t,\widetilde{T}_j))\ ,
\end{aligned}$$

and

$$\begin{aligned}
&\sigma^{w_1}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \sigma^{w_1}_{P^{d,z}}(t,\widetilde{T}_j)\\
&\quad = -\sigma_{w_1}(E_1^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - E_1^{d,zero}(t,\widetilde{T}_j)P^{d,zero}(t,\widetilde{T}_j))\ .
\end{aligned}$$

The components of the vector $\left(\vec{\sigma}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \vec{\sigma}_{P^{d,z}}(t,\widetilde{T}_j)\right)$ can be approximated by assuming $\widetilde{T}_{j-1} \approx \widetilde{T}_j$ with

$$\begin{aligned}
&\sigma^s_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \sigma^s_{P^{d,z}}(t,\widetilde{T}_j)\\
&\quad \approx -\sigma_{s^{zero}} C^{d,zero}(t,\widetilde{T}_j)(P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - P^{d,zero}(t,\widetilde{T}_j))\ ,\\
&\sigma^u_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \sigma^u_{P^{d,z}}(t,\widetilde{T}_j)\\
&\quad \approx -\sigma_u D^{d,zero}(t,\widetilde{T}_j)(P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - P^{d,zero}(t,\widetilde{T}_j))\ ,
\end{aligned}$$

and

$$\begin{aligned}
&\sigma^{w_1}_{P^*}(t,\widetilde{T}_{j-1},\widetilde{T}_j) - \sigma^{w_1}_{P^{d,z}}(t,\widetilde{T}_j)\\
&\quad \approx -\sigma_{w_1} E_1^{d,zero}(t,\widetilde{T}_j)(P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - P^{d,zero}(t,\widetilde{T}_j)).
\end{aligned}$$

where we use (see also page 268 for $E_1^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)$)

$$\begin{aligned}
E_1^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) &= E_1(t,\widetilde{T}_j) + (E_1^{d,zero}(t,\widetilde{T}_{j-1}) - E_1(t,\widetilde{T}_{j-1}))\\
&\approx E_1^{d,zero}(t,\widetilde{T}_j)
\end{aligned}$$

for $\widetilde{T}_{j-1} \approx \widetilde{T}_j$.
The dynamics of the FCDS according to Proposition 5.25 can now be written as

$$
\begin{aligned}
& ds(t,T_0,T_m) \\
&\approx \Bigg(\frac{\big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\big)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} s(t,T_0,T_m) \\
&+ \frac{(1-Z)\sum_{j=1}^n \Big(P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - P^{d,zero}(t,\widetilde{T}_j)\Big)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} \Big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\Big)' \cdot \\
&\Big(\sigma_r B^{d,zero}(t,\widetilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\widetilde{y}), \sigma_u D^{d,zero}(t,\widetilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\widetilde{y})\Big)' \Bigg) dt \\
&+ \Bigg(- \frac{(1-Z)\sum_{j=1}^n \Big(P^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) - P^{d,zero}(t,\widetilde{T}_j)\Big)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \\
&\cdot \Big(\sigma_r B^{d,zero}(t,\widetilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\widetilde{y}), \sigma_u D^{d,zero}(t,\widetilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\widetilde{y})\Big) \\
&- s(t,T_0,T_m) \frac{\big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\big)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \Bigg) d\widetilde{W}(t) \\
&\overset{Th.5.18}{=} s(t,T_0,T_m) \Bigg(\frac{\big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\big)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t,T_i)\right]^2} + \frac{\big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\big)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \cdot \\
&\Big(\sigma_r B^{d,zero}(t,\widetilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\widetilde{y}), \sigma_u D^{d,zero}(t,\widetilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\widetilde{y})\Big)' \Bigg) dt \\
&- \Bigg(\Big(\sigma_r B^{d,zero}(t,\widetilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\widetilde{y}), \sigma_u D^{d,zero}(t,\widetilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\widetilde{y})\Big) \\
&+ \frac{\big(\vec{\sigma}_{\sum_i P^{d,z}}(t)\big)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \Bigg) s(t,T_0,T_m) d\widetilde{W}(t) \, ,
\end{aligned}
$$

with $\tilde{y} \in [T_0, T_m]$. □

Note that we can end up with the same lognormal dynamics no matter which recovery assumptions for the reference entity we make. Also, using the default bucketing approach in order to circumvent certain integrals has no advantages so far, since the critical terms vanish during the above approximations. But, assuming a fine grid for the default bucketing results in another version of lognormal dynamics as shown in the next Corollary.

Corollary 5.27
For fine time steps $T_j - T_{j-1}$ in Proposition 5.25, the dynamics reduce to

$$ds(t, T_0, T_m) = s(t, T_0, T_m)\left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^{m} P^{d,zero}(t, T_i)\right]^2} dt - \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^{m} P^{d,zero}(t, T_i)} d\widetilde{W}(t)\right).$$

Proof:
Assuming the time steps $T_j - T_{j-1}$ in the numerator of Proposition 5.25 to be very fine, we get $P^*(t, \widetilde{T}_{j-1}, \widetilde{T}_j) \approx P^*(t, \widetilde{T}_j, \widetilde{T}_j) = P^{d,zero}(t, \widetilde{T}_j)$. With an analogous approximation as above, we obtain the dynamics.
□

There exists another useful approximation of the dynamics in Proposition 5.24, the so-called shifted-lognormal distribution (see Brigo & Mercurio (2006), page 454ff). It assumes the FCDS spread to be the sum of a lognormally distributed $X(t)$ and a real constant γ, i.e.

$$s(t, T_0, T_m) = X(t) + \gamma.$$

Assumption 5.28
In the SZ4 framework the exact dynamics of the FCDS spread presented in Proposition 5.24 can be approximated by shifted-lognormal dynamcis

$$\begin{aligned} ds(t, T_0, T_m) &= \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^{m} P^{d,zero}(t, T_i)\right]^2} \Big(s(t, T_0, T_m) - (1 - Z)\Delta T_i K\Big) dt \\ &- \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^{m} P^{d,zero}(t, T_i)} \Big(s(t, T_0, T_m) - (1 - Z)\Delta T_i K\Big) d\widetilde{W}(t) \end{aligned}$$

with equidistant $\Delta T_i := T_i - T_{i-1}$ $i = 1 \dots m$, frozen factors r, s^{zero}, u, w_1 in $\vec{\sigma}_{\sum_i P^{d,z}}$ and $P^{d,zero}$ at time t, and a constant K.

Approximation for Proposition 5.24: Within the SZ4 framework we assume the following approximations to hold by fixing $\tilde{y} \in [T_0, T_m]$ and by assuming as before certain integrals to be neglectable (see also Figure 5.3 for an analysis of these integrals).

$$\begin{aligned}
&\sigma^r_{V^{ddp}}(t,T_m) - \sigma^r_{V^{ddp}}(t,T_0) \\
&\quad \approx -\sigma_r \sum_{i=1}^{m} P^{d,zero}(t,T_i) B^{d,zero}(t,T_i) \Big(F(t,T_i) + s^{zero}(t) H(t,T_i) \\
&\quad + u(t) I(t,T_i) + w_1(t) J_1(t,T_i) \Big) (T_i - T_{i-1}) \\
&\quad = \Big(F(t,\tilde{y}) + s^{zero}(t) H(t,\tilde{y}) + u(t) I(t,\tilde{y}) + w_1(t) J_1(t,\tilde{y}) \Big) \Delta T_i \sigma^r_{\sum_i P^{d,z}}(t)
\end{aligned}$$

$$\begin{aligned}
&\sigma^s_{V^{ddp}}(t,T_m) - \sigma^s_{V^{ddp}}(t,T_0) \\
&\quad \approx \Big(F(t,\tilde{y}) + s^{zero}(t) H(t,\tilde{y}) + u(t) I(t,\tilde{y}) + w_1(t) J_1(t,\tilde{y}) \Big) \Delta T_i \sigma^s_{\sum_i P^{d,z}}(t) \\
&\quad + \sigma_{s^{zero}} \int_{T_0}^{T_m} P^{d,zero}(t,y) H(t,y) dy \\
&\quad \approx \Big(F(t,\tilde{y}) + s^{zero}(t) H(t,\tilde{y}) + u(t) I(t,\tilde{y}) + w_1(t) J_1(t,\tilde{y}) \Big) \Delta T_i \sigma^s_{\sum_i P^{d,z}}(t)
\end{aligned}$$

$$\begin{aligned}
&\sigma^u_{V^{ddp}}(t,T_m) - \sigma^u_{V^{ddp}}(t,T_0) \\
&\quad \approx \Big(F(t,\tilde{y}) + s^{zero}(t) H(t,\tilde{y}) + u(t) I(t,\tilde{y}) + w_1(t) J_1(t,\tilde{y}) \Big) \Delta T_i \sigma^u_{\sum_i P^{d,z}}(t) \\
&\quad + \sigma_u \int_{T_0}^{T_m} P^{d,zero}(t,y) I(t,y) dy \\
&\quad \approx \Big(F(t,\tilde{y}) + s^{zero}(t) H(t,\tilde{y}) + u(t) I(t,\tilde{y}) + w_1(t) J_1(t,\tilde{y}) \Big) \Delta T_i \sigma^u_{\sum_i P^{d,z}}(t)
\end{aligned}$$

$$\begin{aligned}
&\sigma^{w_1}_{V^{ddp}}(t,T_m) - \sigma^{w_1}_{V^{ddp}}(t,T_0) \\
&\quad \approx \Big(F(t,\tilde{y}) + s^{zero}(t) H(t,\tilde{y}) + u(t) I(t,\tilde{y}) + w_1(t) J_1(t,\tilde{y}) \Big) \Delta T_i \sigma^{w_1}_{\sum_i P^{d,z}}(t) \\
&\quad + \sigma_{w_1} \int_{T_0}^{T_m} P^{d,zero}(t,y) J_1(t,y) dy \\
&\quad \approx \Big(F(t,\tilde{y}) + s^{zero}(t) H(t,\tilde{y}) + u(t) I(t,\tilde{y}) + w_1(t) J_1(t,\tilde{y}) \Big) \Delta T_i \sigma^{w_1}_{\sum_i P^{d,z}}(t)
\end{aligned}$$

with $\tilde{y} \in [T_0, T_m]$ and $\Delta T_i := T_i - T_{i-1}$ equidistant for $i = 1 \dots m$.
Figure 5.6 and Figure 5.7 compare the exact results of $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$ with the approximations obtained above for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1, 3, 10\}$ and different values of z, $z = 0.1$ (see Figure 5.6) and $z = 0.9$ (see Figure 5.7). We obtain results that are very similar to the previous approximation for $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$ (see page 129).

Analogously to Figures 5.4 and 5.5, the approximations work well for all factors r, w, u, s^{zero} and $z = 0.1$, also for the factors r and w if $z = 0.9$. But the differences in the results increase for $z = 0.9$, especially for u and s^{zero}.

Hence, we obtain for the dynamics with a fixed $\tilde{y} \in [T_0, T_m]$ (see also Proposition 5.24 for $\mu^Z_{\langle\ \rangle}$)

$$\begin{aligned}
&ds(t, T_0, T_m) \\
&= \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} s(t, T_0, T_m) + \mu^Z_{\langle\ \rangle}(t, T_0, T_m) \right) dt \\
&+ \frac{1}{\sum_{i=1}^m P^{d,zero}(t, T_i)} \Big((1 - Z) \Big(\vec{\sigma}_{V^{ddp}}(t, T_m) - \vec{\sigma}_{V^{ddp}}(t, T_0) \Big)' \\
&- s(t, T_0, T_m) \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \Big) d\widetilde{W}(t) \\
&\approx \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} s(t, T_0, T_m) \right. \\
&+ (1 - Z) \Big(F(t, \tilde{y}) + s^{zero}(t) H(t, \tilde{y}) + u(t) I(t, \tilde{y}) + w_1(t) J_1(t, \tilde{y}) \Big) \Delta T_i \\
&\cdot \Big(\sigma^r_{\sum_i P^{d,z}}(t), \sigma^s_{\sum_i P^{d,z}}(t), \sigma^u_{\sum_i P^{d,z}}(t), \sigma^{w_1}_{\sum_i P^{d,z}}(t) \Big) \\
&\left. \cdot \frac{-1}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} \Big(\vec{\sigma}_{\sum_i P^{d,z}}(t) \Big) \right) dt \\
&+ \Big(\Big(F(t, \tilde{y}) + s^{zero}(t) H(t, \tilde{y}) + u(t) I(t, \tilde{y}) + w_1(t) J_1(t, \tilde{y}) \Big) \\
&\cdot (1 - Z) \Delta T_i \Big(\sigma^r_{\sum_i P^{d,z}}(t), \sigma^s_{\sum_i P^{d,z}}(t), \sigma^u_{\sum_i P^{d,z}}(t), \sigma^{w_1}_{\sum_i P^{d,z}}(t) \Big) \\
&- s(t, T_0, T_m) \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \Big) \frac{1}{\sum_{i=1}^m P^{d,zero}(t, T_i)} d\widetilde{W}(t) \\
&\approx \left(\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} s(t, T_0, T_m) \right. \\
&\left. - (1 - Z) \Delta T_i K \cdot \frac{(\vec{\sigma}_{\sum_i P^{d,z}}(t))' (\vec{\sigma}_{\sum_i P^{d,z}}(t))}{\left[\sum_{i=1}^m P^{d,zero}(t, T_i)\right]^2} \right) dt \\
&+ \frac{((1 - Z) \Delta T_i K \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' - s(t, T_0, T_m) \cdot \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)')}{\sum_{i=1}^m P^{d,zero}(t, T_i)} d\widetilde{W}(t)
\end{aligned}$$

$$= \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_i P^{d,z}}(t)}{\left[\sum_{i=1}^{m} P^{d,zero}(t,T_i)\right]^2} \Big(s(t,T_0,T_m) - (1-Z)\Delta T_i K \Big) dt$$
$$- \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^{m} P^{d,zero}(t,T_i)} \Big(s(t,T_0,T_m) - (1-Z)\Delta T_i K \Big) d\widetilde{W}(t)\,,$$

where the last approximation is obtained by substituting $\Big(F(t,\tilde{y}) + s^{zero}(t)H(t,\tilde{y}) + u(t)I(t,\tilde{y}) + w_1(t)J_1(t,\tilde{y})\Big)$ with a constant K. □

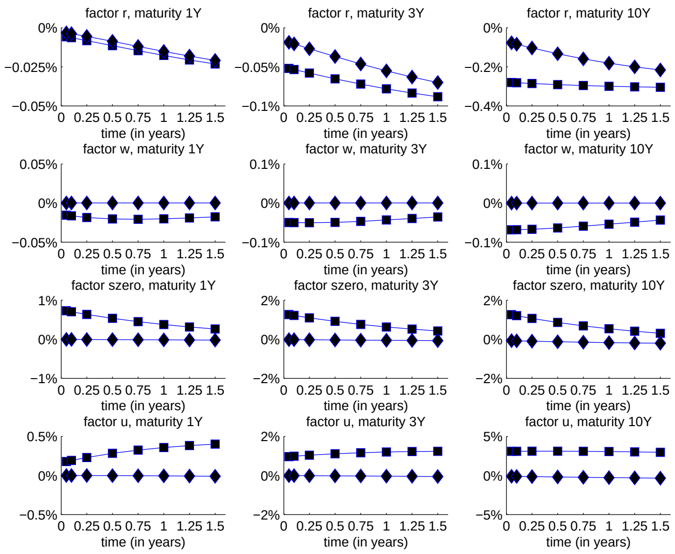

Figure 5.6: Each subplot of this figure gives the exact result (represented by squares) as well as the approximation (represented by diamonds) of $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$, i.e. $\Big(F(t,\tilde{y}) + s^{zero}(t)H(t,\tilde{y}) + u(t)I(t,\tilde{y}) + w_1(t)J_1(t,\tilde{y})\Big) \cdot \Delta T_i \cdot \vec{\sigma}_{\sum_i P^{d,z}}(t)$, for the factors r, w, s^{zero} and u in the SZ4 framework for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1,3,10\}$. The results are based on the parameters of GDPr for rating A2 (cf. Tables 4.4 and 4.7), $\Delta T_i = 0.1$, $\tilde{y} = T_0$ and $z = 0.1$. The x-axis represents the forward starting time, i.e. $T_0 - t$.

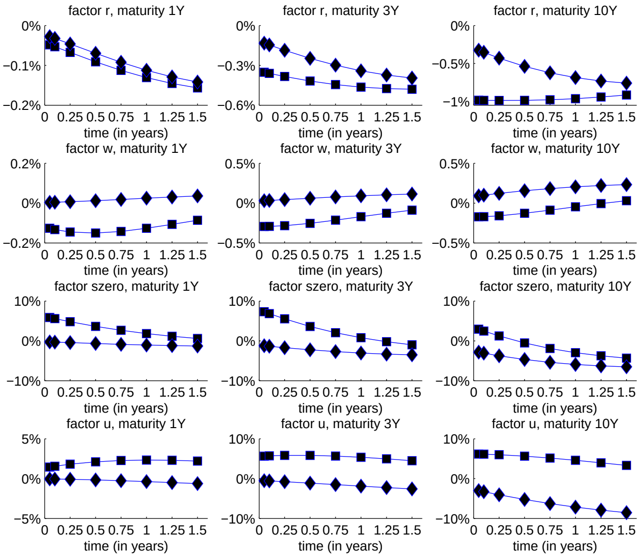

Figure 5.7: Each subplot of this figure gives the exact result (represented by squares) as well as the approximation (represented by diamonds) of $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$, i.e. $\Big(F(t,\tilde{y}) + s^{zero}(t)H(t,\tilde{y}) + u(t)I(t,\tilde{y}) + w_1(t)J_1(t,\tilde{y})\Big) \cdot \Delta T_i \cdot \vec{\sigma}_{\sum_i P^{d,z}}(t)$, for the factors r, w, s^{zero} and u in the SZ4 framework for different maturities (1Y, 3Y, 10Y), i.e. $T_m - T_0 \in \{1, 3, 10\}$. The results are based on the parameters of GDPr for rating A2 (cf. Tables 4.4 and 4.7), $\Delta T_i = 0.1$, $\tilde{y} = T_0$ and $z = 0.9$. The x-axis represents the forward starting time, i.e. $T_0 - t$.

In order to obtain the constant K we match moments as it is proposed by e.g. Brigo & Masetti (2006), page 15. Also, we follow the approach heavily used for LIBOR models (see e.g Mercurio & Morini (2007)) where freezing certain factors at time t_0 has shown to have no major impact on the dynamics. We work with the frozen dynamics of r, w_1, s and u. Therefore, the functions $\vec{\sigma}_{\sum_i P^{d,z}}(x)$, $P^{d,zero}(x,T)$ and $\mu^Z_{\langle\ \rangle}(x,T_0,T_m)$ are now deterministic. Assuming the existence of the following expected values, we equate the dynamics of Proposition 5.24 with the dynamics of the above approximation: [III]

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}[s(\tilde{t},T_0,T_m)\big|\,\mathcal{F}_t] \\
&\quad\approx s(t,T_0,T_m)\underbrace{\left(1+\int_t^{\tilde{t}}\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(x)\right)'\vec{\sigma}_{\sum_i P^{d,z}}(x)}{\left[\sum_{i=1}^m P^{d,zero}(x,T_i)\right]^2}dx\right)}_{(*)} \\
&\quad+\int_t^{\tilde{t}}\mu^Z_{\langle\ \rangle}(x,T_0,T_m)dx \\
&\quad\overset{!}{=}\mathbb{E}_{\widetilde{Q}}[s^{approx}(\tilde{t},T_0,T_m)\big|\,\mathcal{F}_t] \\
&\quad=(1-Z)\Delta T_i K \\
&\quad+\Big(s(t,T_0,T_m)-(1-Z)\Delta T_i K\Big)e^{\int_t^{\tilde{t}}\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}(x)}\right)'\vec{\sigma}_{\sum_i P^{d,z}(x)}}{\left[\sum_{i=1}^m P^{d,zero}(x,T_i)\right]^2}dx}
\end{aligned}
$$

Using the approximation $e^x = 1+x$ for (*), it holds

$$
\begin{aligned}
&s(t,T_0,T_m)e^{\int_t^{\tilde{t}}\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}(x)}\right)'\vec{\sigma}_{\sum_i P^{d,z}(x)}}{\left[\sum_{i=1}^m P^{d,zero}(x,T_i)\right]^2}dx}+\int_t^{\tilde{t}}\mu^Z_{\langle\ \rangle}(x,T_0,T_m)dx \\
&\quad\overset{!}{=}(1-Z)\Delta T_i K \\
&\quad+\Big(s(t,T_0,T_m)-(1-Z)\Delta T_i K\Big)e^{\int_t^{\tilde{t}}\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}(x)}\right)'\vec{\sigma}_{\sum_i P^{d,z}(x)}}{\left[\sum_{i=1}^m P^{d,zero}(x,T_i)\right]^2}dx}.
\end{aligned}
$$

Therefore, the constant K is calculated as

$$
K=\frac{\int_t^{\tilde{t}}\mu^Z_{\langle\ \rangle}(x,T_0,T_m)dx}{(1-Z)\Delta T_i\left(1-e^{\int_t^{\tilde{t}}\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}(x)}\right)'\vec{\sigma}_{\sum_i P^{d,z}(x)}}{\left[\sum_{i=1}^m P^{d,zero}(x,T_i)\right]^2}dx}\right)}.
$$

[III] For the shifted-lognormal dynamics $dX(t)=\alpha(t)(X(t)-\gamma)dt+\beta(t)(X(t)-\gamma)dW(t)$ with a real constant γ, and deterministic functions $\alpha(t)$ and $\beta(t)$, it holds

$$
X(T)=\gamma+(X(t)-\gamma)e^{\int_t^T\alpha(x)dx-\frac{1}{2}\int_t^T\beta^2(x)dx+\int_t^T\beta(x)dW(x)}
$$

(see e.g. Brigo & Mercurio (2006), page 454).

In the following we analyze how well the approximated dynamics work as opposed to the FCDS spread we would get by simulating the factors r, s, u, w_1 and using the formulas for the FCDS spread given in Section 5.4. The analysis is based on the set of parameters called GDPr of Chapter 4 for the rating class A2. We want to emphasize that this analysis is of theoretical nature and therefore assume the parameters as appropriate although they are obtained by means of defaultable bonds. However, parameters that are calibrated on defaultable bonds are not always suitable for pricing credit derivatives. The quoting mechanism and especially the assumptions concerning recovery rates differ within the markets. Hence, it would be appropriate to calibrate on quoted CDS spreads with the help of the closed-form solution of Section 5.4.

The following figures outline the evolution of Forward Credit Default Swaps for different maturities (3Y, 5Y, 7Y, 10Y) of the CDS and for different forward starting times (0.25Y, 0.5Y, 0.75Y, 1Y, 1.5Y). We use 20,000 scenarios and set z to 0.9 in line with the results of Antes et al. (2009) (see Definition 5.1). In Figure 5.8 the FCDS spreads assuming a protection payment of replacement to par are determined by simulating the factors r, s, u, w_1 until end of the forward starting time and then using the formula of Theorem 5.13. In addition to that, the approximated lognormal dynamics of Assumption 5.26 are used with $\tilde{y} = T_0$. For forward starting times under one year, the differences between the FCDS spreads are less than 5bp. For forward starting times over a year, the differences increase up to 10bp for a maturity of ten years. Furthermore, the differences seem to be dependent on the maturity of the underlying CDS. The longer the maturity of the CDS, the more the FCDS spreads deviate - especially for longer forward starting times. Analogously, Figure 5.9 shows the FCDS spreads assuming recovery as a fraction of face value and protection payments at certain dates. The results are obtained by means of Theorem 5.18 and by means of the approximated lognormal dynamics under the recovery assumption $Z = 0.75$. This value is chosen arbitrarily in order to get results of the same dimension as in Figure 5.8. Here, the differences between the FCDS spreads are smaller for shorter forward starting times as in the aforementioned case. But unlike before, the differences (especially for longer forward starting times) become smaller for increasing maturities of the underlying CDS. Figure 5.10 until Figure 5.13 outline the results for the FCDS spreads asuming recovery as a fraction of face value with $Z = 0.75$. Each figure represents the results for a given CDS maturity (3Y, 5Y, 7Y, 10Y) in order to account for the second approximation (shifted-lognormal) of the FCDS spread dynamics (see Assumption 5.28). The differences between the FCDS spread determined by means of Theorem 5.15 and by means of lognormal dynamics behave as in the case

of Figure 5.9. But the FCDS spreads determined with the shifted-lognormal assumptions match extremely well the results obtained via simulating the factors and using Theorem 5.15.
Summarizing the above results, we can claim the approximated dynamics to work well up to a forward starting time of one year no matter which recovery assumptions hold. Further, in case of recovery as a fraction of face value the approximation using shifted-lognormal dynamics even yields promising results for longer forward starting times.

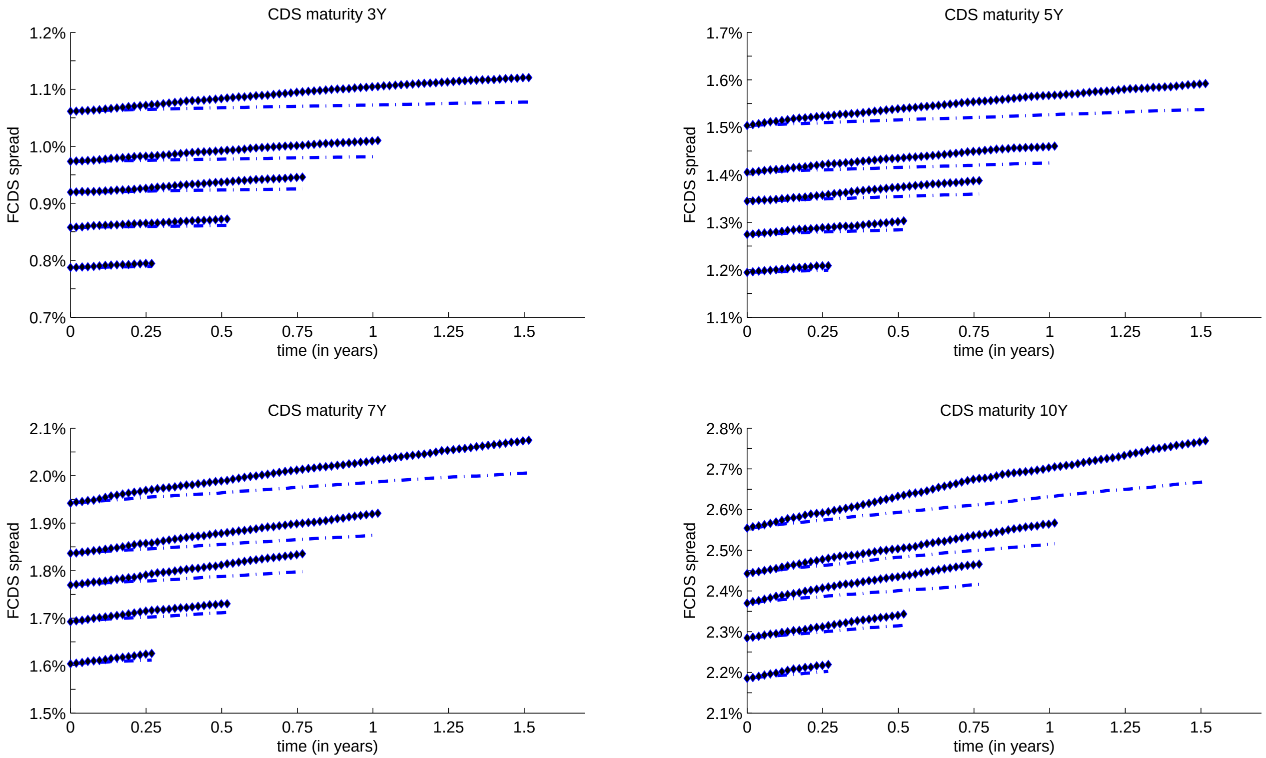

Figure 5.8: Each subplot of this figure shows FCDS spreads for different forward starting times (0.25Y, 0.5Y, 0.75Y, 1Y, 1.5Y) and for a given maturity of the corresponding CDS (3Y, 5Y, 7Y, 10Y). The broken line represents the results obtained by the approximated lognormal dynamics, whereas the diamonds give the solutions of Theorem 5.13. The plot is based on 20,000 scenarios and $z = 0.9$.

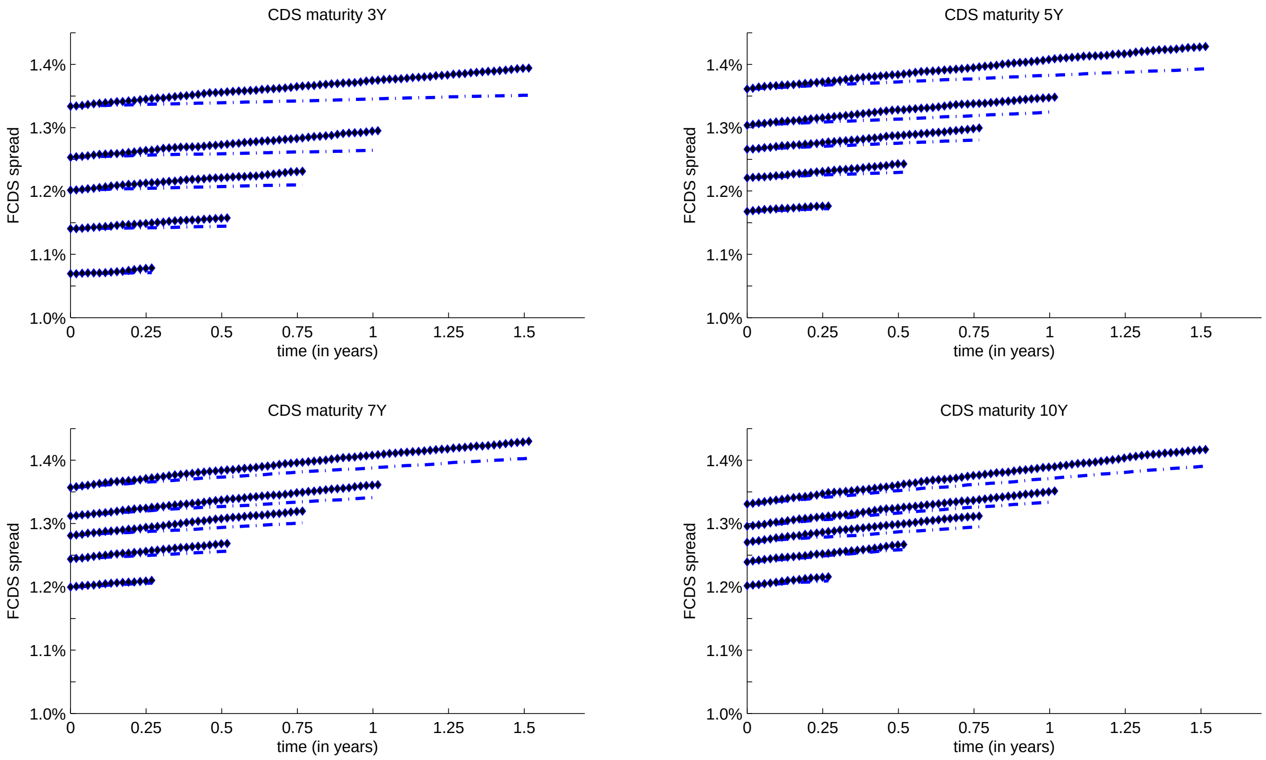

Figure 5.9: Each subplot of this figure shows FCDS spreads for different forward starting times (0.25Y, 0.5Y, 0.75Y, 1Y, 1.5Y) and for a given maturity of the corresponding CDS (3Y, 5Y, 7Y, 10Y). The broken line represents the results obtained by the approximated lognormal dynamics, whereas the diamonds give the solutions of Theorem 5.18. The plot is based on 20,000 scenarios, $Z = 0.75$ and $z = 0.9$.

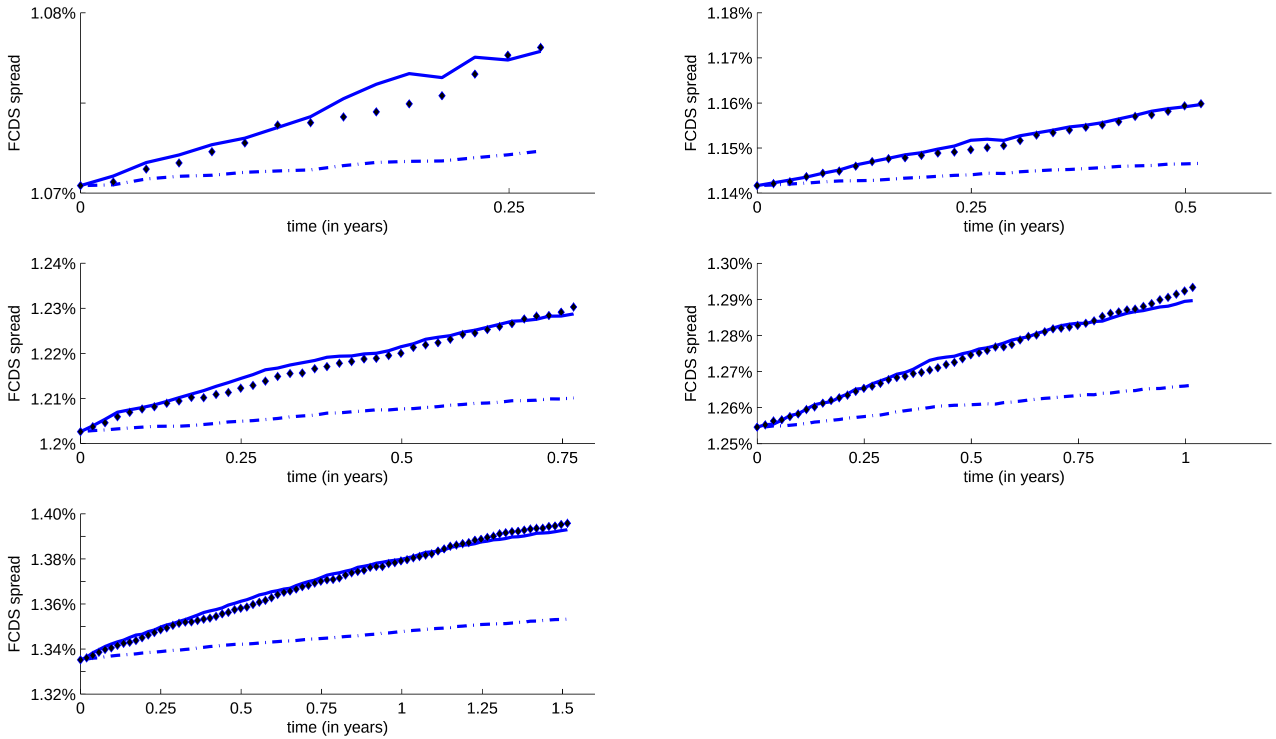

Figure 5.10: Each subplot of this figures shows FCDS spreads for a given forward starting time (0.25Y, 0.5Y, 0.75Y, 1Y, 1.5Y) in case of a CDS with maturity of 3 years. The broken line represents the results obtained by the approximated lognormal dynamics, the solid line shows the results of the shifted-lognormal dynamics, whereas the diamonds give the solutions of Theorem 5.15. The plot is based on 20,000 scenarios, $Z = 0.75$ and $z = 0.9$.

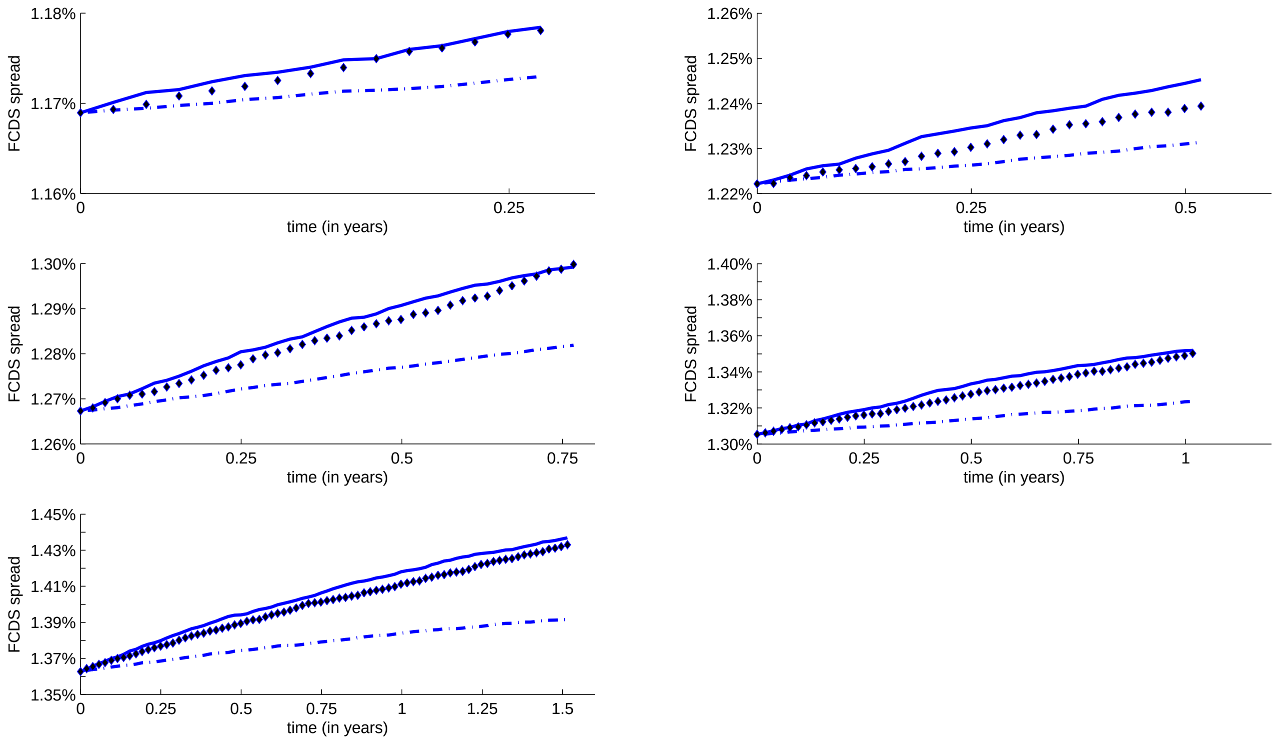

Figure 5.11: Each subplot of this figures shows FCDS spreads for a given forward starting time (0.25Y, 0.5Y, 0.75Y, 1Y, 1.5Y) in case of a CDS with maturity of 5 years. The broken line represents the results obtained by the approximated lognormal dynamics, the solid line shows the results of the shifted-lognormal dynamics, whereas the diamonds give the solutions of Theorem 5.15. The plot is based on 20,000 scenarios, $Z = 0.75$ and $z = 0.9$.

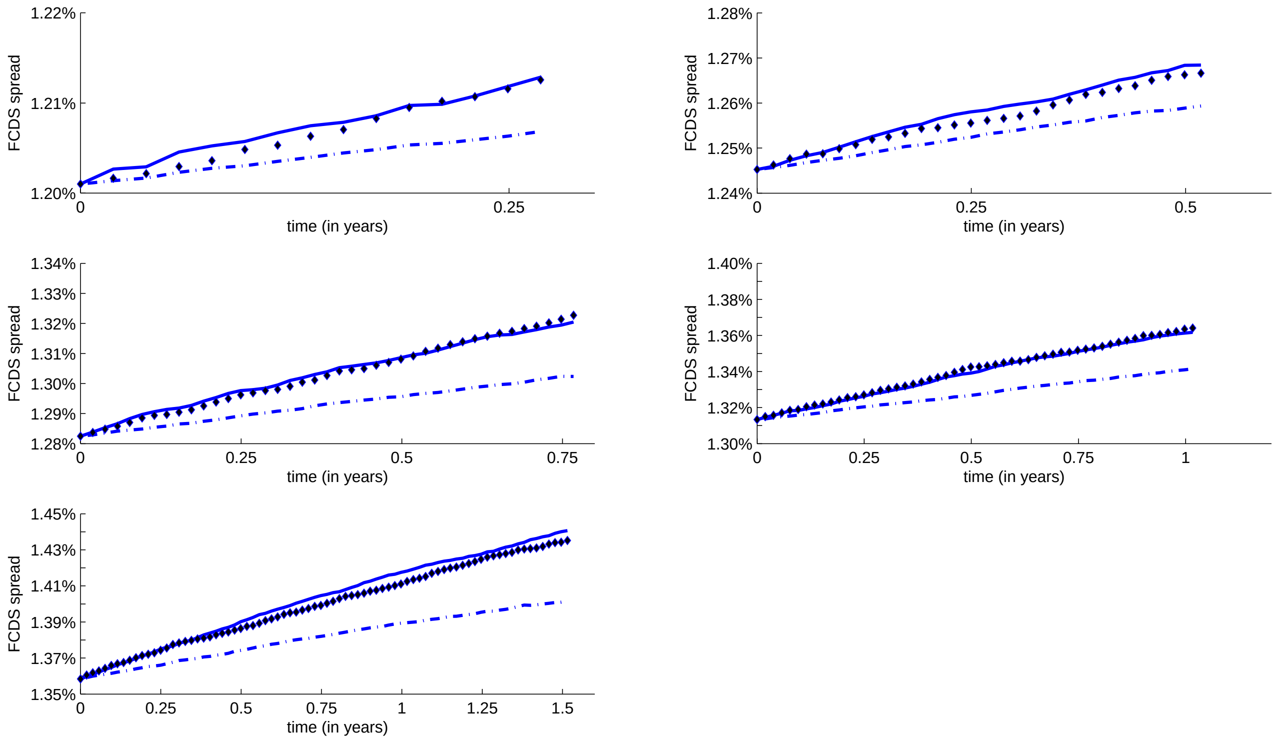

Figure 5.12: Each subplot of this figures shows FCDS spreads for a given forward starting time (0.25Y, 0.5Y, 0.75Y, 1Y, 1.5Y) in case of a CDS with maturity of 7 years. The broken line represents the results obtained by the approximated lognormal dynamics, the solid line shows the results of the shifted-lognormal dynamics, whereas the diamonds give the solutions of Theorem 5.15. The plot is based on 20,000 scenarios, $Z = 0.75$ and $z = 0.9$.

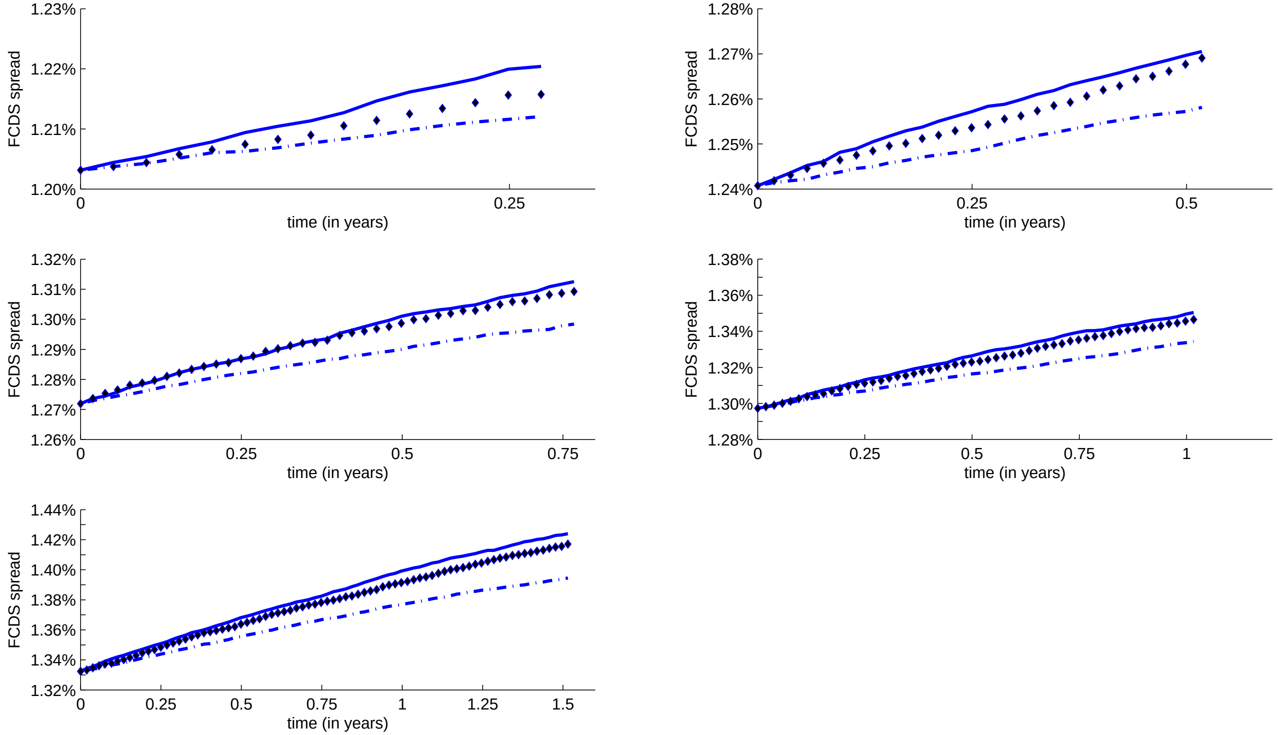

Figure 5.13: Each subplot of this figures shows FCDS spreads for a given forward starting time (0.25Y, 0.5Y, 0.75Y, 1Y, 1.5Y) in case of a CDS with maturity of 10 years. The broken line represents the results obtained by the approximated lognormal dynamics, the solid line shows the results of the shifted-lognormal dynamics, whereas the diamonds give the solutions of Theorem 5.15. The plot is based on 20,000 scenarios, $Z = 0.75$ and $z = 0.9$.

5.4.3 Introducing Counterparty Risk

All previously calculated credit default swaps share the assumption that neither the protection payer nor the protection seller will experience any defaults. Hence, these events are not priced into the credit default swap. But since the collaps of Lehman Brothers in 2008 and the problems of AIG, one of the biggest players in the CDS market, counterparties of OTC transactions are not considered undefaultable anymore.
Schmid (2002) suggested an approach how to price defaultable interest-rate swaps for a predecessor of our framework. In this section we introduce a first step towards pricing counterparty risk inherent in credit default swaps. We adopt the view of the protection buyer who considers her counterparty risky as opposed to herself. But the following calculations can be generalized in order to incorporate both counterparties as risky ones.
In the literature there are several approaches for calculating counterparty risk adjustments (CVA) for credit default swaps. The structural default model is used by e.g. Liang, Zhou, Zhou & Ma (2011) who model the correlation of the counterparty and the reference entity by two correlated geometric Brownian motions assuming a constant interest rate. Lipton & Sepp (2009) propose a multi-dimensional jump-diffusion process that drives the joint dynamics of asset values. Intensity contagion models where default intensities of the surviving firm are dependent on the default of the counterparty are used by e.g. Jarrow & Yu (2001), Leung & Kwok (2005) and Bao, Chen & Li (2012). Brigo & Chourdakis (2009) determine unilateral CVA for CDS assuming the intensities to be CIR processes and using a Gaussian copula for the dependence structure. Brigo & Capponi (2010) generalize the work of Brigo & Chourdakis (2009) in order to calculate bilateral CVA. Bielecki, Crépey, Jeanblanc & Zargari (2012) introduce a Markovian copula set-up in order to model the joint default between counterparty and the reference entity.
We follow the approach of Jarrow & Yu (2001) who use so-called primary and secondary firms in order to model default dependencies, but we do not restrict ourselves to constant interest rates or intensities as it is partially done in the above mentioned approaches. The default intensity of the primary firm is assumed to only depend on the filtration $\mathbb{F}$ which is generated by the state variables, whereas the default intensity of the secondary firm is dependent on the filtration $\mathbb{F}$ as well as the status of the primary firm. In our case, the protection seller is categorized as a secondary firm and the reference asset of the CDS is assumed to be a primary firm. We extend our notation as follows: The superscript ^{cp} indicates that the variable belongs to the counterparty and the superscript ^{ref} refers to variables belonging to

the reference asset. Further, the enlarged filtration $\mathbb{G}^{cp,ref} = \mathbb{F} \vee \mathbb{H}^{cp} \vee \mathbb{H}^{ref}$ is given as $\mathcal{G}_t^{cp,ref} = \mathcal{F}_t \vee \mathcal{H}_t^{cp} \vee \mathcal{H}_t^{ref}$ for every t, whereas $\widetilde{\mathcal{G}}_t^{cp}$ is built by $\widetilde{\mathcal{G}}_t^{cp} = \mathcal{F}_{T^*} \vee \mathcal{H}_t^{cp} \vee \mathcal{H}_{T^*}^{ref}$ for every $t \in [0, T^*]$, with $\widetilde{\mathcal{G}}_0^{cp} = \mathcal{F}_{T^*} \vee \mathcal{H}_{T^*}^{ref}$ for $t = 0$. Hence, the default intensity of the reference asset is adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ and the default intensity of the protection seller is adapted to the filtration $\widetilde{\mathbb{G}}^{cp} = \left(\widetilde{\mathcal{G}}_t^{cp}\right)_{0 \leq t \leq T^*}$. In this context the default intensity of the reference asset $s^{zero,ref}$ takes on the form as in Proposition 5.3 and the default intensity of the protection seller is given by $\widetilde{s}^{zero,cp}(t) = s^{zero,cp}(t) + c\mathbf{1}_{\{T^{d,ref} \leq t\}}$ with $s^{zero,cp}$ as in Proposition 5.3 and c a constant.

Again, we pose ourselves into our general framework introduced in Chapter 4. Therefore, the results obtained in this section can be used for all models derived from the general framework. Further, we take on the view of the protection buyer who assumes herself to be free of default risk and the counterparty to be defaultable with a recovery rate of zero. The reference asset of the CDS is deemed to be a primary firm in a sense that its default intensity is not dependent on the defaults of the protection buyer and seller. However, we assume that the default of the reference asset increases the default intensity of the protection seller, i.e. the protection seller is a secondary firm. This assumption is not unrealistic if we assume the protection seller to be e.g. a big player on the CDS market where contracts on that specific reference asset are traded on a large scale. Hence, a default of the reference asset would lead to protection payments to be made by the protection seller on every sold CDS written on that reference asset.

First, we give the spread of a FCDS according to the assumptions of Theorem 5.18. In the following, we assume the recovery of the counterparty to be zero. Though, this assumption can be easily changed.

Proposition 5.29 ***(cf. Theorem 5.18)***

If the recovery of the reference asset is paid at certain dates and as a fraction of the face value, the swap spread of a Forward Credit Default Swap which incorporates counterparty risk is

$$s^{cp}(t, T_0, T_m)$$
$$= \frac{(1-Z)\sum_{j=1}^{n}\left(e^{-c(\widetilde{T}_j - \widetilde{T}_{j-1})} \cdot \left(P^{d,z,cp,ref}(t, \widetilde{T}_{j-1}, \widetilde{T}_j) - P^{d,z,cp,ref}(t, \widetilde{T}_j)\right)\right)}{\sum_{i=1}^{m} P^{d,zero}(t, T_i)},$$

where we assume the recovery of the counterparty to be zero. The functions $P^{d,z,cp,ref}(t, \widetilde{T}_j)$ *and* $P^{d,z,cp,ref}(t, \widetilde{T}_{j-1}, \widetilde{T}_j)$ *are given in Appendix D, Lemma D.1 and Lemma D.2.*

Proof:

Since we assume the protection buyer to be safe, the payment leg is calculated as before (see Theorem 5.13). However, the protection leg is now given by

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L^{ref}(T_0) \sum_{j=1}^{n} e^{-\int_{T_0}^{\widetilde{T}_j} r(x)dx} (1-Z) \mathbf{1}_{\{\widetilde{T}_{j-1} < T^{d,ref} \leq \widetilde{T}_j\}} L^{cp}(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= (1-Z) \sum_{j=1}^{n} \Big(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_{j-1}) L^{cp}(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&\qquad - \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_j) L^{cp}(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \Big),
\end{aligned}$$

with

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_j) L^{cp}(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_j) L^{cp}(\widetilde{T}_j) \,\middle|\, \widetilde{\mathcal{G}}_t^{cp}\right] \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_j) \mathbb{E}_{\widetilde{Q}}\left[L^{cp}(\widetilde{T}_j) \,\middle|\, \widetilde{\mathcal{G}}_t^{cp}\right] \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_j) e^{-\int_t^{\widetilde{T}_j} \widetilde{s}^{zero,cp}(x)dx} \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_j) e^{-\int_t^{\widetilde{T}_j} s^{zero,cp}(x) + c\mathbf{1}_{\{T^{d,ref} \leq x\}} dx} \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x) + s^{zero,cp}(x)dx} L^{ref}(\widetilde{T}_j) e^{-c(\widetilde{T}_j - T^{d,ref})(1 - L^{ref}(\widetilde{T}_j))} \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x) + s^{zero,cp}(x)dx} L^{ref}(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&\overset{Prop.5.2}{=} L^{ref}(t) \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x) + s^{zero,cp}(x) + s^{zero,ref}(x)dx} \,\middle|\, \mathcal{F}_t\right] \\
&= L^{ref}(t) \cdot P^{d,z,cp,ref}(t, \widetilde{T}_j)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_{j-1}) L^{cp}(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)dx} L^{ref}(\widetilde{T}_{j-1}) L^{cp}(\widetilde{T}_j) \,\middle|\, \widetilde{\mathcal{G}}_t^{cp}\right] \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x) + s^{zero,cp}(x)dx} L^{ref}(\widetilde{T}_{j-1}) e^{-c(\widetilde{T}_j - T^{d,ref})(1 - L^{ref}(\widetilde{T}_j))} \,\middle|\, \mathcal{G}_t^{cp,ref}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)+s^{zero,cp}(x)dx} L^{ref}(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t^{cp,ref} \right] \\
&\quad + \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)+s^{zero,cp}(x)dx} \mathbf{1}_{\{\widetilde{T}_{j-1}<T^{d,ref}\le\widetilde{T}_j\}} e^{-c(\widetilde{T}_j - T^{d,ref})} \,\middle|\, \mathcal{G}_t^{cp,ref} \right] \\
&\approx \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)+s^{zero,cp}(x)dx} L^{ref}(\widetilde{T}_j) \,\middle|\, \mathcal{G}_t^{cp,ref} \right] \\
&\quad + \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)+s^{zero,cp}(x)dx} (L^{ref}(\widetilde{T}_{j-1}) - L^{ref}(\widetilde{T}_j)) e^{-c(\widetilde{T}_j - \widetilde{T}_{j-1})} \,\middle|\, \mathcal{G}_t^{cp,ref} \right]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{Prop. 5.2}{=} L^{ref}(t)\cdot(1-e^{-c(\widetilde{T}_j-\widetilde{T}_{j-1})})\cdot P^{d,z,cp,ref}(t,\widetilde{T}_j) \\
&\quad + L^{ref}(t)\cdot e^{-c(\widetilde{T}_j-\widetilde{T}_{j-1})}\cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)+s^{zero,cp}(x)dx} e^{-\int_t^{\widetilde{T}_{j-1}} s^{zero,ref}(x)dx} \,\middle|\, \mathcal{F}_t \right] \\
&= L^{ref}(t)\cdot(1-e^{-c(\widetilde{T}_j-\widetilde{T}_{j-1})})\cdot P^{d,z,cp,ref}(t,\widetilde{T}_j) \\
&\quad + L^{ref}(t)\cdot e^{-c(\widetilde{T}_j-\widetilde{T}_{j-1})}\cdot P^{d,z,cp,ref}(t,\widetilde{T}_{j-1},\widetilde{T}_j)\,,
\end{aligned}
$$

with $P^{d,z,cp,ref}(t,\widetilde{T}_j)$ and $P^{d,z,cp,ref}(t,\widetilde{T}_{j-1},\widetilde{T}_j)$ of Appendix D. □

If we give up the approximation of the protection leg of Theorem 5.18 but keep all assumptions regarding the recovery rates, we arrive at the framework of Theorem 5.15.

Proposition 5.30 ***(cf. Theorem 5.15)***
If the recovery of the reference asset is paid at default and as a fraction of the face value, the swap spread of a Forward Credit Default Swap which incorporates counterparty risk is

$$
\begin{aligned}
&s^{cp}(t,T_0,T_m) \\
&\quad = \frac{(1-Z)\sum_{j=1}^{n}\left(V^{cp,ref}(t,T_m)-V^{cp,ref}(t,T_0)\right)}{\sum_{i=1}^{m} P^{d,zero}(t,T_i)}\,,
\end{aligned}
$$

where we assume the recovery of the counterparty to be zero. The function $V^{cp,ref}$ is given in Appendix D, Lemma D.3.

Proof:
Again, we assume the protection buyer to be safe. Hence the payment leg looks like before (see Proposition 5.29).
The protection leg is calculated as

$$
(1-Z)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_{T_0}^{l} r(x)dx} L^{cp}(l)dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref} \right]
$$

with [IV]

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_{T_0}^{l} r(x)dx} L^{cp}(l) dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0)\mathbb{E}_{\widetilde{Q}}\left[\int_{T_0}^{T_m} e^{-\int_t^l r(x)dx} L^{cp}(l) dH^{ref}(l) \,\middle|\, \widetilde{\mathcal{G}}_t^{cp}\right] \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_t^l r(x)dx} \mathbb{E}_{\widetilde{Q}}\left[L^{cp}(l) \mid \widetilde{\mathcal{G}}_t^{cp}\right] dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_t^l r(x)dx} e^{-\int_t^l \widetilde{s}^{zero,cp}(x)dx} dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_t^l r(x)+s^{zero,cp}(x)+c\mathbf{1}_{\{T^{d,ref}\leq x\}}dx} dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_t^l r(x)+s^{zero,cp}(x)dx+c\max(0,l-T^{d,ref})} dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_t^l r(x)+s^{zero,cp}(x)dx+0} dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&\overset{Lemma\,D.3}{=} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(x)+s^{zero,cp}(x)dx} L^{ref}(T_0) V^{cp,ref}(T_0,T_m) \,\middle|\, \mathcal{G}_t^{cp,ref}\right] \\
&\overset{Prop.\,5.2}{=} L^{ref}(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} V^{cp,ref}(T_0,T_m) \,\middle|\, \mathcal{F}_t\right] \\
&= L^{ref}(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[\int_{T_0}^{T_m} e^{-\int_t^l r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} s^{zero,ref}(l)dl \,\middle|\, \mathcal{F}_t\right] \\
&\overset{Fubini}{=} L^{ref}(t)\cdot\int_{T_0}^{T_m} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^l r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} s^{zero,ref}(l) \,\middle|\, \mathcal{F}_t\right] dl \\
&= L^{ref}(t)\cdot\int_{t}^{T_m} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^l r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} s^{zero,ref}(l) \,\middle|\, \mathcal{F}_t\right] dl \\
&\quad - L^{ref}(t)\cdot\int_{t}^{T_0} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^l r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} s^{zero,ref}(l) \,\middle|\, \mathcal{F}_t\right] dl \\
&\overset{Lemma\,D.3}{=} L^{ref}(t)\cdot\left(V^{cp,ref}(t,T_m) - V^{cp,ref}(t,T_0)\right)
\end{aligned}
$$

□

Finally, if we remove the assumption regarding the recovery of face value of the reference asset (cf. Theorem 5.13), we arrive at the following proposition.

[IV] For Fubini's Theorem see Duffie (1996), page 282.

Proposition 5.31 ***(cf. Theorem 5.13)***
The swap spread of a Forward Credit Default Swap according to Theorem 5.13 which incorporates counterparty risk is

$$s^{cp}(t,T_0,T_m) \\ = \frac{V^{cp,ref}(t,T_m) - V^{cp,ref}(t,T_0) + P^{*,cp,ref}(t,T_m,T^*) - P^{*,cp,ref}(t,T_0,T^*)}{\sum_{i=1}^m P^{d,zero}(t,T_i)},$$

where we assume the recovery of the counterparty to be zero. The functions $V^{cp,ref}$ and $P^{*,cp,ref}$ are given in Appendix D, Lemma D.3 and D.4.

Proof:
The payment leg is the same as in Proposition 5.30 above, and the protection leg is calculated as

$$\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_{T_0}^l r(x)dx} L^{cp}(l)(1 - Z^{ref}(l))dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right]$$
$$= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_{T_0}^l r(x)dx} L^{cp}(l)dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right]$$
$$-\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_{T_0}^l r(x)dx} L^{cp}(l)Z^{ref}(l)dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right]$$
$$\overset{Prop\,5.30}{=} L^{ref}(t) \cdot (V^{cp,ref}(t,T_m) - V^{cp,ref}(t,T_0))$$
$$-\mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_t^l r(x)dx} \mathbb{E}_{\widetilde{Q}}\left[L^{cp}(l)\middle|\, \widetilde{\mathcal{G}}_t^{cp}\right] Z^{ref}(l)dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right]$$
$$= L^{ref}(t) \cdot (V^{cp,ref}(t,T_m) - V^{cp,ref}(t,T_0))$$
$$-\mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_t^l r(x)dx} e^{-\int_t^l \widetilde{s}^{zero,cp}(x)dx} Z^{ref}(l)dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right]$$
$$= L^{ref}(t) \cdot (V^{cp,ref}(t,T_m) - V^{cp,ref}(t,T_0))$$
$$-\mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0) \int_{T_0}^{T_m} e^{-\int_t^l r(x)+s^{zero,cp}(x)+c\mathbf{1}_{\{T^{d,ref}\leq x\}}dx} Z^{ref}(l)dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right]$$
$$= L^{ref}(t) \cdot (V^{cp,ref}(t,T_m) - V^{cp,ref}(t,T_0))$$

$$
\begin{aligned}
&-\mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_0)\int_{T_0}^{T^*} e^{-\int_t^l r(x)+s^{zero,cp}(x)dx} Z^{ref}(l)dH^{ref}(l)\,\middle|\,\mathcal{G}_t^{cp,ref}\right]\\
&+\mathbb{E}_{\widetilde{Q}}\left[L^{ref}(T_m)\int_{T_m}^{T_*} e^{-\int_t^l r(x)+s^{zero,cp}(x)dx} Z^{ref}(l)dH^{ref}(l)\,\middle|\,\mathcal{G}_t^{cp,ref}\right]\\
&= L^{ref}(t)\cdot(V^{cp,ref}(t,T_m)-V^{cp,ref}(t,T_0))\\
&-\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(x)+s^{zero,cp}(x)dx} L^{ref}(T_0)P^{d,cp,ref}(T_0,T^*)\,\middle|\,\mathcal{G}_t^{cp,ref}\right]\\
&+\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(x)+s^{zero,cp}(x)dx} L^{ref}(T_0)P^{d,z,cp,ref}(T_0,T^*)\,\middle|\,\mathcal{G}_t^{cp,ref}\right]\\
&+\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r(x)+s^{zero,cp}(x)dx} L^{ref}(T_m)P^{d,cp,ref}(T_m,T^*)\,\middle|\,\mathcal{G}_t^{cp,ref}\right]\\
&-\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r(x)+s^{zero,cp}(x)dx} L^{ref}(T_m)P^{d,z,cp,ref}(T_m,T^*)\,\middle|\,\mathcal{G}_t^{cp,ref}\right]\\
&\overset{Prop\,5.2}{=} L^{ref}(t)\cdot(V^{cp,ref}(t,T_m)-V^{cp,ref}(t,T_0))\\
&-L^{ref}(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} P^{d,cp,ref}(T_0,T^*)\,\middle|\,\mathcal{F}_t\right]\\
&+L^{ref}(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} P^{d,z,cp,ref}(T_0,T^*)\,\middle|\,\mathcal{F}_t\right]\\
&+L^{ref}(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} P^{d,cp,ref}(T_m,T^*)\,\middle|\,\mathcal{F}_t\right]\\
&-L^{ref}(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} P^{d,z,cp,ref}(T_m,T^*)\,\middle|\,\mathcal{F}_t\right]\\
&= L^{ref}(t)\cdot(V^{cp,ref}(t,T_m)-V^{cp,ref}(t,T_0))\\
&+L^{ref}(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} P^{d,cp,ref}(T_m,T^*)\,\middle|\,\mathcal{F}_t\right]\\
&-L^{ref}(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} P^{d,cp,ref}(T_0,T^*)\,\middle|\,\mathcal{F}_t\right]\\
&= L^{ref}(t)\cdot(V^{cp,ref}(t,T_m)-V^{cp,ref}(t,T_0)+P^{*,cp,ref}(t,T_m,T^*)\\
&\quad -P^{*,cp,ref}(t,T_0,T^*))
\end{aligned}
$$

where $P^{*,cp,ref}$ is derived in Appendix D, Lemma D.4 and with

$$
\begin{aligned}
&P^{d,cp,ref}(t,T)\\
&:= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(x)+s^{zero,cp}(x)+s^{ref}(x)dx}\,\middle|\,\mathcal{F}_t\right]\\
&= P^{d,cp,ref}(t,T_a,r(t),s^{zero,cp}(t),u^{cp}(t),s^{ref}(t),u^{ref}(t),w_1(t),w_2(t))\,.
\end{aligned}
$$

$P^{d,cp,ref}$ has a similar structure to $P^{d,z,cp,ref}$ of Lemma D.1 where $s^{zero,ref}$ is replaced by s^{ref}. Hence, the factors $\theta_{s^{zero,ref}}$, $b_{s^{zero,ref}u^{ref}}$, $b_{s^{zero,ref}w_1}$, $b_{s^{zero,ref}w_2}$ and $\sigma_{s^{zero,ref}}$ have to be substituted with $\theta_{s^{ref}}$, $b_{s^{ref}u^{ref}}$, $b_{s^{ref}w_1}$, $b_{s^{ref}w_2}$ and $\sigma_{s^{ref}}$ (cf. Proposition 5.3). □

It is remarkable that by incorporating unilateral counterparty risk, we maintain the closed form of the FCDS spread formulas.
In a second step, this approach can be further extended in order to account also for the risk of default for the third party involved in a FCDS contract.

5.5 Credit Default Swaption

In this section we aim to price the optionality to enter into a CDS (called Credit Default Swaption or CDS Option) by means of an analytical formula while staying within our framework.
Alfonsi & Brigo (2005) also introduce an analytical formula for CDS options but under a CIR intensity framework with a deterministic short rate. In this simplified framework, they can use a variant of Jamshidian's decomposition (cf. Jamshidian (1989)) in order to obtain the result. Additionally, they mention but do not further present a possible way to price a CDS option when the short rate also follows a CIR process. According to them, this could be done by mapping the two-dimensional CIR process "in an analogous tractable two-dimensional Gaussian dynamics that preserves as much as possible of the original CIR structure". This, however, again underlines the beauty of our Gaussian framework: the analytical tractability. Brigo & El-Bachir (2010) propose an extension of the work of Alfonsi & Brigo (2005) by using a shifted square-root jump-diffusion model which yields a semi-analytical formula.
Schoenbucher (2000) introduces a credit risk model based on the LIBOR market model using processes for the effective default-free forward rates and effective forward credit spreads. Within this framework an approximate solution for CDS options exists. By changing to a swap-based market model, he shows that an exact formula for CDS options exists. More details to the methods used in this article can be found in Schoenbucher (2003).
In their paper, Krekel & Wenzel (2006) implement Schoenbucher's original model for pricing Credit Default Swaptions and Constant Maturity Credit Default Swaps (see also next section) with Monte Carlo simulation. In addition, they use closed-form solutions derived by Schoenbucher (2003) and Brigo (2005) as control variates in order to increase accuracy. Furthermore, they present a new closed-form solution for CDS options allowing for time varying volatilities and decorrelated discrete default intensities.

Definition 5.32 ***Credit Default Swaption***
A (payer) Credit Default Swaption (CDS Option) gives its holder the right to enter a CDS at time T_0 with swap rate $\widetilde{S}$. In exchange for premiums paid at $T_1, \ldots, T_m$ or until default, this CDS makes a protection payment if default occurs in $[T_0, T_m]$.

The value of a European style CDS Option, where the option expires at T and the swap starts at T_0, is given by:

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl} L(T)\mathbb{E}_{\widetilde{Q}}\left[L(T_0)\left(\int_{T_0}^{T_m} e^{-\int_{T_0}^u r(l)dl}(1-Z(u))dH(u)\right.\right.\right.\\
&\qquad\left.\left.\left.-\widetilde{S}\sum_{i=1}^m e^{-\int_{T_0}^{T_i} r(l)dl} L(T_i)\right)^+ \middle| \mathcal{G}_T\right] \middle| \mathcal{G}_t\right]\\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl} L(T)\mathbb{E}_{\widetilde{Q}}\left[L(T_0)\left(s(T_0,T_0,T_m)\sum_{i=1}^m e^{-\int_{T_0}^{T_i} r(l)dl} L(T_i)\right.\right.\right.\\
&\qquad\left.\left.\left.-\widetilde{S}\sum_{i=1}^m e^{-\int_{T_0}^{T_i} r(l)dl} L(T_i)\right)^+ \middle| \mathcal{G}_T\right] \middle| \mathcal{G}_t\right]\\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl} L(T_0)\left(s(T_0,T_0,T_m)-\widetilde{S}\right)^+ \underbrace{\sum_{i=1}^m e^{-\int_{T_0}^{T_i} r(l)dl} L(T_i)}_{\mathbb{E}_{\widetilde{Q}}\left[\sum_{i=1}^m e^{-\int_{T_0}^{T_i} r(l)dl} L(T_i)\middle|\mathcal{G}_{T_0}\right]} \middle| \mathcal{G}_t\right]\\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl} L(T_0)\left(s(T_0,T_0,T_m)-\widetilde{S}\right)^+ \sum_{i=1}^m P^{d,zero}(T_0,T_i)\middle| \mathcal{G}_t\right]\\
&\overset{Prop.5.2}{=} L(t)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl-\int_t^{T_0} s^{zero}(l)dl}\left(s(T_0,T_0,T_m)-\widetilde{S}\right)^+ \sum_{i=1}^m P^{d,zero}(T_0,T_i)\middle| \mathcal{F}_t\right]\\
&= L(t)\cdot V^{cdso}(t,T,T_0,T_m)\,.
\end{aligned}
$$

If the option maturity coincides with the beginning of the swap (i.e. $T = T_0$), the value of a European style CDS Option simplifies to

$$L(t) \cdot \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} \Big(s(T_0, T_0, T_m) - \widetilde{S} \Big)^+ \sum_{i=1}^{m} P^{d,zero}(T_0, T_i) \middle| \mathcal{F}_t \right]$$
$$= \quad L(t) \cdot V^{cdso}(t, T_0, T_0, T_m) \, .$$

The calculation of this CDS Option's value can be done by means of simulating the factors r, w_1, w_2, u, s up to time T_0 and by using the formula for the Credit Default Swap of Propositions 5.14, 5.16 or Theorem 5.18. Further, we can use the dynamics and its approximations of the Forward Credit Default Swap, which were derived in the previous section, in order to find a closed-form solution. One way is to follow the approach of Brigo & Mercurio (2006) (cf. chapter 23) by performing a change of numéraire. $V^{cdso}(t, T_0, T_0, T_m)$ can be also written as

$$V^{cdso}(t, T_0, T_0, T_m)$$
$$= \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^{T_0} r(l)dl} \left(s(T_0, T_0, T_m) - \widetilde{S} \right)^+ \frac{\widetilde{Q}(T^d > T_0 | \mathcal{F}_{T_0})}{\widetilde{Q}(T^d > t | \mathcal{F}_t)} \sum_{i=1}^{m} P^{d,zero}(T_0, T_i) \middle| \mathcal{F}_t \right]$$

where the term inside the expectation looks like a discount factor times a call option times a new numéraire. This numéraire happens to be the denominator of the FCDS spread if using the approach of Brigo & Mercurio (2006) (see pages 727-731). We arrive at the same denominator if we extend our formulas of the FCDS spread, derived in Theorems 5.13, 5.15 and 5.18, by $\widetilde{Q}(T^d > t|\mathcal{F}_t)$ in the numerator as well as the denominator. Hence, our new numéraire will be the quantity $\widetilde{Q}(T^d > t|\mathcal{F}_t) \sum_{i=1}^{m} P^{d,zero}(t, T_i)$.
If we add the indicator $L(t)$ to this quantity, we would get a tradeable asset, namely the price at time t of a portfolio of defaultable zero-coupon bonds with zero recovery. Using this term as new numéraire would be in spirit with the work of Schoenbucher (2000), who introduced a so-called survival measure. This measure attaches a weight of zero to default events, and therefore it is not equivalent to the measure $\widetilde{Q}$ anymore.
Keeping in line with the argumentation of Brigo & Mercurio (2006), we neglect the indicator $L(t) = \mathbf{1}_{\{T^d > t\}}$. Hence our new numéraire is always strictly positive and we will not end up with a non-equivalent pricing measure.

Theorem 5.33
If the dynamics of the Forward Credit Default Swap are lognormal, the price

at time t of a Credit Default Swaption is given by the formula

$$L(t) \cdot \sum_{i=1}^{m} P^{d,zero}(t,T_i)(s(t,T_0,T_m) \cdot \mathcal{N}(d_1) - \widetilde{S} \cdot \mathcal{N}(d_2)) \,,$$

with

$$d_1 := \frac{\ln\left(\frac{s(t,T_0,T_m)}{\widetilde{S}}\right) + \frac{1}{2}\sigma_Y^2}{\sigma_Y} \,,\; d_2 := d_1 - \sigma_Y \,,$$

and

$$\sigma_Y = \sigma_Y(t,T_0) := \sqrt{\int_t^{T_0} \sigma^2(s)ds} \,.$$

$\mathcal{N}$ denotes the standard normal cumulative distribution function.

Proof:
The price of a CDS Option where the swap starts at the maturity of the option is $L(t) \cdot V^{cdso}(t,T_0,T_0,T_m)$ with

$$V^{cdso}(t,T_0,T_0,T_m) \\ = \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} \left(s(T_0,T_0,T_m) - \widetilde{S}\right)^{+} \frac{\widetilde{Q}(T^d > T_0|\mathcal{F}_{T_0})}{\widetilde{Q}(T^d > t|\mathcal{F}_t)} \sum_{i=1}^{m} P^{d,zero}(T_0,T_i) \middle| \mathcal{F}_t \right].$$

The Radon-Nikodym derivative (cf. Theorem 2.19 and Theorem 2.22) defining the new measure $\widehat{Q}_{0,m}$ is given by

$$\left.\frac{d\widehat{Q}_{0,m}}{d\widetilde{Q}}\right|_{\mathcal{F}_t} = \frac{\widetilde{Q}(T^d > t|\mathcal{F}_t)\sum_{i=1}^{m} P^{d,zero}(t,T_i) \cdot P_0(t_0)}{\widetilde{Q}(T^d > t_0|\mathcal{F}_{t_0})\sum_{i=1}^{m} P^{d,zero}(t_0,T_i) \cdot P_0(t)} \quad t_0 < t < T_i \;\forall i,$$

with P_0 being the non-defaultable cash account (see Chapter 4). The numerator of the FCDS spread $s(t,T_0,T_m)$ is the value of the protection leg. This value can be seen as the price of a CDS where the premium is paid upfront. Therefore, we can interpret the FCDS spread as a ratio between a tradeable asset and our numéraire. Further, the FCDS spread is a martingale under this numéraire's measure.
The change of numéraire yields

$$\begin{aligned} &V^{cdso}(t,T_0,T_0,T_m) \\ &= \sum_{i=1}^{m} P^{d,zero}(t,T_i)\mathbb{E}_{\widehat{Q}_{0,m}}\left[\left(s(T_0,T_0,T_m) - \widetilde{S}\right)^{+} \middle| \mathcal{F}_t \right] \end{aligned}$$

Since we assume the FCDS spread to be lognormal under $\widetilde{Q}$ with $\sigma(t)$ being progessively measurable and satisfying the Novikov condition (cf. Theorem 2.19), the FCDS spread follows a driftless geometric Brownian motion under the new measure $\widehat{Q}_{0,m}$ ($ds(t,T_0,T_m) = s(t,T_0,T_m)\sigma(t)d\widehat{W}$). Hence, we get by means of a variant of Black's formula (cf. Theorem 2.25) the price of a CDS Option at time t

$$\begin{aligned} & L(t) \cdot V^{cdso}(t,T_0,T_0,T_m) \\ = \; & L(t) \cdot \sum_{i=1}^{m} P^{d,zero}(t,T_i)(s(t,T_0,T_m) \cdot \mathcal{N}(d_1) - \widetilde{S} \cdot \mathcal{N}(d_2)) \, , \end{aligned}$$

with

$$d_1 := \frac{\ln\left(\frac{s(t,T_0,T_m)}{\widetilde{S}}\right) + \frac{1}{2}\sigma_Y^2}{\sigma_Y} \; , \; d_2 := d_1 - \sigma_Y \; ,$$

and

$$\sigma_Y = \sigma_Y(t,T_0) := \sqrt{\int_t^{T_0} \sigma^2(s) ds} \; .$$

$\mathcal{N}$ denotes the standard normal cumulative distribution function. □

Theorem 5.34
The price of a Payer Credit Default Swaption in the SZ4 framework (cf. Model 4.4) at time t is approximately given by Theorem 5.33

(a) with $s(t,T_0,T_m)$ determined by Theorem 5.13, Theorem 5.15 or Theorem 5.18 and

$$\begin{aligned} \sigma_Y(t,T_0) = & \\ & \left(\int_t^{T_0} \left(\sigma_r B^{d,zero}(x,\tilde{y}) + \frac{\sigma^r_{\sum_i P^{d,z}}(x)}{\sum_{i=1}^m P^{d,zero}(x,T_i)} \right)^2 \right. \\ & + \left(\sigma_{s^{zero}} C^{d,zero}(x,\tilde{y}) + \frac{\sigma^{s^{zero}}_{\sum_i P^{d,z}}(x)}{\sum_{i=1}^m P^{d,zero}(x,T_i)} \right)^2 \\ & + \left(\sigma_u D^{d,zero}(x,\tilde{y}) + \frac{\sigma^u_{\sum_i P^{d,z}}(x)}{\sum_{i=1}^m P^{d,zero}(x,T_i)} \right)^2 \\ & \left. + \left(\sigma_{w_1} E_1^{d,zero}(x,\tilde{y}) + \frac{\sigma^{w_1}_{\sum_i P^{d,z}}(x)}{\sum_{i=1}^m P^{d,zero}(x,T_i)} \right)^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

where $\tilde{y} \in [T_0, T_m]$*, if the dynamics of the FCDS evolve according to the lognormal approximations of Assumption 5.26. The factors* r, s^{zero}, u, w_1*, which are input for* $\sigma_Y(t, T_0)$*, are frozen at time* t*.*

(b) where in the Theorem's formula $s(t, T_0, T_m)$ *is replaced by* $s(t, T_0, T_m) - (1-Z)\Delta T_i K$*, derived according to Theorem 5.15. Further, the strike* $\widetilde{S}$ *is substituted by* $\widetilde{S} - (1-Z)\Delta T_i K$ *and* σ_Y *is given by*

$$\sigma_Y(t, T_0) = \left(\int_t^{T_0} \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(x)\right)' \left(\vec{\sigma}_{\sum_i P^{d,z}}(x)\right)}{\left(\sum_{i=1}^m P^{d,zero}(x, T_i)\right)^2} dx \right)^{\frac{1}{2}} ,$$

if the dynamics of the FCDS evolve according to the shifted-lognormal approximation of Assumption 5.28. The factors r, s^{zero}, u, w_1*, which are input for* $\sigma_Y(t, T_0)$*, are frozen at time* t*.*

Proof:

(a) We show the application of Theorem 5.33 in case of Theorem 5.13 and the lognormal approximation of the FCDS spread's dynamics of Assumption 5.26. The theorem is analogously applied in case of the FCDS spread being derived by Theorem 5.15 or Theorem 5.18.
As mentioned in the proof of Theorem 5.33, we regard the FCDS spread as a martingale under the measure $\widehat{Q}_{0,m}$. Hence, the lognormal dynamics under the measure $\widetilde{Q}$ are transformed by the change of numéraire to the driftless dynamics

$$\begin{aligned} ds(t, T_0, T_m) = \\ &- \Bigg(\Big(\sigma_r B^{d,zero}(t, \tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t, \tilde{y}), \sigma_u D^{d,zero}(t, \tilde{y}), \sigma_{w_1} E_1^{d,zero}(t, \tilde{y}) \Big) \\ &+ \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^m P^{d,zero}(t, T_i)} \Bigg) s(t, T_0, T_m) d\widehat{W}(t) \ , \end{aligned}$$

with $\tilde{y} \in [T_0, T_m]$.
In the SZ4 framework $\widehat{W}(t)$ is a four-dimensional vector consisting of $(\widehat{W}_r(t), \widehat{W}_{w_1}(t), \widehat{W}_u(t), \widehat{W}_s(t))'$. We simplify the above dynamics by replacing the four-dimensional $\widehat{W}(t)$ with a one-dimensional one, $\widehat{W}_*(t)$, such that we preserve the variance of the process:

$$ds(t, T_0, T_m) = \sigma_*(t) s(t, T_0, T_m) d\widehat{W}_*(t) \ ,$$

where

$$\sigma_*(t) =$$
$$\Bigg(\left(\sigma_r B^{d,zero}(t,\tilde{y}) + \frac{\sigma^r_{\sum_i P^{d,z}}(t)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \right)^2$$
$$+ \left(\sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}) + \frac{\sigma^{s^{zero}}_{\sum_i P^{d,z}}(t)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \right)^2$$
$$+ \left(\sigma_u D^{d,zero}(t,\tilde{y}) + \frac{\sigma^u_{\sum_i P^{d,z}}(t)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \right)^2$$
$$+ \left(\sigma_{w_1} E_1^{d,zero}(t,\tilde{y}) + \frac{\sigma^{w_1}_{\sum_i P^{d,z}}(t)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \right)^2 \Bigg)^{\frac{1}{2}} .$$

Freezing the factors r, s^{zero}, u, w_1 at time t and inserting the above into the formula of Theorem 5.33 will yield the stated result.

(b) If the FCDS spread dynamics evolve according to the shifted-lognormal approximation of the pages 137ff, we end up again with driftless dynamics after changing the numéraire. Under the new measure $\widehat{Q}_{0,m}$ the spread's dynamics are

$$ds(t,T_0,T_m) = -\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)} \Big(s(t,T_0,T_m) - (1-Z)\Delta T_i K \Big) d\widehat{W}(t) ,$$

with K being a constant and ΔT_i being the equidistant interval between spread payments.
As done above, we freeze the factors r, s^{zero}, u, w_1 at time t and replace the four-dimensional vector $\widehat{W}(t)$ with the one-dimensional $\widehat{W}_*(t)$. Hence, the dynamics are

$$ds(t,T_0,T_m) =$$
$$\sqrt{\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)}{\left(\sum_{i=1}^m P^{d,zero}(t,T_i)\right)^2}} \Big(s(t,T_0,T_m) - (1-Z)\Delta T_i K \Big) d\widehat{W}_*(t) .$$

The spread $s(t,T_0,T_m)$ can also be written as

$$s(t,T_0,T_m) = X(t) + (1-Z)\Delta T_i K$$

with

$$dX(t) = \sqrt{\frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)' \left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)}{\left(\sum_{i=1}^m P^{d,zero}(t,T_i)\right)^2}} X(t) d\widehat{W}_*(t) .$$

Therefore, the price of the Credit Default Swaption equals $L(t) \cdot V^{cdso}(t, T_0, T_0, T_m)$ with

$$\begin{aligned} &V^{cdso}(t, T_0, T_0, T_m) \\ &= \sum_{i=1}^{m} P^{d,zero}(t, T_i) \mathbb{E}_{\widehat{Q}_{0,m}} \left[\left(s(T_0, T_0, T_m) - \widetilde{S} \right)^+ \middle| \mathcal{F}_t \right] \\ &= \sum_{i=1}^{m} P^{d,zero}(t, T_i) \mathbb{E}_{\widehat{Q}_{0,m}} \left[\left(X(T_0) - (\widetilde{S} - (1-Z)\Delta T_i K) \right)^+ \middle| \mathcal{F}_t \right] . \end{aligned}$$

Applying Theorem 5.33 for the process $X(t)$ and the strike $\widetilde{S} - (1-Z)\Delta T_i K$, we obtain the pricing formula.

□

As before, we want to analyze how well the proposed approximated dynamics of the FCDS spread work as opposed to the full simulation of the factors r, s, u, w_1. The study is based again on the parameters called GDPr for the rating class A2 of Chapter 4 with $z = 0.9$ and $Z = 0.75$. Since the approximated lognormal dynamics yield equal results in Section 5.4 no matter which recovery assumptions are made, we restrict ourselves to the dynamics with recovery of face value of Proposition 5.24 and Assumption 5.26 in order to compare it with the shifted-lognormal dynamics of Assumption 5.28 and the results obtained via simulation of all factors based on 20,000 scenarios.
Table 5.1 outlines the present value of a CDS Option with strike 0.7%, expiring in 1 year and written on a CDS running for 5 more years if the values for $\sigma_{s^{zero}}$ differ. For small and moderate values of $\sigma_{s^{zero}}$ the formula based on the lognormal dynamics of Theorem 5.34 (a) yields similar results as the formula based on the shifted-lognormal dynamics of Theorem 5.34 (b) since the difference to the simulation results are for both approaches within a few basis points. But for increasing values of $\sigma_{s^{zero}}$ the differences between the lognormal based formula and the simulation (column 5) grow substantially, whereas the differences of the formula assuming shifted-lognormal dynamics to the simulation (columns 6) also increase but much slower. Therefore, for small values of $\sigma_{s^{zero}}$ we can recommend the usage of both pricing formulas of CDS Options and for increasing values of $\sigma_{s^{zero}}$ the formula based on shifted-lognormal dynamics still yields promising results. But for higher values of $\sigma_{s^{zero}}$ we advise the usage of a full simulation.

Table 5.2 contains the present value of a CDS Option for different strikes and different expiries (0.25, 0.75 and 1 year). The underlying CDS of the option is again supposed to run for five years. As in the analysis of the FCDS spreads of Section 5.4, the results based on the shifted-lognormal dynamics are not dependent on the forward starting time of the credit default swap respectively option expiry. However, the difference of the results of Theorem 5.34 (a) to the simulation get bigger for longer option expiries especially if the strike is chosen such that the option is on the border of being out of the money. Again, we can conclude that both approximations work well for options that are in the money. For options not deep in the money and with a longer time period until expiry, we again prefer the usage of the formula based on the shifted-lognormal dynamics.

$\sigma_{s^{zero}}$	Formula (a)	Formula (b)	Simulation	Difference (a)	Difference (b)
0.01	6.26%	6.26%	6.23%	-0.03%	-0.03%
0.05	6.15%	6.15%	6.17%	0.02%	0.02%
0.09	5.90%	5.91%	5.94%	0.03%	0.02%
0.13	5.50%	5.54%	5.56%	0.06%	0.02%
0.15	5.24%	5.31%	5.37%	0.13%	0.06%
0.17	4.93%	5.06%	5.16%	0.23%	0.10%
0.19	4.57%	4.79%	4.89%	0.33%	0.10%
0.20	4.37%	4.65%	4.79%	0.42%	0.14%
0.30	1.81%	3.30%	3.59%	1.77%	0.28%
0.40	0.06%	2.13%	2.44%	2.38%	0.31%

Table 5.1: Present values of CDS Options for different values of $\sigma_{s^{zero}}$ (column 1), expiring in 1 year and written on a CDS running for 5 years. The prices are calculated by formula of Theorem 5.34 (a) (column 2) and (b) (column 3) and via simulation of factors r, s, u, w_1 (column 4) assuming a strike of 0.7%. Column 5 and 6 contain the differences of the results obtained by formula (column 2 and 3) to the simulation results (column 4).

Strike	Formula (a)	Formula (b)	Simulation	Difference (a)	Difference (b)
Expiry 0.25Y					
0.1%	12.45%	12.45%	12.45%	0.00%	0.00%
0.3%	10.12%	10.12%	10.13%	-0.01%	-0.01%
0.5%	7.79%	7.79%	7.81%	-0.02%	-0.02%
0.7%	5.46%	5.46%	5.50%	-0.03%	-0.03%
0.9%	3.13%	3.16%	3.21%	-0.08%	-0.06%
1.1%	0.86%	1.18%	1.27%	-0.41%	-0.09%
1.3%	0.01%	0.21%	0.25%	-0.24%	-0.04%
1.5%	0.00%	0.02%	0.02%	-0.02%	0.00%
1.7%	0.00%	0.00%	0.00%	0.00%	0.00%
Expiry 0.75Y					
0.1%	12.07%	12.07%	12.05%	0.02%	0.02%
0.3%	10.00%	10.00%	9.99%	0.01%	0.01%
0.5%	7.93%	7.93%	7.94%	0.00%	0.00%
0.7%	5.87%	5.87%	5.90%	-0.03%	-0.03%
0.9%	3.80%	3.87%	3.94%	-0.14%	-0.07%
1.1%	1.77%	2.14%	2.24%	-0.47%	-0.10%
1.3%	0.36%	0.94%	1.01%	-0.65%	-0.07%
1.5%	0.02%	0.33%	0.34%	-0.32%	-0.02%
1.7%	0.00%	0.09%	0.08%	-0.08%	0.01%
Expiry 1.00Y					
0.1%	11.71%	11.71%	11.67%	0.04%	0.04%
0.3%	9.77%	9.77%	9.74%	0.03%	0.03%
0.5%	7.83%	7.83%	7.81%	0.02%	0.02%
0.7%	5.88%	5.89%	5.90%	-0.02%	-0.01%
0.9%	3.94%	4.02%	4.09%	-0.15%	-0.06%
1.1%	2.03%	2.39%	2.49%	-0.46%	-0.10%
1.3%	0.58%	1.21%	1.29%	-0.71%	-0.08%
1.5%	0.07%	0.51%	0.54%	-0.47%	-0.02%
1.7%	0.00%	0.19%	0.18%	-0.17%	0.01%

Table 5.2: Present values of CDS Options for different strikes (column 1), expiring in 0.25, 0.75 and 1 year and written on a CDS running for 5 years. The prices are calculated by formula of Theorem 5.34 (a) (column 2) and (b) (column 3) and via simulation of factors r, s, u, w_1 (column 4). Column 5 and 6 contain the differences of the results obtained by formula (column 2 and 3) to the simulation results (column 4).

If the CDS Option expires before the swap starts, i.e. $T < T_0$, we get an additional term in the expected value, namely

$$V^{cdso}(t,T,T_0,T_m) =$$
$$\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} e^{+\int_T^{T_0} r(l)dl} \left(s(T_0,T_0,T_m) - \widetilde{S}\right)^+ \sum_{i=1}^m P^{d,zero}(T_0,T_i) \middle| \mathcal{F}_t \right].$$

Theorem 5.35
The price of a Payer Credit Default Swaption, where the option's expiry does not coincide with the start of the swap, is given by

$$L(t)\cdot V^{cdso}(t,T,T_0,T_m) =$$
$$L(t)\cdot \sum_{i=1}^m P^{d,zero}(t,T_i)$$

$$\cdot \left(Y_0^s \mathcal{N}\left(\frac{\ln\left(\frac{Y^s(t)}{Y^{\widetilde{S}}(t)}\right) + \mu_{Y^s}(t,T_0) - \mu_{Y^{\widetilde{S}}}(t,T_0) + \frac{1}{2}\sigma_+^2(T_0)}{\sqrt{\sigma_+^2(T_0)}} \right) \right.$$
$$\left. - Y_0^{\widetilde{S}} \mathcal{N}\left(\frac{\ln\left(\frac{Y^s(t)}{Y^{\widetilde{S}}(t)}\right) + \mu_{Y^s}(t,T_0) - \mu_{Y^{\widetilde{S}}}(t,T_0) - \frac{1}{2}\sigma_+^2(T_0)}{\sqrt{\sigma_+^2(T_0)}} \right) \right)$$

in the SZ4 framework (cf. Model 4.4), where the terms in the above formula are defined in the proof. $\mathcal{N}$ denotes the standard normal cumulative distribution function.

Proof:
By the change of numéraire we obtain for the price of a Credit Default Swaption

$$L(t)\cdot V^{cdso}(t,T,T_0,T_m) =$$
$$L(t)\cdot \sum_{i=1}^m P^{d,zero}(t,T_i) \mathbb{E}_{\widehat{Q}_{0,m}}\left[e^{+\int_T^{T_0} r(l)dl} \left(s(T_0,T_0,T_m) - \widetilde{S}\right)^+ \middle| \mathcal{F}_t \right].$$

If we assume the time between T and T_0 to be a short period, we can justify the approximation

$$e^{+\int_T^{T_0} r(l)dl} \approx e^{r(T_0)\cdot(T_0-T)} .$$

Using this approximation we get for the price of a Credit Default Swaption

$$\begin{aligned} &L(t)\cdot V^{cdso}(t,T,T_0,T_m) \approx \\ &\quad L(t)\cdot \sum_{i=1}^{m} P^{d,zero}(t,T_i)\mathbb{E}_{\widehat{Q}_{0,m}}\left[e^{r(T_0)\cdot(T_0-T)} \left(s(T_0,T_0,T_m)-\widetilde{S}\right)^+ \middle| \mathcal{F}_t \right] . \end{aligned}$$

It holds for $Y(t) := e^{r(t)\cdot(T_0-T)}$ in the framework of SZ4 under the risk-neutral measure

$$dY(t) = \left(\mu_r(t)(T_0-T)Y(t) + \frac{1}{2}\sigma_r^2(T_0-T)^2Y(t)\right)dt + \sigma_r(T_0-T)Y(t)d\widetilde{W}_r(t).$$

Further, it also holds under $\widetilde{Q}$

$$\begin{aligned} &d\left(\widetilde{Q}(T^d>t|\mathcal{F}_t)\cdot\sum_{i=1}^{m}P^{d,zero}(t,T_i)\right) \\ &= \left(\widetilde{Q}(T^d>t|\mathcal{F}_t)\cdot\mu_{\sum_i P^{d,z}}(t) - s^{zero}(t)\cdot\widetilde{Q}(T^d>t|\mathcal{F}_t)\cdot\sum_{i=1}^{m}P^{d,zero}(t,T_i)\right)dt \\ &\quad + \widetilde{Q}(T^d>t|\mathcal{F}_t)\cdot(\vec{\sigma}_{\sum_i P^{d,z}}(t))'d\widetilde{W}(t) \\ &= \widetilde{Q}(T^d>t|\mathcal{F}_t)\cdot\sum_{i=1}^{m}P^{d,zero}(t,T_i)\left(\left(\frac{\mu_{\sum_i P^{d,z}}(t)}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)} - s^{zero}(t)\right)dt \right. \\ &\quad \left. + \frac{(\vec{\sigma}_{\sum_i P^{d,z}}(t))'}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)}d\widetilde{W}(t)\right) . \end{aligned}$$

Therefore, under the measure $\widehat{Q}_{0,m}$ the dynamics of $Y(t) = e^{r(t)\cdot(T_0-T)}$ evolve in the framework of SZ4 according to

$$\begin{aligned} &dY(t) \\ &\quad = Y(t)\left((T_0-T)\mu_r(t) + \frac{1}{2}(T_0-T)^2\sigma_r^2 + \frac{(T_0-T)\sigma_r\sigma^r_{\sum_i P^{d,z}}(t)}{\sum_{i=1}^{m}P^{d,zero}(t,T_i)}\right)dt \\ &\qquad + (T_0-T)\sigma_r Y(t)d\widehat{W}_r(t) \\ &\quad =: Y(t)\mu_Y(t)dt + Y(t)(\vec{\sigma}_Y)'d\widehat{W}(t) . \end{aligned}$$

Combining the process $Y(t)$ with $s(t,T_0,T_m)$ and $\widetilde{S}$ yields two new processes $Y^s(t)$ and $Y^{\widetilde{S}}(t)$ with the following dynamics under the measure $\widehat{Q}_{0,m}$

$$\begin{aligned}
dY^{\widetilde{S}}(t) &= d(Y(t)\cdot\widetilde{S}) \\
&= \widetilde{S}dY(t) \\
dY^s(t) &= d(Y(t)\cdot s(t,T_0,T_m)) \\
&= s(t,T_0,T_m)dY(t) + Y(t)ds(t,T_0,T_m) + d\left\langle Y(t), s(t,T_0,T_m)\right\rangle \\
&= \Big(\mu_Y(t)dt + (\vec{\sigma_Y})' d\widehat{W}(t)\Big) s(t,T_0,T_m)Y(t) \\
&- \Bigg(\Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y})\Big) \\
&+ \frac{\left(\vec{\sigma}_{\sum_i P^{d,z}}(t)\right)'}{\sum_{i=1}^m P^{d,zero}(t,T_i)}\Bigg) s(t,T_0,T_m)Y(t)d\widehat{W}(t) \\
&- (T_0 - T)\sigma_r \left(\frac{\sigma^r_{\sum_i P^{d,z}}(t)}{\sum_{i=1}^m P^{d,zero}(t,T_i)} + \sigma_r B^{d,zero}(t,\tilde{y})\right) s(t,T_0,T_m)Y(t)dt \\
&=: s(t,T_0,T_m)Y(t)\mu_{Y^s}(t)dt + s(t,T_0,T_m)Y(t)(\vec{\sigma}_{Y^s}(t))' d\widehat{W}(t) \\
&= Y^s(t)\left(\mu_{Y^s}(t)dt + (\vec{\sigma}_{Y^s}(t))' d\widehat{W}(t)\right)
\end{aligned}$$

where $s(t,T_0,T_m)$ is a driftless lognormal process (cf. proof (a) of Theorem 5.34) and $\tilde{y}\in[T_0,T_m]$.
If we freeze all stochastic terms of r, s^{zero}, u, w_1 at time t - as we have already done before for the spread $s(t,T_0,T_m)$ - in order to obtain a deterministic drift, both processes, $Y^{\widetilde{S}}(t)$ and $Y^s(t)$, are geometric Brownian motions with deterministic drift and volatility.
We are left to solve

$$\mathbb{E}_{\widehat{Q}_{0,m}}\left[\left(Y^s(T_0) - Y^{\widetilde{S}}(T_0)\right)^+ \middle| \mathcal{F}_t\right],$$

which looks like some sort of exchange option, for which e.g. Margrabe (1978) and Fischer (1978) proposed closed-form solutions. A more general proof can be found in Li (2008) (see e.g. Proposition 1 on page 6) which we make use of. Again, we replace the four-dimensional process $d\widehat{W}(t)$ with a one-dimensional, such that

$$\begin{aligned}
dY^{\widetilde{S}}(t) &= \widetilde{S}Y(t)(\mu_Y(t)dt + (T_0 - T)\sigma_r d\widehat{W}^1_*(t)) \\
&= Y^{\widetilde{S}}(t)(\mu_Y(t)dt + (T_0 - T)\sigma_r d\widehat{W}^1_*(t)) \\
dY^s(t) &= s(t,T_0,T_m)Y(t)(\mu_{Y^s}(t)dt + \sqrt{(\vec{\sigma}_{Y^s}(t))'\vec{\sigma}_{Y^s}(t)}d\widehat{W}^2_*(t)) \\
&= Y^s(t)(\mu_{Y^s}(t)dt + \sqrt{(\vec{\sigma}_{Y^s}(t))'\vec{\sigma}_{Y^s}(t)}d\widehat{W}^2_*(t))
\end{aligned}$$

Further, we assume ρ to be the correlation between Y^s and $Y^{\widetilde{S}}$. With

$$Y_0^s := \mathbb{E}_{\widehat{Q}_{0,m}}\left[Y^s(T_0)\middle|\,\mathcal{F}_t\right] = Y^s(t)e^{\int_t^{T_0}\mu_{Y^s}(x)dx}\,,$$

$$Y_0^{\widetilde{S}} := \mathbb{E}_{\widehat{Q}_{0,m}}\left[Y^{\widetilde{S}}(T_0)\middle|\,\mathcal{F}_t\right] = Y^{\widetilde{S}}(t)e^{\int_t^{T_0}\mu_{Y}(x)dx}\,,$$

$$\mu_{Y^s}(t,T_0) := \int_t^{T_0}\mu_{Y^s}(x)dx\,,$$

$$\mu_{Y^{\widetilde{S}}}(t,T_0) := \int_t^{T_0}\mu_{Y}(x)dx\,,$$

$$\sigma^2_{Y^s}(t,T_0) := \int_t^{T_0}(\vec{\sigma}_{Y^s}(x))'\vec{\sigma}_{Y^s}(x)dx\,,$$

$$\sigma^2_{Y^{\widetilde{S}}}(t,T_0) := (\sigma_r(T_0-T))^2(T_0-t)\,,$$

$$\sigma^2_+(T_0) := \sigma^2_{Y^s}(t,T_0) + \sigma^2_{Y^{\widetilde{S}}}(t,T_0) + 2\rho\sqrt{\sigma^2_{Y^s}(t,T_0)}\sqrt{\sigma^2_{Y^{\widetilde{S}}}(t,T_0)}$$

we obtain

$$\begin{aligned}
&\mathbb{E}_{\widehat{Q}_{0,m}}\left[\left(Y^s(T_0)-Y^{\widetilde{S}}(T_0)\right)^+\middle|\,\mathcal{F}_t\right]\\
&= Y_0^s\mathcal{N}\left(\frac{\ln\left(\frac{Y^s(t)}{Y^{\widetilde{S}}(t)}\right)+\mu_{Y^s}(t,T_0)-\mu_{Y^{\widetilde{S}}}(t,T_0)+\frac{1}{2}\sigma^2_+(T_0)}{\sqrt{\sigma^2_+(T_0)}}\right)\\
&\quad -Y_0^{\widetilde{S}}\mathcal{N}\left(\frac{\ln\left(\frac{Y^s(t)}{Y^{\widetilde{S}}(t)}\right)+\mu_{Y^s}(t,T_0)-\mu_{Y^{\widetilde{S}}}(t,T_0)-\frac{1}{2}\sigma^2_+(T_0)}{\sqrt{\sigma^2_+(T_0)}}\right)
\end{aligned}$$

□

5.5.1 Big Bang/Small Bang

In 2009 new standards for CDS were introduced. The so-called CDS Big Bang (see Markit (2009a)) proposes global changes in CDS contracts as well as quoting changes for single name CDS in the North American market. Similarly, the CDS Small Bang (see Markit (2009b)) extends the CDS Big Bang to European corporate and sovereign CDS markets. In addition to that, Markit (2009c) provides forthcoming standards for corporate and sovereign CDS markets in Australia, New Zealand, Japan and for Emerging Markets. Those standards are in favour of an upfront payment instead of quoting the

spread which makes the CDS zero at initiation (par spread). The spread payments are fixed at a certain level which is e.g. 100 or 500 basis points per annum for North America, and 25, 100, 500 or 1000 basis points for Europe (plus some additional spreads at 300 and 750 bp).
Reasons for the convention changes are that the market seeks for a standardization in CDS contracts which would lead to a simplified processing of trades and to a netting of offsetting CDS positions. Since investors prefer a small upfront payment and wish to recoupon their existing trades by using the fixed coupons, several coupon options are permitted (cf. the European case of 25, 100, 500 and 1000 bp). But in order to standardize the contracts the number of possible coupons is limited.

Therefore, under the newly introduced CDS standards, the payoff of a CDS Option depends on the dynamics of the upfront payment. I.e. it holds for a European option where the option's maturity coincides with the start of the CDS

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0)\left(\int_{T_0}^{T_m} e^{-\int_{T_0}^{u} r(l)dl}(1-Z(u))dH(u)\right.\right.\\
&\qquad\left.\left.\left.-\widetilde{P}_{uf}-\widetilde{S}\sum_{i=1}^{m} e^{-\int_{T_0}^{T_i} r(l)dl} L(T_i)\right)^+\right|\mathcal{G}_t\right]\\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0)\left(P_{uf}(T_0,T_0,T_m)+\widetilde{S}\sum_{i=1}^{m} e^{-\int_{T_0}^{T_i} r(l)dl}\right.\right.\\
&\qquad\left.\left.\left.-\widetilde{P}_{uf}-\widetilde{S}\sum_{i=1}^{m} e^{-\int_{T_0}^{T_i} r(l)dl} L(T_i)\right)^+\right|\mathcal{G}_t\right]\\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0)\left(P_{uf}(T_0,T_0,T_m)-\widetilde{P}_{uf}\right)^+\middle|\mathcal{G}_t\right]\\
&\overset{Prop.5.2}{=} L(t)\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl}\left(P_{uf}(T_0,T_0,T_m)-\widetilde{P}_{uf}\right)^+\middle|\mathcal{F}_t\right]\\
&= L(t)\cdot V^{ufcdso}(t,T_0,T_0,T_m)\,.
\end{aligned}$$

with $\widetilde{P}_{uf}$ being the fixed contractual upfront payment due at T_0 and $P_{uf}(t,T_0,T_m)$ denoting the upfront payment at time t of a CDS starting at T_0 with the same spread and payment days of the contract.
The value of the upfront payment at time t is the difference between the

protection leg and the payment leg (cf. definition of FCDS in Section 5.4):

$$\begin{aligned}
&P_{uf}(t,T_0,T_m)\\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0) \int_{T_0}^{T_m} e^{-\int_{T_0}^{l} r(x)dx}(1-Z(l))dH(l) \,\middle|\, \mathcal{G}_t \right]\\
&\quad - \mathbb{E}_{\widetilde{Q}}\left[\widetilde{S} \sum_{i=1}^{m} e^{-\int_t^{T_i} r(l)dl} L(T_i) \,\middle|\, \mathcal{G}_t \right]\\
&= L(t)\cdot \left(V_{T^d}^{dp}(t,T_0,T_m,T^*) - \widetilde{S}\sum_{i=1}^{m} P^{d,zero}(t,T_i) \right).
\end{aligned}$$

For $L(t)=1$ we obtain the dynamics of the upfront payment

$$\begin{aligned}
&dP_{uf}(t,T_0,T_m)\\
&= d\left(V_{T^d}^{dp}(t,T_0,T_m,T^*) \right) - d\left(\widetilde{S}\sum_{i=1}^{m} P^{d,zero}(t,T_i) \right)\\
&= (r(t)+s^{zero}(t))\cdot P_{uf}(t,T_0,T_m)dt\\
&\quad + (\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) + \vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)\\
&\quad - \widetilde{S}\cdot \vec{\sigma}_{\sum_i P^{d,z}}(t))' d\widetilde{W}(t)
\end{aligned}$$

with results of the proof of Theorem 5.19 (i)-(iii).
If we pose ourselves in the framework of SZ4 (cf. Model 4.4) and follow the approximations of the pages 126ff we obtain lognormal dynamics for the upfront payment $P_{uf}(t,T_0,T_m)$, i.e. we assume for the components of the vector $\vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)$

$$\begin{aligned}
&\sigma^u_{P^{d*}}(t,T_m,T^*) - \sigma^u_{P^{d*}}(t,T_0,T^*)\\
&\quad \approx -\sigma_u D^{d,zero}(t,\tilde{y})(P^{d*}(t,T_m,T^*) - P^{d*}(t,T_0,T^*))\,,
\end{aligned}$$

(r,s and w_1 analogously) and for components of the vector $\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0)$

$$\begin{aligned}
&\sigma^u_{V^{ddp}}(t,T_m) - \sigma^u_{V^{ddp}}(t,T_0)\\
&\quad \approx -\sigma_u D^{d,zero}(t,\tilde{y})(V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0))\,,
\end{aligned}$$

(s,u and w_1 analogously). Hence, we get with

$$\vec{\sigma}_{V^{ddp}}(t,T_m) - \vec{\sigma}_{V^{ddp}}(t,T_0) + \vec{\sigma}_{P^{d*}}(t,T_m,T^*) - \vec{\sigma}_{P^{d*}}(t,T_0,T^*)$$
$$\approx -\Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y})\Big)$$
$$\cdot \underbrace{(V_{T^d}^{ddp}(t,T_m) - V_{T^d}^{ddp}(t,T_0) + P^{d*}(t,T_m,T^*) - P^{d*}(t,T_0,T^*))}_{V_{T^d}^{dp}(t,T_0,T_m,T^*)}$$

$$\begin{aligned}
&dP_{uf}(t,T_0,T_m)\\
&\approx (r(t)+s^{zero}(t))\cdot P_{uf}(t,T_0,T_m)dt\\
&\quad +\Bigg(-\Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y})\Big)'\\
&\quad \cdot V_{T^d}^{dp}(t,T_0,T_m,T^*) - \widetilde{S}\cdot\vec{\sigma}_{\sum_i P^{d,z}}(t)\Bigg)' d\widetilde{W}(t)\\
&= (r(t)+s^{zero}(t))\cdot P_{uf}(t,T_0,T_m)dt\\
&\quad +\Bigg(-\Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y})\Big)'\\
&\quad \cdot V_{T^d}^{dp}(t,T_0,T_m,T^*) - \widetilde{S}\cdot\Bigg(-\Big(\sigma_r\sum_{i=1}^m P^{d,zero}(t,T_i)B^{d,zero}(t,T_i),\\
&\quad \sigma_{s^{zero}}\sum_{i=1}^m P^{d,zero}(t,T_i)C^{d,zero}(t,T_i), \sigma_u\sum_{i=1}^m P^{d,zero}(t,T_i)D^{d,zero}(t,T_i),\\
&\quad \sigma_{w_1}\sum_{i=1}^m P^{d,zero}(t,T_i)E_1^{d,zero}(t,T_i)\Big)'\Bigg)\Bigg)' d\widetilde{W}(t)\\
&\approx (r(t)+s^{zero}(t))\cdot P_{uf}(t,T_0,T_m)dt\\
&\quad +\Bigg(-\Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y})\Big)\\
&\quad \cdot\Bigg(V_{T^d}^{dp}(t,T_0,T_m,T^*) - \widetilde{S}\sum_{i=1}^m P^{d,zero}(t,T_i)\Bigg)\Bigg) d\widetilde{W}(t)\\
&= P_{uf}(t,T_0,T_m)\cdot\Bigg((r(t)+s^{zero}(t))\,dt -\\
&\quad \Big(\sigma_r B^{d,zero}(t,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t,\tilde{y}), \sigma_u D^{d,zero}(t,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(t,\tilde{y})\Big) d\widetilde{W}(t)\Bigg)\\
&=: P_{uf}(t,T_0,T_m)\cdot(\mu_{uf}(t)dt + (\vec{\sigma}_{uf}(t))' d\widetilde{W}(t))
\end{aligned}$$

with $\tilde{y} \in [T_0, T_m]$. Since r and s^{zero} are Gaussian, $P_{uf}(t, T_0, T_m)$ follows lognormal dynamics.
By means of the above assumptions, the price of a CDS Option is given in closed form.

Proposition 5.36
In the SZ4 framework (cf. Model 4.4) the price at time t of a CDS option is

$$\begin{aligned} & V^{ufcdso}(t, T_0, T_0, T_m) \\ = \;& P_{uf}(t, T_0, T_m) \cdot \mathcal{N}(d_1) - \widetilde{P}_{uf} \cdot e^{-\mu_{\int r+s}(t,T_0) + \frac{1}{2}\sigma^2_{\int r+s}(t,T_0)} \cdot \mathcal{N}(d_2) \end{aligned}$$

with

$$\begin{aligned} d_1 &:= \frac{\ln\left(\frac{P_{uf}(t,T_0,T_m)}{\widetilde{P}_{uf}}\right) + \mu_{\int r+s}(t, T_0) - \frac{1}{2}\sigma^2_{\int r+s}(t, T_0) + \frac{1}{2}\sigma^2_{+}(T_0)}{\sqrt{\sigma^2_{+}(T_0)}}, \\ d_2 &= d_1 - \sqrt{\sigma^2_{+}(T_0)}\,. \end{aligned}$$

$\mu_{\int r+s}(t, T_0)$, $\sigma^2_{+}(T_0)$ and $\sigma^2_{\int r+s}(t, T_0)$ are defined in the proof below. $\mathcal{N}(\cdot)$ denotes the standard normal cumulative distribution function.

Proof:
If $P_{uf}(t, T_0, T_m)$ evolves according to lognormal dynamics we obtain for the price of a CDS option

$$\begin{aligned} & V^{ufcdso}(t, T_0, T_0, T_m) \\ = \;& \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)+s^{zero}(l)dl} \left(P_{uf}(T_0, T_0, T_m) - \widetilde{P}_{uf}\right)^{+} \middle| \mathcal{F}_t \right] \\ = \;& \mathbb{E}_{\widetilde{Q}}\left[\left(Y(T_0) - \widetilde{Y}(T_0)\right)^{+} \middle| \mathcal{F}_t \right] \end{aligned}$$

with

$$\begin{aligned} Y(\tilde{t}) &:= e^{-\int_t^{\tilde{t}} r(l)+s^{zero}(l)dl} \cdot P_{uf}(\tilde{t}, T_0, T_m) \\ \widetilde{Y}(\tilde{t}) &:= e^{-\int_t^{\tilde{t}} r(l)+s^{zero}(l)dl} \cdot \widetilde{P}_{uf}. \end{aligned}$$

Both $Y(\tilde{t})$ and $\widetilde{Y}(\tilde{t})$ follow lognormal dynamics because $P_{uf}(\tilde{t}, T_0, T_m)$ is lognormal (see comment before this Proposition) and

$$\left.\int_t^{\tilde{t}} r(l) + s^{zero}(l)dl\right|_{\mathcal{F}_t} \sim \mathcal{N}(\mu_{\int r+s}(t, \tilde{t}), \sigma^2_{\int r+s}(t, \tilde{t}))$$

is Gaussian in the SZ4 framework (see Appendix E, Lemma E.1 and E.7). Analogous to the proof of Theorem 5.35 (see page 170) we derive the price of a CDS option as

$$
\begin{aligned}
& V^{ufcdso}(t, T_0, T_0, T_m) \\
= \; & Y_0 \cdot \mathcal{N}\left(\frac{\ln\left(\frac{Y(t)}{\widetilde{Y}(t)}\right) + \mu_Y(t, T_0) - \mu_{\widetilde{Y}}(t, T_0) + \frac{1}{2}\sigma_+^2(T_0)}{\sqrt{\sigma_+^2(T_0)}} \right) \\
& - \widetilde{Y}_0 \cdot \mathcal{N}\left(\frac{\ln\left(\frac{Y(t)}{\widetilde{Y}(t)}\right) + \mu_Y(t, T_0) - \mu_{\widetilde{Y}}(t, T_0) - \frac{1}{2}\sigma_+^2(T_0)}{\sqrt{\sigma_+^2(T_0)}} \right)
\end{aligned}
$$

with

$$
Y_0 := \mathbb{E}_{\widetilde{Q}}\left[Y(T_0) \middle| \mathcal{F}_t \right] = P_{uf}(t, T_0, T_m) \,,
$$

$$
\widetilde{Y}_0 := \mathbb{E}_{\widetilde{Q}}\left[\widetilde{Y}(T_0) \middle| \mathcal{F}_t \right] = \widetilde{P}_{uf} \cdot e^{-\mu_{\int r+s}(t,T_0) + \frac{1}{2}\sigma^2_{\int r+s}(t,T_0)} \,,
$$

$$
\mu_Y(t, T_0) := 0 \,,
$$

$$
\mu_{\widetilde{Y}}(t, T_0) := -\mu_{\int r+s}(t, T_0) + \frac{1}{2}\sigma^2_{\int r+s}(t, T_0) \,,
$$

$$
\begin{aligned}
\sigma_+^2(T_0) \; := \; & \int_t^{T_0} (\vec{\sigma}_{uf}(x))' \vec{\sigma}_{uf}(x) dx + \sigma^2_{\int r+s}(t, T_0) \\
& + 2\rho \sqrt{\sigma^2_{\int r+s}(t, T_0) \int_t^{T_0} (\vec{\sigma}_{uf}(x))' \vec{\sigma}_{uf}(x) dx} \,,
\end{aligned}
$$

and ρ being the correlation between Y and $\widetilde{Y}$. □

5.6 Constant Maturity Credit Default Swap

Like in the previous section we want to derive an analytical formula for the price of a credit derivative within our framework. In this section we discuss so-called Constant Maturity Credit Default Swaps (CMCDS) where the premium payments are indexed to the market spread of a CDS. The indexation of the premium payments reduces the mark-to-market exposure to spread volatility (see Pedersen & Sen (2004)).

Pedersen & Sen (2004) present a closed-form expression for valuing CMCDS by assuming an affine model for the hazard rate. At the reset days, they just write the floating premium payments as functions of the hazard rate. By using Taylor's approximation and the risky discount factors, they obtain an approximated closed-form solution. Analogous to the default-free case (cf. LIBOR market model, e.g. Zagst (2002) Chapter 4.7), Brigo (2005) postulates a market model of one- and two-period CDS forward rates and their joint dynamics under a single pricing measure for CMCDS. Further, he presents an approximated valuation formula for CMCDS which only depends on the one-period rates, their volatilities and their correlations. Krekel & Wenzel (2006) who analyze the pricing of credit derivatives within the LIBOR market model with default risk, also derive a closed-form solution for CMCDS. This formula is similar to Brigo's but relaxes the assumption of constant and homogeneous volatilities for the default intensities. They also compare the results of valuing a CMCDS by means of a Monte Carlo simulation as opposed to the closed-form formula of Brigo (2005). They find that the formula is sufficiently accurate but the formula's results deviate heavily from the output of the Monte Carlo simulation for higher volatilities and longer constant maturity periods.

Definition 5.37 ***Constant Maturity Credit Default Swap***
A Constant Maturity Credit Default Swap (CMCDS) is a contract which protects for the time $[T_0, T_m]$ against default of a reference credit. As for normal CDS the protection consists of a payment if default occurs in $[T_0, T_m]$. The premium payments are due on T_i, $i = 1, \ldots, m$, $T_0 < T_i \leq T_m$ except that there has been a default of the reference credit since the last payment day. The premium payment at time T_i is a CDS rate $s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}})$ which is settled at T_{i-1} for a swap starting at this settlement day and maturing in $T_{i-1+\hat{c}}$, $\hat{c} > 0$.

The protection leg of a CMCDS is the same as for an ordinary CDS (cf. Section 5.4)

$$\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0) \int_{T_0}^{T_m} e^{-\int_{T_0}^{u} r(l)dl} (1 - Z(u)) dH(u) \middle| \mathcal{G}_t \right]$$

The premium leg of a CMCDS is built as follows

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_0} r(l)dl} L(T_0) \sum_{i=1}^{m} e^{-\int_{T_0}^{T_i} r(l)dl} L(T_i) s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}}) \middle| \mathcal{G}_t \right] \\
&= \sum_{i=1}^{m} \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^{T_i} r(l)dl} L(T_i) s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}}) \middle| \mathcal{G}_t \right] \\
&= L(t) \sum_{i=1}^{m} \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^{T_i} r(l)+s^{zero}(l)dl} s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}}) \middle| \mathcal{F}_t \right] \\
&= L(t) \sum_{i=1}^{m} \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^{T_{i-1}} r(l)dl} s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}}) \frac{\widetilde{Q}(T^d > T_{i-1} | \mathcal{F}_{T_{i-1}})}{\widetilde{Q}(T^d > t | \mathcal{F}_t)} \right. \\
&\qquad \left. \cdot P^{d,zero}(T_{i-1}, T_i) \middle| \mathcal{F}_t \right] \\
&= L(t) \sum_{i=1}^{m} P^{d,zero}(t, T_i) \mathbb{E}_{\widehat{Q}_{i-1,i}} \left[s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}}) \middle| \mathcal{F}_t \right]
\end{aligned}
$$

where the last equality is derived by a change of numéraire.
The Radon-Nikodym derivative (cf. Theorem 2.19 and Theorem 2.22) defining the new measure $\widehat{Q}_{i-1,i}$ is given by

$$
L_{i-1,i}(t) = \left. \frac{d\widehat{Q}_{i-1,i}}{d\widetilde{Q}} \right|_{\mathcal{F}_t} = \frac{\widetilde{Q}(T^d > t | \mathcal{F}_t) \cdot P^{d,zero}(t, T_i) \cdot P_0(t_0)}{\widetilde{Q}(T^d > t_0 | \mathcal{F}_{t_0}) \cdot P^{d,zero}(t_0, T_i) \cdot P_0(t)} \quad t_0 < t < T_i \ \forall i,
$$

with P_0 being the non-defaultable cash account (see Chapter 4).
Further it holds

$$
\begin{aligned}
&dL_{i-1,i}(t) \\
&= \frac{P_0(t_0)}{\widetilde{Q}(T^d > t_0 | \mathcal{F}_{t_0}) \cdot P^{d,zero}(t_0, T_i)} \left(d\left(\frac{1}{P_0(t)}\right) \cdot \widetilde{Q}(T^d > t | \mathcal{F}_t) \cdot P^{d,zero}(t, T_i) \right. \\
&\qquad \left. + \frac{1}{P_0(t)} \cdot d\left(\widetilde{Q}(T^d > t | \mathcal{F}_t) \cdot P^{d,zero}(t, T_i) \right) \right)
\end{aligned}
$$

$$= \frac{P_0(t_0)}{\widetilde{Q}(T^d > t_0|\mathcal{F}_{t_0}) \cdot P^{d,zero}(t_0,T_i)} \left(d\left(\frac{1}{P_0(t)}\right) \cdot \widetilde{Q}(T^d > t|\mathcal{F}_t) \cdot P^{d,zero}(t,T_i) \right.$$
$$\left. + \frac{1}{P_0(t)} \left(P^{d,zero}(t,T_i) \cdot d\widetilde{Q}(T^d > t|\mathcal{F}_t) + \widetilde{Q}(T^d > t|\mathcal{F}_t) \cdot dP^{d,zero}(t,T_i) \right) \right)$$
$$\stackrel{page\,107ff}{=} L_{i-1,i}(t) \left(-r(t)dt - s^{zero}(t)dt + (r(t) + s^{zero}(t))dt + \frac{\vec{\sigma}_{P^{d,z}}(t,T_i)'}{P^{d,zero}(t,T_i)} d\widetilde{W}(t) \right)$$
$$= L_{i-1,i}(t) \cdot \frac{\vec{\sigma}_{P^{d,z}}(t,T_i)'}{P^{d,zero}(t,T_i)} d\widetilde{W}(t)$$
$$= -L_{i-1,i}(t) \cdot \gamma(t) d\widetilde{W}(t)$$

with $\gamma(t) := -\frac{\vec{\sigma}_{P^{d,z}}(t,T_i)'}{P^{d,zero}(t,T_i)}$.

If we assume - as previously done - lognormal dynamics, we are now able to compute the premium leg of a CMCDS.

Theorem 5.38
If the dynamics of the Forward Credit Default Swap are lognormal under $\widetilde{Q}$ the value of the premium leg of a CMCDS at time t is given as

$$L(t) \sum_{i=1}^{m} P^{d,zero}(t,T_i) \mathbb{E}_{\widehat{Q}_{i-1,i}} \left[s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}}) \right| \mathcal{F}_t]$$
$$= L(t) \sum_{i=1}^{m} P^{d,zero}(t,T_i) \cdot s(t, T_{i-1}, T_{i-1+\hat{c}}) \cdot e^{\int_t^{T_{i-1}} \widehat{\mu}_s(x,T_{i-1},T_{i-1+\hat{c}})dx}$$

with $\widehat{\mu}_s$ defined in the proof.

Proof:
If it holds for the spread of a FCDS under $\widetilde{Q}$

$$ds(t,T_a,T_b) = s(t,T_a,T_b) \cdot \left(\mu_s(t,T_a,T_b)dt + \vec{\sigma}_s(t,T_a,T_b)' d\widetilde{W}(t) \right) ,$$

its dynamics under the new numéraire $\widetilde{Q}(T^d > t|\mathcal{F}_t) \cdot P^{d,zero}(t,T_i)$ are (cf. $dL_{i-1,i}$)

$$\begin{aligned} ds(t,T_a,T_b) &= s(t,T_a,T_b) \cdot \left(\left(\mu_s(t,T_a,T_b) + \vec{\sigma}_s(t,T_a,T_b)' \frac{\vec{\sigma}_{P^{d,z}}(t,T_i)}{P^{d,zero}(t,T_i)} \right) dt \right. \\ &\quad \left. + \vec{\sigma}_s(t,T_a,T_b)' d\widehat{W}(t) \right) \\ &=: s(t,T_a,T_b) \cdot \left(\widehat{\mu}_s(t,T_a,T_b)dt + \vec{\sigma}_s(t,T_a,T_b)' d\widehat{W}(t) \right) . \end{aligned}$$

Hence, the expectation under the measure $\widehat{Q}_{i-1,i}$

$$\mathbb{E}_{\widehat{Q}_{i-1,i}}\left[s(T_{i-1},T_{i-1},T_{i-1+\hat{c}}) \middle| \mathcal{F}_t \right]$$

can be calculated as

$$\begin{aligned} &\mathbb{E}_{\widehat{Q}_{i-1,i}}\left[s(T_{i-1},T_{i-1},T_{i-1+\hat{c}}) \middle| \mathcal{F}_t \right] \\ &\approx \quad s(t,T_{i-1},T_{i-1+\hat{c}}) \cdot e^{\int_t^{T_{i-1}} \widehat{\mu}_s(x,T_{i-1},T_{i-1+\hat{c}})dx} \end{aligned}$$

where we freeze the factors r, s^{zero}, u, w_1, w_2 in $\widehat{\mu}_s$ at time t. □

The exponential term of the above approximation can be considered as a convexity adjustment compensating the fact that the FCDS spread $s(T_{i-1},T_{i-1},T_{i-1+\hat{c}})$ is not a martingale under the applied forward measure. In the SZ4 framework with its lognormal approximations (see Assumption 5.26) this adjustment is calculated as in the next theorem.

Theorem 5.39
The premium leg of a Constant Maturity Credit Default Swap in the SZ4 framework (cf. Model 4.4) is

$$\begin{aligned} &L(t)\sum_{i=1}^{m} P^{d,zero}(t,T_i)\mathbb{E}_{\widehat{Q}_{i-1,i}}\left[s(T_{i-1},T_{i-1},T_{i-1+\hat{c}}) \middle| \mathcal{F}_t \right] \\ &= \quad L(t)\sum_{i=1}^{m} P^{d,zero}(t,T_i)\cdot s(t,T_{i-1},T_{i-1+\hat{c}}) \cdot e^{\int_t^{T_{i-1}} \widehat{\mu}_s(x,T_{i-1},T_{i-1+\hat{c}})dx} \end{aligned}$$

with

$$\begin{aligned} &\widehat{\mu}_s(x,T_{i-1},T_{i-1+\hat{c}}) \\ &= \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(x)\right)' \vec{\sigma}_{\sum_j P^{d,z}}(x)}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(x,T_j)\right]^2} + \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(x)\right)'}{\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(x,T_j)} \\ &\cdot\Bigg(\left(\sigma_r B^{d,zero}(x,\tilde{y}), \sigma_{s^{zero}} C^{d,zero}(x,\tilde{y}), \sigma_u D^{d,zero}(x,\tilde{y}), \sigma_{w_1} E_1^{d,zero}(x,\tilde{y})\right)' \\ &+\left(\sigma_r B^{d,zero}(x,T_i), \sigma_{s^{zero}} C^{d,zero}(x,T_i), \sigma_u D^{d,zero}(x,T_i), \sigma_{w_1} E_1^{d,zero}(x,T_i)\right)'\Bigg) \\ &+\sigma_r^2 B^{d,zero}(x,\tilde{y})B^{d,zero}(x,T_i) + \sigma_{s^{zero}}^2 C^{d,zero}(x,\tilde{y})C^{d,zero}(x,T_i) \\ &+\sigma_u^2 D^{d,zero}(x,\tilde{y})D^{d,zero}(x,T_i) + \sigma_{w_1}^2 E_1^{d,zero}(x,\tilde{y})E_1^{d,zero}(x,T_i) \end{aligned}$$

where $\tilde{y} \in [T_{i-1}, T_{i-1+\hat{c}}]$ *and the factors* r, s^{zero}, u *and* w_1 *in* $P^{d,zero}$ *and in* $\vec{\sigma}_{\sum_j P^{d,z}}$ *are frozen at time* t.

Proof:
Using Assumption 5.26, the approximated lognormal dynamics of the FCDS spread are under $\widetilde{Q}$ given by

$$
\begin{aligned}
& ds(t, T_{i-1}, T_{i-1+\hat{c}}) \\
&\quad = \Bigg(\frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_j P^{d,z}}(t)}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)\right]^2} \\
&\quad + \Big(\sigma_r B^{d,zero}(t, \tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t, \tilde{y}), \sigma_u D^{d,zero}(t, \tilde{y}), \sigma_{w_1} E_1^{d,zero}(t, \tilde{y})\Big) \\
&\quad \cdot \frac{1}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)\right]} \left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)\Bigg) s(t, T_{i-1}, T_{i-1+\hat{c}}) dt \\
&\quad + \Bigg(- \Big(\sigma_r B^{d,zero}(t, \tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t, \tilde{y}), \sigma_u D^{d,zero}(t, \tilde{y}), \sigma_{w_1} E_1^{d,zero}(t, \tilde{y})\Big) \\
&\quad - \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)'}{\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)} \Bigg) s(t, T_{i-1}, T_{i-1+\hat{c}}) d\widetilde{W}(t)
\end{aligned}
$$

with $\tilde{y} \in [T_{i-1}, T_{i-1+\hat{c}}]$,
and the dynamics of the new numéraire $\widetilde{Q}(T^d > t|\mathcal{F}_t) \cdot P^{d,zero}(t, T_i)$ are

$$
\begin{aligned}
& d\widetilde{Q}(T^d > t|\mathcal{F}_t) \cdot P^{d,zero}(t, T_i) \\
& = \widetilde{Q}(T^d > t|\mathcal{F}_t) dP^{d,zero}(t, T_i) - s^{zero}(t) \widetilde{Q}(T^d > t|\mathcal{F}_t) P^{d,zero}(t, T_i) dt \\
& \overset{page\,107ff}{=} \widetilde{Q}(T^d > t|\mathcal{F}_t) P^{d,zero}(t, T_i) \left(r(t)dt + \frac{\vec{\sigma}_{P^{d,z}}(t, T_i)'}{P^{d,zero}(t, T_i)} d\widetilde{W}(t) \right) \\
& = \widetilde{Q}(T^d > t|\mathcal{F}_t) P^{d,zero}(t, T_i) \Big(r(t)dt \\
& - \Big(\sigma_r B^{d,zero}(t, T_i), \sigma_{s^{zero}} C^{d,zero}(t, T_i), \sigma_u D^{d,zero}(t, T_i), \sigma_{w_1} E_1^{d,zero}(t, T_i)\Big) d\widetilde{W}(t) \Big).
\end{aligned}
$$

Hence, we obtain for

$$
\begin{aligned}
&\widehat{\mu}_s(t, T_{i-1}, T_{i-1+\hat{c}}) \\
&\quad = \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_j P^{d,z}}(t)}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)\right]^2} \\
&\quad + \Big(\sigma_r B^{d,zero}(t, \tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t, \tilde{y}), \sigma_u D^{d,zero}(t, \tilde{y}), \sigma_{w_1} E_1^{d,zero}(t, \tilde{y})\Big) \\
&\quad \cdot \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)\right]} \\
&\quad + \Bigg(\Big(\sigma_r B^{d,zero}(t, \tilde{y}), \sigma_{s^{zero}} C^{d,zero}(t, \tilde{y}), \sigma_u D^{d,zero}(t, \tilde{y}), \sigma_{w_1} E_1^{d,zero}(t, \tilde{y})\Big) \\
&\quad + \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)'}{\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)}\Bigg) \\
&\quad \cdot \Big(\sigma_r B^{d,zero}(t, T_i), \sigma_{s^{zero}} C^{d,zero}(t, T_i), \sigma_u D^{d,zero}(t, T_i), \sigma_{w_1} E_1^{d,zero}(t, T_i)\Big)'.
\end{aligned}
$$

In order to calculate the integral of $\widehat{\mu}_s$ the factors r, s^{zero}, u, w_1 need to be frozen at time t. □

Due to the promising results obtained in Section 5.4.2 for the shifted dynamics we also present a closed-form solution based on that assumption.

Theorem 5.40
Assuming the FCDS spread evolves according to the shifted dynamics (see Assumption 5.28), the present value of the premium leg of a Constant Maturity Credit Default Swap in the SZ4 framework (cf. Model 4.4) is

$$
\begin{aligned}
& L(t) \sum_{i=1}^{m} P^{d,zero}(t, T_i) \mathbb{E}_{\widehat{Q}_{i-1,i}} \left[s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}}) \middle| \mathcal{F}_t \right] \\
= \quad & L(t) \sum_{i=1}^{m} P^{d,zero}(t, T_i) \cdot \Bigg((1 - Z) \cdot \Delta T \cdot K \\
& + (s(t, T_{i-1}, T_{i-1+\hat{c}}) - (1 - Z) \cdot \Delta T \cdot K) \cdot e^{\int_t^{T_{i-1}} \widehat{\mu}_{sK}(x, T_{i-1}, T_{i-1+\hat{c}}) dx} \Bigg)
\end{aligned}
$$

with ΔT being the period length of the CDS represented by $s(t, T_{i-1}, T_{i-1+\hat{c}})$ and

$$\begin{aligned}
&\widehat{\mu}_{sK}(t, T_{i-1}, T_{i-1+\hat{c}}) \\
&\quad = \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_j P^{d,z}}(t)}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)\right]^2} + \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)'}{\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)} \\
&\quad \cdot \Big(\sigma_r B^{d,zero}(t, T_i), \sigma_{s^{zero}} C^{d,zero}(t, T_i), \sigma_u D^{d,zero}(t, T_i), \sigma_{w_1} E_1^{d,zero}(t, T_i)\Big)'.
\end{aligned}$$

where the factors r, s^{zero}, u, w_1 are frozen at time t.

Proof:
Assuming the dynamics of the FCDS spread under $\widetilde{Q}$ to be

$$\begin{aligned}
&ds(t, T_{i-1}, T_{i-1+\hat{c}}) \\
&\quad = \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_j P^{d,z}}(t)}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)\right]^2} \Big(s(t, T_{i-1}, T_{i-1+\hat{c}}) - (1 - Z)\Delta T_j K\Big) dt \\
&\quad - \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)'}{\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)} \Big(s(t, T_{i-1}, T_{i-1+\hat{c}}) - (1 - Z)\Delta T_j K\Big) d\widetilde{W}(t)
\end{aligned}$$

with K determined as described on page 140, the change of numéraire results in the following dynamics for s under $\widehat{Q}_{i-1,i}$

$$\begin{aligned}
&ds(t, T_{i-1}, T_{i-1+\hat{c}}) \\
&\quad = \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_j P^{d,z}}(t)}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)\right]^2} \Big(s(t, T_{i-1}, T_{i-1+\hat{c}}) - (1 - Z)\Delta T_j K\Big) dt. \\
&\quad + \left(\frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)'}{\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)} \Big(s(t, T_{i-1}, T_{i-1+\hat{c}}) - (1 - Z)\Delta T_j K\Big)\right) \\
&\quad \cdot \Big(\sigma_r B^{d,zero}(t, T_i), \sigma_{s^{zero}} C^{d,zero}(t, T_i), \sigma_u D^{d,zero}(t, T_i), \sigma_{w_1} E_1^{d,zero}(t, T_i)\Big)' dt \\
&\quad - \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)'}{\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)} \Big(s(t, T_{i-1}, T_{i-1+\hat{c}}) - (1 - Z)\Delta T_j K\Big) d\widehat{W}(t).
\end{aligned}$$

Hence, the expected value of $s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}})$ under $\widehat{Q}_{i-1,i}$ is (see footnote on page 140)

$$\begin{aligned}
&\mathbb{E}_{\widehat{Q}_{i-1,i}}\left[s(T_{i-1}, T_{i-1}, T_{i-1+\hat{c}}) \middle| \mathcal{F}_t \right] \\
&= (1-Z)\cdot \Delta T_j \cdot K \\
&\quad +(s(t, T_{i-1}, T_{i-1+\hat{c}}) - (1-Z)\cdot \Delta T_j \cdot K)\cdot e^{\int_t^{T_{i-1}} \widehat{\mu}_{sK}(x, T_{i-1}, T_{i-1+\hat{c}})dx}
\end{aligned}$$

with

$$\begin{aligned}
&\widehat{\mu}_{sK}(t, T_{i-1}, T_{i-1+\hat{c}}) \\
&= \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)' \vec{\sigma}_{\sum_j P^{d,z}}(t)}{\left[\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)\right]^2} + \frac{\left(\vec{\sigma}_{\sum_j P^{d,z}}(t)\right)'}{\sum_{j=i}^{i-1+\hat{c}} P^{d,zero}(t, T_j)} \\
&\cdot\left(\sigma_r B^{d,zero}(t, T_i), \sigma_{s^{zero}} C^{d,zero}(t, T_i), \sigma_u D^{d,zero}(t, T_i), \sigma_{w_1} E_1^{d,zero}(t, T_i)\right)'.
\end{aligned}$$

In order to calculate the integral of $\widehat{\mu}_{sK}$ the factors r, s, u, w_1 need to be frozen at time t. Further, in order to account for the change of numéraire the factor K needs to be recalculated under the measure $\widehat{Q}_{i-1,i}$ according to page 140. □

In the following we study the behaviour of the CMCDS pricing formulas as opposed to the simulation of the factors r, s, u, w_1. Again we pose ourselves in the framework of SZ4 with the parameters called GDPr for the rating class A2 derived in Chapter 4 with $z = 0.9$ and $Z = 0.75$. We omit the pricing of the protection legs since they are determined the same way for all alternatives. Hence, we compare the present values of the premium legs of a CMCDS with quarterly premium payments determined by the formulas of Theorem 5.39 and Theorem 5.40 assuming a recovery as a fraction of face value with the results of a simulation based on 20,000 scenarios.
Table 5.3 shows the results for CMCDS premium legs where the CMCDS tenor coincides with the swap maturity. As indicated by the results for the FCDS spreads (cf. page 165), the differences of the formulas as opposed to the simulation results increase for longer tenors respectively maturities because longer forward starting periods are needed if the CMCDS maturity lengthens. This effect can be especially seen for the lognormal approximation

(column 5). Our finding is in line with the results of Krekel & Wenzel (2006) who recommend to price by means of simulation for maturities longer than 5 years. In the case of the formula based on the shifted-lognormal dynamics we would also recommend a simulation for tenors longer than 5 years. However, when assuming lognormal dynamics for FCDS spreads we advise to use the CMCDS formula only for short maturites up to 3 years.
In Table 5.4 the results for a CMCDS premium leg with a constant tenor and a swap maturity of 1 year are shown for different values of $\sigma_{s^{zero}}$. Analogously to the results of the CDS option (see Table 5.1) the differences between the closed-form and the simulation-based solutions grow for increasing values of $\sigma_{s^{zero}}$. The results obtained by the formula assuming lognormal dynamics already deviate severely for moderate values of $\sigma_{s^{zero}}$ whereas the closed-form solution based on shifted-lognormal dynamics start to depart heavily for higher values of $\sigma_{s^{zero}}$. Corresponding to the results of Krekel & Wenzel (2006), we recommend to use the formulas up to moderate values of $\sigma_{s^{zero}}$ and a full simulation for CMCDS pricing for high values.

Tenor	Formula	Formula shift	Simulation	Difference	Difference shift
1Y	3.37%	3.39%	3.44%	0.07%	0.05%
2Y	7.59%	7.68%	7.92%	0.33%	0.24%
3Y	11.23%	11.42%	11.53%	0.30%	0.11%
4Y	14.01%	14.33%	14.73%	0.72%	0.40%
5Y	16.03%	16.48%	16.90%	0.87%	0.43%
6Y	17.44%	18.00%	18.90%	1.46%	0.90%
7Y	18.39%	19.06%	20.06%	1.67%	1.00%

Table 5.3: Present values of the premium leg of a CMCDS for different CMCDS tenors (column 1), starting in 0.25 years and running for the same amount of years as the CMCDS tenor with quarterly premium payments. The prices are calculated by formula of Theorem 5.39 (column 2) and Theorem 5.40 (column 3) and via simulation of factors r, s, u, w_1 (column 4). Column 5 and 6 contain the differences of the results obtained by formula (column 2 and 3) to the simulation (column 4).

$\sigma_{s^{zero}}$	Formula	Formula shift	Simulation	Difference	Difference shift
0.05	3.41%	3.42%	3.44%	0.03%	0.02%
0.15	3.27%	3.31%	3.46%	0.19%	0.15%
0.25	2.99%	3.10%	3.39%	0.41%	0.29%
0.35	2.55%	2.78%	3.25%	0.70%	0.47%
0.45	1.96%	2.34%	3.09%	1.13%	0.76%
0.55	1.21%	1.77%	3.22%	2.02%	1.45%
0.65	0.26%	1.07%	3.37%	3.11%	2.30%

Table 5.4: Present values of the premium leg of a CMCDS for different values of $\sigma_{s^{zero}}$ (column 1), starting in 0.25 years and running for 1 year with a CMCDS tenor of 1 year and quarterly premium payments. The prices are calculated by formula of Theorem 5.39 (column 2) and Theorem 5.40 (column 3) and via simulation of factors r, s, u, w_1 (column 4). Column 5 and 6 contain the differences of the results obtained by formula (column 2 and 3) to the simulation (column 4).

Chapter 6

Pricing Inflation-Indexed Derivatives

In the 1980's governments started to issue inflation-linked bonds, e.g. UK GILTS, US TIPS and French OATis. Reasons for those issuances could be (see Dodgson & Kainth (2006)) the wish to better match the liabilities to the future income which tends to rise with inflation, the hope for a cheaper borrowing due to an "inflation-risk premium", or the wish to strengthen "the credibility of economic policy regarding inflation". For investors like pension funds and insurance companies those bonds are attractive since they help to protect their future exposure, e.g. pensions tend to increase with inflation. Since the range of inflation-linked bonds is limited, a market for inflation-indexed derivatives developed in order to fulfill the requirements of the clients. By means of tailor-made inflation derivatives the clients' future liabilities could now be matched more closely. The more complex these derivatives become the more grows the need for pricing models. A popular approach is based on a foreign-currency analogy where the valuation of an inflation derivative becomes equivalent to the pricing of cross-currency interest rate derivatives. In this approach the inflation index acts as an exchange rate between the nominal and the real economy. Articles introducing such a framework are those of Hughston (1998), and Jarrow & Yildirim (2003). For example, Jarrow & Yildirim (2003) model the instantaneous nominal and real forward rates in a Heath-Jarrow-Morton framework as correlated one factor processes and the inflation rate as a lognormal exchange rate. The equivalent formulation of that framework in terms of short rates can be found in Mercurio (2005). An important advantage of that approach is the analytical tractability due to the normal respectively lognormal distributions. However, drawbacks are the possibility of negative rates, and the fact that the real interest rates are unobservable, therefore making it difficult to estimate the parameters for its process and for the correlations with nominal interest rates and inflation rates. A second way to tackle the pricing issue are market models which are built similarly to the interest-rate markets. Mercurio (2005) presents two market models in his article, where the first one assumes that nominal and real forward rates follow a lognormal LIBOR market model and that the forward CPI evolves according to lognormal dynamics. As for the above mentioned models a major disadvantage is still the difficulty in estimating the parameters for the real rates. The second market model which is equivalent to the models of Belgrade, Benhamou & Koehler (2004) and Belgrade & Benhamou (2004) overcomes that issue by modelling the dynamics of forward CPIs as geometric Brownian motions under their associated forward measures. These frameworks do not rely on unobservable real rates but do still have many free parameters. All the above models have in common that a strong smile in the prices of caps and floors cannot be captured. Therefore, more sophisticated approaches try to yield a satisfactory calibration to all market quotes. For example, the model of Mercurio & Moreni (2009) is based on forward CPI dynamics in a multi-factor volatility setting that leads to SABR-like dynamics for forward inflation rates. However, as noted by Bonneton & Jaeckel (2010) it is arguable if such models are currently of any use as long as the inflation option market is still illiquid.

In Section 4.4 we introduced another approach for pricing inflation-related products which is based on short-rate modelling without using a foreign-currency analogy. Papers which also resort to that are e.g. Dodgson & Kainth (2006) using a Hull-White set-up for nominal and inflation rates, Korn & Kruse (2004) modelling the inflation index as a lognormal process with a drift that preserves the Fisher Equation (cf. Fisher (1930)) and Leung & Wu (2011) using a HJM framework for forward nominal and forward inflation rates.

The real and inflation short-rate model (see Model 4.8) decomposes the short rate r into the real short rate r_R and the inflation short rate r_I, i.e.

$$r(t) = r_R(t) + r_I(t)$$

where the real short rate evolves according to a two-factor Hull-White model under the risk-neutral measure $\widetilde{Q}$

$$dr_R(t) = (\theta_R(t) + b_{Rw}w(t) - \hat{a}_R r_R(t))\, dt + \sigma_R d\widetilde{W}_R(t),$$

with positive constants $b_{Rw}, \hat{a}_R, \sigma_R$ and a continuous, deterministic function $\theta_R(t)$. The inflation short rate is correlated with the macroeconomic factor w, which also drives the real short rate r_R, and follows a Vasicek process

$$dr_I(t) = (\theta_I - \hat{a}_I r_I(t))\, dt + \sigma_I \rho_{wI} d\widetilde{W}_w(t) + \sigma_I \sqrt{1 - \rho_{wI}^2} d\widetilde{W}_I(t),$$

with positive constants $\hat{a}_I, \sigma_I$, a non-negative constant θ_I and independent Brownian motions $\widetilde{W}_R$, $\widetilde{W}_I$ and $\widetilde{W}_w$.
Our set-up is inspired by the so-called Fisher Equation (cf. Fisher (1930)) which states that today's nominal interest rate for the period up to time T is the sum of the real interest rate for that period and the expected inflation up to time T. The true inflation for that period as well as the true real rate will be only known after the end of that period. Furthermore, real interest rates are not directly observable in the market. Therefore, we do not explicitly model the real economy as it is done in Hughston (1998) and Jarrow & Yildirim (2003). Instead, we assume that the nominal short rate is driven by two unobservable processes capturing the market's expectations for the real interest rate and the inflation rate. This framework is similar to that of Dodgson & Kainth (2006) who "ignore the existence of a "real" economy" by only modelling the inflation rate and the nominal short rate as correlated Hull-White processes. The advantage of our set-up and the one of Dodgson & Kainth (2006) is the analytical tractability due to the Gaussian processes. But shortcomings are that smiles of market prices as a function of strikes and products depending on the correlation of different forward rates cannot be captured very well.
For notational convenience we omit the subscript for the macroeconomic factor w and refer to the inflation short rate as r_I instead of w_2 (cf. Model 4.8). Further, the risk-free bond of Lemma 4.9 simplifies to

$$P(t,T,r,w,r_I) = e^{A(t,T)-B^R(t,T)r_R-E_1(t,T)w-B^I(t,T)r_I}$$

since it holds $\hat{b}_{rw_2} := \hat{a}_R - \hat{a}_I$ and $E_1(t,T)+B^R(t,T) = B^I(t,T)$. Additionally, within this chapter we use the notation

$$B^{X+/-Y}(t,T) := \frac{1}{\hat{a}_X + / - \hat{a}_Y}\left(1 - e^{-(\hat{a}_x+/-\hat{a}_Y)(T-t)}\right) .$$

6.1 Inflation-Indexed Swaps

Definition 6.1
An Inflation-Indexed Swap consists of two legs where one is paying a fixed rate and the other is paying the inflation rate calculated over a predefined period. The inflation rate is determined as the percentage return of the CPI index over the respective time interval.

In a zero-coupon swap, the inflation-indexed leg pays at maturity T_m the inflation rate of the time interval $[T_0, T_m]$ times the nominal value N

$$N \cdot \left(\frac{CPI(T_m)}{CPI(T_0)} - 1 \right) ,$$

whereas the other leg pays the percentage return of the fixed rate c

$$N \cdot \left((1+c)^{(T_m - T_0)} - 1 \right) .$$

For a Year-On-Year (YOY) Swap, the fixed leg pays the amount

$$N \cdot c \cdot \Delta_i \,, \;\; \Delta_i := T_i - T_{i-1} \,, \; i = 1 \dots m$$

at the set of dates $T_1, \dots, T_m$. The amount payable at times $T_1, \dots, T_m$ of the inflation-indexed leg is derived as

$$N \cdot \Delta_i \cdot \left(\frac{CPI(T_i)}{CPI(T_{i-1})} - 1 \right) .$$

The inflation-indexed derivatives market is dominated by zero-coupon inflation-indexed swaps which are used as hedges for inflation bond exposures (see Bonneton & Jaeckel (2010)).

Theorem 6.2 ***(Zero-Coupon Inflation-Indexed Swap)***
In the real and inflation short-rate model, the price at time t of a zero-coupon inflation-indexed swap is given by

$$V^{zciis}(t, T_0, T_m) = N \cdot \left(\frac{CPI(t)}{CPI(T_0)} P^R(t, T_m) - P(t, T_m) \cdot (1+c)^{(T_m - T_0)} \right) .$$

P^R denotes the price of a real zero-coupon bond which is calculated similarly to the risk-free bond in the four factor framework of Schmid and Zagst (cf. Theorem 4.5).

Proof:
At time $t > T_0$, the value of the zero-coupon inflation-indexed swap is

$$
\begin{aligned}
&V^{zciis}(t, T_0, T_m) \\
&= \mathbb{E}_{\widetilde{Q}}\left[N \cdot e^{-\int_t^{T_m} r(x)dx} \cdot \left(\left(\frac{CPI(T_m)}{CPI(T_0)} - 1 \right) - \left((1+c)^{(T_m - T_0)} - 1 \right) \right) \middle| \mathcal{F}_t \right] \\
&= N \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r(x)dx} \cdot \frac{CPI(T_m)}{CPI(T_0)} \middle| \mathcal{F}_t \right] - N \cdot P(t, T_m) \cdot (1+c)^{(T_m - T_0)} \\
&= N \cdot \left(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r(x)dx} \cdot \frac{CPI(t)}{CPI(T_0)} \cdot e^{\int_t^{T_m} r_I(x)dx} \middle| \mathcal{F}_t \right] - P(t, T_m) \cdot (1+c)^{(T_m - T_0)} \right) \\
&= N \cdot \left(\frac{CPI(t)}{CPI(T_0)} \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_m} r_R(x)dx} \middle| \mathcal{F}_t \right] - P(t, T_m) \cdot (1+c)^{(T_m - T_0)} \right) \\
&= N \cdot \left(\frac{CPI(t)}{CPI(T_0)} P^R(t, T_m) - P(t, T_m) \cdot (1+c)^{(T_m - T_0)} \right)
\end{aligned}
$$

□

The term $\mathbb{E}_{\widetilde{Q}}\left[CPI(T_m) \cdot e^{-\int_t^{T_m} r(x)dx} \middle| \mathcal{F}_t \right] = CPI(t) \cdot P^R(t, T_m)$ denotes a zero-coupon inflation bond which "is generally not available directly but a useful theoretical tool" (see Dodgson & Kainth (2006)). With the help of Theorem 6.2, real discount factors P^R respectively zero-coupon inflation bonds can be stripped from quotes of zero-coupon inflation-indexed swaps which are given in terms of the fixed rate c for maturities T_m: For example, at $t = T_0$ it holds $P^R(T_0, T_m) = P(T_0, T_m) \cdot (1+c)^{(T_m - T_0)}$.
However, the inflation market is still not liquid as Dodgson & Kainth (2006) note in their paper. Extracting prices of zero-coupon inflation bonds from inflation-linked swaps and bonds show discrepancies, especially for the US market.

Year-on-year (YoY) inflation swaps are traded at much lower volumes than zero-coupon inflation swaps, but are important for building more exotic products. Since the inflation payments do not reference the inflation index from the start date (T_0) to the coupon date but have different accrual periods, the valuation of the YoY inflation swap is model-dependent.

Theorem 6.3 ***(Year-on-Year Inflation-Indexed Swap)***
In the real and inflation short-rate model, the price at time t of a year-on-year inflation-indexed swap is

$$\begin{aligned} V^{yoyiis}(t,T_m) &= N \cdot \sum_{i=1}^{m} \Delta_i \cdot \frac{P^R(t,T_i)\cdot P(t,T_{i-1})}{P^R(t,T_{i-1})} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \\ &\quad - N \cdot (1+c) \cdot \sum_{i=1}^{m} \Delta_i \cdot P(t,T_i) \ . \end{aligned}$$

The correlation adjustment factor C^{yoy} is defined as

$$\begin{aligned} C^{yoy}(t,T_{i-1},T_i) = \frac{b_{Rw}\sigma_w\sigma_I\rho_{wI}}{(\hat{a}_R-\hat{a}_w)\hat{a}_I} \cdot \Big(& B^w(T_{i-1},T_i)\cdot(B^w(t,T_{i-1}) - B^{I+w}(t,T_{i-1})) \\ & - B^R(T_{i-1},T_i)\cdot(B^R(t,T_{i-1}) - B^{I+R}(t,T_{i-1}))\Big) \ . \end{aligned}$$

Proof:
At time t, the value of the year-on-year inflation-indexed swap can be written as

$$\begin{aligned} & V^{yoyiis}(t,T_m) \\ &= \mathbb{E}_{\widetilde{Q}}\left[N \cdot \sum_{i=1}^{m} e^{-\int_t^{T_i} r(x)dx} \cdot \Delta_i \cdot \left(\frac{CPI(T_i)}{CPI(T_{i-1})} - 1 \right) \middle| \mathcal{F}_t \right] \\ &\quad - \mathbb{E}_{\widetilde{Q}}\left[N \cdot c \cdot \sum_{i=1}^{m} e^{-\int_t^{T_i} r(x)dx} \cdot \Delta_i \middle| \mathcal{F}_t \right] \\ &= N \cdot \sum_{i=1}^{m} \Delta_i \cdot \left(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx} \cdot e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} \middle| \mathcal{F}_t \right] - P(t,T_i) \right) \\ &\quad - N \cdot c \cdot \sum_{i=1}^{m} \Delta_i \cdot P(t,T_i) \ . \end{aligned}$$

Since the integrals over the short rate r and the inflation short rate r_I are normal, the expectation $\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx} e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} \middle| \mathcal{F}_t \right]$ can be also written as

$$\begin{aligned} & \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_{i-1}} r(x)dx} e^{-\int_{T_{i-1}}^{T_i} r_R(x)dx} \middle| \mathcal{F}_t \right] \\ & \stackrel{I}{=} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_{i-1}} r(x)dx} \middle| \mathcal{F}_t \right] \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_{T_{i-1}}^{T_i} r_R(x)dx} \middle| \mathcal{F}_t \right] \\ & \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_I(x)+r_R(x)dx, \int_{T_{i-1}}^{T_i} r_R(x)dx \middle| \mathcal{F}_t \right]} \end{aligned}$$

$$
\begin{aligned}
&= P(t,T_{i-1}) \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_{T_{i-1}}^{T_i} r_R(x)dx} \middle| \mathcal{F}_t \right] \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_I(x)dx\,,\int_{T_{i-1}}^{T_i} r_R(x)dx \middle| \mathcal{F}_t\right]} \\
&\quad \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_R(x)dx\,,\int_{T_{i-1}}^{T_i} r_R(x)dx \middle| \mathcal{F}_t\right]} \\
&= P(t,T_{i-1}) \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r_R(x)dx + \int_t^{T_{i-1}} r_R(x)dx} \middle| \mathcal{F}_t \right] \\
&\quad \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_R(x)dx\,,\int_t^{T_i} r_R(x)dx - \int_t^{T_{i-1}} r_R(x)dx \middle| \mathcal{F}_t\right]} \\
&= P(t,T_{i-1}) \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r_R(x)dx} \middle| \mathcal{F}_t \right] \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_{i-1}} r_R(x)dx} \middle| \mathcal{F}_t \right] \\
&\quad \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_R(x)dx\,,\int_t^{T_i} r_R(x)dx \middle| \mathcal{F}_t\right]} \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_R(x)dx\,,-\int_t^{T_{i-1}} r_R(x)dx \middle| \mathcal{F}_t\right]} \\
&\quad \cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r_R(x)dx\,,\int_t^{T_{i-1}} r_R(x)dx \middle| \mathcal{F}_t\right]} \\
&= P(t,T_{i-1}) \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot P^R(t,T_i) \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_{i-1}} r_R(x)dx} \middle| \mathcal{F}_t \right] \cdot e^{-Var_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_R(x)dx \middle| \mathcal{F}_t\right]} \\
&\overset{II}{=} \frac{P^R(t,T_i) \cdot P(t,T_{i-1})}{\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_{i-1}} r_R(x)dx} \middle| \mathcal{F}_t \right]} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \\
&= \frac{P^R(t,T_i) \cdot P(t,T_{i-1})}{P^R(t,T_{i-1})} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \ ,
\end{aligned}
$$

with the rules for expectations of normal random variables[I] [II] and

$$
C^{yoy}(t,T_{i-1},T_i) := Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_I(x)dx\,, \int_{T_{i-1}}^{T_i} r_R(x)dx \middle| \mathcal{F}_t \right] .
$$

The calculation of the covariance is straightforward (see Lemma E.1 of Appendix E). □

In general, there is a lag between the date for the index and the payment day. This lag could be a few days, a few weeks or even longer. Therefore, another correlation adjustment is needed to correct this discrepancy. For example, if the inflation-indexed payments of a year-on-year swap are due after the inflation index is fixed, i.e. $T_{i^{pay}} > T_i$, a second adjustment factor is introduced.

[I] It holds for $X_i \sim \mathcal{N}(\mu_{X_i}, \sigma^2_{X_i})$ $i=1,2$: $\mathbb{E}\left[e^{X_i}\right] = e^{\mu_{X_i} + \frac{\sigma^2_{X_i}}{2}}$ $i = 1,2$ and $\mathbb{E}\left[e^{X_1+X_2}\right] = e^{\mu_{X_1}+\mu_{X_2}+\frac{Var[X_1+X_2]}{2}} = E\left[e^{X_1}\right] \cdot E\left[e^{X_2}\right] \cdot e^{Covar[X_1,X_2]}$ since $Var[X_1+X_2] = \sigma^2_{X_1} + \sigma^2_{X_2} + 2 \cdot Covar[X_1,X_2]$ (see e.g. Mueller (1991), page 552).

[II] It holds for $X \sim \mathcal{N}(\mu_X, \sigma^2_X)$: $\mathbb{E}\left[e^X\right] = e^{\mu_X + \frac{\sigma^2_X}{2}}$ and $\mathbb{E}\left[e^{-X}\right] = e^{-\mu_X + \frac{\sigma^2_X}{2}}$. Hence, it also holds $\frac{1}{\mathbb{E}[e^{-X}]} = e^{\mu_X - \frac{\sigma^2_X}{2}} = \mathbb{E}\left[e^X\right] \cdot e^{-\sigma^2_X}$.

Proposition 6.4 ***(Delayed Payment)***
The inflation-indexed leg of a year-on-year swap with delayed payments at time $T_{i^{pay}} > T_i$, $\forall\, i$ is valued according to

$$\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_{i^{pay}}} r(x)dx} \cdot \frac{CPI(T_i)}{CPI(T_{i-1})} \middle| \mathcal{F}_t \right]$$
$$= \frac{P^R(t,T_i)\cdot P(t,T_{i-1})\cdot P(t,T_{i^{pay}})}{P^R(t,T_{i-1})\cdot P(t,T_i)} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot e^{C^{del}(t,T_{i-1},T_i,T_{i^{pay}})} .$$

The correlation adjustment factor $C^{del}(t,T_{i-1},T_i,T_{i^{pay}})$ is defined at the end of the proof.

Proof:
The payment of the ith period of the inflation-indexed leg is based on

$$\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_{i^{pay}}} r(x)dx} \cdot \frac{CPI(T_i)}{CPI(T_{i-1})} \middle| \mathcal{F}_t \right]$$
$$= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_{T_i}^{T_{i^{pay}}} r(x)dx} \cdot e^{-\int_t^{T_i} r(x)dx} \cdot e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} \middle| \mathcal{F}_t \right]$$
$$\overset{footn.\,I}{=} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx} \cdot e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} \middle| \mathcal{F}_t \right] \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_{T_i}^{T_{i^{pay}}} r(x)dx} \middle| \mathcal{F}_t \right]$$
$$\cdot\, e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r(x)dx + \int_{T_{i-1}}^{T_i} r_I(x)dx\, ,\, -\int_{T_i}^{T_{i^{pay}}} r(x)dx \middle| \mathcal{F}_t \right]}$$
$$\overset{Th.6.3}{=} \frac{P^R(t,T_i)\cdot P(t,T_{i-1})}{P^R(t,T_{i-1})} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_{i^{pay}}} r(x)dx + \int_t^{T_i} r(x)dx} \middle| \mathcal{F}_t \right]$$
$$\cdot\, e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r(x)dx\, ,\, -\int_{T_i}^{T_{i^{pay}}} r(x)dx \middle| \mathcal{F}_t \right]} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\, ,\, -\int_{T_i}^{T_{i^{pay}}} r(x)dx \middle| \mathcal{F}_t \right]}$$
$$= \frac{P^R(t,T_i)\cdot P(t,T_{i-1})}{P^R(t,T_{i-1})} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_{i^{pay}}} r(x)dx} \middle| \mathcal{F}_t \right]$$
$$\cdot\, \mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_i} r(x)dx} \middle| \mathcal{F}_t \right] \cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_{i^{pay}}} r(x)dx\, ,\, \int_t^{T_i} r(x)dx \middle| \mathcal{F}_t \right]}$$
$$\cdot\, e^{Covar_{\widetilde{Q}}\left[\int_t^{T_i} r(x)dx\, ,\, \int_{T_i}^{T_{i^{pay}}} r(x)dx \middle| \mathcal{F}_t \right]} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\, ,\, -\int_{T_i}^{T_{i^{pay}}} r(x)dx \middle| \mathcal{F}_t \right]}$$
$$= \frac{P^R(t,T_i)\cdot P(t,T_{i-1})\cdot P(t,T_{i^{pay}})}{P^R(t,T_{i-1})} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot e^{C^{del}(t,T_{i-1},T_i,T_{i^{pay}})}$$
$$\cdot\, \mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_i} r(x)dx} \middle| \mathcal{F}_t \right] \cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r(x)dx\, ,\, \int_t^{T_i} r(x)dx \middle| \mathcal{F}_t \right]}$$
$$\overset{footn.\,II}{=} \frac{P^R(t,T_i)\cdot P(t,T_{i-1})\cdot P(t,T_{i^{pay}})}{P^R(t,T_{i-1})\cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx} \middle| \mathcal{F}_t \right]} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot e^{C^{del}(t,T_{i-1},T_i,T_{i^{pay}})}$$

with

$$
\begin{aligned}
&C^{del}(t, T_{i-1}, T_i, T_{i^{pay}}) := Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,, -\int_{T_i}^{T_{i^{pay}}} r(x)dx\middle|\,\mathcal{F}_t\right]\\
&= \; -Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,, \int_{T_i}^{T_{i^{pay}}} r_I(x)dx\middle|\,\mathcal{F}_t\right]\\
&\quad -Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,, \int_{T_i}^{T_{i^{pay}}} r_R(x)dx\middle|\,\mathcal{F}_t\right].
\end{aligned}
$$

The covariance terms are determined in Lemma E.3.

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,, \int_{T_i}^{T_{i^{pay}}} r_I(x)dx\middle|\,\mathcal{F}_t\right]\\
&= \frac{\sigma_I^2}{\hat{a}_I^2}\cdot\big(\hat{a}_I^2\cdot e^{-\hat{a}_I(T_i-T_{i-1})}\cdot B^{I+I}(t,T_{i-1})\cdot B^I(T_{i-1},T_i)\cdot B^I(T_i,T_{i^{pay}})\\
&\quad +\hat{a}_I\cdot B^I(T_i,T_{i^{pay}})\cdot(B^I(T_{i-1},T_i)-B^{I+I}(T_{i-1},T_i))\big)
\end{aligned}
$$

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,, \int_{T_i}^{T_{i^{pay}}} r_R(x)dx\middle|\,\mathcal{F}_t\right]\\
&= \frac{b_{Rw}\sigma_w\sigma_I\rho_{wI}}{(\hat{a}_R-\hat{a}_w)\hat{a}_I}\cdot\Big(B^w(T_i,T_{i^{pay}})\cdot(B^w(T_{i-1},T_i)-B^{I+w}(T_{i-1},T_i)\\
&\quad +\hat{a}_I\cdot e^{-\hat{a}_w(T_i-T_{i-1})}\cdot B^I(T_{i-1},T_i)\cdot B^{I+w}(t,T_{i-1}))\\
&\quad -B^R(T_i,T_{i^{pay}})\cdot(B^R(T_{i-1},T_i)-B^{I+R}(T_{i-1},T_i)\\
&\quad +\hat{a}_I\cdot e^{-\hat{a}_R(T_i-T_{i-1})}\cdot B^I(T_{i-1},T_i)\cdot B^{I+R}(t,T_{i-1}))\Big)
\end{aligned}
$$

□

For convexity adjustments due to delayed payments see also the article of Brody, Crosby & Li (2008). They determined those adjustments for zero-coupon inflation swaps and YoY inflation swaps in a multi-factor version of the Hughston (1998) model and the Jarrow & Yildirim (2003) model.

6.2 Inflation-Indexed Options

In this section we present how inflation-indexed options are valued. Exemplary, we will show the pricing of an inflation-indexed caplet which is a call option on the CPI-implied inflation rate.

A source of optionality in the market are the deflation floors on the principal of inflation-indexed bonds. For instance, the government-issued US TIPS and French OATis have a floor which guarantees a repayment of the principal at par. Further, according to Dodgson & Kainth (2006) commercially issued European inflation bonds have coupons which are based on the YoY inflation and are floored at zero.
In general, period-on-period swaps (e.g. YoY) are combined with inflation-linked caps and floors for partial indexation (see Kerkhof (2005)).

Theorem 6.5 ***(Inflation-Indexed Caplet)***
The value at time t of an inflation-indexed caplet is

$$\begin{aligned} V^{iicaplet}(t,T_i) &= \frac{P^R(t,T_i)\cdot P(t,T_{i-1})}{P^R(t,T_{i-1})}\cdot e^{C^{yoy}(t,T_{i-1},T_i)}\cdot\mathcal{N}(d_1) \\ &\quad -(1+K)\cdot P(t,T_i)\cdot\mathcal{N}(d_2)\ , \end{aligned}$$

with

$$d_{1,2} = \frac{\ln\left(\frac{P^R(t,T_i)\cdot P(t,T_{i-1})\cdot e^{C^{yoy}(t,T_{i-1},T_i)}}{P^R(t,T_{i-1})\cdot P(t,T_i)\cdot(1+K)}\right)\pm\frac{V_I(t,T_{i-1},T_i)}{2}}{\sqrt{V_I(t,T_{i-1},T_i)}}\ ,$$

and

$$\begin{aligned} V_I(t,T_{i-1},T_i) &= \frac{\sigma_I^2}{\hat{a}_I^2}\cdot\big(T_i-T_{i-1}+\hat{a}_I^2\cdot B^{I+I}(t,T_{i-1})\cdot(B^I(T_{i-1},T_i))^2 \\ &\quad -2\cdot B^I(T_{i-1},T_i)+B^{I+I}(T_{i-1},T_i)\big)\ , \end{aligned}$$

where $\mathcal{N}(\cdot)$ denotes the standard normal cumulative distribution function and K is the cap rate, i.e. the strike of the call option on the inflation rate of the period $[T_{i-1},T_i]$.

Proof:
At time t it holds that

$$\begin{aligned} &V^{iicaplet}(t,T_i) \\ &= \mathbb{E}_{\tilde{Q}}\left[e^{-\int_t^{T_i}r(x)dx}\left(\frac{CPI(T_i)}{CPI(T_{i-1})}-1-K\right)^+\middle|\mathcal{F}_t\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx} \left(e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} - (1+K) \right)^+ \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[\left(e^{-\int_t^{T_i} r(x)dx} e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} - (1+K) \cdot e^{-\int_t^{T_i} r(x)dx} \right)^+ \middle| \mathcal{F}_t \right] \\
&\overset{Th. 6.3}{=} \frac{P^R(t,T_i)\cdot P(t,T_{i-1})}{P^R(t,T_{i-1})} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot \mathcal{N}(d_1) - (1+K)\cdot P(t,T_i)\cdot \mathcal{N}(d_2)\,,
\end{aligned}
$$

with

$$
d_{1,2} = \frac{\ln\left(\frac{P^R(t,T_i)\cdot P(t,T_{i-1})\cdot e^{C^{yoy}(t,T_{i-1},T_i)}}{P^R(t,T_{i-1})\cdot P(t,T_i)\cdot (1+K)} \right) \overset{+}{-} \frac{V_I(t,T_{i-1},T_i)}{2}}{\sqrt{V_I(t,T_{i-1},T_i)}}\;,
$$

and

$$
\begin{aligned}
V_I(t,T_{i-1},T_i) &= Var_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right] \\
&\overset{Lemma\,E.1}{=} \sigma_I^2 \cdot \left(\int_t^{T_{i-1}} \left(B^I(y,T_i) - B^I(y,T_{i-1}) \right)^2 dy + \int_{T_{i-1}}^{T_i} (B^I(y,T_i))^2 dy \right) \\
&= \frac{\sigma_I^2}{\hat{a}_I^2} \cdot \Big(T_i - T_{i-1} + \hat{a}_I^2 \cdot B^{I+I}(t,T_{i-1}) \cdot (B^I(T_{i-1},T_i))^2 - 2\cdot B^I(T_{i-1},T_i) \\
&\qquad + B^{I+I}(T_{i-1},T_i) \Big)\;.
\end{aligned}
$$

The derivation of the Margrabe-like formula is done analogously to page 170 based on two lognormal distributions. For a detailed derivation of $C^{yoy}(t,T_{i-1},T_i)$ please refer to Lemma E.1. □

The prices of inflation-indexed caps and floors can now be obtained easily since these derivatives are sums of caplets respectively floorlets. Inflation-indexed floorlets are put options on the inflation rate and are, therefore, priced analogously to inflation-indexed caplets.

6.3 Inflation Hybrids

The pricing of inflation hybrids is straightforward if the asset can be decomposed into interest-related and inflation-related terms which are then priced independently. But if we consider more exotic and complex products, the valuation can become quite difficult. As in Dodgson & Kainth (2006) we have a closer look on an interest-rate caplet which is linked to the inflation rate. The payoff of this product is

$$\max\left(\Delta_i \cdot \left(L(T_{i-1}, T_i) + K_1 - K_2 \cdot \left(\frac{CPI(T_i)}{CPI(T_{i-1})} - 1\right)\right), 0\right)$$

with $\Delta_i := T_i - T_{i-1}$, the LIBOR rate L and constants K_1, K_2. This hybrid caplet pays the difference - if positive - between the LIBOR rate L for lending at time T_{i-1} for the period $[T_{i-1}, T_i]$ added to a constant K_1 and K_2-times the inflation rate for the same period. In that sense, it is a bet on the real interest rate leveraged by the factors K_1 and K_2.
The value of this caplet at time t is just the discounted expectation of the above payoff

$$\begin{aligned}
&V^{iihybrid}(t, T_{i-1}, T_i) \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx} \cdot \Delta_i \cdot \left(L(T_{i-1}, T_i) + K_1 - K_2 \cdot \left(\frac{CPI(T_i)}{CPI(T_{i-1})} - 1\right)\right)^+ \middle| \mathcal{F}_t\right] \\
&= P(t, T_i) \\
&\cdot \mathbb{E}_{P(\cdot, T_i)}\left[\left(\frac{1}{P(T_{i-1}, T_i)} - 1 + \Delta_i \cdot \left(K_1 + K_2 \cdot \left(1 - e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\right)\right)\right)^+ \middle| \mathcal{F}_t\right],
\end{aligned}$$

where the last row is derived by changing the measure to the T_i-forward measure.
One way to determine the value of that hybrid asset is by simulating the paths for the real short rate, the macroeconomic factor and the inflation short rate. A faster way is to approximate the sum of the two lognormals with a normal distribution and matching mean and variance. According to Dodgson & Kainth (2006) this method is supposed to work quite well for small variances.

Theorem 6.6 ***(Inflation-Linked Hybrid Caplet)***
At time t the value of the above described hybrid caplet can be approximated by

$$\begin{aligned}
&V^{iihybrid}(t, T_{i-1}, T_i) \\
&= P(t, T_i) \cdot \left(\sigma^{hybrid} f_{\mathcal{N}}(\widetilde{X}) + \left(\mu^{hybrid} - 1 + \Delta_i \cdot (K_1 + K_2)\right) \cdot \mathcal{N}(-\widetilde{X})\right)
\end{aligned}$$

where $f_{\mathcal{N}}$ denotes the standard normal density function and

$$\widetilde{X} := \frac{1 - \Delta_i \cdot (K_1 + K_2) - \mu^{hybrid}}{\sigma^{hybrid}} \ .$$

μ^{hybrid} and $(\sigma^{hybrid})^2$ are given at the end of the proof.

Proof:
The Radon-Nikodym derivative (cf. Theorem 2.19 and Theorem 2.22) defining the T_i-forward measure $P(\cdot, T_i)$ is given by

$$L_i(t) = \frac{dP(\cdot, T_i)}{d\widetilde{Q}}\bigg|_{\mathcal{F}_t} = \frac{P(t,T_i) \cdot P_0(t_0)}{P(t_0,T_i) \cdot P_0(t)} \quad t_0 < t < T_i \ \forall i,$$

with P_0 being the non-defaultable cash account (see Chapter 4) and $P(t, T_i)$ being the price at time t of a non-defaultable zero-coupon bond with maturity T_i. Hence, it follows with $P(t,T_i) = e^{A(t,T_i) - B^R(t,T_i) r_R(t) - E_1(t,T_i) w(t) - B^I(t,T_i) r_I(t)}$

$$\begin{aligned} dL_i(t) &= \frac{P_0(t_0)}{P(t_0,T_i)} \left(P(t,T_i) \cdot d\left(\frac{1}{P_0(t)}\right) + \frac{1}{P_0(t)} \cdot dP(t,T_i) \right) \\ &= \frac{P_0(t_0)}{P(t_0,T_i)} \bigg(\frac{P(t,T_i)}{P_0(t)} \big(-\sigma_R B^R(t,T_i), -\sigma_I \sqrt{1-\rho_{wI}^2} B^I(t,T_i), \\ &\qquad -\sigma_w E_1(t,T_i) - \sigma_I \rho_{wI} B^I(t,T_i) \big) d\widetilde{W}(t) \bigg) \\ &= L_i(t) \frac{(\vec{\sigma}_{P(t,T_i)})'}{P(t,T_i)} d\widetilde{W}(t) = -L_i(t) \gamma(t) d\widetilde{W}(t) \ , \end{aligned}$$

where $\widetilde{W}$ is defined as $\widetilde{W} := (\widetilde{W}_R, \widetilde{W}_I, \widetilde{W}_w)'$, $\gamma(t) := \frac{-(\vec{\sigma}_{P(t,T_i)})'}{P(t,T_i)}$ and $\vec{\sigma}_{P(t,T_i)} = (-\sigma_R B^R(t,T_i), -\sigma_I \sqrt{1-\rho_{wI}^2} B^I(t,T_i), -\sigma_w E_1(t,T_i) - \sigma_I \rho_{wI} B^I(t,T_i))' P(t,T_i)$ is derived analogously to page 107. Since it holds under $\widetilde{Q}$

$$\begin{aligned} dw(t) &= (\theta_w - \hat{a}_w w(t))\, dt + \sigma_w d\widetilde{W}_w(t) \ , \\ dr_I(t) &= (\theta_I - \hat{a}_I r_I(t))\, dt + \sigma_I \rho_{wI} d\widetilde{W}_w(t) + \sigma_I \sqrt{1-\rho_{wI}^2} d\widetilde{W}_I(t) \ , \\ dr_R(t) &= (\theta_R(t) + b_{Rw} w(t) - \hat{a}_R r_R(t))\, dt + \sigma_R d\widetilde{W}_R(t) \ , \end{aligned}$$

it follows for the T_i-forward measure

$$\begin{aligned} dw(t) &= \big(\theta_w - \sigma_w \sigma_I \rho_{wI} \cdot B^I(t,T_i) - \sigma_w^2 E_1(t,T_i) - \hat{a}_w w(t)\big)\, dt + \sigma_w d\widehat{W}_w(t) \ , \\ dr_I(t) &= \big(\theta_I - \sigma_I^2 \cdot B^I(t,T_i) - \sigma_w \sigma_I \rho_{wI} E_1(t,T_i) - \hat{a}_I r_I(t)\big)\, dt \\ &\qquad + \sigma_I \rho_{wI} d\widehat{W}_w(t) + \sigma_I \sqrt{1-\rho_{wI}^2} d\widehat{W}_I(t) \ , \\ dr_R(t) &= \big(\theta_R(t) + b_{Rw} w(t) - \sigma_R^2 B^R(t,T_i) - \hat{a}_R r_R(t)\big)\, dt + \sigma_R d\widehat{W}_R(t) \ . \end{aligned}$$

In order to approximate the lognormals $\frac{1}{P(T_{i-1},T_i)}$ and $e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}$, we need to determine their means and variances.
Since it holds for $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ $E[\frac{1}{e^X}] = \frac{1}{E[e^X]} \cdot e^{\sigma_X^2}$ (cf. footnotes on page 193), the mean of the lognormal $\frac{1}{P(T_{i-1},T_i)}$ can be obtained by

$$\mathbb{E}_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)}\middle|\mathcal{F}_t\right] = \frac{1}{\mathbb{E}_{P(\cdot,T_i)}\left[P(T_{i-1},T_i)|\,\mathcal{F}_t\right]}$$
$$\cdot e^{Var_{P(\cdot,T_i)}\left[A(T_{i-1},T_i)-B^R(T_{i-1},T_i)r_R(T_{i-1})-E_1(T_{i-1},T_i)w(T_{i-1})-B^I(T_{i-1},T_i)r_I(T_{i-1})\middle|\mathcal{F}_t\right]}\,.$$

Further, we can determine the expected value of
$P(T_{i-1},T_i) = e^{A(T_{i-1},T_i)-B^R(T_{i-1},T_i)r_R(T_{i-1})-E_1(T_{i-1},T_i)w(T_{i-1})-B^I(T_{i-1},T_i)r_I(T_{i-1})}$ under the T_i-forward measure, by first calculating the expected value under the measure $\widetilde{Q}$ and then adjusting the expectation by the terms which entered the drift of $dw(t), dr_I(t)$ and $dr_R(t)$ due to the change of measure. Analogously as in the proof of Lemma E.1, it now holds under the T_i-forward measure for r_I

$$\begin{aligned} r_I(x)\big|_{\mathcal{F}_t} & \\ = \;& r_I(t)\cdot e^{-\hat{a}_I(x-t)} + \theta_I \cdot B^I(t,x) \\ - \;& \sigma_I^2 \int_t^x e^{-\hat{a}_I(x-y)} B^I(y,T_i)dy - \sigma_w\sigma_I\rho_{wI}\int_t^x e^{-\hat{a}_I(x-y)}E_1(y,T_i)dy \\ + \;& \sigma_I \cdot \rho_{wI}\int_t^x e^{-\hat{a}_I(x-y)}d\widehat{W}_w(y) + \sigma_I\sqrt{1-\rho_{wI}^2}\int_t^x e^{-\hat{a}_I(x-y)}d\widehat{W}_I(y)\;, \end{aligned}$$

and for w

$$\begin{aligned} w(x)\big|_{\mathcal{F}_t} & \\ = \;& w(t)\cdot e^{-\hat{a}_w(x-t)} + \theta_I \cdot B^w(t,x) \\ - \;& \sigma_w\sigma_I\rho_{wI}\int_t^x e^{-\hat{a}_w(x-y)}B^I(y,T_i)dy - \sigma_w^2\int_t^x e^{-\hat{a}_w(x-y)}E_1(y,T_i)dy \\ + \;& \sigma_w\int_t^x e^{-\hat{a}_w(x-y)}d\widehat{W}_w(y)\;, \end{aligned}$$

and for r_R

$$\begin{aligned} r_R(x)\big|_{\mathcal{F}_t} & \\ = \;& r_R(t)\cdot e^{-\hat{a}_R(x-t)} + \int_t^x \theta_R(y)e^{-\hat{a}_R(x-y)}dy \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{b_{Rw}}{\hat{a}_R - \hat{a}_w}\left(w(t) - \frac{\theta_w}{\hat{a}_w}\right) \cdot \left(e^{-\hat{a}_w(x-t)} - e^{-\hat{a}_R(x-t)}\right) + \frac{b_{Rw} \cdot \theta_w}{\hat{a}_w} B^R(t,x) \\
&+ \quad b_{Rw} \int_t^x e^{-\hat{a}_R(x-y)} \left(-\sigma_w \sigma_I \rho_{wI} \int_t^y e^{-\hat{a}_w(y-z)} B^I(z,T_i) dz\right. \\
&- \quad \left.\sigma_w^2 \int_t^y e^{-\hat{a}_w(y-z)} E_1(z,T_i) dz\right) dy - \sigma_R^2 \int_t^x e^{-\hat{a}_R(x-y)} B^R(y,T_i) dy \\
&+ \quad \frac{b_{Rw} \cdot \sigma_w}{\hat{a}_R - \hat{a}_w} \int_t^x (e^{-\hat{a}_w(x-y)} - e^{-\hat{a}_R(x-y)}) d\widehat{W}_w(y) \\
&+ \quad \sigma_R \int_t^x e^{-\hat{a}_R(x-y)} d\widehat{W}_R(y) \ ,
\end{aligned}
$$

Hence, the expectation under $\widetilde{Q}$ needs to be adjusted as follows

$$
\begin{aligned}
&\mathbb{E}_{P(\cdot,T_i)}\left[P(T_{i-1},T_i) \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}_{P(\cdot,T_i)}\left[e^{A(T_{i-1},T_i) - B^R(T_{i-1},T_i) r_R(T_{i-1}) - E_1(T_{i-1},T_i) w(T_{i-1}) - B^I(T_{i-1},T_i) r_I(T_{i-1})} \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{A(T_{i-1},T_i) - B^R(T_{i-1},T_i) r_R(T_{i-1}) - E_1(T_{i-1},T_i) w(T_{i-1}) - B^I(T_{i-1},T_i) r_I(T_{i-1})} \middle| \mathcal{F}_t\right] \\
&\cdot e^{E_1(T_{i-1},T_i)\cdot(W_1(T_{i-1}) + W_2(T_{i-1}))} \cdot e^{B^I(T_{i-1},T_i)\cdot(I_1(T_{i-1}) + I_2(T_{i-1},I))} \\
&\cdot e^{B^R(T_{i-1},T_i)\cdot(R_1(T_{i-1}) + R_2(T_{i-1}) + R_3(T_{i-1}))} ,
\end{aligned}
$$

with

$$
\begin{aligned}
&E_1(T_{i-1},T_i) \cdot (W_1(T_{i-1}) + W_2(T_{i-1})) \\
&\quad := \quad E_1(T_{i-1},T_i)\left(\sigma_w^2 \int_t^{T_{i-1}} e^{-\hat{a}_w(T_{i-1}-y)} E_1(y,T_i) dy\right. \\
&\qquad\qquad \left. + \sigma_w \sigma_I \rho_{wI} \int_t^{T_{i-1}} e^{-\hat{a}_w(T_{i-1}-y)} B^I(y,T_i) dy\right)
\end{aligned}
$$

being the adjustment for the term $-E_1(T_{i-1},T_i) w(T_{i-1})$, and

$$
\begin{aligned}
&B^I(T_{i-1},T_i) \cdot (I_1(T_{i-1}) + I_2(T_{i-1},I)) \\
&\quad := \quad B^I(T_{i-1},T_i)\left(\sigma_w \sigma_I \rho_{wI} \int_t^{T_{i-1}} e^{-\hat{a}_I(T_{i-1}-y)} E_1(y,T_i) dy\right. \\
&\qquad\qquad \left. + \sigma_I^2 \int_t^{T_{i-1}} e^{-\hat{a}_I(T_{i-1}-y)} B^I(y,T_i) dy\right)
\end{aligned}
$$

the adjustment for the term $-B^I(T_{i-1},T_i) r_I(T_{i-1})$, and with

$$B^R(T_{i-1},T_i)\cdot(R_1(T_{i-1})+R_2(T_{i-1})+R_3(T_{i-1}))$$
$$:= B^R(T_{i-1},T_i)\Big(\sigma_R^2\int_t^{T_{i-1}} e^{-\hat{a}_R(T_{i-1}-y)}B^R(y,T_i)dy$$
$$+b_{rw}\int_t^{T_{i-1}} e^{-\hat{a}_R(T_{i-1}-y)}W_1(y)dy+b_{rw}\int_t^{T_{i-1}} e^{-\hat{a}_R(T_{i-1}-y)}W_2(y)dy\Big)$$

being the adjustment for the term $-B^R(T_{i-1},T_i)r_R(T_{i-1})$, where the last two terms (R_1 and R_2) are due to the fact that the factor w also enters the drift of r_R.
Closed-form solutions for these adjustment terms are given in Lemma E.5 in the Appendix.
Further, the mean of $P(T_{i-1},T_i)$ under $\widetilde{Q}$ can be decomposed into

$$\mathbb{E}_{\widetilde{Q}}\left[P(T_{i-1},T_i)\middle|\mathcal{F}_t\right]$$
$$=\mathbb{E}_{\widetilde{Q}}\left[\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_{T_{i-1}}^{T_i}r(x)dx}\middle|\mathcal{F}_{T_{i-1}}\right]\middle|\mathcal{F}_t\right]$$
$$=\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_{T_{i-1}}^{T_i}r(x)dx}\middle|\mathcal{F}_t\right]$$
$$=\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i}r(x)dx}e^{+\int_t^{T_{i-1}}r(x)dx}\middle|\mathcal{F}_t\right]$$
$$\overset{footn.\,I}{=}\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i}r(x)dx}\middle|\mathcal{F}_t\right]\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_{i-1}}r(x)dx}\middle|\mathcal{F}_t\right]\cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i}r(x)dx,\int_t^{T_{i-1}}r(x)dx\middle|\mathcal{F}_t\right]}$$
$$=P(t,T_i)\cdot\mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_{i-1}}r(x)dx}\middle|\mathcal{F}_t\right]\cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_{i-1}}r(x)dx,\int_t^{T_{i-1}}r(x)dx\middle|\mathcal{F}_t\right]}$$
$$\cdot e^{Covar_{\widetilde{Q}}\left[-\int_{T_{i-1}}^{T_i}r(x)dx,\int_t^{T_{i-1}}r(x)dx\middle|\mathcal{F}_t\right]}$$
$$\overset{footn.\,II}{=}\frac{P(t,T_i)}{\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_{i-1}}r(x)dx}\middle|\mathcal{F}_t\right]}\cdot e^{-Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i}r(x)dx,\int_t^{T_{i-1}}r(x)dx\middle|\mathcal{F}_t\right]}$$
$$=\frac{P(t,T_i)}{P(t,T_{i-1})}\cdot e^{-Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i}r(x)dx,\int_t^{T_{i-1}}r(x)dx\middle|\mathcal{F}_t\right]},$$

with the help of the footnotes of page 193. The mean of $e^{\int_{T_{i-1}}^{T_i}r_I(x)dx}$ can be determined analogously by adjusting the expectation under the measure $\widetilde{Q}$

$$\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_{i-1}}^{T_i}r_I(x)dx}\middle|\mathcal{F}_t\right]=\mathbb{E}_{\widetilde{Q}}\left[e^{\int_{T_{i-1}}^{T_i}r_I(x)dx}\middle|\mathcal{F}_t\right]\cdot e^{-I_3(T_{i-1},T_i)-I_4(T_{i-1},T_i,I)}$$

with

$$I_3(T_{i-1},T_i):=\int_{T_{i-1}}^{T_i}\sigma_w\sigma_I\rho_{wI}\int_t^x e^{-\hat{a}_I(x-y)}E_1(y,T_i)dydx$$

and

$$I_4(T_{i-1},T_i,I) := \int_{T_{i-1}}^{T_i} \sigma_I^2 \int_t^x e^{-\hat{a}_I(x-y)} B^I(y,T_i)dydx$$

being the adjustments term for r_I due to the change of measure (cf. above dynamics for r_I under the T_i-forward measure).
Under $\widetilde{Q}$ it holds

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx} e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} e^{\int_t^{T_i} r(x)dx}\middle|\mathcal{F}_t\right]\\
&\overset{footn. I}{=} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx} e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right] \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_i} r(x)dx}\middle|\mathcal{F}_t\right]\\
&\cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r(x)dx + \int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r(x)dx\middle|\mathcal{F}_t\right]}\\
&\overset{Th. 6.3}{=} \frac{P(t,T_{i-1})\cdot P^R(t,T_i)}{P^R(t,T_{i-1})} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_i} r(x)dx}\middle|\mathcal{F}_t\right]\\
&\cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r(x)dx + \int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r(x)dx\middle|\mathcal{F}_t\right]}\\
&= \frac{P(t,T_{i-1})\cdot P^R(t,T_i)}{P^R(t,T_{i-1})} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{\int_t^{T_i} r(x)dx}\middle|\mathcal{F}_t\right]\\
&\cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r(x)dx, \int_t^{T_i} r(x)dx\middle|\mathcal{F}_t\right]} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r(x)dx\middle|\mathcal{F}_t\right]}\\
&\overset{footn. II}{=} \frac{P(t,T_{i-1})\cdot P^R(t,T_i)\cdot e^{C^{yoy}(t,T_{i-1},T_i)}}{P^R(t,T_{i-1})\cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx}\middle|\mathcal{F}_t\right]} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r(x)dx\middle|\mathcal{F}_t\right]}\\
&= \frac{P(t,T_{i-1})\cdot P^R(t,T_i)}{P^R(t,T_{i-1})\cdot P(t,T_i)} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r(x)dx\middle|\mathcal{F}_t\right]}\\
&= \frac{P(t,T_{i-1})\cdot P^R(t,T_i)}{P^R(t,T_{i-1})\cdot P(t,T_i)} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r_I(x)dx\middle|\mathcal{F}_t\right]}\\
&\cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r_R(x)dx\middle|\mathcal{F}_t\right]}\\
&= \frac{P(t,T_{i-1})\cdot P^R(t,T_i)}{P^R(t,T_{i-1})\cdot P(t,T_i)} \cdot e^{C^{yoy}(t,T_{i-1},T_i)} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r_I(x)dx\middle|\mathcal{F}_t\right]}\\
&\cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_{i-1}} r_R(x)dx\middle|\mathcal{F}_t\right]} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_{T_{i-1}}^{T_i} r_R(x)dx\middle|\mathcal{F}_t\right]}\\
&= \frac{P(t,T_{i-1})\cdot P^R(t,T_i)}{P^R(t,T_{i-1})\cdot P(t,T_i)} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_i} r_I(x)dx\middle|\mathcal{F}_t\right]}\\
&\cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx, \int_t^{T_{i-1}} r_R(x)dx\middle|\mathcal{F}_t\right]} \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{T_i} r_I(x)dx, \int_{T_{i-1}}^{T_i} r_R(x)dx\middle|\mathcal{F}_t\right]}
\end{aligned}$$

with

$$C^{yoy}(t,T_{i-1},T_i) = Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_I(x)dx\,,\int_{T_{i-1}}^{T_i} r_R(x)dx\middle|\,\mathcal{F}_t\right]\ .$$

The covariances and the additional terms needed for adjusting the expected value under the risk-neutral measure in order to determine it under the T_i-forward measure are given in Lemma E.5 in the Appendix.

Now, we can approximate the discounted payoff by

$$V^{iihybrid}(t,T_{i-1},T_i) = P(t,T_i)\cdot\mathbb{E}_{P(\cdot,T_i)}\left[\left(X^{hybrid}-1+\Delta_i\cdot(K_1+K_2)\right)^+\middle|\,\mathcal{F}_t\right]\ ,$$

where X^{hybrid} is normal with the same mean and variance as $\frac{1}{P(T_{i-1},T_i)} - K_2\cdot\Delta_i\cdot e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}$.
The variance for the lognormal $\frac{1}{P(T_{i-1},T_i)}$ is given by [III]

$$\begin{aligned}
&Var_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)}\middle|\,\mathcal{F}_t\right]\\
&= Var_{P(\cdot,T_i)}\left[e^{-A(T_{i-1},T_i)+B^R(T_{i-1},T_i)r_R(T_{i-1})+E_1(T_{i-1},T_i)w(T_{i-1})+B^I(T_{i-1},T_i)r_I(T_{i-1})}\middle|\,\mathcal{F}_t\right]\\
&\overset{footn.\,III}{=}\left(\mathbb{E}_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)}\middle|\,\mathcal{F}_t\right]\right)^2\\
&\cdot\left(e^{Var_{P(\cdot,T_i)}\left[B^R(T_{i-1},T_i)r_R(T_{i-1})+E_1(T_{i-1},T_i)w(T_{i-1})+B^I(T_{i-1},T_i)r_I(T_{i-1})\middle|\mathcal{F}_t\right]}-1\right)
\end{aligned}$$

where the variance is obtained with the help of the equations for $r_R(x)\big|_{\mathcal{F}_t}$, $r_I(x)\big|_{\mathcal{F}_t}$ and $w(x)\big|_{\mathcal{F}_t}$ (cf. pages 200ff)

$$\begin{aligned}
&Var_{P(\cdot,T_i)}\left[-A(T_{i-1},T_i)+B^R(T_{i-1},T_i)\cdot r_R(T_{i-1})+E_1(T_{i-1},T_i)\cdot w(T_{i-1})\right.\\
&\qquad\left.+B^I(T_{i-1},T_i)\cdot r_I(T_{i-1})\middle|\,\mathcal{F}_t\right]\\
&=(B^R(T_{i-1},T_i))^2\cdot Var_{\widetilde{Q}}\left[\sigma_R\int_t^{T_{i-1}} e^{-\hat{a}_R(T_{i-1}-l)}d\widetilde{W}_R(l)\middle|\,\mathcal{F}_t\right]\\
&+(B^I(T_{i-1},T_i))^2\cdot Var_{\widetilde{Q}}\left[\sigma_I\sqrt{1-\rho_{wI}^2}\int_t^{T_{i-1}} e^{-\hat{a}_I(T_{i-1}-l)}d\widetilde{W}_I(l)\middle|\,\mathcal{F}_t\right]
\end{aligned}$$

[III] It holds for $X_i\sim\mathcal{N}(\mu_{X_i},\sigma_{X_i}^2)$, $i=1,2$: $Var[e^{X_i}] = (\mathbb{E}[e^{X_i}])^2\cdot(e^{\sigma_{X_i}^2}-1)$ and $Var[e^{X_1+X_2}] = \mathbb{E}[e^{X_1}]\cdot\mathbb{E}[e^{X_2}]\cdot(e^{Covar[X_1+X_2]}-1)$ (see e.g. Mueller (1991), page 552).

$$
\begin{aligned}
&+Var_{\widetilde{Q}}\Bigg[\int_t^{T_{i-1}}\Big(B^R(T_{i-1},T_i)\cdot\frac{\sigma_w b_{Rw}}{\hat{a}_R-\hat{a}_w}\cdot\big(e^{-\hat{a}_w(T_{i-1}-l)}-e^{-\hat{a}_R(T_{i-1}-l)}\big)\\
&+E_1(T_{i-1},T_i)\cdot\sigma_w\cdot e^{-\hat{a}_w(T_{i-1}-l)}+B^I(T_{i-1},T_i)\cdot\sigma_I\rho_{wI}\cdot e^{-\hat{a}_I(T_{i-1}-l)}\Big)d\widetilde{W}_w(l)\Big|\,\mathcal{F}_t\Bigg]\\
&=(B^R(T_{i-1},T_i)\cdot\sigma_R)^2B^{R+R}(t,T_{i-1})+(B^I(T_{i-1},T_i)\cdot\sigma_I)^2(1-\rho_{wI}^2)B^{I+I}(t,T_{i-1})\\
&+\int_t^{T_{i-1}}\Bigg(B^R(T_{i-1},T_i)\cdot\frac{\sigma_w b_{Rw}}{\hat{a}_R-\hat{a}_w}\cdot\big(e^{-\hat{a}_w(T_{i-1}-l)}-e^{-\hat{a}_R(T_{i-1}-l)}\big)\Bigg)^2dl\\
&+\int_t^{T_{i-1}}\big(E_1(T_{i-1},T_i)\cdot\sigma_w\cdot e^{-\hat{a}_w(T_{i-1}-l)}\big)^2dl\\
&+\int_t^{T_{i-1}}\big(B^I(T_{i-1},T_i)\cdot\sigma_I\rho_{wI}\cdot e^{-\hat{a}_I(T_{i-1}-l)}\big)^2dl\\
&+2\int_t^{T_{i-1}}(E_1(T_{i-1},T_i)\cdot\sigma_w\cdot e^{-\hat{a}_w(T_{i-1}-l)})(B^I(T_{i-1},T_i)\cdot\sigma_I\rho_{wI}\cdot e^{-\hat{a}_I(T_{i-1}-l)})dl\\
&+2\int_t^{T_{i-1}}B^R(T_{i-1},T_i)\cdot\frac{\sigma_w b_{Rw}}{\hat{a}_R-\hat{a}_w}e^{-\hat{a}_w(T_{i-1}-l)}(E_1(T_{i-1},T_i)\cdot\sigma_w\cdot e^{-\hat{a}_w(T_{i-1}-l)})dl\\
&-2\int_t^{T_{i-1}}B^R(T_{i-1},T_i)\cdot\frac{\sigma_w b_{Rw}}{\hat{a}_R-\hat{a}_w}e^{-\hat{a}_R(T_{i-1}-l)}(E_1(T_{i-1},T_i)\cdot\sigma_w\cdot e^{-\hat{a}_w(T_{i-1}-l)})dl\\
&+2\int_t^{T_{i-1}}B^R(T_{i-1},T_i)\cdot\frac{\sigma_w b_{Rw}}{\hat{a}_R-\hat{a}_w}e^{-\hat{a}_w(T_{i-1}-l)}(B^I(T_{i-1},T_i)\cdot\sigma_I\rho_{wI}\cdot e^{-\hat{a}_I(T_{i-1}-l)})dl\\
&-2\int_t^{T_{i-1}}B^R(T_{i-1},T_i)\cdot\frac{\sigma_w b_{Rw}}{\hat{a}_R-\hat{a}_w}e^{-\hat{a}_R(T_{i-1}-l)}(B^I(T_{i-1},T_i)\cdot\sigma_I\rho_{wI}\cdot e^{-\hat{a}_I(T_{i-1}-l)})dl\\
&=(B^R(T_{i-1},T_i)\cdot\sigma_R)^2\cdot B^{R+R}(t,T_{i-1})+(B^I(T_{i-1},T_i)\cdot\sigma_I)^2\cdot B^{I+I}(t,T_{i-1})\\
&+\Bigg(\frac{B^R(T_{i-1},T_i)\cdot\sigma_w b_{Rw}}{\hat{a}_R-\hat{a}_w}\Bigg)^2\cdot\big(B^{w+w}(t,T_{i-1})-2\cdot B^{w+R}(t,T_{i-1})+B^{R+R}(t,T_{i-1})\big)\\
&+\ (E_1(T_{i-1},T_i)\cdot\sigma_w)^2\cdot B^{w+w}(t,T_{i-1})\\
&+2\cdot E_1(T_{i-1},T_i)\cdot\sigma_w\cdot B^I(T_{i-1},T_i)\cdot\sigma_I\rho_{wI}\cdot B^{I+w}(t,T_{i-1})\\
&+2\cdot B^R(T_{i-1},T_i)\frac{\sigma_w b_{Rw}}{\hat{a}_R-\hat{a}_w}\cdot\Big(E_1(T_{i-1},T_i)\cdot\sigma_w\cdot\big(B^{w+w}(t,T_{i-1})-B^{w+R}(t,T_{i-1})\big)\\
&+B^I(T_{i-1},T_i)\cdot\sigma_I\rho_{wI}\cdot\big(B^{I+w}(t,T_{i-1})-B^{I+R}(t,T_{i-1})\big)\Big)\ .
\end{aligned}
$$

The variance is the same under both measures since the drift only is affected by the change of measure.

Analogously, the variance of the lognormal $e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}$ is

$$Var_{P(\cdot,T_i)}\left[e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]$$
$$\stackrel{footn.\,III}{=}\left(\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]\right)^2\cdot\left(e^{Var_{P(\cdot,T_i)}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\middle|\mathcal{F}_t\right]}-1\right)$$

where it holds (cf. Theorem 6.5)

$$Var_{P(\cdot,T_i)}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\middle|\mathcal{F}_t\right]=Var_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\middle|\mathcal{F}_t\right]=V_I(t,T_{i-1},T_i)$$

For the variance of the sum of these lognormals, we also need to know the covariance. We obtain this covariance with the help of the equations for $r_R(x)\big|_{\mathcal{F}_t}$, $r_I(x)\big|_{\mathcal{F}_t}$ and $w(x)\big|_{\mathcal{F}_t}$ (cf. pages 200ff) and $\int_{T_{i-1}}^{T_i} r_I(x)dx\Big|_{\mathcal{F}_t}$ (cf. Lemma E.1 in the Appendix).

$$\begin{aligned}
&C^{P,\int r_I}\\
&= Covar_{P(\cdot,T_i)}\big[B^R(T_{i-1},T_i)\cdot r_R(T_{i-1})+E_1(T_{i-1},T_i)\cdot w(T_{i-1})\\
&\quad +B^I(T_{i-1},T_i)\cdot r_I(T_{i-1})\,,\int_{T_{i-1}}^{T_i} r_I(x)dx\bigg|\,\mathcal{F}_t\bigg]\\
&= Covar_{\widetilde{Q}}\big[B^R(T_{i-1},T_i)\cdot r_R(T_{i-1})+E_1(T_{i-1},T_i)\cdot w(T_{i-1})\\
&\quad +B^I(T_{i-1},T_i)\cdot r_I(T_{i-1})\,,\int_{T_{i-1}}^{T_i} r_I(x)dx\bigg|\,\mathcal{F}_t\bigg]\\
&= Covar_{\widetilde{Q}}\bigg[B^R(T_{i-1},T_i)\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\int_t^{T_{i-1}}(e^{-\hat{a}_w(T_{i-1}-y)}-e^{-\hat{a}_R(T_{i-1}-y)})d\widetilde{W}_w(y)\\
&\quad +E_1(T_{i-1},T_i)\sigma_w\int_t^{T_{i-1}} e^{-\hat{a}_w(T_{i-1}-y)}d\widetilde{W}_w(y)\\
&\quad +B^I(T_{i-1},T_i)\sigma_I\rho_{wI}\int_t^{T_{i-1}} e^{-\hat{a}_I(T_{i-1}-y)}d\widetilde{W}_w(y)\,,\\
&\quad \sigma_I\rho_{wI}\left(\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_w(y)+\int_{T_{i-1}}^{T_i}B^I(y,T_i)d\widetilde{W}_w(y)\right)\bigg|\,\mathcal{F}_t\bigg]\\
&+Covar_{\widetilde{Q}}\bigg[B^I(T_{i-1},T_i)\sigma_I\sqrt{1-\rho_{wI}^2}\int_t^{T_{i-1}} e^{-\hat{a}_I(T_{i-1}-y)}d\widetilde{W}_I(y)\,,\\
&\quad \sigma_I\sqrt{1-\rho_{wI}^2}\left(\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_I(y)+\int_{T_{i-1}}^{T_i}B^I(y,T_i)d\widetilde{W}_I(y)\right)\bigg|\,\mathcal{F}_t\bigg]
\end{aligned}$$

$$= Covar_{\widetilde{Q}}\Bigg[B^R(T_{i-1},T_i)\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\int_t^{T_{i-1}}(e^{-\hat{a}_w(T_{i-1}-y)}-e^{-\hat{a}_R(T_{i-1}-y)})d\widetilde{W}_w(y)$$

$$+E_1(T_{i-1},T_i)\sigma_w\int_t^{T_{i-1}}e^{-\hat{a}_w(T_{i-1}-y)}d\widetilde{W}_w(y)$$

$$+B^I(T_{i-1},T_i)\sigma_I\rho_{wI}\int_t^{T_{i-1}}e^{-\hat{a}_I(T_{i-1}-y)}d\widetilde{W}_w(y)\,,$$

$$\sigma_I\rho_{wI}\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_w(y)\bigg|\mathcal{F}_t\Bigg]$$

$$+Covar_{\widetilde{Q}}\Bigg[B^I(T_{i-1},T_i)\sigma_I\sqrt{1-\rho_{wI}^2}\int_t^{T_{i-1}}e^{-\hat{a}_I(T_{i-1}-y)}d\widetilde{W}_I(y)\,,$$

$$\sigma_I\sqrt{1-\rho_{wI}^2}\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_I(y)\bigg|\mathcal{F}_t\Bigg]$$

$$=B^R(T_{i-1},T_i)\frac{b_{Rw}\sigma_w\sigma_I\rho_{wI}}{\hat{a}_R-\hat{a}_w}\int_t^{T_{i-1}}B^I(T_{i-1},T_i)e^{-\hat{a}_I(T_{i-1}-y)}(e^{-\hat{a}_w(T_{i-1}-y)}$$

$$-e^{-\hat{a}_R(T_{i-1}-y)})dy+E_1(T_{i-1},T_i)\sigma_w\sigma_I\rho_{wI}\int_t^{T_{i-1}}e^{-\hat{a}_w(T_{i-1}-y)}B^I(T_{i-1},T_i)e^{-\hat{a}_I(T_{i-1}-y)}dy$$

$$+B^I(T_{i-1},T_i)\sigma_I^2\rho_{wI}^2\int_t^{T_{i-1}}e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i)e^{-\hat{a}_I(T_{i-1}-y)}dy$$

$$+B^I(T_{i-1},T_i)\sigma_I^2(1-\rho_{wI}^2)\int_t^{T_{i-1}}e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i)e^{-\hat{a}_I(T_{i-1}-y)}dy$$

$$=B^I(T_{i-1},T_i)\cdot\Bigg[\frac{\sigma_w b_{Rw}\sigma_I\rho_{wI}}{\hat{a}_R-\hat{a}_w}\cdot B^R(T_{i-1},T_i)\cdot(B^{I+w}(t,T_{i-1})-B^{I+R}(t,T_{i-1}))$$

$$+E_1(T_{i-1},T_i)\cdot\sigma_w\sigma_I\rho_{wI}\cdot B^{I+w}(t,T_{i-1})+B^I(T_{i-1},T_i)\cdot\sigma_I^2\cdot B^{I+I}(t,T_{i-1})\Bigg],$$

such that

$$Covar_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)}\,,e^{\int_{T_{i-1}}^{T_i}r_I(x)dx}\bigg|\mathcal{F}_t\right]$$

$$\overset{footn.\,III}{=}\mathbb{E}_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)}\bigg|\mathcal{F}_t\right]\cdot\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_{i-1}}^{T_i}r_I(x)dx}\bigg|\mathcal{F}_t\right]\cdot\left(e^{C^{P,\int r_I}}-1\right)$$

Finally, we obtain the hybrid payoff by normal approximation

$$V^{iihybrid}(t,T_{i-1},T_i)$$

$$=P(t,T_i)\cdot\mathbb{E}_{P(\cdot,T_i)}\left[\left(X^{hybrid}-1+\Delta_i\cdot(K_1+K_2)\right)^+\bigg|\mathcal{F}_t\right]\;,$$

$$= P(t,T_i)\int_{\{y\geq\widetilde{X}\}}\Big(\sigma^{hybrid}\cdot y + \mu^{hybrid} - 1 + \Delta_i\cdot(K_1+K_2)\Big)\cdot f_{\mathcal{N}}(y)dy$$
$$= P(t,T_i)\cdot\Big(\sigma^{hybrid} f_{\mathcal{N}}(\widetilde{X}) + \big(\mu^{hybrid} - 1 + \Delta_i\cdot(K_1+K_2)\big)\cdot\mathcal{N}(-\widetilde{X})\Big)$$

where $f_{\mathcal{N}}$ denotes the standard normal density function and $\widetilde{X} := \frac{1-\Delta_i\cdot(K_1+K_2)-\mu^{hybrid}}{\sigma^{hybrid}}$. Further, μ^{hybrid} and $(\sigma^{hybrid})^2$ denote the mean and variance of $\frac{1}{P(T_{i-1},T_i)} - K_2\cdot\Delta_i\cdot e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}$ under the $P(\cdot,T_i)$-forward measure:

$$\mu^{hybrid}$$
$$= \mathbb{E}_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)} - K_2\cdot\Delta_i\cdot e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]$$
$$= \mathbb{E}_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)}\middle|\mathcal{F}_t\right] - K_2\cdot\Delta_i\cdot\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]$$

and

$$(\sigma^{hybrid})^2$$
$$= Var_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)} - K_2\cdot\Delta_i\cdot e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]$$
$$= Var_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)}\middle|\mathcal{F}_t\right] + (K_2\cdot\Delta_i)^2\cdot Var_{P(\cdot,T_i)}\left[e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]$$
$$-2\cdot K_2\cdot\Delta_i\cdot Covar_{P(\cdot,T_i)}\left[\frac{1}{P(T_{i-1},T_i)}, e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]$$

□

Figure 6.1 compares the dependence on σ_I for the normal approximation of Theorem 6.6 with the results obtained via simulation with 10,000 scenarios. We used the parameters of the INFcorr framework given in Chapter 4 (see Section 4.7 with Tables 4.3 and 4.6) with $K_1 = 0.005$ and $K_2 \in \{0.5, 1.0, 1.5\}$. The first row of Figure 6.1 contains the results for $K_2 = 0.5$, the second row for $K_2 = 1.0$ and the third for $K_2 = 1.5$. Further, the left-hand side of the figure shows the result for $\sigma_R = 0.005$ whereas the right-hand side for $\sigma_R = 0.1$. The results in Figure 6.1 indicate that the normal approximation overestimates the true value. Especially for increasing σ_I and decreasing K_2, the approximation deviates from the simulated results. However, the approximated values approach the results obtained by simulation for growing K_2

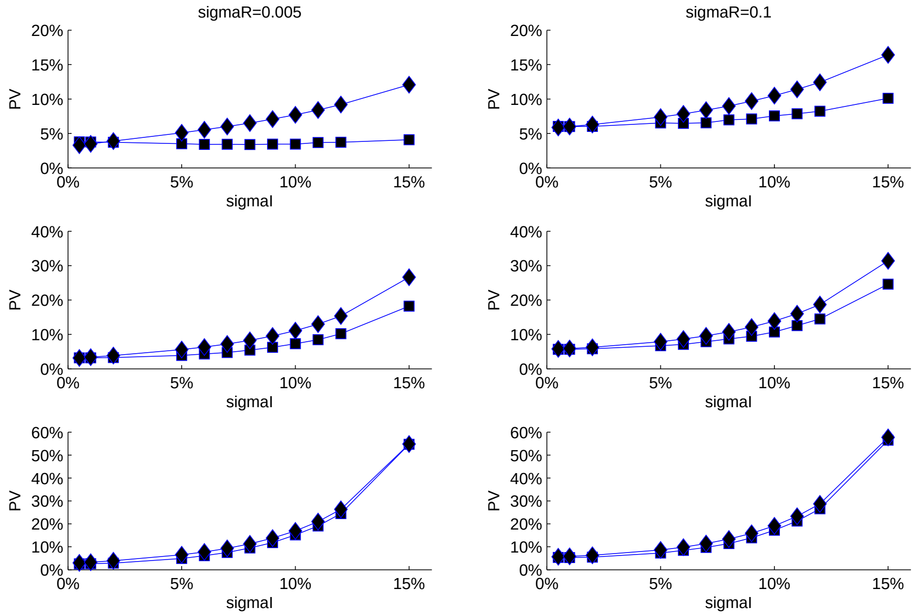

Figure 6.1: The present value of a hybrid inflation caplet (cf. Theorem 6.6) with a tenor of 1 year and maturity in 10 years for varying values of σ_I. The results are obtained via normal approximation (given by diamonds) and via simulation for 10,000 paths (given by squares). The parameters are taken from the model INFcorr of Section 4.7 with $K_1 = 0.005$ and $K_2 \in \{0.5, 1.0, 1.5\}$ where the rows of the figure represent K_2 in increasing order, i.e. the first row contains $K_2 = 0.5$. The left part of the figure is obtained by setting $\sigma_R = 0.005$, the right part by $\sigma_R = 0.1$.

even for higher values of σ_I. The reason for this is that for small values of K_2 the inflation as well as the interest rate impact the value of the hybrid caplet. For growing values of K_2, the caplet is mainly driven by the inflation and its volatility. We can confirm - as noted by Dodgson & Kainth (2006) - that the normal approximation yields extremely accurate results for small values of σ_I and σ_R. Dependent on the factors K_1 and K_2, the approximation even holds for higher volatilities. Realistic parameters in a Hull-White inflation set-up are according to Dodgson & Kainth (2006) $\sigma_I, \sigma_R \leq 0.02$. Hence, we can conclude that the normal approximation is a fast way to approximate hybrid payoffs for realistic parameters within our framework.

6.3.1 Inflation-Linked Equity Options

Investors who want the real value of their equity portfolio to remain at least equal to its original value, can use inflation-linked equity options to do so (see Kerkhof (2005)). For example, buying the stock and a spread option which pays the difference (if positive) between the inflation and the stock return, guarantees the real value of the equity. Furthermore, according to Dodgson & Kainth (2006) bonds paying the maximum of an equity index and a price index are "an increasingly popular retail product". Based on that observation Hammarlid (2010) introduces a European call option with an inflation-linked strike and prices this product under certain assumptions, namely a log-normal distribution of the ratio of the stock return and the inflation-linked bond price. Both papers refer to a payoff of the form

$$\max\left(\frac{CPI(T)}{CPI(T_0)} - \frac{S(T)}{S(T_0)}, 0\right)$$

with S being an equity index. While Dodgson & Kainth (2006) do not provide a pricing for such payoffs, Hammarlid (2010) rewrites the payoff such that the non-traded inflation index is substituted by a zero-coupon inflation-linked bond. Hence, this payoff can be regarded as an exchange option and can be priced by means of a change of numéraire (cf. Margrabe (1978)).

Following the approach of Hammarlid (2010) we introduce an additional process describing continuous stock returns into our set-up. This process was previously introduced in Meyer (2005) and Hagedorn et al. (2007) who analyzed interest rate models and equtiy models which incorporate macroeconomic information. We assume that under the risk-neutral measure $\widetilde{Q}$ the process of the continuous stock return evolves according to the following equation

$$dR_E(t) = (\alpha_E + b_{ER}\, r_R(t) - b_{EI}\, r_I(t) + b_{Ew}\, w(t))\, dt + \sigma_E d\widetilde{W}_E(t) \ ,$$

with an uncorrelated Brownian motion $\widetilde{W}_E$, $\alpha_E \in \mathbb{R}$ and positive constants $b_{ER}, b_{EI}, b_{Ew}, \sigma_E$. Further, the equity index $S(T)$ as well as the inflation index $CPI(T)$ are lognormal with $S(T) = S(0)e^{\int_0^T R_E(x)dx}$ and $CPI(T) = CPI(0)e^{\int_0^T r_I(x)dx}$. Hence, we pose ourselves in a similar set-up as in Hammarlid (2010), but start with modelling different processes, e.g. inflation short rate vs. proportional change of index.

Theorem 6.7 ***(Inflation-Linked Equity Option)***
The value at time t ($t \leq T_0 \leq T_i$) of a European option referring to an equity index and having an inflation-linked strike with payoff

$$\max\left(\frac{CPI(T)}{CPI(T_0)} - \frac{S(T)}{S(T_0)}, 0\right)$$

is

$$V^{iieopt}(t,T_0,T_i) = P(t,T_i)\cdot\left(\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_0}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]\cdot\mathcal{N}(d_1) - \mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_0}^{T_i} R_E(x)dx}\middle|\mathcal{F}_t\right]\cdot\mathcal{N}(d_2)\right)$$

with

$$d_1 = \frac{1}{\sqrt{V_{EI}(t,T_0,T_i)}}\cdot\left(\ln\left(\frac{\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_0}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]}{\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_0}^{T_i} R_E(x)dx}\middle|\mathcal{F}_t\right]}\right)+\frac{1}{2}V_{EI}(t,T_0,T_i)\right),$$

$$d_2 = d_1 - \sqrt{V_{EI}(t,T_0,T_i)},$$

and

$$V_{EI}(t,T_0,T_i) = Var_{P(\cdot,T_i)}\left[\int_{T_0}^{T_i} r_I(x)dx\middle|\mathcal{F}_t\right] + Var_{P(\cdot,T_i)}\left[\int_{T_0}^{T_i} R_E(x)dx\middle|\mathcal{F}_t\right] +2\,Covar_{P(\cdot,T_i)}\left[\int_{T_0}^{T_i} r_I(x)dx\,,\int_{T_0}^{T_i} R_E(x)dx\middle|\mathcal{F}_t\right].$$

Proof:
By a change of measure the value of the European option at time t, $(t \le T_0 \le T_i)$, is

$$\begin{aligned}
&V^{iieopt}(t,T_0,T_i)\\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)dx}\left(e^{\int_{T_0}^{T_i} r_I(x)dx} - e^{\int_{T_0}^{T_i} R_E(x)dx}\right)^+\middle|\mathcal{F}_t\right]\\
&= P(t,T_i)\cdot\mathbb{E}_{P(\cdot,T_i)}\left[\left(e^{\int_{T_0}^{T_i} r_I(x)dx} - e^{\int_{T_0}^{T_i} R_E(x)dx}\right)^+\middle|\mathcal{F}_t\right]\\
&= P(t,T_i)\cdot\left(\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_0}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]\cdot\mathcal{N}(d_1) - \mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_0}^{T_i} R_E(x)dx}\middle|\mathcal{F}_t\right]\cdot\mathcal{N}(d_2)\right)
\end{aligned}$$

with

$$d_1 = \frac{1}{\sqrt{V_{EI}(t,T_0,T_i)}}\cdot\left(\ln\left(\frac{\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_0}^{T_i} r_I(x)dx}\middle|\mathcal{F}_t\right]}{\mathbb{E}_{P(\cdot,T_i)}\left[e^{\int_{T_0}^{T_i} R_E(x)dx}\middle|\mathcal{F}_t\right]}\right)+\frac{1}{2}V_{EI}(t,T_0,T_i)\right),$$

$$d_2 = d_1 - \sqrt{V_{EI}(t,T_0,T_i)},$$

and

$$
\begin{aligned}
V_{EI}(t,T_0,T_i) &= Var_{P(\cdot,T_i)}\left[\int_{T_0}^{T_i} r_I(x)dx \middle| \mathcal{F}_t\right] + Var_{P(\cdot,T_i)}\left[\int_{T_0}^{T_i} R_E(x)dx \middle| \mathcal{F}_t\right] \\
&\quad +2\,Covar_{P(\cdot,T_i)}\left[\int_{T_0}^{T_i} r_I(x)dx\,,\int_{T_0}^{T_i} R_E(x)dx \middle| \mathcal{F}_t\right] .
\end{aligned}
$$

The formula is based as before on two lognormal distributions. The term $V_I(t,T_0,T_i) = Var_{P(\cdot,T_i)}\left[\int_{T_0}^{T_i} r_I(x)dx \middle| \mathcal{F}_t\right] = Var_{\widetilde{Q}}\left[\int_{T_0}^{T_i} r_I(x)dx \middle| \mathcal{F}_t\right]$ is given in Theorem 6.5. Also, a detailed derivation of the above terms can be found in the Appendix (Lemma E.6). □

The most challenging part in pricing such products will be the choice of parameters - especially for correlation - as long as there is no liquid market. We somehow circumvent this problematic part by letting $\widetilde{W}_E$ be independent of the Brownian motions of the other processes. However, if we omit this assumption such hybrid products like this are needed to calibrate the additional correlation parameters.

6.3.2 Inflation-Indexed Credit Default Swap

In addition to the above mentioned hybrid products there exist approaches linking not only inflation and interest rates, but inflation and credit. Kerkhof (2005) mentions in his survey of inflation derivatives inflation-linked CDO and CDS. By means of an inflation-linked CDS an investor can trade on his view on inflation and credit correlation. Avogaro (2006) introduces a CDS whose premium payments are based on the inflation rate. We will adopt the payoffs of that approach but will model the underlying factors according to our set-up.

Definition 6.8
An Inflation-Indexed Credit Default Swap is a CDS where the premium payments are a combination of the CDS spread and the inflation rate represented by the ratio of the CPI at the beginning and the end of each payment period.

The premium due at the end of the period $[T_{i-1}, T_i]$ is calculated as

$$N \cdot (T_i - T_{i-1}) \cdot \left(\left(\frac{CPI(T_i)}{CPI(T_{i-1})} - 1 \right)^+ + s \right) \cdot 1_{\{T^d > T_i\}}$$

with the deal's notional N and the CDS spread s.
The protection leg resembles that of a general CDS but can also be extended by further payments, e.g. dependent on the six month LIBOR. We define the protection payment for the period $[T_{i-1}, T_i]$ as a normal CDS protection in case of default plus the six month LIBOR if there occured no default in the respective period.

Avogaro (2006) uses a Jarrow-Yildirim framework where the CPI acts as an exchange rate between the nominal and the real interest rates. Both interest rates evolve according to the Hull-White model whereas the CPI follows a log-normal diffusion process. The main difference to our set-up is the model used for the intensity process: the Cox-Ingersoll-Ross (CIR) framework that guarantees a non-negative intensity process. But on the other hand, an intensity following a CIR diffusion process will make it impossible to determine an analytical formula for an inflation-indexed CDS under the assumption of correlation between the interest rates, CPI and the default intensity. Hence, Avogaro (2006) has to simulate the underlying factors when introducing correlated processes.
Although our set-up only assumes an indirect correlation for the zero-recovery short-rate spread s^{zero} by incorporating the macroeconomic factor w into the drift, it can be extended to a framework with correlated Brownian motions without losing the analytical tractability. Therefore, by assuming a default intensity which evolves according to a Vasicek process, we are able to price the inflation-indexed CDS without simulations by means of the following formula.

Theorem 6.9 ***(Inflation-Indexed Credit Default Swap)***
At time t, the fair CDS spread $s^{iicds}(t, T_0, T_m)$ of an Inflation-Indexed Credit Default Swap with notional N and maturity $T_m = \widetilde{T}_n$, with protection payments which are based on the LIBOR rate and on the tenor $[\widetilde{T}_{j-1}, \widetilde{T}_j]$, $j = 1 \ldots n$, and with premium payments which are based on the inflation rate and

on the tenor $[T_{i-1}, T_i]$, $i = 1 \dots m$, *is*

$$\begin{aligned}
& s^{iicds}(t, T_0, T_m) \\
= \quad & \frac{1}{\sum_{i=1}^{m}(T_i - T_{i-1})P^{d,zero}(t, T_i)} \cdot \Bigg[\\
& \sum_{j=1}^{n} \Bigg(P(t, \widetilde{T}_{j-1}) \cdot P^S(t, \widetilde{T}_j) \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{\widetilde{T}_{j-1}} r_R(x)dx\, , \int_t^{\widetilde{T}_j} s^{zero}(x)dx \middle| \mathcal{F}_t\right]} \\
& \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{\widetilde{T}_{j-1}} r_I(x)dx\, , \int_t^{\widetilde{T}_j} s^{zero}(x)dx \middle| \mathcal{F}_t\right]} - P^{d,zero}(t, \widetilde{T}_j) \Bigg) \\
& + V_{T^d}^{ddp}(t, T_m) - V_{T^d}^{ddp}(t, T_0) - P^{d,*}(t, T_0, T^*) + P^{d,*}(t, T_m, T^*) \\
& - \sum_{i=1}^{m}(T_i - T_{i-1}) \cdot \Big(E_{\widetilde{Q}}^{cds}(t, T_{i-1}, T_i) \cdot \mathcal{N}(d_1) - P^{d,zero}(t, T_i) \cdot \mathcal{N}(d_2) \Big) \Bigg]
\end{aligned}$$

with

$$d_{1,2} = \frac{\ln\left(\frac{E_{\widetilde{Q}}^{cds}(t, T_{i-1}, T_i)}{P^{d,zero}(t, T_i)} \right) \pm \frac{V_I(t, T_{i-1}, T_i)}{2}}{\sqrt{V_I(t, T_{i-1}, T_i)}} .$$

The other terms in the above formula are given in the proof.

Proof:
The value at time t of the premium leg

$$\begin{aligned}
& \mathbb{E}_{\widetilde{Q}}\left[N \cdot \sum_{i=1}^{m}(T_i - T_{i-1}) \cdot e^{-\int_t^{T_i} r(x)dx} \left(\left(\frac{CPI(T_i)}{CPI(T_{i-1})} - 1 \right)^+ + s^{iicds} \right) \cdot 1_{\{T^d > T_i\}} \middle| \mathcal{G}_t \right] \\
& = L(t) \cdot N \cdot \sum_{i=1}^{m}(T_i - T_{i-1}) \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x) + s^{zero}(x)dx} \cdot \left(e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} - 1 \right)^+ \middle| \mathcal{F}_t \right] \\
& \quad + L(t) \cdot N \cdot \sum_{i=1}^{m}(T_i - T_{i-1}) \cdot s^{iicds} \cdot P^{d,zero}(t, T_i)
\end{aligned}$$

depends on the term

$$\mathbb{E}_{\widetilde{Q}}\left[\left(e^{-\int_t^{T_i} r(x) + s^{zero}(x)dx + \int_{T_{i-1}}^{T_i} r_I(x)dx} - e^{-\int_t^{T_i} r(x) + s^{zero}(x)dx} \right)^+ \middle| \mathcal{F}_t \right]$$

for whose derivation we need the following components. The variance $V_I(t, T_{i-1}, T_i) := Var_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right]$ which is already given in Theorem

6.5. The expected value

$$
\begin{aligned}
E^{cds}_{\widetilde{Q}}(t,T_{i-1},T_i) &:= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)+s^{zero}(x)dx+\int_{T_{i-1}}^{T_i} r_I(x)dx} \middle| \mathcal{F}_t \right] \\
&\overset{footn.\,I}{=} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)+s^{zero}(x)dx} \middle| \mathcal{F}_t \right] \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} \middle| \mathcal{F}_t \right] \\
&\quad \cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r_R(x)+r_I(x)+s^{zero}(x)dx\,,\int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right]} \\
&= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_i} r(x)+s^{zero}(x)dx} \middle| \mathcal{F}_t \right] \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} \middle| \mathcal{F}_t \right] \\
&\quad \cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r_R(x)dx\,,\int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right]} \cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} r_I(x)dx\,,\int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right]} \\
&\quad \cdot e^{Covar_{\widetilde{Q}}\left[-\int_t^{T_i} s^{zero}(x)dx\,,\int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right]}
\end{aligned}
$$

can be decomposed into $P^{d,zero}(t,T_i)$ and into the expected value of $e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}$ and related covariance terms.
The expected value under the measure $\widetilde{Q}$ is already determined in Section 6.3 (cf. proof of Theorem 6.6):

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[e^{\int_{T_{i-1}}^{T_i} r_I(x)dx} \middle| \mathcal{F}_t \right] \\
&= \frac{P(t,T_{i-1})\cdot P^R(t,T_i)}{P(t,T_i)\cdot P^R(t,T_{i-1})} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_t^{T_{i-1}} r_R(x)dx \middle| \mathcal{F}_t \right]} \\
&\quad \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_t^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right]} \cdot e^{Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_R(x)dx\,,\int_t^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right]}
\end{aligned}
$$

where the covariance terms under the measure $\widetilde{Q}$ are equivalent to those given in the proof of Theorem 6.6.
The first component of

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\left[\int_t^{T_i} r_I(x)dx\,, \int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right] \\
&= Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_I(x)dx\,, \int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t \right] + V_I(t,T_{i-1},T_i)
\end{aligned}
$$

is given in Proposition 6.4, and the remaining covariance terms in Lemma E.7.
Hence, it holds (cf. Theorem 6.5)

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[\left(e^{-\int_t^{T_i} r(x)+s^{zero}(x)dx+\int_{T_{i-1}}^{T_i} r_I(x)dx} - e^{-\int_t^{T_i} r(x)+s^{zero}(x)dx} \right)^+ \middle| \mathcal{F}_t \right] \\
&= E^{cds}_{\widetilde{Q}}(t,T_{i-1},T_i)\cdot \mathcal{N}(d_1) - P^{d,zero}(t,T_i)\cdot \mathcal{N}(d_2)
\end{aligned}
$$

with

$$d_{1,2} = \frac{\ln\left(\frac{E^{cds}_{\widetilde{Q}}(t,T_{i-1},T_i)}{P^{d,zero}(t,T_i)}\right) \pm \frac{V_I(t,T_{i-1},T_i)}{2}}{\sqrt{V_I(t,T_{i-1},T_i)}} .$$

The value of the protection leg at time t is

$$\begin{aligned}
&\mathbb{E}_{\widetilde{Q}}\left[N \cdot \sum_{j=1}^{n}(\widetilde{T}_j - \widetilde{T}_{j-1}) \cdot e^{-\int_t^{\widetilde{T}_j} r(x)dx} \cdot L(\widetilde{T}_{j-1},\widetilde{T}_j) \cdot 1_{\{T^d>\widetilde{T}_j\}}\middle|\mathcal{G}_t\right] \\
&+\mathbb{E}_{\widetilde{Q}}\left[N \cdot \int_{\widetilde{T}_0}^{\widetilde{T}_n} e^{-\int_t^l r(x)dx}(1-Z(l))dH(l)\middle|\mathcal{G}_t\right] \\
&= L(t)\cdot N \cdot \sum_{j=1}^{n} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} r(x)+s^{zero}(x)dx} \cdot \left(e^{\int_{\widetilde{T}_{j-1}}^{\widetilde{T}_j} r(x)dx} - 1\right)\middle|\mathcal{F}_t\right] \\
&+L(t)\cdot N \cdot (V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0) - P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*)) \\
&= L(t)\cdot N \cdot \sum_{j=1}^{n}\left(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_{j-1}} r(x)dx} \cdot e^{-\int_t^{\widetilde{T}_j} s^{zero}(x)dx}\middle|\mathcal{F}_t\right] - P^{d,zero}(t,\widetilde{T}_j)\right) \\
&+L(t)\cdot N \cdot (V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0) - P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*)) \\
&\overset{footn.\,I}{=} L(t)\cdot N \cdot \sum_{j=1}^{n}\left(\mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_{j-1}} r(x)dx}\middle|\mathcal{F}_t\right] \cdot \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{\widetilde{T}_j} s^{zero}(x)dx}\middle|\mathcal{F}_t\right]\right. \\
&\left.\cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{\widetilde{T}_{j-1}} r_R(x)+r_I(x)dx,\int_t^{\widetilde{T}_j} s^{zero}(x)dx\middle|\mathcal{F}_t\right]} - P^{d,zero}(t,\widetilde{T}_j)\right) \\
&+L(t)\cdot N \cdot (V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0) - P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*)) \\
&= L(t)\cdot N \cdot \sum_{j=1}^{n}\left(P(t,\widetilde{T}_{j-1}) \cdot P^S(t,\widetilde{T}_j) \cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{\widetilde{T}_{j-1}} r_R(x)dx,\int_t^{\widetilde{T}_j} s^{zero}(x)dx\middle|\mathcal{F}_t\right]}\right. \\
&\left.\cdot e^{Covar_{\widetilde{Q}}\left[\int_t^{\widetilde{T}_{j-1}} r_I(x)dx,\int_t^{\widetilde{T}_j} s^{zero}(x)dx\middle|\mathcal{F}_t\right]} - P^{d,zero}(t,\widetilde{T}_j)\right) \\
&+L(t)\cdot N \cdot (V^{ddp}_{T^d}(t,T_m) - V^{ddp}_{T^d}(t,T_0) - P^{d,*}(t,T_0,T^*) + P^{d,*}(t,T_m,T^*))
\end{aligned}$$

with $\widetilde{T}_n = T_m$ and $\widetilde{T}_0 = T_0$. The covariance terms are given in Lemma E.7 in the Appendix. The first equivalence is due to Theorem 5.13 assuming a zero-coupon bond with maturity T^* as underlying reference asset. The third equivalence follows from the footnote introduced in the proof of Theorem 6.3. The last equivalence is based on a decomposition of the covariance terms and on the survival probability of Theorem 5.4. □

In the following, we compare CDS spreads obtained by the above formula with plain vanilla CDS spreads of Theorem 5.13 for a forward starting CDS that matures in five years. We use the parameters of the uncorrelated inflation short-rate framework of Chapter 4 (see model INF with data set BBB1 in Table 4.3, Table 4.6, Table 4.9 and Table 4.10). Further, we assume the premium payments to be quarterly and the tenor of the LIBOR-based protection payments to be six months. We analyze the impact of higher levels of the inflation short rate r_I and the short-rate spread s, i.e. θ_s and θ_I, as well as the dependence of the spreads on σ_I. Figure 6.2 presents in the upper row the spreads of a plain vanilla CDS, whereas the lower row gives the spreads of a structured CDS as given in Theorem 6.9. The plots in the first column are obtained with $\theta_s = 0.001$ and the ones in the second column with $\theta_s = 0.005$. In each subplot, diamonds refer to results based on $\theta_I = 0.01$ and squares refer to results based on $\theta_I = 0.005$.

As expected, all spreads increase for higher values of θ_s indicating an increased risk of default. The spreads of the structured CDS (given in the second row) are on a high level for small values of σ_I and drop for increasing σ_I. As opposed to the spreads of the structured CDS, the plain vanilla CDS spreads start at a lower level and increase slightly for greater values of σ_I.
In order to compare these results, it must be noted that the premium payments of the structured CDS consists of two parts: the CDS spread s^{iicds} given in Figure 6.2 and the inflation rate - if positive - for the respective period which has to be paid on top of the CDS spread. Further, the protection payments of the structured CDS differ from the plain vanilla CDS since a second term based on the LIBOR rate is included. In Tables 6.1 -6.4 we compare the performance of the CDS spreads by introducing two simplified versions of the structured CDS. First, we eliminate the LIBOR-based protection payments such that the protection leg is equal to the plain vanilla case but keep the premium payments linked to the inflation rate. This structured CDS is labelled CPI in the Tables 6.1 - 6.4. Second, we reverse these changes by keeping the LIBOR-based protection payments but eliminate the premium payments based on CPI. This structured CDS is labelled LIBOR in the Tables 6.1 - 6.4. The plain vanilla CDS is denoted by PLAIN and the set-up of Theorem 6.9 is called CPI&LIBOR.
Comparing Tables 6.1 - 6.4, we see at first glance that the structures PLAIN and CPI with equivalent protection legs but differing premium legs obtain similar results throughout all combinations of θ_s, θ_I and σ_I. The CDS spreads of the structures PLAIN and CPI increase for greater values of σ_I. Ana-

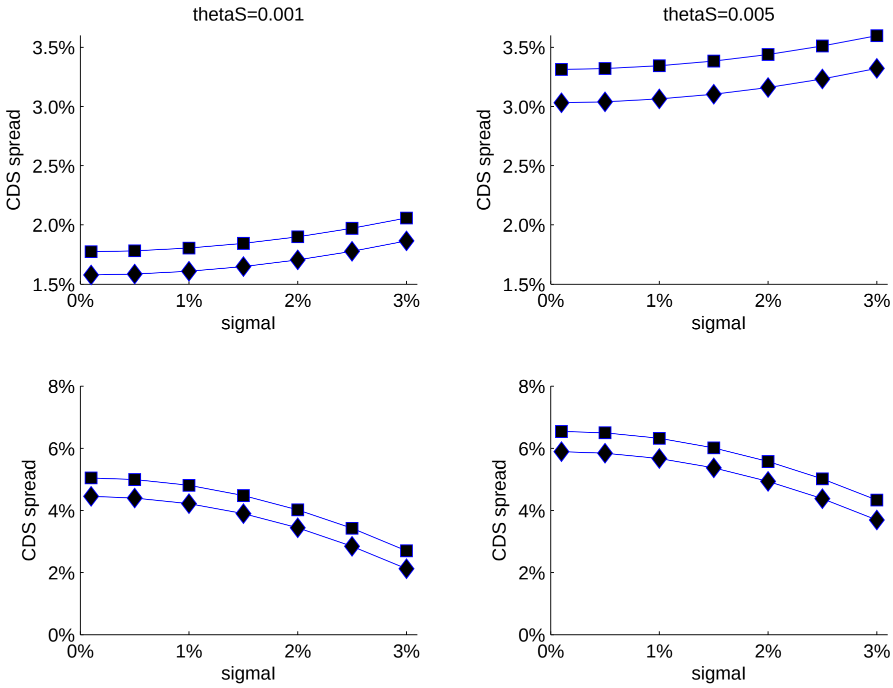

Figure 6.2: CDS spreads calculated within the inflation short-rate framework ($\rho_{wI} = 0$, cf. model INF in Chapter 4) by means of the formulas given in Theorem 5.13 and Theorem 6.9 for a CDS starting in 1 year and maturing in 5 years. The first row gives the results of Theorem 5.13 and the second row the spreads obtained by Theorem 6.9. The results in the left column are based on $\theta_s = 0.001$, the results in the right column on $\theta_s = 0.005$. In each subplot, diamonds refer to results based on $\theta_I = 0.01$ and squares refer to results based on $\theta_I = 0.005$. Premium payments are quarterly, whereas the tenor of the LIBOR-based protection payment of the CDS given in Theorem 6.9 is 6 months.

logously, the performances of the structures LIBOR and CPI&LIBOR, which also share the same protection leg but differ in the set-up of the premium legs, show similarities. Both structures have CDS spreads that start on a high level but decrease for greater values of σ_I.

Table 6.1 and Table 6.2 contain CDS spreads that are determined for $\theta_s = 0.001$, whereas the Tables 6.3 and 6.4 are based on a higher mean reversion level ($\theta_s = 0.005$). As expected, we see that all spreads given in Table 6.3 and Table 6.4 are higher than the spreads of the first two tables, accounting

for the increased default probabilities.
If we look in detail at the structures PLAIN and CPI, we can conclude that both structures yield higher CDS spreads for increasing σ_I and lower CDS spreads for greater values of θ_I. An increase in the volatility of the inflation short rate r_I impacts directly the default probabilities as well as the prices of riskless bonds, since r_I enters the SDEs of the short rate r and the short-rate spread s. Therefore increasing values of σ_I imply a higher risk of default and higher prices of riskless bonds. The influence of θ_I is opposite to σ_I, since the spreads decline for higher values of θ_I. This is mainly due to the fact that the inflation short-rate r_I reduces the drift of the short-rate spread s ($b_{sI} > 0$) resulting in lower probabilities of default. For the given set of parameters, the same relationship holds for the short rate r since the parameter $\hat{b}_{rI}$ is negative resulting in lower interest rates for higher values of r_I.
However, there are differences in the behaviour of the structures PLAIN and CPI. For small values of σ_I the spreads of the structure CPI are lower than the ones of the plain vanilla CDS (PLAIN). For increasing values of σ_I the structure CPI yields higher spreads than the plain vanilla case. Since the premium payments of the structure CPI consists of the spread s^{iicds} and an additional inflation premium, the expected value of this second term influences the spread. The additional inflation premium is only due if the inflation rate of the respective period is positive, hence the additional term is an optionality on the inflation rate and is therefore driven by the volatility of the inflation short rate r_I. For small values of σ_I, the expectation about the future inflation rate is that it is mainly increasing. Hence, it is expected that the additional inflation premium has to be paid for the majority of the periods. This expectation reduces the CDS spread of the structure CPI as opposed to the structure PLAIN because the protection payments are assumed to be equal. For higher values of σ_I, the probability of the additional inflation premium not being paid increases. Therefore, the CDS spreads of the structure CPI approach and even exceed the results of the plain vanilla CDS due to the costs of the optionality.
The structures LIBOR and CPI&LIBOR differ from the previously discussed structures by the set-up of the protection leg. Here, the proctection leg has an additional term that is based on the LIBOR rate for a specified tenor. The impact of this additional term can be observed by the higher level of the CDS spreads of the structures LIBOR and CPI&LIBOR throughout all combinations of θ_I, θ_s and σ_I. The influence of θ_I on the CDS spreads of the structures LIBOR and CPI&LIBOR is similar to the structures PLAIN and CPI: A higher mean reversion level of the inflation short rate implies lower probabilities of default and therefore lower CDS spreads that have to be paid. The influence of σ_I on the CDS spreads of the structures LIBOR

and CPI&LIBOR is opposite to the structures PLAIN and CPI: The CDS spreads decline for increasing values of σ_I. As noted above, the inflation short rate r_I influences also the riskless interest rates. Therefore, a higher volatility in the interest rates increases the probability of the additional protection payments becoming smaller. The behaviour of the CDS spreads of the structure CPI&LIBOR is analogous to the structure CPI with respect to σ_I. The premium payments of the structure CPI&LIBOR contain an additional inflation premium as opposed to the premium leg of the structure LIBOR. For small values of σ_I, the expected value of this inflation premium reduces the value of the constant CDS spread s^{iicds} to be paid in the structure CPI&LIBOR. However, for increasing values of σ_I, the probability of the inflation premium being positive decreases. Therefore the CDS spreads of the structure CPI&LIBOR approach the ones of the structure LIBOR. Finally, for all four structures it can be observed that for increased mean reversion level and volatility of the inflation short rate, i.e. $\theta_I = 0.01$ and $\sigma_I = 0.03$, the CDS spreads become closer.
Summarizing the above analysis, we can conclude that including an additional protection payment can increase the CDS spreads substantially as opposed to the plain vanilla case. In addition, the introduction of an optional premium payment can decrease the CDS spread but not in all cases. The value of such a structured CDS stronlgy depends on the correlation of the driving factors for the default probability and for the additional premium and protection cash flows.

Structure	σ_I						
	0.1%	0.5%	1%	1.5%	2%	2.5%	3%
PLAIN	1.7732	1.7808	1.8046	1.8443	1.8997	1.9709	2.0579
CPI	1.7477	1.7694	1.8050	1.8500	1.9088	1.9825	2.0716
LIBOR	5.0681	5.0042	4.8046	4.4722	4.0077	3.4117	2.6852
CPI&LIBOR	5.0426	4.9927	4.8049	4.4779	4.0167	3.4232	2.6989

Table 6.1: CDS spreads (in %) of a forward starting CDS with maturity in 5 years within the inflation short-rate framework for $\theta_s = 0.001$, $\theta_I = 0.005$ and different values for σ_I. PLAIN denotes a plain vanilla CDS. CPI refers to a CDS with premium payments linked to the inflation rate. LIBOR refers to a CDS where the proctection payments include a LIBOR-based term. CPI&LIBOR denotes a CDS where the premium leg is equivalent to the structure named CPI and the protection leg is equivalent to the structure called LIBOR.

Structure	σ_I						
	0.1%	0.5%	1%	1.5%	2%	2.5%	3%
PLAIN	1.5765	1.5842	1.6083	1.6485	1.7047	1.7769	1.8651
CPI	1.5628	1.5801	1.6219	1.6759	1.7413	1.8200	1.9131
LIBOR	4.4653	4.4011	4.2006	3.8668	3.4002	2.8016	2.0720
CPI&LIBOR	4.4516	4.3970	4.2142	3.8942	3.4368	2.8447	2.1200

Table 6.2: CDS spreads (in %) of a forward starting CDS with maturity in 5 years within the inflation short-rate framework for $\theta_s = 0.001$, $\theta_I = 0.01$ and different values for σ_I. PLAIN denotes a plain vanilla CDS. CPI refers to a CDS with premium payments linked to the inflation rate. LIBOR refers to a CDS where the proctection payments include a LIBOR-based term. CPI&LIBOR denotes a CDS where the premium leg is equivalent to the structure named CPI and the protection leg is equivalent to the structure called LIBOR.

Structure	σ_I						
	0.1%	0.5%	1%	1.5%	2%	2.5%	3%
PLAIN	3.3125	3.3201	3.3440	3.3838	3.4395	3.5111	3.5984
CPI	3.2844	3.3059	3.3411	3.3861	3.4450	3.5189	3.6084
LIBOR	6.5684	6.5081	6.3198	6.0064	5.5682	5.0059	4.3204
CPI&LIBOR	6.5403	6.4939	6.3170	6.0086	5.5736	5.0138	4.3304

Table 6.3: CDS spreads (in %) of a forward starting CDS with maturity in 5 years within the inflation short-rate framework for $\theta_s = 0.005$, $\theta_I = 0.005$ and different values for σ_I. PLAIN denotes a plain vanilla CDS. CPI refers to a CDS with premium payments linked to the inflation rate. LIBOR refers to a CDS where the proctection payments include a LIBOR-based term. CPI&LIBOR denotes a CDS where the premium leg is equivalent to the structure named CPI and the protection leg is equivalent to the structure called LIBOR.

Structure	σ_I						
	0.1%	0.5%	1%	1.5%	2%	2.5%	3%
PLAIN	3.0315	3.0393	3.0635	3.1039	3.1605	3.2331	3.3218
CPI	3.0163	3.0340	3.0753	3.1284	3.1934	3.2720	3.3654
LIBOR	5.9051	5.8445	5.6554	5.3406	4.9006	4.3359	3.6475
CPI&LIBOR	5.8898	5.8393	5.6672	5.3651	4.9335	4.3749	3.6911

Table 6.4: CDS spreads (in %) of a forward starting CDS with maturity in 5 years within the inflation short-rate framework for $\theta_s = 0.005$, $\theta_I = 0.01$ and different values for σ_I. PLAIN denotes a plain vanilla CDS. CPI refers to a CDS with premium payments linked to the inflation rate. LIBOR refers to a CDS where the proctection payments include a LIBOR-based term. CPI&LIBOR denotes a CDS where the premium leg is equivalent to the structure named CPI and the protection leg is equivalent to the structure called LIBOR.

Chapter 7

Summary and Conclusion

In this thesis we studied the determinants of non-defaultable and defaultable bond prices within a defaultable term structure model by directly including the chosen factors into the pricing framework. We tested the inclusion of observable as well as unobservable macroeconomic factors into the pricing framework and found that observable macroeconomic factors are capable of improving the pricing. But the impact of observable macroeconomic factors is limited, since there is still a large portion of systematic behaviour that cannot be explained. In line with these findings, we also analyzed the combined incorporation of observable and unobservable macroeconomic factors and obtaind promising results. Further, we used this defaultable term structure model for pricing credit as well as inflation-linked derivatives. We derived and analyzed closed-form solutions for a range of complex derivatives like credit default swaptions, constant maturity credit default swaps and hybrid inflation-linked derivatives, enabling its usage in valuation and risk management.

In Chapter 4 we introduced a family of models based on a defaultable term structure model that incorporates macroeconomic and firm-specific factors. This approach is influenced by the literature of determinants of sovereign and bond prices which indicates that both prices are driven by common factors and that defaultable bond prices can be also explained by firm-specific factors. Since the availability of firm-specific data is sparse, its disclosure often comes with a delay of up to one year (annual reports) and its content is additionally disturbed by regulatory requirements, we assumed the firm-specific data entering our model to be unobservable. However, there is a wide range of macroeconomic factors published on a regular basis that are supposed to drive sovereign as well as defaultable bonds. We tested a collection of observable factors for its ability to explain bond prices, including well-known factors as well as factors that, so far, have not been widely used in literature.

Also, we analyzed whether there is an additional impact by including several observable as well as unobservable macroeconomic factors. We led the reader in detail through the modelling, the selection of macroeconomic factors, the calibration process and the analysis of the results. We found that the gross domestic product as well as the composite indices of leading and coincident indicators performed best with respect to pricing errors and fitting abilities. Further, we concluded that the usage of more than one macroeconomic factor in both the non-defaultable and the defaultable pricing framework increased the pricing ability and stabilized the estimation procedure. Especially, if we followed the findings in the literature and combined an observable and an unobservable macroeconomic factor we obtained good fitting abilities. So far, research indicates that bond prices are explained by macroeconomic factors like gross domestic product (GDP), but also that the major part in the variance in bond prices can be explained by a common factor which is still unknown. Our set-up is therefore in line with current research and is flexible enough to react on future developments by additionally including more factors or changing the factors from being unobservable to observable or vice versa.

In Chapter 5 we used the defaultable term structure model for pricing in closed form complex credit derivatives as credit default swaps, credit default swaptions and constant maturity credit default swaps while still keeping the link to macroeconomic and firm-specific factors. This approach allowed us to use the pricing framework for bonds of Chapter 4 and extend it to credit derivatives. Research indicates that the determinants of bond prices are still valid for credit derivatives. Therefore, we started with the defaultable term structure framework and derived in a consistent way the dynamics of the credit default swap spread. An approximated version of these dynamics allowed us to price credit default swaptions and constant maturity credit default swaps in closed form without having to use time-consuming simulations of risk factors. We studied the results of these semi-analytical solutions in comparison to a full simulation approach of all factors defined in the term structure model. The semi-analytical formulas yielded promising results up to mid-term maturities and moderate volatilities of credit spreads. These findings are in line with research (see e.g. Krekel & Wenzel (2006)) that tested other approaches for pricing such derivatives and obtained similar results. Further, we extended the pricing of credit default swaptions to the case where the credit default swap spread is fixed in advance instead of being determined at par. This new requirement of setting up credit default swaps is proposed by the market in order to increase the standardization of these over-the-counter (OTC) contracts. Due to recent developments, we also outlined how the pricing of credit default swaps changed if additionally

the credit risk of the counterparty had to be included. Given certain assumptions, we showed that the pricing was still possible in closed form.
In Chapter 6 we applied a variant of our term structure framework to the pricing of inflation-linked assets. We used a framework that decomposes the short rate into a real short rate and an inflation short rate. Starting with standard inflation-linked derivatives like zero-coupon inflation-linked swaps and year-on-year inflation-linked swaps, we further extended our framework to the pricing of inflation-indexed caplets. Since inflation-linked assets help to protect future exposures and lock in real values their popularity as well as the market for tailor-made solutions grows rapidly. Therefore, we outlined the pricing of complex hybrid inflation-linked derivatives incorporating interest rate, equity or credit components. We derived closed-form solutions for inflation-linked equity options and credit default swaps. Also, we presented a feasible approximation for pricing hybrid inflation-linked derivatives in closed form enabling a fast and accurate pricing for such complex derivatives given moderate market conditions.

Altogether, the contribution of this thesis to academic literature and practice is twofold: We help to expand the analysis of determinants of bond prices to the field of term structure models respectively hybrid models by generalizing the work of Antes et al. (2008) and by testing several macroeconomic factors as well as composite indices for their ability to explain sovereign and bond spreads. The majority of studies concerning the topic of credit spread determinants is based on a structural framework although structural models display severe shortcomings in pricing credit risk. Therefore, this thesis promotes the usage and further development of hybrid models because of its explicit linkage to macroeconomic factors. Further, our analysis is carried out by incorporating the candidate factors into the pricing model and not by simply regressing the credit spread data against macroeconomic and firm-specific data. Also, by generalizing the work of Antes et al. (2008) and by extending the works of Antes et al. (2009) and Hagedorn et al. (2007) to pricing complex credit derivatives respectively hybrid inflation-linked derivatives, we provide a framework that is capable of pricing bonds as well as complex derivatives in closed form, therefore enabling its usage especially in risk management with its challenges in determining potential future exposure and counterparty value adjustments on a single asset basis as well as on portfolio or netting-set level. These challenges ask for pricing frameworks that are capable of pricing a wide range of assets while covering the driving factors of the state of economy. Additionally, our framework is able to speed up the computation of such risk management figures. By means of its (approximated) closed-form solutions for a variety of complex derivatives our framework reduces the need to simulate all factors until the final time horizon and/or helps to improve accuracy of simulation-based pricing by serving as control variates.

Appendix A

Determination of θ_r

We follow the proposal of Hull & White (1990) and determine the deterministic function θ_r by adjusting it to the term structure at time $t = 0$. This is done by means of the forward short rate (see Definition 2.26). With the help of the Leibniz integral rule [I] we get for the forward short rate

$$\begin{aligned}
-f(t,T) &= \frac{\partial}{\partial T} \ln P(t,T) \\
&= \frac{\partial}{\partial T} \left(A(t,T) - B(t,T)r(t) - E_1(t,T)w_1(t) - E_2(t,T)w_2(t)\right) \\
&= \frac{1}{2}\sigma_r^2 B(T,T)^2 + \frac{1}{2}\sigma_{w_1}^2 E_1(T,T)^2 + \frac{1}{2}\sigma_{w_2}^2 E_2(T,T)^2 \\
&\quad + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2} E_1(T,T)E_2(T,T) + \sigma_r\sigma_{w_1}\rho_{rw_1} B(T,T)E_1(T,T) \\
&\quad + \sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) B(T,T)E_2(T,T) \\
&\quad - \theta_r(T)B(T,T) - \theta_{w_1}E_1(T,T) - \theta_{w_2}E_2(T,T) \\
&\quad + \int_t^T \Big(\sigma_r^2 B(l,T)(B(l,T))_T + \sigma_{w_1}^2 E_1(l,T)(E_1(l,T))_T \\
&\quad + \sigma_{w_2}^2 E_2(l,T)(E_2(l,T))_T \\
&\quad + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}((E_1(l,T))_T E_2(l,T) + E_1(l,T)(E_2(l,T))_T) \\
&\quad + \sigma_r\sigma_{w_1}\rho_{rw_1}((B(l,T))_T E_1(l,T) + B(l,T)(E_1(l,T))_T)
\end{aligned}$$

[I] Leibniz Integral Rule:

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}dx + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z}$$

$$
\begin{aligned}
&+\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2}+\rho_{rw_2}\sqrt{1-\rho^2_{w_1w_2}}\right)((B(l,T))_T E_2(l,T)\\
&+B(l,T)(E_2(l,T))_T)\\
&\left.-\theta_r(l)(B(l,T))_T-\theta_{w_1}(E_1(l,T))_T-\theta_{w_2}(E_2(l,T))_T\right)dl\\
&-(B(t,T))_T r(t)-(E_1(t,T))_T w_1(t)-(E_2(t,T))_T w_2(t)\ ,
\end{aligned}
$$

where the first four rows of the last equality vanish since $B(T,T)=E_1(T,T)=E_2(T,T)=0$. Furthermore, using again the Leibniz integral rule we obtain

$$
\begin{aligned}
-f_T(t,T) &:= -\frac{\partial}{\partial T}f(t,T)\\
&= \sigma_r^2 B(T,T)(B(T,T))_T+\sigma^2_{w_1}E_1(T,T)(E_1(T,T))_T\\
&\quad+\sigma^2_{w_2}E_2(T,T)(E_2(T,T))_T\\
&\quad+\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}((E_1(T,T))_T E_2(T,T)+E_1(T,T)(E_2(T,T))_T)\\
&\quad+\sigma_r\sigma_{w_1}\rho_{rw_1}((B(T,T))_T E_1(T,T)+B(T,T)(E_1(T,T))_T)\\
&\quad+\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2}+\rho_{rw_2}\sqrt{1-\rho^2_{w_1w_2}}\right)((B(T,T))_T E_2(T,T)\\
&\quad+B(T,T)(E_2(T,T))_T)\\
&\quad-\theta_r(T)(B(T,T))_T-\theta_{w_1}(E_1(T,T))_T-\theta_{w_2}(E_2(T,T))_T\\
&\quad+\int_t^T\Big(\sigma_r^2((B(l,T))_T^2+B(l,T)(B(l,T))_{TT})\\
&\quad+\sigma^2_{w_1}((E_1(l,T))_T^2+E_1(l,T)(E_1(l,T))_{TT})\\
&\quad+\sigma^2_{w_2}((E_2(l,T))_T^2+E_2(l,T)(E_2(l,T))_{TT})\\
&\quad+\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}((E_1(l,T))_{TT}E_2(l,T)+2(E_1(l,T))_T(E_2(l,T))_T\\
&\quad+E_1(l,T)(E_2(l,T))_{TT})\\
&\quad+\sigma_r\sigma_{w_1}\rho_{rw_1}((B(l,T))_{TT}E_1(l,T)+2(B(l,T))_T(E_1(l,T))_T\\
&\quad+B(l,T)(E_1(l,T))_{TT})\\
&\quad+\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2}+\rho_{rw_2}\sqrt{1-\rho^2_{w_1w_2}}\right)((B(l,T))_{TT}E_2(l,T)\\
&\quad+2(B(l,T))_T(E_2(l,T))_T+B(l,T)(E_2(l,T))_{TT})\\
&\quad-\theta_r(l)(B(l,T))_{TT}-\theta_{w_1}(E_1(l,T))_{TT}-\theta_{w_2}(E_2(l,T))_{TT}\Big)dl\\
&\quad-(B(t,T))_{TT}r(t)-(E_1(t,T))_{TT}w_1(t)-(E_2(t,T))_{TT}w_2(t)\ ,
\end{aligned}
$$

where terms in the first seven rows can be cancelled using $B(T,T)=E_1(T,T)=E_2(T,T)=(E_1(T,T))_T=(E_2(T,T))_T=0$. We can rearrange this equation with the help of $(B(T,T))_T=1$,

$$(B(t,T))_{TT}=-\hat{a}_r(B(t,T))_T,$$

$$(E_1(t,T))_{TT} = b_{rw_1} e^{-\hat{a}_{w_1}(T-t)} - \hat{a}_r (E_1(t,T))_T$$

and

$$(E_2(t,T))_{TT} = \hat{b}_{rw_2} e^{-\hat{a}_{w_2}(T-t)} - \hat{a}_r (E_2(t,T))_T,$$

and end with

$$\begin{aligned}
-f_T(t,T) \;=\; & -\theta_r(T) + \hat{a}_r f(t,T) - b_{rw_1} e^{-\hat{a}_{w_1}(T-t)} w_1(t) \\
& - \hat{b}_{rw_2} e^{-\hat{a}_{w_2}(T-t)} w_2(t) \\
& + \int_t^T \Big(\sigma_r^2 (B(l,T))_T^2 + \sigma_{w_1}^2 (E_1(l,T))_T^2 + \sigma_{w_2}^2 (E_2(l,T))_T^2 \\
& + 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2} (E_1(l,T))_T (E_2(l,T))_T \\
& + 2\sigma_r\sigma_{w_1}\rho_{rw_1} (B(l,T))_T (E_1(l,T))_T \\
& + 2\sigma_r\sigma_{w_2} \left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2} \right) (B(l,T))_T (E_2(l,T))_T \\
& + b_{rw_1} e^{-\hat{a}_{w_1}(T-l)} \left(\sigma_{w_1}^2 E_1(l,T) + \sigma_r\sigma_{w_1}\rho_{rw_1} B(l,T) - \theta_{w_1} \right) \\
& + \hat{b}_{rw_2} e^{-\hat{a}_{w_2}(T-l)} \Big(\sigma_{w_2}^2 E_2(l,T) \\
& + \sigma_r\sigma_{w_2} \left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2} \right) B(l,T) - \theta_{w_2} \Big) \\
& + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2} (b_{rw_1} e^{-\hat{a}_{w_1}(T-l)} E_2(l,T) \\
& + \hat{b}_{rw_2} e^{-\hat{a}_{w_2}(T-l)} E_1(l,T)) \Big) dl
\end{aligned}$$

As it holds that

$$\int_t^T \sigma_r^2 (B(l,T))_T^2 dl = \sigma_r^2 \left(\frac{\hat{a}_r}{2} B(t,T)^2 + B(t,T)(B(t,T))_T \right),$$

$$\begin{aligned}
& \int_t^T b_{rw_1} e^{-\hat{a}_{w_1}(T-l)} \sigma_{w_1}^2 E_1(l,T) + \sigma_{w_1}^2 (E_1(l,T))_T^2 dl \\
& = \sigma_{w_1}^2 \left(\frac{\hat{a}_r}{2} E_1(t,T)^2 + E_1(t,T)(E_1(t,T))_T \right)
\end{aligned}$$

and

$$\begin{aligned}
& \int_t^T \hat{b}_{rw_2} e^{-\hat{a}_{w_2}(T-l)} \sigma_{w_2}^2 E_2(l,T) + \sigma_{w_2}^2 (E_2(l,T))_T^2 dl \\
& = \sigma_{w_2}^2 \left(\frac{\hat{a}_r}{2} E_2(t,T)^2 + E_2(t,T)(E_2(t,T))_T \right)
\end{aligned}$$

we obtain for θ_r

$$
\begin{aligned}
&\theta_r(T)\\
&= f_T(t,T) + \hat{a}_r f(t,T) + \Phi_T(t,T) + \hat{a}_r \Phi(t,T)\\
&\quad - b_{rw_1} e^{-\hat{a}_{w_1}(T-t)} w_1(t) - \hat{b}_{rw_2} e^{-\hat{a}_{w_2}(T-t)} w_2(t)\\
&\quad - b_{rw_1} \frac{\theta_{w_1}}{\hat{a}_{w_1}} (1 - e^{-\hat{a}_{w_1}(T-t)}) - \hat{b}_{rw_2} \frac{\theta_{w_2}}{\hat{a}_{w_2}} (1 - e^{-\hat{a}_{w_2}(T-t)})\\
&\quad + \frac{b_{rw_1}\sigma_r\sigma_{w_1}\rho_{rw_1}}{\hat{a}_r\hat{a}_{w_1}(\hat{a}_{w_1}-\hat{a}_r)} \Big((\hat{a}_{w_1} - \hat{a}_r)(1 - e^{-\hat{a}_{w_1}(T-t)}) + \hat{a}_{w_1}(e^{-(\hat{a}_r+\hat{a}_{w_1})(T-t)}\\
&\quad - e^{-2\hat{a}_r(T-t)}) \Big)\\
&\quad + \frac{\hat{b}_{rw_2}\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)}{\hat{a}_r\hat{a}_{w_2}(\hat{a}_{w_2}-\hat{a}_r)} \Big((\hat{a}_{w_2} - \hat{a}_r)(1 - e^{-\hat{a}_{w_2}(T-t)})\\
&\quad + \hat{a}_{w_2}(e^{-(\hat{a}_r+\hat{a}_{w_2})(T-t)} - e^{-2\hat{a}_r(T-t)}) \Big)\\
&\quad + \frac{b_{rw_1}\hat{b}_{rw_2}\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}}{(\hat{a}_{w_1}-\hat{a}_r)(\hat{a}_{w_2}-\hat{a}_r)\hat{a}_r\hat{a}_{w_1}\hat{a}_{w_2}} \Big(\hat{a}_{w_1}\hat{a}_{w_2}(-e^{-2\hat{a}_r(T-t)} + e^{-(\hat{a}_r+\hat{a}_{w_1})(T-t)}\\
&\quad + e^{-(\hat{a}_r+\hat{a}_{w_2})(T-t)}) - \hat{a}_r(\hat{a}_{w_1} + \hat{a}_{w_2} - \hat{a}_r)e^{-(\hat{a}_{w_1}+\hat{a}_{w_2})(T-t)}\\
&\quad + (\hat{a}_{w_1} - \hat{a}_r)(\hat{a}_{w_2} - \hat{a}_r)(1 - e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_{w_2}(T-t)}) \Big)
\end{aligned}
$$

with $\Phi(t,T) := \frac{1}{2}\sigma_r^2 B(t,T)^2 + \frac{1}{2}\sigma_{w_1}^2 E_1(t,T)^2 + \frac{1}{2}\sigma_{w_2}^2 E_2(t,T)^2$.

By setting $t := 0, T := t$ we get $\theta_r(t)$. In order to determine $f(0,t)$ and $f_T(0,t)$ for every t we need to know the term structure at time 0. We derive the term structure with the help of non-defaultable bonds which we only have for certain maturities. Therefore we fit the data on a Nelson-Siegel curve (see Nelson & Siegel (1987)) by non-linear regression. Thus, we assume the zero rates to follow

$$R(0,t) = \beta_0 + (\beta_1 + \beta_2)\frac{1 - e^{-\frac{t}{\beta_3}}}{\frac{t}{\beta_3}} - \beta_2 e^{-\frac{t}{\beta_3}}.$$

Then, within the Nelson-Siegel framework it holds for the instantaneous forward rate

$$f(0,t) = \beta_0 + \beta_1 e^{-\frac{t}{\beta_3}} + \beta_2 \frac{t}{\beta_3} e^{-\frac{t}{\beta_3}}$$

and for its derivative

$$f_T(0,t) = -\frac{\beta_1}{\beta_3} e^{-\frac{t}{\beta_3}} + \beta_2 (\frac{1}{\beta_3} - \frac{t}{(\beta_3)^2}) e^{-\frac{t}{\beta_3}} \ .$$

Appendix B

Results of the Calibration

This chapter contains the result of the models analyzed in Chapter 4, Section 4.8. The analysis is based on an insample period from January 1 1999 to December 27 2002 and an out-of-sample period from January 3 2003 to December 31 2004.
Tables B.1 to B.4 contain the pricing errors between market and model implied prices for the different four and five factor models.
Tables B.5 to B.16 show the resulting confidence intervalls as well as the constants of the linear regression.
Tables B.17 to B.28 outline the tests of the standardized innovations with respect to the requirements of the state space model.

Model	Treasury Strips		US Industrials A2		US Industrials BBB1	
	insample	out-of-s.	insample	out-of-s.	insample	out-of-s.
GDPn	0.4829	0.4946	0.1217	0.1090	0.0614	0.0943
	(0.3129)	(0.1604)	(0.0815)	(0.0924)	(0.0264)	(0.0319)
GDPr	0.4548	0.5332	0.0650	0.0720	0.0609	0.0444
	(0.3265)	(0.2035)	(0.0328)	(0.0322)	(0.0212)	(0.0286)
CPI	0.4535	0.5919	0.0936	0.0366	0.0558	0.0773
	(0.2804)	(0.2443)	(0.0488)	(0.0197)	(0.0332)	(0.0396)
IP	0.4577	0.4809	0.1013	0.1669	0.0684	0.0520
	(0.3607)	(0.1770)	(0.0710)	(0.1159)	(0.0302)	(0.0322)
Prod	0.4623	0.4950	0.0856	0.1067	0.0593	0.0372
	(0.2916)	(0.1417)	(0.0552)	(0.0613)	(0.0251)	(0.0151)
CILI	0.4393	0.4358	0.0620	0.0605	0.0510	0.0387
	(0.2806)	(0.1334)	(0.0230)	(0.0233)	(0.0134)	(0.0118)
CICI	0.4448	0.5072	0.0526	0.0490	0.0423	0.0434
	(0.2831)	(0.1836)	(0.0233)	(0.0203)	(0.0129)	(0.0157)
SZ5	0.5625	1.3842	0.0627	0.0565	0.0548	0.0359
	(0.3274)	(1.2490)	(0.0229)	(0.0219)	(0.0153)	(0.0115)
SZ5u	0.3559	0.8900	0.0747	0.0360	0.0520	0.0313
	(0.0625)	(0.0979)	(0.0345)	(0.0223)	(0.0162)	(0.0118)
5corr	0.3069	0.8045	0.0838	0.0549	0.0573	0.0448
	(0.0859)	(0.1649)	(0.0215)	(0.0153)	(0.0126)	(0.0135)
INF	0.2946	0.4940	0.0520	0.0722	0.0453	0.0399
	(0.1830)	(0.2937)	(0.0181)	(0.0259)	(0.0096)	(0.0153)
INFcorr	0.3936	0.5939	0.0500	0.0488	0.0407	0.0481
	(0.2689)	(0.4392)	(0.0193)	(0.0198)	(0.0128)	(0.0173)

Table B.1: Average absolute deviations (in %) of the model and market prices over all maturities and for the maturities from 1 year to 10 years (in brackets). The insample period is from January 1 1999 to December 27 2002 and the out-of-sample period from January 3 2003 to December 31 2004.

Treasury Strips (insample)											
Model	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y	20Y	25Y
GDPn	1.0016	0.9036	0.6277	0.2955	0.0967	0.1322	0.2266	0.3544	0.4573	0.2654	0.9504
GDPr	0.7892	0.6663	0.3996	0.1170	0.1648	0.2739	0.3585	0.4481	0.5234	0.2888	0.9733
CPI	0.9314	0.8770	0.6320	0.3153	0.1035	0.0492	0.1452	0.3042	0.4138	0.2858	0.9309
IP	0.6330	0.5348	0.2805	0.0904	0.2441	0.3514	0.4332	0.5306	0.5950	0.3308	1.0113
Prod	0.9669	0.8536	0.5652	0.2423	0.0321	0.1329	0.2333	0.3684	0.4673	0.2671	0.9561
CILI	0.8523	0.7769	0.5248	0.2216	0.0252	0.1331	0.2286	0.3677	0.4629	0.2847	0.9542
CICI	0.8529	0.7855	0.5284	0.2182	0.0070	0.1328	0.2306	0.3824	0.4822	0.3004	0.9729
SZ5	0.4258	0.2697	0.0048	0.2185	0.3211	0.3702	0.4151	0.4570	0.5052	0.5555	2.6450
SZ5u	0.2917	0.1805	0.0690	0.0714	0.0550	0.0245	0.0105	0.0707	0.1362	0.5792	2.4261
5corr	0.2364	0.0723	0.1010	0.1048	0.0667	0.0349	0.0132	0.0914	0.1897	0.5009	1.9651
INF	0.5979	0.4716	0.2394	0.0715	0.0795	0.1249	0.1763	0.2553	0.3341	0.4157	0.4748
INFcorr	0.7377	0.6968	0.4716	0.1872	0.0501	0.1402	0.2234	0.3521	0.4577	0.5230	0.4898

Treasury Strips (out-of-sample)											
Model	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y	20Y	25Y
GDPn	0.5678	0.4456	0.2720	0.1367	0.0913	0.1127	0.1295	0.1672	0.2133	0.2947	3.0095
GDPr	0.6918	0.5272	0.2561	0.0887	0.1540	0.2121	0.2244	0.2354	0.2541	0.2594	2.9622
CPI	0.5454	0.5277	0.4827	0.3075	0.1171	0.0420	0.1466	0.2738	0.3402	0.5103	3.2172
IP	0.4304	0.3116	0.1328	0.1021	0.1460	0.1823	0.1977	0.2228	0.2551	0.3014	3.0076
Prod	0.6330	0.5023	0.2934	0.1019	0.0294	0.0784	0.1110	0.1647	0.2133	0.3006	3.0176
CILI	0.3449	0.2767	0.1955	0.0987	0.0237	0.0812	0.1138	0.1891	0.2316	0.2547	2.9842
CICI	0.3537	0.3239	0.2733	0.1444	0.0068	0.1035	0.1774	0.2677	0.3123	0.4516	3.1644
SZ5	0.7673	0.4859	0.0106	0.6361	1.0602	1.3838	1.6083	1.9230	2.1208	0.5640	4.6661
SZ5u	0.4513	0.3088	0.1038	0.0822	0.0736	0.0490	0.0105	0.1103	0.2558	1.8247	6.5202
5corr	0.2959	0.1400	0.1064	0.2143	0.1649	0.0910	0.0177	0.1825	0.3777	1.6011	5.6575
INF	0.4960	0.4375	0.3142	0.1365	0.0750	0.1824	0.2946	0.4384	0.6148	1.1204	1.3244
INFcorr	0.7792	0.7353	0.6260	0.3138	0.0246	0.2241	0.4179	0.6483	0.8194	0.9619	0.9830

Table B.2: Average absolute deviations (in %) of the model prices and market data given by US Treasury Strips for different maturities.

US Industrials A2 (insample)											
Model	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y	20Y	25Y
GDPn	0.0990	0.0751	0.0521	0.0283	0.0230	0.0499	0.0816	0.1383	0.1971	0.2658	0.3285
GDPr	0.0394	0.0199	0.0029	0.0113	0.0057	0.0113	0.0277	0.0621	0.1086	0.1916	0.2349
CPI	0.0489	0.0286	0.0211	0.0194	0.0127	0.0238	0.0434	0.0841	0.1370	0.2690	0.3411
IP	0.0391	0.0148	0.0210	0.0219	0.0020	0.0350	0.0707	0.1366	0.2099	0.2849	0.2786
Prod	0.0504	0.0197	0.0450	0.0537	0.0319	0.0041	0.0264	0.0810	0.1443	0.2346	0.2513
CILI	0.0965	0.0782	0.0507	0.0201	0.0066	0.0002	0.0060	0.0234	0.0540	0.1410	0.2051
CICI	0.1127	0.0908	0.0590	0.0246	0.0127	0.0099	0.0071	0.0115	0.0381	0.0877	0.1243
SZ5	0.0730	0.0541	0.0260	0.0004	0.0050	2e-5	0.0105	0.0380	0.0800	0.1712	0.2315
5corr	0.1095	0.0906	0.0584	0.0231	0.0061	0.0006	0.0010	0.0126	0.0483	0.2277	0.3433
SZ5u	0.0281	0.0140	0.0191	0.0276	0.0213	0.0095	0.0136	0.0498	0.1009	0.2273	0.3102
INF	0.0867	0.0639	0.0327	0.0028	0.0081	0.0067	5e-5	0.0208	0.0554	0.1250	0.1696
INFcorr	0.0801	0.0588	0.0288	0.0003	0.0076	0.0050	0.0049	0.0275	0.0611	0.1132	0.1627
US Industrials A2 (out-of-sample)											
Model	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y	20Y	25Y
GDPn	0.0754	0.0612	0.0472	0.0300	0.0212	0.0622	0.1011	0.1639	0.2215	0.2244	0.1907
GDPr	0.0291	0.0118	0.0034	0.0098	0.0059	0.0080	0.0238	0.0605	0.1139	0.2428	0.2830
CPI	0.0322	0.0183	0.0180	0.0102	0.0088	0.0121	0.0175	0.0283	0.0430	0.0958	0.1183
IP	0.0503	0.0149	0.0263	0.0334	0.0026	0.0576	0.1175	0.2273	0.3466	0.4798	0.4798
Prod	0.0514	0.0165	0.0448	0.0525	0.0343	0.0046	0.0284	0.0922	0.1722	0.3219	0.3550
CILI	0.0728	0.0763	0.0581	0.0224	0.0056	0.0002	0.0035	0.0184	0.0550	0.1626	0.1907
CICI	0.0819	0.0844	0.0642	0.0256	0.0080	0.0066	0.0071	0.0058	0.0249	0.0999	0.1303
SZ5	0.0368	0.0360	0.0226	0.0004	0.0048	1e-5	0.0099	0.0364	0.0791	0.1831	0.2121
SZ5u	0.0159	0.0122	0.0135	0.0167	0.0148	0.0070	0.0082	0.0323	0.0639	0.1029	0.1090
5corr	0.0556	0.0610	0.0465	0.0170	0.0039	0.0005	0.0007	0.0080	0.0306	0.1539	0.2262
INF	0.0554	0.0548	0.0372	0.0042	0.0087	0.0084	3e-5	0.0307	0.0919	0.2431	0.2592
INFcorr	0.0440	0.0419	0.0267	0.0002	0.0080	0.0055	0.0032	0.0273	0.0677	0.1502	0.1623

Table B.3: Average absolute deviations (in %) of the model prices and market data given by US Industrials A2 for different maturities.

US Industrials BBB1 (insample)											
Model	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y	20Y	25Y
GDPn	0.0548	0.0377	0.0241	0.0207	0.0179	0.0101	0.0019	0.0316	0.0784	0.1841	0.2136
GDPr	0.0364	0.0190	0.0021	0.0117	0.0087	0.0005	0.0123	0.0382	0.0751	0.1943	0.2720
CPI	0.0341	0.0127	0.0167	0.0415	0.0478	0.0435	0.0333	0.0069	0.0427	0.1497	0.1851
IP	0.0283	0.0111	0.0153	0.0274	0.0239	0.0125	0.0034	0.0386	0.0899	0.2203	0.2819
Prod	0.0631	0.0335	0.0068	0.0269	0.0253	0.0143	0.0009	0.0307	0.0711	0.1668	0.2133
CILI	0.0715	0.0530	0.0300	0.0086	0.0007	0.0021	0.0009	0.0124	0.0390	0.1445	0.1986
CICI	0.0651	0.0483	0.0276	0.0106	0.0054	0.0033	0.0001	0.0108	0.0323	0.1068	0.1555
SZ5	0.0645	0.0447	0.0208	0.0018	0.0044	0.0026	0.0037	0.0214	0.0527	0.1622	0.2244
SZ5u	0.0690	0.0494	0.0238	0.0047	0.0133	0.0167	0.0157	0.0102	0.0292	0.1405	0.1993
5corr	0.0699	0.0508	0.0281	0.0083	0.0018	0.0001	0.0023	0.0115	0.0359	0.1696	0.2516
INF	0.0570	0.0391	0.0182	0.0021	0.0040	0.0033	5e-5	0.0101	0.0294	0.1283	0.2066
INFcorr	0.0613	0.0436	0.0225	0.0048	0.0007	0.0001	0.0039	0.0173	0.0403	0.1054	0.1483
US Industrials BBB1 (out-of-sample)											
Model	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y	20Y	25Y
GDPn	0.0627	0.0476	0.0273	0.0156	0.0170	0.0112	0.0017	0.0403	0.1102	0.3131	0.3905
GDPr	0.0387	0.0211	0.0018	0.0160	0.0127	0.0006	0.0180	0.0547	0.0965	0.1177	0.1109
CPI	0.0287	0.0118	0.0132	0.0429	0.0552	0.0545	0.0448	0.0104	0.0559	0.2353	0.2979
IP	0.0293	0.0110	0.0155	0.0327	0.0289	0.0148	0.0038	0.0421	0.0878	0.1467	0.1594
Prod	0.0362	0.0185	0.0064	0.0147	0.0139	0.0080	0.0007	0.0176	0.0440	0.1115	0.1374
CILI	0.0606	0.0480	0.0292	0.0082	0.0007	0.0019	0.0008	0.0105	0.0314	0.0983	0.1362
CICI	0.0714	0.0574	0.0367	0.0129	0.0048	0.0033	9e-5	0.0130	0.0391	0.1057	0.1333
SZ5	0.0466	0.0345	0.0174	0.0012	0.0037	0.0018	0.0032	0.0160	0.0370	0.1017	0.1313
SZ5u	0.0490	0.0368	0.0188	0.0037	0.0097	0.0114	0.0100	0.0073	0.0214	0.0740	0.1024
5corr	0.0746	0.0587	0.0354	0.0103	0.0015	0.0001	0.0029	0.0131	0.0313	0.1108	0.1540
INF	0.0631	0.0484	0.0266	0.0030	0.0056	0.0053	6e-5	0.0180	0.0485	0.0982	0.1222
INFcorr	0.0595	0.0466	0.0276	0.0067	0.0008	0.0002	0.0051	0.0234	0.0574	0.1394	0.1622

Table B.4: Average absolute deviations (in %) of the model prices and market data given by US Industrials BBB1 for different maturities.

	Maturity	β_0	β_1	R^2
TS	3M	[0.0007, 0.0055]	[0.713, 0.804]	83.89%
	6M	[-0.0028, 0.0018]	[0.800, 0.889]	87.16%
	1Y	[-0.0040, -0.0009]	[0.895, 0.956]	94.58%
	2Y	[-0.0025, -0.0008]	[0.959, 0.991]	98.56%
	3Y	[-0.0017, -0.0005]	[0.998, 1.021]	99.33%
	4Y	[-0.0035, -0.0020]	[1.059, 1.089]	98.95%
	5Y	[-0.0062, -0.0038]	[1.116, 1.164]	97.68%
	7Y	[-0.0084, -0.0041]	[1.152, 1.237]	93.67%
	10Y	[-0.0157, -0.0076]	[1.234, 1.389]	84.35%
	20Y	[-0.0041, 0.0117]	[0.835, 1.107]	48.88%
	25Y	[0.0428, 0.0475]	[0.229, 0.326]	39.47%
A2	3M	[-0.0026, -0.0006]	[1.094, 1.332]	66.33%
	6M	[-0.0023, -0.0009]	[1.121, 1.295]	78.42%
	1Y	[-0.0006, 0.0004]	[0.960, 1.080]	84.53%
	2Y	[0.0005, 0.0010]	[0.890, 0.945]	95.48%
	3Y	[-0.0005, -0.0001]	[0.999, 1.047]	97.09%
	4Y	[-0.0025, -0.0017]	[1.157, 1.245]	93.34%
	5Y	[-0.0051, -0.0036]	[1.345, 1.485]	88.59%
	7Y	[-0.0113, -0.0084]	[1.774, 2.037]	79.85%
	10Y	[-0.0221, -0.0165]	[2.455, 2.949]	69.30%
	20Y	[-0.0520, -0.0330]	[3.790, 5.323]	40.00%
	25Y	[-0.0549, -0.0267]	[3.309, 5.550]	22.77%
BBB1	3M	[-0.0002, 0.0009]	[0.906, 0.998]	88.87%
	6M	[-0.0004, 0.0003]	[0.963, 1.025]	95.07%
	1Y	[-0.0005, -6e-6]	[0.999, 1.038]	98.07%
	2Y	[-0.0002, 0.0003]	[0.982, 1.017]	98.37%
	3Y	[0.0001, 0.0005]	[0.969, 0.995]	99.05%
	4Y	[0.0002, 0.0003]	[0.978, 0.991]	99.78%
	5Y	[-6e-5, -3e-5]	[1.002, 1.004]	99.99%
	7Y	[-0.0013, -0.0008]	[1.045, 1.079]	98.70%
	10Y	[-0.0036, -0.0023]	[1.124, 1.206]	93.91%
	20Y	[-0.0107, -0.0057]	[1.258, 1.530]	66.49%
	25Y	[-0.0125, -0.0049]	[1.201, 1.596]	48.59%

Table B.5: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the GDPn framework.

	Maturity	β_0	β_1	R^2
TS	3M	[0.0029, 0.0066]	[0.732, 0.806]	89.04%
	6M	[-0.0007, 0.0027]	[0.826, 0.896]	91.90%
	1Y	[-0.0024, -0.0004]	[0.928, 0.969]	97.55%
	2Y	[-0.0024, -0.0014]	[1.009, 1.030]	99.49%
	3Y	[-0.0035, -0.0017]	[1.055, 1.093]	98.37%
	4Y	[-0.0070, -0.0043]	[1.136, 1.191]	97.10%
	5Y	[-0.0111, -0.0072]	[1.211, 1.292]	94.78%
	7Y	[-0.0156, -0.0093]	[1.277, 1.405]	89.30%
	10Y	[-0.0261, -0.0147]	[1.387, 1.608]	77.55%
	20Y	[-0.0110, 0.0087]	[0.891, 1.234]	42.12%
	25Y	[0.0426, 0.0475]	[0.230, 0.330]	37.19%
A2	3M	[-0.0007, 2e-5]	[1.012, 1.097]	92.03%
	6M	[-0.0005, -0.0001]	[1.014, 1.057]	97.86%
	1Y	[-3e-5, 6e-5]	[0.992, 1.002]	99.86%
	2Y	[0.0001, 0.0004]	[0.960, 0.984]	99.18%
	3Y	[8e-6, 0.0001]	[0.988, 0.9998]	99.81%
	4Y	[-0.0004, -0.0002]	[1.018, 1.038]	99.49%
	5Y	[-0.0009, -0.0004]	[1.040, 1.085]	97.71%
	7Y	[-0.0019, -0.0009]	[1.078, 1.171]	91.67%
	10Y	[-0.0033, -0.0014]	[1.110, 1.267]	81.21%
	20Y	[-0.0049, -0.0002]	[0.994, 1.319]	48.82%
	25Y	[-0.0033, 0.0031]	[0.787, 1.207]	29.80%
BBB1	3M	[-0.0003, 0.0006]	[0.925, 1.002]	92.34%
	6M	[-0.0002, 0.0002]	[0.973, 1.010]	98.18%
	1Y	[-5e-5, -7e-6]	[1.001, 1.005]	99.98%
	2Y	[9e-6, 0.0002]	[0.987, 1.005]	99.55%
	3Y	[6e-5, 0.0002]	[0.987, 0.999]	99.81%
	4Y	[-1e-5, 1e-6]	[0.9996, 1.0004]	100%
	5Y	[-0.0004, -0.0002]	[1.007, 1.022]	99.72%
	7Y	[-0.0013, -0.0007]	[1.037, 1.081]	97.74%
	10Y	[-0.0029, -0.0016]	[1.093, 1.183]	92.42%
	20Y	[-0.0063, -0.0018]	[1.188, 1.460]	64.09%
	25Y	[-0.0062, 0.0001]	[1.124, 1.517]	46.06%

Table B.6: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the GDPr framework.

	Maturity	β_0	β_1	R^2
TS	3M	[-0.0087, -0.0025]	[0.880, 1.004]	81.36%
	6M	[-0.0122, -0.0062]	[0.967, 1.086]	84.86%
	1Y	[-0.0120, -0.0080]	[1.041, 1.121]	93.16%
	2Y	[-0.0066, -0.0046]	[1.031, 1.071]	98.08%
	3Y	[-0.0019, -0.0012]	[1.005, 1.019]	99.73%
	4Y	[-0.0002, 0.0004]	[0.9997, 1.012]	99.81%
	5Y	[0.0005, 0.0019]	[0.990, 1.018]	98.98%
	7Y	[0.0042, 0.0070]	[0.921, 0.976]	95.81%
	10Y	[0.0056, 0.0105]	[0.877, 0.972]	87.76%
	20Y	[0.0173, 0.0268]	[0.571, 0.735]	54.55%
	25Y	[0.0446, 0.0485]	[0.207, 0.286]	42.72%
A2	3M	[-0.0003, 0.0005]	[0.974, 1.070]	89.56%
	6M	[-0.0004, 4e-5]	[1.004, 1.060]	96.33%
	1Y	[-0.0003, 0.0002]	[0.977, 1.027]	96.83%
	2Y	[-2e-5, 0.0004]	[0.950, 0.991]	97.68%
	3Y	[-0.0002, 4e-5]	[0.992, 1.023]	98.83%
	4Y	[-0.0008, -0.0003]	[1.046, 1.090]	97.76%
	5Y	[-0.0014, -0.0007]	[1.095, 1.165]	95.15%
	7Y	[-0.0025, -0.0012]	[1.173, 1.297]	88.08%
	10Y	[-0.0036, -0.0015]	[1.233, 1.433]	76.99%
	20Y	[-0.0033, 0.0014]	[1.061, 1.456]	43.46%
	25Y	[-0.0008, 0.0054]	[0.784, 1.280]	24.61%
BBB1	3M	[-0.0001, 0.0006]	[0.918, 0.979]	94.80%
	6M	[-0.0001, 0.0001]	[0.981, 1.001]	99.48%
	1Y	[-0.0002, 0.0001]	[1.003, 1.031]	99.02%
	2Y	[0.0001, 0.0007]	[0.968, 1.024]	96.05%
	3Y	[0.0005, 0.0012]	[0.937, 0.991]	95.92%
	4Y	[0.0007, 0.0013]	[0.929, 0.973]	97.21%
	5Y	[0.0006, 0.0011]	[0.940, 0.971]	98.59%
	7Y	[0.0001, 0.0002]	[0.987, 0.995]	99.91%
	10Y	[-0.0017, -0.0011]	[1.047, 1.085]	98.27%
	20Y	[-0.0069, -0.0033]	[1.140, 1.335]	75.21%
	25Y	[-0.0080, -0.0024]	[1.091, 1.386]	57.08%

Table B.7: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the CPI framework.

	Maturity	β_0	β_1	R^2
TS	3M	[0.0002, 0.0036]	[0.814, 0.887]	91.16%
	6M	[-0.0032, -0.0001]	[0.906, 0.972]	93.79%
	1Y	[-0.0038, -0.0023]	[0.992, 1.024]	98.67%
	2Y	[-0.0019, -0.0009]	[1.023, 1.042]	99.53%
	3Y	[-0.0010, 0.0013]	[1.015, 1.065]	97.12%
	4Y	[-0.0024, 0.0009]	[1.049, 1.118]	94.97%
	5Y	[-0.0041, 0.0004]	[1.078, 1.171]	91.52%
	7Y	[-0.0045, 0.0020]	[1.067, 1.200]	84.54%
	10Y	[-0.0080, 0.0026]	[1.066, 1.274]	70.45%
	20Y	[0.0054, 0.0220]	[0.665, 0.956]	37.04%
	25Y	[0.0437, 0.0484]	[0.213, 0.308]	36.16%
A2	3M	[-0.0007, 2e-5]	[0.982, 1.062]	92.58%
	6M	[-0.0004, -0.0002]	[1.012, 1.039]	99.09%
	1Y	[0.0002, 0.0005]	[0.957, 0.994]	98.12%
	2Y	[0.0006, 0.0008]	[0.922, 0.952]	98.68%
	3Y	[-8e-5, -5e-5]	[1.004, 1.007]	99.99%
	4Y	[-0.0017, -0.0013]	[1.102, 1.141]	98.43%
	5Y	[-0.0036, -0.0028]	[1.219, 1.298]	95.12%
	7Y	[-0.0085, -0.0065]	[1.478, 1.645]	86.77%
	10Y	[-0.0168, -0.0126]	[1.871, 2.198]	74.51%
	20Y	[-0.0405, -0.0250]	[2.576, 3.599]	40.73%
	25Y	[-0.0429, -0.0195]	[2.206, 3.701]	22.76%
BBB1	3M	[-0.0002, 0.0004]	[0.950, 1.005]	95.90%
	6M	[-0.0002, 3e-5]	[0.996, 1.014]	99.57%
	1Y	[-0.0002, 0.0001]	[0.994, 1.023]	98.94%
	2Y	[0.0001, 0.0006]	[0.956, 0.998]	97.63%
	3Y	[0.0003, 0.0007]	[0.946, 0.978]	98.60%
	4Y	[0.0003, 0.0004]	[0.967, 0.982]	99.70%
	5Y	[-0.0001, -5e-5]	[1.003, 1.008]	99.97%
	7Y	[-0.0017, -0.0011]	[1.076, 1.120]	97.95%
	10Y	[-0.0046, -0.0029]	[1.210, 1.319]	90.98%
	20Y	[-0.0132, -0.0066]	[1.503, 1.910]	57.07%
	25Y	[-0.0150, -0.0052]	[1.444, 2.051]	38.51%

Table B.8: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the IP framework.

	Maturity	β_0	β_1	R^2
TS	3M	[0.0021, 0.0068]	[0.702, 0.793]	83.47%
	6M	[-0.0014, 0.0031]	[0.788, 0.877]	86.85%
	1Y	[-0.0027, 0.0003]	[0.884, 0.944]	94.56%
	2Y	[-0.0013, 0.0002]	[0.949, 0.978]	98.81%
	3Y	[-0.0003, 7e-5]	[0.995, 1.004]	99.91%
	4Y	[-0.0023, -0.0013]	[1.052, 1.072]	99.50%
	5Y	[-0.0050, -0.0030]	[1.106, 1.147]	98.31%
	7Y	[-0.0072, -0.0034]	[1.141, 1.219]	94.63%
	10Y	[-0.0144, -0.0069]	[1.224, 1.369]	85.83%
	20Y	[-0.0038, 0.0114]	[0.841, 1.104]	50.88%
	25Y	[0.0429, 0.0475]	[0.228, 0.322]	39.50%
A2	3M	[-0.0001, 0.0006]	[0.914, 0.992]	91.95%
	6M	[-0.0004, -0.0001]	[1.005, 1.047]	97.85%
	1Y	[0.0003, 0.0009]	[0.905, 0.986]	91.03%
	2Y	[0.0015, 0.0021]	[0.779, 0.849]	90.94%
	3Y	[0.0010, 0.0014]	[0.861, 0.897]	97.82%
	4Y	[8e-5, 0.0001]	[0.987, 0.993]	99.95%
	5Y	[-0.0013, -0.0010]	[1.094, 1.122]	99.17%
	7Y	[-0.0044, -0.0034]	[1.288, 1.378]	94.29%
	10Y	[-0.0090, -0.0067]	[1.527, 1.713]	85.05%
	20Y	[-0.0196, -0.0118]	[1.788, 2.327]	52.37%
	25Y	[-0.0200, -0.0082]	[1.528, 2.305]	31.47%
BBB1	3M	[0.0014, 0.0022]	[0.793, 0.873]	89.21%
	6M	[0.0006, 0.0011]	[0.902, 0.948]	96.90%
	1Y	[-0.0002, -2e-5]	[1.002, 1.013]	99.83%
	2Y	[-0.0003, 0.0002]	[0.980, 1.022]	97.79%
	3Y	[-2e-5, 0.0004]	[0.963, 0.999]	98.24%
	4Y	[0.0001, 0.0003]	[0.976, 0.995]	99.50%
	5Y	[-3e-5, -1e-5]	[1.001, 1.002]	100%
	7Y	[-0.0008, -0.0003]	[1.020, 1.058]	98.26%
	10Y	[-0.0018, -0.0004]	[1.031, 1.119]	91.89%
	20Y	[-0.0017, 0.0021]	[0.883, 1.100]	61.16%
	25Y	[0.0001, 0.0050]	[0.736, 1.012]	43.17%

Table B.9: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the Prod framework.

	Maturity	β_0	β_1	R^2
	3M	[-0.0040, 0.0012]	[0.824, 0.930]	83.78%
	6M	[-0.0074, -0.0023]	[0.908, 1.011]	86.81%
	1Y	[-0.0077, -0.0041]	[0.983, 1.055]	93.89%
	2Y	[-0.0038, -0.0021]	[1.000, 1.035]	98.47%
T	3Y	[-0.0005, -0.0001]	[1.001, 1.008]	99.93%
S	4Y	[-0.0002, 0.0010]	[1.006, 1.030]	99.27%
	5Y	[-0.0005, 0.0016]	[1.012, 1.054]	97.84%
	7Y	[0.0017, 0.0051]	[0.971, 1.039]	94.23%
	10Y	[0.0014, 0.0074]	[0.946, 1.062]	85.06%
	20Y	[0.0126, 0.0237]	[0.627, 0.820]	51.61%
	25Y	[0.0437, 0.0479]	[0.221, 0.305]	42.78%
	3M	[-0.0017, 0.0001]	[1.007, 1.234]	64.85%
	6M	[-0.0011, 0.0005]	[0.952, 1.145]	68.94%
	1Y	[-0.0006, 0.0005]	[0.943, 1.073]	82.03%
	2Y	[-0.0002, 0.0003]	[0.970, 1.019]	96.92%
A	3Y	[-0.0001, 0.0001]	[0.990, 1.005]	99.72%
2	4Y	[-2e-6, 1e-6]	[0.9999, 1.0002]	100%
	5Y	[-0.0001, 3e-5]	[0.997, 1.008]	99.84%
	7Y	[-0.0003, 0.0001]	[0.990, 1.024]	98.53%
	10Y	[-0.0006, 0.0002]	[0.979, 1.046]	94.66%
	20Y	[-0.0007, 0.0016]	[0.888, 1.053]	72.27%
	25Y	[0.0004, 0.0039]	[0.766, 1.002]	51.35%
	3M	[0.0014, 0.0027]	[0.741, 0.861]	77.08%
	6M	[0.0009, 0.0019]	[0.824, 0.912]	88.17%
	1Y	[0.0004, 0.0010]	[0.915, 0.962]	96.72%
B	2Y	[0.0001, 0.0002]	[0.983, 0.996]	99.77%
B	3Y	[-1e-5, 4e-6]	[0.9998, 1.001]	100%
B	4Y	[-3e-5, 1e-5]	[0.999, 1.002]	99.99%
1	5Y	[-1e-5, 8e-6]	[0.999, 1.001]	100%
	7Y	[-0.0002, 4e-5]	[0.999, 1.014]	99.70%
	10Y	[-0.0006, 0.0001]	[1.003, 1.047]	97.62%
	20Y	[-0.0011, 0.0016]	[0.940, 1.099]	75.64%
	25Y	[-0.0004, 0.0036]	[0.846, 1.072]	57.69%

Table B.10: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the CILI framework.

	Maturity	β_0	β_1	R^2
T S	3M	[-0.0045, 0.0010]	[0.827, 0.939]	82.53%
	6M	[-0.0079, -0.0026]	[0.914, 1.021]	86.06%
	1Y	[-0.0082, -0.0046]	[0.993, 1.064]	94.05%
	2Y	[-0.0039, -0.0022]	[1.003, 1.037]	98.62%
	3Y	[-0.0001, -4e-5]	[1.000, 1.002]	100%
	4Y	[0.0003, 0.0012]	[1.002, 1.021]	99.53%
	5Y	[0.0001, 0.0019]	[1.007, 1.044]	98.36%
	7Y	[0.0024, 0.0057]	[0.963, 1.028]	94.63%
	10Y	[0.0019, 0.0077]	[0.945, 1.057]	85.72%
	20Y	[0.0133, 0.0243]	[0.620, 0.811]	51.54%
	25Y	[0.0444, 0.0485]	[0.209, 0.292]	40.51%
A 2	3M	[-0.0027, -0.0005]	[1.059, 1.321]	60.88%
	6M	[-0.0022, -0.0003]	[1.031, 1.254]	66.58%
	1Y	[-0.0017, -0.0004]	[1.037, 1.186]	80.77%
	2Y	[-0.0007, -0.0002]	[1.017, 1.076]	95.93%
	3Y	[-0.0002, 0.0001]	[0.989, 1.018]	98.87%
	4Y	[2e-5, 0.0002]	[0.980, 0.999]	99.54%
	5Y	[0.0001, 0.0002]	[0.985, 0.995]	99.84%
	7Y	[-0.0002, 1e-5]	[0.998, 1.015]	99.60%
	10Y	[-0.0008, -0.0002]	[1.012, 1.055]	97.67%
	20Y	[-0.0012, 0.0002]	[0.972, 1.073]	88.60%
	25Y	[-0.0007, 0.0014]	[0.913, 1.052]	79.15%
B B B 1	3M	[0.0003, 0.0017]	[0.814, 0.932]	80.42%
	6M	[3e-5, 0.0010]	[0.885, 0.969]	90.36%
	1Y	[-0.0001, 0.0004]	[0.954, 0.997]	97.47%
	2Y	[-0.0001, 0.0001]	[0.992, 1.008]	99.66%
	3Y	[-0.0001, 3e-5]	[0.998, 1.008]	99.88%
	4Y	[-4e-5, 3e-5]	[0.999, 1.004]	99.96%
	5Y	[-1e-6, 7e-7]	[0.9999, 1.0001]	100%
	7Y	[-0.0002, 2e-5]	[0.994, 1.007]	99.76%
	10Y	[-0.0005, 1e-5]	[0.986, 1.022]	98.34%
	20Y	[-0.0007, 0.0014]	[0.913, 1.031]	83.60%
	25Y	[0.0001, 0.0031]	[0.842, 1.008]	70.00%

Table B.11: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the CICI framework.

	Maturity	β_0	β_1	R^2
TS	3M	[0.0054, 0.0076]	[0.807, 0.857]	95.43%
	6M	[0.0017, 0.0036]	[0.902, 0.944]	97.33%
	1Y	[-0.0001, -1e-6]	[0.9996, 1.0007]	100%
	2Y	[-0.0032, -0.001]	[1.038, 1.085]	97.50%
	3Y	[-0.0070, -0.0026]	[1.076, 1.168]	91.87%
	4Y	[-0.0142, -0.0077]	[1.174, 1.307]	86.82%
	5Y	[-0.0223, -0.0124]	[1.257, 1.453]	78.34%
	7Y	[-0.0279, -0.0089]	[1.174, 1.532]	51.88%
	10Y	[0.0193, 0.0494]	[0.122, 0.660]	3.84%
	20Y	[0.0613, 0.0831]	[-0.422, -0.025]	2.34%
	25Y	[0.0485, 0.0517]	[0.220, 0.317]	36.82%
A2	3M	[-0.0007, 0.0006]	[0.949, 1.101]	77.56%
	6M	[-0.0006, 0.0004]	[0.955, 1.074]	84.51%
	1Y	[-0.0004, 0.0001]	[0.986, 1.050]	95.10%
	2Y	[-9e-6, -5e-7]	[0.99999, 1.001]	100%
	3Y	[4e-5, 0.0001]	[0.985, 0.996]	99.85%
	4Y	[-1e-7, 2e-7]	[0.99998, 1.000]	100%
	5Y	[-0.0003, -0.0001]	[1.010, 1.027]	99.63%
	7Y	[-0.0009, -0.0004]	[1.030, 1.084]	96.64%
	10Y	[-0.0019, -0.0007]	[1.045, 1.151]	89.06%
	20Y	[-0.0022, 0.0013]	[0.899, 1.143]	56.96%
	25Y	[-0.0003, 0.0046]	[0.702, 1.023]	35.29%
BBB1	3M	[0.0012, 0.0023]	[0.788, 0.892]	83.05%
	6M	[0.0008, 0.0016]	[0.867, 0.938]	92.55%
	1Y	[0.0003, 0.0007]	[0.949, 0.979]	98.70%
	2Y	[-6e-6, 2e-5]	[0.998, 1.001]	99.99%
	3Y	[-0.0001, 9e-7]	[0.999, 1.005]	99.94%
	4Y	[-4e-5, 5e-6]	[0.999, 1.002]	99.99%
	5Y	[-0.0001, 2e-5]	[0.999, 1.005]	99.96%
	7Y	[-0.0003, 0.0001]	[0.999, 1.025]	99.14%
	10Y	[-0.0007, 0.0003]	[0.995, 1.054]	95.77%
	20Y	[0.0002, 0.0030]	[0.866, 1.032]	71.24%
	25Y	[0.0019, 0.0057]	[0.743, 0.962]	53.36%

Table B.12: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the SZ5 framework.

	Maturity	β_0	β_1	R^2
TS	3M	[0.0043, 0.0060]	[0.861, 0.901]	97.38%
	6M	[0.0006, 0.0020]	[0.950, 0.983]	98.49%
	1Y	[-0.0011, -0.0005]	[1.009, 1.022]	99.77%
	2Y	[-0.0011, -0.0004]	[1.010, 1.025]	99.70%
	3Y	[-0.0004, 0.0004]	[0.994, 1.010]	99.66%
	4Y	[-0.0005, -0.0002]	[1.004, 1.011]	99.93%
	5Y	[-0.0003, -0.0002]	[1.002, 1.005]	99.99%
	7Y	[0.0043, 0.0053]	[0.905, 0.925]	99.36%
	10Y	[0.0101, 0.0119]	[0.801, 0.833]	98.03%
	20Y	[0.0353, 0.0387]	[0.390, 0.452]	77.79%
	25Y	[0.0523, 0.0538]	[0.145, 0.187]	54.00%
A2	3M	[-0.0007, -0.0002]	[1.043, 1.095]	96.98%
	6M	[-0.0003, 0.0001]	[0.993, 1.031]	98.24%
	1Y	[0.0003, 0.0006]	[0.918, 0.960]	97.36%
	2Y	[0.0006, 0.0010]	[0.871, 0.920]	96.13%
	3Y	[0.0004, 0.0007]	[0.911, 0.946]	98.17%
	4Y	[0.0001, 0.0002]	[0.973, 0.990]	99.61%
	5Y	[-0.0005, -0.0002]	[1.027, 1.049]	99.43%
	7Y	[-0.0017, -0.0010]	[1.117, 1.183]	95.80%
	10Y	[-0.0036, -0.0021]	[1.237, 1.374]	87.40%
	20Y	[-0.0079, -0.0032]	[1.401, 1.787]	56.26%
	25Y	[-0.0076, -0.0005]	[1.241, 1.802]	35.72%
BBB1	3M	[0.0011, 0.0023]	[0.788, 0.894]	82.76%
	6M	[0.0007, 0.0015]	[0.868, 0.941]	92.21%
	1Y	[0.0002, 0.0006]	[0.953, 0.986]	98.51%
	2Y	[-0.0001, -7e-6]	[1.002, 1.012]	99.87%
	3Y	[-0.0002, 0.0001]	[0.992, 1.012]	99.48%
	4Y	[-0.0001, 0.0002]	[0.981, 1.003]	99.36%
	5Y	[-4e-5, 0.0002]	[0.979, 0.998]	99.52%
	7Y	[-0.0001, 4e-5]	[0.994, 1.007]	99.77%
	10Y	[-0.0008, -0.0002]	[1.020, 1.054]	98.62%
	20Y	[-0.0023, 0.0003]	[1.030, 1.183]	79.78%
	25Y	[-0.0021, 0.0018]	[0.967, 1.196]	62.80%

Table B.13: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the SZ5u framework.

	Maturity	β_0	β_1	R^2
TS	3M	[0.0047, 0.0057]	[0.884, 0.908]	99.03%
	6M	[0.0005, 0.0011]	[0.982, 0.995]	99.77%
	1Y	[-0.0023, -0.0015]	[1.034, 1.051]	99.64%
	2Y	[-0.0030, -0.0020]	[1.039, 1.059]	99.50%
	3Y	[-0.0018, -0.0010]	[1.016, 1.034]	99.61%
	4Y	[-0.0013, -0.0008]	[1.013, 1.022]	99.89%
	5Y	[-0.0002, -2e-5]	[1.000, 1.004]	99.97%
	7Y	[0.0054, 0.0065]	[0.890, 0.910]	99.35%
	10Y	[0.0115, 0.0134]	[0.783, 0.818]	97.49%
	20Y	[0.0340, 0.0380]	[0.395, 0.467]	73.25%
	25Y	[0.0509, 0.0528]	[0.154, 0.200]	52.20%
A2	3M	[-0.0015, 0.0008]	[0.942, 1.233]	51.36%
	6M	[-0.0007, 0.0013]	[0.872, 1.118]	55.15%
	1Y	[-0.0003, 0.0011]	[0.888, 1.049]	73.22%
	2Y	[4e-5, 0.0005]	[0.947, 1.003]	95.78%
	3Y	[5e-5, 0.0002]	[0.985, 0.997]	99.8%
	4Y	[-1e-5, -6e-7]	[1.000, 1.001]	100%
	5Y	[-2e-5, -8e-6]	[1.000, 1.002]	100%
	7Y	[0.0001, 0.0003]	[0.973, 0.991]	99.58%
	10Y	[0.0008, 0.0013]	[0.907, 0.953]	96.97%
	20Y	[0.0045, 0.0059]	[0.670, 0.784]	75.42%
	25Y	[0.0068, 0.0088]	[0.535, 0.691]	53.82%
BBB1	3M	[-0.0001, 0.0014]	[0.831, 0.966]	77.00%
	6M	[-0.0003, 0.0008]	[0.898, 0.995]	87.89%
	1Y	[-0.0003, 0.0004]	[0.955, 1.007]	96.50%
	2Y	[-5e-5, 0.0001]	[0.986, 1.001]	99.69%
	3Y	[5e-6, 4e-5]	[0.997, 1.000]	99.99%
	4Y	[-3e-6, -9e-7]	[1.0001, 1.0003]	100%
	5Y	[5e-6, 5e-5]	[0.995, 0.998]	99.98%
	7Y	[0.0002, 0.0005]	[0.965, 0.981]	99.64%
	10Y	[0.0010, 0.0016]	[0.897, 0.934]	97.87%
	20Y	[0.0047, 0.0063]	[0.673, 0.769]	80.79%
	25Y	[0.0066, 0.0088]	[0.573, 0.699]	65.52%

Table B.14: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the 5corr framework.

	Maturity	β_0	β_1	R^2
TS	3M	[0.0027, 0.0068]	[0.821, 0.915]	86.34%
	6M	[-0.0010, 0.0029]	[0.913, 1.002]	89.78%
	1Y	[-0.0026, -0.0002]	[0.996, 1.049]	96.63%
	2Y	[-0.0009, -0.0001]	[1.007, 1.024]	99.61%
	3Y	[0.0006, 0.0018]	[0.969, 0.993]	99.25%
	4Y	[0.0007, 0.0027]	[0.947, 0.985]	97.95%
	5Y	[0.0011, 0.0041]	[0.913, 0.970]	95.37%
	7Y	[0.0066, 0.0111]	[0.783, 0.864]	88.69%
	10Y	[0.0145, 0.0206]	[0.626, 0.733]	75.36%
	20Y	[0.0350, 0.0417]	[0.316, 0.431]	44.26%
	25Y	[0.0379, 0.0441]	[0.269, 0.381]	38.69%
A2	3M	[-0.0018, -0.0003]	[1.078, 1.272]	73.51%
	6M	[-0.0015, -0.0003]	[1.060, 1.209]	81.29%
	1Y	[-0.0010, -0.0003]	[1.044, 1.123]	93.46%
	2Y	[-0.0001, -4e-5]	[1.004, 1.010]	99.95%
	3Y	[4e-5, 0.0002]	[0.977, 0.994]	99.61%
	4Y	[3e-5, 0.0002]	[0.9837, 0.996]	99.80%
	5Y	[-7e-7, 2e-7]	[1.0000, 1.0001]	100%
	7Y	[-0.0004, -0.0001]	[1.003, 1.032]	98.98%
	10Y	[-0.0009, -0.0001]	[0.999, 1.063]	95.15%
	20Y	[-0.0017, 0.0004]	[0.956, 1.100]	79.28%
	25Y	[-0.0012, 0.0019]	[0.891, 1.097]	63.61%
BBB1	3M	[-0.0002, 0.0011]	[0.889, 1.004]	83.60%
	6M	[-0.0004, 0.0005]	[0.950, 1.026]	92.76%
	1Y	[-0.0003, 0.0001]	[0.995, 1.027]	98.70%
	2Y	[-0.0001, -2e-5]	[1.002, 1.005]	99.99%
	3Y	[8e-6, 0.0001]	[0.993, 1.000]	99.95%
	4Y	[2e-5, 0.0001]	[0.994, 0.999]	99.97%
	5Y	[-1e-6, -3e-7]	[1.00001, 1.0001]	100%
	7Y	[-0.0003, -4e-5]	[1.003, 1.017]	99.75%
	10Y	[-0.0006, 3e-5]	[0.999, 1.036]	98.25%
	20Y	[0.0002, 0.0023]	[0.912, 1.034]	82.67%
	25Y	[0.0017, 0.0046]	[0.829, 1.002]	67.90%

Table B.15: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the INF framework.

	Maturity	β_0	β_1	R^2
TS	3M	[-0.0143, -0.0073]	[1.041, 1.190]	80.77%
	6M	[-0.0174, -0.0108]	[1.127, 1.266]	84.77%
	1Y	[-0.0156, -0.0114]	[1.171, 1.260]	93.39%
	2Y	[-0.0068, -0.0050]	[1.084, 1.120]	98.63%
	3Y	[0.0007, 0.0013]	[0.982, 0.995]	99.79%
	4Y	[0.0046, 0.0058]	[0.904, 0.928]	99.12%
	5Y	[0.0078, 0.0098]	[0.833, 0.873]	97.12%
	7Y	[0.0157, 0.0188]	[0.674, 0.732]	91.70%
	10Y	[0.0238, 0.0281]	[0.520, 0.598]	79.46%
	20Y	[0.0390, 0.0443]	[0.265, 0.354]	47.69%
	25Y	[0.0401, 0.0452]	[0.235, 0.322]	43.14%
A2	3M	[-0.0008, 0.0005]	[0.960, 1.123]	75.52%
	6M	[-0.0008, 0.0003]	[0.974, 1.100]	83.51%
	1Y	[-0.0006, -1e-5]	[1.004, 1.071]	94.75%
	2Y	[-7e-6, 2e-6]	[0.9997, 1.001]	100%
	3Y	[0.0001, 0.0002]	[0.973, 0.987]	99.73%
	4Y	[0.00007, 0.0001]	[0.985, 0.993]	99.91%
	5Y	[-0.0001, -6e-5]	[1.006, 1.014]	99.91%
	7Y	[-0.0008, -0.0005]	[1.041, 1.077]	98.53%
	10Y	[-0.0019, -0.0010]	[1.081, 1.153]	94.79%
	20Y	[-0.0027, -0.0007]	[1.064, 1.203]	83.36%
	25Y	[-0.0019, 0.0008]	[0.992, 1.177]	72.11%
BBB1	3M	[0.0002, 0.0016]	[0.837, 0.953]	81.66%
	6M	[-0.0001, 0.0009]	[0.910, 0.990]	91.39%
	1Y	[-0.0002, 0.0002]	[0.975, 1.013]	98.10%
	2Y	[-0.0001, -4e-5]	[1.002, 1.009]	99.94%
	3Y	[-6e-6, 1e-5]	[0.999, 1.001]	100%
	4Y	[3e-6, 5e-6]	[0.9996, 0.9998]	100%
	5Y	[-0.0001, -7e-5]	[1.004, 1.009]	99.97%
	7Y	[-0.0007, -0.0004]	[1.023, 1.042]	99.55%
	10Y	[-0.0016, -0.0009]	[1.053, 1.096]	97.85%
	20Y	[-0.0032, -0.0009]	[1.060, 1.192]	84.69%
	25Y	[-0.0029, 0.0003]	[1.019, 1.203]	73.44%

Table B.16: R^2 and confidence intervals with respect to a significance level of 5% for the parameters β_0 and β_1 of the linear regression model $P^{Market}(t,T) = \beta_0 + \beta_1 P^{Model}(t,T) + \epsilon$ applied to zero rates within the INFcorr framework.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	x	-	-	-	x^{***}	x^{*}	x	x^{*}_{+}	x^{**}	x^{**}	x^{*}
6M	-	x	x_{+}	-	-	x^{***}	x^{*}	x	x^{***}	x^{**}	x	x
1Y	-	x^{*}	x	-	-	x	x^{*}_{+}	x	x^{**}	x^{**}	x^{*}	x
2Y	-	-	x	-	-	x	x^{*}	x	-	x^{***}_{+}	x^{*}	x^{*}
3Y	-	x^{***}_{+}	x^{*}	x	-	x^{**}	x^{*}	-	x^{**}	x^{**}_{+}	x^{*}	x_{+}
4Y	-	x	x	-	-	x	-	-	-	x^{*}	x^{*}	x
5Y	-	x	x	-	-	x	-	-	-	x^{**}_{+}	x	-
7Y	-	x	x	-	-	x	-	x	-	x^{**}_{+}	x_{+}	-
10Y	-	x	x	-	-	x	-	x	-	-	-	-
20Y	-	x	x	-	-	x_{+}	-	-	-	-	x^{*}_{+}	-
25Y	-	-	-	-	-	x	x^{**}_{+}	-	-	-	x^{***}	-

Table B.17: Test of the standardized innovations within the GDPn framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	x	x_{+}	-	x	x_{+}^{***}	x^{*}	-	x^{*}	x^{***}	x^{**}	x^{*}
6M	-	x	x_{+}	-	-	-	x_{+}^{***}	x	x^{*}	x^{**}	x^{*}	x
1Y	-	x^{**}	x	-	-	-	x^{***}	x^{*}	-	-	-	-
2Y	-	-	x^{*}	-	x^{*}	x^{**}	x^{***}	x	x	x^{*}	x^{**}	x
3Y	-	x	x	-	-	-	x^{**}	x	x^{**}	x^{***}	x^{**}	-
4Y	-	x	x_{+}	-	-	x^{***}	x^{*}	x	-	x^{**}	x_{+}^{***}	-
5Y	-	x	x	-	-	x	x^{*}	x	-	x^{**}	x^{*}	-
7Y	-	x	x	-	-	x	x^{*}	x	-	x^{**}	x	-
10Y	-	x	x	-	-	x	x_{+}^{*}	x	-	-	x	x^{*}
20Y	-	x	x	-	-	-	x	x	-	x	x_{+}	-
25Y	-	-	x_{+}^{***}	-	-	-	x	x	-	x	x_{+}	-

Table B.18: Test of the standardized innovations within the GDPr framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	x^{**}	x	-	x	x^{***}_{+}	x^{**}	x^{*}_{+}	x	x^{**}_{+}	x^{**}	x^{***}
6M	-	x^{***}	x	-	x	x^{***}	x^{*}	x	x^{*}	x^{***}	x	x
1Y	-	-	x^{***}	-	-	x^{***}	x^{*}	x^{**}_{+}	-	-	-	-
2Y	-	-	x	-	x_{+}	x^{**}_{+}	x^{*}	-	-	-	x	-
3Y	x^{***}_{+}	x^{***}_{+}	x^{*}	x_{+}	x	x^{**}	x^{**}	x	-	x^{***}_{+}	x^{*}	-
4Y	-	x	x	-	-	x^{**}_{+}	x	-	x	x_{+}	x^{*}	-
5Y	-	x	x	-	-	-	x	-	x^{***}	x^{*}_{+}	x^{*}	x
7Y	-	x^{*}	x_{+}	-	-	-	x^{**}	-	-	-	x^{*}	-
10Y	-	x	x_{+}	-	-	x^{***}	x^{*}_{+}	-	-	-	x_{+}	-
20Y	-	x^{***}	-	-	-	x	x	-	-	-	x^{***}	-
25Y	-	-	-	-	-	x	x	-	-	-	x^{***}	x^{***}

Table B.19: Test of the standardized innovations within the CPI framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	x	x_+	-	x	x^{***}	x^*	x	x	x^{**}_+	x^*	x
6M	-	x^*	x	-	x	x^{**}	x^{**}	x	x^*	x^{**}	x^*	x^*
1Y	-	-	x^*	-	-	x	-	-	-	-	x^{***}	x^{**}
2Y	-	x	x	-	x^*	x^*	x^{***}	-	-	-	x	x
3Y	-	x^*	x	-	-	x	-	-	x	x^{**}_+	x^{**}	x^*_+
4Y	-	x	x_+	-	-	x	-	-	-	x^{***}_+	x^*	-
5Y	-	x	x_+	-	-	x	-	-	-	-	x	x^{***}_+
7Y	-	x	x_+	-	-	x	-	-	-	-	-	x
10Y	-	x	x_+	-	-	x	-	-	-	-	-	x^{**}
20Y	-	x	x	-	-	x^{***}	-	-	-	x	-	-
25Y	-	-	-	-	-	-	-	x^{***}	-	x	-	-

Table B.20: Test of the standardized innovations within the IP framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	x	-	-	x	x^{**}	x^{*}	x	-	-	x^{*}	x
6M	-	x	x	-	-	x^{**}	x_{+}^{***}	x	x^{**}	x^{**}	x	x
1Y	-	x^{*}	x	-	-	x	-	-	-	-	x^{*}	x
2Y	-	-	x	-	-	x	-	-	-	-	x	x
3Y	-	x^{*}	x^{*}	-	-	x^{*}	x^{*}	x_{+}	-	-	x	x
4Y	-	x	x_{+}	-	-	x	-	-	x	x^{*}	x^{*}	x
5Y	-	x	x	-	-	x	-	-	-	-	x	x
7Y	-	x	x^{***}	-	-	x	-	-	-	-	x_{+}	x
10Y	-	x	x	-	-	x	-	-	-	-	-	x
20Y	-	x	x	-	-	-	-	-	-	-	-	x
25Y	-	-	-	-	-	-	x_{+}	x	-	x_{+}	-	x

Table B.21: Test of the standardized innovations within the Prod framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	x^{**}	x	-	-	-	x	x_{+}	-	x^{***}	x^{**}	x
6M	-	x^{***}	x	-	-	-	x	x	x^{**}	x^{**}	x^{**}	x
1Y	-	-	x^{***}	-	-	x^{*}	x^{*}	x	x	x^{***}	x^{*}	x
2Y	-	-	x^{*}	-	x_{+}	-	x^{*}	x	x^{*}	-	x^{***}	x
3Y	-	x^{*}	x^{*}	-	-	x^{***}	x^{*}	x	x_{+}^{***}	x_{+}^{***}	x^{*}	x
4Y	-	x	x	-	x	x_{+}^{***}	x^{*}	x	x	x	x^{**}	x
5Y	-	x	x_{+}	-	-	-	x^{*}	x	x	x	x^{**}	x
7Y	-	x	x	-	-	x_{+}^{***}	-	x	-	-	x_{+}	-
10Y	-	x	x_{+}	-	-	x^{**}	-	x	-	-	x_{+}	x_{+}^{***}
20Y	-	x	x	-	-	x	x	x	-	x	-	-
25Y	-	-	-	-	-	x	x_{+}	x^{*}	-	x	-	-

Table B.22: Test of the standardized innovations within the CILI framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	x^{*}	x	-	-	-	x	x	x^{**}_{+}	x^{**}_{+}	x^{***}	-
6M	-	x^{***}	x	-	-	x^{*}	x	x	x^{*}	x^{**}	x^{**}	x^{***}_{+}
1Y	-	-	x^{***}	-	-	x	x^{*}	x	x	x^{***}	x^{*}	x^{***}
2Y	-	-	x^{*}	-	x	x	x^{*}	x	x	x^{***}_{+}	x^{**}	x
3Y	-	x^{*}	x^{*}	-	-	x	x^{*}	x	x	x^{***}	x^{*}	x^{**}_{+}
4Y	-	x	-	-	-	x	x^{*}	x	x	x	x^{*}	x
5Y	-	x	-	-	x	x	x^{*}	x	x^{*}	x^{**}	x^{**}	-
7Y	-	x	x_{+}	-	-	x	x	x^{**}	-	x^{***}_{+}	x^{**}	-
10Y	-	x	x_{+}	-	-	x	x	x^{*}_{+}	-	-	x^{*}	-
20Y	-	x	-	-	-	x	x^{*}	x^{*}	-	x	x	x
25Y	-	-	-	-	-	x	x^{*}	x	-	x	x	x^{***}

Table B.23: Test of the standardized innovations within the CICI framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	x	x	x	-	-	x^{***}	-	-	x^{**}	x^{**}_{+}	x_{+}
6M	-	x	x^{*}	-	x	x^{***}	x^{***}	x	x^{*}_{+}	x^{**}	x^{**}	-
1Y	-	x^{***}	x^{*}	x	-	-	x^{***}	x	x	x^{***}	x^{**}	-
2Y	-	x^{*}	-	-	-	-	x^{*}	x	-	-	x^{***}	x_{+}
3Y	-	x^{**}	-	-	x	-	x^{*}	x	x	x^{*}	x^{**}	-
4Y	-	x^{*}	-	x_{+}	-	-	x^{*}	x	-	-	x^{***}	x
5Y	-	x^{*}	x^{***}	x	-	x^{***}	x^{**}	x	-	-	x^{**}	-
7Y	-	x	x^{***}	x	-	x^{***}	x^{***}	x	-	-	x_{+}	-
10Y	-	x	x^{**}	x	-	x^{***}_{+}	x^{***}	x_{+}	-	-	x_{+}	-
20Y	-	x	x	-	-	-	x	x	-	x	-	-
25Y	-	-	-	-	-	x	x^{***}	x	-	x	-	-

Table B.24: Test of the standardized innovations within the SZ5 framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	-	x	x	x	x^{***}	x^{**}	x_{+}	-	x^{**}	x^{**}	x
6M	-	-	x^{*}	x^{**}	-	-	x^{***}	x^{***}	x^{*}	x^{**}_{+}	x^{*}	x^{*}_{+}
1Y	-	x^{***}	x^{*}	x^{***}	-	-	x^{***}	-	x^{**}	-	x^{***}	x^{*}
2Y	-	-	x	x	-	x^{***}_{+}	x^{*}	x^{***}_{+}	-	-	x^{***}_{+}	x
3Y	-	-	x^{*}	x	x^{*}	x^{*}	x^{*}	x	-	x^{***}	x	x^{***}_{+}
4Y	-	x^{**}	x^{*}	x	x^{*}	x^{*}	x^{**}	x	x_{+}	x^{*}	x^{*}	-
5Y	-	x^{***}_{+}	x	x	-	x^{*}_{+}	x^{*}	x^{***}	-	x^{*}	x^{*}	x^{**}_{+}
7Y	-	x^{***}	x^{***}	x^{***}	-	x^{**}	x^{*}_{+}	x^{***}	-	-	x^{**}	x
10Y	-	x^{***}	x^{***}	-	-	x	x^{***}_{+}	-	-	-	x^{*}_{+}	-
20Y	-	-	x^{**}	-	-	x	x^{***}	-	-	x	x_{+}	-
25Y	-	-	-	-	-	x	x	-	-	x	x_{+}	-

Table B.25: Test of the standardized innovations within the SZ5u. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	Treasury Strips				US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	-	x^{***}	-	-	-	x	-	x^{***}_{+}	-	x^{***}	-
6M	-	-	x^{**}	-	-	-	x	x	x^{*}	-	x^{*}	-
1Y	-	x^{**}	x^{*}	x^{*}	x^{*}	x^{***}	x^{**}	x	x	x^{**}_{+}	x^{**}	x
2Y	-	-	x	x^{**}	-	-	x^{*}	x_{+}	-	x^{***}	-	-
3Y	-	-	x^{*}	x	-	-	x^{*}	x_{+}	-	x^{*}	x^{**}	x^{***}_{+}
4Y	x^{***}	x^{*}	x^{*}	x	x^{*}	x^{***}	x^{**}	x	x^{*}	x^{**}	x^{***}	-
5Y	-	x^{***}_{+}	x^{*}	x^{***}_{+}	-	x^{***}_{+}	x^{**}	x^{**}	-	-	-	-
7Y	-	x^{***}_{+}	x^{***}	-	-	x_{+}	x	-	-	-	-	x^{**}
10Y	-	-	x^{***}	-	-	x	x	-	-	x^{*}	-	x
20Y	-	-	x^{*}	-	-	x	x	-	-	x	-	-
25Y	-	-	-	-	-	x	x_{+}	-	-	x	-	-

Table B.26: Test of the standardized innovations within the 5corr framework. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	-	x^{*}	x^{*}_{+}	x^{*}	x^{***}	x^{**}	x^{*}
6M	x	x^{***}	x^{***}	x	x	x^{**}	x^{*}	x
1Y	-	x^{***}	x^{**}	x	x	-	x^{***}	x^{***}
2Y	-	x^{***}	x^{*}	x	-	x^{***}_{+}	x^{**}	x^{**}
3Y	x	-	x^{**}	x	x	x^{*}	x^{**}	x
4Y	-	x^{***}	x^{*}	x	-	x^{**}	x^{**}	x^{***}_{+}
5Y	-	x	x^{*}	x	-	x	x	-
7Y	-	x	x_{+}	x	-	x	x	-
10Y	-	x	x_{+}	x	-	x	-	-
20Y	-	x	x	x	-	x	x^{**}	x
25Y	-	x	x	-	-	x	x	-

Table B.27: Test of the standardized innovations within the INFcorr framework for defaultable bonds. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Maturity	US Industrials A2				US Industrials BBB1			
	NAC	HS	ND	M0	NAC	HS	ND	M0
3M	-	-	x^{*}	-	x^{*}	x^{**}	x^{***}	x
6M	-	-	x^{*}	-	x^{*}	x^{***}	x^{**}	x_{+}
1Y	-	-	x^{**}	x^{*}	x	x^{***}_{+}	x^{***}	-
2Y	-	-	x	-	-	-	x^{**}	x
3Y	x^{*}	x^{***}_{+}	x^{**}	x	x	x^{*}	x^{***}	x^{*}_{+}
4Y	-	-	x^{*}	x^{*}	-	x^{***}	x^{***}	x
5Y	-	x^{***}	x^{***}	x^{*}	-	x^{***}	x^{*}	x
7Y	-	x	-	x^{*}	-	x_{+}	x^{*}	x
10Y	-	x	-	-	-	x_{+}	x	x
20Y	-	x	x	x	-	x	x	-
25Y	-	x	x	x^{*}	-	x	x	-

Table B.28: Test of the standardized innovations within the INF framework for defaultable bonds. The residuals of a given maturity are tested for the hypotheses of no autocorrelation (NAC), of homoscedasticity (HS), if they are drawn from a normal distribution (ND) and if they have a mean of 0 (M0). x indicates that the hypothesis cannot be rejected at a 5% level. The superscript * signifies that 2.5% of the biggest outliers are removed from the data set. ** stands for a removal of 5% of the outliers and *** for 10%. The subscript + indicates that we use 1% as the level of significance.

Appendix C

Credit Derivatives

In order to improve the readability of Chapter 5 certain proofs are given in this chapter.

Proof of Theorem 5.7:

According to Feynman-Kac (see Theorem 2.14) $v(r, s^{zero}, u, w_1, w_2, t, T)$ solves the following equation:

$$\begin{aligned}
(r + s^{zero})v \;=\;& v_t + [\theta_r(t) + b_{rw_1} w_1 + \hat{b}_{rw_2} w_2 - \hat{a}_r r] v_r \\
&+ [\theta_{s^{zero}} + b_{s^{zero}u} u - b_{s^{zero}w_1} w_1 - b_{s^{zero}w_2} w_2 - \hat{a}_s s^{zero}] v_{s^{zero}} \\
&+ [\theta_u - \hat{a}_u u] v_u + [\theta_{w_1} - \hat{a}_{w_1} w_1] v_{w_1} + [\theta_{w_2} - \hat{a}_{w_2} w_2] v_{w_2} \\
&+ \frac{1}{2}\Big(\sigma_r^2 v_{rr} + \sigma_{s^{zero}}^2 v_{s^{zero}s^{zero}} + \sigma_u^2 v_{uu} + \sigma_{w_1}^2 v_{w_1w_1} + \sigma_{w_2}^2 v_{w_2w_2} \\
&+ 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2} v_{w_1w_2} + 2\sigma_r\sigma_{w_1}\rho_{rw_1} v_{rw_1} \\
&+ 2\sigma_r\sigma_{s^{zero}}(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2}) v_{rs^{zero}} \\
&+ 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) v_{rw_2} \\
&+ 2\sigma_{s^{zero}}\sigma_u\rho_{su} v_{s^{zero}u} + 2\sigma_{s^{zero}}\sigma_{w_1}\rho_{sw_1} v_{s^{zero}w_1} \\
&+ 2\sigma_{s^{zero}}\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) v_{s^{zero}w_2}\Big)
\end{aligned}$$

with terminal condition $v(r, s^{zero}, u, w_1, w_2, T, T) = s^{zero}(T)$.
Also, it is assumed that v takes on the form
$v(t,T) = P^{d,zero}(t,T) \cdot (F(t,T) + G(t,T)r(t) + H(t,T)s^{zero}(t) + I(t,T)u(t) + J_1(t,T)w_1(t) + J_2(t,T)w_2(t))$.

Inserting the partial derivatives

$$
\begin{aligned}
v_t &= (A_t^{d,zero} - B_t^{d,zero} r - C_t^{d,zero} s^{zero} - D_t^{d,zero} u - (E_1^{d,zero})_t w_1 - (E_2^{d,zero})_t w_2) \cdot v \\
&\quad + (F_t + G_t r + H_t s^{zero} + I_t u + (J_1)_t w_1 + (J_2)_t w_2) \cdot P^{d,zero}\,, \\
v_{rr} &= (B^{d,zero})^2 \cdot v - 2GB^{d,zero} \cdot P^{d,zero}\,, \quad v_r = -B^{d,zero} \cdot v + G \cdot P^{d,zero}\,, \\
v_{s^{zero}s^{zero}} &= (C^{d,zero})^2 \cdot v - 2HC^{d,zero} \cdot P^{d,zero}\,, \quad v_{s^{zero}} = -C^{d,zero} \cdot v + H \cdot P^{d,zero}\,, \\
v_{uu} &= (D^{d,zero})^2 \cdot v - 2ID^{d,zero} \cdot P^{d,zero}\,, \quad v_u = -D^{d,zero} \cdot v + I \cdot P^{d,zero}\,, \\
v_{w_1w_1} &= (E_1^{d,zero})^2 \cdot v - 2J_1E_1^{d,zero} \cdot P^{d,zero}\,, \quad v_{w_1} = -E_1^{d,zero} \cdot v + J_1 \cdot P^{d,zero}\,, \\
v_{w_2w_2} &= (E_2^{d,zero})^2 \cdot v - 2J_2E_2^{d,zero} \cdot P^{d,zero}\,, \quad v_{w_2} = -E_2^{d,zero} \cdot v + J_2 \cdot P^{d,zero}\,, \\
v_{w_1w_2} &= E_1^{d,zero} E_2^{d,zero} \cdot v - (E_2^{d,zero} J_1 + E_1^{d,zero} J_2) \cdot P^{d,zero}\,, \\
v_{rw_1} &= B^{d,zero} E_1^{d,zero} \cdot v - (B^{d,zero} J_1 + E_1^{d,zero} G) \cdot P^{d,zero}\,, \\
v_{rw_2} &= B^{d,zero} E_2^{d,zero} \cdot v - (B^{d,zero} J_2 + E_2^{d,zero} G) \cdot P^{d,zero}\,, \\
v_{rs^{zero}} &= B^{d,zero} C^{d,zero} \cdot v - (B^{d,zero} H + C^{d,zero} G) \cdot P^{d,zero}\,, \\
v_{s^{zero}w_1} &= C^{d,zero} E_1^{d,zero} \cdot v - (C^{d,zero} J_1 + E_1^{d,zero} H) \cdot P^{d,zero}\,, \\
v_{s^{zero}w_2} &= C^{d,zero} E_2^{d,zero} \cdot v - (C^{d,zero} J_2 + E_2^{d,zero} H) \cdot P^{d,zero}\,, \\
v_{s^{zero}u} &= C^{d,zero} D^{d,zero} \cdot v - (D^{d,zero} H + C^{d,zero} I) \cdot P^{d,zero}\,,
\end{aligned}
$$

dividing by $P^{d,zero} > 0$ and canceling terms with the help of the PDEs for $A_t^{d,zero}$, $B_t^{d,zero}$, $C_t^{d,zero}$, $D_t^{d,zero}$, $(E_1^{d,zero})_t$ and $(E_2^{d,zero})_t$ (see Proposition 5.3 and Theorem 4.3), we end up with

$$
\begin{aligned}
& F_t + \theta_r(t) G + \theta_{w_1} J_1 + \theta_{w_2} J_2 + \theta_u I + \theta_{s^{zero}} H - \sigma_{s^{zero}}^2 C^{d,zero} H - \sigma_r^2 B^{d,zero} G \\
& - \sigma_{w_1}^2 E_1^{d,zero} J_1 - \sigma_{w_2}^2 E_2^{d,zero} J_2 - \sigma_u^2 D^{d,zero} I \\
& - \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}(E_1^{d,zero} J_2 + E_2^{d,zero} J_1) - \sigma_r\sigma_{w_1}\rho_{rw_1}(B^{d,zero} J_1 + E_1^{d,zero} G) \\
& - \sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)(B^{d,zero} J_2 + E_2^{d,zero} G) \\
& - \sigma_r\sigma_{s^{zero}}(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})(B^{d,zero} H + C^{d,zero} G) \\
& - \sigma_{s^{zero}}\sigma_{w_1}\rho_{sw_1}(E_1^{d,zero} H + C^{d,zero} J_1) - \sigma_{s^{zero}}\sigma_u\rho_{su}(C^{d,zero} I + D^{d,zero} H) \\
& - \sigma_{s^{zero}}\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)(E_2^{d,zero} H + C^{d,zero} J_2) \\
& + r(G_t - \hat{a}_r G) \\
& + s^{zero}(H_t - \hat{a}_s H) \\
& + u(I_t - \hat{a}_u I + b_{s^{zero}u} H) \\
& + w_1((J_1)_t + b_{rw_1} G - \hat{a}_{w_1} J_1 - b_{s^{zero}w_1} H) \\
& + w_2((J_2)_t + \hat{b}_{rw_2} G - \hat{a}_{w_2} J_2 - b_{s^{zero}w_2} H) \\
& = 0.
\end{aligned}
$$

This equation results in the following PDEs

$$\begin{aligned}
G_t &= \hat{a}_r G,\\
H_t &= \hat{a}_s H,\\
I_t &= \hat{a}_u I - b_{s^{zero}u} H,\\
(J_1)_t &= \hat{a}_{w_1} J_1 + b_{s^{zero}w_1} H - b_{rw_1} G,\\
(J_2)_t &= \hat{a}_{w_2} J_2 + b_{s^{zero}w_2} H - \hat{b}_{rw_2} G,\\
F_t &= -\theta_r(t)G - \theta_{w_1} J_1 - \theta_{w_2} J_2 - \theta_u I - \theta_{s^{zero}} H + \sigma^2_{s^{zero}} C^{d,zero} H\\
&\quad + \sigma_r^2 B^{d,zero} G + \sigma_{w_1}^2 E_1^{d,zero} J_1 + \sigma_{w_2}^2 E_2^{d,zero} J_2 + \sigma_u^2 D^{d,zero} I\\
&\quad + \sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}(E_1^{d,zero} J_2 + E_2^{d,zero} J_1)\\
&\quad + \sigma_r\sigma_{w_1}\rho_{rw_1}(B^{d,zero} J_1 + E_1^{d,zero} G)\\
&\quad + \sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho^2_{w_1w_2}}\right)(B^{d,zero} J_2 + E_2^{d,zero} G)\\
&\quad + \sigma_r\sigma_{s^{zero}}(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})(B^{d,zero} H + C^{d,zero} G)\\
&\quad + \sigma_{s^{zero}}\sigma_{w_1}\rho_{sw_1}(E_1^{d,zero} H + C^{d,zero} J_1) + \sigma_{s^{zero}}\sigma_u\rho_{su}(C^{d,zero} I + D^{d,zero} H)\\
&\quad + \sigma_{s^{zero}}\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho^2_{w_1w_2}}\right)(E_2^{d,zero} H + C^{d,zero} J_2).
\end{aligned}$$

By means of the terminal condition ($G(T,T)=0$, $F(T,T)=0$, $H(T,T)=1$, $I(T,T)=0$, $J_1(T,T)=0$, $J_2(T,T)=0$) and since it holds that

$$\begin{aligned}
-C_t &= H,\\
-D_t &= I,\\
-(E_1^{d,zero} - E_1)_t &= J_1,\\
-(E_2^{d,zero} - E_2)_t &= J_2,
\end{aligned}$$

we obtain the stated solutions with $G(t,T)=0$ and Theorem 2.15. □

Proof of Proposition 5.10:

Applying Feynman-Kac (see Theorem 2.14) we get the following equation:

$$\begin{aligned}
&(r + s^{zero})P^{d,*}\\
&= P_t^{d,*}\\
&\quad + \left(\theta_r(t) + b_{rw_1}w_1 + \hat{b}_{rw_2}w_2 - \hat{a}_r r\right)P_r^{d,*}\\
&\quad + (\theta_{w_1} - \hat{a}_{w_1}w_1)\,P_{w_1}^{d,*}
\end{aligned}$$

$$
\begin{aligned}
&+ (\theta_{w_2} - \hat{a}_{w_2} w_2)\, P^{d,*}_{w_2} \\
&+ (\theta_u - \hat{a}_u u)\, P^{d,*}_u \\
&+ (\theta_s + b_{su} u - b_{sw_1} w_1 - b_{sw_2} w_2 - \hat{a}_s s)\, P^{d,*}_s \\
&+ \frac{1}{2}\Big(\sigma_r^2 P^{d,*}_{rr} + \sigma_s^2 P^{d,*}_{ss} + \sigma_u^2 P^{d,*}_{uu} + \sigma_{w_1}^2 P^{d,*}_{w_1 w_1} + \sigma_{w_2}^2 P^{d,*}_{w_2 w_2} \\
&+ 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1 w_2} P^{d,*}_{w_1 w_2} + 2\sigma_r \sigma_{w_1}\rho_{rw_1} P^{d,*}_{rw_1} \\
&+ 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1 w_2} + \rho_{rw_2}\sqrt{1-\rho^2_{w_1 w_2}}\right) P^{d,*}_{rw_2} \\
&+ 2\sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2}) P^{d,*}_{sr} \\
&+ 2\sigma_s\sigma_u\rho_{su} P^{d,*}_{su} + 2\sigma_s\sigma_{w_1}\rho_{sw_1} P^{d,*}_{sw_1} \\
&+ 2\sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1 w_2} + \rho_{sw_2}\sqrt{1-\rho^2_{w_1 w_2}}\right) P^{d,*}_{sw_2}\Big).
\end{aligned}
$$

Inserting the partial derivatives (see e.g. Theorem 4.3), using $s^{zero} = \frac{s}{1-z}$, regrouping the terms and dividing by $P^{d,*} > 0$, we obtain the system of differential equations:

$$
\begin{aligned}
B^{d,*}_t &= \hat{a}_r B^{d,*} - 1 \\
C^{d,*}_t &= \hat{a}_s C^{d,*} - \frac{1}{1-z} \\
D^{d,*}_t &= \hat{a}_u D^{d,*} - b_{su} C^{d,*} \\
(E^{d,*}_1)_t &= \hat{a}_{w_1} E^{d,*}_1 - b_{rw_1} B^{d,*} + b_{sw_1} C^{d,*} \\
(E^{d,*}_2)_t &= \hat{a}_{w_2} E^{d,*}_2 - \hat{b}_{rw_2} B^{d,*} + b_{sw_2} C^{d,*}
\end{aligned}
$$

$$
\begin{aligned}
-A^{d,*}_t = \; & \frac{1}{2}\Big(\sigma_r^2 (B^{d,*})^2 + \sigma_s^2 (C^{d,*})^2 + \sigma_u^2 (D^{d,*})^2 + \sigma_{w_1}^2 (E^{d,*}_1)^2 + \sigma_{w_2}^2 (E^{d,*}_2)^2 \\
&+ 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1 w_2} E^{d,*}_1 E^{d,*}_2 + 2\sigma_r\sigma_{w_1}\rho_{rw_1} B^{d,*} E^{d,*}_1 \\
&+ 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1 w_2} + \rho_{rw_2}\sqrt{1-\rho^2_{w_1 w_2}}\right) B^{d,*} E^{d,*}_2 \\
&+ 2\sigma_r\sigma_s(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2}) B^{d,*} C^{d,*} \\
&+ 2\sigma_s\sigma_u\rho_{su} C^{d,*} D^{d,*} + 2\sigma_s\sigma_{w_1}\rho_{sw_1} C^{d,*} E^{d,*}_1 \\
&+ 2\sigma_s\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1 w_2} + \rho_{sw_2}\sqrt{1-\rho^2_{w_1 w_2}}\right) C^{d,*} E^{d,*}_2\Big) \\
&- \theta_r(t) B^{d,*} - \theta_s C^{d,*} - \theta_u D^{d,*} - \theta_{w_1} E^{d,*}_1 - \theta_{w_2} E^{d,*}_2.
\end{aligned}
$$

Since it must hold that $P^{d,*}(T,T,T^*) = P^d(T,T^*)$, the boundary conditions are $A^{d,*}(T,T,T^*) = A^d(T,T^*)$, $B^{d,*}(T,T,T^*) = B^d(T,T^*)$, $C^{d,*}(T,T,T^*) = C^d(T,T^*)$, $D^{d,*}(T,T,T^*) = D^d(T,T^*)$, $E^{d,*}_1(T,T,T^*) = E^d_1(T,T^*)$ and $E^{d,*}_2(T,T,T^*) = E^d_2(T,T^*)$.

Hence the differential equations result in (cf. Theorem 2.15 and Theorem 4.3)

$$\begin{aligned}
A^{d,*}(t,T,T^*) &= A^{d,*}(T,T,T^*) - \int_t^T A_t^{d,*}(l,T,T^*)dl, \\
B^{d,*}(t,T,T^*) &= e^{-\hat{a}_r(T-t)}\Big(B^{d,*}(T,T,T^*) + \int_0^{T-t} e^{\hat{a}_r l} dl\Big) \\
&= \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(T^*-t)}\right) = B^d(t,T^*) = B(t,T^*), \\
C^{d,*}(t,T,T^*) &= e^{-\hat{a}_s(T-t)}\Big(C^{d,*}(T,T,T^*) + \frac{1}{1-z}\int_0^{T-t} e^{\hat{a}_s l} dl\Big) \\
&= e^{-\hat{a}_s(T-t)}C^d(T,T^*) + \frac{1}{1-z}C^d(t,T),
\end{aligned}$$

$$\begin{aligned}
&D^{d,*}(t,T,T^*) \\
&= e^{-\hat{a}_u(T-t)}\Big(D^{d,*}(T,T,T^*) + \int_0^{T-t} e^{\hat{a}_u l} b_{su} C^{d,*}(0,l,l+T^*-T)dl\Big) \\
&= e^{-\hat{a}_u(T-t)}D^d(T,T^*) - b_{su}C^d(T,T^*)\left(\frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_u(T-t)}}{\hat{a}_s - \hat{a}_u}\right) \\
&\quad + \frac{1}{1-z}D^d(t,T),
\end{aligned}$$

$$\begin{aligned}
&E_1^{d,*}(t,T,T^*) \\
&= e^{-\hat{a}_{w_1}(T-t)}\Big(E_1^{d,*}(T,T,T^*) + \int_0^{T-t} e^{\hat{a}_{w_1} l}(b_{rw_1}B^{d,*}(0,l,l+T^*-T) \\
&\quad - b_{sw_1}C^{d,*}(0,l,l+T^*-T))dl\Big) \\
&= e^{-\hat{a}_{w_1}(T-t)}E_1^d(T,T^*) \\
&\quad + \frac{b_{rw_1}}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_{w_1}} + e^{-\hat{a}_r(T^*-T)}\frac{e^{-\hat{a}_{w_1}(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_{w_1} - \hat{a}_r}\right) \\
&\quad + b_{sw_1}C^d(T,T^*)\left(\frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_{w_1}(T-t)}}{\hat{a}_s - \hat{a}_{w_1}}\right) \\
&\quad + \frac{1}{1-z}(E_1^d(t,T) - E_1(t,T)),
\end{aligned}$$

$E_2^{d,*}(t,T,T^*)$ is determined analogously. □

Proof of Proposition 5.17:

According to Feynman-Kac (see Theorem 2.14) the following differential equation must hold:

$$\begin{aligned}
&(r + s^{zero})P^* \\
&= P_t^* \\
&\quad + \left(\theta_r(t) + b_{rw_1} w_1 + \hat{b}_{rw_2} w_2 - \hat{a}_r r\right) P_r^* \\
&\quad + (\theta_{w_1} - \hat{a}_{w_1} w_1)\, P_{w_1}^* \\
&\quad + (\theta_{w_2} - \hat{a}_{w_2} w_2)\, P_{w_2}^* \\
&\quad + (\theta_u - \hat{a}_u u)\, P_u^* \\
&\quad + (\theta_{s^{zero}} + b_{s^{zero}u} u - b_{s^{zero}w_1} w_1 - b_{s^{zero}w_2} w_2 - \hat{a}_s s^{zero})\, P_{s^{zero}}^* \\
&\quad + \frac{1}{2}\Big(\sigma_r^2 P_{rr}^* + \sigma_{s^{zero}}^2 P_{s^{zero}s^{zero}}^* + \sigma_u^2 P_{uu}^* + \sigma_{w_1}^2 P_{w_1w_1}^* + \sigma_{w_2}^2 P_{w_2w_2}^* \\
&\quad + 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2} P_{w_1w_2}^* + 2\sigma_r\sigma_{w_1}\rho_{rw_1} P_{rw_1}^* \\
&\quad + 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) P_{rw_2}^* \\
&\quad + 2\sigma_r\sigma_{s^{zero}}(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2}) P_{s^{zero}r}^* \\
&\quad + 2\sigma_{s^{zero}}\sigma_u\rho_{su} P_{s^{zero}u}^* + 2\sigma_{s^{zero}}\sigma_{w_1}\rho_{sw_1} P_{s^{zero}w_1}^* \\
&\quad + 2\sigma_{s^{zero}}\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right) P_{s^{zero}w_2}^*\Big).
\end{aligned}$$

Inserting the partial derivatives (see e.g. Theorem 4.3), regrouping the terms and dividing by $P^* > 0$, we obtain the solution analogously to Theorem 4.3:

$$\begin{aligned}
B_t^* &= \hat{a}_r B^* - 1 \\
C_t^* &= \hat{a}_s C^* - 1 \\
D_t^* &= \hat{a}_u D^* - b_{s^{zero}u} C^* \\
(E_1^*)_t &= \hat{a}_{w_1} E_1^* - b_{rw_1} B^* + b_{s^{zero}w_1} C^* \\
(E_2^*)_t &= \hat{a}_{w_2} E_2^* - \hat{b}_{rw_2} B^* + b_{s^{zero}w_2} C^*
\end{aligned}$$

$$\begin{aligned}
-A_t^* &= \frac{1}{2}\Big(\sigma_r^2(B^*)^2 + \sigma_{s^{zero}}^2(C^*)^2 + \sigma_u^2(D^*)^2 + \sigma_{w_1}^2(E_1^*)^2 + \sigma_{w_2}^2(E_2^*)^2 \\
&\quad + 2\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^*E_2^* + 2\sigma_r\sigma_{w_1}\rho_{rw_1}B^*E_1^* \\
&\quad + 2\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2} + \rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)B^*E_2^* \\
&\quad + 2\sigma_r\sigma_{s^{zero}}(\rho_{rw_1}\rho_{sw_1} + \rho_{rw_2}\rho_{sw_2})B^*C^* \\
&\quad + 2\sigma_{s^{zero}}\sigma_u\rho_{s^{zero}u}C^*D^* + 2\sigma_{s^{zero}}\sigma_{w_1}\rho_{s^{zero}w_1}C^*E_1^* \\
&\quad + 2\sigma_{s^{zero}}\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2} + \rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)C^*E_2^*\Big) \\
&\quad - \theta_r(t)B^* - \theta_{s^{zero}}C^* - \theta_u D^* - \theta_{w_1}E_1^* - \theta_{w_2}E_2^*.
\end{aligned}$$

Since it must hold that $P^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) = P(\widetilde{T}_{j-1},\widetilde{T}_j)$, the boundary conditions $A^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) = A(\widetilde{T}_{j-1},\widetilde{T}_j)$, $B^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) = B(\widetilde{T}_{j-1},\widetilde{T}_j)$, $C^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) = 0$, $D^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) = 0$, $E_1^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) = E_1(\widetilde{T}_{j-1},\widetilde{T}_j)$ and $E_2^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) = E_2(\widetilde{T}_{j-1},\widetilde{T}_j)$ must be fulfilled. Using Theorem 2.15, we finally obtain

$$\begin{aligned}
B^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) &= e^{-\hat{a}_r(\widetilde{T}_{j-1}-t)}\Big(B^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) + \int_0^{\widetilde{T}_{j-1}-t} e^{\hat{a}_r l}dl\Big) \\
&= \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(\widetilde{T}_j-t)}\right) = B(t,\widetilde{T}_j), \\
C^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) &= e^{-\hat{a}_s(\widetilde{T}_{j-1}-t)}\Big(C^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) + \int_0^{\widetilde{T}_{j-1}-t} e^{\hat{a}_s l}dl\Big) \\
&= \frac{1}{\hat{a}_s}\left(1 - e^{-\hat{a}_s(\widetilde{T}_{j-1}-t)}\right) = C^{d,zero}(t,\widetilde{T}_{j-1}), \\
D^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) &= e^{-\hat{a}_u(\widetilde{T}_{j-1}-t)}\Big(D^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) + \int_0^{\widetilde{T}_{j-1}-t} e^{\hat{a}_u l}b_{s^{zero}u}C^*(0,l,\widetilde{T}_j)dl\Big) \\
&= \frac{b_{s^{zero}u}}{\hat{a}_s}\left(\frac{1 - e^{-\hat{a}_u(\widetilde{T}_{j-1}-t)}}{\hat{a}_u} + \frac{e^{-\hat{a}_u(\widetilde{T}_{j-1}-t)} - e^{-\hat{a}_s(\widetilde{T}_{j-1}-t)}}{\hat{a}_u - \hat{a}_s}\right) \\
&= D^{d,zero}(t,\widetilde{T}_{j-1}),
\end{aligned}$$

$$\begin{aligned}
&E_1^*(t,\widetilde{T}_{j-1},\widetilde{T}_j) \\
&\quad = e^{-\hat{a}_{w_1}(\widetilde{T}_{j-1}-t)}\Big(E_1^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j) + \int_0^{\widetilde{T}_{j-1}-t} e^{\hat{a}_{w_1}l}(b_{rw_1}B^*(0,l,l+\widetilde{T}_j-\widetilde{T}_{j-1}) \\
&\qquad - b_{s^{zero}w_1}C^*(0,l,\widetilde{T}_j))dl\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b_{rw_1}}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_1}(\widetilde{T}_j-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(\widetilde{T}_j-t)}-e^{-\hat{a}_r(\widetilde{T}_j-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right)\\
&\quad -\frac{b_{s^{zero}w_1}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_1}(\widetilde{T}_{j-1}-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}(\widetilde{T}_{j-1}-t)}-e^{-\hat{a}_s(\widetilde{T}_{j-1}-t)}}{\hat{a}_{w_1}-\hat{a}_s}\right),
\end{aligned}
$$

$$
\begin{aligned}
&E_2^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&= e^{-\hat{a}_{w_2}(\widetilde{T}_{j-1}-t)}\Big(E_2^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j)+\int_0^{\widetilde{T}_{j-1}-t} e^{\hat{a}_{w_2}l}(\hat{b}_{rw_2}B^*(0,l,l+\widetilde{T}_j-\widetilde{T}_{j-1})\\
&\quad -b_{s^{zero}w_2}C^*(0,l,\widetilde{T}_j))dl\Big)\\
&= \frac{\hat{b}_{rw_2}}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_{w_2}(\widetilde{T}_j-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(\widetilde{T}_j-t)}-e^{-\hat{a}_r(\widetilde{T}_j-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right)\\
&\quad -\frac{b_{s^{zero}w_2}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_{w_2}(\widetilde{T}_{j-1}-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}(\widetilde{T}_{j-1}-t)}-e^{-\hat{a}_s(\widetilde{T}_{j-1}-t)}}{\hat{a}_{w_2}-\hat{a}_s}\right),
\end{aligned}
$$

$$
\begin{aligned}
&A^*(t,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&= A^*(\widetilde{T}_{j-1},\widetilde{T}_{j-1},\widetilde{T}_j)-\int_t^{\widetilde{T}_{j-1}} A_t^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)dl\\
&= A(\widetilde{T}_{j-1},\widetilde{T}_j)+\int_t^{\widetilde{T}_{j-1}}\frac{1}{2}\Big(\sigma_r^2(B^*(l,\widetilde{T}_{j-1},\widetilde{T}_j))^2+\sigma_{s^{zero}}^2(C^*(l,\widetilde{T}_{j-1},\widetilde{T}_j))^2\\
&\quad +\sigma_u^2(D^*(l,\widetilde{T}_{j-1},\widetilde{T}_j))^2+\sigma_{w_1}^2(E_1^*(l,\widetilde{T}_{j-1},\widetilde{T}_j))^2+\sigma_{w_2}^2(E_2^*(l,\widetilde{T}_{j-1},\widetilde{T}_j))^2\\
&\quad +\sigma_{w_1}\sigma_{w_2}\rho_{w_1w_2}E_1^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)E_2^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&\quad +\sigma_r\sigma_{w_1}\rho_{rw_1}B^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)E_1^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&\quad +\sigma_{s^{zero}}\sigma_u\rho_{su}C^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)D^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&\quad +\sigma_{s^{zero}}\sigma_{w_1}\rho_{sw_1}C^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)E_1^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&\quad +\sigma_r\sigma_{w_2}\left(\rho_{rw_1}\rho_{w_1w_2}+\rho_{rw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)B^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)E_2^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&\quad +\sigma_{s^{zero}}\sigma_{w_2}\left(\rho_{sw_1}\rho_{w_1w_2}+\rho_{sw_2}\sqrt{1-\rho_{w_1w_2}^2}\right)C^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)E_2^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&\quad +\sigma_r\sigma_{s^{zero}}(\rho_{rw_1}\rho_{sw_1}+\rho_{rw_2}\rho_{sw_2})B^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)C^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&\quad -\theta_r(l)B^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)-\theta_{s^{zero}}C^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)-\theta_uD^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)\\
&\quad -\theta_{w_1}E_1^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)-\theta_{w_2}E_2^*(l,\widetilde{T}_{j-1},\widetilde{T}_j)dl.
\end{aligned}
$$

□

Appendix D

FCDS Counterparty Risk

In this chapter we calculate certain terms needed for pricing counterparty risk of Forward Credit Default Swaps (FCDS). First, we determine the terms $P^{d,z,cp,ref}(t,T_a)$ and $P^{d,z,cp,ref}(t,T_a,T_b)$, $t \leq T_a \leq T_b$ which are introduced in Proposition 5.29.

Lemma D.1
For $t \leq T_a$

$$\begin{aligned}
&P^{d,z,cp,ref}(t,T_a)\\
&:= \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_a} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} \,\middle|\, \mathcal{F}_t \right]\\
&= P^{d,z,cp,ref}(t,T_a,r(t),s^{zero,cp}(t),u^{cp}(t),s^{zero,ref}(t),u^{ref}(t),w_1(t),w_2(t))
\end{aligned}$$

is given by

$$\begin{aligned}
&P^{d,z,cp,ref}(t,T_a)\\
&= e^{A^{d,z,cp,ref}(t,T_a)-B^{d,z,cp,ref}(t,T_a)r-C_1^{d,z,cp,ref}(t,T_a)s^{zero,cp}-D_1^{d,z,cp,ref}(t,T_a)u^{cp}}\\
&\quad \cdot e^{-C_2^{d,z,cp,ref}(t,T_a)s^{zero,ref}-D_2^{d,z,cp,ref}(t,T_a)u^{ref}-E_1^{d,z,cp,ref}(t,T_a)w_1-E_2^{d,z,cp,ref}(t,T_a)w_2}
\end{aligned}$$

with the functions $A^{d,z,cp,ref}(t,T_a)$, $B^{d,z,cp,ref}(t,T_a)$, $C_1^{d,z,cp,ref}(t,T_a)$, $C_2^{d,z,cp,ref}(t,T_a)$, $D_1^{d,z,cp,ref}(t,T_a)$, $D_2^{d,z,cp,ref}(t,T_a)$, $E_1^{d,z,cp,ref}(t,T_a)$, and $E_2^{d,z,cp,ref}(t,T_a)$ defined at the end of the proof.

Proof:
In the following, we assume all Brownian motions to be uncorrelated.
With Feynman-Kac (see Theorem 2.14) we obtain

$$
\begin{aligned}
&(r + s^{zero,cp} + s^{zero,ref}) P^{d,z,cp,ref} \\
= \;& P_t^{d,z,cp,ref} \\
+ \;& \Big(\theta_r(t) + b_{rw_1} \cdot w_1 + \hat{b}_{rw_2} \cdot w_2 - \hat{a}_r \cdot r\Big) \cdot P_r^{d,z,cp,ref} \\
+ \;& (\theta_{w_1} - \hat{a}_{w_1} \cdot w_1) \cdot P_{w_1}^{d,z,cp,ref} + (\theta_{w_2} - \hat{a}_{w_2} \cdot w_2) \cdot P_{w_2}^{d,z,cp,ref} \\
+ \;& \Big(\theta_{s^{zero,cp}} + b_{s^{zero,cp} u^{cp}} \cdot u^{cp} - b_{s^{zero,cp} w_1} \cdot w_1 - b_{s^{zero,cp} w_2} \cdot w_2 \\
- \;& \hat{a}_{s^{cp}} \cdot s^{zero,cp}\Big) \cdot P_{s^{zero,cp}}^{d,z,cp,ref} + (\theta_{u^{cp}} - \hat{a}_{u^{cp}} \cdot u^{cp}) \cdot P_{u^{cp}}^{d,z,cp,ref} \\
+ \;& \Big(\theta_{s^{zero,ref}} + b_{s^{zero,ref} u^{ref}} \cdot u^{ref} - b_{s^{zero,ref} w_1} \cdot w_1 - b_{s^{zero,ref} w_2} \cdot w_2 \\
- \;& \hat{a}_{s^{ref}} \cdot s^{zero,ref}\Big) \cdot P_{s^{zero,ref}}^{d,z,cp,ref} + \big(\theta_{u^{ref}} - \hat{a}_{u^{ref}} \cdot u^{ref}\big) \cdot P_{u^{ref}}^{d,z,cp,ref} \\
+ \;& \frac{1}{2} \cdot \Big(\sigma_r^2 \cdot P_{rr}^{d,z,cp,ref} + \sigma_{s^{zero,cp}}^2 \cdot P_{s^{zero,cp} s^{zero,cp}}^{d,z,cp,ref} + \sigma_{s^{zero,ref}}^2 \cdot P_{s^{zero,ref} s^{zero,ref}}^{d,z,cp,ref} \\
+ \;& \sigma_{u^{cp}}^2 \cdot P_{u^{cp} u^{cp}}^{d,z,cp,ref} + \sigma_{u^{ref}}^2 \cdot P_{u^{ref} u^{ref}}^{d,z,cp,ref} + \sigma_{w_1}^2 \cdot P_{w_1 w_1}^{d,z,cp,ref} + \sigma_{w_2}^2 \cdot P_{w_2 w_2}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_{w_1} \cdot \sigma_{w_2} \cdot \rho_{w_1 w_2} \cdot P_{w_1 w_2}^{d,z,cp,ref} + 2 \cdot \sigma_r \cdot \sigma_{w_1} \cdot \rho_{rw_1} \cdot P_{rw_1}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_r \cdot \sigma_{w_2} \cdot \Big(\rho_{rw_1} \cdot \rho_{w_1 w_2} + \rho_{rw_2} \cdot \sqrt{1 - \rho_{w_1 w_2}^2}\Big) \cdot P_{rw_2}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_r \cdot \sigma_{s^{zero,cp}} \cdot \big(\rho_{rw_1} \cdot \rho_{s^{cp} w_1} + \rho_{rw_2} \cdot \rho_{s^{cp} w_2}\big) \cdot P_{s^{zero,cp} r}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_r \cdot \sigma_{s^{zero,ref}} \cdot \big(\rho_{rw_1} \cdot \rho_{s^{ref} w_1} + \rho_{rw_2} \cdot \rho_{s^{ref} w_2}\big) \cdot P_{s^{zero,ref} r}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_{s^{zero,cp}} \cdot \sigma_{u^{cp}} \cdot \rho_{s^{cp} u^{cp}} \cdot P_{s^{zero,cp} u^{cp}}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_{s^{zero,cp}} \cdot \sigma_{w_1} \cdot \rho_{s^{cp} w_1} \cdot P_{s^{zero,cp} w_1}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_{s^{zero,ref}} \cdot \sigma_{u^{ref}} \cdot \rho_{s^{ref} u^{ref}} \cdot P_{s^{zero,ref} u^{ref}}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_{s^{zero,ref}} \cdot \sigma_{w_1} \cdot \rho_{s^{ref} w_1} \cdot P_{s^{ref} w_1}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_{s^{zero,cp}} \cdot \sigma_{w_2} \cdot \Big(\rho_{s^{cp} w_1} \cdot \rho_{w_1 w_2} + \rho_{s^{cp} w_2} \cdot \sqrt{1 - \rho_{w_1 w_2}^2}\Big) \cdot P_{s^{zero,cp} w_2}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_{s^{zero,ref}} \cdot \sigma_{w_2} \cdot \Big(\rho_{s^{ref} w_1} \cdot \rho_{w_1 w_2} + \rho_{s^{ref} w_2} \cdot \sqrt{1 - \rho_{w_1 w_2}^2}\Big) \cdot P_{s^{zero,ref} w_2}^{d,z,cp,ref} \\
+ \;& 2 \cdot \sigma_{s^{zero,cp}} \cdot \sigma_{s^{zero,ref}} \cdot \big(\rho_{s^{cp} w_1} \cdot \rho_{s^{ref} w_1} + \rho_{s^{cp} w_2} \cdot \rho_{s^{ref} w_2}\big)\Big),
\end{aligned}
$$

with boundary condition $P^{d,z,cp,ref}(T_a, T_a) = 1$, i.e.
$A^{d,z,cp,ref}(T_a, T_a) = B^{d,z,cp,ref}(T_a, T_a) = C_1^{d,z,cp,ref}(T_a, T_a) = \cdots = 0$.
Inserting the partial derivatives (see e.g. Theorem 4.3), regrouping the terms and dividing by $P^{d,z,cp,ref} > 0$, we arrive at the below system of differential

equations.

$$
\begin{aligned}
&(B^{d,z,cp,ref})_t = \hat{a}_r \cdot B^{d,z,cp,ref} - 1\\
&(C_1^{d,z,cp,ref})_t = \hat{a}_{s^{cp}} \cdot C_1^{d,z,cp,ref} - 1\\
&(C_2^{d,z,cp,ref})_t = \hat{a}_{s^{ref}} \cdot C_2^{d,z,cp,ref} - 1\\
&(D_1^{d,z,cp,ref})_t = \hat{a}_{u^{cp}} \cdot D_1^{d,z,cp,ref} - b_{s^{zero,cp}u^{cp}} \cdot C_1^{d,z,cp,ref}\\
&(D_2^{d,z,cp,ref})_t = \hat{a}_{u^{ref}} \cdot D_2^{d,z,cp,ref} - b_{s^{zero,ref}u^{ref}} \cdot C_2^{d,z,cp,ref}\\
&(E_1^{d,z,cp,ref})_t\\
&\quad = \hat{a}_{w_1} \cdot E_1^{d,z,cp,ref} - b_{rw_1} \cdot B^{d,z,cp,ref} + b_{s^{zero,cp}w_1} \cdot C_1^{d,z,cp,ref}\\
&\qquad + b_{s^{zero,ref}w_1} \cdot C_2^{d,z,cp,ref}\\
&(E_2^{d,z,cp,ref})_t\\
&\quad = \hat{a}_{w_2} \cdot E_2^{d,z,cp,ref} - \hat{b}_{rw_2} \cdot B^{d,z,cp,ref} + b_{s^{zero,cp}w_2} \cdot C_1^{d,z,cp,ref}\\
&\qquad + b_{s^{zero,ref}w_2} \cdot C_2^{d,z,cp,ref}
\end{aligned}
$$

and

$$
\begin{aligned}
&-(A^{d,z,cp,ref})_t =\\
&\frac{1}{2} \cdot \Big(\sigma_r^2 \cdot (B^{d,z,cp,ref})^2 + \sigma_{s^{zero,cp}}^2 \cdot (C_1^{d,z,cp,ref})^2 + \sigma_{s^{zero,ref}}^2 \cdot (C_2^{d,z,cp,ref})^2\\
&+\sigma_{u^{cp}}^2 \cdot (D_1^{d,z,cp,ref})^2 + \sigma_{u^{ref}}^2 \cdot (D_2^{d,z,cp,ref})^2 + \sigma_{w_1}^2 \cdot (E_1^{d,z,cp,ref})^2\\
&+\sigma_{w_2}^2 \cdot (E_2^{d,z,cp,ref})^2 + 2 \cdot \sigma_{w_1} \cdot \sigma_{w_2} \cdot \rho_{w_1w_2} \cdot E_1^{d,z,cp,ref} \cdot E_2^{d,z,cp,ref}\\
&+2 \cdot \sigma_r \cdot \sigma_{w_1} \cdot \rho_{rw_1} \cdot B^{d,z,cp,ref} \cdot E_1^{d,z,cp,ref}\\
&+2 \cdot \sigma_r \cdot \sigma_{w_2} \cdot \left(\rho_{rw_1} \cdot \rho_{w_1w_2} + \rho_{rw_2} \cdot \sqrt{1-\rho_{w_1w_2}^2}\right) \cdot B^{d,z,cp,ref} \cdot E_2^{d,z,cp,ref}\\
&+2 \cdot \sigma_r \cdot \sigma_{s^{zero,cp}} \cdot \left(\rho_{rw_1} \cdot \rho_{s^{cp}w_1} + \rho_{rw_2} \cdot \rho_{s^{cp}w_2}\right) \cdot B^{d,z,cp,ref} \cdot C_1^{d,z,cp,ref}\\
&+2 \cdot \sigma_r \cdot \sigma_{s^{zero,ref}} \cdot \left(\rho_{rw_1} \cdot \rho_{s^{ref}w_1} + \rho_{rw_2} \cdot \rho_{s^{ref}w_2}\right) \cdot B^{d,z,cp,ref} \cdot C_2^{d,z,cp,ref}\\
&+2 \cdot \sigma_{s^{zero,cp}} \cdot \sigma_{u^{cp}} \cdot \rho_{s^{cp}u^{cp}} \cdot C_1^{d,z,cp,ref} \cdot D_1^{d,z,cp,ref}\\
&+2 \cdot \sigma_{s^{zero,cp}} \cdot \sigma_{w_1} \cdot \rho_{s^{cp}w_1} \cdot C_1^{d,z,cp,ref} \cdot E_1^{d,z,cp,ref}\\
&+2 \cdot \sigma_{s^{zero,ref}} \cdot \sigma_{u^{ref}} \cdot \rho_{s^{ref}u^{ref}} \cdot C_2^{d,z,cp,ref} \cdot D_2^{d,z,cp,ref}\\
&+2 \cdot \sigma_{s^{zero,ref}} \cdot \sigma_{w_1} \cdot \rho_{s^{ref}w_1} \cdot C_2^{d,z,cp,ref} \cdot E_1^{d,z,cp,ref}\\
&+2 \cdot \sigma_{s^{zero,cp}} \cdot \sigma_{w_2} \left(\rho_{s^{cp}w_1} \cdot \rho_{w_1w_2} + \rho_{s^{cp}w_2} \cdot \sqrt{1-\rho_{w_1w_2}^2}\right) \cdot C_1^{d,z,cp,ref} \cdot E_2^{d,z,cp,ref}\\
&+2 \cdot \sigma_{s^{zero,ref}} \cdot \sigma_{w_2} \cdot \left(\rho_{s^{ref}w_1} \cdot \rho_{w_1w_2} + \rho_{s^{ref}w_2} \cdot \sqrt{1-\rho_{w_1w_2}^2}\right) \cdot C_2^{d,z,cp,ref} \cdot E_2^{d,z,cp,ref}\\
&+2 \cdot \sigma_{s^{zero,cp}} \cdot \sigma_{s^{zero,ref}} \cdot \left(\rho_{s^{cp}w_1} \cdot \rho_{s^{ref}w_1} + \rho_{s^{cp}w_2} \cdot \rho_{s^{ref}w_2}\right) \cdot C_1^{d,z,cp,ref} \cdot C_2^{d,z,cp,ref}\Big)
\end{aligned}
$$

$-\theta_r(t)\cdot B^{d,z,cp,ref} - \theta_{s^{zero,cp}}\cdot C_1^{d,z,cp,ref} - \theta_{s^{zero,ref}}\cdot C_2^{d,z,cp,ref} - \theta_{u^{cp}}\cdot D_1^{d,z,cp,ref}$
$-\theta_{u^{ref}}\cdot D_2^{d,z,cp,ref} - \theta_{w_1}\cdot E_1^{d,z,cp,ref} - \theta_{w_2}\cdot E_2^{d,z,cp,ref}.$

The solution of this system of differential equations is (cf. Theorem 2.15 and the proof of Theorem 4.3)

$$B^{d,z,cp,ref}(t,T_a) = B(t,T_a),$$

$$C_1^{d,z,cp,ref}(t,T_a) = \frac{1}{\hat{a}_{s^{cp}}}\cdot\left(1-e^{-\hat{a}_{s^{cp}}\cdot(T_a-t)}\right) = C^{d,zero,cp}(t,T_a),$$

$$C_2^{d,z,cp,ref}(t,T_a) = \frac{1}{\hat{a}_{s^{ref}}}\cdot\left(1-e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}\right) = C^{d,zero,ref}(t,T_a),$$

$$\begin{aligned}
&D_1^{d,z,cp,ref}(t,T_a)\\
&= \frac{b_{s^{zero,cp}u^{cp}}}{\hat{a}_{s^{cp}}}\cdot\left(\frac{1-e^{-\hat{a}_{u^{cp}}\cdot(T_a-t)}}{\hat{a}_{u^{cp}}} + \frac{e^{-\hat{a}_{u^{cp}}\cdot(T_a-t)}-e^{-\hat{a}_{s^{cp}}\cdot(T_a-t)}}{\hat{a}_{u^{cp}}-\hat{a}_{s^{cp}}}\right)\\
&= D^{d,zero,cp}(t,T_a),
\end{aligned}$$

$$\begin{aligned}
&D_2^{d,z,cp,ref}(t,T_a)\\
&= \frac{b_{s^{zero,ref}u^{ref}}}{\hat{a}_{s^{ref}}}\cdot\left(\frac{1-e^{-\hat{a}_{u^{ref}}\cdot(T_a-t)}}{\hat{a}_{u^{ref}}} + \frac{e^{-\hat{a}_{u^{ref}}\cdot(T_a-t)}-e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}}{\hat{a}_{u^{ref}}-\hat{a}_{s^{ref}}}\right)\\
&= D^{d,zero,ref}(t,T_a),
\end{aligned}$$

$$\begin{aligned}
&E_1^{d,z,cp,ref}(t,T_a)\\
&= -\frac{b_{s^{zero,cp}w_1}}{\hat{a}_{s^{cp}}}\cdot\left(\frac{1-e^{-\hat{a}_{w_1}\cdot(T_a-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}\cdot(T_a-t)}-e^{-\hat{a}_{s^{cp}}\cdot(T_a-t)}}{\hat{a}_{w_1}-\hat{a}_{s^{cp}}}\right)\\
&\quad -\frac{b_{s^{zero,ref}w_1}}{\hat{a}_{s^{ref}}}\cdot\left(\frac{1-e^{-\hat{a}_{w_1}\cdot(T_a-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}\cdot(T_a-t)}-e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}}{\hat{a}_{w_1}-\hat{a}_{s^{ref}}}\right)\\
&\quad +\frac{b_{rw_1}}{\hat{a}_r}\cdot\left(\frac{1-e^{-\hat{a}_{w_1}\cdot(T_a-t)}}{\hat{a}_{w_1}} + \frac{e^{-\hat{a}_{w_1}\cdot(T_a-t)}-e^{-\hat{a}_r\cdot(T_a-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),
\end{aligned}$$

$$E_2^{d,z,cp,ref}(t,T_a)$$
$$= -\frac{b_{s^{zero,cp}w_2}}{\hat{a}_{s^{cp}}} \cdot \left(\frac{1-e^{-\hat{a}_{w_2}\cdot(T_a-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}\cdot(T_a-t)} - e^{-\hat{a}_{s^{cp}}\cdot(T_a-t)}}{\hat{a}_{w_2}-\hat{a}_{s^{cp}}} \right)$$
$$-\frac{b_{s^{zero,ref}w_2}}{\hat{a}_{s^{ref}}} \cdot \left(\frac{1-e^{-\hat{a}_{w_2}\cdot(T_a-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}\cdot(T_a-t)} - e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}}{\hat{a}_{w_2}-\hat{a}_{s^{ref}}} \right)$$
$$+\frac{\hat{b}_{rw_2}}{\hat{a}_r} \cdot \left(\frac{1-e^{-\hat{a}_{w_2}\cdot(T_a-t)}}{\hat{a}_{w_2}} + \frac{e^{-\hat{a}_{w_2}\cdot(T_a-t)} - e^{-\hat{a}_r\cdot(T_a-t)}}{\hat{a}_{w_2}-\hat{a}_r} \right),$$

and

$$A^{d,z,cp,ref}(t,T_a) =$$
$$\int_t^{T_a} \frac{1}{2} \cdot \Big(\sigma_r^2 \cdot (B^{d,z,cp,ref}(l,T_a))^2 + \sigma_{s^{zero,cp}}^2 \cdot (C_1^{d,z,cp,ref}(l,T_a))^2$$
$$+\sigma_{s^{zero,ref}}^2 \cdot (C_2^{d,z,cp,ref}(l,T_a))^2 + \sigma_{u^{cp}}^2 \cdot (D_1^{d,z,cp,ref}(l,T_a))^2$$
$$+\sigma_{u^{ref}}^2 \cdot (D_2^{d,z,cp,ref}(l,T_a))^2 + \sigma_{w_1}^2 \cdot (E_1^{d,z,cp,ref}(l,T_a))^2 + \sigma_{w_2}^2 \cdot (E_2^{d,z,cp,ref}(l,T_a))^2 \Big)$$
$$+\sigma_{w_1} \cdot \sigma_{w_2} \cdot \rho_{w_1 w_2} \cdot E_1^{d,z,cp,ref}(l,T_a) \cdot E_2^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_r \cdot \sigma_{w_1} \cdot \rho_{rw_1} \cdot B^{d,z,cp,ref}(l,T_a) \cdot E_1^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_{s^{zero,cp}} \cdot \sigma_{u^{cp}} \cdot \rho_{s^{cp}u^{cp}} \cdot C_1^{d,z,cp,ref}(l,T_a) \cdot D_1^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_{s^{zero,cp}} \cdot \sigma_{w_1} \cdot \rho_{s^{cp}w_1} \cdot C_1^{d,z,cp,ref}(l,T_a) \cdot E_1^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_{s^{zero,ref}} \cdot \sigma_{u^{ref}} \cdot \rho_{s^{ref}u^{ref}} \cdot C_2^{d,z,cp,ref}(l,T_a) \cdot D_2^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_{s^{zero,ref}} \cdot \sigma_{w_1} \cdot \rho_{s^{ref}w_1} \cdot C_2^{d,z,cp,ref}(l,T_a) \cdot E_1^{d,z,cp,ref}(l,T_a)$$
$$+E_2^{d,z,cp,ref}(l,T_a) \cdot \Big(\sigma_r \cdot \sigma_{w_2} \cdot \left(\rho_{rw_1} \cdot \rho_{w_1 w_2} + \rho_{rw_2} \cdot \sqrt{1-\rho_{w_1 w_2}^2} \right) \cdot B^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_{s^{zero,cp}} \cdot \sigma_{w_2} \cdot \left(\rho_{s^{cp}w_1} \cdot \rho_{w_1 w_2} + \rho_{s^{cp}w_2} \cdot \sqrt{1-\rho_{w_1 w_2}^2} \right) \cdot C_1^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_{s^{zero,ref}} \cdot \sigma_{w_2} \cdot \left(\rho_{s^{ref}w_1} \cdot \rho_{w_1 w_2} + \rho_{s^{ref}w_2} \cdot \sqrt{1-\rho_{w_1 w_2}^2} \right) \cdot C_2^{d,z,cp,ref}(l,T_a) \Big)$$
$$+\sigma_r \cdot \sigma_{s^{zero,cp}} \cdot (\rho_{rw_1} \cdot \rho_{s^{cp}w_1} + \rho_{rw_2} \cdot \rho_{s^{cp}w_2}) \cdot B^{d,z,cp,ref}(l,T_a) \cdot C_1^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_r \cdot \sigma_{s^{zero,ref}} \cdot (\rho_{rw_1} \cdot \rho_{s^{ref}w_1} + \rho_{rw_2} \cdot \rho_{s^{ref}w_2}) \cdot B^{d,z,cp,ref}(l,T_a) \cdot C_2^{d,z,cp,ref}(l,T_a)$$
$$+\sigma_{s^{zero,cp}} \cdot \sigma_{s^{zero,ref}} \cdot (\rho_{s^{cp}w_1} \cdot \rho_{s^{ref}w_1} + \rho_{s^{cp}w_2} \cdot \rho_{s^{ref}w_2}) \cdot C_1^{d,z,cp,ref}(l,T_a) \cdot C_2^{d,z,cp,ref}(l,T_a)$$
$$-\theta_r(l) \cdot B^{d,z,cp,ref}(l,T_a) - \theta_{s^{zero,cp}} \cdot C_1^{d,z,cp,ref}(l,T_a) - \theta_{s^{zero,ref}} \cdot C_2^{d,z,cp,ref}(l,T_a)$$
$$-\theta_{u^{cp}} \cdot D_1^{d,z,cp,ref}(l,T_a) - \theta_{u^{ref}} \cdot D_2^{d,z,cp,ref}(l,T_a) - \theta_{w_1} \cdot E_1^{d,z,cp,ref}(l,T_a)$$
$$-\theta_{w_2} \cdot E_2^{d,z,cp,ref}(l,T_a) dl.$$

□

Lemma D.2
For $t \leq T_a \leq T_b$

$$\begin{aligned} & P^{d,z,cp,ref}(t,T_a,T_b) \\ := \;& \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_b} r(x)+s^{zero,cp}(x)dx} e^{-\int_t^{T_a} s^{zero,ref}(x)dx} \,\middle|\, \mathcal{F}_t \right] \\ = \;& P^{d,z,cp,ref}(t,T_a,T_b,r(t),s^{zero,cp}(t),u^{cp}(t),s^{zero,ref}(t),u^{ref}(t),w_1(t),w_2(t)) \end{aligned}$$

is given by

$$\begin{aligned} & P^{d,z,cp,ref}(t,T_a,T_b) \\ = \;& e^{A^{d,z,cp,ref}(t,T_a,T_b)-B^{d,z,cp,ref}(t,T_a,T_b)r-C_1^{d,z,cp,ref}(t,T_a,T_b)s^{zero,cp}-D_1^{d,z,cp,ref}(t,T_a,T_b)u^{cp}} \\ & \cdot e^{-C_2^{d,z,cp,ref}(t,T_a,T_b)s^{zero,ref}-D_2^{d,z,cp,ref}(t,T_a,T_b)u^{ref}-E_1^{d,z,cp,ref}(t,T_a,T_b)w_1} \\ & \cdot e^{-E_2^{d,z,cp,ref}(t,T_a,T_b)w_2} \end{aligned}$$

with the functions $A^{d,z,cp,ref}(t,T_a,T_b)$, $B^{d,z,cp,ref}(t,T_a,T_b)$, $C_1^{d,z,cp,ref}(t,T_a,T_b)$, $C_2^{d,z,cp,ref}(t,T_a,T_b)$, $D_1^{d,z,cp,ref}(t,T_a,T_b)$, $D_2^{d,z,cp,ref}(t,T_a,T_b)$, $E_1^{d,z,cp,ref}(t,T_a,T_b)$, *and* $E_2^{d,z,cp,ref}(t,T_a,T_b)$ *defined at the end of the proof.*

Proof:
With Feynman-Kac (see Theorem 2.14) we arrive for $t \leq T_a$ at the same system of differential equations as in the proof of Lemma D.1. Since it holds for $P^{d,z,cp,ref}(t,T_a,T_b)$ that

$$\begin{aligned} & P^{d,z,cp,ref}(t,T_a,T_b) \\ = \;& \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_a} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_{T_a}^{T_b} r(x)+s^{zero,cp}(x)dx} \,\middle|\, \mathcal{F}_{T_a} \right] \,\middle|\, \mathcal{F}_t \right] \\ = \;& \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_a} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} P^{d,zero,cp}(T_a,T_b) \,\middle|\, \mathcal{F}_t \right], \end{aligned}$$

the boundary condition is $P^{d,z,cp,ref}(T_a,T_a,T_b) = P^{d,zero,cp}(T_a,T_b)$. This boundary condition translates into the following equations
$A^{d,z,cp,ref}(T_a,T_a,T_b) = A^{d,zero,cp}(T_a,T_b)$, $B^{d,z,cp,ref}(T_a,T_a,T_b) = B^{d,zero,cp}(T_a,T_b)$,
$C_1^{d,z,cp,ref}(T_a,T_a,T_b) = C^{d,zero,cp}(T_a,T_b)$, $C_2^{d,z,cp,ref}(T_a,T_a,T_b) = 0$,
$D_1^{d,z,cp,ref}(T_a,T_a,T_b) = D^{d,zero,cp}(T_a,T_b)$, $D_2^{d,z,cp,ref}(T_a,T_a,T_b) = 0$,
$E_1^{d,z,cp,ref}(T_a,T_a,T_b) = E_1^{d,zero,cp}(T_a,T_b)$,
and $E_2^{d,z,cp,ref}(T_a,T_a,T_b) = E_2^{d,zero,cp}(T_a,T_b)$. With these conditions, we fi-

nally obtain with Theorem 2.15

$$
\begin{aligned}
B^{d,z,cp,ref}(t,T_a,T_b) &= e^{-\hat{a}_r\cdot(T_a-t)}\cdot\Big(B^{d,z,cp,ref}(T_a,T_a,T_b)+\int_0^{T_a-t}e^{\hat{a}_r\cdot l}dl\Big)\\
&= \frac{1}{\hat{a}_r}\cdot\big(1-e^{-\hat{a}_r\cdot(T_b-t)}\big)=B(t,T_b),\\
C_1^{d,z,cp,ref}(t,T_a,T_b) &= e^{-\hat{a}_{s^{cp}}\cdot(T_a-t)}\cdot\Big(C_1^{d,z,cp,ref}(T_a,T_a,T_b)+\int_0^{T_a-t}e^{\hat{a}_{s^{cp}}\cdot l}dl\Big)\\
&= \frac{1}{\hat{a}_{s^{cp}}}\cdot\big(1-e^{-\hat{a}_{s^{cp}}\cdot(T_b-t)}\big)=C^{d,zero,cp}(t,T_b),\\
C_2^{d,z,cp,ref}(t,T_a,T_b) &= e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}\cdot\Big(C_2^{d,z,cp,ref}(T_a,T_a,T_b)+\int_0^{T_a-t}e^{\hat{a}_{s^{ref}}\cdot l}dl\Big)\\
&= \frac{1}{\hat{a}_{s^{ref}}}\cdot\big(1-e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}\big)=C^{d,zero,ref}(t,T_a),
\end{aligned}
$$

$$
\begin{aligned}
&D_1^{d,z,cp,ref}(t,T_a,T_b)\\
&= e^{-\hat{a}_{u^{cp}}(T_a-t)}\cdot\Big(D_1^{d,z,cp,ref}(T_a,T_a,T_b)\\
&\quad+\int_0^{T_a-t}e^{\hat{a}_{u^{cp}}\cdot l}\cdot b_{s^{zero,cp}u^{cp}}\cdot C_1^{d,z,cp,ref}(0,l,T_b-T_a+l)dl\Big)\\
&= \frac{b_{s^{zero,cp}u^{cp}}}{\hat{a}_{s^{cp}}}\cdot\left(\frac{1-e^{-\hat{a}_{u^{cp}}\cdot(T_b-t)}}{\hat{a}_{u^{cp}}}+\frac{e^{-\hat{a}_{u^{cp}}\cdot(T_b-t)}-e^{-\hat{a}_{s^{cp}}\cdot(T_b-t)}}{\hat{a}_{u^{cp}}-\hat{a}_{s^{cp}}}\right)\\
&= D^{d,zero,cp}(t,T_b),
\end{aligned}
$$

$$
\begin{aligned}
&D_2^{d,z,cp,ref}(t,T_a,T_b)\\
&= e^{-\hat{a}_{u^{ref}}(T_a-t)}\cdot\Big(D_2^{d,z,cp,ref}(T_a,T_a,T_b)\\
&\quad+\int_0^{T_a-t}e^{\hat{a}_{u^{ref}}\cdot l}\cdot b_{s^{zero,ref}u^{ref}}\cdot C_2^{d,z,cp,ref}(0,l,T_b-T_a+l)dl\Big)\\
&= \frac{b_{s^{zero,ref}u^{ref}}}{\hat{a}_{s^{ref}}}\cdot\left(\frac{1-e^{-\hat{a}_{u^{ref}}\cdot(T_a-t)}}{\hat{a}_{u^{ref}}}+\frac{e^{-\hat{a}_{u^{ref}}\cdot(T_a-t)}-e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}}{\hat{a}_{u^{ref}}-\hat{a}_{s^{ref}}}\right)\\
&= D^{d,zero,ref}(t,T_a),
\end{aligned}
$$

$$\begin{aligned}
&E_1^{d,z,cp,ref}(t,T_a,T_b)\\
&= e^{-\hat{a}_{w_1}\cdot(T_a-t)}\cdot\Big(E_1^{d,z,cp,ref}(T_a,T_a,T_b)\\
&\quad+\int_0^{T_a-t} e^{\hat{a}_{w_1}\cdot l}\cdot(b_{rw_1}\cdot B^{d,z,cp,ref}(0,l,l+T_b-T_a)\\
&\quad-b_{s^{zero,cp}w_1}\cdot C_1^{d,z,cp,ref}(0,l,l+T_b-T_a)\\
&\quad-b_{s^{zero,ref}w_1}\cdot C_2^{d,z,cp,ref}(0,l,l+T_b-T_a))dl\Big)\\
&= -\frac{b_{s^{zero,cp}w_1}}{\hat{a}_{s^{cp}}}\cdot\left(\frac{1-e^{-\hat{a}_{w_1}\cdot(T_b-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}\cdot(T_b-t)}-e^{-\hat{a}_{s^{cp}}\cdot(T_b-t)}}{\hat{a}_{w_1}-\hat{a}_{s^{cp}}}\right)\\
&\quad-\frac{b_{s^{zero,ref}w_1}}{\hat{a}_{s^{ref}}}\cdot\left(\frac{1-e^{-\hat{a}_{w_1}\cdot(T_a-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}\cdot(T_a-t)}-e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}}{\hat{a}_{w_1}-\hat{a}_{s^{ref}}}\right)\\
&\quad+\frac{b_{rw_1}}{\hat{a}_r}\cdot\left(\frac{1-e^{-\hat{a}_{w_1}\cdot(T_b-t)}}{\hat{a}_{w_1}}+\frac{e^{-\hat{a}_{w_1}\cdot(T_b-t)}-e^{-\hat{a}_r\cdot(T_b-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right),
\end{aligned}$$

$$\begin{aligned}
&E_2^{d,z,cp,ref}(t,T_a,T_b)\\
&= e^{-\hat{a}_{w_2}\cdot(T_a-t)}\cdot\Big(E_2^{d,z,cp,ref}(T_a,T_a,T_b)\\
&\quad+\int_0^{T_a-t} e^{\hat{a}_{w_2}\cdot l}\cdot(\hat{b}_{rw_2}\cdot B^{d,z,cp,ref}(0,l,l+T_b-T_a)\\
&\quad-b_{s^{zero,cp}w_2}\cdot C_1^{d,z,cp,ref}(0,l,l+T_b-T_a)\\
&\quad-b_{s^{zero,ref}w_2}\cdot C_2^{d,z,cp,ref}(0,l,l+T_b-T_a))dl\Big)\\
&= -\frac{b_{s^{zero,cp}w_2}}{\hat{a}_{s^{cp}}}\cdot\left(\frac{1-e^{-\hat{a}_{w_2}\cdot(T_b-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}\cdot(T_b-t)}-e^{-\hat{a}_{s^{cp}}\cdot(T_b-t)}}{\hat{a}_{w_2}-\hat{a}_{s^{cp}}}\right)\\
&\quad-\frac{b_{s^{zero,ref}w_2}}{\hat{a}_{s^{ref}}}\cdot\left(\frac{1-e^{-\hat{a}_{w_2}\cdot(T_a-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}\cdot(T_a-t)}-e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}}{\hat{a}_{w_2}-\hat{a}_{s^{ref}}}\right)\\
&\quad+\frac{\hat{b}_{rw_2}}{\hat{a}_r}\cdot\left(\frac{1-e^{-\hat{a}_{w_2}\cdot(T_b-t)}}{\hat{a}_{w_2}}+\frac{e^{-\hat{a}_{w_2}\cdot(T_b-t)}-e^{-\hat{a}_r\cdot(T_b-t)}}{\hat{a}_{w_2}-\hat{a}_r}\right),
\end{aligned}$$

and

$$\begin{aligned}
&A^{d,z,cp,ref}(t,T_a,T_b)=A^{d,z,cp,ref}(T_a,T_a,T_b)-\int_t^{T_a}A_t^{d,z,cp,ref}(l,T_a,T_b)dl\\
&=A^{d,zero,cp}(T_a,T_b)+\int_t^{T_a}\frac{1}{2}\cdot\Big(\sigma_r^2\cdot(B^{d,z,cp,ref}(l,T_a,T_b))^2\\
&+\sigma^2_{s^{zero,cp}}\cdot(C_1^{d,z,cp,ref}(l,T_a,T_b))^2+\sigma^2_{s^{zero,ref}}\cdot(C_2^{d,z,cp,ref}(l,T_a,T_b))^2\\
&+\sigma^2_{u^{cp}}\cdot(D_1^{d,z,cp,ref}(l,T_a,T_b))^2+\sigma^2_{u^{ref}}\cdot(D_2^{d,z,cp,ref}(l,T_a,T_b))^2
\end{aligned}$$

$$
\begin{aligned}
&+\sigma_{w_1}^2 \cdot (E_1^{d,z,cp,ref}(l,T_a,T_b))^2 + \sigma_{w_2}^2 \cdot (E_2^{d,z,cp,ref}(l,T_a,T_b))^2\Big) \\
&+\sigma_{w_1} \cdot \sigma_{w_2} \cdot \rho_{w_1 w_2} \cdot E_1^{d,z,cp,ref}(l,T_a,T_b) \cdot E_2^{d,z,cp,ref}(l,T_a,T_b) \\
&+\sigma_r \cdot \sigma_{w_1} \cdot \rho_{rw_1} \cdot B^{d,z,cp,ref}(l,T_a,T_b) \cdot E_1^{d,z,cp,ref}(l,T_a,T_b) \\
&+\sigma_{s^{zero,cp}} \cdot \sigma_{u^{cp}} \cdot \rho_{s^{cp}u^{cp}} \cdot C_1^{d,z,cp,ref}(l,T_a,T_b) \cdot D_1^{d,z,cp,ref}(l,T_a,T_b) \\
&+\sigma_{s^{zero,cp}} \cdot \sigma_{w_1} \cdot \rho_{s^{cp}w_1} \cdot C_1^{d,z,cp,ref}(l,T_a,T_b) \cdot E_1^{d,z,cp,ref}(l,T_a,T_b) \\
&+\sigma_{s^{zero,ref}} \cdot \sigma_{u^{ref}} \cdot \rho_{s^{ref}u^{ref}} \cdot C_2^{d,z,cp,ref}(l,T_a,T_b) \cdot D_2^{d,z,cp,ref}(l,T_a,T_b) \\
&+\sigma_{s^{zero,ref}} \cdot \sigma_{w_1} \cdot \rho_{s^{ref}w_1} \cdot C_2^{d,z,cp,ref}(l,T_a,T_b) \cdot E_1^{d,z,cp,ref}(l,T_a,T_b) \\
&+E_2^{d,z,cp,ref}(l,T_a,T_b) \cdot \sigma_{w_2} \cdot \\
&\Big(\sigma_r \cdot \Big(\rho_{rw_1} \cdot \rho_{w_1 w_2} + \rho_{rw_2} \cdot \sqrt{1-\rho_{w_1 w_2}^2}\Big) \cdot B^{d,z,cp,ref}(l,T_a,T_b) \\
&+\sigma_{s^{zero,cp}} \cdot \Big(\rho_{s^{cp}w_1} \cdot \rho_{w_1 w_2} + \rho_{s^{cp}w_2} \cdot \sqrt{1-\rho_{w_1 w_2}^2}\Big) \cdot C_1^{d,z,cp,ref}(l,T_a,T_b) \\
&+\sigma_{s^{zero,ref}} \cdot \Big(\rho_{s^{ref}w_1} \cdot \rho_{w_1 w_2} + \rho_{s^{ref}w_2} \cdot \sqrt{1-\rho_{w_1 w_2}^2}\Big) \cdot C_2^{d,z,cp,ref}(l,T_a,T_b)\Big) \\
&+B^{d,z,cp,ref}(l,T_a,T_b) \cdot \sigma_r \cdot \\
&\Big(\sigma_{s^{zero,cp}} \cdot (\rho_{rw_1} \cdot \rho_{s^{cp}w_1} + \rho_{rw_2} \cdot \rho_{s^{cp}w_2}) \cdot C_1^{d,z,cp,ref}(l,T_a,T_b) \\
&+\sigma_{s^{zero,ref}} \cdot (\rho_{rw_1} \cdot \rho_{s^{ref}w_1} + \rho_{rw_2} \cdot \rho_{s^{ref}w_2}) \cdot C_2^{d,z,cp,ref}(l,T_a,T_b)\Big) \\
&+\sigma_{s^{zero,cp}} \cdot \sigma_{s^{zero,ref}} \cdot (\rho_{s^{cp}w_1} \cdot \rho_{s^{ref}w_1} + \rho_{s^{cp}w_2} \cdot \rho_{s^{ref}w_2}) \cdot C_1^{d,z,cp,ref}(l,T_a,T_b) \\
&\quad \cdot C_2^{d,z,cp,ref}(l,T_a,T_b) \\
&-\theta_r(l) \cdot B^{d,z,cp,ref}(l,T_a,T_b) - \theta_{s^{zero,cp}} \cdot C_1^{d,z,cp,ref}(l,T_a,T_b) \\
&-\theta_{s^{zero,ref}} \cdot C_2^{d,z,cp,ref}(l,T_a,T_b) - \theta_{u^{cp}} \cdot D_1^{d,z,cp,ref}(l,T_a,T_b) \\
&-\theta_{u^{ref}} \cdot D_2^{d,z,cp,ref}(l,T_a,T_b) - \theta_{w_1} \cdot E_1^{d,z,cp,ref}(l,T_a,T_b) \\
&-\theta_{w_2} \cdot E_2^{d,z,cp,ref}(l,T_a,T_b)dl.
\end{aligned}
$$

□

For Proposition 5.30, which assumes recovery as a fraction of the face value, we need the following lemma.

Lemma D.3
For $t < T$

$$
\mathbb{E}_{\widetilde{Q}}\left[\int_t^T e^{-\int_t^l r(x)+s^{zero,cp}(x)dx} dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref}\right]
$$

is determined by

$$\begin{aligned} L^{ref}(t) \quad \cdot \quad & \int_t^T \Big(P^{d,z,cp,ref}(t,l) \cdot (K(t,l) + M^{ref}(t,l) \cdot s^{zero,ref}(t) \\ & + N^{ref}(t,l) \cdot u^{ref}(t) + O_1(t,l) \cdot w_1(t) + O_2(t,l) \cdot w_2(t)) \Big) dl \end{aligned}$$

with the functions $K(t,T)$*,* $M^{ref}(t,T)$*,* $N^{ref}(t,T)$*,* $O_1(t,T)$ *and* $O_2(t,T)$ *as given in the proof.*

Proof:
It holds (c.f. Theorem 5.6)

$$\mathbb{E}_{\widetilde{Q}} \left[\int_t^T e^{-\int_t^l r(x) + s^{zero,cp}(x)dx} dH^{ref}(l) \,\middle|\, \mathcal{G}_t^{cp,ref} \right] = L^{ref}(t) \cdot V^{cp,ref}(t,T)$$

with

$$\begin{aligned} V^{cp,ref}(t,T) &= \mathbb{E}_{\widetilde{Q}} \left[\int_t^T e^{-\int_t^l r(x) + s^{zero,cp}(x) + s^{zero,ref}(x)dx} s^{zero,ref}(l) dl \,\middle|\, \mathcal{F}_t \right] \\ &= \int_t^T \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^l r(x) + s^{zero,cp}(x) + s^{zero,ref}(x)dx} s^{zero,ref}(l) \,\middle|\, \mathcal{F}_t \right] dl. \end{aligned}$$

Further, using Feynman-Kac (see Theorem 2.14) for

$$\begin{aligned} & v^{cp,ref}(t,T,r,s^{zero,cp},u^{zero,cp},s^{zero,ref},u^{zero,ref},w_1,w_2) \\ := \; & \mathbb{E}_{\widetilde{Q}} \left[e^{-\int_t^T r(x) + s^{zero,cp}(x) + s^{zero,ref}(x)dx} s^{zero,ref}(T) \,\middle|\, \mathcal{F}_t \right] \\ = \; & P^{d,z,cp,ref}(t,T) \cdot (K(t,T) + L(t,T) \cdot r(t) + M^{cp}(t,T) \cdot s^{zero,cp}(t) \\ & + M^{ref}(t,T) \cdot s^{zero,ref}(t) + N^{cp}(t,T) \cdot u^{cp}(t) + N^{ref}(t,T) \cdot u^{ref}(t) \\ & + O_1(t,T) \cdot w_1(t) + O_2(t,T) \cdot w_2(t)) \end{aligned}$$

we obtain the same equation as in Lemma D.1 with $P^{d,z,cp,ref}$ replaced by $v^{cp,ref}$. Since it must hold $v^{cp,ref}(T,T) = s^{zero,ref}(T)$, the boundary conditions are $M^{ref}(T,T) = 1$ and $K(T,T) = L(T,T) = \cdots = 0$. Now, we use the same procedure as it is done in Theorem 5.7. We insert the respective derivatives of $v^{cp,ref}$ which all contain $P^{d,z,cp,ref}$, we divide the whole equation by $P^{d,z,cp,ref} > 0$ and cancel terms with the help of the PDEs for $A^{d,z,cp,ref}$, $B^{d,z,cp,ref}$, $C_1^{d,z,cp,ref}$, $C_2^{d,z,cp,ref}$, $D_1^{d,z,cp,ref}$, $D_2^{d,z,cp,ref}$, $E_1^{d,z,cp,ref}$, and $E_2^{d,z,cp,ref}$

(see Lemma D.1). Hence, the equation is reduced to

$$
\begin{aligned}
0 \;=\;& \\
& (K_t + L_t \cdot r + M_t^{cp} \cdot s^{zero,cp} + M_t^{ref} \cdot s^{zero,ref} + N_t^{cp} \cdot u^{cp} + N_t^{ref} \cdot u^{ref} \\
& +(O_1)_t \cdot w_1 + (O_2)_t \cdot w_2) + \Big(\theta_r(t) + b_{rw_1} \cdot w_1 + \hat{b}_{rw_2} \cdot w_2 - \hat{a}_r \cdot r\Big) \cdot L \\
& + (\theta_{w_1} - \hat{a}_{w_1} \cdot w_1) \cdot O_1 + (\theta_{w_2} - \hat{a}_{w_2} \cdot w_2) \cdot O_2 \\
& +\Big(\theta_{s^{zero,cp}} + b_{s^{zero,cp}u^{cp}} \cdot u^{cp} - b_{s^{zero,cp}w_1} \cdot w_1 - b_{s^{zero,cp}w_2} \cdot w_2 \\
& -\hat{a}_{s^{cp}} \cdot s^{zero,cp}\Big) \cdot M^{cp} + (\theta_{u^{cp}} - \hat{a}_{u^{cp}} \cdot u^{cp}) \cdot N^{cp} \\
& +\Big(\theta_{s^{zero,ref}} + b_{s^{zero,ref}u^{ref}} \cdot u^{ref} - b_{s^{zero,ref}w_1} \cdot w_1 - b_{s^{zero,ref}w_2} \cdot w_2 \\
& -\hat{a}_{s^{ref}} \cdot s^{zero,ref}\Big) \cdot M^{ref} + \big(\theta_{u^{ref}} - \hat{a}_{u^{ref}} \cdot u^{ref}\big) \cdot N^{ref} \\
& -\Big[\sigma_r^2 \cdot B^{d,z,cp,ref} \cdot L + \sigma_{s^{zero,cp}}^2 \cdot C_1^{d,z,cp,ref} \cdot M^{cp} + \sigma_{s^{zero,ref}}^2 \cdot C_2^{d,z,cp,ref} \cdot M^{ref} \\
& +\sigma_{u^{cp}}^2 \cdot D_1^{d,z,cp,ref} \cdot N^{cp} + \sigma_{u^{ref}}^2 \cdot D_2^{d,z,cp,ref} \cdot N^{ref} + \sigma_{w_1}^2 \cdot E_1^{d,z,cp,ref} \cdot O_1 \\
& +\sigma_{w_2}^2 \cdot E_2^{d,z,cp,ref} \cdot O_2 + \sigma_{w_1} \cdot \sigma_{w_2} \cdot \rho_{w_1w_2} \cdot (E_1^{d,z,cp,ref} \cdot O_2 + E_2^{d,z,cp,ref} \cdot O_1) \\
& +\sigma_r \cdot \sigma_{w_1} \cdot \rho_{rw_1} \cdot (B^{d,z,cp,ref} \cdot O_1 + L \cdot E_1^{d,z,cp,ref}) \\
& +\sigma_r \cdot \sigma_{w_2} \cdot \left(\rho_{rw_1} \cdot \rho_{w_1w_2} + \rho_{rw_2} \cdot \sqrt{1-\rho_{w_1w_2}^2}\right) \\
& \cdot(B^{d,z,cp,ref} \cdot O_2 + L \cdot E_2^{d,z,cp,ref}) \\
& +\sigma_r \cdot \sigma_{s^{zero,cp}} \cdot (\rho_{rw_1} \cdot \rho_{s^{cp}w_1} + \rho_{rw_2} \cdot \rho_{s^{cp}w_2}) \cdot (B^{d,z,cp,ref} \cdot M^{cp} + L \cdot C_1^{d,z,cp,ref}) \\
& +\sigma_r \cdot \sigma_{s^{zero,ref}} \cdot (\rho_{rw_1} \cdot \rho_{s^{ref}w_1} + \rho_{rw_2} \cdot \rho_{s^{ref}w_2}) \cdot (B^{d,z,cp,ref} \cdot M^{ref} + L \cdot C_2^{d,z,cp,ref}) \\
& +\sigma_{s^{zero,cp}} \cdot \sigma_{u^{cp}} \cdot \rho_{s^{cp}u^{cp}} \cdot (C_1^{d,z,cp,ref} \cdot N^{cp} + M^{cp} \cdot D_1^{d,z,cp,ref}) \\
& +\sigma_{s^{zero,cp}} \cdot \sigma_{w_1} \cdot \rho_{s^{cp}w_1} \cdot (C_1^{d,z,cp,ref} \cdot O_1 + M^{cp} \cdot E_1^{d,z,cp,ref}) \\
& +\sigma_{s^{zero,ref}} \cdot \sigma_{u^{ref}} \cdot \rho_{s^{ref}u^{ref}} \cdot (C_2^{d,z,cp,ref} \cdot N^{ref} + M^{ref} \cdot D_2^{d,z,cp,ref}) \\
& +\sigma_{s^{zero,ref}} \cdot \sigma_{w_1} \cdot \rho_{s^{ref}w_1} \cdot (C_2^{d,z,cp,ref} \cdot O_1 + M^{ref} \cdot E_1^{d,z,cp,ref}) \\
& +\sigma_{s^{zero,cp}} \cdot \sigma_{w_2} \cdot \left(\rho_{s^{cp}w_1} \cdot \rho_{w_1w_2} + \rho_{s^{cp}w_2} \cdot \sqrt{1-\rho_{w_1w_2}^2}\right) \\
& \cdot(C_1^{d,z,cp,ref} \cdot O_2 + M^{cp} \cdot E_2^{d,z,cp,ref}) \\
& +\sigma_{s^{zero,ref}} \cdot \sigma_{w_2} \left(\rho_{s^{ref}w_1} \cdot \rho_{w_1w_2} + \rho_{s^{ref}w_2} \cdot \sqrt{1-\rho_{w_1w_2}^2}\right) \\
& \cdot(C_2^{d,z,cp,ref} \cdot O_2 + M^{ref} \cdot E_2^{d,z,cp,ref}) \\
& +\sigma_{s^{zero,cp}} \cdot \sigma_{s^{zero,ref}} \cdot (\rho_{s^{cp}w_1} \cdot \rho_{s^{ref}w_1} + \rho_{s^{cp}w_2} \cdot \rho_{s^{ref}w_2}) \\
& \cdot(C_1^{d,z,cp,ref} \cdot M^{ref} + M^{cp} \cdot C_2^{d,z,cp,ref})\Big]
\end{aligned}
$$

After rearranging the terms, we arrive at the following differential equations

$$\begin{aligned}
L_t - \hat{a}_r \cdot L &= 0\,,\\
M_t^{cp} - \hat{a}_{s^{cp}} \cdot M^{cp} &= 0\,,\\
M_t^{ref} - \hat{a}_{s^{ref}} \cdot M^{ref} &= 0\,,\\
N_t^{cp} + b_{s^{zero,cp}u^{cp}} \cdot M^{cp} - \hat{a}_{u^{cp}} \cdot N^{cp} &= 0\,,\\
N_t^{ref} + b_{s^{zero,ref}u^{ref}} \cdot M^{ref} - \hat{a}_{u^{ref}} \cdot N^{ref} &= 0\,,\\
(O_1)_t + b_{rw_1} \cdot L - b_{s^{zero,cp}w_1} \cdot M^{cp} - b_{s^{zero,ref}w_1} \cdot M^{ref} - \hat{a}_{w_1} \cdot O_1 &= 0\,,\\
(O_2)_t + \hat{b}_{rw_2} \cdot L - b_{s^{zero,cp}w_2} \cdot M^{cp} - b_{s^{zero,ref}w_2} \cdot M^{ref} - \hat{a}_{w_2} \cdot O_2 &= 0\,,\\
K_t + \theta_r(t) \cdot L + \theta_{w_1} \cdot O_1 + \theta_{w_2} \cdot O_2 + \theta_{u^{cp}} \cdot N^{cp} + \theta_{u^{ref}} \cdot N^{ref} &\\
+\theta_{s^{zero,cp}} \cdot M^{cp} + \theta_{s^{zero,ref}} \cdot M^{ref} - [\ldots] &= 0\,.
\end{aligned}$$

Considering the boundary conditions, the solutions of these differential equations are (cf. Theorem 2.15)

$$\begin{aligned}
L(t,T) &= 0\,,\\
M^{cp}(t,T) &= 0\,,\\
M^{ref}(t,T) &= e^{-\hat{a}_{s^{ref}} \cdot (T-t)}\,,\\
N^{cp}(t,T) &= 0\,,\\
N^{ref}(t,T) &= -b_{s^{zero,ref}u^{ref}} \cdot \frac{e^{-\hat{a}_{s^{ref}} \cdot (T-t)} - e^{-\hat{a}_{u^{ref}} \cdot (T-t)}}{\hat{a}_{s^{ref}} - \hat{a}_{u^{ref}}}\,,\\
O_1(t,T) &= b_{s^{zero,ref}w_1} \cdot \frac{e^{-\hat{a}_{s^{ref}} \cdot (T-t)} - e^{-\hat{a}_{w_1} \cdot (T-t)}}{\hat{a}_{s^{ref}} - \hat{a}_{w_1}}\,,\\
O_2(t,T) &= b_{s^{zero,ref}w_2} \cdot \frac{e^{-\hat{a}_{s^{ref}} \cdot (T-t)} - e^{-\hat{a}_{w_2} \cdot (T-t)}}{\hat{a}_{s^{ref}} - \hat{a}_{w_2}}\,,\\
K(t,T) &= -\int_t^T K_t(l,T)dl\,.
\end{aligned}$$

□

The lemma below is needed for Proposition 5.31.

Lemma D.4

For $t < T_a < T_b$ and $t < \min(T^{d,ref}, T^{d,cp})$, it holds that

$$P^{*,cp,ref}(t,T_a,T_b) := \mathbb{E}_{\widetilde{Q}}\left[e^{-\int_t^{T_a} r(x)+s^{zero,cp}(x)+s^{zero,ref}(x)dx} P^{d,cp,ref}(T_a,T_b) \,\middle|\, \mathcal{F}_t \right]$$

with

$$\begin{aligned} &P^{*,cp,ref}(t,T_a,T_b) \\ &= e^{A^{*,cp,ref}(t,T_a,T_b)-B^{*,cp,ref}(t,T_a,T_b)r-C_1^{*,cp,ref}(t,T_a,T_b)s^{zero,cp}-D_1^{*,cp,ref}(t,T_a,T_b)u^{cp}} \\ &\quad \cdot e^{-C_2^{*,cp,ref}(t,T_a,T_b)s^{ref}-D_2^{*,cp,ref}(t,T_a,T_b)u^{ref}-E_1^{*,cp,ref}(t,T_a,T_b)w_1-E_2^{*,cp,ref}(t,T_a,T_b)w_2} \end{aligned}$$

with the functions $A^{*,cp,ref}(t,T_a,T_b)$, $B^{*,cp,ref}(t,T_a,T_b)$, $C_1^{*,cp,ref}(t,T_a,T_b)$, $C_2^{*,cp,ref}(t,T_a,T_b)$, $D_1^{*,cp,ref}(t,T_a,T_b)$, $D_2^{*,cp,ref}(t,T_a,T_b)$, $E_1^{*,cp,ref}(t,T_a,T_b)$, and $E_2^{*,cp,ref}(t,T_a,T_b)$ defined at the end of the proof.

Proof:

Applying Feynman-Kac (see Theorem 2.14) leads to the same equation as in the proof of Lemma D.1 except that all entries referring to $s^{zero,ref}$ on the right hand side of the equation have to be substituted with s^{ref}. If we now insert the partial derivatives, use $s^{zero,ref} = \frac{s^{ref}}{1-z^{ref}}$, regroup the terms and divide by $P^{*,cp,ref} > 0$, we obtain the following system of differential equations:

$$\begin{aligned} (B^{*,cp,ref})_t &= \hat{a}_r \cdot B^{*,cp,ref} - 1 \\ (C_1^{*,cp,ref})_t &= \hat{a}_{s^{cp}} \cdot C_1^{*,cp,ref} - 1 \\ (C_2^{*,cp,ref})_t &= \hat{a}_{s^{ref}} \cdot C_2^{*,cp,ref} - \frac{1}{1-z^{ref}} \\ (D_1^{*,cp,ref})_t &= \hat{a}_{u^{cp}} \cdot D_1^{*,cp,ref} - b_{s^{zero,cp}u^{cp}} \cdot C_1^{*,cp,ref} \\ (D_2^{*,cp,ref})_t &= \hat{a}_{u^{ref}} \cdot D_2^{*,cp,ref} - b_{s^{ref}u^{ref}} \cdot C_2^{*,cp,ref} \\ &(E_1^{*,cp,ref})_t \\ &= \hat{a}_{w_1} \cdot E_1^{*,cp,ref} - b_{rw_1} \cdot B^{*,cp,ref} + b_{s^{zero,cp}w_1} \cdot C_1^{*,cp,ref} \\ &\quad + b_{s^{ref}w_1} \cdot C_2^{*,cp,ref} \\ &(E_2^{*,cp,ref})_t \\ &= \hat{a}_{w_2} \cdot E_2^{*,cp,ref} - \hat{b}_{rw_2} \cdot B^{*,cp,ref} + b_{s^{zero,cp}w_2} \cdot C_1^{*,cp,ref} \\ &\quad + b_{s^{ref}w_2} \cdot C_2^{*,cp,ref} \end{aligned}$$

where the differential equations for $B^{*,cp,ref}$, $C_2^{*,cp,ref}$, and $D_2^{*,cp,ref}$ are analogous to the proof of Proposition 5.10 (see pages 263ff) and $(A^{*,cp,ref})_t$ has the same structure as $(A^{d,z,cp,ref})_t$ in Lemma D.1 with $\theta_{s^{zero,ref}}$ and $\sigma_{s^{zero,ref}}$

replaced by $\theta_{s^{ref}}$ and $\sigma_{s^{ref}}$. Since it must hold that $P^{*,cp,ref}(T_a,T_a,T_b) = P^{d,cp,ref}(T_a,T_b)$, the boundary conditions are
$A^{*,cp,ref}(T_a,T_a,T_b) = A^{d,cp,ref}(T_a,T_b)$, $B^{*,cp,ref}(T_a,T_a,T_b) = B^{d,cp,ref}(T_a,T_b)$,
$C_1^{*,cp,ref}(T_a,T_a,T_b) = C_1^{d,cp,ref}(T_a,T_b)$, $C_2^{*,cp,ref}(T_a,T_a,T_b) = C_2^{d,cp,ref}(T_a,T_b)$,
$D_1^{*,cp,ref}(T_a,T_a,T_b) = D_1^{d,cp,ref}(T_a,T_b)$, $D_2^{*,cp,ref}(T_a,T_a,T_b) = D_2^{d,cp,ref}(T_a,T_b)$,
$E_1^{*,cp,ref}(T_a,T_a,T_b) = E_1^{d,cp,ref}(T_a,T_b)$ and $E_2^{*,cp,ref}(T_a,T_a,T_b) = E_2^{d,cp,ref}(T_a,T_b)$.
Hence, the differential equations result in (c.f. Theorem 2.15 and Proposition 5.10)

$$\begin{aligned}
B^{*,cp,ref}(t,T_a,T_b) &= e^{-\hat{a}_r\cdot(T_a-t)}\cdot\Big(B^{*,cp,ref}(T_a,T_a,T_b)+\int_0^{T_a-t}e^{\hat{a}_r\cdot l}dl\Big)\\
&= \frac{1}{\hat{a}_r}\cdot\big(1-e^{-\hat{a}_r\cdot(T_b-t)}\big)=B(t,T_b),
\end{aligned}$$

$$\begin{aligned}
C_1^{*,cp,ref}(t,T_a,T_b) &= e^{-\hat{a}_{s^{cp}}\cdot(T_a-t)}\cdot\Big(C_1^{*,cp,ref}(T_a,T_a,T_b)+\int_0^{T_a-t}e^{\hat{a}_{s^{cp}}\cdot l}dl\Big)\\
&= \frac{1}{\hat{a}_{s^{cp}}}\cdot\big(1-e^{-\hat{a}_{s^{cp}}\cdot(T_b-t)}\big)=C^{d,cp}(t,T_b),
\end{aligned}$$

$$\begin{aligned}
&C_2^{*,cp,ref}(t,T_a,T_b)\\
&= e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}\cdot\Big(C_2^{*,cp,ref}(T_a,T_a,T_b)+\frac{1}{1-z^{ref}}\cdot\int_0^{T_a-t}e^{\hat{a}_{s^{ref}}\cdot l}dl\Big)\\
&= e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}\cdot C^{d,ref}(T_a,T_b)+\frac{1}{1-z^{ref}}\cdot C^{d,ref}(t,T_a),
\end{aligned}$$

$$\begin{aligned}
&D_1^{*,cp,ref}(t,T_a,T_b)\\
&= e^{-\hat{a}_{u^{cp}}(T_a-t)}\cdot\Big(D_1^{*,cp,ref}(T_a,T_a,T_b)\\
&\quad+\int_0^{T_a-t}e^{\hat{a}_{u^{cp}}\cdot l}\cdot b_{s^{zero,cp}u^{cp}}\cdot C_1^{*,cp,ref}(0,l,T_b-T_a+l)dl\Big)\\
&= \frac{b_{s^{zero,cp}u^{cp}}}{\hat{a}_{s^{cp}}}\cdot\Big(\frac{1-e^{-\hat{a}_{u^{cp}}\cdot(T_b-t)}}{\hat{a}_{u^{cp}}}+\frac{e^{-\hat{a}_{u^{cp}}\cdot(T_b-t)}-e^{-\hat{a}_{s^{cp}}\cdot(T_b-t)}}{\hat{a}_{u^{cp}}-\hat{a}_{s^{cp}}}\Big)\\
&= D^{d,zero,cp}(t,T_b),
\end{aligned}$$

$$
\begin{aligned}
&D_2^{*,cp,ref}(t,T_a,T_b)\\
&\quad= e^{-\hat{a}_{u^{ref}}(T_a-t)}\cdot\Big(D_2^{*,cp,ref}(T_a,T_a,T_b)\\
&\qquad+\int_0^{T_a-t} e^{\hat{a}_{u^{ref}}\cdot l}\cdot b_{s^{ref}u^{ref}}\cdot C_2^{*,cp,ref}(0,l,T_b-T_a+l)dl\Big)\\
&\quad= e^{-\hat{a}_{u^{ref}}\cdot(T_a-t)}\cdot D^{d,ref}(T_a,T_b)\\
&\qquad-b_{s^{ref}u^{ref}}\cdot C^{d,ref}(T_a,T_b)\cdot\left(\frac{e^{-\hat{a}_{s^{ref}}\cdot(T_a-t)}-e^{-\hat{a}_{u^{ref}}\cdot(T_a-t)}}{\hat{a}_{s^{ref}}-\hat{a}_{u^{ref}}}\right)\\
&\qquad+\frac{1}{1-z^{ref}}\cdot D^{d,ref}(t,T_a),
\end{aligned}
$$

$$
\begin{aligned}
&E_1^{*,cp,ref}(t,T_a,T_b)\\
&\quad= e^{-\hat{a}_{w_1}\cdot(T_a-t)}\Big(E_1^{*,cp,ref}(T_a,T_a,T_b)\\
&\qquad+\int_0^{T_a-t} e^{\hat{a}_{w_1}\cdot l}(b_{rw_1}\cdot B^{*,cp,ref}(0,l,l+T_b-T_a)\\
&\qquad-b_{s^{zero,cp}w_1}\cdot C_1^{*,cp,ref}(0,l,l+T_b-T_a)\\
&\qquad-b_{s^{ref}w_1}\cdot C_2^{*,cp,ref}(0,l,l+T_b-T_a))dl\Big)\\
&\quad= e^{-\hat{a}_{w_1}\cdot(T_a-t)}\cdot E_1^{d,cp,ref}(T_a,T_b)\\
&\qquad+\frac{b_{rw_1}}{\hat{a}_r}\cdot\left(\frac{1-e^{-\hat{a}_{w_1}\cdot(T_a-t)}}{\hat{a}_{w_1}}+e^{-\hat{a}_r\cdot(T_b-T_a)}\frac{e^{-\hat{a}_{w_1}\cdot(T_a-t)}-e^{-\hat{a}_r\cdot(T_a-t)}}{\hat{a}_{w_1}-\hat{a}_r}\right)\\
&\qquad-\frac{b_{s^{zero,cp}w_1}}{\hat{a}_{s^{cp}}}\cdot\left(\frac{1-e^{-\hat{a}_{w_1}\cdot(T_a-t)}}{\hat{a}_{w_1}}+e^{-\hat{a}_{s^{cp}}\cdot(T_b-T_a)}\frac{e^{-\hat{a}_{w_1}\cdot(T_a-t)}-e^{-\hat{a}_{s^{cp}}\cdot(T_a-t)}}{\hat{a}_{w_1}-\hat{a}_{s^{cp}}}\right)\\
&\qquad+b_{s^{ref}w_1}C^{d,ref}(T_a,T_b)\left(\frac{e^{-\hat{a}_{s^{ref}}(T_a-t)}-e^{-\hat{a}_{w_1}(T_a-t)}}{\hat{a}_{s^{ref}}-\hat{a}_{w_1}}\right)\\
&\qquad+\frac{1}{1-z^{ref}}(E_1^{d,ref}(t,T_a)-E_1^{ref}(t,T_a)),
\end{aligned}
$$

$E_2^{*,cp,ref}(t,T_a,T_b)$ is determined analogously.

$$
A^{*,cp,ref}(t,T_a,T_b)=A^{*,cp,ref}(T_a,T_a,T_b)-\int_t^{T_a} A_t^{*,cp,ref}(l,T_a,T_b)dl\,.
$$

□

Appendix E

Inflation-Indexed Derivatives

This chapter contains certain terms that are needed for pricing inflation-indexed products.

Lemma E.1
The correlation adjustment $C^{yoy}(t, T_{i-1}, T_i)$ *for year-on-year inflation-indexed swaps is calculated as*

$$C^{yoy}(t, T_{i-1}, T_i) := Covar_{\widetilde{Q}} \left[\int_t^{T_{i-1}} r_I(x)dx \, , \int_{T_{i-1}}^{T_i} r_R(x)dx \middle| \mathcal{F}_t \right]$$
$$= \frac{b_{Rw} \cdot \sigma_w \cdot \sigma_I \cdot \rho_{wI}}{(\hat{a}_R - \hat{a}_w) \cdot \hat{a}_I} \Big(B^w(T_{i-1}, T_i) \cdot (B^w(t, T_{i-1}) - B^{I+w}(t, T_{i-1}))$$
$$-B^R(T_{i-1}, T_i) \cdot (B^R(t, T_{i-1}) - B^{I+R}(t, T_{i-1})) \Big) \; .$$

Proof:
Under the risk-neutral measure $\widetilde{Q}$, it holds for $t \leq x$ and $t \leq T_{i-1} \leq T_i$:

$$\begin{aligned} r_I(x)\big|_{\mathcal{F}_t} & \\ = \; & r_I(t) \cdot e^{-\hat{a}_I(x-t)} + \theta_I \cdot B^I(t, x) \\ + \; & \sigma_I \cdot \rho_{wI} \int_t^x e^{-\hat{a}_I(x-y)} d\widetilde{W}_w(y) + \sigma_I \sqrt{1 - \rho_{wI}^2} \int_t^x e^{-\hat{a}_I(x-y)} d\widetilde{W}_I(y) \; , \end{aligned}$$

$$\begin{aligned}
&\int_{T_{i-1}}^{T_i} r_I(x)dx\bigg|_{\mathcal{F}_t} \\
&= r_I(t)\int_{T_{i-1}}^{T_i} e^{-\hat{a}_I(x-t)}dx + \theta_I \int_{T_{i-1}}^{T_i} B^I(t,x)dx \\
&+ \sigma_I \cdot \rho_{wI} \int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_I(x-y)}d\widetilde{W}_w(y)dx \\
&+ \sigma_I\sqrt{1-\rho_{wI}^2}\int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_I(x-y)}d\widetilde{W}_I(y)dx \\
&= r_I(t)\cdot e^{-\hat{a}_I(T_{i-1}-t)}\cdot B^I(T_{i-1},T_i) \\
&+ \frac{\theta_I}{\hat{a}_I}\left(T_i - T_{i-1} - e^{-\hat{a}_I(T_{i-1}-t)}\cdot B^I(T_{i-1},T_i)\right) \\
&+ \sigma_I\cdot\rho_{wI}\left(\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_w(y) + \int_{T_{i-1}}^{T_i} B^I(y,T_i)d\widetilde{W}_w(y)\right) \\
&+ \sigma_I\sqrt{1-\rho_{wI}^2}\left(\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_I(y) + \int_{T_{i-1}}^{T_i} B^I(y,T_i)d\widetilde{W}_I(y)\right) ,
\end{aligned}$$

and

$$\begin{aligned}
&r_R(x)\big|_{\mathcal{F}_t} \\
&= r_R(t)\cdot e^{-\hat{a}_R(x-t)} + \int_t^x \theta_R(y)e^{-\hat{a}_R(x-y)}dy \\
&+ b_{Rw}\int_t^x w(y)e^{-\hat{a}_R(x-y)}dy + \sigma_R\int_t^x e^{-\hat{a}_R(x-y)}d\widetilde{W}_R(y) \\
&= r_R(t)\cdot e^{-\hat{a}_R(x-t)} + \int_t^x \theta_R(y)e^{-\hat{a}_R(x-y)}dy \\
&+ \frac{b_{Rw}}{\hat{a}_R-\hat{a}_w}\left(w(t)-\frac{\theta_w}{\hat{a}_w}\right)\cdot\left(e^{-\hat{a}_w(x-t)}-e^{-\hat{a}_R(x-t)}\right) + \frac{b_{Rw}\cdot\theta_w}{\hat{a}_w}B^R(t,x) \\
&+ \frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\int_t^x(e^{-\hat{a}_w(x-y)}-e^{-\hat{a}_R(x-y)})d\widetilde{W}_w(y) \\
&+ \sigma_R\int_t^x e^{-\hat{a}_R(x-y)}d\widetilde{W}_R(y) ,
\end{aligned}$$

$$\begin{aligned}
&\int_{T_{i-1}}^{T_i} r_R(x)dx\bigg|_{\mathcal{F}_t} \\
&= r_R(t)\int_{T_{i-1}}^{T_i} e^{-\hat{a}_R(x-t)}dx + \int_{T_{i-1}}^{T_i}\int_t^x \theta_R(y)e^{-\hat{a}_R(x-y)}dydx
\end{aligned}$$

$$+\frac{b_{Rw}}{\hat{a}_R-\hat{a}_w}\left(w(t)-\frac{\theta_w}{\hat{a}_w}\right)\int_{T_{i-1}}^{T_i}\left(e^{-\hat{a}_w(x-t)}-e^{-\hat{a}_R(x-t)}\right)dx$$
$$+\frac{b_{Rw}\cdot\theta_w}{\hat{a}_w}\int_{T_{i-1}}^{T_i}B^R(t,x)dx$$
$$+\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\int_{T_{i-1}}^{T_i}\int_t^x(e^{-\hat{a}_w(x-y)}-e^{-\hat{a}_R(x-y)})d\widetilde{W}_w(y)dx$$
$$+\sigma_R\int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_R(x-y)}d\widetilde{W}_R(y)dx$$
$$=r_R(t)\cdot e^{-\hat{a}_R(T_{i-1}-t)}\cdot B^R(T_{i-1},T_i)$$
$$+\int_t^{T_{i-1}}\theta_R(y)e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i)dy+\int_{T_{i-1}}^{T_i}\theta_R(y)B^R(y,T_i)dy$$
$$+\frac{b_{Rw}}{\hat{a}_R-\hat{a}_w}\left(w(t)-\frac{\theta_w}{\hat{a}_w}\right)\cdot\left(e^{-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)-e^{-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)\right)$$
$$+\frac{b_{Rw}\cdot\theta_w}{\hat{a}_w\cdot\hat{a}_R}\left(T_i-T_{i-1}-e^{-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)\right)$$
$$+\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\Big(\int_t^{T_{i-1}}(e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i)-e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))d\widetilde{W}_w(y)\Big)$$
$$+\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\Big(\int_{T_{i-1}}^{T_i}(B^w(y,T_i)-B^R(y,T_i))d\widetilde{W}_w(y)\Big)$$
$$+\sigma_R\Big(\int_t^{T_{i-1}}e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i)d\widetilde{W}_R(y)+\int_{T_{i-1}}^{T_i}B^R(y,T_i)d\widetilde{W}_R(y)\Big)\ .$$

Hence, the covariance $Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}}r_I(x)dx\,,\int_{T_{i-1}}^{T_i}r_R(x)dx\middle|\,\mathcal{F}_t\right]$ is calculated as follows:

$$C^{yoy}(t,T_{i-1},T_i)$$
$$=Covar_{\widetilde{Q}}\Bigg[\sigma_I\cdot\rho_{wI}\int_t^{T_{i-1}}B^I(y,T_{i-1})d\widetilde{W}_w(y)\,,$$
$$\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\int_t^{T_{i-1}}(e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i)-e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))d\widetilde{W}_w(y)\Bigg|\,\mathcal{F}_t\Bigg]$$
$$=\frac{b_{Rw}\cdot\sigma_w\cdot\sigma_I\cdot\rho_{wI}}{\hat{a}_R-\hat{a}_w}\int_t^{T_{i-1}}B^I(y,T_{i-1})\cdot(e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i)$$
$$-e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))dy$$

$$= \frac{b_{Rw} \cdot \sigma_w \cdot \sigma_I \cdot \rho_{wI}}{(\hat{a}_R - \hat{a}_w) \cdot \hat{a}_I} \Big(B^w(T_{i-1}, T_i) \cdot (B^w(t, T_{i-1}) - B^{I+w}(t, T_{i-1}))$$
$$- B^R(T_{i-1}, T_i) \cdot (B^R(t, T_{i-1}) - B^{I+R}(t, T_{i-1})) \Big) \,.$$

□

Corollary E.2
The following general equations hold:

$$\int_{T_1}^{T_2} B^x(y, T_3) B^v(y, T_4) dy$$
$$= \int_{T_1}^{T_2} \frac{1}{\hat{a}_x} \left(1 - e^{-\hat{a}_x(T_3 - y)}\right) \frac{1}{\hat{a}_v} \left(1 - e^{-\hat{a}_v(T_4 - y)}\right) dy$$
$$= \frac{1}{\hat{a}_x \hat{a}_v} \left[T_2 - T_1 - e^{-\hat{a}_x(T_3 - T_2)} B^x(T_1, T_2) - e^{-\hat{a}_v(T_4 - T_2)} B^v(T_1, T_2) \right.$$
$$\left. + e^{-\hat{a}_x(T_3 - T_2) - \hat{a}_v(T_4 - T_2)} B^{x+v}(T_1, T_2)\right] \,,$$

and

$$\int_{T_1}^{T_2} e^{-\hat{a}_x(T_3 - y)} B^v(y, T_4) dy$$
$$= \frac{1}{\hat{a}_v} \left[e^{-\hat{a}_x(T_3 - T_2)} B^x(T_1, T_2) - e^{-\hat{a}_x(T_3 - T_2) - \hat{a}_v(T_4 - T_2)} B^{x+v}(T_1, T_2)\right] \,,$$

and

$$\int_{T_1}^{T_2} (B^x(y, T_3) - B^x(y, T_2))(B^x(y, T_4) - B^x(y, T_3)) dy$$
$$= \int_{T_1}^{T_2} e^{-\hat{a}_x(T_2 - y)} B^x(T_2, T_3) e^{-\hat{a}_x(T_3 - y)} B^x(T_3, T_4) dy$$
$$= B^x(T_2, T_3) B^x(T_3, T_4) \frac{1}{2\hat{a}_x} \left[e^{-\hat{a}_x(T_3 - T_2)} - e^{-\hat{a}_x(T_3 + T_2 - 2T_1)}\right]$$
$$= B^x(T_2, T_3) B^x(T_3, T_4) e^{-\hat{a}_x(T_3 - T_2)} B^{x+x}(T_1, T_2) \,.$$

Lemma E.3
The covariance terms for the correlation adjustment $C^{del}(t, T_{i-1}, T_i)$ for de-

layed payments of Proposition 6.4 are

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_{T_i}^{T_{i^{pay}}} r_I(x)dx\middle|\,\mathcal{F}_t\right]\\
&=\frac{\sigma_I^2}{\hat{a}_I^2}\cdot\big(\hat{a}_I^2\cdot e^{-\hat{a}_I(T_i-T_{i-1})}\cdot B^{I+I}(t,T_{i-1})\cdot B^I(T_{i-1},T_i)\cdot B^I(T_i,T_{i^{pay}})\\
&\qquad+\hat{a}_I\cdot B^I(T_i,T_{i^{pay}})\cdot(B^I(T_{i-1},T_i)-B^{I+I}(T_{i-1},T_i))\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_{T_i}^{T_{i^{pay}}} r_R(x)dx\middle|\,\mathcal{F}_t\right]\\
&=\frac{b_{Rw}\sigma_w\sigma_I\rho_{wI}}{(\hat{a}_R-\hat{a}_w)\hat{a}_I}\cdot\Big(B^w(T_i,T_{i^{pay}})\cdot(B^w(T_{i-1},T_i)-B^{I+w}(T_{i-1},T_i)\\
&\qquad+\hat{a}_I\cdot e^{-\hat{a}_w(T_i-T_{i-1})}\cdot B^I(T_{i-1},T_i)\cdot B^{I+w}(t,T_{i-1}))\\
&\qquad-B^R(T_i,T_{i^{pay}})\cdot(B^R(T_{i-1},T_i)-B^{I+R}(T_{i-1},T_i)\\
&\qquad+\hat{a}_I\cdot e^{-\hat{a}_R(T_i-T_{i-1})}\cdot B^I(T_{i-1},T_i)\cdot B^{I+R}(t,T_{i-1}))\Big).
\end{aligned}
$$

Proof:
According to Lemma E.1 it holds

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\Big[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_{T_i}^{T_{i^{pay}}} r_I(x)dx\Big|\,\mathcal{F}_t\Big]=\\
&Covar_{\widetilde{Q}}\Big[\sigma_I\cdot\rho_{wI}\left(\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_w(y)+\int_{T_{i-1}}^{T_i}B^I(y,T_i)d\widetilde{W}_w(y)\right),\\
&\sigma_I\cdot\rho_{wI}\left(\int_t^{T_i}(B^I(y,T_{i^{pay}})-B^I(y,T_i))d\widetilde{W}_w(y)+\int_{T_i}^{T_{i^{pay}}}B^I(y,T_{i^{pay}})d\widetilde{W}_w(y)\right)\Big|\,\mathcal{F}_t\Big]+\\
&Covar_{\widetilde{Q}}\Big[\sigma_I\cdot\sqrt{1-\rho_{wI}^2}\left(\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_I(y)+\int_{T_{i-1}}^{T_i}B^I(y,T_i)d\widetilde{W}_I(y)\right),\\
&\sigma_I\cdot\sqrt{1-\rho_{wI}^2}\left(\int_t^{T_i}(B^I(y,T_{i^{pay}})-B^I(y,T_i))d\widetilde{W}_I(y)+\int_{T_i}^{T_{i^{pay}}}B^I(y,T_{i^{pay}})d\widetilde{W}_I(y)\right)\Big|\,\mathcal{F}_t\Big]\\
&=\sigma_I^2\int_t^{T_{i-1}}\big(B^I(y,T_i)-B^I(y,T_{i-1})\big)\cdot\big(B^I(y,T_{i^{pay}})-B^I(y,T_i)\big)\,dy\\
&\qquad+\sigma_I^2\int_{T_{i-1}}^{T_i}B^I(y,T_i)\cdot\big(B^I(y,T_{i^{pay}})-B^I(y,T_i)\big)\,dy\,.
\end{aligned}
$$

The integrals within the above covariance can be calculated with the help of the equations of Corollary E.2. Hence, it holds for

$$
\begin{aligned}
& Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx \, , \int_{T_i}^{T_{i^{pay}}} r_I(x)dx \,\middle|\, \mathcal{F}_t \right] \\
&= \frac{\sigma_I^2}{\hat{a}_I^2} \Big[\hat{a}_I^2 e^{-\hat{a}_I(T_i - T_{i-1})} B^I(T_i, T_{i^{pay}}) B^I(T_{i-1}, T_i) B^{I+I}(t, T_{i-1}) \\
&\qquad + (1 - e^{-\hat{a}_I(T_{i^{pay}} - T_i)}) B^I(T_{i-1}, T_i) + (e^{-\hat{a}_I(T_{i^{pay}} - T_i)} - 1) B^{I+I}(T_{i-1}, T_i) \Big] \\
&= \frac{\sigma_I^2}{\hat{a}_I^2} \Big[\hat{a}_I^2 e^{-\hat{a}_I(T_i - T_{i-1})} B^I(T_i, T_{i^{pay}}) B^I(T_{i-1}, T_i) B^{I+I}(t, T_{i-1}) \\
&\qquad + \hat{a}_I B^I(T_i, T_{i^{pay}}) (B^I(T_{i-1}, T_i) - B^{I+I}(T_{i-1}, T_i)) \Big] \, .
\end{aligned}
$$

With the help of Lemma E.1 and Corollary E.2 it also holds

$$
\begin{aligned}
& Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx \, , \int_{T_i}^{T_{i^{pay}}} r_R(x)dx \,\middle|\, \mathcal{F}_t \right] = \\
& Covar_{\widetilde{Q}}\Big[\sigma_I \cdot \rho_{wI} \left(\int_t^{T_{i-1}} (B^I(y, T_i) - B^I(y, T_{i-1})) d\widetilde{W}_w(y) + \int_{T_{i-1}}^{T_i} B^I(y, T_i) d\widetilde{W}_w(y) \right) , \\
& \frac{b_{Rw} \cdot \sigma_w}{\hat{a}_R - \hat{a}_w} \left(\int_t^{T_i} (e^{-\hat{a}_w(T_i - y)} B^w(T_i, T_{i^{pay}}) - e^{-\hat{a}_R(T_i - y)} B^R(T_i, T_{i^{pay}})) d\widetilde{W}_w(y) \right. \\
& \left. + \int_{T_i}^{T_{i^{pay}}} (B^w(y, T_{i^{pay}}) - B^R(y, T_{i^{pay}})) d\widetilde{W}_w(y) \right) \Big| \mathcal{F}_t \Big] \\
&= \frac{\sigma_I \cdot \rho_{wI} \cdot b_{Rw} \cdot \sigma_w}{\hat{a}_R - \hat{a}_w} \int_t^{T_{i-1}} (B^I(y, T_i) - B^I(y, T_{i-1}))(e^{-\hat{a}_w(T_i - y)} B^w(T_i, T_{i^{pay}}) \\
& - e^{-\hat{a}_R(T_i - y)} B^R(T_i, T_{i^{pay}})) dy \\
& + \int_{T_{i-1}}^{T_i} B^I(y, T_i)(e^{-\hat{a}_w(T_i - y)} B^w(T_i, T_{i^{pay}}) - e^{-\hat{a}_R(T_i - y)} B^R(T_i, T_{i^{pay}})) dy \\
&= \frac{\sigma_I \cdot \rho_{wI} \cdot b_{Rw} \cdot \sigma_w}{\hat{a}_I \cdot (\hat{a}_R - \hat{a}_w)} \Big[B^w(T_i, T_{i^{pay}})(e^{-\hat{a}_w(T_i - T_{i-1})}(1 - e^{-\hat{a}_I(T_i - T_{i-1})}) B^{w+I}(t, T_{i-1}) \\
& + B^w(T_{i-1}, T_i) - B^{w+I}(T_{i-1}, T_i)) - B^R(T_i, T_{i^{pay}})(B^R(T_{i-1}, T_i) - B^{R+I}(T_{i-1}, T_i) \\
& + e^{-\hat{a}_R(T_i - T_{i-1})}(1 - e^{-\hat{a}_I(T_i - T_{i-1})}) B^{R+I}(t, T_{i-1})) \Big]
\end{aligned}
$$

$$= \frac{\sigma_I \cdot \rho_{wI} \cdot b_{Rw} \cdot \sigma_w}{\hat{a}_I \cdot (\hat{a}_R - \hat{a}_w)} \Big[B^w(T_i, T_{i^{pay}})(\hat{a}_I e^{-\hat{a}_w(T_i - T_{i-1})} B^I(T_{i-1}, T_i) B^{w+I}(t, T_{i-1})$$
$$+ B^w(T_{i-1}, T_i) - B^{w+I}(T_{i-1}, T_i)) - B^R(T_i, T_{i^{pay}})($$
$$\hat{a}_I e^{-\hat{a}_R(T_i - T_{i-1})} B^I(T_{i-1}, T_i) B^{R+I}(t, T_{i-1}) + B^R(T_{i-1}, T_i) - B^{R+I}(T_{i-1}, T_i)) \Big]$$

□

Corollary E.4
The following general equations hold:

(a)

$$\int_t^x e^{-\hat{a}_1(x-l)} e^{-\hat{a}_2(T_i - l)} dl = e^{-\hat{a}_2(T_i - x)} B^{1+2}(t, x)$$

(b)

$$\int_t^x e^{-\hat{a}_1(x-l)} B^2(l, T_i) dl = \frac{1}{\hat{a}_2} [B^1(t, x) - e^{-\hat{a}_2(T_i - x)} B^{1+2}(t, x)]$$

(c)

$$\begin{aligned} &\int_t^x e^{-\hat{a}_1(x-l)} \int_t^l e^{-\hat{a}_2(l-y)} dy dl \\ = &\int_t^x e^{-\hat{a}_1(x-l)} B^2(t, l) dl = \frac{1}{\hat{a}_2} [B^1(t, x) - e^{-\hat{a}_2(x-t)} B^{1-2}(t, x)] \end{aligned}$$

(d)

$$\begin{aligned} &\int_t^x e^{-\hat{a}_1(x-l)} \int_t^l e^{-\hat{a}_2(l-y)} e^{-\hat{a}_3(T_i - y)} dy dl \\ = &\int_t^x e^{-\hat{a}_1(x-l)} \frac{1}{\hat{a}_2 + \hat{a}_3} (e^{-\hat{a}_3(T_i - l)} - e^{-\hat{a}_2(l-t) - \hat{a}_3(T_i - t)}) dl \\ = &\frac{1}{\hat{a}_2 + \hat{a}_3} [e^{-\hat{a}_3(T_i - x)} B^{1+3}(t, x) - e^{-\hat{a}_3(T_i - t) - \hat{a}_2(x-t)} B^{1-2}(t, x)] \\ = &\frac{1}{\hat{a}_2 + \hat{a}_3} [e^{-\hat{a}_3(T_i - x)} B^{1+3}(t, x) - e^{-\hat{a}_3(T_i - t) - \hat{a}_1(x-t)} B^{2-1}(t, x)] \end{aligned}$$

(e)

$$
\begin{aligned}
&\int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_1(x-l)}e^{-\hat{a}_2(T_i-l)}dldx \\
&\stackrel{(a)}{=} \int_{T_{i-1}}^{T_i} e^{-\hat{a}_2(T_i-x)}B^{1+2}(t,x)dx \\
&= \int_{T_{i-1}}^{T_i} \frac{1}{\hat{a}_1+\hat{a}_2}(e^{-\hat{a}_2(T_i-x)} - e^{-\hat{a}_2(T_i-t)}e^{-\hat{a}_1(x-t)})dx \\
&= \frac{1}{\hat{a}_1+\hat{a}_2}(B^2(T_{i-1},T_i) - e^{-\hat{a}_1(T_{i-1}-t)-\hat{a}_2(T_i-t)}B^1(T_{i-1},T_i))
\end{aligned}
$$

(f)

$$
\begin{aligned}
&\int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_1(x-l)}B^2(l,T_i)dldx \\
&\stackrel{(b)}{=} \int_{T_{i-1}}^{T_i} \frac{1}{\hat{a}_2}[B^1(t,x) - e^{-\hat{a}_2(T_i-x)}B^{1+2}(t,x)]dx \\
&\stackrel{(e)}{=} \frac{1}{\hat{a}_2}[\frac{1}{\hat{a}_1}(T_i - T_{i-1} - e^{-\hat{a}_1(T_{i-1}-t)}B^1(T_{i-1},T_i)) - \frac{1}{\hat{a}_1+\hat{a}_2}(B^2(T_{i-1},T_i) \\
&\quad -e^{-\hat{a}_1(T_{i-1}-t)-\hat{a}_2(T_i-t)}B^1(T_{i-1},T_i))]
\end{aligned}
$$

(g)

$$
\begin{aligned}
&\int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_1(T_i-l)}dldx \\
&= \int_{T_{i-1}}^{T_i} \frac{1}{\hat{a}_1}(e^{-\hat{a}_1(T_i-x)} - e^{-\hat{a}_1(T_i-t)})dx \\
&= \frac{1}{\hat{a}_1}[B^1(T_{i-1},T_i) - e^{-\hat{a}_1(T_i-t)}(T_i - T_{i-1})]
\end{aligned}
$$

Lemma E.5
The terms needed for the approximation of an inflation hybrid's price in Theorem 6.6 are given in this lemma's proof.

Proof:
In order to determine the expected value of $\frac{1}{P(T_{i-1},T_i)}$ under the T_i-forward measure we need to adjust the risk-neutral value by the following terms

resulting from the change of measure.

$$I_1(x) := \sigma_w \sigma_I \rho_{wI} \int_t^x e^{-\hat{a}_I(x-l)} E_1(l, T_i) dl$$
$$\overset{Cor.E.4(a)(b)}{=} \frac{\sigma_w \sigma_I \rho_{wI} b_{Rw}}{\hat{a}_R} \Big(\frac{1}{\hat{a}_w} \big(B^I(t,x) - e^{-\hat{a}_w(T_i - x)} B^{I+w}(t,x) \big)$$
$$+ \frac{1}{\hat{a}_w - \hat{a}_R} \big(e^{-\hat{a}_w(T_i-x)} B^{I+w}(t,x) - e^{-\hat{a}_R(T_i-x)} B^{I+R}(t,x) \big) \Big)$$

$$I_2(x, I) := \sigma_I^2 \int_t^x e^{-\hat{a}_I(x-l)} B^I(l, T_i) dl$$
$$\overset{Cor.E.4(b)}{=} \frac{\sigma_I^2}{\hat{a}_I} \big(B^I(t,x) - e^{-\hat{a}_I(T_i-x)} B^{I+I}(t,x) \big)$$

$$W_1(x) := \sigma_w^2 \int_t^x e^{-\hat{a}_w(x-l)} E_1(l, T_i) dl$$
$$\overset{Cor.E.4(a)(b)}{=} \frac{\sigma_w^2 b_{Rw}}{\hat{a}_R} \Big(\frac{1}{\hat{a}_w} \big(B^w(t,x) - e^{-\hat{a}_w(T_i-x)} B^{w+w}(t,x) \big)$$
$$+ \frac{1}{\hat{a}_w - \hat{a}_R} \big(e^{-\hat{a}_w(T_i-x)} B^{w+w}(t,x) - e^{-\hat{a}_R(T_i-x)} B^{w+R}(t,x) \big) \Big)$$

$$W_2(x) := \sigma_w \sigma_I \rho_{wI} \int_t^x e^{-\hat{a}_w(x-l)} B^I(l, T_i) dl$$
$$\overset{Cor.E.4(b)}{=} \frac{\sigma_w \sigma_I \rho_{wI}}{\hat{a}_I} \big(B^w(t,x) - e^{-\hat{a}_I(T_i-x)} B^{I+w}(t,x) \big)$$

$$R_1(x) := \sigma_R^2 \int_t^x e^{-\hat{a}_R(x-l)} B^R(l, T_i) dl$$
$$\overset{Cor.E.4(b)}{=} \frac{\sigma_R^2}{\hat{a}_R} \Big(B^R(t,x) - e^{-\hat{a}_R(T_i-x)} B^{R+R}(t,x) \Big)$$

$$R_2(x) := b_{Rw} \int_t^x e^{-\hat{a}_R(x-l)} W_1(l) dl$$
$$= \sigma_w^2 b_{Rw} \int_t^x e^{-\hat{a}_R(x-l)} \int_t^l e^{-\hat{a}_w(l-y)} E_1(y, T_i) dy dl$$
$$\overset{Cor.E.4(c)(d)}{=} \frac{\sigma_w^2 b_{Rw}^2}{\hat{a}_R} \Big[\frac{1}{\hat{a}_w^2} \Big(B^R(t,x) - e^{-\hat{a}_w(x-t)} B^{R-w}(t,x) \Big)$$

$$
\begin{aligned}
&- \frac{1}{2\hat{a}_w^2}\Big(e^{-\hat{a}_w(T_i-x)}B^{R+w}(t,x) - e^{-\hat{a}_w(T_i+x-2t)}B^{R-w}(t,x)\Big)\\
&+ \frac{1}{\hat{a}_w-\hat{a}_R}\Big(\frac{1}{2\hat{a}_w}\Big(e^{-\hat{a}_w(T_i-x)}B^{R+w}(t,x) - e^{-\hat{a}_w(T_i+x-2t)}B^{R-w}(t,x)\Big)\\
&- \frac{1}{\hat{a}_w+\hat{a}_R}\Big(e^{-\hat{a}_R(T_i-x)}B^{R+R}(t,x) - e^{-\hat{a}_R(T_i+x-2t)}B^{w-R}(t,x)\Big)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&R_3(x) := b_{Rw}\int_t^x e^{-\hat{a}_R(x-l)}W_2(l)dl\\
&= \sigma_w\sigma_I\rho_{wI}b_{Rw}\int_t^x e^{-\hat{a}_R(x-l)}\int_t^l e^{-\hat{a}_w(l-y)}B^I(y,T_i)dydl\\
&\overset{Cor.E.4(c)(d)}{=} \frac{\sigma_w\sigma_I\rho_{wI}b_{Rw}}{\hat{a}_I}\Bigg(\frac{1}{\hat{a}_w}\left(B^R(t,x) - e^{-\hat{a}_w(x-t)}B^{R-w}(t,x)\right)\\
&\quad - \frac{1}{\hat{a}_w+\hat{a}_I}\left(e^{-\hat{a}_I(T_i-x)}B^{R+I}(t,x) - e^{-\hat{a}_I(T_i-t)-\hat{a}_w(x-t)}B^{R-w}(t,x)\right)\Bigg)
\end{aligned}
$$

The terms needed for the expected value of $e^{\int_{T_{i-1}}^{T_i} r_I(x)dx}$ under the T_i-forward measure are determined analogously.

$$
\begin{aligned}
&I_3(T_{i-1},T_i) := \int_{T_{i-1}}^{T_i} I_1(x)dx\\
&= \sigma_w\sigma_I\rho_{wI}\int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_I(x-l)}E_1(l,T_i)dldx\\
&\overset{Cor.E.4(e)(f)}{=} \frac{\sigma_w\sigma_I\rho_{wI}b_{Rw}}{\hat{a}_R}\Big[\frac{1}{\hat{a}_w}\Big[\frac{1}{\hat{a}_I}\left(T_i - T_{i-1} - e^{-\hat{a}_I(T_{i-1}-t)}B^I(T_{i-1},T_i)\right)\\
&\quad - \frac{1}{\hat{a}_w+\hat{a}_I}\left(B^w(T_{i-1},T_i) - e^{-\hat{a}_w(T_i-t)-\hat{a}_I(T_{i-1}-t)}B^I(T_{i-1},T_i)\right)\Big]\\
&\quad + \frac{1}{\hat{a}_w-\hat{a}_R}\Big[\frac{1}{\hat{a}_I+\hat{a}_w}\left(B^w(T_{i-1},T_i) - e^{-\hat{a}_w(T_i-t)-\hat{a}_I(T_{i-1}-t)}B^I(T_{i-1},T_i)\right)\\
&\quad - \frac{1}{\hat{a}_R+\hat{a}_I}\left(B^R(T_{i-1},T_i) - e^{-\hat{a}_R(T_i-t)-\hat{a}_I(T_{i-1}-t)}B^I(T_{i-1},T_i)\right)\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&I_4(T_{i-1},T_i,I) := \int_{T_{i-1}}^{T_i} I_2(x,I)dx\\
&= \sigma_I^2\int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_I(x-l)}B^I(l,T_i)dldx\\
&\overset{Cor.E.4(f)}{=} \frac{\sigma_I^2}{\hat{a}_I^2}\Bigg(T_i - T_{i-1} - B^I(T_{i-1},T_i)(e^{-\hat{a}_I(T_{i-1}-t)} + \frac{1}{2}(1 - e^{-\hat{a}_I(T_i+T_{i-1}-2t)}))\Bigg)
\end{aligned}
$$

The covariance in the expected value of $\frac{1}{P(T_{i-1},T_i)}$ can be decomposed into

$$\begin{aligned}
& Covar_{P(\cdot,T_i)}\left[\int_{T_{i-1}}^{T_i} r(x)dx\,,\int_t^{T_{i-1}} r(x)dx\,\middle|\,\mathcal{F}_t\right] \\
&= Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r(x)dx\,,\int_t^{T_{i-1}} r(x)dx\,\middle|\,\mathcal{F}_t\right] \\
&= Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_R(x)dx\,,\int_t^{T_{i-1}} r_R(x)dx\,\middle|\,\mathcal{F}_t\right] \\
&\quad +Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_R(x)dx\,,\int_t^{T_{i-1}} r_I(x)dx\,\middle|\,\mathcal{F}_t\right] \\
&\quad +Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_t^{T_{i-1}} r_R(x)dx\,\middle|\,\mathcal{F}_t\right] \\
&\quad +Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_t^{T_{i-1}} r_I(x)dx\,\middle|\,\mathcal{F}_t\right]
\end{aligned}$$

where the second term is already given in Lemma E.1 and the last term in Lemma E.3. The remaining terms are obtained by combining the terms with $\widetilde{W}_i$, $i = w$ or $i = R$, of the integrals $\int r_R(x)dx$ and $\int r_I(x)dx$ of Lemma E.1.

$$\begin{aligned}
& Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_R(x)dx\,,\int_{T_{i-1}}^{T_i} r_R(x)dx\,\middle|\,\mathcal{F}_t\right] \\
&= Covar_{\widetilde{Q}}\left[\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\Big(\int_t^{T_{i-1}}(B^w(y,T_{i-1})-B^R(y,T_{i-1}))d\widetilde{W}_w(y)\right. \\
&+\sigma_R\int_t^{T_{i-1}} B^R(y,T_{i-1})d\widetilde{W}_R(y)\Big)\,,\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\Big(\int_{T_{i-1}}^{T_i}(B^w(y,T_i)-B^R(y,T_i))d\widetilde{W}_w(y) \\
&+\int_t^{T_{i-1}}(e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i)-e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))d\widetilde{W}_w(y)\Big) \\
&\left.+\sigma_R\Big(\int_t^{T_{i-1}} e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i)d\widetilde{W}_R(y)+\int_{T_{i-1}}^{T_i} B^R(y,T_i)d\widetilde{W}_R(y)\Big)\,\middle|\,\mathcal{F}_t\right] \\
&= \int_t^{T_{i-1}} \sigma_R^2 B^R(y,T_{i-1})e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i)dy \\
&\quad + \int_t^{T_{i-1}} \left(\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\right)^2\big(B^w(y,T_{i-1})-B^R(y,T_{i-1})\big) \\
&\quad \cdot\big(e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i)-e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i)\big)\,dy
\end{aligned}$$

$$\overset{Cor.E.2}{=} B^R(T_{i-1},T_i)\Bigg(\big(B^R(t,T_{i-1}) - B^{R+R}(t,T_{i-1})\big)\,\frac{1}{\hat{a}_R}\left(\sigma_R^2 + \left(\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\right)^2\right)$$
$$- \big(B^R(t,T_{i-1}) - B^{R+w}(t,T_{i-1})\big)\,\frac{1}{\hat{a}_w}\left(\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\right)^2\Bigg)$$
$$+ B^w(T_{i-1},T_i)\Bigg(\big(B^w(t,T_{i-1}) - B^{w+w}(t,T_{i-1})\big)\,\frac{1}{\hat{a}_w}\left(\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\right)^2$$
$$- \big(B^w(t,T_{i-1}) - B^{R+w}(t,T_{i-1})\big)\,\frac{1}{\hat{a}_R}\left(\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\right)^2\Bigg)$$

and

$$Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_R(x)dx\,,\int_{T_{i-1}}^{T_i} r_I(x)dx\,\middle|\,\mathcal{F}_t\right]$$
$$= Covar_{\widetilde{Q}}\Bigg[\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\int_t^{T_{i-1}}(B^w(y,T_{i-1}) - B^R(y,T_{i-1}))d\widetilde{W}_w(y)\ \ ,$$
$$\sigma_I\cdot\rho_{wI}\left(\int_t^{T_{i-1}}(B^I(y,T_i) - B^I(y,T_{i-1}))d\widetilde{W}_w(y) + \int_{T_{i-1}}^{T_i}B^I(y,T_i)d\widetilde{W}_w(y)\right)\Bigg|\,\mathcal{F}_t\Bigg]$$
$$= \int_t^{T_{i-1}}\sigma_I\rho_{wI}\left(B^I(y,T_i) - B^I(y,T_{i-1})\right)\cdot\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}$$
$$\cdot\left(B^w(y,T_{i-1}) - B^R(y,T_{i-1})\right)dy$$
$$\overset{Cor.E.2}{=} \frac{b_{Rw}\sigma_w\sigma_I\rho_{wI}}{\hat{a}_R-\hat{a}_w}B^I(T_{i-1},T_i)\Big(\frac{1}{\hat{a}_w}(B^I(t,T_{i-1}) - B^{I+w}(t,T_{i-1}))$$
$$-\frac{1}{\hat{a}_R}(B^I(t,T_{i-1}) - B^{I+R}(t,T_{i-1}))\Big)\ .$$

Furthermore, it holds

$$Covar_{\widetilde{Q}}\left[\int_t^{T_i} r_I(x)dx\,,\int_{T_{i-1}}^{T_i} r_R(x)dx\,\middle|\,\mathcal{F}_t\right]$$
$$=\ Covar_{\widetilde{Q}}\left[\int_t^{T_{i-1}} r_I(x)dx\,,\int_{T_{i-1}}^{T_i} r_R(x)dx\,\middle|\,\mathcal{F}_t\right]$$
$$+Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_{T_{i-1}}^{T_i} r_R(x)dx\,\middle|\,\mathcal{F}_t\right]$$

where the first covariance is given in Lemma E.1 and the second covariance is obtained with the help of the previously calculated covariance and Lemma

E.7, i.e.

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_{T_{i-1}}^{T_i} r_R(x)dx\middle|\,\mathcal{F}_t\right]\\
&= \quad Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_{t}^{T_i} r_R(x)dx\middle|\,\mathcal{F}_t\right]\\
&\quad\; - Covar_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_{t}^{T_{i-1}} r_R(x)dx\middle|\,\mathcal{F}_t\right]\ .
\end{aligned}
$$

□

Lemma E.6
The terms needed for the European option with an inflation-linked strike of Theorem 6.7 are given in this lemma's proof.

Proof:
It holds under the risk-neutral measure $\widetilde{Q}$ that

$$
\begin{aligned}
&R_E(x)\big|_{\mathcal{F}_t}\\
&= \quad R_E(t) + \alpha_E(x-t) + b_{ER}\int_t^x r_R(y)dy\\
&\quad\; -b_{EI}\int_t^x r_I(y)dy + b_{Ew}\int_t^x w(y)dy + \int_t^x \sigma_E d\widetilde{W}_E(y)\\
&= \quad R_E(t) + \alpha_E(x-t) + b_{ER}\Bigg(r_R(t)B^R(t,x) + \int_t^x \theta_R(y)B^R(y,x)dy\\
&\quad\; +\frac{b_{Rw}}{\hat{a}_R-\hat{a}_w}\left(w(t)-\frac{\theta_w}{\hat{a}_w}\right)\big(B^w(t,x)-B^R(t,x)\big)\\
&\quad\; +\frac{b_{Rw}\theta_w}{\hat{a}_R\hat{a}_w}\big(x-t-B^R(t,x)\big)\\
&\quad\; +\frac{b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\left(\int_t^x B^w(y,x)-B^R(y,x)d\widetilde{W}_w(y)\right)\\
&\quad\; +\sigma_R\int_t^x B^R(y,x)d\widetilde{W}_R(y)\Bigg)\\
&\quad\; -b_{EI}\Bigg(r_I(t)B^I(t,x) + \frac{\theta_I}{\hat{a}_I}\big(x-t-B^I(t,x)\big)\\
&\quad\; +\sigma_I\rho_{wI}\int_t^x B^I(y,x)d\widetilde{W}_w(y) + \sigma_I\sqrt{1-\rho_{wI}^2}\int_t^x B^I(y,x)d\widetilde{W}_I(y)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+b_{Ew}\Bigg(w(t)B^w(t,x)+\frac{\theta_w}{\hat{a}_w}\left(x-t-B^w(t,x)\right) \\
&+\sigma_w\int_t^x B^w(y,x)d\widetilde{W}_w(y)\Bigg)+\int_t^x \sigma_E d\widetilde{W}_E(y)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{T_{i-1}}^{T_i} R_E(x)dx\Big|_{\mathcal{F}_t} \\
=\;& R_E(t)(T_i-T_{i-1})+\alpha_E(\frac{1}{2}T_i^2-\frac{1}{2}T_{i-1}^2-t(T_i-T_{i-1})) \\
&+b_{ER}r_R(t)\frac{1}{\hat{a}_R}\left(T_i-T_{i-1}-e^{-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)\right) \\
&+\frac{b_{ER}b_{Rw}}{\hat{a}_R-\hat{a}_w}\left(w(t)-\frac{\theta_w}{\hat{a}_w}\right)\Bigg(\frac{1}{\hat{a}_w}\left(T_i-T_{i-1}-e^{-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)\right) \\
&-\frac{1}{\hat{a}_R}\left(T_i-T_{i-1}-e^{-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)\right)\Bigg) \\
&+\frac{b_{ER}b_{Rw}\theta_w}{\hat{a}_R\hat{a}_w}\Bigg(\frac{1}{2}T_i^2-\frac{1}{2}T_{i-1}^2-t(T_i-T_{i-1}) \\
&-\frac{1}{\hat{a}_R}\left(T_i-T_{i-1}-e^{-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)\right)\Bigg) \\
&-b_{EI}r_I(t)\frac{1}{\hat{a}_I}\Bigg(T_i-T_{i-1}-e^{-\hat{a}_I(T_{i-1}-t)}B^I(T_{i-1},T_i)\Bigg) \\
&-\frac{b_{EI}\theta_I}{\hat{a}_I}\Bigg(\frac{1}{2}T_i^2-\frac{1}{2}T_{i-1}^2-t(T_i-T_{i-1}) \\
&-\frac{1}{\hat{a}_I}\left(T_i-T_{i-1}-e^{-\hat{a}_I(T_{i-1}-t)}B^I(T_{i-1},T_i)\right)\Bigg) \\
&+b_{Ew}w(t)\frac{1}{\hat{a}_w}\Bigg(T_i-T_{i-1}-e^{-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)\Bigg) \\
&+\frac{b_{Ew}\theta_w}{\hat{a}_w}\Bigg(\frac{1}{2}T_i^2-\frac{1}{2}T_{i-1}^2-t(T_i-T_{i-1}) \\
&-\frac{1}{\hat{a}_w}\left(T_i-T_{i-1}-e^{-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)\right)\Bigg) \\
&+b_{ER}\Bigg(\int_t^{T_i}\theta_R(y)\frac{1}{\hat{a}_R}(T_i-y-B^R(y,T_i))dy \\
&-\int_t^{T_{i-1}}\theta_R(y)\frac{1}{\hat{a}_R}(T_{i-1}-y-B^R(y,T_{i-1}))dy\Bigg)
\end{aligned}
$$

$$+\int_{T_{i-1}}^{T_i}\Big(\frac{b_{ER}b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\left(\frac{1}{\hat{a}_w}(T_i-y-B^w(y,T_i))-\frac{1}{\hat{a}_R}(T_i-y-B^R(y,T_i))\right)$$
$$-b_{EI}\sigma_I\rho_{wI}\frac{1}{\hat{a}_I}(T_i-y-B^I(y,T_i))+b_{Ew}\sigma_w\frac{1}{\hat{a}_w}(T_i-y-B^w(y,T_i))\Big)d\widetilde{W}_w(y)$$
$$+\int_{t}^{T_{i-1}}\Big(\frac{b_{ER}b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\Big(\frac{1}{\hat{a}_w}(T_i-T_{i-1}-e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i))$$
$$-\frac{1}{\hat{a}_R}(T_i-T_{i-1}-e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))\Big)$$
$$-b_{EI}\sigma_I\rho_{wI}\frac{1}{\hat{a}_I}(T_i-T_{i-1}-e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i))$$
$$+b_{Ew}\sigma_w\frac{1}{\hat{a}_w}(T_i-T_{i-1}-e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i))\Big)d\widetilde{W}_w(y)$$
$$+\int_{T_{i-1}}^{T_i}b_{ER}\sigma_R\frac{1}{\hat{a}_R}(T_i-y-B^R(y,T_i))d\widetilde{W}_R(y)$$
$$+\int_{t}^{T_{i-1}}b_{ER}\sigma_R\frac{1}{\hat{a}_R}(T_i-T_{i-1}-e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))d\widetilde{W}_R(y)$$
$$-\int_{T_{i-1}}^{T_i}b_{EI}\sigma_I\sqrt{1-\rho_{wI}^2}\frac{1}{\hat{a}_I}(T_i-y-B^I(y,T_i))d\widetilde{W}_I(y)$$
$$-\int_{t}^{T_{i-1}}b_{EI}\sigma_I\sqrt{1-\rho_{wI}^2}\frac{1}{\hat{a}_I}(T_i-T_{i-1}-e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i))d\widetilde{W}_I(y)$$
$$+\int_{T_{i-1}}^{T_i}\sigma_E(T_i-y)d\widetilde{W}_E(y)+\int_{t}^{T_{i-1}}\sigma_E(T_i-T_{i-1})d\widetilde{W}_E(y)$$

The variance of $\int_{T_{i-1}}^{T_i}R_E(x)dx$ is determined as follows:

$$Var_{P(\cdot,T_i)}\left[\int_{T_{i-1}}^{T_i}R_E(x)dx\middle|\mathcal{F}_t\right]$$
$$=Var_{\widetilde{Q}}\left[\int_{T_{i-1}}^{T_i}R_E(x)dx\middle|\mathcal{F}_t\right]$$
$$=\int_{T_{i-1}}^{T_i}\Big(\frac{b_{ER}b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\left(\frac{1}{\hat{a}_w}(T_i-y-B^w(y,T_i))-\frac{1}{\hat{a}_R}(T_i-y-B^R(y,T_i))\right)$$
$$-b_{EI}\sigma_I\rho_{wI}\frac{1}{\hat{a}_I}(T_i-y-B^I(y,T_i))+b_{Ew}\sigma_w\frac{1}{\hat{a}_w}(T_i-y-B^w(y,T_i))\Big)^2dy$$
$$+\int_{t}^{T_{i-1}}\Big(\frac{b_{ER}b_{Rw}\sigma_w}{\hat{a}_R-\hat{a}_w}\Big(\frac{1}{\hat{a}_w}(T_i-T_{i-1}-e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i))$$

$$
-\frac{1}{\hat{a}_R}(T_i - T_{i-1} - e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))\Big)
$$
$$
-b_{EI}\sigma_I\rho_{wI}\frac{1}{\hat{a}_I}(T_i - T_{i-1} - e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i))
$$
$$
+b_{Ew}\sigma_w\frac{1}{\hat{a}_w}(T_i - T_{i-1} - e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i))\Big)^2 dy
$$
$$
+\int_{T_{i-1}}^{T_i}\left(b_{ER}\sigma_R\frac{1}{\hat{a}_R}(T_i - y - B^R(y,T_i))\right)^2 dy
$$
$$
+\int_{t}^{T_{i-1}}\left(b_{ER}\sigma_R\frac{1}{\hat{a}_R}(T_i - T_{i-1} - e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))\right)^2 dy
$$
$$
+\int_{T_{i-1}}^{T_i}\left(b_{EI}\sigma_I\sqrt{1-\rho_{wI}^2}\frac{1}{\hat{a}_I}(T_i - y - B^I(y,T_i))\right)^2 dy
$$
$$
+\int_{t}^{T_{i-1}}\left(b_{EI}\sigma_I\sqrt{1-\rho_{wI}^2}\frac{1}{\hat{a}_I}(T_i - T_{i-1} - e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i))\right)^2 dy
$$
$$
+\int_{T_{i-1}}^{T_i}(\sigma_E(T_i - y))^2\,dy + \int_{t}^{T_{i-1}}(\sigma_E(T_i - T_{i-1}))^2\,dy
$$

The first equivalence is due to the fact that the change of measure only affects the drift terms. Therefore, the variances and covariances are the same for the risk-neutral measure $\widetilde{Q}$ and the T_i-forward measure.
Analogously, the covariance between $\int_{T_{i-1}}^{T_i} r_I(x)dx$ and $\int_{T_{i-1}}^{T_i} R_E(x)dx$ can be obtained by means of the following integrals:

$$
Covar_{P(\cdot,T_i)}\left[\int_{T_{i-1}}^{T_i} r_I(x)dx\,,\int_{T_{i-1}}^{T_i} R_E(x)dx\middle|\mathcal{F}_t\right]
$$
$$
= Covar_{\widetilde{Q}}\Big[\sigma_I\cdot\rho_{wI}\left(\int_t^{T_{i-1}}(B^I(y,T_i) - B^I(y,T_{i-1}))d\widetilde{W}_w(y) + \int_{T_{i-1}}^{T_i} B^I(y,T_i)d\widetilde{W}_w(y)\right)
$$
$$
+\sigma_I\sqrt{1-\rho_{wI}^2}\left(\int_t^{T_{i-1}}(B^I(y,T_i) - B^I(y,T_{i-1}))d\widetilde{W}_I(y) + \int_{T_{i-1}}^{T_i} B^I(y,T_i)d\widetilde{W}_I(y)\right)\,,
$$
$$
+\int_{T_{i-1}}^{T_i}\Big(\frac{b_{ER}b_{Rw}\sigma_w}{\hat{a}_R - \hat{a}_w}\left(\frac{1}{\hat{a}_w}(T_i - y - B^w(y,T_i)) - \frac{1}{\hat{a}_R}(T_i - y - B^R(y,T_i))\right)
$$
$$
-b_{EI}\sigma_I\rho_{wI}\frac{1}{\hat{a}_I}(T_i - y - B^I(y,T_i)) + b_{Ew}\sigma_w\frac{1}{\hat{a}_w}(T_i - y - B^w(y,T_i))\Big)d\widetilde{W}_w(y)
$$
$$
+\int_t^{T_{i-1}}\Big(\frac{b_{ER}b_{Rw}\sigma_w}{\hat{a}_R - \hat{a}_w}\Big(\frac{1}{\hat{a}_w}(T_i - T_{i-1} - e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i))
$$

$$
\begin{aligned}
&-\frac{1}{\hat{a}_R}(T_i - T_{i-1} - e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))\Big) \\
&-b_{EI}\sigma_I\rho_{wI}\frac{1}{\hat{a}_I}(T_i - T_{i-1} - e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i)) \\
&+b_{Ew}\sigma_w\frac{1}{\hat{a}_w}(T_i - T_{i-1} - e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i))\Big)d\widetilde{W}_w(y) \\
&-\int_{T_{i-1}}^{T_i} b_{EI}\sigma_I\sqrt{1-\rho_{wI}^2}\frac{1}{\hat{a}_I}(T_i - y - B^I(y,T_i))d\widetilde{W}_I(y) \\
&-\int_t^{T_{i-1}} b_{EI}\sigma_I\sqrt{1-\rho_{wI}^2}\frac{1}{\hat{a}_I}(T_i - T_{i-1} - e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i))d\widetilde{W}_I(y)\Big|\mathcal{F}_t\Big] \\
&= -b_{EI}\sigma_I^2(1-\rho_{wI}^2)\frac{1}{\hat{a}_I}\Big(\int_t^{T_{i-1}}(T_i - T_{i-1} - e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i)) \\
&\cdot(B^I(y,T_i) - B^I(y,T_{i-1}))dy + \int_{T_{i-1}}^{T_i}(T_i - y - B^I(y,T_i))B^I(y,T_i)dy\Big) \\
&+\sigma_I\rho_{wI}\Big(\int_{T_{i-1}}^{T_i} B^I(y,T_i)\Big(\frac{b_{ER}b_{Rw}\sigma_w}{\hat{a}_R - \hat{a}_w}\Big(\frac{1}{\hat{a}_w}(T_i - y - B^w(y,T_i)) \\
&-\frac{1}{\hat{a}_R}(T_i - y - B^R(y,T_i))\Big) - b_{EI}\sigma_I\rho_{wI}\frac{1}{\hat{a}_I}(T_i - y - B^I(y,T_i)) \\
&+b_{Ew}\sigma_w\frac{1}{\hat{a}_w}(T_i - y - B^w(y,T_i))\Big)dy \\
&+\int_t^{T_{i-1}}(B^I(y,T_i) - B^I(y,T_{i-1}))\Big(\frac{b_{ER}b_{Rw}\sigma_w}{\hat{a}_R - \hat{a}_w}\Big(\frac{1}{\hat{a}_w}(T_i - T_{i-1} \\
&-e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i)) - \frac{1}{\hat{a}_R}(T_i - T_{i-1} - e^{-\hat{a}_R(T_{i-1}-y)}B^R(T_{i-1},T_i))\Big) \\
&-b_{EI}\sigma_I\rho_{wI}\frac{1}{\hat{a}_I}(T_i - T_{i-1} - e^{-\hat{a}_I(T_{i-1}-y)}B^I(T_{i-1},T_i)) \\
&+b_{Ew}\sigma_w\frac{1}{\hat{a}_w}(T_i - T_{i-1} - e^{-\hat{a}_w(T_{i-1}-y)}B^w(T_{i-1},T_i))\Big)dy\Big)
\end{aligned}
$$

In order to calculate all these integrals we need in addition to the previously introduce building blocks of Corollary E.2, the following ones:

$$
\begin{aligned}
&\int_{T_1}^{T_2} yB^x(y,T_3)dy \\
&\quad = \frac{1}{\hat{a}_x}[\frac{1}{2}(T_2^2 - T_1^2) - \frac{1}{\hat{a}_x}(T_2 e^{-\hat{a}_x(T_3-T_2)} - T_1 e^{-\hat{a}_x(T_3-T_1)}) + e^{-\hat{a}_x(T_3-T_2)}B^x(T_1,T_2)]\,,
\end{aligned}
$$

and

$$\int_{T_1}^{T_2} B^x(y,T_3)dy$$
$$= \frac{1}{\hat{a}_x}[T_2 - T_1 - e^{-\hat{a}_x(T_3-T_2)}B^x(T_1,T_2)]\,.$$

Analogously to the proof of Theorem 6.6 we need to adjust the drift of $dR_E(t)$ when changing the measure from $\widetilde{Q}$ to the T_i-forward measure. Since $dP(\cdot,T_i)$ is driven by $d\widetilde{W}_i$, $i = w, R, I$ and it holds for R_E under $\widetilde{Q}$

$$\begin{aligned} R_E(x)\big|_{\mathcal{F}_t} & \\ = \quad & R_E(t) + \alpha_E(x-t) + b_{ER}\int_t^x r_R(y)dy \\ & -b_{EI}\int_t^x r_I(y)dy + b_{Ew}\int_t^x w(y)dy + \int_t^x \sigma_E d\widetilde{W}_E(y) \end{aligned}$$

we need to adjust the integrals $+b_{ER}\int_t^x r_R(y)dy$, $-b_{EI}\int_t^x r_I(y)dy$ and $+b_{Ew}\int_t^x w(y)dy$ when changing to the T_i-forward measure. Hence, for the expectation $\mathbb{E}_{P(\cdot,T_i)}\left[\int_{T_0}^{T_i} R_E(x)dx\middle|\mathcal{F}_t\right]$ we need to adjust the expected value under the risk-neutral measure $\mathbb{E}_{\widetilde{Q}}\left[\int_{T_0}^{T_i} R_E(x)dx\middle|\mathcal{F}_t\right]$ with three additional terms.

Since under the T_i-forward measure the drift of the inflation short rate r_I is extended by $-\sigma_I^2 \cdot B^I(t,T_i) - \sigma_w\sigma_I\rho_{wI}E_1(t,T_i)$, the expectation under the risk-neutral measure needs to be adjusted by the following term (see also page 200):

$$\begin{aligned} & b_{EI}\int_{T_{i-1}}^{T_i}\int_t^x\int_t^y e^{-\hat{a}_I(y-l)}\left(\sigma_I^2 B^I(l,T_i) + \sigma_w\sigma_I\rho_{wI}E_1(l,T_i)\right)dldydx \\ &= b_{EI}\int_{T_{i-1}}^{T_i}\int_t^x\int_l^x e^{-\hat{a}_I(y-l)}\left(\sigma_I^2 B^I(l,T_i) + \sigma_w\sigma_I\rho_{wI}E_1(l,T_i)\right)dydldx \\ &= b_{EI}\int_{T_{i-1}}^{T_i}\int_t^x B^I(l,x)\left(\sigma_I^2 B^I(l,T_i) + \sigma_w\sigma_I\rho_{wI}E_1(l,T_i)\right)dldx \\ &\overset{Lemma\,E.5}{=} -\frac{b_{EI}}{\hat{a}_I}\left(I_3(T_{i-1},T_i) + I_4(T_{i-1},T_i,I) - \sigma_w\sigma_I\rho_{wI}I_5(T_{i-1},T_i) - \sigma_I^2 I_6(T_{i-1},T_i,I)\right) \end{aligned}$$

with

$$
\begin{aligned}
I_5(T_{i-1}, T_i) &:= \int_{T_{i-1}}^{T_i} \int_t^x E_1(l, T_i) dl dx \\
&\overset{Cor.E.4(g)}{=} \frac{b_{Rw}}{\hat{a}_R} \Bigg(\frac{1}{\hat{a}_w} (\frac{1}{2}(T_i^2 - T_{i-1}^2) - t(T_i - T_{i-1}) - \frac{1}{\hat{a}_w}(B^w(T_{i-1}, T_i) \\
&\quad - e^{-\hat{a}_w(T_i - t)}(T_i - T_{i-1}))) + \frac{1}{\hat{a}_w - \hat{a}_R}(\frac{1}{\hat{a}_w}(B^w(T_{i-1}, T_i) \\
&\quad - e^{-\hat{a}_w(T_i - t)}(T_i - T_{i-1})) - \frac{1}{\hat{a}_R}(B^R(T_{i-1}, T_i) - e^{-\hat{a}_R(T_i - t)}(T_i - T_{i-1}))) \Bigg) ,
\end{aligned}
$$

and

$$
\begin{aligned}
I_6(T_{i-1}, T_i, I) &:= \int_{T_{i-1}}^{T_i} \int_t^x B^I(l, T_i) dl dx \\
&\overset{Cor.E.4(g)}{=} \frac{1}{\hat{a}_I} \left(\frac{1}{2}(T_i^2 - T_{i-1}^2) - \frac{1}{\hat{a}_I}(B^I(T_{i-1}, T_i) + (\hat{a}_I t - e^{-\hat{a}_I(T_i - t)})(T_i - T_{i-1})) \right) ,
\end{aligned}
$$

Under the T_i-forward measure the drift for the macroeconomic factor w shows additional terms as opposed to the risk-neutral measure, i.e. $-\sigma_w \sigma_I \rho_{wI} \cdot B^I(t, T_i) - \sigma_w^2 E_1(t, T_i)$. Therefore the adjustment for the expectation of $\int_{T_0}^{T_i} R_E(x) dx$ with respect to the macroeconomic factor w (see also page 200) is determined by

$$
\begin{aligned}
&-b_{Ew} \int_{T_{i-1}}^{T_i} \int_t^x \int_t^y e^{-\hat{a}_w(y-l)} \left(\sigma_w^2 E_1(l, T_i) + \sigma_w \sigma_I \rho_{wI} B^I(l, T_i) \right) dl dy dx \\
&= -b_{Ew} \int_{T_{i-1}}^{T_i} \int_t^x \int_l^x e^{-\hat{a}_w(y-l)} \left(\sigma_w^2 E_1(l, T_i) + \sigma_w \sigma_I \rho_{wI} B^I(l, T_i) \right) dy dl dx \\
&= -b_{Ew} \int_{T_{i-1}}^{T_i} \int_t^x B^w(l, x) \left(\sigma_w^2 E_1(l, T_i) + \sigma_w \sigma_I \rho_{wI} B^I(l, T_i) \right) dl dx \\
&= \frac{b_{Ew}}{\hat{a}_w} \Bigg(\sigma_I \sigma_w \rho_{wI} I_7(T_{i-1}, T_i) + \sigma_w^2 I_8(T_{i-1}, T_i) \\
&\quad - \sigma_I \sigma_w \rho_{wI} I_6(T_{i-1}, T_i, I) - \sigma_w^2 I_5(T_{i-1}, T_i) \Bigg)
\end{aligned}
$$

with

$$
\begin{aligned}
I_7(T_{i-1},T_i) &:= \int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_w(x-l)}B^I(l,T_i)dldx \\
&\overset{Cor.E.4(f)}{=} \frac{1}{\hat{a}_I}\Bigg(\frac{1}{\hat{a}_w}(T_i-T_{i-1}-e^{-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)) \\
&\quad -\frac{1}{\hat{a}_w+\hat{a}_I}(B^I(T_{i-1},T_i)-e^{-\hat{a}_w(T_{i-1}-t)-\hat{a}_I(T_i-t)}B^w(T_{i-1},T_i))\Bigg) ,
\end{aligned}
$$

and

$$
\begin{aligned}
I_8(T_{i-1},T_i) &:= \int_{T_{i-1}}^{T_i}\int_t^x e^{-\hat{a}_w(x-l)}E_1(l,T_i)dldx \\
&\overset{Cor.E.4(e)(f)}{=} \frac{b_{Rw}}{\hat{a}_R}\Bigg(\frac{1}{\hat{a}_w}(\frac{1}{\hat{a}_w}(T_i-T_{i-1}-e^{-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)) \\
&\quad -\frac{1}{2\hat{a}_w}B^w(T_{i-1},T_i)(1-e^{-\hat{a}_w(T_i+T_{i-1}-2t)})) \\
&\quad +\frac{1}{\hat{a}_w-\hat{a}_R}(\frac{1}{2\hat{a}_w}B^w(T_{i-1},T_i)(1-e^{-\hat{a}_w(T_i+T_{i-1}-2t)}) \\
&\quad -\frac{1}{\hat{a}_w+\hat{a}_R}(B^R(T_{i-1},T_i)-e^{-\hat{a}_R(T_i-t)-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)))\Bigg) ,
\end{aligned}
$$

For the real short rate r_R we need to consider the new term in the drift under the T_i-forward measure, i.e. $-\sigma_R^2 B^R(t,T_i)$, as well as the influence of $w(t)$ on the drift of r_R (i.e. $b_{rw}w(t)$) which adds an additional term to the adjustment:

$$
\begin{aligned}
&-b_{ER}\int_{T_{i-1}}^{T_i}\int_t^x\int_t^y e^{-\hat{a}_R(y-l)}\Bigg(\sigma_R^2 B^R(l,T_i) \\
&+b_{Rw}\int_t^l e^{-\hat{a}_w(l-s)}\left(\sigma_w^2 E_1(s,T_i)+\sigma_w\sigma_I\rho_{wI}B^I(s,T_i)\right)ds\Bigg)dldydx \\
&= -b_{ER}\int_{T_{i-1}}^{T_i}\int_t^x\int_l^x e^{-\hat{a}_R(y-l)}\sigma_R^2 B^R(l,T_i)dydldx \\
&-b_{ER}\int_{T_{i-1}}^{T_i}\int_t^x\int_t^y e^{-\hat{a}_R(y-l)}\Bigg(\\
&+b_{Rw}\int_t^l e^{-\hat{a}_w(l-s)}\left(\sigma_w^2 E_1(s,T_i)+\sigma_w\sigma_I\rho_{wI}B^I(s,T_i)\right)ds\Bigg)dldydx
\end{aligned}
$$

$$
\begin{aligned}
&= -b_{ER}\int_{T_{i-1}}^{T_i}\int_t^x B^R(l,x)\sigma_R^2 B^R(l,T_i)dldx \\
&-b_{ER}\int_{T_{i-1}}^{T_i}\int_t^x\int_t^y e^{-\hat{a}_R(y-l)}\Bigg(\\
&+b_{Rw}\int_t^l e^{-\hat{a}_w(l-s)}\left(\sigma_w^2 E_1(s,T_i)+\sigma_w\sigma_I\rho_{wI}B^I(s,T_i)\right)ds\Bigg)dldydx \\
&\overset{Lemma\ E.5}{=} b_{ER}\Bigg(\frac{1}{\hat{a}_R}\left(I_4(T_{i-1},T_i,R)-\sigma_R^2 I_6(T_{i-1},T_i,R)\right)-b_{Rw}\sigma_w\sigma_I\rho_{wI}I_9(T_{i-1},T_i,I) \\
&-\frac{b_{Rw}^2\sigma_w^2}{\hat{a}_R}(I_9(T_{i-1},T_i,w)+\frac{1}{\hat{a}_w-\hat{a}_R}(I_{10}(T_{i-1},T_i,w)-I_{10}(T_{i-1},T_i,R)))\Bigg)\ ,
\end{aligned}
$$

with

$$
\begin{aligned}
I_9(T_{i-1},T_i,I) &:= \int_{T_{i-1}}^{T_i}\int_t^x\int_t^y\int_t^l e^{-\hat{a}_R(y-l)}e^{-\hat{a}_w(l-s)}B^I(s,T_i)dsdldydx \\
&= \frac{1}{\hat{a}_I}\Bigg(\int_{T_{i-1}}^{T_i}\int_t^x\int_t^y\int_t^l e^{-\hat{a}_R(y-l)}e^{-\hat{a}_w(l-s)}dsdldydx - I_{10}(T_{i-1},T_i,I)\Bigg) \\
&= \frac{1}{\hat{a}_I}\Bigg(\int_{T_{i-1}}^{T_i}\int_t^x\int_t^y e^{-\hat{a}_R(y-l)}B^w(t,l)dldydx - I_{10}(T_{i-1},T_i,I)\Bigg) \\
&= \frac{1}{\hat{a}_I}\Bigg(\int_{T_{i-1}}^{T_i}\int_t^x\frac{1}{\hat{a}_w}\left(B^R(t,y)-\frac{1}{\hat{a}_R-\hat{a}_w}(e^{-\hat{a}_w(y-t)}-e^{-\hat{a}_R(y-t)})\right)dydx \\
&\quad -I_{10}(T_{i-1},T_i,I)\Bigg) \\
&= \frac{1}{\hat{a}_I}\Bigg(\int_{T_{i-1}}^{T_i}\frac{1}{\hat{a}_w}\left(\frac{1}{\hat{a}_R}(x-t-B^R(t,x))-\frac{1}{\hat{a}_R-\hat{a}_w}(B^w(t,x)-B^R(t,x))\right)dx \\
&\quad -I_{10}(T_{i-1},T_i,I)\Bigg) \\
&= \frac{1}{\hat{a}_I}\Bigg(\frac{1}{\hat{a}_w}\Bigg(\frac{1}{\hat{a}_R}\Bigg(\frac{1}{2}(T_i^2-T_{i-1}^2)-t(T_i-T_{i-1}) \\
&\quad +\frac{1}{\hat{a}_R}\left(e^{-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)-(T_i-T_{i-1})\right)\Bigg) \\
&\quad -\frac{1}{\hat{a}_R-\hat{a}_w}\Bigg(\frac{1}{\hat{a}_R}\left(e^{-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)-(T_i-T_{i-1})\right) \\
&\quad -\frac{1}{\hat{a}_w}\left(e^{-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)-(T_i-T_{i-1})\right)\Bigg)\Bigg)-I_{10}(T_{i-1},T_i,I)\Bigg)
\end{aligned}
$$

and

$$I_{10}(T_{i-1},T_i,x) := \int_{T_{i-1}}^{T_i}\int_t^x\int_t^y\int_t^l e^{-\hat{a}_R(y-l)}e^{-\hat{a}_w(l-s)}e^{-\hat{a}_x(T_i-s)}dsdldydx$$
$$= \int_{T_{i-1}}^{T_i}\int_t^x\int_t^y e^{-\hat{a}_R(y-l)}e^{-\hat{a}_x(T_i-l)}B^{w+x}(t,l)dldydx$$
$$= \int_{T_{i-1}}^{T_i}\int_t^x \frac{1}{\hat{a}_w+\hat{a}_x}\left(e^{-\hat{a}_x(T_i-y)}B^{R+x}(t,y) - \frac{e^{-\hat{a}_x(T_i-t)}}{\hat{a}_R-\hat{a}_w}(e^{-\hat{a}_w(y-t)}-e^{-\hat{a}_R(y-t)})\right)dydx$$
$$= \int_{T_{i-1}}^{T_i}\frac{1}{\hat{a}_w+\hat{a}_x}\left(\frac{1}{\hat{a}_R+\hat{a}_x}(\frac{1}{\hat{a}_x}(e^{-\hat{a}_x(T_i-x)}-e^{-\hat{a}_x(T_i-t)}) - e^{-\hat{a}_x(T_i-t)}B^R(t,x))\right.$$
$$\left. - \frac{1}{\hat{a}_R-\hat{a}_w}(e^{-\hat{a}_x(T_i-t)}B^w(t,x)-e^{-\hat{a}_x(T_i-t)}B^R(t,x))\right)dx$$
$$= \frac{1}{\hat{a}_w+\hat{a}_x}\left(\frac{1}{\hat{a}_R+\hat{a}_x}\left(\frac{1}{\hat{a}_x}\left(B^x(T_{i-1},T_i)-e^{-\hat{a}_x(T_i-t)}(T_i-T_{i-1})\right)\right.\right.$$
$$\left. + \frac{1}{\hat{a}_R}\left(e^{-\hat{a}_x(T_i-t)-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)-e^{-\hat{a}_x(T_i-t)}(T_i-T_{i-1})\right)\right)$$
$$-\frac{1}{\hat{a}_R-\hat{a}_w}\left(-\frac{1}{\hat{a}_w}\left(e^{-\hat{a}_x(T_i-t)-\hat{a}_w(T_{i-1}-t)}B^w(T_{i-1},T_i)-e^{-\hat{a}_x(T_i-t)}(T_i-T_{i-1})\right)\right.$$
$$\left.\left. + \frac{1}{\hat{a}_R}\left(e^{-\hat{a}_x(T_i-t)-\hat{a}_R(T_{i-1}-t)}B^R(T_{i-1},T_i)-e^{-\hat{a}_x(T_i-t)}(T_i-T_{i-1})\right)\right)\right) .$$

□

Lemma E.7
The covariance terms needed for the inflation-indexed CDS of Theorem 6.9 are given in this lemma's proof.

Proof:

$$Covar_{\tilde{Q}}\left[\int_t^{T_i} r_R(x)dx\,,\int_{T_{i-1}}^{T_i} r_I(x)dx\middle|\mathcal{F}_t\right]$$
$$\overset{page\,286}{=} Covar_{\tilde{Q}}\left[\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R-\hat{a}_w}\int_t^{T_i}(B^w(y,T_i)-B^R(y,T_i))d\widetilde{W}_w(y)\ ,\right.$$
$$\left.\sigma_I\cdot\rho_{wI}\left(\int_t^{T_{i-1}}(B^I(y,T_i)-B^I(y,T_{i-1}))d\widetilde{W}_w(y)+\int_{T_{i-1}}^{T_i}B^I(y,T_i)d\widetilde{W}_w(y)\right)\middle|\mathcal{F}_t\right]$$

$$= \frac{b_{Rw}\sigma_w\sigma_I\rho_{wI}}{\hat{a}_R - \hat{a}_w} \int_t^{T_{i-1}} \left(B^I(y,T_i) - B^I(y,T_{i-1})\right) \cdot \left(B^w(y,T_i) - B^R(y,T_i)\right) dy$$
$$+ \frac{b_{Rw}\sigma_w\sigma_I\rho_{wI}}{\hat{a}_R - \hat{a}_w} \int_{T_{i-1}}^{T_i} B^I(y,T_i) \cdot \left(B^w(y,T_i) - B^R(y,T_i)\right) dy$$
$$\stackrel{Cor.E.2}{=} \frac{b_{Rw}\sigma_w\sigma_I\rho_{wI}}{\hat{a}_R - \hat{a}_w} \Big(\frac{1}{\hat{a}_w\hat{a}_I} \Big(T_i - T_{i-1} - B^w(t,T_i) - B^I(t,T_i) + B^{I+w}(t,T_i)$$
$$+ B^I(t,T_{i-1}) + e^{-\hat{a}_w(T_i - T_{i-1})} \big(B^w(t,T_{i-1}) - B^{I+w}(t,T_{i-1})\big)\Big)$$
$$- \frac{1}{\hat{a}_R\hat{a}_I} \Big(T_i - T_{i-1} - B^R(t,T_i) - B^I(t,T_i) + B^{I+R}(t,T_i)$$
$$+ B^I(t,T_{i-1}) + e^{-\hat{a}_R(T_i - T_{i-1})} \big(B^R(t,T_{i-1}) - B^{I+R}(t,T_{i-1})\big)\Big)\Big).$$

This follows from using $B^x(t,T_i) = B^x(T_{i-1},T_i) + e^{-\hat{a}_x(T_i - T_{i-1})} B^x(t,T_{i-1})$ and with:

$$\int_t^{T_{i-1}} \left(B^x(y,T_i) - B^x(y,T_{i-1})\right) B^z(y,T_i))dy$$
$$= \int_t^{T_{i-1}} e^{-\hat{a}_x(T_{i-1}-y)} B^x(T_{i-1},T_i) B^z(y,T_i))dy$$
$$= \frac{1}{\hat{a}_z} B^x(T_{i-1},T_i) \left[B^x(t,T_{i-1}) - e^{-\hat{a}_z(T_i - T_{i-1})} B^{x+z}(t,T_{i-1})\right]$$
$$= \frac{1}{\hat{a}_z\hat{a}_x} (1 - e^{-\hat{a}_x(T_i - T_{i-1})}) \left[B^x(t,T_{i-1}) - e^{-\hat{a}_z(T_i - T_{i-1})} B^{x+z}(t,T_{i-1})\right]$$
$$= \frac{1}{\hat{a}_z\hat{a}_x} [B^x(t,T_{i-1}) - (B^x(t,T_i) - B^x(T_{i-1},T_i)) - e^{-\hat{a}_z(T_i - T_{i-1})} B^{x+z}(t,T_{i-1})$$
$$+ (B^{x+z}(t,T_i) - B^{x+z}(T_{i-1},T_i))].$$

Under the risk-neutral measure $\widetilde{Q}$, it holds for $t \leq x$ and $t \leq T_{i-1} \leq T_i$:

$$s^{zero}(x)\big|_{\mathcal{F}_t}$$
$$= s^{zero}(t) \cdot e^{-\hat{a}_s(x-t)} + \int_t^x \theta_{s^{zero}} e^{-\hat{a}_s(x-y)} dy$$
$$- b_{s^{zero}w} \int_t^x w(y) e^{-\hat{a}_s(x-y)} dy - b_{s^{zero}I} \int_t^x r_I(y) e^{-\hat{a}_s(x-y)} dy$$
$$+ b_{s^{zero}u} \int_t^x u(y) e^{-\hat{a}_s(x-y)} dy + \sigma_{s^{zero}} \int_t^x e^{-\hat{a}_s(x-y)} d\widetilde{W}_s(y)$$

$$
\begin{aligned}
&= s^{zero}(t) \cdot e^{-\hat{a}_s(x-t)} + \theta_{s^{zero}} B^s(t,x) \\
&- \frac{b_{s^{zero}w}}{\hat{a}_s - \hat{a}_w} \left(w(t) - \frac{\theta_w}{\hat{a}_w} \right) \left(e^{-\hat{a}_w(x-t)} - e^{-\hat{a}_s(x-t)} \right) - \frac{b_{s^{zero}w} \cdot \theta_w}{\hat{a}_w} B^s(t,x) \\
&- \frac{b_{s^{zero}I}}{\hat{a}_s - \hat{a}_I} \left(r_I(t) - \frac{\theta_I}{\hat{a}_I} \right) \left(e^{-\hat{a}_I(x-t)} - e^{-\hat{a}_s(x-t)} \right) - \frac{b_{s^{zero}I} \cdot \theta_I}{\hat{a}_I} B^s(t,x) \\
&+ \frac{b_{s^{zero}u}}{\hat{a}_s - \hat{a}_u} \left(u(t) - \frac{\theta_u}{\hat{a}_u} \right) \left(e^{-\hat{a}_u(x-t)} - e^{-\hat{a}_s(x-t)} \right) + \frac{b_{s^{zero}u} \cdot \theta_u}{\hat{a}_u} B^s(t,x) \\
&+ \sigma_{s^{zero}} \int_t^x e^{-\hat{a}_s(x-y)} d\widetilde{W}_s(y) \\
&+ \frac{b_{s^{zero}u} \cdot \sigma_u}{\hat{a}_s - \hat{a}_u} \int_t^x (e^{-\hat{a}_u(x-y)} - e^{-\hat{a}_s(x-y)}) d\widetilde{W}_u(y) \\
&- \frac{b_{s^{zero}w} \cdot \sigma_w}{\hat{a}_s - \hat{a}_w} \int_t^x (e^{-\hat{a}_w(x-y)} - e^{-\hat{a}_s(x-y)}) d\widetilde{W}_w(y) \\
&- \frac{b_{s^{zero}I} \cdot \sigma_I \rho_{wI}}{\hat{a}_s - \hat{a}_I} \int_t^x (e^{-\hat{a}_I(x-y)} - e^{-\hat{a}_s(x-y)}) d\widetilde{W}_w(y) \\
&- \frac{b_{s^{zero}I} \cdot \sigma_I \sqrt{1-\rho_{wI}^2}}{\hat{a}_s - \hat{a}_I} \int_t^x (e^{-\hat{a}_I(x-y)} - e^{-\hat{a}_s(x-y)}) d\widetilde{W}_I(y)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left. \int_t^{T_i} s^{zero}(x) dx \right|_{\mathcal{F}_t} \\
&= s^{zero}(t) \cdot B^s(t,T_i) + \frac{\theta_{s^{zero}}}{\hat{a}_s} (T_i - t - B^s(t,T_i)) \\
&- \frac{b_{s^{zero}w}}{\hat{a}_s - \hat{a}_w} \left(w(t) - \frac{\theta_w}{\hat{a}_w} \right) \left(B^w(t,T_i) - B^s(t,T_i) \right) \\
&- \frac{b_{s^{zero}I}}{\hat{a}_s - \hat{a}_I} \left(r_I(t) - \frac{\theta_I}{\hat{a}_I} \right) \left(B^I(t,T_i) - B^s(t,T_i) \right) \\
&+ \frac{b_{s^{zero}u}}{\hat{a}_s - \hat{a}_u} \left(u(t) - \frac{\theta_u}{\hat{a}_u} \right) \left(B^u(t,T_i) - B^s(t,T_i) \right) \\
&+ \left(T_i - t - B^s(t,T_i) \right) \cdot \left(\frac{b_{s^{zero}u}\theta_u}{\hat{a}_s \hat{a}_u} - \frac{b_{s^{zero}w}\theta_w}{\hat{a}_s \hat{a}_w} - \frac{b_{s^{zero}I}\theta_I}{\hat{a}_s \hat{a}_I} \right) \\
&+ \sigma_{s^{zero}} \int_t^{T_i} B^s(y,T_i) d\widetilde{W}_s(y)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{b_{s^{zero}u} \cdot \sigma_u}{\hat{a}_s - \hat{a}_u} \int_t^{T_i} B^u(y,T_i) - B^s(y,T_i) d\widetilde{W}_u(y) \\
&- \quad \frac{b_{s^{zero}w} \cdot \sigma_w}{\hat{a}_s - \hat{a}_w} \int_t^{T_i} B^w(y,T_i) - B^s(y,T_i) d\widetilde{W}_w(y) \\
&- \quad \frac{b_{s^{zero}I} \cdot \sigma_I \rho_{wI}}{\hat{a}_s - \hat{a}_I} \int_t^{T_i} B^I(y,T_i) - B^s(y,T_i) d\widetilde{W}_w(y) \\
&- \quad \frac{b_{s^{zero}I} \cdot \sigma_I \sqrt{1-\rho_{wI}^2}}{\hat{a}_s - \hat{a}_I} \int_t^{T_i} B^I(y,T_i) - B^s(y,T_i) d\widetilde{W}_I(y) \; .
\end{aligned}
$$

Hence, the covariance $Covar_{\widetilde{Q}}\left[\int_t^{T_i} s^{zero}(x)dx, \int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t\right]$ is calculated analogously to the last determined covariance $Covar_{\widetilde{Q}}\left[\int_t^{T_i} r_R(x)dx, \int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t\right]$ with the help of Corollary E.2 and the comments on page 307:

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\left[\int_t^{T_i} s^{zero}(x)dx \,, \int_{T_{i-1}}^{T_i} r_I(x)dx \middle| \mathcal{F}_t\right] \\
&\overset{page\,286}{=} Covar_{\widetilde{Q}}\Bigg[-\frac{b_{s^{zero}w} \cdot \sigma_w}{\hat{a}_s - \hat{a}_w} \int_t^{T_i} B^w(y,T_i) - B^s(y,T_i) d\widetilde{W}_w(y) \\
&- \quad \frac{b_{s^{zero}I} \cdot \sigma_I \rho_{wI}}{\hat{a}_s - \hat{a}_I} \int_t^{T_i} B^I(y,T_i) - B^s(y,T_i) d\widetilde{W}_w(y) \\
&- \quad \frac{b_{s^{zero}I} \cdot \sigma_I \sqrt{1-\rho_{wI}^2}}{\hat{a}_s - \hat{a}_I} \int_t^{T_i} B^I(y,T_i) - B^s(y,T_i) d\widetilde{W}_I(y) \,, \\
&+ \quad \sigma_I \sqrt{1-\rho_{wI}^2} \left(\int_t^{T_{i-1}} (B^I(y,T_i) - B^I(y,T_{i-1})) d\widetilde{W}_I(y) + \int_{T_{i-1}}^{T_i} B^I(y,T_i) d\widetilde{W}_I(y)\right) \\
&+ \quad \sigma_I \cdot \rho_{wI} \left(\int_t^{T_{i-1}} (B^I(y,T_i) - B^I(y,T_{i-1})) d\widetilde{W}_w(y) + \int_{T_{i-1}}^{T_i} B^I(y,T_i) d\widetilde{W}_w(y)\right) \Bigg| \mathcal{F}_t\Bigg] \\
&= -\frac{b_{s^{zero}I} \cdot \sigma_I^2}{\hat{a}_s - \hat{a}_I} \Big(\int_t^{T_{i-1}} \big(B^I(y,T_i) - B^s(y,T_i)\big) \cdot \big(B^I(y,T_i) - B^I(y,T_{i-1})\big) dy \\
&+ \quad \int_{T_{i-1}}^{T_i} \big(B^I(y,T_i) - B^s(y,T_i)\big) \cdot B^I(y,T_i) dy \Big) \\
&- \quad \frac{b_{s^{zero}w} \cdot \sigma_w \cdot \sigma_I \cdot \rho_{wI}}{\hat{a}_s - \hat{a}_w} \Big(\int_{T_{i-1}}^{T_i} \big(B^w(y,T_i) - B^s(y,T_i)\big) \cdot B^I(y,T_i) dy \\
&+ \quad \int_t^{T_{i-1}} \big(B^w(y,T_i) - B^s(y,T_i)\big) \cdot \big(B^I(y,T_i) - B^I(y,T_{i-1})\big) dy \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{Cor.E.2}{=} - \frac{b_{s^{zero}w}\sigma_w\sigma_I\rho_{wI}}{\hat{a}_s-\hat{a}_w}\Big(\frac{1}{\hat{a}_w\hat{a}_I}\Big(T_i - T_{i-1} - B^w(t,T_i) - B^I(t,T_i) + B^{I+w}(t,T_i)\\
&+ \quad B^I(t,T_{i-1}) + e^{-\hat{a}_w(T_i-T_{i-1})}\big(B^w(t,T_{i-1}) - B^{I+w}(t,T_{i-1})\big)\Big)\\
&- \quad \frac{1}{\hat{a}_s\hat{a}_I}\Big(T_i - T_{i-1} - B^s(t,T_i) - B^I(t,T_i) + B^{I+s}(t,T_i)\\
&+ \quad B^I(t,T_{i-1}) + e^{-\hat{a}_s(T_i-T_{i-1})}\big(B^s(t,T_{i-1}) - B^{I+s}(t,T_{i-1})\big)\Big)\Big)\\
&- \quad \frac{b_{s^{zero}I}\sigma_I^2}{\hat{a}_s-\hat{a}_I}\Big(\frac{1}{(\hat{a}_I)^2}\Big(T_i - T_{i-1} - 2B^I(t,T_i) + B^{I+I}(t,T_i)\\
&+ \quad B^I(t,T_{i-1}) + e^{-\hat{a}_I(T_i-T_{i-1})}\big(B^I(t,T_{i-1}) - B^{I+I}(t,T_{i-1})\big)\Big)\\
&- \quad \frac{1}{\hat{a}_s\hat{a}_I}\Big(T_i - T_{i-1} - B^s(t,T_i) - B^I(t,T_i) + B^{I+s}(t,T_i)\\
&+ \quad B^I(t,T_{i-1}) + e^{-\hat{a}_s(T_i-T_{i-1})}\big(B^s(t,T_{i-1}) - B^{I+s}(t,T_{i-1})\big)\Big)\Big)
\end{aligned}
$$

Furthermore, the following covariance terms can also be calculated by means of Corollary E.2:

$$
\begin{aligned}
&Covar_{\widetilde{Q}}\left[\int_t^{T_i} s^{zero}(x)dx\,, \int_t^{T_{i-1}} r_I(x)dx\middle|\,\mathcal{F}_t\right]\\
&\overset{page\,286}{=} Covar_{\widetilde{Q}}\Big[-\frac{b_{s^{zero}w}\cdot\sigma_w}{\hat{a}_s-\hat{a}_w}\int_t^{T_i} B^w(y,T_i) - B^s(y,T_i)d\widetilde{W}_w(y)\\
&- \quad \frac{b_{s^{zero}I}\cdot\sigma_I\rho_{wI}}{\hat{a}_s-\hat{a}_I}\int_t^{T_i} B^I(y,T_i) - B^s(y,T_i)d\widetilde{W}_w(y)\\
&- \quad \frac{b_{s^{zero}I}\cdot\sigma_I\sqrt{1-\rho_{wI}^2}}{\hat{a}_s-\hat{a}_I}\int_t^{T_i} B^I(y,T_i) - B^s(y,T_i)d\widetilde{W}_I(y)\,,\\
&\quad \sigma_I\ \cdot\rho_{wI}\left(\int_t^{T_{i-1}} B^I(y,T_{i-1})d\widetilde{W}_w(y)\right) + \sigma_I\sqrt{1-\rho_{wI}^2}\left(\int_t^{T_{i-1}} B^I(y,T_{i-1})d\widetilde{W}_I(y)\right)\middle|\,\mathcal{F}_t\Big]\\
&= -\frac{b_{s^{zero}I}\cdot\sigma_I^2}{\hat{a}_s-\hat{a}_I}\Big(\int_t^{T_{i-1}}\big(B^I(y,T_i) - B^s(y,T_i)\big)\cdot B^I(y,T_{i-1})dy\Big)\\
&- \quad \frac{b_{s^{zero}w}\cdot\sigma_w\cdot\sigma_I\cdot\rho_{wI}}{\hat{a}_s-\hat{a}_w}\Big(\int_t^{T_{i-1}}\big(B^w(y,T_i) - B^s(y,T_i)\big)\cdot B^I(y,T_{i-1})dy\Big)\\
&\overset{Cor.E.2}{=} -\frac{b_{s^{zero}I}\sigma_I^2}{\hat{a}_s-\hat{a}_I}\Big(\frac{1}{(\hat{a}_I)^2}\Big(T_{i-1} - t - B^I(t,T_{i-1})\\
&+ \quad e^{-\hat{a}_I(T_i-T_{i-1})}\big(B^{I+I}(t,T_{i-1}) - B^I(t,T_{i-1})\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
- & \quad \frac{1}{\hat{a}_s \hat{a}_I}\Big(T_{i-1} - t - B^I(t,T_{i-1}) + e^{-\hat{a}_s(T_i - T_{i-1})}\big(B^{I+s}(t,T_{i-1}) - B^s(t,T_{i-1})\big)\Big)\Big) \\
- & \quad \frac{b_{s^{zero}w}\sigma_w\sigma_I\rho_{wI}}{\hat{a}_s - \hat{a}_w}\Big(\frac{1}{\hat{a}_w\hat{a}_I}\Big(T_{i-1} - t - B^I(t,T_{i-1}) \\
+ & \quad e^{-\hat{a}_w(T_i - T_{i-1})}\big(B^{I+w}(t,T_{i-1}) - B^w(t,T_{i-1})\big)\Big) \\
- & \quad \frac{1}{\hat{a}_s \hat{a}_I}\Big(T_{i-1} - t - B^I(t,T_{i-1}) + e^{-\hat{a}_s(T_i - T_{i-1})}\big(B^{I+s}(t,T_{i-1}) - B^s(t,T_{i-1})\big)\Big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
& Covar_{\widetilde{Q}}\left[\int_t^{T_i} s^{zero}(x)dx\,, \int_t^{T_{i-1}} r_R(x)dx \middle| \mathcal{F}_t\right] \\
& \overset{page\,286}{=} Covar_{\widetilde{Q}}\Big[-\frac{b_{s^{zero}w}\cdot\sigma_w}{\hat{a}_s - \hat{a}_w}\int_t^{T_i} B^w(y,T_i) - B^s(y,T_i)d\widetilde{W}_w(y) \\
& - \quad \frac{b_{s^{zero}I}\cdot\sigma_I\rho_{wI}}{\hat{a}_s - \hat{a}_I}\int_t^{T_i} B^I(y,T_i) - B^s(y,T_i)d\widetilde{W}_w(y)\,, \\
& \frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R - \hat{a}_w}\Big(\int_t^{T_{i-1}}(B^w(y,T_{i-1}) - B^R(y,T_{i-1}))d\widetilde{W}_w(y)\Big)|\mathcal{F}_t\Big] \\
& = -\frac{b_{Rw}\cdot\sigma_w}{\hat{a}_R - \hat{a}_w}\cdot\Big(\\
& \frac{b_{s^{zero}I}\cdot\sigma_I\cdot\rho_{wI}}{\hat{a}_s - \hat{a}_I}\Big(\int_t^{T_{i-1}}\big(B^w(y,T_{i-1}) - B^R(y,T_{i-1})\big)\cdot\big(B^I(y,T_i) - B^s(y,T_i)\big)dy\Big) \\
& +\frac{b_{s^{zero}w}\cdot\sigma_w}{\hat{a}_s - \hat{a}_w}\Big(\int_t^{T_{i-1}}\big(B^w(y,T_{i-1}) - B^R(y,T_{i-1})\big)\cdot\big(B^w(y,T_i) - B^s(y,T_i)\big)dy\Big)\Big) \\
& \overset{Cor.E.2}{=} -\frac{b_{s^{zero}w}\cdot b_{Rw}\cdot\sigma_w^2}{(\hat{a}_s - \hat{a}_w)(\hat{a}_R - \hat{a}_w)}\Big(\frac{1}{(\hat{a}_w)^2}\Big(T_{i-1} - t - B^w(t,T_{i-1}) \\
& +e^{-\hat{a}_w(T_i - T_{i-1})}\big(B^{w+w}(t,T_{i-1}) - B^w(t,T_{i-1})\big)\Big) \\
& -\frac{1}{\hat{a}_w\hat{a}_s}\Big(T_{i-1} - t - B^w(t,T_{i-1}) + e^{-\hat{a}_s(T_i - T_{i-1})}\big(B^{w+s}(t,T_{i-1}) - B^s(t,T_{i-1})\big)\Big) \\
& -\frac{1}{\hat{a}_w\hat{a}_R}\Big(T_{i-1} - t - B^R(t,T_{i-1}) + e^{-\hat{a}_w(T_i - T_{i-1})}\big(B^{w+R}(t,T_{i-1}) - B^w(t,T_{i-1})\big)\Big) \\
& +\frac{1}{\hat{a}_s\hat{a}_R}\Big(T_{i-1} - t - B^R(t,T_{i-1}) + e^{-\hat{a}_s(T_i - T_{i-1})}\big(B^{s+R}(t,T_{i-1}) - B^s(t,T_{i-1})\big)\Big)\Big)
\end{aligned}
$$

$$
-\frac{b_{s^{zero}I} \cdot b_{Rw} \cdot \sigma_w \cdot \sigma_I \cdot \rho_{wI}}{(\hat{a}_R - \hat{a}_w)(\hat{a}_s - \hat{a}_I)} \Big(\frac{1}{\hat{a}_w \hat{a}_I} \Big(T_{i-1} - t - B^w(t, T_{i-1})
$$

$$
+ e^{-\hat{a}_I (T_i - T_{i-1})} \big(B^{w+I}(t, T_{i-1}) - B^I(t, T_{i-1}) \big) \Big)
$$

$$
- \frac{1}{\hat{a}_w \hat{a}_s} \Big(T_{i-1} - t - B^w(t, T_{i-1}) + e^{-\hat{a}_s (T_i - T_{i-1})} \big(B^{w+s}(t, T_{i-1}) - B^s(t, T_{i-1}) \big) \Big)
$$

$$
- \frac{1}{\hat{a}_I \hat{a}_R} \Big(T_{i-1} - t - B^R(t, T_{i-1}) + e^{-\hat{a}_I (T_i - T_{i-1})} \big(B^{I+R}(t, T_{i-1}) - B^I(t, T_{i-1}) \big) \Big)
$$

$$
+ \frac{1}{\hat{a}_s \hat{a}_R} \Big(T_{i-1} - t - B^R(t, T_{i-1}) + e^{-\hat{a}_s (T_i - T_{i-1})} \big(B^{s+R}(t, T_{i-1}) - B^s(t, T_{i-1}) \big) \Big) \Big)
$$

□

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