KPMG Center of Excellence
in Risk Management

Titel: Robustness of quadratic hedging strategies via backward stochastic differential equations

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Abstract. We consider a backward stochastic differential equation with jumps (BSDEJ) which is driven by a Brownian motion and a Poisson random measure. We present two candidate-approximations to this BSDEJ and we prove that the solution of each candidate-approximation converges to the solution of the original BSDEJ in a space which we specify. We use this result to investigate in further detail the consequences of the choice of the model to (partial) hedging in incomplete markets in finance. As an application, we consider models in which the small variations in the price dynamics are modeled with a Poisson random measure with infinite activity and models in which these small variations are modeled with a Brownian motion. Using the convergence results on BSDEJs, we show that quadratic hedging strategies are robust towards the choice of the model and we derive an estimation of the model risk.

Since Bismut [6] introduced the theory of backward stochastic differential equations (BSDEs), there has been a wide range of literature about this topic. Researchers have kept on developing results on these equations and recently, many papers have studied BSDEs driven by Lévy processes (see, e.g., El Otmani [19], Carbone et al. [9], and Øksendal and Zhang [34]).

In this paper we consider a BSDE which is driven by a Brownian motion and a Poisson random measure (BSDEJ). We present two candidate-approximations to this BSDEJ and we prove that the solution of each candidate-approximation converges to the solution of the BSDEJ in a space which we specify. Our aim from considering such approximations is to investigate the effect of the small jumps of the Lévy process in quadratic hedging strategies in incomplete markets in finance (see, e.g., Föllmer and Schweizer [20] and Vandaele and Vanmaele [33] for more about quadratic hedging strategies in incomplete markets). These strategies are related to the study of the Föllmer-Schweizer decomposition (FS) or/and the Galtchouk-Kunita-Watanabe (GKW) decomposition which are both backward stochastic differential equations (see Choulli et al. [12] for more about these decompositions).

The two most popular types of quadratic hedging strategies are the locally risk-minimizing strategies and the mean-variance hedging strategies. To explain, let us consider a market in which the risky asset is modelled by a jump-diffusion process $S(t)_{t\geq 0}$. Let ξ be a contingent claim. A locally risk-minimizing strategy is a non self-financing strategy that allows a small cost process $C(t)_{t\geq 0}$ and insists on the fact that the terminal condition of the value of the portfolio is equal to the contingent claim (see Schweizer [31]). In other words the existence of the local risk-minimizing strategy for ξ is related to the Föllmer-Schweizer (FS) decomposition, i.e.,

$$\xi = \xi^{(0)} + \int_0^T \chi^{FS}(s)dS(s) + \phi^{FS}(T), \tag{0.1}$$

where $\chi^{FS}(t)_{t\geq 0}$ is a process such that the integral in (0.1) exists and $\phi^{FS}(t)_{t\geq 0}$ is a martingale which has to satisfy certain conditions that we will show in the next sections of the paper. The financial importance of the FS decomposition lies in the fact that it directly provides the locally risk-minimizing strategy for ξ. In fact at each time t the number of risky assets is given by $\chi^{FS}(t)$ and the cost $C(t)$ is given by $\phi^{FS}(t) + \xi^{(0)}$. The mean-variance hedging strategy is a self-financing strategy which minimizes the hedging error in mean square sense (see Föllmer and Sondermann [21]).

In this paper we study the robustness of these two latter hedging strategies towards the approximation of the market prices. Hereto we assume that the process $S(t)_{t\geq 0}$ is a jump-diffusion with stochastic factors and driven by a pure jump term with infinite activity and a Brownian motion $W(t)_{t\geq 0}$. We consider three approximations to $S(t)_{t\geq 0}$. In the first approximation $S_{0,\varepsilon}(t)_{t\geq 0}$, we truncate the small jumps and rescale the Brownian motion $W(t)_{t\geq 0}$ to justify the variance of the small jumps. In the second approximation $S_{1,\varepsilon}(t)_{t\geq 0}$, we truncate the small jumps and replace them by a Brownian motion $B(t)_{t\geq 0}$ independent of $W(t)_{t\geq 0}$ and scaled with the standard deviation of the small jumps. In the third approximation $S_{2,\varepsilon}(t)_{t\geq 0}$, we truncate the small jumps.

This idea of shifting from a model with small jumps to another where those variations are represented by some appropriately scaled continuous component goes back to Asmussen and Rosinsky [1] who proved that the second model approximates the first one. This explains our choice of the two models $S_{0,\varepsilon}(t)_{t\geq 0}$ and $S_{1,\varepsilon}(t)_{t\geq 0}$. This kind of approximation results is here considered for the purpose of a study of robustness of the model. Hence it is interesting from the modeling point of view. In addition, it is also interesting from a simulation point of view. In fact no easy algorithms are available for simulating general Lévy processes. In the present paper the approximating processes we obtain contain a compound Poisson process and a Brownian motion which are both easy to simulate (see Cont and Tankov [13]). For numerical solutions to BSDEs driven by a Brownian motion and a compound Poisson process, we refer to the paper by Bouchard and Elie [7]. This latter paper is in fact an extension of the work by Bouchard and Touzi [8] written for Brownian noise where time discretisation is studied to solve BSDEs with an Euler type scheme. In a forthcoming paper by Khedher et al. [24], BSDEs driven by Brownian motion and jumps with infinite activity are considered. There the combined effect of approximation and time-discretisation is studied together with a numerical scheme to solve such BSDEs. This then will be used to prove the robustness of the locally risk-minimizing strategies to model risk and numerical discretisation.

We do not discuss in this paper any preferences for the choice of the model. We leave this to further studies. For instance Daveloose et al. [14] have this type of discussion about the model choice, in the case the dynamics are given by an exponential Lévy process. Benth et al. [4, 5] investigated the consequences of this approximation to option pricing in finance. They consider option prices written in exponential Lévy processes and they proved the robustness of the option prices after a change of measure where the measure depends on the model choice. For this purpose the authors used Fourier transform techniques.

In this paper we focus mostly on the locally risk-minimizing strategies and we show that under some conditions on the parameters of the stock price process, the value of the portfolio, the amount of wealth, and the cost process in a locally risk-minimizing strategy are robust to the choice of the model. Moreover, we prove the robustness of the value of the portfolio and the amount of wealth in a mean-variance hedging strategy, where we assume that the parameters of the jump-diffusion are deterministic. To prove these results we use the convergence results on BSDEJs and we exploit the relation between BSDEJs and quadratic hedging strategies. In this context, we refer to a paper by Jeanblanc et al. [23] in which the authors exploit the relation between BSDEJs and mean variance hedging strategies in a general semimartingale setting.

This robustness study is a continuation and a generalization of the results by Benth et al. [5]. In fact we consider more general dynamics and we prove that indeed the locally risk-minimizing strategy and the mean-variance hedging strategy are robust to the risk of model choice. For the special choice of dynamics for the price process, namely an exponential Lévy process, Daveloose et al. [14] study robustness of quadratic hedging strategies using a Fourier approach.

The paper is organised as follows: in Section 2 we introduce the notations and we make a short introduction to BSDEJs. In Section 3 we present the two candidate-approximations to the original BSDEJ and we prove the robustness. In Section 4 we prove the robustness of quadratic hedging strategies towards the choice of the model. In Section 5 we conclude.

1. Some mathematical preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We fix $T > 0$. Let $W = W(t)$ and $B = B(t)$, $t \in [0,T]$, be two independent standard Wiener processes and $\widetilde{N} = \widetilde{N}(dt,dz)$, $t, z \in [0,T] \times \mathbb{R}_0$ ($\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$) be a centered Poisson random measure, i.e. $\widetilde{N}(dt,dz) = N(dt,dz) - \ell(dz)dt$, where $\ell(dz)$ is the jump measure and $N(dt,dz)$ is the Poisson random measure independent of the Brownian motions W and B and such that $\mathbb{E}[N(dt,dz)] = \ell(dz)dt$. Define $\mathcal{B}(\mathbb{R}_0)$ as the σ-algebra generated by the Borel sets $\bar{U} \subset \mathbb{R}_0$. We assume that the jump measure has a finite second moment. Namely $\int_{\mathbb{R}_0} z^2 \ell(dz) < \infty$. We introduce the $\mathbb{P}$-augmented filtrations $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$, $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T}$, respectively by

$$\mathcal{F}_t = \sigma\Big\{W(s), \int_0^s \int_A \widetilde{N}(du,dz), \quad s \le t, \quad A \in \mathcal{B}(\mathbb{R}_0)\Big\} \vee \mathcal{N},$$

$$\mathcal{G}_t = \sigma\Big\{W(s), B(s), \int_0^s \int_A \widetilde{N}(du,dz), \quad s \le t, \quad A \in \mathcal{B}(\mathbb{R}_0)\Big\} \vee \mathcal{N},$$

where $\mathcal{N}$ represents the set of $\mathbb{P}$-null events in $\mathcal{F}$. We introduce the notation $\mathbb{H} = (\mathcal{H}_t)_{0 \le t \le T}$, such that $\mathcal{H}_t$ will be given either by the σ-algebra $\mathcal{F}_t$ or $\mathcal{G}_t$ depending on our analysis later.

Define the following spaces;

- L^2_T: the space of all $\mathcal{H}_T$-measurable random variables $X : \Omega \to \mathbb{R}$ such that

$$\|X\|^2 = \mathbb{E}[X^2] < \infty.$$

- H_T^2: the space of all $\mathbb{H}$-predictable processes $\phi : \Omega \times [0,T] \to \mathbb{R}$, such that
$$\|\phi\|^2_{H_T^2} = \mathbb{E}\Big[\int_0^T |\phi(t)|^2 dt\Big] < \infty.$$
- $\widetilde{H}_T^2$: the space of all $\mathbb{H}$-adapted, càdlàg processes $\psi : \Omega \times [0,T] \to \mathbb{R}$ such that
$$\|\psi\|^2_{\widetilde{H}_T^2} = \mathbb{E}\Big[\int_0^T |\psi(t)|^2 dt\Big] < \infty.$$
- $\widehat{H}_T^2$: the space of all $\mathbb{H}$-predictable mappings $\theta : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$, such that
$$\|\theta\|^2_{\widehat{H}_T^2} = \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} |\theta(t,z)|^2 \ell(dz)dt\Big] < \infty.$$
- S_T^2: the space of all $\mathbb{H}$-adapted, càdlàg processes $\gamma : \Omega \times [0,T] \to \mathbb{R}$ such that
$$\|\gamma\|^2_{S_T^2} = \mathbb{E}[\sup_{0\le t\le T} |\gamma^2(t)|] < \infty.$$
- $\nu = S_T^2 \times H_T^2 \times \widehat{H}_T^2$.
- $\widetilde{\nu} = S_T^2 \times H_T^2 \times \widehat{H}_T^2 \times H_T^2$.
- $\widehat{L}_T^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \ell)$: the space of all $\mathcal{B}(\mathbb{R}_0)$-measurable mappings $\psi : \mathbb{R}_0 \to \mathbb{R}$ such that
$$\|\psi\|^2_{\widehat{L}_T^2(\mathbb{R}_0,\mathcal{B}(\mathbb{R}_0),\ell)} = \int_{\mathbb{R}_0} |\psi(z)|^2 \ell(dz) < \infty.$$

The following result is crucial in the study of the existence and uniqueness of the backward stochastic differential equations we are interested in. Indeed it is an application of the decomposition of a random variable $\xi \in L_T^2$ with respect to orthogonal martingale random fields as integrators. See Kunita and Watanabe [26], Cairoli and Walsh [10], and Di Nunno and Eide [17] for the essential ideas. In Di Nunno [15, 16], and Di Nunno and Eide [17], explicit representations of the integrands are given in terms of the non-anticipating derivative.

Theorem 1.1. *Let $\mathbb{H} = \mathbb{G}$. Every $\mathcal{G}_T$-measurable random variable $\xi \in L_T^2$ has a unique representation of the form*

$$\xi = \xi^{(0)} + \sum_{k=1}^{3} \int_0^T \int_{\mathbb{R}} \varphi_k(t,z)\mu_k(dt,dz), \tag{1.1}$$

where the stochastic integrators

$$\mu_1(dt,dz) = W(dt) \times \delta_0(dz), \quad \mu_2(dt,dz) = B(dt) \times \delta_0(dz),$$
$$\mu_3(dt,dz) = \widetilde{N}(dt,dz)\mathbf{1}_{[0,T]\times\mathbb{R}_0}(t,z),$$

are orthogonal martingale random fields on $[0,T] \times \mathbb{R}_0$ and the stochastic integrands are φ_1, $\varphi_2 \in H_T^2$ and $\varphi_3 \in \widehat{H}_T^2$. Moreover $\xi^{(0)} = \mathbb{E}[\xi]$.
Let $\mathbb{H} = \mathbb{F}$. Then for every $\mathcal{F}_T$-measurable random variable $\xi \in L_T^2$, (1.1) *holds with $\mu_2(dt,dz) = 0$.*

As we shall see the above result plays a central role in the analysis that follows. Let us now consider a pair (ξ, f), where ξ is called the terminal condition and f the driver such that

Assumptions 1.
(A) $\xi \in L_T^2$ *is* $\mathcal{H}_T$*-measurable*
(B) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *such that*

- $f(\cdot, x, y, z)$ *is* $\mathbb{H}$*-progressively measurable for all* x, y, z*,*
- $f(\cdot, 0, 0, 0) \in H_T^2$*,*
- $f(\cdot, x, y, z)$ *satisfies a uniform Lipschitz condition in* (x, y, z)*, i.e. there exists a constant* C *such that for all* $(x_i, y_i, z_i) \in \mathbb{R} \times \mathbb{R} \times \widehat{L}_T^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \ell)$*,* $i = 1, 2$ *we have*

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \\ \leq C\Big(|x_1 - x_2| + |y_1 - y_2| + \|z_1 - z_2\|\Big), \quad \textit{for all } t.$$

We consider the following backward stochastic differential equation with jumps (in short BSDEJ)

$$\begin{cases} -dX(t) & = f(t, X(t), Y(t), Z(t, \cdot))dt - Y(t)dW(t) - \displaystyle\int_{\mathbb{R}_0} Z(t, z)\widetilde{N}(dt, dz), \\ X(T) & = \xi. \end{cases} \tag{1.2}$$

Definition 1.2. *A solution to the BSDEJ* (1.2) *is a triplet of* $\mathbb{H}$*-adapted or predictable processes* $(X, Y, Z) \in \nu$ *satisfying*

$$X(t) = \xi + \int_t^T f(s, X(s), Y(s), Z(s, \cdot))ds - \int_t^T Y(s)dW(s) \\ - \int_t^T \int_{\mathbb{R}_0} Z(s, z)\widetilde{N}(ds, dz), \qquad 0 \leq t \leq T.$$

The existence and uniqueness result for the solution of the BSDEJ (1.2) is guaranteed by the following result proved in Tang and Li [32].

Theorem 1.3. *Given a pair* (ξ, f) *satisfying Assumptions 1(A) and (B), there exists a unique solution* $(X, Y, Z) \in \nu$ *to the BSDEJ* (1.2)*.*

2. Two candidate-approximating BSDEJs and robustness

2.1. **Two candidate-approximating BSDEJs.** In this subsection we present two candidate-approximations of the BSDEJ (1.2). Let $\mathbb{H} = \mathbb{F}$ and f_ε^0 be a function satisfying Assumptions 1(B), for all $\varepsilon \in [0, 1]$. In the first candidate-approximation, we approximate the terminal condition ξ of the BSDEJ (1.2) by a sequence of random variables $\xi_\varepsilon^0 \in L_T^2$, $\mathcal{F}_T$-measurable such that

$$\lim_{\varepsilon \to 0} \xi_\varepsilon^0 = \xi, \quad \text{in } L_T^2.$$

We obtain the following approximation

$$
\text{(2.1)}\quad \begin{cases} -dX_\varepsilon(t) &= f^0_\varepsilon(t, X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t,\cdot))dt - Y_\varepsilon(t)dW(t) - \int_{\mathbb{R}_0} Z_\varepsilon(t,z)\widetilde{N}(dt,dz), \\ X_\varepsilon(T) &= \xi^0_\varepsilon. \end{cases}
$$

We present the following condition on f^0_ε, which we need to impose when we study the robustness results in the next section. For all $(x_i, y_i, z_i) \in \mathbb{R}\times\mathbb{R}\times\widehat{L}^2_T(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \ell)$, $i = 1, 2$, and for all $t \in [0, T]$, it holds that

$$
\begin{aligned}
&|f(t, x_1, y_1, z_1) - f^0_\varepsilon(t, x_2, y_2, z_2)| \\
&\qquad \le C\Big(|x_1 - x_2| + |y_1 - y_2| + \|z_1 - z_2\| + \widetilde{G}(\varepsilon)|y_2| + \widetilde{G}(\varepsilon)\|z_2\|\Big), \qquad (2.2)
\end{aligned}
$$

for C and $\widetilde{G}(\varepsilon)$ positive constants and $\widetilde{G}(\varepsilon)$ vanishing when ε goes to 0.

In the next theorem we state the existence and uniqueness of the solution $(X_\varepsilon, Y_\varepsilon, Z_\varepsilon) \in \nu$ of the BSDEJ (2.1). This result on existence and uniqueness of the solution to (2.1) is along the same lines as the proof of Theorem 1.3, see also Tang and Li [32].

Theorem 2.1. *Let $\mathbb{H} = \mathbb{F}$. Given a pair $(\xi^0_\varepsilon, f^0_\varepsilon)$ such that $\xi^0_\varepsilon \in L^2_T$ is $\mathcal{F}_T$-measurable and f^0_ε satisfies Assumptions 1(B), then there exists a unique solution $(X_\varepsilon, Y_\varepsilon, Z_\varepsilon) \in \nu$ to the BSDEJ* (2.1).

Let $\mathbb{H} = \mathbb{G}$. We present the second candidate-approximation to (1.2). Hereto we introduce a sequence of random variables $\mathcal{G}_T$-measurable $\xi^1_\varepsilon \in L^2_T$ such that

$$
\lim_{\varepsilon\to 0} \xi^1_\varepsilon = \xi
$$

and a function f^1_ε satisfying

Assumptions 2. $f^1_\varepsilon : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is such that for all* $\varepsilon \in [0, 1]$,

- $f^1_\varepsilon(\cdot, x, y, z, \zeta)$ *is $\mathbb{G}$-progressively measurable for all* x, y, z, ζ,
- $f^1_\varepsilon(\cdot, 0, 0, 0, 0) \in H^2_T$,
- $f^1_\varepsilon(\cdot, x, y, z, \zeta)$ *satisfies a uniform Lipschitz condition in* (x, y, z, ζ).

Besides Assumptions 2 which we impose on f^1_ε, we need moreover to assume the following condition in the robustness analysis later on. For all $(x_i, y_i, z_i, \zeta) \in \mathbb{R} \times \mathbb{R} \times \widehat{L}^2_T(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \ell) \times \mathbb{R}$, $i = 1, 2$, and for all $t \in [0, T]$, it holds that

$$
\begin{aligned}
&|f(t, x_1, y_1, z_1) - f^1_\varepsilon(t, x_2, y_2, z_2, \zeta)| \\
&\qquad \le C\Big(|x_1 - x_2| + |y_1 - y_2| + \|z_1 - z_2\| + |\zeta| + \widetilde{G}(\varepsilon)\|z_1\||\Big), \qquad (2.3)
\end{aligned}
$$

for C and $\widetilde{G}(\varepsilon)$ positive constants and $\widetilde{G}(\varepsilon)$ vanishing when ε goes to 0.
We introduce the second candidate BSDEJ approximation to (1.2) which reads as follows
(2.4)

$$\begin{cases} -dX_\varepsilon(t) &= f^1_\varepsilon(t, X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t,\cdot), \zeta_\varepsilon(t))dt - Y_\varepsilon(t)dW(t) - \int_{\mathbb{R}_0} Z_\varepsilon(t,z)\widetilde{N}(dt,dz) \\ & \qquad -\zeta_\varepsilon(t)dB(t), \\ X_\varepsilon(T) &= \xi^1_\varepsilon, \end{cases}$$

where we use the same notations as in (2.1). B is a Brownian motion independent of W. Because of the presence of the additional noise B the solution processes are expected to be $\mathbb{G}$-adapted (or predictable). Notice that the solution of such equation is given by $(X_\varepsilon, Y_\varepsilon, Z_\varepsilon, \zeta_\varepsilon) \in \widetilde{\nu}$. In the next theorem we state the existence and uniqueness of the solution of the equation (2.4). The proof is very similar to the proof of Theorem 2.1. However we work under the σ-algebra $\mathcal{G}_t$.

Theorem 2.2. *Let $\mathbb{H} = \mathbb{G}$. Given a pair $(\xi^1_\varepsilon, f^1_\varepsilon)$ such that $\xi^1_\varepsilon \in L^2_T$ is $\mathcal{G}_T$-measurable and f^1_ε satisfies Assumptions 2, then there exists a unique solution $(X_\varepsilon, Y_\varepsilon, Z_\varepsilon, \zeta_\varepsilon) \in \widetilde{\nu}$ to the BSDEJ* (2.1).

It is expected that when (2.3) holds, the process ζ_ε vanishes when ε goes to 0. This will be shown in the next subsection in which we also prove the robustness of the BSDEJs.

2.2. **Robustness of the BSDEJs.** Before we show the convergence of the two equations (2.1) and (2.4) to the BSDEJ (1.2) when ε goes to 0, we present the following lemma in which we prove the boundedness of the solution of (1.2) and of that of (2.1). We need this lemma in Theorem 2.4 and for our analysis in the next section.

Lemma 2.3. *Let (X, Y, Z), $(X_\varepsilon, Y_\varepsilon, Z_\varepsilon)$ be the solution of* (1.2) *and* (2.1)*, respectively. Then we have for all $t \in [0,T]$,*

$$\begin{aligned} &\mathbb{E}\Big[\int_t^T X^2(s)ds\Big] + \mathbb{E}\Big[\int_t^T Y^2(s)ds\Big] + \mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} Z^2(s,z)\ell(dz)ds\Big] \\ &\qquad \le C\Big(\mathbb{E}[\xi^2] + \mathbb{E}\Big[\int_t^T |f(s,0,0,0)|^2 ds\Big]\Big), \end{aligned}$$

respectively,

$$\begin{aligned} &\mathbb{E}\Big[\int_t^T X^2_\varepsilon(s)ds\Big] + \mathbb{E}\Big[\int_t^T Y^2_\varepsilon(s)ds\Big] + \mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} Z^2_\varepsilon(s,z)\ell(dz)ds\Big] \\ &\qquad \le C\Big(\mathbb{E}[|\xi^0_\varepsilon|^2] + \mathbb{E}\Big[\int_t^T |f^0_\varepsilon(s,0,0,0)|^2 ds\Big]\Big), \end{aligned}$$

where C is a positive constant.

Proof. Recall the expression of X given by (1.2). Applying the Itô formula to $e^{\beta t}X^2(t)$ and taking the expectation, we get

$$\begin{aligned}\mathbb{E}[e^{\beta t}X^2(t)] &= \mathbb{E}[e^{\beta T}X^2(T)] - \beta\mathbb{E}\Big[\int_t^T e^{\beta s}X^2(s)ds\Big] - \mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}Y^2(s)ds\Big]\\ &\quad + 2\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}X(s)\Big(f(s,X(s),Y(s),Z(s,.)) - f(s,0,0,0)\Big)ds\Big]\\ &\quad + 2\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}X(s)f(s,0,0,0)ds\Big] - \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}Z^2(s,z)\ell(dz)ds\Big].\end{aligned}$$

Thus by the Lipschitz property of f we find

$$\begin{aligned}&\mathbb{E}[e^{\beta t}X^2(t)] + \mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}Y^2(s)ds\Big] + \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}Z^2(s,z)\ell(dz)ds\Big]\\ &\le \mathbb{E}[e^{\beta T}X^2(T)] - \beta\mathbb{E}\Big[\int_t^T e^{\beta s}X^2(s)ds\Big]\\ &\quad + 2C\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}X(s)\Big(|X(s)| + |Y(s)| + |\int_{\mathbb{R}_0} Z^2(s,z)\ell(dz)|^{\frac{1}{2}}\Big)ds\Big]\\ &\quad + 2\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}X(s)f(s,0,0,0)ds\Big].\end{aligned}$$

Using the fact that for every $k > 0$ and $a, b \in \mathbb{R}$ we have that $2ab \le ka^2 + \frac{b^2}{k}$ and $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, choosing $\beta = 6C^2 + 2$, and noticing that $\beta > 0$, the result follows for (X, Y, Z). The same computations lead to the result for the approximation $(X_\varepsilon, Y_\varepsilon, Z_\varepsilon)$. □

From now on we use a unified notation for both BSDEJs (2.1) and (2.4) in the BSDEJ

$$\text{(2.5)}\qquad \begin{cases} -dX^\rho_\varepsilon(t) &= f^\rho_\varepsilon(t)dt - Y^\rho_\varepsilon(t)dW(t) - \displaystyle\int_{\mathbb{R}_0} Z^\rho_\varepsilon(t,z)\widetilde{N}(dt,dz) - \zeta^\rho_\varepsilon(t)dB(t),\\ X^\rho_\varepsilon(T) &= \xi^\rho_\varepsilon, \qquad \text{for } \rho = 0 \text{ and } \rho = 1,\end{cases}$$

where

$$f^\rho_\varepsilon(t) = \begin{cases} f^0_\varepsilon(t, X^0_\varepsilon(t), Y^0_\varepsilon(t), Z^0_\varepsilon(t)), & \rho = 0,\\ f^1_\varepsilon(t, X^1_\varepsilon(t), Y^1_\varepsilon(t), Z^1_\varepsilon(t), \zeta^1_\varepsilon(t)), & \rho = 1\end{cases}$$

and

$$\zeta^\rho_\varepsilon(t) = \begin{cases} 0, & \rho = 0,\\ \zeta^1_\varepsilon(t), & \rho = 1.\end{cases}$$

Notice that the BSDEJ (2.5) has the same solution as (2.1) and (2.4) respectively for $\rho = 0$ and $\rho = 1$. We state the following theorem in which we prove the convergence of both BSDEJs (2.1) and (2.4) to the BSDEJ (1.2).

Theorem 2.4. *Assume that f_ε^0 and f_ε^1 satisfy (2.2) and (2.3) respectively. Let (X,Y,Z) be the solution of (1.2) and $(X_\varepsilon^\rho, Y_\varepsilon^\rho, Z_\varepsilon^\rho, \zeta_\varepsilon^\rho)$ be the solution of (2.5). Then we have for $t\in[0,T]$, $\rho=0$ and $\rho=1$*

$$\begin{aligned}
&\mathbb{E}\Big[\int_t^T |X(s)-X_\varepsilon^\rho(s)|^2 ds\Big] + \mathbb{E}\Big[\int_t^T |Y(s)-Y_\varepsilon^\rho(s)|^2 ds\Big] \\
&\quad + \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} |Z(s,z)-Z_\varepsilon^\rho(s,z)|^2\ell(dz)ds\Big] + \mathbb{E}\Big[\int_t^T |\zeta_\varepsilon^\rho(s)|^2 ds\Big] \\
&\qquad \le K\mathbb{E}[|\xi-\xi_\varepsilon^\rho|^2] + \widetilde{K}\widetilde{G}^2(\varepsilon)(1-\rho)\Big(\mathbb{E}[|\xi_\varepsilon^0|^2] + \mathbb{E}[\int_t^T |f_\varepsilon^0(s,0,0,0)|^2 ds]\Big) \\
&\qquad\quad + \widehat{K}\widetilde{G}^2(\varepsilon)\rho\Big(\mathbb{E}[|\xi|^2] + \mathbb{E}[\int_t^T |f(s,0,0,0)|^2 ds]\Big),
\end{aligned}$$

where K, $\widetilde{K}$, $\widehat{K}$ and $\widetilde{G}(\varepsilon)$ are positive constants and with $\widetilde{G}(\varepsilon)$ vanishing when ε goes to 0.

Proof. Let

$$\begin{aligned}
&\bar{X}_\varepsilon^\rho(t) = X(t)-X_\varepsilon^\rho(t), \quad \bar{Y}_\varepsilon^\rho(t) = Y(t)-Y_\varepsilon^\rho(t), \quad \bar{Z}_\varepsilon^\rho(t,z) = Z(t,z)-Z_\varepsilon^\rho(t,z), \\
&\bar{f}_\varepsilon^\rho(t) = f(t,X(t),Y(t),Z(t,.)) - f_\varepsilon^\rho(t).
\end{aligned} \tag{2.6}$$

Applying the Itô formula to $\mathrm{e}^{\beta t}|\bar{X}_\varepsilon^\rho(t)|^2$, we get

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{\beta t}|\bar{X}_\varepsilon^\rho(t)|^2] + \mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{Y}_\varepsilon^\rho(s)|^2 ds\Big] + \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|\bar{Z}_\varepsilon^\rho(s,z)|^2\ell(dz)ds\Big] \\
&\quad + \mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\zeta_\varepsilon^\rho(s)|^2 ds\Big] \\
&\qquad = \mathbb{E}[e^{\beta T}|\bar{X}_\varepsilon^\rho(T)|^2] - \beta\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{X}_\varepsilon^\rho(s)|^2 ds\Big] + 2\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{X}_\varepsilon^\rho(s)||\bar{f}_\varepsilon^\rho(s)|ds\Big].
\end{aligned} \tag{2.7}$$

Using conditions (2.2) and (2.3), we get

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{\beta t}|\bar{X}_\varepsilon^\rho(t)|^2] + \mathbb{E}[\int_t^T \mathrm{e}^{\beta s}|\bar{Y}_\varepsilon^\rho(s)|^2 ds] + \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|\bar{Z}_\varepsilon^\rho(s,z)|^2\ell(dz)ds\Big] \\
&\quad + \mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\zeta_\varepsilon^\rho(s)|^2 ds\Big] \\
&\le \mathbb{E}[e^{\beta T}|\bar{X}_\varepsilon^\rho(T)|^2] - \beta\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{X}_\varepsilon^\rho(s)|^2 ds\Big] \\
&\quad + 2C\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{X}_\varepsilon^\rho(s)|\Big(|\bar{X}_\varepsilon^\rho(s)| + |\bar{Y}_\varepsilon^\rho(s)| + |\zeta_\varepsilon^\rho(s)| + (1-\rho)\widetilde{G}(\varepsilon)|Y_\varepsilon^0(s)| \\
&\qquad + \rho\widetilde{G}(\varepsilon)(\int_{\mathbb{R}_0} |Z(s,z)|^2\ell(dz))^{\frac{1}{2}} + (1-\rho)\widetilde{G}(\varepsilon)(\int_{\mathbb{R}_0} |Z_\varepsilon^0(s,z)|^2\ell(dz))^{\frac{1}{2}}
\end{aligned}$$

$$+ \Big(\int_{\mathbb{R}_0} |\bar{Z}_\varepsilon^\rho(s,z)|^2 \ell(dz))^{\frac{1}{2}}\Big)ds\Big].$$

Using the fact that for every $k > 0$ and $a, b \in \mathbb{R}$ we have that $2ab \leq ka^2 + \frac{b^2}{k}$ and $\left(\sum_{i=1}^7 a_i\right)^2 \leq 7\sum_{i=1}^7 a_i^2$, we obtain

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{\beta t}|\bar{X}_\varepsilon^\rho(t)|^2] + \mathbb{E}[\int_t^T \mathrm{e}^{\beta s}|\bar{Y}_\varepsilon^\rho(s)|^2 ds] + \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|\bar{Z}_\varepsilon^\rho(s,z)|^2\ell(dz)ds\Big] \\
&\qquad + \mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\zeta_\varepsilon^\rho(s)|^2 ds\Big] \\
&\quad \leq \mathbb{E}[e^{\beta T}|\bar{X}_\varepsilon^\rho(T)|^2] - \beta\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{X}_\varepsilon^\rho(s)|^2 ds\Big] + 14C^2\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{X}_\varepsilon^\rho(s)|^2 ds\Big] \\
&\qquad + \frac{1}{2}\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{X}_\varepsilon^\rho(s)|^2 ds\Big] + \frac{1}{2}\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\zeta_\varepsilon^\rho(s)|^2 ds\Big] \\
&\qquad + \frac{1}{2}\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{Y}_\varepsilon^\rho(s)|^2 ds\Big] + \frac{1}{2}\mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|\bar{Z}_\varepsilon^\rho(s,z)|^2\ell(dz)ds\Big] \\
&\qquad + \frac{1}{2}(1-\rho)\widetilde{G}^2(\varepsilon)\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|Y_\varepsilon^0(s)|^2 ds\Big] + \frac{1}{2}\rho\widetilde{G}^2(\varepsilon)\mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|Z(s,z)|^2\ell(dz)ds\Big] \\
&\qquad + \frac{1}{2}(1-\rho)\widetilde{G}^2(\varepsilon)\mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|Z_\varepsilon^0(s,z)|^2\ell(dz)ds\Big].
\end{aligned}$$

Choosing $\beta = 14C^2 + 1$ and since $\mathbb{E}[\mathrm{e}^{\beta t}|\bar{X}_\varepsilon^\rho(t)|^2] > 0$, we get

$$\begin{aligned}
&\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{X}_\varepsilon^\rho(s)|^2 ds\Big] + \mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\bar{Y}_\varepsilon^\rho(s)|^2 ds\Big] + \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|\bar{Z}_\varepsilon^\rho(s,z)|^2\ell(dz)ds\Big] \\
&\qquad + \mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|\zeta_\varepsilon^\rho(s)|^2 ds\Big] \\
&\quad \leq K\mathbb{E}[\mathrm{e}^{\beta T}|\bar{X}_\varepsilon^\rho(T)|^2] + \frac{1}{2}(1-\rho)\widetilde{G}^2(\varepsilon)\Big(\mathbb{E}\Big[\int_t^T \mathrm{e}^{\beta s}|Y_\varepsilon^0(s)|^2 ds\Big] \\
&\qquad + \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|Z_\varepsilon^0(s,z)|^2\ell(dz)ds\Big]\Big) + \frac{1}{2}\rho\widetilde{G}^2(\varepsilon)\mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0} \mathrm{e}^{\beta s}|Z(s,z)|^2\ell(dz)ds\Big],
\end{aligned}$$

where K is a positive constant and the result follows using Lemma 2.3 and the fact that $\beta > 0$. □

Remark 2.5. *Since $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0,T]$, the solution of* (1.2) *is also $\mathbb{G}$-adapted. This fact allowed us to compare the solution of* (1.2) *with the solution of* (2.4).

Notice that in the case $\rho = 0$, the condition (2.2) implies that for $(x,y,z) \in \mathbb{R} \times \mathbb{R} \times \widehat{L}_T^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \ell)$

$$\lim_{\varepsilon \to 0} f_\varepsilon^0(t,x,y,z) = f(t,x,y,z), \qquad \mathbb{P}\text{-a.s.}, \forall t \in [0,T].$$

Thus the convergence of the solution of (2.1) to the solution of (1.2) in the space $\widetilde{H}_T^2 \times H_T^2 \times \widehat{H}_T^2$, follows directly from Proposition 2.1 in El Karoui, Peng, and Quenez [18]. We presented the proof for the sake of completeness. In the latter theorem, we proved the convergence of the solution of (2.1) respectively (2.4) to the solution of (1.2) in the space $\widetilde{H}_T^2 \times H_T^2 \times \widehat{H}_{T,}^2$ respectively $\widetilde{H}_T^2 \times H_T^2 \times \widehat{H}_T^2 \times H_T^2$. In the next theorem we prove the convergence in ν, respectively $\widetilde{\nu}$.

Theorem 2.6. *Assume that* (2.2) *and* (2.3) *hold. Let* X*,* X_ε^ρ *be the solution of* (1.2)*,* (2.5)*, respectively. Then we have for* $\rho = 0$ *and* $\rho = 1$

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X(t)-X_\varepsilon^\rho(t)|^2\Big] \le C\mathbb{E}[|\xi-\xi_\varepsilon^\rho|^2] + \widehat{K}\widetilde{G}^2(\varepsilon)\rho\Big(\mathbb{E}[|\xi|^2]+\mathbb{E}[\int_t^T |f(s,0,0,0)|^2 ds]\Big)$$
$$+\widetilde{K}\widetilde{G}^2(\varepsilon)(1-\rho)\Big(\mathbb{E}[|\xi_\varepsilon^0|^2]+\mathbb{E}[\int_t^T |f_\varepsilon^0(s,0,0,0)|^2 ds]\Big),$$

where C*,* $\widetilde{K}$*, and* $\widehat{K}$ *are positive constants.*

Proof. Let $\bar{X}_\varepsilon^\rho$, $\bar{Y}_\varepsilon^\rho$, $\bar{Z}_\varepsilon^\rho$, and $\bar{f}_\varepsilon^\rho$ be as in (2.6). Then applying Hölder's inequality, we have for $K > 0$

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\bar{X}_\varepsilon^\rho(t)|^2\Big] \le K\Big(\mathbb{E}\Big[|\bar{X}_\varepsilon^\rho(T)|^2\Big]+\mathbb{E}\Big[\int_0^T|\bar{f}_\varepsilon^\rho(s)|^2ds\Big]+\mathbb{E}\Big[\sup_{0\le t\le T}|\int_t^T\bar{Y}_\varepsilon^\rho(s)dW(s)|^2\Big]$$
$$+\mathbb{E}\Big[\sup_{0\le t\le T}|\int_t^T\int_{\mathbb{R}_0}\bar{Z}_\varepsilon^\rho(s,z)\widetilde{N}(ds,dz)|^2\Big]$$
$$+\mathbb{E}\Big[\sup_{0\le t\le T}|\int_t^T\zeta_\varepsilon^\rho(s)dB(s)|^2\Big]\Big).$$

However from Burkholder's inequality we can prove that for $C > 0$, we have (for more details see Tang and Li [32])

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\int_t^T\int_{\mathbb{R}_0}\bar{Z}_\varepsilon^\rho(s,z)\widetilde{N}(ds,dz)|^2\Big] \le C\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0}|\bar{Z}_\varepsilon^\rho(s,z)|^2\ell(dz)ds\Big],$$
$$\mathbb{E}\Big[\sup_{0\le t\le T}|\int_t^T\bar{Y}_\varepsilon^\rho(s)dW(s)|^2\Big] \le C\mathbb{E}\Big[\int_0^T|\bar{Y}_\varepsilon^\rho(s)|^2ds\Big],$$
$$\mathbb{E}\Big[\sup_{0\le t\le T}|\int_t^T\zeta_\varepsilon^\rho(s)dB(s)|^2\Big] \le C\mathbb{E}\Big[\int_0^T|\zeta_\varepsilon^\rho(s)|^2ds\Big].$$

Thus from the estimates on f^0 and f_ε^1 in equations (2.2) and (2.3), Lemma 2.3 and Theorem 2.4 we get the result. □

Notice that we proved the convergence of the two candidate approximating BSDEJs (2.1), (2.4) to the BSDEJ (1.2) in the space ν, $\widetilde{\nu}$ respectively. This type of convergence is stronger than the L^2-convergence.

3. Robustness of quadratic hedging strategies

We assume we have two assets. One of them is a riskless asset with price $S^{(0)}$ given by

$$dS^{(0)}(t) = S^{(0)}(t)r(t)dt,$$

where the short rate $r(t) = r(t,\omega) \in \mathbb{R}$ is $\mathbb{F}$-adapted. The dynamics of the risky asset are given by

$$\begin{cases} dS^{(1)}(t) &= S^{(1)}(t)\Big\{a(t)dt + b(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(dt,dz)\Big\}, \\ S^{(1)}(0) &= x \in \mathbb{R}_+\,, \end{cases}$$

where $a(t) = a(t,\omega) \in \mathbb{R}$, $b(t) = b(t,\omega) \in \mathbb{R}$, and $\gamma(t,z) = \gamma(t,z,\omega) \in \mathbb{R}$ for $t \geq 0$, $z \in \mathbb{R}_0$ are $\mathbb{F}$-adapted processes. We assume that $\gamma(t,z,\omega) = g(z)\widetilde{\gamma}(t,\omega)$, such that

$$G^2(\varepsilon) := \int_{|z|\leq\varepsilon} g^2(z)\ell(dz) < \infty. \tag{3.1}$$

The dynamics of the discounted price process $\widetilde{S} = \frac{S^{(1)}}{S^{(0)}}$ are given by

$$d\widetilde{S}(t) = \widetilde{S}(t)\Big[(a(t) - r(t))dt + b(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(dt,dz)\Big]. \tag{3.2}$$

For $\widetilde{S}$ to be positive, we assume $\gamma(t,z) > -1$, a.e. in (t,z,ω). We further assume that the semimartingale $\widetilde{S}$ is locally square integrable (in the sense of Definition 2.27 in Jacod and Shiryaev [22]). We can decompose $\widetilde{S}$ into a locally square integrable local martingale M starting at zero in zero and a predictable finite variation process A, with $A(0) = 0$, where M and A have the following expressions

$$M(t) = \int_0^t b(s)\widetilde{S}(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\widetilde{S}(s)\widetilde{N}(ds,dz), \tag{3.3}$$

$$A(t) = \int_0^t (a(s) - r(s))\widetilde{S}(s)ds.$$

We denote the predictable compensator associated to M (see Protter [28]) by

$$\langle M\rangle(t) = \int_0^t b^2(s)\widetilde{S}^2(s)ds + \int_0^t \int_{\mathbb{R}_0} \widetilde{S}^2(s)\gamma^2(s,z)\ell(dz)ds$$

and we can represent the process A as follows

$$A(t) = \int_0^t \alpha(s)d\langle M\rangle(s)\,, \tag{3.4}$$

where

$$\alpha(t) := \frac{a(t) - r(t)}{\widetilde{S}(t)\big(b^2(t) + \int_{\mathbb{R}_0} \gamma^2(t,z)\ell(dz)\big)}, \qquad 0 \leq t \leq T. \tag{3.5}$$

We define a process K by means of α as follows

$$K(t) = \int_0^t \alpha^2(s)d\langle M\rangle(s) = \int_0^t \frac{(a(s)-r(s))^2}{b^2(s)+\int_{\mathbb{R}_0}\gamma^2(s,z)\ell(dz)}ds. \tag{3.6}$$

The process K is called the mean-variance tradeoff (MVT) process.

Since the stock price fluctuations are modeled by jump-diffusion, the market is incomplete and not every contingent claim can be replicated by a self-financing strategy and there is no perfect hedge. However, one can adopt a partial hedging strategy according to some optimality criteria minimizing the risk. Föllmer and Schweizer [20] introduced the so-called quadratic hedging strategies. The study of such strategies heavily depends on the Föllmer-Schweizer (FS) decomposition. This decomposition was first introduced by Föllmer and Schweizer [20] for the continuous case and extended to the discontinuous case by Ansel and Stricker [2].

In order to formulate our robustness study for the quadratic hedging strategies, we present in the sequel the relation between BSDEs and the FS decomposition. We denote by $L(\widetilde{S})$, the $\widetilde{S}$-integrable processes, that is the class of predictable processes for which we can determine the stochastic integral with respect to $\widetilde{S}$. We define the space Θ by

$$\Theta := \Big\{\theta \in L(\widetilde{S}) \,|\, \mathbb{E}\Big[\int_0^T \theta^2(s)d\langle M\rangle(s) + \big(\int_0^T |\theta(s)dA(s)|\big)^2\Big] < \infty\Big\}. \tag{3.7}$$

Consider a process $\chi^{FS} \in \Theta$. Let ξ be a square integrable contingent claim and $\widetilde{\xi} = \xi/S^{(0)}(T)$ its discounted value. Define the process $\widetilde{V}$ as follows

$$\widetilde{V}(t) := \mathbb{E}\left[\widetilde{\xi} - \int_t^T \chi^{FS}(s)dA(s)|\mathcal{F}_t\right], \qquad 0 \le t \le T.$$

Then applying the Galtchouk-Kunita-Watanabe decomposition (see, e.g., Ansel and Stricker [3]) to the random variable $U(T) := \widetilde{\xi} - \int_0^T \chi^{FS}(s)dA(s)$, we get

$$U(T) = \mathbb{E}\left[\widetilde{\xi} - \int_0^T \chi^{FS}(s)dA(s)\right] + \int_0^T \widetilde{\chi}(s)dM(s) + \phi^{FS}(T), \tag{3.8}$$

where $\widetilde{\chi} \in \Theta$ and ϕ^{FS} is a square integrable martingale such that $[\phi^{FS}, M]$ is a local martingale. Taking conditional expectations in (3.8), we obtain

$$\mathbb{E}\left[U(T)|\mathcal{F}_t\right] = \mathbb{E}\left[\widetilde{\xi} - \int_0^T \chi^{FS}(s)dA(s)\right] + \int_0^t \widetilde{\chi}(s)dM(s) + \phi^{FS}(t), \qquad 0 \le t \le T,$$

which implies

$$\widetilde{V}(t) = \widetilde{V}(0) + \int_0^t \widetilde{\chi}(s)dM(s) + \int_0^t \chi^{FS}(s)dA(s) + \phi^{FS}(t).$$

In Proposition 14 in Schweizer [29], it is shown that $\widetilde{\chi} = \chi^{FS}$ in $L^2(M)$ under the condition

$$\frac{|a(t)-r(t)|}{\sqrt{\kappa(t)}} \le C, \qquad \mathbb{P}\text{-a.s.}, \qquad \forall \quad 0 \le t \le T, \tag{3.9}$$

where $\kappa(t) = b^2(t) + \int_{\mathbb{R}_0} \gamma^2(t,z)\ell(dz)$ and C is a positive constant. Thus we obtain the following decomposition for the process $\widetilde{V}$

$$\widetilde{V}(t) = \widetilde{V}(0) + \int_0^t \chi^{FS}(s) d\tilde{S}(s) + \phi^{FS}(t)\,, \qquad 0 \leq t \leq T\,. \tag{3.10}$$

The latter decomposition is called the FS decomposition of the value process $\widetilde{V}$ and in particular of $\widetilde{\xi}$ for $t = T$. This explains the superscript FS in χ^{FS} and ϕ^{FS}. Notice that (3.9) is a sufficient condition for the existence of decomposition (3.10). The most general result concerning the existence and uniqueness of the FS decomposition is given by Choulli et al. [11].

The financial importance of such decomposition lies in the fact that it directly provides the locally risk-minimizing strategy in our setting. In fact, $\widetilde{V}(t)$ is the value of the portfolio in a locally risk-minimizing strategy at time t, the component $\chi^{FS}(t)$ is the number of risky assets to invest in the stock at time t, and $\phi^{FS} + \widetilde{V}(0)$ is the cost process in a locally risk-minimizing strategy (see Proposition 3.4 in Schweizer [31]). These components will be identified solving some BSDEJs of the type presented in Section 2. We refer also to Jeanblanc et al. [23] for a discussion about the relation between BSDEJs and quadratic hedging strategies in the context of general semimartingales.

Now substituting the dynamics (3.2) of $\widetilde{S}$ in (3.10) we get

$$\begin{cases} d\widetilde{V}(t) &= \widetilde{\pi}(t)(a(t) - r(t))dt + \widetilde{\pi}(t)b(t)dW(t) \\ & \qquad + \int_{\mathbb{R}_0} \widetilde{\pi}(t)\gamma(t,z)\widetilde{N}(dt,dz) + d\phi^{FS}(t), \\ \widetilde{V}(T) &= \widetilde{\xi}, \end{cases} \tag{3.11}$$

where $\widetilde{\pi} = \chi^{FS}\widetilde{S}$. The process $\widetilde{\pi}$ is interpreted as the amount of wealth $\widetilde{V}(t)$ to invest in the stock at time t in a locally risk-minimizing strategy.

Since $\phi^{FS}(T)$ is a $\mathcal{F}_T$-measurable square integrable random variable, applying Theorem 1.1 with $\mathbb{H} = \mathbb{F}$ and the $\mathbb{P}$-martingale property of ϕ^{FS} we know that there exist stochastic integrands Y^{FS}, Z^{FS}, such that

$$\phi^{FS}(t) = \mathbb{E}[\phi^{FS}(T)] + \int_0^t Y^{FS}(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} Z^{FS}(s,z)\widetilde{N}(ds,dz).$$

Since ϕ^{FS} is a martingale, we have $\mathbb{E}[\phi^{FS}(T)] = \mathbb{E}[\phi^{FS}(0)]$. However from (3.10) we deduce that $\phi^{FS}(0) = 0$. Therefore

$$\phi^{FS}(t) = \int_0^t Y^{FS}(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} Z^{FS}(s,z)\widetilde{N}(ds,dz). \tag{3.12}$$

In view of the orthogonality of ϕ^{FS} and M, we get

$$Y^{FS}(t)b(t) + \int_{\mathbb{R}_0} Z^{FS}(t,z)\gamma(t,z)\ell(dz) = 0. \tag{3.13}$$

In that case, the set of equations (3.11) are equivalent to

$$
\left\{
\begin{array}{ll}
d\widetilde{V}(t) & = \widetilde{\pi}(t)(a(t)-r(t))dt + \big(\widetilde{\pi}(t)b(t) + Y^{FS}(t)\big)dW(t) \\
& \quad + \int_{\mathbb{R}_0} \big(\widetilde{\pi}(t)\gamma(t,z) + Z^{FS}(t,z)\big)\widetilde{N}(dt,dz), \\
\widetilde{V}(T) & = \widetilde{\xi}.
\end{array}
\right.
\tag{3.14}
$$

3.1. **First candidate-approximation to S.** Now we consider an approximation to the price of the risky asset. In this model we approximate the small jumps by a Brownian motion B which is independent of W and which we scale with the standard deviation of the small jumps. That is

$$
\left\{
\begin{array}{ll}
dS^{(1)}_{1,\varepsilon}(t) & = S^{(1)}_{1,\varepsilon}(t)\Big\{a(t)dt + b(t)dW(t) + \int_{|z|>\varepsilon} \gamma(t,z)\widetilde{N}(dt,dz) + G(\varepsilon)\widetilde{\gamma}(t)dB(t)\Big\}, \\
S^{(1)}_{1,\varepsilon}(0) & = S^{(1)}(0) = x\,.
\end{array}
\right.
$$

The discounted price process is given by

$$
d\widetilde{S}_{1,\varepsilon}(t) = \widetilde{S}_{1,\varepsilon}(t)\Big\{(a(t)-r(t))dt + b(t)dW(t) + \int_{|z|>\varepsilon} \gamma(t,z)\widetilde{N}(dt,dz) + G(\varepsilon)\widetilde{\gamma}(t)dB(t)\Big\}.
$$

It was proven in Benth et al. [4], that the process $\widetilde{S}_{1,\varepsilon}$ converges to $\widetilde{S}$ in L^2 when ε goes to 0 with rate of convergence $G^2(\varepsilon)$ defined in (3.1).

In the following we study the robustness of the quadratic hedging strategies towards approximations where the price processes are modeled by $\widetilde{S}$ and $\widetilde{S}_{1,\varepsilon}$. We will first show that considering the approximation $\widetilde{S}_{1,\varepsilon}$, the value of the portfolio in a quadratic hedging strategy will be written as a solution of a BSDEJ of type (2.5) with $\rho = 1$. That is what explains our choice of the index 1 in $\widetilde{S}_{1,\varepsilon}$. Here we choose to start with the approximation $\widetilde{S}_{1,\varepsilon}$ because it involves another Brownian motion B besides the Brownian motion W. The approximations in which we truncate the small jumps in the underlying price process and the one in which we truncate the small jumps and replace them by scaling the Brownian motion W are studied in the next two subsections.

The locally square integrable local martingale $M_{1,\varepsilon}$ in the semimartingale decomposition of $\widetilde{S}_{1,\varepsilon}$ is given by

$$
\begin{aligned}
M_{1,\varepsilon}(t) &= \int_0^t b(s)\widetilde{S}_{1,\varepsilon}(s)dW(s) + \int_0^t \int_{|z|>\varepsilon} \gamma(s,z)\widetilde{S}_{1,\varepsilon}(s)\widetilde{N}(ds,dz) \\
&\quad + G(\varepsilon)\int_0^t \widetilde{\gamma}(s)\widetilde{S}_{1,\varepsilon}(s)dB(s)
\end{aligned}
\tag{3.15}
$$

and the predictable finite variation process $A_{1,\varepsilon}$ is given by

$$
A_{1,\varepsilon}(t) = \int_0^t \alpha_{1,\varepsilon}(s)d\langle M_{1,\varepsilon}\rangle(s)\,,
\tag{3.16}
$$

where

$$\alpha_{1,\varepsilon}(t) := \frac{a(t)-r(t)}{\widetilde{S}_{1,\varepsilon}(t)\big(b^2(t)+G^2(\varepsilon)\widetilde{\gamma}^2(t)+\int_{|z|>\varepsilon}\gamma^2(t,z)\ell(dz)\big)}, \qquad 0\le t\le T. \tag{3.17}$$

Thus the mean-variance tradeoff process $K_{1,\varepsilon}$ is given by

$$\begin{aligned} K_{1,\varepsilon}(t) = \int_0^t \alpha_{1,\varepsilon}^2(s)d\langle M_{1,\varepsilon}\rangle(s) = \int_0^t \frac{(a(s)-r(s))^2}{b^2(s)+G^2(\varepsilon)\widetilde{\gamma}^2(s)+\int_{|z|>\varepsilon}\gamma^2(s,z)\ell(dz)}ds \\ = K(t), \end{aligned} \tag{3.18}$$

in view of the definition of $G(\varepsilon)$, equation (3.1). Hence the assumption (3.9) ensures the existence of the FS decomposition with respect to $\widetilde{S}_{1,\varepsilon}$ for any square integrable $\mathcal{G}_T$-measurable random variable.

Let ξ_ε^1 be a square integrable contingent claim as a financial derivative with underlying $S_{1,\varepsilon}^{(1)}$ and maturity T. We denote its discounted payoff by $\widetilde{\xi}_\varepsilon^1 = \xi_\varepsilon^1/S^{(0)}(T)$. Consider $\chi_{1,\varepsilon}^{FS}\in\Theta$ and define

$$\widetilde{V}_{1,\varepsilon} := \mathbb{E}\left[\widetilde{\xi}_\varepsilon^1 - \int_t^T \chi_{1,\varepsilon}^{FS}(s)dA_{1,\varepsilon}(s)\,|\mathcal{G}_t\right], \qquad 0\le t\le T\,.$$

Then following the same steps as before and imposing the condition (3.9), we prove the FS decomposition for the value process $\widetilde{V}_{1,\varepsilon}$ written under the world measure $\mathbb{P}$ to be as follows

$$\widetilde{V}_{1,\varepsilon}(t) = \widetilde{V}_{1,\varepsilon}(0) + \int_0^t \chi_{1,\varepsilon}^{FS}(s)d\widetilde{S}_{1,\varepsilon}(s) + \phi_{1,\varepsilon}^{FS}(t), \tag{3.19}$$

where $\phi_{1,\varepsilon}^{FS}$ is a $\mathbb{P}$-martingale such that $[\phi_{1,\varepsilon}^{FS}, M_{1,\varepsilon}]$ is a local martingale. Replacing $\widetilde{S}_{1,\varepsilon}$ by its expression in (3.19), we get

$$\begin{cases} d\widetilde{V}_{1,\varepsilon}(t) &= \widetilde{\pi}_{1,\varepsilon}(t)(a(t)-r(t))dt + \widetilde{\pi}_{1,\varepsilon}(t)b(t)dW(t) + \widetilde{\pi}_{1,\varepsilon}(t)G(\varepsilon)\widetilde{\gamma}(t)dB(t) \\ &\qquad + \int_{|z|>\varepsilon}\widetilde{\pi}_{1,\varepsilon}(t)\gamma(t,z)\widetilde{N}(dt,dz) + d\phi_{1,\varepsilon}^{FS}(t), \\ \widetilde{V}_{1,\varepsilon}(T) &= \widetilde{\xi}_\varepsilon^1, \end{cases}$$

where $\widetilde{\pi}_{1,\varepsilon} = \chi_{1,\varepsilon}^{FS}\widetilde{S}_{1,\varepsilon}$. Notice that $\phi_{1,\varepsilon}^{FS}(T)$ is a $\mathcal{G}_T$-measurable square integrable random variable. Thus applying Theorem 1.1 with $\mathbb{H}=\mathbb{G}$ and using the $\mathbb{P}$-martingale property of $\phi_{1,\varepsilon}^{FS}$ we know that there exist stochastic integrands $Y_{1,\varepsilon}^{FS}$, $Y_{2,\varepsilon}^{FS}$, and Z_ε^{FS}, such that

$$\begin{aligned} \phi_{1,\varepsilon}^{FS}(t) = \mathbb{E}[\phi_{1,\varepsilon}^{FS}(T)] + \int_0^t Y_{1,\varepsilon}^{FS}(s)dW(s) + \int_0^t Y_{2,\varepsilon}^{FS}(s)dB(s) \\ + \int_0^t\int_{\mathbb{R}_0} Z_\varepsilon^{FS}(s,z)\widetilde{N}(ds,dz). \end{aligned}$$

Using the martingale property of $\phi_{1,\varepsilon}^{FS}$ and equation (3.19), we get $\mathbb{E}[\phi_{1,\varepsilon}^{FS}(T)] = \mathbb{E}[\phi_{1,\varepsilon}^{FS}(0)] = 0$. Therefore we deduce

$$\phi_{1,\varepsilon}^{FS}(t) = \int_0^t Y_{1,\varepsilon}^{FS}(s)dW(s) + \int_0^t Y_{2,\varepsilon}^{FS}(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} Z_{\varepsilon}^{FS}(s,z)\widetilde{N}(ds,dz). \tag{3.20}$$

In view of the orthogonality of $\phi_{1,\varepsilon}^{FS}$ with respect to $M_{1,\varepsilon}$, we have

$$0 = Y_{1,\varepsilon}^{FS}(t)b(t) + Y_{2,\varepsilon}^{FS}(t)G(\varepsilon)\widetilde{\gamma}(t) + \int_{\mathbb{R}_0} Z_{\varepsilon}^{FS}(t,z)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}\ell(dz). \tag{3.21}$$

The equation we obtain for the approximating problem is thus given by

$$\begin{cases} d\widetilde{V}_{1,\varepsilon}(t) &= \widetilde{\pi}_{1,\varepsilon}(t)(a(t)-r(t))dt + (\widetilde{\pi}_{1,\varepsilon}(t)b(t) + Y_{1,\varepsilon}^{FS}(t))dW(t) \\ & \qquad +(\widetilde{\pi}_{1,\varepsilon}(t)G(\varepsilon)\widetilde{\gamma}(t) + Y_{2,\varepsilon}^{FS}(t))dB(t) \\ & \qquad + \int_{\mathbb{R}_0} \big(\widetilde{\pi}_{1,\varepsilon}(t)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z) + Z_{\varepsilon}^{FS}(t,z)\big)\widetilde{N}(dt,dz), \\ \widetilde{V}_{1,\varepsilon}(T) &= \widetilde{\xi}_{\varepsilon}^{1}. \end{cases} \tag{3.22}$$

In order to apply the robustness results studied in Section 2, we have to prove that $\widetilde{V}$ and $\widetilde{V}_{1,\varepsilon}$ are respectively equations of type (1.2) and (2.4). This is the purpose of the next lemma. Notice that the processes $\widetilde{V}_{1,\varepsilon}$, $\widetilde{\pi}_{1,\varepsilon}$, and $\phi_{1,\varepsilon}^{FS}$ are all $\mathbb{G}$-adapted.

Lemma 3.1. *Assume* (3.9) *holds. Let $\widetilde{V}$, $\widetilde{V}_{1,\varepsilon}$ be given by* (3.14), (3.22), *respectively. Then $\widetilde{V}$ satisfies a BSDEJ of type* (1.2) *and $\widetilde{V}_{1,\varepsilon}$ satisfies a BSDEJ of type* (2.4).

Proof. From the expression of $\widetilde{V}$, we deduce

$$\begin{cases} d\widetilde{V}(t) &= -f(t,\widetilde{V}(t),\widetilde{Y}(t),\widetilde{Z}(t,.))dt + \widetilde{Y}(t)dW(t) + \int_{\mathbb{R}_0} \widetilde{Z}(t,z)\widetilde{N}(dt,dz), \\ \widetilde{V}(T) &= \widetilde{\xi}, \end{cases}$$

where

$$\begin{aligned} &\widetilde{Y}(t) = \widetilde{\pi}(t)b(t) + Y^{FS}(t), \quad \widetilde{Z}(t,z) = \widetilde{\pi}(t)\gamma(t,z) + Z^{FS}(t,z), \\ &f(t,\widetilde{V}(t),\widetilde{Y}(t),\widetilde{Z}(t,.)) = -\widetilde{\pi}(t)(a(t)-r(t)). \end{aligned} \tag{3.23}$$

We have to show that f satisfies Assumptions 1(B). We first express $\widetilde{\pi}$ in terms of $\widetilde{V}$, $\widetilde{Y}$, and $\widetilde{Z}$. Inspired by (3.13), we combine $\widetilde{Y}$ and $\widetilde{Z}$ to get

$$\begin{aligned} \widetilde{Y}(t)b(t) + \int_{\mathbb{R}_0} \widetilde{Z}(t,z)\gamma(t,z)\ell(dz) = \widetilde{\pi}(t)\Big(b^2(t) + \int_{\mathbb{R}_0} \gamma^2(t,z)\ell(dz)\Big) + Y^{FS}(t)b(t) \\ + \int_{\mathbb{R}_0} Z^{FS}(t,z)\gamma(t,z)\ell(dz). \end{aligned}$$

From (3.13), we deduce that

$$\widetilde{\pi}(t) = \frac{1}{\kappa(t)}\Big(\widetilde{Y}(t)b(t) + \int_{\mathbb{R}_0} \widetilde{Z}(t,z)\gamma(t,z)\ell(dz)\Big). \tag{3.24}$$

Hence

$$f(t,\widetilde{V}(t),\widetilde{Y}(t),\widetilde{Z}(t,.)) = -\frac{a(t)-r(t)}{\kappa(t)}\Big(\widetilde{Y}(t)b(t) + \int_{\mathbb{R}_0}\widetilde{Z}(t,z)\gamma(t,z)\ell(dz)\Big). \tag{3.25}$$

Now we have to prove that f is Lipschitz. Let

$$h(t) = \frac{a(t)-r(t)}{\kappa(t)}, \qquad t\in[0,T]. \tag{3.26}$$

We have

$$\begin{aligned}|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| &\le |h(t)|\Big[|y_1-y_2||b(t)| + \int_{\mathbb{R}_0}|z_1-z_2||\gamma(t,z)|\ell(dz)\Big]\\ &\le |h(t)|[|y_1-y_2||b(t)|\\ &\quad + (\int_{\mathbb{R}_0}|z_1-z_2|^2\ell(dz))^{\frac12}(\int_{\mathbb{R}_0}|\gamma(t,z)|^2\ell(dz))^{\frac12}]\\ &\le \sqrt{\kappa(t)}|h(t)|\Big(|y_1-y_2| + \|z_1-z_2\|\Big).\end{aligned}$$

Thus f is Lipschitz if there exists a positive constant C such that

$$\sqrt{\kappa(t)}|h(t)| = \frac{|a(t)-r(t)|}{\sqrt{\kappa(t)}} \le C \qquad \forall t\in[0,T]$$

and we prove the statement for $\widetilde{V}$.

From equation (3.22), we have

$$\begin{cases} d\widetilde{V}_{1,\varepsilon}(t) &= -f^1_\varepsilon(t,\widetilde{V}_{1,\varepsilon}(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,.),\widetilde{\zeta}_\varepsilon(t))dt + \widetilde{Y}_\varepsilon(t)dW(t) + \widetilde{\zeta}_\varepsilon(t)dB(t)\\ &\qquad + \int_{\mathbb{R}_0}\widetilde{Z}_\varepsilon(t,z)\widetilde{N}(dt,dz),\\ \widetilde{V}_{1,\varepsilon}(T) &= \widetilde{\xi}^1_\varepsilon,\end{cases}$$

where

$$\begin{aligned} \widetilde{Y}_\varepsilon(t) = \widetilde{\pi}_{1,\varepsilon}(t)b(t) + Y^{FS}_{1,\varepsilon}(t), \quad \widetilde{\zeta}_\varepsilon(t) = \widetilde{\pi}_{1,\varepsilon}(t)G(\varepsilon)\widetilde{\gamma}(t) + Y^{FS}_{2,\varepsilon}(t),\\ \widetilde{Z}_\varepsilon(t,z) = \widetilde{\pi}_{1,\varepsilon}(t)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z) + Z^{FS}_\varepsilon(t,z),\\ f^1_\varepsilon(t,\widetilde{V}_{1,\varepsilon}(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,.),\widetilde{\zeta}_\varepsilon(t)) = -\widetilde{\pi}_{1,\varepsilon}(t)(a(t)-r(t)).\end{aligned} \tag{3.27}$$

With the same arguments as above and using (3.21) we can prove that

$$\widetilde{\pi}_{1,\varepsilon}(t) = \frac{1}{\kappa(t)}\Big\{\widetilde{Y}_\varepsilon(t)b(t) + \widetilde{\zeta}_\varepsilon(t)G(\varepsilon)\widetilde{\gamma}(t) + \int_{\mathbb{R}_0}\widetilde{Z}_\varepsilon(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z)\gamma(t,z)\ell(dz)\Big\}. \tag{3.28}$$

Hence

$$\begin{aligned} f^1_\varepsilon(t,\widetilde{V}_{1,\varepsilon}(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,.),\widetilde{\zeta}_\varepsilon(t)) = -\frac{a(t)-r(t)}{\kappa(t)}\Big(&\widetilde{Y}_\varepsilon(t)b(t) + \widetilde{\zeta}_\varepsilon(t)G(\varepsilon)\widetilde{\gamma}(t)\\ &+ \int_{\mathbb{R}_0}\widetilde{Z}_\varepsilon(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z)\gamma(t,z)\ell(dz)\Big)\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}&|f_\varepsilon^1(t,x_1,y_1,z_1,\zeta_1)-f_\varepsilon^1(t,x_2,y_2,z_2,\zeta_2)|\\&\qquad\le |h(t)|\Big[|y_1-y_2||b(t)|+\int_{\mathbb{R}_0}\mathbf{1}_{\{|z|>\varepsilon\}}(z)|z_1-z_2||\gamma(t,z)|\ell(dz)\\&\qquad\qquad+G(\varepsilon)|\widetilde{\gamma}(t)||\zeta_1-\zeta_2|\Big]\\&\quad\le\sqrt{\kappa(t)}|h(t)|\Big(|y_1-y_2|+|\zeta_1-\zeta_2|+\|z_1-z_2\|\Big)\end{aligned}$$

and we prove the statement. □

Now we present the following main result in which we prove the robustness of the value of the portfolio.

Theorem 3.2. *Assume that* (3.9) *holds and that for all* $t\in[0,T]$,

$$|\frac{\widetilde{\gamma}(t)(a(t)-r(t))}{\kappa(t)}|\le K\,,\qquad \mathbb{P}\text{-}a.s. \tag{3.30}$$

Let $\widetilde{V}$, $\widetilde{V}_{1,\varepsilon}$ *be given by* (3.14), (3.22), *respectively. Then*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\widetilde{V}(t)-\widetilde{V}_{1,\varepsilon}(t)|^2\Big]\le C\mathbb{E}[|\widetilde{\xi}-\widetilde{\xi}_\varepsilon^1|^2]+\widetilde{C}G^2(\varepsilon)\,.$$

Proof. This is an immediate result of Theorem 2.6 with $\rho=1$ and noticing that $f(t,0,0,0)=0$. We only have to prove the assumption (2.3) on the drivers f and f_ε^1 given by (3.25) and (3.29). We have for all $t\in[0,T]$, recalling (3.26)

$$\begin{aligned}&|f(t,\widetilde{V}(t),\widetilde{Y}(t),\widetilde{Z}(t,.))-f_\varepsilon^1(t,\widetilde{V}_{1,\varepsilon}(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,.),\widetilde{\zeta}_\varepsilon(t))|\\&\quad=\Big|h(t)\Big\{(\widetilde{Y}(t)-\widetilde{Y}_\varepsilon(t))b(t)-\widetilde{\zeta}_\varepsilon(t)G(\varepsilon)\widetilde{\gamma}(t)\\&\qquad+\int_{|z|>\varepsilon}(\widetilde{Z}(t,z)-\widetilde{Z}_\varepsilon(t,z))\gamma(t,z)\ell(dz)+\int_{|z|\le\varepsilon}\widetilde{Z}(t,z)\gamma(t,z)\ell(dz)\Big\}\Big|\\&\quad\le|h(t)|(\sqrt{\kappa(t)}+|\widetilde{\gamma}(t)|)\Big\{|\widetilde{Y}(t)-\widetilde{Y}_\varepsilon(t)|+\|\widetilde{Z}(t,.)-\widetilde{Z}_\varepsilon(t,.)\|\\&\qquad+G(\varepsilon)\|\widetilde{Z}(t,.)\|+|\widetilde{\zeta}_\varepsilon(t)|\Big\},\end{aligned}$$

which proves the statement. □

Remark 3.3. *We used the expectation* $\mathbb{E}[|\widetilde{\xi}-\widetilde{\xi}_\varepsilon^\rho|^2]$ *to dominate the convergence results. In finance the discounted contingent claim* $\widetilde{\xi}=\xi/S^{(0)}(T)$ *is given by the payoff function* $\xi=g(S^{(1)}(T))$. *Thus we have*

$$\mathbb{E}[|\widetilde{\xi}-\widetilde{\xi}_\varepsilon^\rho|^2]=\mathbb{E}\Big[\Big|\frac{g(S^{(1)}(T))}{S^{(0)}(T)}-\frac{g(S^{(1)}_{\rho,\varepsilon}(T))}{S^{(0)}(T)}\Big|^2\Big],\qquad \rho=0,1,2,$$

where the case $\rho=0$ *refers to the second candidate-approximation of Section 3.3 and* $\rho=2$ *refers to the one in Section 3.2. The convergence of the latter quantity when* ε

goes to 0 *was studied in Benth et al.* [4] *using Fourier transform techniques. It was also studied in Kohatsu-Higa and Tankov* [25] *in which the authors show that adding a small variance Brownian motion to the big jumps gives better convergence results than when we only truncate the small jumps. For this purpose the authors consider a discretisation of the price models.*

The next theorem contains the robustness result for the amount of wealth to invest in the stock in a locally risk-minimizing strategy.

Theorem 3.4. *Assume that* (3.9) *holds and that for all* $t \in [0,T]$,

$$\sup_{t\leq s\leq T} \widetilde{\gamma}^2(s) \leq K, \qquad \inf_{t\leq s\leq T} \kappa(s) \geq \widetilde{K}, \qquad \mathbb{P}\text{-}a.s., \tag{3.31}$$

where K *is a positive constant and* $\widetilde{K}$ *is a strictly positive constant. Let* $\widetilde{\pi}$, $\widetilde{\pi}_{1,\varepsilon}$ *be given by* (3.24), (3.28), *respectively. Then for all* $t \in [0,T]$,

$$\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{1,\varepsilon}(s)|^2 ds\Big] \leq C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + \widetilde{C}G^2(\varepsilon),$$

where C *and* $\widetilde{C}$ *are positive constants.*

Proof. Using (3.24) and (3.28), we have

$$\begin{aligned}
|\widetilde{\pi}(s) - \widetilde{\pi}_{1,\varepsilon}(s)|^2 &= \frac{1}{\kappa^2(s)}\Big\{(\widetilde{Y}(s) - \widetilde{Y}_\varepsilon(s))b(s) - \widetilde{\zeta}_\varepsilon(s)G(\varepsilon)\widetilde{\gamma}(s) \\
&\quad + \int_{|z|>\varepsilon}(\widetilde{Z}(s,z) - \widetilde{Z}_\varepsilon(s,z))\gamma(s,z)\ell(dz) + \int_{|z|\leq\varepsilon}\widetilde{Z}(s,z)\gamma(s,z)\ell(dz)\Big\}^2 \\
&\leq \frac{C}{\kappa(s)}\Big\{|\widetilde{Y}(s) - \widetilde{Y}_\varepsilon(s)|^2 + |\widetilde{\zeta}_\varepsilon(s)|^2 \\
&\quad + G^2(\varepsilon)|\widetilde{\gamma}^2(t)|\int_{\mathbb{R}_0}|\widetilde{Z}(s,z)|^2\ell(dz) + \int_{\mathbb{R}_0}|\widetilde{Z}(s,z) - \widetilde{Z}_\varepsilon(s,z)|^2\ell(dz)\Big\}.
\end{aligned}$$

Hence from Lemma 2.3, Theorem 2.4 and Lemma 3.1, we deduce

$$\begin{aligned}
\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{1,\varepsilon}(s)|^2 ds\Big] &\leq \frac{C}{\inf_{t\leq s\leq T}\kappa(s)}\Big\{\mathbb{E}\Big[\int_t^T |\widetilde{Y}(s) - \widetilde{Y}_\varepsilon(s)|^2 ds\Big] + \mathbb{E}\Big[\int_t^T |\widetilde{\zeta}_\varepsilon(s)|^2 ds\Big] \\
&\quad + G^2(\varepsilon)\sup_{t\leq s\leq T}\widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0}|\widetilde{Z}(s,z)|^2\ell(dz)ds\Big] \\
&\quad + \mathbb{E}\Big[\int_t^T\int_{\mathbb{R}_0}|\widetilde{Z}(s,z) - \widetilde{Z}_\varepsilon(s,z)|^2\ell(dz)ds\Big]\Big\} \\
&\leq \widetilde{C}\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + \widehat{C}G^2(\varepsilon)\mathbb{E}[\xi^2]
\end{aligned}$$

and we prove the statement. □

The robustness of the process ϕ^{FS} defined in (3.12) is shown in the next theorem.

Theorem 3.5. *Assume that* (3.9) *and* (3.31) *hold and for all* $t \in [0,T]$,

$$\sup_{t \le s \le T} \kappa(s) \le \widehat{K} < \infty, \qquad \mathbb{P}\text{-}a.s. \tag{3.32}$$

Let ϕ^{FS}, $\phi^{FS}_{1,\varepsilon}$ *be given by* (3.12), (3.20), *respectively. Then for all* $t \in [0,T]$, *we have*

$$\mathbb{E}\Big[|\phi^{FS}(t) - \phi^{FS}_{1,\varepsilon}(t)|^2\Big] \le C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + C'G^2(\varepsilon),$$

where C *and* C' *are positive constants.*

Proof. From (3.23), (3.27), Theorem 2.6, and Theorem 3.4, we have

$$\begin{aligned}\mathbb{E}\Big[\int_t^T |Y^{FS}(s) - Y^{FS}_{1,\varepsilon}(s)|^2 ds\Big] &\le C\Big\{\mathbb{E}\Big[\int_t^T |\widetilde{Y}(s) - \widetilde{Y}_{1,\varepsilon}(s)|^2 ds\Big] \\ &\qquad + \sup_{t \le s \le T} \kappa(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{1,\varepsilon}(s)|^2 ds\Big]\Big\} \\ &\le \widetilde{C}\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + KG^2(\varepsilon)\mathbb{E}[|\widetilde{\xi}|^2].\end{aligned} \tag{3.33}$$

Moreover, starting again from (3.27) we arrive at

$$\begin{aligned}\mathbb{E}\Big[\int_t^T |Y^{FS}_{2,\varepsilon}(s)|^2 ds\Big] \le C\Big\{&\mathbb{E}[\int_t^T |\widetilde{\zeta}_\varepsilon(s)|^2 ds + \sup_{t \le s \le T} \kappa(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{1,\varepsilon}(s)|^2 ds\Big] \\ &+ G^2(\varepsilon) \sup_{t \le s \le T} \widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s)|^2 ds\Big].\end{aligned}$$

However from (3.24) and Lemma 2.3, we get

$$\begin{aligned}\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s)|^2 ds\Big] &\le \frac{1}{\inf_{t \le s \le T} \kappa(s)}\Big\{\mathbb{E}\Big[\int_t^T \widetilde{Y}^2(s) ds\Big] + \mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} \widetilde{Z}^2(s,z)\ell(dz)ds\Big]\Big\} \\ &\le C\mathbb{E}[\widetilde{\xi}^2].\end{aligned} \tag{3.34}$$

Thus from Theorem 2.4 and Theorem 3.4 we conclude in view of assumption (3.32)

$$\mathbb{E}\Big[\int_t^T |Y^{FS}_{2,\varepsilon}(s)|^2 ds\Big] \le C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + C'G^2(\varepsilon)\mathbb{E}[\widetilde{\xi}^2]. \tag{3.35}$$

Let $G^2(\infty) = \int_{\mathbb{R}_0} g^2(z)\ell(dz)$. From (3.23), (3.27), Theorem 2.4, Theorem 3.4 and (3.34), we obtain

$$\begin{aligned}&\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} |Z^{FS}(s,z) - Z^{FS}_\varepsilon(s,z)|^2 \ell(dz)ds\Big] \\ &\quad \le C\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} |\widetilde{Z}(s,z) - \widetilde{Z}_\varepsilon(s,z)|^2 \ell(dz)ds\Big] \\ &\qquad + G^2(\infty) \sup_{t \le s \le T} \widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{1,\varepsilon}(s)|^2 ds\Big]\end{aligned}$$

$$+ G^2(\varepsilon) \sup_{t\le s\le T} \widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s)|^2 ds\Big]$$

$$\le C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + C'G^2(\varepsilon)\mathbb{E}[\widetilde{\xi}^2]. \tag{3.36}$$

Finally from (3.12) and (3.20), we infer

$$\mathbb{E}[|\phi^{FS}(t) - \phi^{FS}_{1,\varepsilon}(t)|^2] \le \mathbb{E}\Big[\int_0^T |Y^{FS}(s) - Y^{FS}_{1,\varepsilon}(s)|^2 ds\Big] + \mathbb{E}\Big[\int_0^T |Y^{FS}_{2,\varepsilon}(s)|^2 ds\Big]$$
$$+ \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} |Z^{FS}(s,z) - Z^{FS}_\varepsilon(s,z)|^2 \ell(dz)ds\Big]$$

and combining with the relations (3.33), (3.35) and (3.36) the result follows. □

Let $C(t) = \phi^{FS}(t) + \widetilde{V}(0)$ and $C_{1,\varepsilon}(t) = \phi^{FS}_{1,\varepsilon}(t) + \widetilde{V}_{1,\varepsilon}(0)$. Then the processes C and $C_{1,\varepsilon}$ are the cost processes in a locally risk-minimizing strategy for $\widetilde{\xi}$ and $\widetilde{\xi}^1_\varepsilon$. In the next corollary we prove the robustness of this cost process.

Corollary 3.6. *Assume that* (3.9), (3.30), (3.31), *and* (3.32) *hold. Then for all* $t \in [0,T]$, *we have*

$$\mathbb{E}[|C(t) - C_{1,\varepsilon}(t)|^2] \le \widetilde{K}\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + K'G^2(\varepsilon),$$

where $\widetilde{K}$ *and* K' *are two positive constants.*

Proof. From Theorem 3.2, we deduce

$$\mathbb{E}\Big[|\widetilde{V}_{1,\varepsilon}(0) - \widetilde{V}(0)|^2\Big] \le C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + \widetilde{C}G^2(\varepsilon).$$

Applying the latter together with Theorem 3.5 we get

$$\mathbb{E}[|C(t) - C_{1,\varepsilon}(t)|^2] = \mathbb{E}\Big[|(\widetilde{V}_{1,\varepsilon}(0) + \phi^{FS}_{1,\varepsilon}(t)) - (\widetilde{V}(0) + \phi^{FS}(t))|^2\Big]$$
$$\le 2\Big(\mathbb{E}\Big[|\widetilde{V}_{1,\varepsilon}(0) - \widetilde{V}(0)|^2\Big] + \mathbb{E}[|\phi^{FS}_{1,\varepsilon}(t) - \phi^{FS}(t)|^2]\Big)$$
$$\le \widetilde{K}\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + K'G^2(\varepsilon).$$

□

In the next section we present a second candidate-approximation to S and we study the robustness of the quadratic hedging strategies.

3.2. **Second candidate-approximation to S.** In this model we truncate the small jumps in S. We obtain

$$\begin{cases} dS^{(1)}_{2,\varepsilon}(t) &= S^{(1)}_{2,\varepsilon}(t)\Big\{a(t)dt + b(t)dW(t) + \int_{|z|>\varepsilon} \gamma(t,z)\widetilde{N}(dt,dz)\Big\}, \\ S^{(1)}_{2,\varepsilon}(0) &= S^{(1)}(0) = x\,. \end{cases}$$

The discounted price process is given by

$$d\widetilde{S}_{2,\varepsilon}(t) = \widetilde{S}_{2,\varepsilon}(t)\Big\{(a(t)-r(t))dt + b(t)dW(t) + \int_{|z|>\varepsilon} \gamma(t,z)\widetilde{N}(dt,dz)\Big\}.$$

It is easy to show that $\widetilde{S}_{2,\varepsilon}$ converges to $\widetilde{S}$ in L^2 when ε goes to 0 with rate of convergence $G^2(\varepsilon)$. Notice that this second choice of the approximating process $S_{2,\varepsilon}^{(1)}$ allows to work under the same filtration $\mathbb{F}$ as for the original process. However it involves a different variance.

Let $\widetilde{\pi}_{2,\varepsilon} = \chi_{2,\varepsilon}^{FS}\widetilde{S}_{2,\varepsilon}$, where $\chi_{2,\varepsilon}^{FS} \in \Theta$, (3.7). Without going through details and since the computations are similar to the previous section, we claim that the discounted value of the portfolio associated with $\widetilde{S}_{2,\varepsilon}$ is given by

$$\begin{cases} d\widetilde{V}_{2,\varepsilon}(t) &= \widetilde{\pi}_{2,\varepsilon}(t)(a(t)-r(t))dt + \widetilde{\pi}_{2,\varepsilon}(t)b(t)dW(t) \\ &\qquad + \displaystyle\int_{|z|>\varepsilon} \widetilde{\pi}_{2,\varepsilon}(t)\gamma(t,z)\widetilde{N}(dt,dz) + d\phi_{2,\varepsilon}^{FS}(t), \\ \widetilde{V}_{2,\varepsilon}(T) &= \widetilde{\xi}_{\varepsilon}^{2}, \end{cases}$$

where $\phi_{2,\varepsilon}^{FS}$ is a $\mathbb{P}$-martingale such that $[\phi_{2,\varepsilon}^{FS}, M_{2,\varepsilon}]$ is a local martingale with $M_{2,\varepsilon}$ being the locally square integrable local martingale part in $\widetilde{S}_{2,\varepsilon}$ and where $\widetilde{\xi}_{\varepsilon}^{2}$ is the discounted value of the contingent claim. Moreover, $\phi_{2,\varepsilon}^{FS}(T)$ is a $\mathcal{F}_T$-measurable square integrable random variable. Thus applying Theorem 1.1 with $\mathbb{H} = \mathbb{F}$ and using the $\mathbb{P}$-martingale property of $\phi_{2,\varepsilon}^{FS}$ we know that there exist stochastic integrands Y_{ε}^{FS} and Z_{ε}^{FS}, such that

$$\phi_{2,\varepsilon}^{FS}(t) = \int_0^t Y_{\varepsilon}^{FS}(s)dW(s) + \int_0^t\int_{\mathbb{R}_0} Z_{\varepsilon}^{FS}(s,z)\widetilde{N}(ds,dz).$$

Thus the equation we obtain for the approximating problem $\widetilde{V}_{2,\varepsilon}$ is given by

$$(3.37)\qquad \begin{cases} d\widetilde{V}_{2,\varepsilon}(t) &= \widetilde{\pi}_{2,\varepsilon}(t)(a(t)-r(t))dt + (\widetilde{\pi}_{2,\varepsilon}(t)b(t) + Y_{\varepsilon}^{FS}(t))dW(t) \\ &\qquad + \displaystyle\int_{\mathbb{R}_0} \big(\widetilde{\pi}_{2,\varepsilon}(t)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z) + Z_{\varepsilon}^{FS}(t,z)\big)\widetilde{N}(dt,dz), \\ \widetilde{V}_{2,\varepsilon}(T) &= \widetilde{\xi}_{\varepsilon}^{2}. \end{cases}$$

To prove similar convergence results as in Section 3.1, we identify (3.37) with the BSDEJ (2.1). In that case the driver of (3.37) is given by

$$(3.38)\qquad f_{\varepsilon}^{0}(t,\widetilde{V}_{2,\varepsilon}(t),\widetilde{Y}_{\varepsilon}(t),\widetilde{Z}_{\varepsilon}(t,\cdot)) = -h_{\varepsilon}(t)\Big[b(t)\widetilde{Y}_{\varepsilon}(t) + \int_{|z|>\varepsilon}\widetilde{Z}_{\varepsilon}(t,z)\gamma(t,z)\ell(dz)\Big],$$

where

$$(3.39)\qquad h_{\varepsilon}(t) = \frac{a(t)-r(t)}{\kappa_{\varepsilon}(t)} \quad \text{and} \quad \kappa_{\varepsilon}(t) = b^2(t) + \int_{|z|>\varepsilon}\gamma^2(t,z)\ell(dz).$$

In the following lemma we prove that under some conditions on the parameters of the price process, f_{ε}^{0} is Lipschitz and satisfies (2.2).

Lemma 3.7. *Define* $\kappa_1(t) = b^2(t) + \int_{|z|>1} \gamma^2(t,z)\ell(dz)$. *Assume that for all* $t \in [0,T]$, *we have*

$$\frac{|a(t)-r(t)|}{\sqrt{\kappa_1(t)}} \leq K, \qquad \mathbb{P}\text{-}a.s., \tag{3.40}$$

where K *is a positive constant. Then* $f^0_\varepsilon(t,x,y,z)$, $t \in [0,T]$, *satisfies a uniform Lipschitz condition in* (x,y,z), *for all* $\varepsilon \in [0,1]$.
Moreover, assume

$$\inf_{t\leq s\leq T} \kappa(s) \geq \widehat{K}, \quad \textit{and} \quad \sup_{t\leq s\leq T} \widetilde{\gamma}^2(t) \leq \widetilde{K} \qquad \mathbb{P}\text{-}a.s., \tag{3.41}$$

where $\widehat{K}$ *and* $\widetilde{K}$ *are positive constants. Then* $f^0_\varepsilon(t,x,y,z)$, $t \in [0,T]$, *satisfies condition* (2.2).

Proof. We have

$$\begin{aligned}|f^0_\varepsilon(t,x_1,y_1,z_1) - f^0_\varepsilon(t,x_2,y_2,z_2)| &\leq |h_\varepsilon(t)|\Big[b(t)|y_1-y_2| + \int_{|z|>\varepsilon} |z_1-z_2|\gamma(t,z)\ell(dz)\Big] \\ &\leq \frac{|a(t)-r(t)|}{\sqrt{\kappa_1(t)}}\Big[|y_1-y_2| + (\int_{\mathbb{R}_0} |z_1-z_2|^2\ell(dz))^{\frac{1}{2}}\Big].\end{aligned}$$

Thus f^0_ε is Lipschitz requiring (3.40) is satisfied.
Recall the expressions of h and h_ε in (3.26) and (3.39), respectively. Then we have using (3.25) and (3.38)

$$\begin{aligned}&|f(t,\widetilde{V}(t),\widetilde{Y}(t),\widetilde{Z}(t,\cdot)) - f^0_\varepsilon(t,\widetilde{V}_\varepsilon(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,\cdot))| \\ &\leq |h(t)b(t)||\widetilde{Y}(t)-\widetilde{Y}_\varepsilon(t)| + |h(t)-h_\varepsilon(t)||b(t)\widetilde{Y}_\varepsilon(t)| \\ &\quad + |h(t)|\int_{\mathbb{R}_0} |\widetilde{Z}(t,z)-\widetilde{Z}_\varepsilon(t,z)||\gamma(t,z)|\ell(dz) \\ &\quad + |h(t)-h_\varepsilon(t)|\int_{\mathbb{R}_0} |\widetilde{Z}_\varepsilon(t,z)\gamma(t,z)|\ell(dz) + |h_\varepsilon(t)|\int_{|z|\leq\varepsilon} |\widetilde{Z}_\varepsilon(t,z)\gamma(t,z)|\ell(dz).\end{aligned}$$

Notice that

$$\begin{aligned}|h(t)-h_\varepsilon(t)| &= |a(t)-r(t)|\frac{|\kappa_\varepsilon(t)-\kappa(t)|}{|\kappa(t)\kappa_\varepsilon(t)|} \\ &\leq \frac{|a(t)-r(t)|}{|\kappa(t)\kappa_1(t)|}\widetilde{\gamma}^2(t)G^2(\varepsilon).\end{aligned}$$

Thus

$$\begin{aligned}&|f(t,\widetilde{V}(t),\widetilde{Y}(t),\widetilde{Z}(t,\cdot)) - f^0_\varepsilon(t,\widetilde{V}_\varepsilon(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,\cdot))| \\ &\leq \frac{|a(t)-r(t)|}{\sqrt{\kappa(t)}}\Big(|\widetilde{Y}(t)-\widetilde{Y}_\varepsilon(t)| + (\int_{\mathbb{R}_0} |\widetilde{Z}(t,z)-\widetilde{Z}_\varepsilon(t,z)|^2\ell(dz))^{\frac{1}{2}}\Big)\end{aligned}$$

$$+ G^2(\varepsilon)\frac{|a(t)-r(t)|}{|\kappa_1(t)|}\widetilde{\gamma}^2(t)\Big[\frac{1}{\sqrt{\kappa(t)}}+1\Big]\Big(|\widetilde{Y}_\varepsilon(t)| + (\int_{\mathbb{R}_0}|\widetilde{Z}_\varepsilon(t,z)|^2\ell(dz))^{\frac{1}{2}}\Big)$$

and the statement of the lemma follows providing that conditions (3.40) and (3.41) hold. □

We do not prove in this section the convergence results since they follow the same lines as the latter section. However we claim that, considering the approximation $\widetilde{S}_{2,\varepsilon}$, the value of the portfolio, the amount of wealth to invest in the stock, and the cost process in the locally risk-minimizing strategy are robust when imposing certain boundedness conditions on the parameters of the price process.

3.3. **Third candidate-approximation to S.** In the candidate-approximation $S_{1,\varepsilon}$, the variance of the continuous part is given by $b^2(t) + G^2(\varepsilon)\widetilde{\gamma}^2(t)$, which is the same as the sum of the variance of the small jumps and the variance of the continuous part in S. We studied this approximation by embedding the original model solution into a larger filtration $\mathbb{G}$. If one insists on working under the filtration $\mathbb{F}$, then one could also select a third candidate-approximation $S_{0,\varepsilon}^{(1)}$ in the following way.

$$\begin{cases} dS_{0,\varepsilon}^{(1)}(t) &= S_{0,\varepsilon}^{(1)}(t)\Big\{a(t)dt + (b(t)+\widetilde{G}(\varepsilon)\widetilde{\gamma}(t))dW(t) + \int_{|z|>\varepsilon}\gamma(t,z)\widetilde{N}(dt,dz)\Big\}, \\ S_{0,\varepsilon}^{(1)}(0) &= S^{(1)}(0) = x\,, \end{cases}$$

where $\widetilde{G}(\varepsilon)$ satisfies the relation

$$(b(t)+\widetilde{G}(\varepsilon)\widetilde{\gamma}(t))^2 = b^2(t) + G^2(\varepsilon)\widetilde{\gamma}^2(t).$$

We choose

$$\widetilde{G}(\varepsilon) = \frac{-b(t)+\operatorname{sgn}(b(t))\big(b^2(t)+\widetilde{\gamma}^2(t)G^2(\varepsilon)\big)^{\frac{1}{2}}}{\widetilde{\gamma}(t)}, \tag{3.42}$$

which is clearly vanishing when ε goes to 0.

Notice that we obtain this third candidate-approximation $S_{0,\varepsilon}^{(1)}$ by truncating the small jumps of the jump-diffusion and replacing them by the Brownian motion W which is scaled with $\widetilde{G}(\varepsilon)\widetilde{\gamma}(t)$. $\widetilde{G}(\varepsilon)$ is chosen in a way to keep the same variance as the original model $S^{(1)}$.

The discounted price process is given by

$$d\widetilde{S}_{0,\varepsilon}(t) = \widetilde{S}_{0,\varepsilon}(t)\Big\{(a(t)-r(t))dt + (b(t)+\widetilde{G}(\varepsilon)\widetilde{\gamma}(t))dW(t) + \int_{|z|>\varepsilon}\gamma(t,z)\widetilde{N}(dt,dz)\Big\}.$$

It is easy to show that $\widetilde{S}_{0,\varepsilon}(t)$ converges to $\widetilde{S}(t)$ in L^2 when ε goes to 0 with rate of convergence $\widetilde{G}(\varepsilon)$.

The locally square integrable local martingale $M_{0,\varepsilon}$ in the semimartingale decomposition of $\widetilde{S}_{0,\varepsilon}$ is given by

$$M_{0,\varepsilon}(t) = \int_0^t (b(s) + \widetilde{G}(\varepsilon)\widetilde{\gamma}(s))\widetilde{S}_{0,\varepsilon}(s)dW(s) + \int_0^t \int_{|z|>\varepsilon} \gamma(s,z)\widetilde{S}_{0,\varepsilon}(s)\widetilde{N}(ds,dz)\,.$$

We define the process $\alpha_{0,\varepsilon}$ by

$$\alpha_{0,\varepsilon}(t) := \frac{a(t)-r(t)}{\widetilde{S}_{0,\varepsilon}(t)\big(b^2(t) + \int_{\mathbb{R}_0} \gamma^2(t,z)\ell(dz)\big)}\,, \qquad 0 \le t \le T.$$

Thus the mean-variance tradeoff process $K_{0,\varepsilon}$ is given by

$$K_{0,\varepsilon}(t) = \int_0^t \alpha^2_{0,\varepsilon}(s)d\langle M_{0,\varepsilon}\rangle(s) = \int_0^t \frac{(a(s)-r(s))^2}{b^2(s) + \int_{\mathbb{R}_0}\gamma^2(s,z)\ell(dz)}ds = K(t)\,.$$

Let ξ^0_ε be a square integrable contingent claim as a financial derivative with underlying $\widetilde{S}_{0,\varepsilon}$. We denote the discounted payoff of ξ^0_ε by $\widetilde{\xi}^0_\varepsilon = \xi^0_\varepsilon / S^{(0)}(T)$. Following the same steps as before, we get the following equation for the value of the portfolio

$$\begin{cases} d\widetilde{V}_{0,\varepsilon}(t) &= \widetilde{\pi}_{0,\varepsilon}(t)(a(t)-r(t))dt + \widetilde{\pi}_{0,\varepsilon}(t)(b(t) + \widetilde{G}(\varepsilon)\widetilde{\gamma}(t))dW(t) \\ &\qquad + \int_{|z|>\varepsilon} \widetilde{\pi}_{0,\varepsilon}(t)\gamma(t,z)\widetilde{N}(dt,dz) + d\phi^{FS}_{0,\varepsilon}(t), \\ \widetilde{V}_{0,\varepsilon}(T) &= \widetilde{\xi}^0_\varepsilon, \end{cases}$$

where $\widetilde{\pi}_{0,\varepsilon} = \chi^{FS}_{0,\varepsilon}\widetilde{S}_{0,\varepsilon}$ and $\chi^{FS}_{0,\varepsilon} \in \Theta$, (3.7). Since $\phi^{FS}_{0,\varepsilon}(T)$ is a $\mathcal{F}_T$-measurable square integrable random variable, then applying Theorem 1.1 with $\mathbb{H} = \mathbb{F}$ and using the $\mathbb{P}$-martingale property of $\phi^{FS}_{0,\varepsilon}$ we know that there exist stochastic integrands Y^{FS}_ε and Z^{FS}_ε, such that

$$\phi^{FS}_{0,\varepsilon}(t) = \mathbb{E}[\phi^{FS}_{0,\varepsilon}(T)] + \int_0^t Y^{FS}_\varepsilon(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} Z^{FS}_\varepsilon(s,z)\widetilde{N}(ds,dz). \tag{3.43}$$

Using the same arguments as for $\phi^{FS}_{1,\varepsilon}$ we can prove that $\mathbb{E}[\phi^{FS}_{0,\varepsilon}(T)] = \mathbb{E}[\phi^{FS}_{0,\varepsilon}(0)] = 0$. In view of the orthogonality of $\phi^{FS}_{0,\varepsilon}$ with respect to $M_{0,\varepsilon}$, we have

$$0 = Y^{FS}_\varepsilon(t)[b(t) + \widetilde{G}(\varepsilon)\widetilde{\gamma}(t)] + \int_{\mathbb{R}_0} Z^{FS}_\varepsilon(t,z)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z)\ell(dz). \tag{3.44}$$

The equation we obtain for the approximating problem is thus given by

$$\begin{cases} d\widetilde{V}_{0,\varepsilon}(t) &= \widetilde{\pi}_{0,\varepsilon}(t)(a(t)-r(t))dt + \big(\widetilde{\pi}_{0,\varepsilon}(t)[b(t) + \widetilde{G}(\varepsilon)\widetilde{\gamma}(t)] + Y^{FS}_\varepsilon(t)\big)dW(t) \\ &\qquad + \int_{\mathbb{R}_0} \big(\widetilde{\pi}_{0,\varepsilon}(t)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z) + Z^{FS}_\varepsilon(t,z)\big)\widetilde{N}(dt,dz), \\ \widetilde{V}_{0,\varepsilon}(T) &= \widetilde{\xi}^0_\varepsilon. \end{cases} \tag{3.45}$$

In the next lemma we prove that $\widetilde{V}_{0,\varepsilon}$ satisfies the set of equations of type (2.1).

Lemma 3.8. *Assume that* (3.9) *holds. Let $\widetilde{V}_{0,\varepsilon}$ be given by* (3.45)*. Then $\widetilde{V}_{0,\varepsilon}$ satisfies a BSDEJ of type* (2.1).

Proof. We rewrite equation (3.45) as

$$\begin{cases} d\widetilde{V}_{0,\varepsilon}(t) &= -f_\varepsilon^0(t,\widetilde{V}_{0,\varepsilon}(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,.))dt+\widetilde{Y}_\varepsilon(t)dW(t)+\int_{\mathbb{R}_0}\widetilde{Z}_\varepsilon(t,z)\widetilde{N}(dt,dz),\\ \widetilde{V}_{0,\varepsilon}(T) &= \widetilde{\xi}_\varepsilon^0,\end{cases}$$

where we introduce the processes $\widetilde{Y}_\varepsilon$, $\widetilde{Z}_\varepsilon$ and the function f_ε^0 by

$$\begin{aligned}\widetilde{Y}_\varepsilon(t) &= \widetilde{\pi}_{0,\varepsilon}(t)[b(t)+\widetilde{G}(\varepsilon)\widetilde{\gamma}(t)]+Y_\varepsilon^{FS}(t),\\ \widetilde{Z}_\varepsilon(t,z) &= \widetilde{\pi}_{0,\varepsilon}(t)\gamma(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z)+Z_\varepsilon^{FS}(t,z),\\ f_\varepsilon^0(t,\widetilde{V}_{0,\varepsilon}(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,.)) &= -\widetilde{\pi}_{0,\varepsilon}(t)(a(t)-r(t)).\end{aligned} \tag{3.46}$$

With the same arguments as above and using (3.44) we can prove that

$$\widetilde{\pi}_{0,\varepsilon}(t)=\frac{1}{\kappa(t)}\Big\{\widetilde{Y}_\varepsilon(t)\big[b(t)+\widetilde{G}(\varepsilon)\widetilde{\gamma}(t)\big]+\int_{\mathbb{R}_0}\widetilde{Z}_\varepsilon(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z)\gamma(t,z)\ell(dz)\Big\}. \tag{3.47}$$

Hence

$$\begin{aligned}f_\varepsilon^0(t,\widetilde{V}_{0,\varepsilon}(t),\widetilde{Y}_\varepsilon(t),\widetilde{Z}_\varepsilon(t,.)) = -\frac{a(t)-r(t)}{\kappa(t)}\Big(&\widetilde{Y}_\varepsilon(t)[b(t)+\widetilde{G}(\varepsilon)\widetilde{\gamma}(t)]\\ &+\int_{\mathbb{R}_0}\widetilde{Z}_\varepsilon(t,z)\mathbf{1}_{\{|z|>\varepsilon\}}(z)\gamma(t,z)\ell(dz)\Big)\end{aligned}$$

and along the same lines as in the proof of Lemma 3.7 it is easy to show that f_ε^0 is Lipschitz when (3.9) holds. This proves the statement. □

Now we present the following theorem in which we prove the robustness of the value of the portfolio.

Theorem 3.9. *Assume that* (3.9) *and* (3.30) *hold. Let* $\widetilde{V}$, $\widetilde{V}_{0,\varepsilon}$ *be given by* (3.14), (3.45), *respectively. Then we have*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\widetilde{V}(t)-\widetilde{V}_{0,\varepsilon}(t)|^2\Big]\le C\mathbb{E}[|\widetilde{\xi}-\widetilde{\xi}_\varepsilon^0|^2]+\widetilde{C}[\widetilde{G}^2(\varepsilon)+G^2(\varepsilon)]\mathbb{E}[|\widetilde{\xi}_\varepsilon^0|^2],$$

where C *and* $\widetilde{C}$ *are positive constants and* $\widetilde{G}(\varepsilon)$ *is given by* (3.42).

Proof. Following the same steps as in the proof of Theorem 3.2, we can show that f_ε^0 satisfies condition (2.2). Indeed

$$\begin{aligned}&|f(t,x_1,y_1,z_1)-f_\varepsilon^0(t,x_2,y_2,z_2)|\\ &\quad\le |h(t)|\Big[|y_1-y_2||b(t)|+\widetilde{G}(\varepsilon)|y_2||\widetilde{\gamma}(t)|+\int_{|z|\le\varepsilon}|z_2||\gamma(t,z)|\ell(dz)\\ &\qquad+\int_{\mathbb{R}_0}|z_1-z_2||\gamma(t,z)|\ell(dz)\Big]\\ &\quad\le |h(t)|\Big[|y_1-y_2||b(t)|+\widetilde{G}(\varepsilon)|y_2||\widetilde{\gamma}(t)|+(\int_{|z|\le\varepsilon}|\gamma(t,z)|^2\ell(dz))^{\frac{1}{2}}(\int_{\mathbb{R}_0}|z_2|^2\ell(dz))^{\frac{1}{2}}\end{aligned}$$

$$+ (\int_{\mathbb{R}_0} |\gamma(t,z)|^2 \ell(dz))^{\frac{1}{2}} (\int_{\mathbb{R}_0} |z_1 - z_2|^2 \ell(dz))^{\frac{1}{2}}\Big]$$
$$\leq |h(t)|(\sqrt{\kappa(t)} + |\widetilde{\gamma}(t)|)\Big[|y_1 - y_2| + \widetilde{G}(\varepsilon)|y_2| + G(\varepsilon)\|z_2\| + \|z_1 - z_2\|\Big].$$

From (3.30) and (3.9) and noticing that $f^0_\varepsilon(t,0,0,0) = 0$, we prove the statement by applying Theorem 2.6. □

Remark 3.10. *In the convergence result in the latter theorem, the term* $\mathbb{E}[|\widetilde{\xi}^0_\varepsilon|^2]$ *appears. It is given by*

$$\mathbb{E}[|\widetilde{\xi}^0_\varepsilon|^2] = \mathbb{E}\Big[\Big|\frac{g(S^{(1)}_{0,\varepsilon}(T))}{S^{(0)}(T)}\Big|^2\Big],$$

where g *is the payoff function. In case* g *is Lipschitz with* K *being the Lipschitz coefficient and* $g(0) = 0$, *then we have*

$$\mathbb{E}\Big[\Big|\frac{g(S^{(1)}_{0,\varepsilon}(T))}{S^{(0)}(T)}\Big|^2\Big] \leq K\mathbb{E}\Big[\Big|\frac{S^{(1)}_{0,\varepsilon}(T)}{S^{(0)}(T)}\Big|^2\Big].$$

This latter quantity is bounded in ε *by a constant (see Lemma 3.2 in Benth et al.* [4]*). In case* g *is not Lipschitz, one can still prove the boundedness of* $\mathbb{E}[|g(S^{(1)}_{0,\varepsilon}(T))/S^{(0)}(T)|^2]$ *using Fourier transforms as in Benth et al.* [4].

In the next theorem we prove the robustness of the amount of wealth to invest in a locally risk-minimizing strategy.

Theorem 3.11. *Assume that* (3.9) *and* (3.41) *hold. Let* $\widetilde{\pi}$, $\widetilde{\pi}_{0,\varepsilon}$ *be given by* (3.24), (3.47), *respectively. Then*

$$\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{0,\varepsilon}(s)|^2 ds\Big] \leq C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^0_\varepsilon|^2] + \widetilde{C}[\widetilde{G}^2(\varepsilon) + G^2(\varepsilon)]\mathbb{E}[|\widetilde{\xi}^0_\varepsilon|^2],$$

where C *and* $\widetilde{C}$ *are positive constants.*

Proof. We have

$$\begin{aligned}
|\widetilde{\pi}(s) - \widetilde{\pi}_{0,\varepsilon}(s)|^2 &= \frac{1}{\kappa^2(s)}\Big\{(\widetilde{Y}(s) - \widetilde{Y}_\varepsilon(s))b(s) - \widetilde{Y}_\varepsilon(s)\widetilde{G}(\varepsilon)\widetilde{\gamma}(s) \\
&\quad + \int_{\mathbb{R}_0} (\widetilde{Z}(s,z) - \widetilde{Z}_\varepsilon(s,z))\gamma(s,z)\ell(dz) + \int_{|z|\leq\varepsilon} \widetilde{Z}_\varepsilon(s,z)\gamma(s,z)\ell(dz)\Big\}^2 \\
&\leq \frac{C}{\kappa(s)}\Big\{|\widetilde{Y}(s) - \widetilde{Y}_\varepsilon(s)|^2 + \widetilde{G}^2(\varepsilon)\widetilde{\gamma}^2(s)|\widetilde{Y}_\varepsilon(s)|^2 \\
&\quad + \int_{\mathbb{R}_0} |\widetilde{Z}(s,z) - \widetilde{Z}_\varepsilon(s,z)|^2\ell(dz) + G^2(\varepsilon)\widetilde{\gamma}^2(s)\int_{\mathbb{R}_0} |\widetilde{Z}_\varepsilon(s,z)|^2\ell(dz)\Big\},
\end{aligned}$$

where C is a positive constant. Hence from Theorem 2.4 and Lemma 2.3, we deduce

$$\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{0,\varepsilon}(s)|^2 ds\Big] \leq \frac{C}{\inf_{t\leq s\leq T}\kappa(s)}\Big\{\mathbb{E}\Big[\int_t^T |\widetilde{Y}(s) - \widetilde{Y}_\varepsilon(s)|^2 ds\Big]$$

$$
\begin{aligned}
&+ \widetilde{G}^2(\varepsilon) \sup_{t\le s\le T} \widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T |\widetilde{Y}_\varepsilon(s)|^2 ds\Big] \\
&+ G^2(\varepsilon) \sup_{t\le s\le T} \widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} |\widetilde{Z}_\varepsilon(s,z)|^2 \ell(dz)ds\Big] \\
&+ \mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} |\widetilde{Z}(s,z) - \widetilde{Z}_\varepsilon(s,z)|^2 \ell(dz)ds\Big]\Big\} \\
\le\ & \widetilde{C}\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^0_\varepsilon|^2] + C'(\widetilde{G}^2(\varepsilon) + G^2(\varepsilon))\mathbb{E}[|\widetilde{\xi}^0_\varepsilon|^2]
\end{aligned}
$$

and we prove the statement. □

In the next theorem we deal with the robustness of the process ϕ^{FS}.

Theorem 3.12. *Assume that* (3.9) *and* (3.41) *hold. Let* ϕ^{FS}, $\phi^{FS}_{0,\varepsilon}$ *be given by* (3.12), (3.43)*, respectively. Then for all* $t \in [0,T]$ *we have*

$$
\mathbb{E}\Big[|\phi^{FS}(t) - \phi^{FS}_{0,\varepsilon}(t)|^2\Big] \le C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^0_\varepsilon|^2] + \widetilde{C}[\widetilde{G}^2(\varepsilon) + G^2(\varepsilon)]\mathbb{E}[|\widetilde{\xi}^0_\varepsilon|^2] + C'G^2(\varepsilon),
$$

where C*,* $\widetilde{C}$*, and* C' *are positive constants.*

Proof. From (3.23), (3.46), Lemma 2.3, Theorem 2.4 and Theorem 3.11, we have

$$
\begin{aligned}
\mathbb{E}\Big[\int_t^T |Y^{FS}(s) - Y^{FS}_\varepsilon(s)|^2 ds\Big] &\le C\Big\{\mathbb{E}\Big[\int_t^T |\widetilde{Y}(s) - \widetilde{Y}_\varepsilon(s)|^2 ds\Big] \\
&\quad + \sup_{t\le s\le T} \kappa(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{0,\varepsilon}(s)|^2 ds\Big]\Big\} \\
&\quad + \widetilde{G}^2(\varepsilon) \sup_{t\le s\le T} \widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}_{0,\varepsilon}(s)|^2 ds\Big] \\
&\le \widetilde{C}\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^0_\varepsilon|^2] + C'[\widetilde{G}^2(\varepsilon) + G^2(\varepsilon)]\mathbb{E}[|\widetilde{\xi}^0_\varepsilon|^2].
\end{aligned}
$$

Combining (3.23), (3.46), Lemma 2.3, Theorem 2.4 and Theorem 3.11, we arrive at

$$
\begin{aligned}
&\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} |Z^{FS}(s,z) - Z^{FS}_\varepsilon(s,z)|^2 \ell(dz)ds\Big] \\
&\quad \le C\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}_0} |\widetilde{Z}(s,z) - \widetilde{Z}_\varepsilon(s,z)|^2 \ell(dz)ds\Big] \\
&\qquad + G^2(\infty) \sup_{t\le s\le T} \widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s) - \widetilde{\pi}_{0,\varepsilon}(s)|^2 ds\Big] \\
&\qquad + G^2(\varepsilon) \sup_{t\le s\le T} \widetilde{\gamma}^2(s)\mathbb{E}\Big[\int_t^T |\widetilde{\pi}(s)|^2 ds\Big] \\
&\quad \le C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^0_\varepsilon|^2] + C'G^2(\varepsilon)\mathbb{E}[\widetilde{\xi}^2] + \widetilde{C}[\widetilde{G}^2(\varepsilon) + G^2(\varepsilon)]\mathbb{E}[|\widetilde{\xi}^0_\varepsilon|^2]
\end{aligned}
$$

and the result follows. □

Define the cost process in the risk-minimizing strategy for $\widetilde{\xi}^0_\varepsilon$ by

$$C_{0,\varepsilon}(t) = \phi^{FS}_{0,\varepsilon}(t) + \widetilde{V}_{0,\varepsilon}(0).$$

Then an obvious implication of the last theorem is the robustness of the cost process and it is easy to show that under the same conditions of the last theorem we have for all $t \in [0,T]$,

$$\mathbb{E}[|C(t) - C_{0,\varepsilon}(t)|^2] \leq K\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^0_\varepsilon|^2] + K'G^2(\varepsilon) + \widetilde{K}[\widetilde{G}^2(\varepsilon) + G^2(\varepsilon)]\mathbb{E}[|\widetilde{\xi}^0_\varepsilon|^2],$$

where K, K', and $\widetilde{K}$ are positive constants.

Analogously and using similar computations, one can prove the robustness of the amount invested in the riskless asset in locally risk-minimizing strategies.

3.4. **A note on the robustness of the mean-variance hedging strategies.** A mean-variance hedging (MVH) strategy is a self-financing strategy for which we do not impose the replication requirement. However we insist on the self-financing constraint. In this case we define the shortfall or loss from hedging $\widetilde{\xi}$ by

$$\widetilde{\xi} - \widetilde{V}(0) - \int_0^T \widetilde{\Gamma}(s)d\widetilde{S}(s), \quad \widetilde{V}(0) \in \mathbb{R}, \quad \widetilde{\Gamma} \in \Theta\,.$$

In order to obtain the MVH strategy one has to minimize the latter quantity in the L^2-norm by choosing $(\widetilde{V}(0), \widetilde{\Gamma}) \in (\mathbb{R}, \Theta)$. Schweizer [29] gives a formula for the number of risky assets in a MVH strategy where he assumes that the so-called extended mean-variance tradeoff process is deterministic.

In this paper, given the dynamics of the stock price process S, the process A defined in (3.4) is continuous. Thus the mean-variance tradeoff process and the extended mean-variance tradeoff process defined in Schweizer [29] coincide. Therefore applying Theorem 3 and Corollary 10 in Schweizer [29] and assuming that the mean-variance tradeoff process K is deterministic, the discounted number of risky assets in a MVH strategy is given by

$$\widetilde{\Gamma}(t) = \widetilde{\chi}^{FS}(t) + \alpha(t)\Big(\widetilde{V}(t-) - \widetilde{V}(0) - \int_0^t \widetilde{\Gamma}(s)d\widetilde{S}(s)\Big), \tag{3.48}$$

where α and $\widetilde{\chi}^{FS}$ are as defined in (3.5) and (3.10), and $\widetilde{V}$ is the value of the portfolio in a locally risk-minimizing strategy. Multiplying (3.48) by $\widetilde{S}$ we obtain the following equation for the amount of wealth in a MVH hedging strategy

$$\widetilde{\Upsilon}(t) = \widetilde{\pi}(t) + h(t)\Big(\widetilde{V}(t-) - \widetilde{V}(0) - \int_0^t \frac{\widetilde{\Upsilon}(s)}{\widetilde{S}(s)}d\widetilde{S}(s)\Big),$$

where h is given by (3.26). Since K is deterministic then a, b, r, γ, and thus h should be deterministic. We consider the approximating stock process $\widetilde{S}_{1,\varepsilon}$. The amount of wealth in a MVH strategy associated to $\widetilde{S}_{1,\varepsilon}$ is given by

$$\widetilde{\Upsilon}_{1,\varepsilon}(t) = \widetilde{\pi}_{1,\varepsilon}(t) + h(t)\Big(\widetilde{V}_{1,\varepsilon}(t-) - \widetilde{V}_{1,\varepsilon}(0) - \int_0^t \frac{\widetilde{\Upsilon}_{1,\varepsilon}(s)}{\widetilde{S}_{1,\varepsilon}(s)}d\widetilde{S}_{1,\varepsilon}(s)\Big).$$

Before we show the robustness of the mean-variance hedging strategies. We present the following lemma in which we show the boundedness in L^2 of $\widetilde{\Upsilon}$.

Lemma 3.13. *Assume that the mean-variance tradeoff process* K (3.6) *is deterministic and that* (3.9) *holds true. Then for all* $t \in [0,T]$*,*

$$\mathbb{E}[\widetilde{\Upsilon}^2(t)] \le C(T)\mathbb{E}[\widetilde{\xi}^2],$$

where $C(T)$ *is a positive constant depending on* T*.*

Proof. Applying Itô isometry and Hölder inequality, we get

$$\begin{aligned}\mathbb{E}[\widetilde{\Upsilon}^2(t)] &\le \mathbb{E}[\widetilde{\pi}^2(t)] + C'h^2(t)\Big(\mathbb{E}[\widetilde{V}^2(t)] + \mathbb{E}[\widetilde{V}^2(0)] \\ &\quad + \int_0^t \mathbb{E}[\widetilde{\Upsilon}^2(s)]\{(a(s)-r(s))^2 + b^2(s) + \int_{\mathbb{R}_0} \gamma^2(s,z)\ell(dz)\}ds\Big),\end{aligned}$$

where C' is a positive constant. Using Lemma 2.3, Lemma 3.1, and equation (3.24), the result follows applying Gronwall's inequality. □

In the following theorem we prove the robustness of the amount of wealth in a MVH strategy.

Theorem 3.14. *Assume the mean-variance tradeoff process is deterministic and that* (3.9) *and* (3.31) *hold. Then for all* $t \in [0,T]$*,*

$$\mathbb{E}\Big[|\widetilde{\Upsilon}(t) - \widetilde{\Upsilon}_{1,\varepsilon}(t)|^2] \le C\mathbb{E}[|\widetilde{\xi} - \widetilde{\xi}^1_\varepsilon|^2] + \widetilde{C}G^2(\varepsilon).$$

Proof. We have

$$\begin{aligned}&|\widetilde{\Upsilon}(t) - \widetilde{\Upsilon}_{1,\varepsilon}(t)| \\ &\le |\widetilde{\pi}(t) - \widetilde{\pi}_{1,\varepsilon}(t)| + |h(t)|\Big(|\widetilde{V}(t-) - \widetilde{V}_{1,\varepsilon}(t-)| + |\widetilde{V}(0) - \widetilde{V}_{1,\varepsilon}(0)| \\ &\quad + \int_0^t |\widetilde{\Upsilon}(s) - \widetilde{\Upsilon}_{1,\varepsilon}(s)||a(s)-r(s)|ds + |\int_0^t (\widetilde{\Upsilon}(s) - \widetilde{\Upsilon}_{1,\varepsilon}(s))b(s)dW(s)| \\ &\quad + |\int_0^t \int_{|z|>\varepsilon} (\widetilde{\Upsilon}(s) - \widetilde{\Upsilon}_{1,\varepsilon}(s))\gamma(s,z)\widetilde{N}(ds,dz)| \\ &\quad + G(\varepsilon)|\int_0^t (\widetilde{\Upsilon}_{1,\varepsilon}(s) - \widetilde{\Upsilon}(s)\widetilde{\gamma}(s)dB(s)| \\ &\quad + |\int_0^t \int_{|z|\le\varepsilon} \widetilde{\Upsilon}(s)\gamma(s,z)\widetilde{N}(ds,dz)| + G(\varepsilon)|\int_0^t \widetilde{\Upsilon}(s)\widetilde{\gamma}(s)dB(s)|\Big).\end{aligned}$$

Using Itô isometry and Hölder inequality, we get

$$\begin{aligned}&\mathbb{E}[|\widetilde{\Upsilon}(t) - \widetilde{\Upsilon}_{1,\varepsilon}(t)|^2] \\ &\quad \le \mathbb{E}[|\widetilde{\pi}(t) - \widetilde{\pi}_{1,\varepsilon}(t)|^2] + \widetilde{C}h^2(t)\Big(\mathbb{E}[|\widetilde{V}(t) - \widetilde{V}_{1,\varepsilon}(t)|^2] + \mathbb{E}[|\widetilde{V}(0) - \widetilde{V}_{1,\varepsilon}(0)|^2]\end{aligned}$$

$$+\int_0^t \mathbb{E}[|\widetilde{\Upsilon}(s)-\widetilde{\Upsilon}_{1,\varepsilon}(s)|^2]\Big(|a(s)-r(s)|^2+|b(s)|^2+\int_{\mathbb{R}_0}|\gamma(s,z)|^2\ell(dz)\Big)ds$$
$$+G^2(\varepsilon)\int_0^t \mathbb{E}[\widetilde{\Upsilon}^2(s)]\widetilde{\gamma}^2(s)ds\Big),$$

where $\widetilde{C}$ is a positive constant. Using Theorem 3.2, Theorem 3.4, and Lemma 3.13 the result follows applying Gronwall's inequality. □

We proved in this section that when the mean-variance tradeoff process K defined in (3.6) is deterministic, then the amount of wealth in a MVH strategy is robust towards the choice of the model. It follows immediately that the value of the portfolio and the amount invested in the riskless asset are also robust for the mean-variance hedging strategy. The same results hold true when we consider the stock price process $\widetilde{S}_{0,\varepsilon}$ or $\widetilde{S}_{2,\varepsilon}$. We do not present these results since they follow the same lines as for the approximation $\widetilde{S}_{1,\varepsilon}$.

4. Conclusion

In this paper we consider different models for the price process. Then using BSDEJs we proved that the locally risk-minimizing and the mean-variance hedging strategies are robust towards the choice of the model. Our results are given in terms of estimates containing $\mathbb{E}[|\widetilde{\xi}-\widetilde{\xi}^{\rho}_{\varepsilon}|^2]$, which is a quantity well studied by Benth et al. [4] and Kohatsu-Higa and Tankov [25]

We have specifically studied three types of approximations of the price S and we considered the role of the filtration in our study of these approximations. It is also possible to consider other approximations to the price S. For example we can add to the Lévy process a scaled Brownian motion. In that case, based on the robustness of the BSDEJs, we can also prove the robustness of quadratic hedging strategies. This type of approximation was discussed and justified in a paper by Benth et al. [4].

As far as further investigations are concerned, we consider in another paper a time-discretisation of these different price models and study the convergence of the quadratic hedging strategies related to each of these time-discretised price models to the quadratic hedging strategies related to the original continuous time model. Moreover, we are concerned with the characterization of the approximating models which give the best convergence rates when the robustness of quadratic hedging strategies is taken into account.

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