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***Abstract*—We address the linear precoder design problem based on chance constrained *quality-of-service* (QoS) power minimization in the vector *broadcast channel* (BC). We divide the problem into a two step optimization that separates the precoder design from the power allocation. For the power allocation, we propose a map that fits into the framework of standard interference functions. Therefore, we can compute the optimal power allocation for given beamformers and detect whether a tuple of beamformers is feasible. This allows us to test conservative and non-conservative approaches for the beamformer design, e.g., a design based on a rank-one channel approximation is used. Numerical results show that the approximation is adequate for the non-conservative calculation approach, i.e., the post-processing power allocation is capable of compensating for the suboptimal beamforming. Thereby, a wider range of rate targets is achieved than with the conservative beamformer designs.**

***Index Terms*—QoS power minimization; power allocation; statistical CSI; chance-constrained requirements; rank-one channels**

I. Introduction

In this work, we focus on linear transmit beamforming for decreasing the required power at a multi-antenna transmitter, while reliably serving several mobiles with a pre-defined data rate. For perfect *channel state information* (CSI) at the transmitter, this QoS optimization problem is well explored (e.g., [1]–[3]). Recent advances for QoS optimization (e.g., [4], [5]) also take into account that the transmitter is not fully aware of the channel states in reality. Only a statistical model for the channels can be acquired, so that formulations have to be used that are robust w.r.t. these uncertainties.

In this context, we address the linear precoder design problem based on chance constrained QoS power minimization, where certain rate targets shall be achieved with predefined probabilities. Unfortunately, these chance constraints are non-convex in the beamformers and their closed-form expressions are intractable for state of the art optimization methods (e.g., [6], [7]). Therefore, the recent literature concentrates on conservative convex approximations of the chance constraints to efficiently compute 'robust' beamformers with standard convex interior-point solvers [4], [5], [8]–[11] (more detailed explanations are given in Section IV).

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For these approaches, we observed two important drawbacks. First, the conservative optimizations might propose a transmit power that considerably exceeds the optimum of the actual problem. Second, the methods might detect infeasibility even though the original problem formulation is feasible.

To overcome the first issue, we present a twofold solution approach similar to [12] that separates the beamformer design from the power allocation. However, we give up the fixed precoding and allow for an adaptive design, e.g., with either of the conservative approaches. To compensate for the suboptimal beamforming, we suggest the application of a power allocation based on the fixed point framework of standard interference functions [13]. This power allocation minimizes the transmit power subject to the probabilistic constraints for given feasible beamformers. As a consequence, the proposed transmit power can be considerably reduced.

Unfortunately, the suggested power allocation is unable to extend the feasible set of the conservative approaches. These approaches fail to deliver reasonable beamformers in case they detect infeasibility. This motivates to test adequate non-robust beamformer designs as an input for the optimal power allocation. For example, we can use a rank-one channel approximation to propose a non-conservative beamformer calculation. To detect whether the power allocation can compensate for the non-robust beamforming and fulfill the original chance constraints, we establish a feasibility test based on the fixed point map of the power allocation that was introduced in a general form in [14]. In Section VII, numerical results show that the rank-one channel approximations are adequate for the beamformer calculation in scenarios with a dominant channel mean. That is, the power allocation is capable of compensating for the suboptimal beamforming. Thereby, we achieve a wider range of rate targets than with existing conservative designs.

II. System Model

In the considered vector BC, K mobiles are simultaneously served in the same frequency band by an N-antenna base station. The base station linearly precodes the independent data signals $s_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$ with the beamforming vectors $\boldsymbol{t}_k \in \mathbb{C}^N$, $k \in \{1,\ldots,K\}$, and simultaneously transmits the superimposed outcomes $\boldsymbol{x} = \sum_{k=1}^K \boldsymbol{t}_k s_k$ over the frequency flat fading channels $\boldsymbol{h}_k^{\mathrm{H}} \in \mathbb{C}^{1\times N}$, $k \in \{1,\ldots,K\}$, to the K mobiles. Besides the channel distortion, each mobile suffers from zero-mean additive Gaussian noise $n_k \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_k^2)$ with variance $\sigma_k^2 > 0$, $k \in \{1, \ldots, K\}$, such that the received signal of mobile k is given by $y_k = \boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{t}_k s_k + \boldsymbol{h}_k^{\mathrm{H}} \sum_{i \neq k} \boldsymbol{t}_i s_i + n_k$.

As mentioned in the introduction, the base station is not aware of the channel states $\boldsymbol{h}_k$, while the mobiles are assumed to have sufficiently accurate knowledge. The base station models the channel states as complex Gaussian vectors

$$\boldsymbol{h}_k = \boldsymbol{m}_k + \boldsymbol{C}_k^{1/2} \boldsymbol{w}_k \tag{1}$$

with mean $\boldsymbol{m}_k$, covariance matrix $\boldsymbol{C}_k$, and the white error $\boldsymbol{w}_k \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_N)$, $k \in \{1, \ldots, K\}$.

III. Quality of Service Optimization

We focus on a QoS based beamformer design. The QoS metrics of interest are the achievable user rates. For user k,

$$r_k = I(y_k; s_k) = \log_2(1 + \mathrm{SINR}_k) \tag{2}$$

with the *signal-to-interference-plus-noise-ratio* (SINR)

$$\mathrm{SINR}_k = \frac{|\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{t}_k|^2}{\sigma_k^2 + \sum_{i \neq k} |\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{t}_i|^2} \tag{3}$$

is required to lie reliably above a threshold $\rho_k \in \mathbb{R}_+$, $k \in \{1, \ldots, K\}$. The objective is to minimize the base station's resources, which are represented by the transmit power

$$P_{\mathrm{tx}} = \sum_{k=1}^{K} \|\boldsymbol{t}_k\|_2^2. \tag{4}$$

Note that this problem is non-convex in above formulation and might become infeasible for $N < K$. However, equivalently rewriting it into a standard SINR form, feasibility of the problem can be determined in closed-form [15] and efficient algorithmic solutions can be found either via a convex problem reformulation or via uplink-downlink duality and fixed point iterations (e.g., see [1], [2]) for the perfect CSI setup.

The feasibility issue and finding tractable problem reformulations become more difficult when assuming only statistical channel knowledge at the base station, i.e., the mean $\boldsymbol{m}_k$ and the covariance $\boldsymbol{C}_k$ of the Gaussian channels $\boldsymbol{h}_k$. Then, we cannot directly rely on (2) for the QoS power minimization. Instead, a robust formulation is considered, where the probability of an outage shall not exceed a predefined percentage $\varepsilon_k \in (0, 1)$. This *chance-constrained* optimization reads as

$$\begin{aligned} \underset{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_K}{\text{minimize}} \quad & \sum_{i=1}^{K} \|\boldsymbol{t}_i\|_2^2 \\ \text{subject to} \quad & \Pr(r_k \geq \rho_k) \geq 1 - \varepsilon_k \quad \forall\, k \in \{1, \ldots, K\}. \end{aligned} \tag{P.1}$$

Note that this problem might already become infeasible for sufficiently large targets ρ_k, $k \in \{1, \ldots, K\}$, even for $N \geq K$ since we are not aware of the channel states.

To see the nature of the probabilistic requirements in (P.1), we equivalently rewrite them via inserting (2) and (3):

$$\Pr(r_k \geq \rho_k) = \Pr\big(\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{B}_k \boldsymbol{h}_k \geq \sigma_k^2\big) \geq 1 - \varepsilon_k \tag{5}$$

where $\boldsymbol{B}_k = \frac{1}{2^{\rho_k} - 1} \boldsymbol{t}_k \boldsymbol{t}_k^{\mathrm{H}} - \sum_{i \neq k} \boldsymbol{t}_i \boldsymbol{t}_i^{\mathrm{H}}$. Due to the Hermitian form in $\boldsymbol{h}_k$ of the stochastic constraint, the probability in (5) is strictly positive, only if $\boldsymbol{B}_k$ has a positive eigenvalue, otherwise it is zero for sure. In the previous case, the probability is a *cumulative distribution function* (CDF) of a non-central indefinite quadratic form in complex Gaussian random variables and can be computed as in [6], [7], for example.

IV. Conservative Solution Approaches

It is well known that above chance constraints are non-convex in the optimization parameters and their closed-form expressions are intractable for the state of the art optimization methods (e.g., see references in [7]). Therefore, several tractable convex approximations were proposed for the beamformer design that are robust in the sense that their solutions fulfill the actual chance constraints in (P.1).

- Most prominent are the approaches in [5], [8]. Therein, the probabilistic rate constraints are rewritten into affinely perturbed linear matrix inequalities whose probability is approximated in [16].
- In [9], a semidefinite relaxation approach is considered, where a Bernstein's type inequality is employed according to [17] to remove the uncertainty expression and obtain a convex formulation.
- In the approximations of [4], [10], [11], worst-case constraints replace the chance constraints. That is, the probabilistic constraints in (P.1) are replaced by the deterministic requirements $r_k \geq \rho_k$ that have to be fulfilled for all $\boldsymbol{w}_k$ in (1) that lie in set $\mathcal{W}_k$, e.g., a simple sphere, where $\Pr(\boldsymbol{w}_k \in \mathcal{W}_k) = 1 - \varepsilon_k$, $k \in \{1, \ldots, K\}$.

Due to the conservatism, the actually achieved probabilities (5) of these approaches strongly depend on the given requirements. If the different rate targets ρ_k, $k \in \{1, \ldots, K\}$, are close to being infeasible, a conservative approach might result in an objective that considerably exceeds the optimum of (P.1) or detect infeasibility even though (P.1) is feasible. These issues motivated us to propose a twofold solution approach with separate beamformer design and power allocation. In contrast to the power allocation in [12], the following power allocation is optimal and applicable for any suboptimal but feasible set of fixed beamformers.

V. Post-Processing Power Allocation

Motivated by the conservatism of above mentioned beamformer designs, we next propose a (post-processing) power allocation that compensates for the suboptimal beamforming. To this end, we recast the *given* beamformer tuple as

$$\boldsymbol{t}_k = \boldsymbol{\tau}_k \sqrt{p_k} \tag{6}$$

with unit-norm $\boldsymbol{\tau}_k \in \mathbb{C}^N$ and the per-user transmit powers $p_i \in \mathbb{R}_+$, $i \in \{1, \ldots, K\}$. Inserting (6) into (5), the reliability probability of mobile k reads as

$$F_k(\boldsymbol{p}, \sigma_k^2) = \Pr\Big(\beta_{k,k} p_k - \sum_{i \neq k} \beta_{k,i} p_i \geq \sigma_k^2\Big) \tag{7}$$

with $\beta_{k,j} = |\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{\tau}_j|^2$ for $j \neq k$ and $\beta_{k,k} = \frac{1}{2^{\rho_k} - 1} |\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{\tau}_k|^2$ being correlated (non-central) chi-square distributed with degree 2 for Gaussian $\boldsymbol{h}_k$ and $\boldsymbol{p} = [p_1, \ldots, p_K]^{\mathrm{T}}$ comprises the per-user power allocation at the base station. The transmit power (4) becomes $P_{\text{tx}} = \sum_{i=1}^{K} p_i = \mathbf{1}^{\mathrm{T}}\boldsymbol{p}$, such that the power allocation optimization reads as

$$\begin{aligned} \underset{\boldsymbol{p}}{\text{minimize}} \quad & \mathbf{1}^{\mathrm{T}}\boldsymbol{p} && \text{(P.2)} \\ \text{subject to} \quad & F_k(\boldsymbol{p}, \sigma_k^2) \geq 1 - \varepsilon_k \quad \forall\, k \in \{1, \dots, K\}, \\ & \boldsymbol{p} \geq \mathbf{0}. \end{aligned}$$

Note that the solution to (P.2) depends on the distribution of $\boldsymbol{h}_k$ and the beamformer directions $\boldsymbol{\tau}_k$, $k \in \{1, \dots, K\}$. The proposed approach can be applied for a quite general class of channel distributions, besides Gaussian ones.

Definition 1. We say that a channel distributions is *well-behaved* if $F_k(\boldsymbol{p}, \sigma_k^2)$ in (7) is continuously

(i) increasing in p_k for fixed p_i, $i \neq k$ and σ_k^2;
(ii) decreasing in each p_i, $i \neq k$ for fixed p_k and σ_k^2;
(iii) decreasing in σ_k^2, for fixed p_j, $j \in \{1, \dots, K\}$.

for all probability values $F_k(\boldsymbol{p}, \sigma_k^2) \in (0, 1)$.

Examples for well-behaved channel distributions are all distributions with continuous probability density functions (PDF) that are non-zero in $\mathbb{C}^N$, e.g., the Gaussian distribution.

Note that all reliability requirements in (P.2) are satisfied with equality in the optimum due to the given monotonicity properties (i)-(iii) in Definition 1. That is, the optimizer $\boldsymbol{p}^\star$ of (P.2) satisfies $F_k(\boldsymbol{p}^\star, \sigma_k^2) = 1 - \varepsilon_k$ for all $k \in \{1, \dots, K\}$. Moreover, whenever a power allocation $\boldsymbol{p}$ satisfies the probability constraints in (P.2), they are also satisfied by $\boldsymbol{p}' = \alpha\boldsymbol{p}$ with $\alpha > 1$ since $F_k(\alpha\boldsymbol{p}, \sigma_k^2) = F_k(\boldsymbol{p}, \frac{\sigma_k^2}{\alpha})$ [cf. (7)] that increases for decreasing $\frac{\sigma_k^2}{\alpha}$ [see (iii) of Definition 1].

A. Fixed-Point Framework

These properties motivate a fixed-point iteration based solution approach to (P.2). To this end, we define the functions $f_k : \mathbb{R}_+^K \to \mathbb{R}_+$, $\boldsymbol{p} \mapsto f_k(\boldsymbol{p})$, $k \in \{1, \dots, K\}$, where

$$f_k(\boldsymbol{p}) = \min\Big\{x : \Pr\Big(\beta_{k,k}x - \sum_{i\neq k}\beta_{k,i}p_i \geq \sigma_k^2\Big) = 1 - \varepsilon_k\Big\}. \quad (8)$$

Proposition 1. *The map* $\boldsymbol{f} : \mathbb{R}_+^K \to \mathbb{R}_+^K$ *with* $\boldsymbol{p} \mapsto \boldsymbol{f}(\boldsymbol{p}) = [f_1(\boldsymbol{p}), \dots, f_K(\boldsymbol{p})]^{\mathrm{T}}$ *and* $f_k(\boldsymbol{p})$ *defined in* (8) *is a standard interference function according to* [13].

In other words, $\boldsymbol{f}(\boldsymbol{p})$ satisfies the three properties:

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{p}) &> \mathbf{0} && (\textit{positivity}) \\ \boldsymbol{f}(\boldsymbol{p}) &\geq \boldsymbol{f}(\boldsymbol{p}') \text{ for } \boldsymbol{p} \geq \boldsymbol{p}' && (\textit{monotonicity}) \\ \alpha\boldsymbol{f}(\boldsymbol{p}) &> \boldsymbol{f}(\alpha\boldsymbol{p}) \text{ for } \alpha > 1, && (\textit{scalability}) \end{aligned}$$

where the vector inequalities are component-wise. Therefore, the simple fixed-point iteration

$$\boldsymbol{p}^{(n+1)} = \boldsymbol{f}(\boldsymbol{p}^{(n)}) \quad (9)$$

converges to the unique global optimizer $\boldsymbol{p}^\star$ of (P.2) for any initial $\boldsymbol{p}^{(0)}$ if (P.2) is feasible [13].

Proof of Proposition 1: Elementwise *positivity* of $\boldsymbol{f}(\boldsymbol{p})$ follows directly by the definition of $f_k(\boldsymbol{p})$ in (8) and the monotonicity property (i) in Definition 1 for $\varepsilon_k \in (0, 1)$. Similarly, using property (ii) in Definition 1, we have that

$$\Pr\Big(\beta_{k,k}x - \sum_{i\neq k}\beta_{k,i}p_i' \geq \sigma_k^2\Big) \geq \Pr\Big(\beta_{k,k}x - \sum_{i\neq k}\beta_{k,i}p_i \geq \sigma_k^2\Big)$$

for all $x > 0$ when $\boldsymbol{p} \geq \boldsymbol{p}' \geq \mathbf{0}$. Therefore, we can conclude with (i) in Definition 1 that $f_k(\boldsymbol{p}) \geq f_k(\boldsymbol{p}')$, where equality holds only if $p_k' > p_k$ and $p_l' = p_l$ for all $l \neq k$ and strict inequality holds if there is one $p_l' > p_l$ with $l \neq k$. This proves elementwise *monotonicity* of $\boldsymbol{f}(\boldsymbol{p})$.

To prove the *scalability* property elementwise, we write the following inequality that is valid for $\alpha > 1$:

$$\begin{aligned} f_k(\alpha\boldsymbol{p}) &= \min\Big\{x : \Pr\Big(\beta_{k,k}\frac{x}{\alpha} - \sum_{i\neq k}\beta_{k,i}p_i \geq \frac{\sigma_k^2}{\alpha}\Big) = 1 - \varepsilon_k\Big\} \\ &= \alpha\min\Big\{z : \Pr\Big(\beta_{k,k}z - \sum_{i\neq k}\beta_{k,i}p_i \geq \frac{\sigma_k^2}{\alpha}\Big) = 1 - \varepsilon_k\Big\} \\ &\overset{\text{(iii)}}{<} \alpha\min\Big\{z : \Pr\Big(\beta_{k,k}z - \sum_{i\neq k}\beta_{k,i}p_i \geq \sigma_k^2\Big) = 1 - \varepsilon_k\Big\} \\ &= \alpha f_k(\boldsymbol{p}), \end{aligned}$$

where the first equality follows from the definition of f_k in (8), the second equality follows from substituting $z = x/\alpha$, and the inequality is due to property (iii) of Definition 1. ■

B. Feasibility Detection

To test whether (P.2) is feasible, we can exploit that (9) results in a component-wise monotonically increasing sequence when starting from the all zero vector [13, Lemma 2], i.e., $\mathbf{0} = \boldsymbol{p}^{(0)} \leq \boldsymbol{p}^{(1)} \leq \ldots \leq \boldsymbol{p}^{(n)}$. If the sequence converges before exceeding some threshold for P_{tx}, the convergence point is the optimal solution. Otherwise, infeasibility can be declared.

However, since the transmit power is unbounded in (P.2), above test fails when the optimum lies above the transmit power threshold. For an alternative test, we consider the high power limit, where $\sigma_k^2 \to 0$, $k \in \{1, \dots, K\}$ and the continuous monotone vector map $\boldsymbol{f}(\boldsymbol{p})$ comprises the elements

$$f_k(\boldsymbol{p}) = \min\Big\{x : \Pr\big(\beta_{k,k}x - \sum_{i\neq k}\beta_{k,i}p_i \geq 0\big) = 1 - \varepsilon_k\Big\}. \quad (10)$$

In contrast to *positivity* and *monotonicity*, that are still valid for this $\boldsymbol{f}(\boldsymbol{p})$, the *scalability* property changes to

$$\alpha\boldsymbol{f}(\boldsymbol{p}) = \boldsymbol{f}(\alpha\boldsymbol{p}) \text{ for } \alpha > 0. \qquad (\textit{scale-invariance})$$

Therefore, we can normalize $\boldsymbol{p}$ to $\mathbf{1}^{\mathrm{T}}\boldsymbol{p} = 1$. In other words, $\boldsymbol{f}(\boldsymbol{p})$ is now a general interference function according to [1].

To test whether the probabilistic constraints $F_k(\boldsymbol{p}, 0) \geq 1 - \varepsilon_k$, $k \in \{1, \dots, K\}$ can be satisfied simultaneously, the generalized *signal-to-interference-ratio* (SIR) balancing approach of [14] can be used. That is, we search for the $\boldsymbol{p}$ that balances the SIRs $p_k/f_k(\boldsymbol{p})$ at their maximum value, i.e.,

$$C^{-1} = \max_{\mathbf{1}^{\mathrm{T}}\boldsymbol{p}=1} \min_l \frac{p_l}{f_l(\boldsymbol{p})}. \quad (11)$$

If $C < 1$ the probability constraints in (P.2) can be satisfied with finite $\boldsymbol{p}$ and for $C = 1$ infinite transmit power is required.

For $C > 1$ the probability requirements are infeasible with the given beamformers. To find the optimizer of (11), we can use the (scaled) fixed point iteration [14]

$$\boldsymbol{p}^{(n+1)} = \boldsymbol{f}(\boldsymbol{p}^{(n)})/\mathbf{1}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{p}^{(n)}). \tag{12}$$

If the sequence has converged to the optimum $\boldsymbol{p}^\star$ of (11), the balancing level C^{-1} is simply found via $C\boldsymbol{p}^\star = \boldsymbol{f}(\boldsymbol{p}^\star)$.

Convergence of (12) to the unique fixed point $\boldsymbol{p}^\star$ of (11) is ensured if the interference function $\boldsymbol{f}(\boldsymbol{p})$ is primitive (strongly order-preserving), i.e., if $\boldsymbol{p}' \geq \boldsymbol{p}$ with $\boldsymbol{p}' \neq \boldsymbol{p}$ implies $\boldsymbol{f}^m(\boldsymbol{p}') > \boldsymbol{f}^m(\boldsymbol{p})$ for some positive integer m (cf. [14, Lemma 2]). Since the considered beamformers and channel distributions are well-behaved (see Definition 1) and $f_k(\boldsymbol{p})$ is strictly increasing in all p_l with $l \neq k$, it is straightforward to show that primitivity is satisfied with at most $m = 2$.

VI. Non-Conservative Beamformer Design

Now we are aware of optimally allocating power to the users for given (feasible) beamformers. However, the power allocation does not extend the feasible set for the conservative QoS optimizations, that fail to deliver a reasonable beamformer whenever they detect infeasibility, even though (P.1) is feasible. This motivates the use of non-conservative approximations for the chance constraints in (P.1), e.g., we use a simple rank-one approximation of the channel for this purpose.

We approximate $\boldsymbol{h}_k$ with $\tilde{\boldsymbol{h}}_k = \boldsymbol{v}_k a_k(\boldsymbol{h}_k)$, where the parameters $\boldsymbol{v}_k$ and $a_k(\boldsymbol{h}_k)$ are chosen to minimize the mean square error $\mathrm{E}\big[\|\boldsymbol{h}_k - \tilde{\boldsymbol{h}}_k\|_2^2\big]$, i.e., $a_k(\boldsymbol{h}_k) = \boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{h}_k$ with unit-norm $\boldsymbol{v}_k$, that denotes the dominant eigenvector of $\boldsymbol{m}_k\boldsymbol{m}_k^{\mathrm{H}} + \boldsymbol{C}_k$. This model is a good approximation when either the entries in $\boldsymbol{h}_k$ are strongly correlated, e.g., in satellite mobile communications, or when the mean $\boldsymbol{m}_k$ is dominant over the covariance.

We first approximate (5) by replacing $\boldsymbol{h}_k$ with $\tilde{\boldsymbol{h}}_k$. If we rewrite the result in terms of the *outage probability*, we obtain

$$\Pr\left(\tilde{\boldsymbol{h}}_k^{\mathrm{H}}\boldsymbol{B}_k\tilde{\boldsymbol{h}}_k < \sigma_k^2\right) = 1 - \Pr\left(\tilde{\boldsymbol{h}}_k^{\mathrm{H}}\boldsymbol{B}_k\tilde{\boldsymbol{h}}_k \geq \sigma_k^2\right) \leq \varepsilon_k \tag{13}$$

with matrix $\boldsymbol{B}_k$ defined below (5). This stochastic requirement contains the quadratic form $\tilde{\boldsymbol{h}}_k^{\mathrm{H}}\boldsymbol{B}_k\tilde{\boldsymbol{h}}_k = X_k\boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{B}_k\boldsymbol{v}_k$ where $X_k = |a_k(\boldsymbol{h}_k)|^2$ is non-central χ^2-distributed with two degrees of freedom and non-centrality parameter $\lambda_k = \frac{2|\mathrm{E}[a_k(\boldsymbol{h}_k)]|^2}{\mathrm{var}(a_k(\boldsymbol{h}_k))}$. Substituting this quadratic form into (13), we can write

$$\Pr\left(X_k < \frac{\sigma_k^2}{\boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{B}_k\boldsymbol{v}_k}\right) = \mathrm{F}_{\chi^2_{2,\lambda_k}}\left(\frac{\sigma_k^2}{\boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{B}_k\boldsymbol{v}_k}\right) \leq \varepsilon_k$$

where $\mathrm{F}_{\chi^2_{2,\lambda_k}}$ is the CDF of X_k. Since the stochastic event is now separated into a random left-hand side and a deterministic right-hand side, we can equivalently reformulate the probabilistic constraint into the deterministic requirement

$$\frac{\sigma_k^2}{\boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{B}_k\boldsymbol{v}} \leq q \tag{14}$$

where $q = \mathrm{F}^{-1}_{\chi^2_{2,\lambda_k}}(\varepsilon_k)$ is the inverse CDF of the non-central χ^2-distributed random variable X_k (e.g., see [18]) that is evaluated at ε_k and independent of the optimization variables.

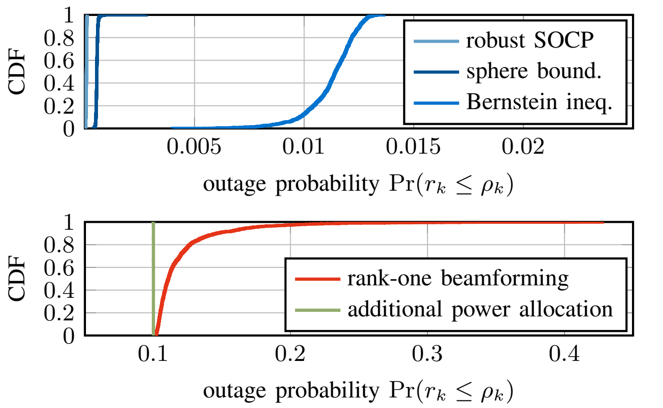

Figure 1. CDF of the Achieved Outage Probability

Via inserting $\boldsymbol{B}_k = \frac{1}{2^{\rho_k}-1}\boldsymbol{t}_k\boldsymbol{t}_k^{\mathrm{H}} - \sum_{i\neq k}\boldsymbol{t}_i\boldsymbol{t}_i^{\mathrm{H}}$ [cf. (5) and below] into (14) and reformulating the result, we approximate the chance constraints in (P.1) to obtain a standard QoS power minimization with SINR-like requirements [cf. (3)], i.e.,

$$\underset{\boldsymbol{t}_1,\ldots,\boldsymbol{t}_K}{\text{minimize}} \sum_{i=1}^{K} \|\boldsymbol{t}_i\|_2^2 \tag{P.3}$$

$$\text{subject to } \frac{|\boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{t}_k|^2}{\frac{\sigma_k^2}{q} + \sum_{i\neq k}|\boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{t}_i|^2} \geq 2^{\rho_k} - 1 \ \forall\, k \in \{1,\ldots,K\}.$$

This optimization problem can for example be solved with the methods and algorithms in [1], [2]. Moreover, feasibility of (P.3) can be declared whenever $\sum_{k=1}^{K} 2^{-\rho_k} > K - N$ [15]. Note, however, that feasibility of (P.3) does not imply feasibility of (P.1) because of the non-conservatism of the rank-one channel approximation. Whether the optimal beamformers of (P.3) are feasible for (P.2) or not has to be detected with the test in Subsection V-B. If the test is positive, the power allocation in Subsection V-A results in a feasible solution for (P.1). Otherwise, we fail to detect feasibility of (P.1).

VII. Numerical Results

In this section, we present simulation results to compare the non-conservative rank-one approximation based beamformer design with the following conservative approaches: the *robust SOCP* formulation from [8] and the two semidefinite relaxation approaches from [11] and [9] that are based on a worst-case (*sphere bounding*) design reformulation and a probability approximation with *Bernstein's type inequality*, respectively. Moreover, we show that the power allocation is able to compensate for the suboptimal beamforming, i.e., it reduces the proposed minimal transmit power for the conservative approaches and ensures reliability of the optimistic beamforming based on the rank-one approximation.

The simulations were performed for the following setup: the number of users and transmit antennas are $K = N = 3$, $\sigma_k^2 = 0.1$, ten percent outage is allowed, i.e., $\varepsilon_k = 0.1$, and the channel covariance matrices are fixed to $\boldsymbol{C}_k = \frac{0.01}{N}\boldsymbol{I}_N$, $k \in \{1,2,3\}$. We use the rate targets $\rho_1 = \rho_3 = 0.1\rho$ and $\rho_2 = 0.2\rho$, where the common factor ρ is successively increased. The channel means are drawn from a standard complex Gaussian distribution of appropriate dimensions.

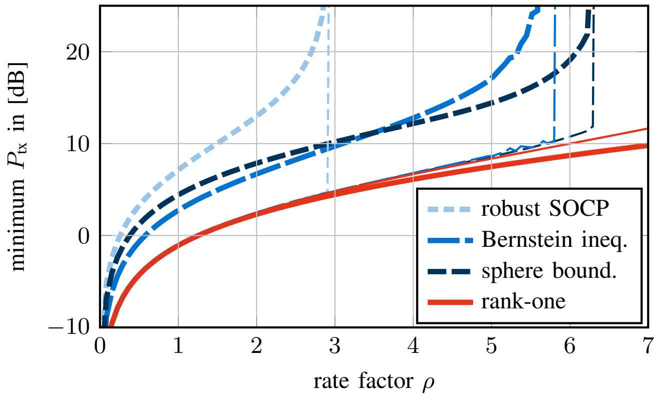

Figure 2. Minimal transmit power before power allocation (thick lines) and after the post-processing power allocation (thin lines)

In Fig. 1, we plotted the CDF curves for the achieved outage probabilities of the conservative methods and the rank-one approximation approach. We created these curves by drawing 500 channel mean realizations, applying above beamformer designs for $\rho = 2$, and calculating the resulting outage probabilities for all three users. As can be seen, the actual outage probabilities of the conservative designs are below 0.015. This promises considerable performance gains for the post-processing. In contrast, the rank-one channel approximation results in beamforming that violates the outage requirements (see Fig. 1). However, the outage requirement $\varepsilon_k = 0.1$ is met after the additional power allocation. The curves support our expectation that the rank-one approximation is optimistic and additional power has to be spent for meeting the requirements.

We observe these behaviors also in Fig. 2, where the minimal transmit power P_{tx} is plotted over ρ for an exemplary scenario. Therein, the transmit powers of the conservative approaches lie considerably above the optimistic rank-one channel approximation approach (thick lines). After the additional power allocation (thin lines), all four beamformer designs achieve similar P_{tx} within their feasible set. For the conservative approaches, we applied standard interior-point solvers that fail to deliver a precoder above the infeasibility thresholds: about $\rho = 3$ for the robust SOCP and about $\rho = 6$ for the semidefinite relaxation approaches. Therefore, the power allocation fails above these values for ρ. The rank-one approximation method is however applicable over the whole range for ρ. Its results are already close to the post-processing transmit powers for $\rho \leq 3$. For $\rho \geq 3$, its beamformers are still adequate, so that the additional power allocation meets the outage requirements. Hence, a wider range of ρ values can be supported than for the conservative beamformer designs.

The power gain due to the additional power allocation is more than 3dB for the Bernstein's inequality approach, 5dB for the sphere bounding method, and about 7dB for the robust SOCP formulation at $\rho = 1$. The gain is even larger, when ρ is close to the infeasibility bound of these methods. The power loss for reaching the required reliability with the beamformers of the rank-one channel approximation appears to be small but increasing with ρ (see Fig. 2). In other words, the larger the rate requirements, the less accurate is this approximation.

VIII. Conclusion

We have shown that the proposed minimal transmit power of state-of-the-art chance-constrained beamformer designs can be considerably reduced with an additional power allocation. Moreover, when the power allocation is able to compensate for the suboptimal beamforming, non-conservative designs may extend the achievable feasibility range of the conservative ones. These results motivate us to investigate upper bounds for chance constraints in future works. The bounds will serve as the basis for necessary feasibility tests and optimistic performance characterizations in probabilistic QoS optimizations.

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