Optimizing the Number of Feedback Users

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***Abstract*—In a multiuser broadcast channel with M antennas at the base station and $K \geq M$ i.i.d. users, multiuser diversity can be exploited by optimally choosing a set of M users to serve simultaneously with linear beamforming. In order to perform the user selection, the base station requires *channel state information* (CSI) of all K users which is obtained through limited feedback in *frequency division duplex* (FDD) systems. In this work we assume the feedback consists solely of the *channel direction information* (CDI) of the users. Although most of the literature assumes a per-user limited feedback, a constraint on the *system* limited feedback is actually more appropriate. Given a constraint on the total amount of channel uses reserved for the feedback of all the users, we experience a tradeoff between the attainable degree of multiuser diversity and the CDI feedback quality. The optimum number of feedback users are determined based on a closed-form approximation of a lower bound on the sum rate achieved with user selection. It is shown that the optimum number of feedback users increases linearly with the total amount of channel uses reserved for the feedback of all the users.**

I. Introduction

In a multiuser downlink, a *base station* (BS) equipped with M antennas can serve in the downlink M single-antenna users simultaneously employing linear beamforming. With $K \geq M$ i.i.d. users, multiuser diversity can be exploited by performing user selection to optimally choose a set of M users to serve. To this end, the base station requires *channel state information* (CSI) of all K users. In a *time division duplex* (TDD) system, the reciprocity between the downlink and uplink channels can be exploited, enabling the base station to have access to the downlink CSI of the users based on uplink channel estimation. In an FDD system, however, the transmit CSI for the downlink is obtained through limited feedback in the uplink [1]. In this work we assume the feedback consists solely of the *channel direction information* (CDI) of the users.

In an FDD system, the CSI at the base station becomes available in a three step process. First, each of the $K \geq M$ users estimate their M-dimensional downlink channel vector with the aid of a common pilot. Each user afterwards quantizes with B bits the CDI of its channel estimate, i.e. the normalized version of its channel estimate, using the *random vector quantization* (RVQ) scheme. Finally, B bits per user are fed back to the base station. We assume error- and delay-free transmission of the feedback to focus on the effects of the CDI quantization.

With the quantized CDI of the K users, the base station can perform user selection to optimally choose a set of M users to serve simultaneously [4], for instance, by choosing the set of users which are most orthogonal to one another. The base station can afterwards serve the selected set of users employing *zero forcing* (ZF) beamforming [2] or *minimum mean square error* (MMSE) beamforming [3] based on the fed back CDI of the selected users.

User selection in order to maximize the sum rate with zero-forcing beamforming based on limited feedback has been considered in the literature [4], [6]–[8]. In such works, however, the K users must report a *channel quality indicator* (CQI) in addition to their quantized CDI. The CQI includes information about the *channel magnitude information* (CMI) and/or about the CDI quantization error, to aid the user selection at the base station. Such a CQI feedback is necessary to exploit multiuser diversity for very large values of K. Nevertheless, it is usually assumed in such works that the base station has access to an *unquantized* version of the CQI of the users. In practice, however, the limited feedback reserved for each user has to be shared for reporting back the quantized CDI and the quantized CQI. In this paper we consider only CDI feedback of the uers, i.e., the entire limited feedback of B bits per user is employed for quantizing the CDI.

In most of the literature, a per-user limited feedback load is assumed, which implies that given K users the total available feedback load is given by KB bits, i.e. it increases with K. Despite this fact, a *system* limited feedback is actually more appropriate. Given a constraint on the total amount of channel uses reserved for the feedback of all the K users, we experience a tradeoff between the attainable degree of multiuser diversity and the CDI feedback quality. In this case, the question arises whether it is desired that many users feedback a small number of feedback bits, i.e. potentially higher multiuser diversity, in contrast to having few users feed back more feedback bits, i.e. to have better CDI feedback quality (with smaller quantization error). Such tradeoff has also been discussed in [11] in the context of minimizing the sum MSE of the selected users and in [13] for maximizing the sum rate of the selected users with ZF beamforming but assuming in the latter a constraint on the total number of feedback bits.

With the user selection and MMSE beamforming based on CDI feedback of the users, we derive a very good approximation for a lower bound on the sum rate with limited feedback and user selection. Based on this lower bound we are able to find the optimum number of feedback users given a constraint on the system limited feedback, by taking into account the inherent tradeoff between potentially higher multiuser diversity and a smaller CDI quantization error. To this end, this work is organized as follows. Section II presents the system model and the MMSE beamforming based on the CDI feedback from the users. User selection based on the CDI feedback of the users is discussed in Section III. The approximation of the lower bound on the average sum rate is derived in Section IV. The system limited feedback is treated in Section V, along with some simulation results. We conclude the paper in Section VI.

II. System Model

We consider an FDD downlink system in a single-cell with a base station equipped with M antennas and K i.i.d. single-antenna users. The available power at the base station is P_{DL} and the transmit signals of the users are zero-mean unit-variance complex Gaussian, i.e. we assume Gaussian signalling. The downlink channel of user k is denoted by $\boldsymbol{h}_k \in \mathbb{C}^M$, whose entries are assumed to independent zero-mean unit-variance complex Gaussian random variables. As discussed before, the CSI of the K users is obtained in a three step process

1) *Training Phase:* Each user first obtains a *minimum mean square error* (MMSE) estimate $\hat{\boldsymbol{h}}_k \in \mathbb{C}^M$ of its downlink channel employing a common downlink pilot consisting of T_{DL} pilot symbols. We assume $T_{\mathrm{DL}} \geq M$ such that the users can obtain a meaningful estimate of its channel vector. With $\boldsymbol{e}_k \in \mathbb{C}^M$ as the error vector of user k, the downlink channel for user k is given by

$$\boldsymbol{h}_k = \hat{\boldsymbol{h}}_k + \boldsymbol{e}_k, \tag{1}$$

where the elements of $\boldsymbol{e}_k$ are zero-mean complex Gaussian random variables with variance σ_e^2 while the elements of $\boldsymbol{h}_k$ are zero-mean complex Gaussian random variables with variance $(1-\sigma_e^2)$. The variance of the estimation error is given by [9]

$$\sigma_e^2 = \frac{1}{1 + \frac{P_{\mathrm{DL}}}{M\sigma_{\mathrm{n}}^2} T_{\mathrm{DL}}}, \tag{2}$$

where σ_{n}^2 is the variance of the AWGN at the users.

2) *Quantization:* Each user computes its estimated *channel direction information* (CDI) $\hat{\boldsymbol{h}}_{k,\mathrm{n}} = \frac{\hat{\boldsymbol{h}}_k}{\|\hat{\boldsymbol{h}}_k\|_2}$. The estimated CDI is quantized with B bits employing random vector quantization (RVQ) [10] as follows

$$\boldsymbol{h}_{k,\mathrm{q}} = \underset{\boldsymbol{t}_{k,j} \in \mathcal{C}_k}{\operatorname{argmax}} \ |\boldsymbol{t}_{k,j}^{\mathrm{H}} \hat{\boldsymbol{h}}_{k,\mathrm{n}}|, \tag{3}$$

where each user k has a different codebook $\mathcal{C}_k$ consisting of 2^B unit-norm random beamforming vectors $\boldsymbol{t}_{k,j} \in \mathcal{C}_k$, which is also available at the BS[1]. Let us further define

$$c_k \quad = \quad \boldsymbol{h}_{k,\mathrm{q}}^{\mathrm{H}} \hat{\boldsymbol{h}}_{k,\mathrm{n}} \in \mathbb{C} \tag{4}$$

such that

$$|c_k| \quad = \quad \|\boldsymbol{h}_{k,\mathrm{q}}\|_2 \, \|\hat{\boldsymbol{h}}_{k,\mathrm{n}}\|_2 \, \cos\theta_k = \cos\theta_k \tag{5}$$

where θ_k is the angle between $\boldsymbol{h}_{k,\mathrm{q}}$ and $\hat{\boldsymbol{h}}_{k,\mathrm{n}}$ and we have that $\boldsymbol{h}_{k,\mathrm{q}}$ and $\hat{\boldsymbol{h}}_{k,\mathrm{n}}$ have unit norm. The downlink channel for user k can be now expressed as

$$\boldsymbol{h}_k = \|\hat{\boldsymbol{h}}_k\|_2 \, (c_k \, \boldsymbol{h}_{k,\mathrm{q}} + \boldsymbol{e}_{k,\mathrm{q}}) + \boldsymbol{e}_k, \tag{6}$$

with the estimated CDI as

$$\hat{\boldsymbol{h}}_{k,\mathrm{n}} = c_k \, \boldsymbol{h}_{k,\mathrm{q}} + \boldsymbol{e}_{k,\mathrm{q}}, \tag{7}$$

where we denote $\boldsymbol{e}_{k,\mathrm{q}}$ as the quantization error, which is orthogonal to $\boldsymbol{h}_{k,\mathrm{q}}$:

$$\boldsymbol{h}_{k,\mathrm{q}}^{\mathrm{H}} \boldsymbol{e}_{k,\mathrm{q}} \quad = \quad 0. \tag{8}$$

This implies $\boldsymbol{e}_{\mathrm{q},k}$ lies in the nullspace of $\boldsymbol{h}_{k,\mathrm{q}}^{\mathrm{H}}$ and is uniformly distributed in the $M-1$ dimensional nullspace, due to the i.i.d. assumption of the channel elements. In addition, $\boldsymbol{e}_{k,\mathrm{q}}$ has also zero mean and since $\|\boldsymbol{h}_{k,\mathrm{n}}\|_2^2 = \|\boldsymbol{h}_{k,\mathrm{q}}\|_2^2 = 1$, from (7) with (5), we obtain

$$\|\boldsymbol{e}_{\mathrm{q},k}\|_2^2 = \sin^2\theta_k \tag{9}$$

3) *Feedback:* Afterwards, each user feeds back to the base station the B bits corresponding to the index of the quantized CDI $\boldsymbol{h}_{k,\mathrm{q}}$. We assume the feedback bits are sent using QPSK symbols in the uplink during a *system* feedback phase consisting of T_{F} channel uses. Since the base station is equipped with M antennas, we can exploit the spatial dimension for the feedback link, such that at each channel use during the feedback phase at most M users can simultaneously send one QPSK feedback symbol. The base station can receive and detect simultaneously the feedback of M users in the uplink by employing, for instance, MMSE receive beamforming. To this end, we assume that the base station knows when the users transmit their feedback and in addition, the base station has estimated the uplink channel of each of the K users[2]. Hence, given T_{F} channel uses reserved for the feedback of all K users and assuming the same number of feedback bits B per user, each user can feedback

$$B = 2 \left\lfloor \frac{T_{\mathrm{F}} M}{K} \right\rfloor \quad \text{bits}, \tag{10}$$

where $\lfloor \bullet \rfloor$ is the floor operator. We assume error- and delay-free transmission of the feedback, to focus on the effects of the quantization.

After error-free feedback of the B bits from each user, the BS would have access to the K users' CDI as transmit CSI for the downlink, with which it can perform user selection. In the following, we present the MMSE beamforming scheme and the user selection based on CDI feedback of the K users.

[1]We assume a different codebook for each user since otherwise there exists a non-zero probability that two or more users feed back the same channel vector which can lead to numerical problems when computing the beamforming vectors.

[2]Note that the uplink channel estimation of the K users does not really impose an additional overhead for the feedback detection if all the K users intend to transmit data in the uplink and multiuser diversity is to be exploited in the uplink data transmission of the users. In this case, we point out that the base station would need to estimate the uplink channels of the K users *anyhow* in order to perform user selection in the uplink.

A. CDI-based MMSE Beamforming

Define the set of $K_\mathrm{s} \leq M$ selected users as $\mathcal{S}$ with

$$\mathcal{S} = \{\pi(1), \pi(2), \cdots, \pi(K_\mathrm{s})\}, \tag{11}$$

where $\pi(m) \in \{1, 2, \cdots, K\}$ for $m = 1, \ldots, K_\mathrm{S}$. We define $\boldsymbol{p}_{\pi(m)}$ as the beamforming vector for the selected user $\pi(m)$, such that the power constraint is given by

$$\sum_{m=1}^{K_\mathrm{s}} \|\boldsymbol{p}_{\pi(m)}\|_2^2 = P_\mathrm{DL}. \tag{12}$$

The SINR for user $\pi(m)$ is given by

$$\mathrm{SINR}_{\pi(m)} = \frac{\left|\boldsymbol{h}_{\pi(m)}^\mathrm{T} \boldsymbol{p}_{\pi(m)}\right|^2}{\sigma_\mathrm{n}^2 + \sum\limits_{n=1, n\neq m}^{K_\mathrm{s}} \left|\boldsymbol{h}_{\pi(m)}^\mathrm{T} \boldsymbol{p}_{\pi(n)}\right|^2}. \tag{13}$$

Let us collect the fed back CDI of the selected users in the matrix

$$\boldsymbol{H}_\mathrm{q} = \left[\hat{\boldsymbol{h}}_{\mathrm{q},\pi(1)}, \hat{\boldsymbol{h}}_{\mathrm{q},\pi(2)}, \cdots, \hat{\boldsymbol{h}}_{\mathrm{q},\pi(K_\mathrm{s})}\right]^\mathrm{T} \in \mathbb{C}^{K_\mathrm{s} \times M}, \tag{14}$$

whose rows correspond to the CDIs of the K_s selected users.

We collect the beamforming vectors of the selected users in the set $\mathcal{S}$ in the matrix $\boldsymbol{P} = \left[\boldsymbol{p}_{\pi(1)}, \ldots \boldsymbol{p}_{\pi(K_\mathrm{s})}\right] \in \mathbb{C}^{M \times K_\mathrm{s}}$. The MMSE beamforming vectors for the selected set of users based on the fed back CDIs are given by [3]

$$\boldsymbol{P} \;=\; \frac{1}{g}\left((1-\kappa)\,\boldsymbol{H}_\mathrm{q}^\mathrm{H}\boldsymbol{H}_\mathrm{q} + \xi_\mathrm{q}\mathbf{1}_M\right)^{-1}\boldsymbol{H}_\mathrm{q}^\mathrm{H}, \tag{15}$$

where g is chosen in order to satisfy the power constraint $\mathrm{tr}\left(\boldsymbol{P}^\mathrm{H}\boldsymbol{P}\right) = P_\mathrm{DL}$ from (12), such that

$$g = \sqrt{\frac{\mathrm{tr}\left(\left((1-\kappa)\boldsymbol{H}_\mathrm{q}^\mathrm{H}\boldsymbol{H}_\mathrm{q} + \xi_\mathrm{q}\mathbf{1}_M\right)^{-2}\boldsymbol{H}_\mathrm{q}^\mathrm{H}\boldsymbol{H}_\mathrm{q}\right)}{P_\mathrm{DL}}}.$$

The constants κ and ξ_q depend on the quantization and estimation error and are defined as [3]

$$\kappa \;=\; \frac{\mathrm{E}\left[\tan^2\theta_k\right]}{M-1} \tag{16}$$

$$\xi_\mathrm{q} \;=\; K_\mathrm{s}\kappa + \frac{K_\mathrm{s}\mathrm{E}\left[\cos^{-2}\theta_k\right]}{(M-1)\left(1-\sigma_{e_\mathrm{DL}}^2\right)}\left(\sigma_{e_\mathrm{DL}}^2 + \frac{\sigma_\mathrm{n}^2}{P_\mathrm{DL}}\right), \tag{17}$$

where we can use the approximation $\mathrm{E}\left[\cos^{-2}\theta_k\right] \approx \frac{1}{\mathrm{E}\left[\cos^2\theta_k\right]}$ and where $\mathrm{E}\left[\cos^2\theta_k\right] = 1 - \mathrm{E}\left[\sin^2\theta_k\right]$, with [2]

$$\mathrm{E}\left[\sin^2\theta_k\right] = 2^B\,\mathrm{Beta}\left(2^B, \frac{M}{M-1}\right), \tag{18}$$

where $\mathrm{Beta}(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}\,\mathrm{d}y$ is the Beta function and $\mathrm{E}\left[\tan^2\theta_k\right] \approx \frac{\mathrm{E}\left[\sin^2\theta_k\right]}{\mathrm{E}\left[\cos^2\theta_k\right]}$. Both approximations represent lower bounds for very small B but become quite tight as B increases.

III. User Selection based on CDI Feedback

In the following we discuss a user selection scheme where $K_\mathrm{s} = M$ users are selected solely based on CDI feedback of the K users, i.e. no CQI is employed for the user selection at the base station. With K users and $K_\mathrm{s} = M$ selected users, the optimum set $\mathcal{S}$ has to be chosen among $\frac{K!}{M!(K-M)!}$ possible sets. Similarly as how $\boldsymbol{H}_\mathrm{q}$ from (14) for the optimum user set $\mathcal{S}$ was defined, let us define the matrix $\boldsymbol{H}_{\mathrm{q},\ell} \in \mathbb{C}^{M\times M}$, whose rows correspond to the CDIs of the $K_\mathrm{s} = M$ users from the ℓ-th possible set. Based on the quantized CDI of the users in ℓ-th set, i.e. $\boldsymbol{H}_{\mathrm{q},\ell}$, we can express the *sum mean square error* (SMSE) of the users in the ℓ-th user set as [11]

$$\mathrm{SMSE}_\ell = \frac{\left(-M\kappa + \xi_\mathrm{q}\mathrm{tr}\left(\left((1-\kappa)\,\boldsymbol{H}_{\mathrm{q},\ell}\boldsymbol{H}_{\mathrm{q},\ell}^\mathrm{H} + \xi_\mathrm{q}\mathbf{1}_M\right)^{-1}\right)\right)}{1-\kappa}. \tag{19}$$

Note that for every possible set ℓ, the entries on the diagonal of $\boldsymbol{H}_{\mathrm{q},\ell}\boldsymbol{H}_{\mathrm{q},\ell}^\mathrm{H}$ are always 1 since the rows of $\boldsymbol{H}_{\mathrm{q},\ell}$ correspond to users' CDIs, which by definition have unit norm. Hence with the user selection we can only influence the off-diagonal terms of $\boldsymbol{H}_{\mathrm{q},\ell}\boldsymbol{H}_{\mathrm{q},\ell}^\mathrm{H}$. If we are interested in selecting the optimum user set $\mathcal{S}$, which minimizes the SMSE based on the quantized CDI of the served users, we need to choose the users' CDIs which are most orthogonal to one another. Based on the CDI feedback of the users, this implies the best decision is to find the optimum set ℓ^* among all sets which most closely fulfills

$$\boldsymbol{H}_{\mathrm{q},\ell^*}\boldsymbol{H}_{\mathrm{q},\ell^*}^\mathrm{H} \approx \mathbf{1}_M. \tag{20}$$

where let us recall that we have defined the CDIs of the selected set of users in (14) and hence, we have that

$$\boldsymbol{H}_\mathrm{q} = \boldsymbol{H}_{\mathrm{q},\ell^*}.$$

The optimum set of users $\mathcal{S}$ corresponds to the set with the smallest SMSE among all $\frac{K!}{M!(K-M)!}$ possible sets. The optimum set can be obtained for instance via an exhaustive search among the SMSE of all possible sets.

Having found orthogonal quantized CDIs, i.e.,

$$\boldsymbol{H}_\mathrm{q}\boldsymbol{H}_\mathrm{q}^\mathrm{H} = \mathbf{1}_M \tag{21}$$

does not obviously imply that the actual channels of the users in the selected $\mathcal{S}$ are orthogonal, due to the quantization error! In fact as $K \to \infty$, such that the base station is able to find an optimum set of users with orthogonal CDIs such that (21) holds, the average sum rate of the selected users would nevertheless saturate since the average SINRs of the selected users cease to increase with K, i.e. multiuser diversity can no longer be exploited as in the case with perfect CSI. This indicates indeed that the user selection based solely on the CDI of the users does not benefit from multiuser diversity for large K. However, for finite K we can benefit from multiuser diversity by selecting $K_\mathrm{s} = M$ users whose quantized CDIs are the most orthogonal to each other, such that, for instance, the selected set of users achieves the smallest sum MSE (19) based on the quantized CDI feedback among all possible sets.

In order to observe the effect of the quantization, let us compute an approximation for the SINR as $K \to \infty$ without considering estimation errors. From (6), the downlink channel can be expressed without estimation errors as

$$\boldsymbol{h}_k = \|\boldsymbol{h}_k\|_2 \left(c_k \, \boldsymbol{h}_{k,\mathrm{q}} + \boldsymbol{e}_{k,\mathrm{q}}\right). \tag{22}$$

With $K_\mathrm{s} = M$ and assuming an optimum user set with M orthogonal CDIs, i.e. that (21) is fulfilled, implies that with the power constraint (12) the matrix (15) with the beamforming vectors of the selected set of users is given by $\boldsymbol{P} = \sqrt{\frac{P_\mathrm{DL}}{M}} \boldsymbol{H}_\mathrm{q}^\mathrm{H}$ and hence, the beamforming vectors of the selected set of users are

$$\boldsymbol{p}_{\pi(m)} = \sqrt{\frac{P_\mathrm{DL}}{M}} \, \boldsymbol{h}^*_{\pi(m),\mathrm{q}} \tag{23}$$

for $m = 1, \ldots, M$ and in addition since the selected set of CDIs are orthogonal

$$\boldsymbol{h}^\mathrm{T}_{\pi(m),\mathrm{q}}, \boldsymbol{p}_{\pi(i)} = 0 \quad \forall n \neq m.$$

For this case, the terms in the numerator and denominator of the SINR expression (13) are

$$\begin{aligned} \left|\boldsymbol{h}^\mathrm{T}_{\pi(m)} \boldsymbol{p}_{\pi(m)}\right|^2 &= \frac{P_\mathrm{DL}}{M} \|\boldsymbol{h}_{\pi(m)}\|_2^2 \, \cos^2\theta_{\pi(m)} \\ \left|\boldsymbol{h}^\mathrm{T}_{\pi(m)} \boldsymbol{p}_{\pi(n)}\right|^2 &= \frac{P_\mathrm{DL}}{M} \|\boldsymbol{h}_{\pi(m)}\|_2^2 \sin^2\theta_{\pi(m)} \left|\bar{\boldsymbol{e}}^\mathrm{T}_{\pi(m),\mathrm{q}} \boldsymbol{h}^*_{\pi(n),\mathrm{q}}\right|^2 \end{aligned}$$

for $n \neq m$, which result from (5), (22), (8) (9) and (23) with $\boldsymbol{e}_{\pi(m),\mathrm{q}} = \sin\theta_{\pi(m)} \; \bar{\boldsymbol{e}}_{\pi(m),\mathrm{q}}$ where $\|\bar{\boldsymbol{e}}_{\pi(m),\mathrm{q}}\|_2^2 = 1$. Based on these expressions, the SINR from (13) can be written as

$$\begin{aligned} \mathrm{SINR}_{\pi(m),\mathrm{sat}} &= \frac{\frac{P_\mathrm{DL}}{M} \|\boldsymbol{h}_{\pi(m)}\|_2^2 \cos^2\theta_{\pi(m)}}{\sigma_\mathrm{n}^2 + \frac{P_\mathrm{DL}}{M} \|\boldsymbol{h}_{\pi(m)}\|_2^2 \, \sin^2\theta_{\pi(m)} \sum\limits_{n=1, n\neq m}^{M} \left|\bar{\boldsymbol{e}}^\mathrm{T}_{\pi(m),\mathrm{q}} \boldsymbol{h}^*_{\pi(n),\mathrm{q}}\right|^2} \\ &\approx \frac{\frac{P_\mathrm{DL}}{M} \|\boldsymbol{h}_{\pi(m)}\|_2^2 \cos^2\theta_{\pi(m)}}{\sigma_\mathrm{n}^2 + \frac{P_\mathrm{DL}}{M} \|\boldsymbol{h}_{\pi(m)}\|_2^2 \, \sin^2\theta_{\pi(m)}} \end{aligned} \tag{24}$$

where under the assumption of orthogonal CDIs, $\left|\bar{\boldsymbol{e}}^\mathrm{T}_{\pi(m),\mathrm{q}} \boldsymbol{h}^*_{\pi(n),\mathrm{q}}\right|^2$ is the magnitude square of the product of two independent unit norm vectors, which lie in the $(M-1)$-dimensional nullspace of $\boldsymbol{h}^\mathrm{H}_{\pi(m),\mathrm{q}}$ and hence is a random variable distributed according to the beta distribution with parameters $(1, M-2)$ and independent of $\theta_\pi(m)$ [2, Lemma 2]. The approximation in the second step results from the fact that the mean of $\left|\bar{\boldsymbol{e}}^\mathrm{T}_{\pi(m),\mathrm{q}} \boldsymbol{h}^*_{\pi(n),\mathrm{q}}\right|^2$ is $\frac{1}{M-1}$ [12] and therefore, $\sum_{n=1,n\neq m}^{M} \left|\bar{\boldsymbol{e}}^\mathrm{T}_{\pi(m),\mathrm{q}} \boldsymbol{h}^*_{\pi(n),\mathrm{q}}\right|^2 \approx (M-1) \cdot \frac{1}{M-1}$.

As $K \to \infty$, such that the base station is always able to select an orthogonal set of CDIs, the SINR of the selected users will be given by the previous expression. Since the user selection is not based on the CMI $\|\boldsymbol{h}_{\pi(m)}\|_2$ or on the quantization error $\theta_\pi(m)$, the average SINR of the selected users will not increase with K once the base station has been able to find an orthogonal set of CDIs. This means that the sum rate saturates for large values of K to the following rate

$$\begin{aligned} R_\mathrm{sat} &= \mathrm{E}_{\boldsymbol{H}} \left[\sum_{m=1}^{M} \log_2\left(1 + \mathrm{SINR}_{\pi(m),\mathrm{sat}}\right)\right] \\ &= M \; \mathrm{E}_{\boldsymbol{H}} \left[\log_2\left(1 + \mathrm{SINR}_{\pi(m),\mathrm{sat}}\right)\right], \end{aligned} \tag{25}$$

where $\mathrm{E}_{\boldsymbol{H}}[\bullet]$ is the expectation over the channels of the selected users and the second step follows from the fact that the users are i.i.d. Thus, no multiuser diversity can be exploited for large values of K. This saturation could be avoided if the users report a CQI including the CMI and the quantization error [4]. However, the quantization of the CQI has to be taken into account. Due to simplicity and due to the fact that CQI feedback is only beneficial for very large values of K, we focus only on the CDI feedback with which multiuser diversity can be exploited for up to moderate values of K.

IV. Lower Bound on Sum Rate with User Selection

As a figure of merit of the FDD downlink system under consideration, we consider the average sum rate with Gaussian signalling. With $\epsilon_{\pi(m)}$ as the *mean square error* (MSE) of the m-th selected user based on the quantized CDI of the selected users, i.e. $\boldsymbol{H}_\mathrm{q}$, we can express the *sum mean square error* (SMSE) of the selected users (19) as

$$\sum_{m=1}^{M} \epsilon_{\pi(m)} = \frac{\left(-M\kappa + \mathrm{tr}\left(\left(\gamma \boldsymbol{H}_\mathrm{q} \boldsymbol{H}_\mathrm{q}^\mathrm{H} + \mathbf{1}_M\right)^{-1}\right)\right)}{1-\kappa} \tag{26}$$

where $\gamma = \frac{1-\kappa}{\xi_\mathrm{q}}$. The SMSE of the selected set of users defined in (26) can be employed to obtain a lower bound on the sum rate R with limited feedback MMSE beamforming and user selection with quantized CDI of the K users as transmit CSI available at the base station

$$\begin{aligned} R &\overset{(a)}{\geq} \mathrm{E}_{\boldsymbol{H}} \left[\sum_{m=1}^{M} \log_2\left(1 + \mathrm{SINR}_{\pi(m)}\right)\right] && (27) \\ &\overset{(b)}{=} -\mathrm{E}_{\boldsymbol{H}} \left[\sum_{m=1}^{M} \log_2 \epsilon'_{\pi(m)}\right] \\ &\overset{(c)}{\geq} -\sum_{m=1}^{M} \log_2 \mathrm{E}_{\boldsymbol{H}} \left[\epsilon'_{\pi(m)}\right] \\ &\overset{(d)}{\geq} -\sum_{m=1}^{M} \log_2 \mathrm{E}_{\boldsymbol{H}_\mathrm{q}} \left[\epsilon_{\pi(m)}\right] && (28) \\ &\overset{(e)}{=} -M \log_2 \frac{\mathrm{E}_{\boldsymbol{H}_\mathrm{q}} \left[\sum_{m=1}^{M} \epsilon_{\pi(m)}\right]}{M} && (29) \end{aligned}$$

where step (a) follows by asuming Gaussian signalling which does not need to be capacity achieving for this case. In addition we assume the MMSE beamforming scheme discussed before. Step (b) results assuming the selected users employ an MMSE receiver with the resulting MSE for the m-th selected user defined as $\epsilon'_{\pi(m)}$. Jensen's inequality is applied in step (c). Let us define $\mathrm{E}_{\boldsymbol{H}}[\bullet]$ as the expectation over the CDIs of the selected users. Step (d) results by taking the expectation over all the random variables except the quantized CDIs. Additionally we employ the fact that $\mathrm{E}_{\boldsymbol{H}}\left[\epsilon'_{\pi(m)}\right] \leq \mathrm{E}_{\boldsymbol{H}_\mathrm{q}}\left[\epsilon_{\pi(m)}\right]$, since the MSE $\epsilon'_{\pi(m)}$ is a convex function of the selected users' channels. In step (e), we use the fact that the users are i.i.d. and hence the average SMSE can be written as

$$\mathrm{E}_{\boldsymbol{H}_\mathrm{q}} \left[\sum_{m=1}^{M} \epsilon_{\pi(m)}\right] = M \; \mathrm{E}_{\boldsymbol{H}_\mathrm{q}} \left[\epsilon_{\pi(m)}\right].$$

In order to provide a closed-form for the lower bound on the sum rate with user selection given in (29), we require the expectation of the SMSE (26) over the quantized CDI $\boldsymbol{H}_{\text{q}}$ of the selected users, which implies we need $\mathrm{E}_{\boldsymbol{H}_{\text{q}}}\left[\mathrm{tr}\left(\left(\gamma \boldsymbol{H}_{\text{q}}\boldsymbol{H}_{\text{q}}^{\text{H}}+\mathbf{1}_M\right)^{-1}\right)\right]$ which can be approximated as $\mathrm{E}_{\boldsymbol{H}_{\text{q}}}\left[\mathrm{tr}\left(\left(\gamma \boldsymbol{H}_{\text{q}}\boldsymbol{H}_{\text{q}}^{\text{H}}+\mathbf{1}_M\right)^{-1}\right)\right]$

$$\begin{aligned}
&\overset{\text{(a)}}{\approx} \mathrm{E}_{\boldsymbol{H}_{\text{q,all}}}\left[\mathrm{tr}\left(\left(\gamma \frac{M}{K}\boldsymbol{H}_{\text{q,all}}^{\text{H}}\boldsymbol{H}_{\text{q,all}}+\mathbf{1}_M\right)^{-1}\right)\right]\\
&\overset{\text{(b)}}{=} \mathrm{E}_{\boldsymbol{H}_{\text{q}}'}\left[\mathrm{tr}\left(\left(\gamma \boldsymbol{H}_{\text{q}}^{',\text{H}}\boldsymbol{H}_{\text{q}}'+\mathbf{1}_M\right)^{-1}\right)\right]\\
&\overset{\text{(c)}}{\approx} M\left(1-\frac{\left(\sqrt{\gamma\left(1+\sqrt{\mu}\right)^2+1}-\sqrt{\gamma\left(1-\sqrt{\mu}\right)^2+1}\right)^2}{4\mu\gamma}\right)
\end{aligned} \tag{30}$$

where step (a) follows by denoting the matrix

$$\boldsymbol{H}_{\text{q,all}}=\left[\boldsymbol{h}_{\text{q},1},\ldots,\boldsymbol{h}_{\text{q},K}\right]^{\text{T}}\in\mathbb{C}^{K\times M},$$

which contains the quantized CDI of *all* the K users and from the following argument. Consider the case when $K\rightarrow\infty$, such that the optimum user selection picks out M users whose CDIs are all perfectly orthogonal to one another, i.e. $\boldsymbol{H}_{\text{q}}\boldsymbol{H}_{\text{q}}^{\text{H}}=\mathbf{1}_M$ since the rows of $\boldsymbol{H}_{\text{q}}$ contain unit norm vectors (quantized CDIs). In this case we can show that

$$\frac{1}{M}\lambda_i\left(\boldsymbol{H}_{\text{q}}\boldsymbol{H}_{\text{q}}^{\text{H}}\right)\approx\frac{1}{K}\lambda_i\left(\boldsymbol{H}_{\text{q,all}}^{\text{H}}\boldsymbol{H}_{\text{q,all}}\right) \tag{31}$$

holds, where $\lambda_i\left(\boldsymbol{A}\right)$ denotes the i-th ordered eigenvalue of $\boldsymbol{A}$. For $K\rightarrow\infty$, the M eigenvalues of $\boldsymbol{H}_{\text{q}}\boldsymbol{H}_{\text{q}}^{\text{H}}\in\mathbb{C}^{M\times M}$ are equal to one and the M eigenvalues of $\boldsymbol{H}_{\text{q,all}}^{\text{H}}\boldsymbol{H}_{\text{q,all}}\in\mathbb{C}^{M\times M}$ are all equal to $\frac{K}{M}$, since

$$\mathrm{tr}\left(\boldsymbol{H}_{\text{q,all}}^{\text{H}}\boldsymbol{H}_{\text{q,all}}\right)=\mathrm{tr}\left(\boldsymbol{H}_{\text{q,all}}\boldsymbol{H}_{\text{q,all}}^{\text{H}}\right)=K,$$

since all the diagonal elements of the matrix $\boldsymbol{H}_{\text{q,all}}\boldsymbol{H}_{\text{q,all}}^{\text{H}}$ are equal to one, since the rows of $\boldsymbol{H}_{\text{q,all}}$ correspond to CDIs which by definition have unit norm. Albeit (31) holds strictly only when $K\rightarrow\infty$, simulation results indicate that it is still indeed a good approximation for finite values of K such that the approximation of the expected value given in step (a) of the derivation of (30) is also still very good. In step (b) we have simply employed the substitution $\boldsymbol{H}_{\text{q}}'=\sqrt{\frac{M}{K}}\boldsymbol{H}_{\text{q,all}}\in\mathbb{C}^{K\times M}$, such that the elements of $\boldsymbol{H}_{\text{q}}'$ have zero mean and variance $\frac{1}{K}$ since the variance of the elements of $\boldsymbol{H}_{\text{q,all}}$ is $\frac{1}{M}$ due to the fact that the rows have unit norm, i.e. $\|\boldsymbol{h}_{\text{q},k}\|_2^2=1\ \forall\, k$.

The final approximation (c) in the derivation of (30) follows from applying with $\mu=\frac{M}{K}$ a central result in random matrix theory [14, Th. 2.39], that states that when the entries of $\boldsymbol{H}_{\text{q}}'$ are zero-mean i.i.d. with variance $\frac{1}{K}$, the empirical distribution of the eigenvalues of $\boldsymbol{H}_{\text{q}}'\boldsymbol{H}_{\text{q}}^{',\text{H}}$ converges almost surely, as $K,M\rightarrow\infty$ with $\frac{M}{K}=\mu$, to the Marčenko-Pastur law [14, (1.16)]. The approximation given in [14] uses the concepts of large system analysis and the fact that if $K,M\rightarrow\infty$ with $\frac{M}{K}=\mu$, the spread of the eigenvalues of $\boldsymbol{H}_{\text{q}}\boldsymbol{H}_{\text{q}}^{\text{H}}$ decreases with increasing K and M. Let us recall that $\boldsymbol{H}_{\text{q}}$ includes the CDIs of the optimum selected users (c.f. (14)), which best fulfill (20). As $K,M\rightarrow\infty$ with $\frac{M}{K}=\mu$, the eigenvalues converge to deterministic values. Although the elements of $\boldsymbol{H}_{\text{q}}$ are not completely independent because of the fact that the rows of $\boldsymbol{H}_{\text{q}}$ have unit norm, (c) it is still very good approximation for small K and M as verified by simulation results in the following. With (30) and also (26) and (29), we have a closed-form approximation for the lower bound on the sum rate with user selection, which is given by

$$R_{\text{lb}}\approx -M\log_2\left(1-\frac{\left(\sqrt{\gamma\left(1+\sqrt{\mu}\right)^2+1}-\sqrt{\gamma\left(1-\sqrt{\mu}\right)^2+1}\right)^2}{4\mu\gamma(1-\kappa)}\right) \tag{32}$$

In order to verify the previous approximation for the lower bound of the sum rate with MMSE beamforming we depict in Figure 1, the average sum rate given by (27), the lower bound on the average sum rate given by (28) and the approximation of the lower bound R_{lb} given in (32). To this end, we have assumed that the user selection is performed via an exhaustive search in order to find the optimum user set which corresponds to the set with the smallest SMSE based on the quantized CDI. For Figure 1 we assume $M=4$ antennas, $B=12$ feedback bits per user, $K_{\text{s}}=M$ users served with the MMSE beamforming scheme based on CDI feedback presented in Section II-A. The SNR $\frac{P_{\text{DL}}}{\sigma_{\text{n}}^2}=10$ and no estimation errors are considered. In addition, we include the upper bound on the average sum rate (25), to which the average sum rate (27) converges as $K\rightarrow\infty$. The average values required in (27), (28) and (25) are computed via Monte Carlo simulations. From the figure, we can observe that (32) is a very good approximation for the lower bound on the average sum rate (28), except for small values of K. Therefore we state that the average sum rate R with limited feedback beamforming and user selection is bounded by

$$R_{\text{lb}}\leq R\leq R_{\text{ub}}. \tag{33}$$

V. System Limited Feedback

As discussed in Section II, we assume a system limited feedback phase consisting of T_{F} channel uses, which are reserved for the feedback of all K users. As shown before, in this case each user can feedback

$$B=2\left\lfloor\frac{T_{\text{F}}M}{K}\right\rfloor \tag{34}$$

feedback bits. In this work, we are interested in finding the optimal number of users $K_{\text{F,opt}}$ which should feed back in order to maximize a average sum rate with CDI-based limited feedback and user selection. From the expression in (34), we can observe a tradeoff between multiuser diversity and the accuracy of the quantization. During the available T_{F} channel uses of the feedback phase at most $K=MT_{\text{F}}$ users can feed back $B=2$ bits or at least $K=M$ users can feed back $B=2T_{\text{F}}$ bits, i.e.

$$M\leq K_{\text{F,opt}}\leq MT_{\text{F}}. \tag{35}$$

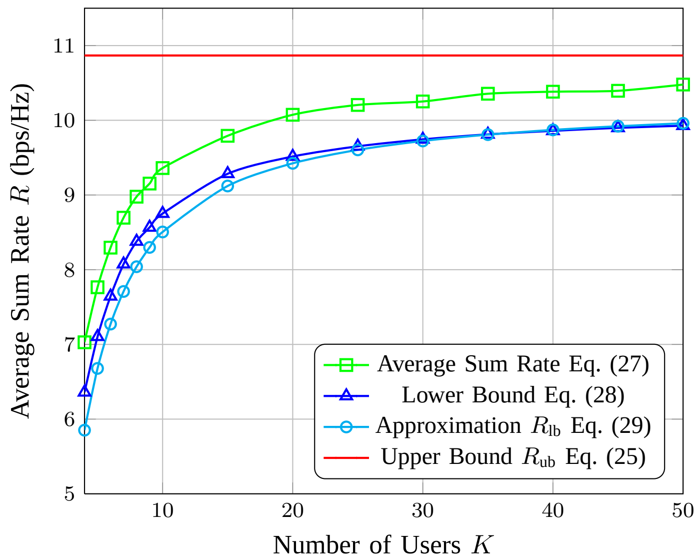

Figure 1. Average Sum Rate vs. Number of Users

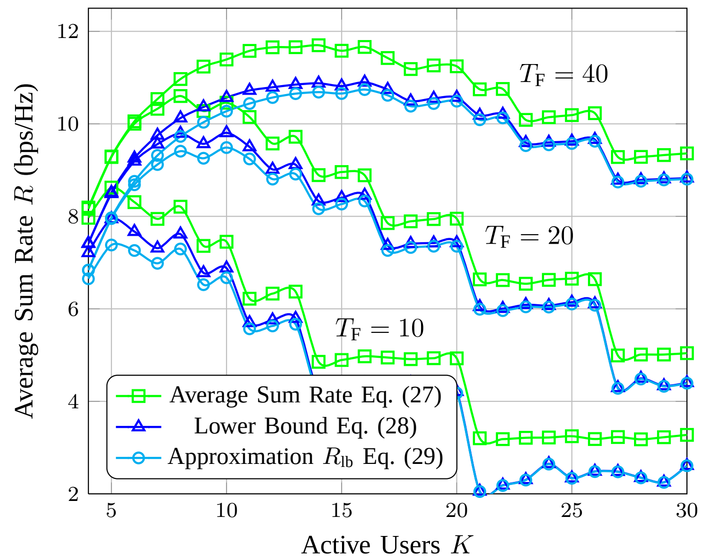

Figure 2. Tradeoff between Multiuser Diversity and CDI Quality

In order to observe the tradeoff between potentially higher multiuser diversity (many users feeding back coarsely quantized CDIs) and better CDI quality (few users feeding back quantized CDIs with smaller quantization error), we depict in Figure 2 the average sum rate (27), the lower bound on the average sum rate (28) and the approximation R_{lb} from (32) as a function of the number of users K that could feedback. For the figure, we assume $M = 4$, $K_{\text{s}} = M$, $\frac{P_{\text{DL}}}{\sigma_{\text{n}}^2} = 10$ and no estimation errors are considered. The curves are plotted for different lengths of the system feedback phase: $T_{\text{F}} = \{10, 20, 40\}$ channel uses. For each K, the number of feedback bits per user is given by (34). The sudden jumps in the curves is a result of the fact that the number of feedback bits B can only take even values and that not at every instance during the feedback phase are M users relaying their feedback to the base station. This is because we assume the same amount of limited feedback per user. We can observe that for each T_{F} the maximum of each of the three curves is achieved with approximately the same value of K. Hence, this implies that we can employ the approximation of the lower bound R_{lb} from (32) in order to find the optimum number of feedback users $K_{\text{F,opt}}$, without needing to perform exhaustive Monte Carlo simulations to determine the average sum rate (27).

It is clear that the optimal number of feedback users $K_{\text{F,opt}}$ which maximize the sum rate with user selection increases with T_{F}. This can be seen in Figure 3, where we have plotted the optimum number of feedback users based on the average sum rate (27), the lower bound on the average sum rate (28) and the approximation R_{lb} (32) as a function of T_{F} for the same scenario as in the previous figure. We can see that the optimum number of feedback users for the average sum rate is almost the same as the optimum number of feedback users based on the approximation of the lower bound. The optimum number of feedback users $K_{\text{F,opt}}$ is able to increase linearly with T_{F}. The number of feedback bits with which the optimum number of feedback users should relay their quantized CDI is given by (34) and is depicted in Figure 4.

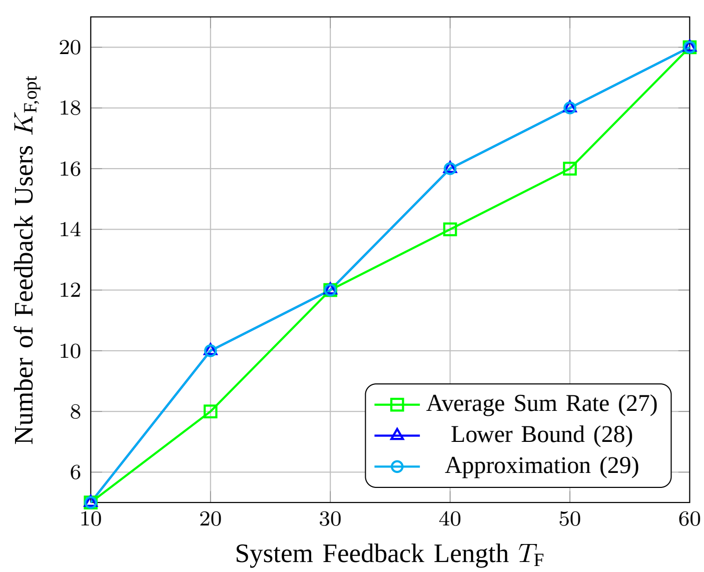

Figure 3. Optimum Number of Feedback Users

The sum rate as a function of T_{F} achieved with the optimum number of feedback users $K_{\text{F,opt}}$ feeding back their CDI with the corresponding number of feedback bits is depicted in Figure 5. We can observe that the average sum rate increases with T_{F}.

VI. Summary and Conclusion

In this paper, we have derived an approximation for the lower bound of the sum rate with limited feedback beamforming and user selection based solely on the quantized CDI of the users. Given a constraint on the system feedback, we have a tradeoff between the CDI quantization quality and the mulituser diversity. We have shown that the derived approximation can be employed to find the optimum number

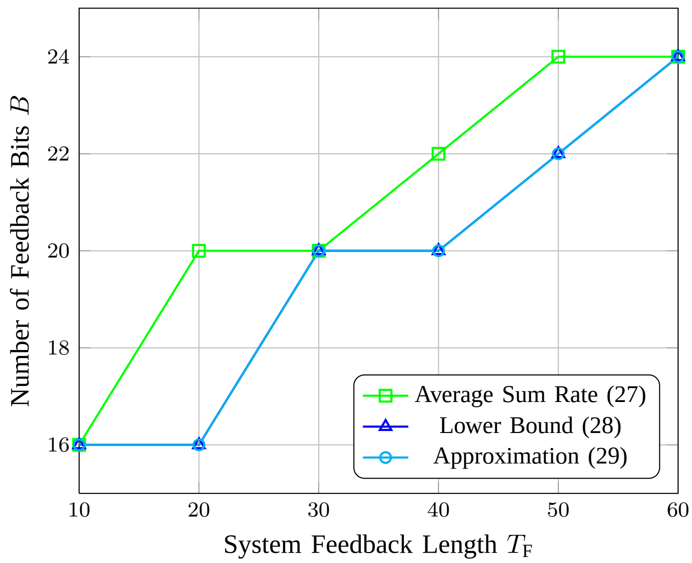

Figure 4. Optimum Number of Feedback Bits B per User

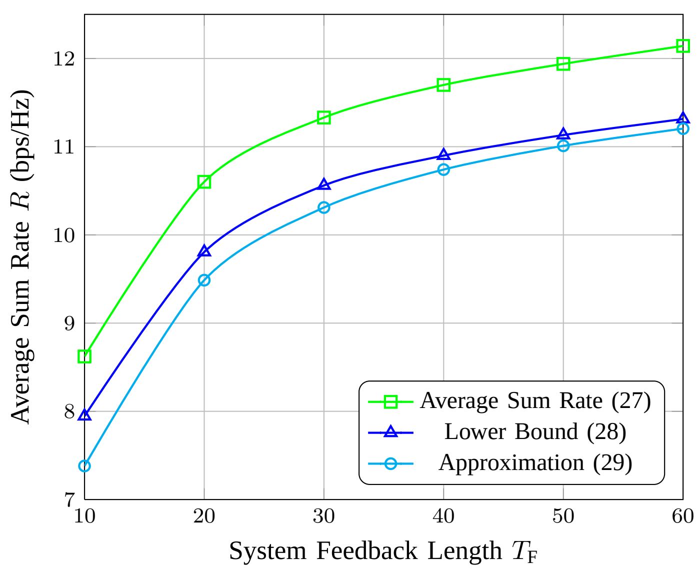

Figure 5. Average Sum Rate vs. T_{F}

of feedback users, i.e. to find the optimum operating point considering the tradeoff. Although it was shown that with the presented user selection scheme based solely on quantized CDI feedback, multiuser diversity is not able to be exploited for large values of K, this happens for very large values of K. In addition, the notion that the sum rate with perfect CSI grows unbonded as a function of the number of users K is a consequence of the assumed channel model, since in practice the sum rate would saturate as $K \rightarrow \infty$ even with perfect CSI. Hence, a user selection scheme based solely on the CDI feedback is still able to benefit from multiuser diversity.

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