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Lower Bounds on the Infima in Some $\mathcal{H}_\infty$ Optimization Problems

Sander Wahls*, Member, IEEE*, and Holger Boche*, Fellow, IEEE*

***Abstract*—We consider three $\mathcal{H}_\infty$ optimization problems: system inversion, model matching with probably unstable plant, and full information control. A common theme in numerical solution of these problems is that the infimal $\mathcal{H}_\infty$ performance among solutions has to be approximated before a suboptimal solution can be found. Recently, a sequence of lower bounds that converges monotonously towards the exact infimum has been established for the system inversion problem. The goal of this technical note is twofold. First, we show that except for some rare cases these lower bounds converge at least root-exponentially fast, i.e., we establish the good-naturedness of this approximation method. Second, we show how the approximation method can be extended such that also arbitrarily good lower bounds on the infima in the model matching problem and the full information control problem can be obtained.**

***Index Terms*—Finite sections, full information control, gap metric, infima, model matching, system inversion, Toeplitz operators.**

NOMENCLATURE

$\mathbb{N}$: natural numbers (including zero); $\mathbb{Z}$: integers; $\mathbb{R}$: real numbers; $\mathbb{C}$: complex numbers; $i := \sqrt{-1}$; $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$; $\Re(\cdot)$: real part; $\mathbb{C}^{q\times p}$: complex $q \times p$ matrices; $\mathbb{C}^q := \mathbb{C}^{q\times 1}$; $0 \leq \sigma_{\min}(A) = \sigma_1(A) \leq \ldots \leq \sigma_{\min(q,p)}(A) = \sigma_{\max}(A)$: singular values of $A \in \mathbb{C}^{q\times p}$; $\|A\| := \sigma_{\max}(A)$: spectral norm; $\det(A)$: determinant; A^T: transpose; A^*: conjugate transpose; $\mathcal{R}^{q\times p}$: rational $q \times p$ matrices; $\|F\|_\infty := \operatorname{esssup}_{|z|=1}\|F(z)\|$: infinity norm of $F \in \mathcal{R}^{q\times p}$; $\operatorname{esssup}_{z\in\mathbb{C}}\operatorname{rank}[F(z)]$: normal rank; $F^\sim(z) := F(1/z^*)^*$: para-hermitian; $F^\dagger := F^\sim(FF^\sim)^{-1}$: para-pseudoinverse (requires full row normal rank); $\mathcal{RH}_\infty^{q\times p}$: rational $q \times p$ matrices without poles in $1 \leq |z| \leq \infty$; $\dim(\mathcal{V})$: dimension of a vector space $\mathcal{V}$; $(\ell_2^q, \|\cdot\|_{\ell_2^q})$: Banach space of vector sequences $x = \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^q$ with $\|x\|_{\ell_2^q}^2 := \sum_{k=0}^\infty x_k^* x_k < \infty$; $(\mathcal{L}(\ell_2^p, \ell_2^q), \|\cdot\|_{op})$: Banach space of linear operators $T : \ell_2^p \to \ell_2^q$ with finite operator norm $\|T\|_{op} := \sup_{x\in\ell_2^p, \|x\|_{\ell_2^p}=1} \|Tx\|_{\ell_2^q}$; T^+: Moore-Penrose pseudoinverse of $T \in \mathcal{L}(\ell_2^p, \ell_2^q)$; $\ker(T)$: kernel/null space; $O(\cdot)$: Landau symbol;

I. INTRODUCTION

We consider three common $\mathcal{H}_\infty$ optimization problems: system inversion, model matching with unstable plant, and full information control. Numerical solutions for these problems are well-known. See, e.g., [1]–[6]. All numerical approaches have in common that only suboptimal solutions are computed, i.e., given some $\gamma > 0$, a solution with performance measure lower than γ is sought. Of course, γ should be chosen small but still admissible (i.e., the solution set has to be nonempty). The usual way to obtain a good γ are so-called γ-iterations.

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The authors are with the Technische Universität München, Lehrstuhl für Theoretische Informationstechnik, München 80333, Germany (e-mail: sander.wahls@tum.de; boche@tum.de).

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There, a test to check whether some γ is admissible or not is designed, and then bisection is used to get an approximation of the infimum over all admissible γ, say, γ_{opt}. Of course, initial lower and upper bounds on γ_{opt} are required to start the bisection procedure. Usually, the lower bound is simply chosen to be zero. However, improved lower bounds can help to increase precision or to decrease the necessary number of iterations in the γ-iteration, respectively. The goal of this technical note is to derive such bounds. Our contributions are the following.

We first analyze a recently proposed approximation method for the infimum in the system inversion problem [7]. The method consists in a sequence of finite eigenvalue problems of increasing dimension, where each eigenvalue problem gives a lower bound on the exact infimum. It was found in [7] that the lower bounds converge monotonously towards the exact infimum. However, the usefulness of these bounds depends crucially on the speed of convergence. The first contribution of this technical note is to provide estimates on the speed with which the bounds converge. We show that the convergence is at least root-exponential in most cases. The second contribution is to show that the infima in the model matching and full information control problems can be reduced to infima in system inversion problems, i.e., we extend both the approximation method for the infimum and the results on its speed of convergence to these two problems.

The technical note is structured as follows. In Section II we review some results on the finite section method and on the sequence of lower bounds in the system inversion problem that has been recently established in [7]. Then, we derive our estimates on the speed of convergence of these lower bounds in Section III. Connections to the infima in model matching and full information control are given in the Sections IV and V. Finally, we give a numerical example in Section VI and close the technical note with a conclusion (Section VII).

We finally note that all problems and results in this technical note are stated in discrete-time. Continuous-time problems can be treated by using the well-known bilinear transform approach [6].

II. Preliminaries

A. Finite Sections of Block Toeplitz Operators

In this subsection, we recall a recent result on the finite sections of block Toeplitz operators. Let $a : \mathbb{T} \to \mathbb{C}^{q\times p}$ denote a continuous function with Fourier coefficients $\{a_k\}_{k\in\mathbb{Z}}$, i.e., $a_k := \int_{\theta=0}^{2\pi} a(e^{i\theta})e^{-ik\theta}(d\theta/2\pi)$. Then, $\mathfrak{T}_a \in \mathcal{L}(\ell_2^p, \ell_2^q)$ denotes the block Toeplitz operator with symbol a. This operator maps any $x = \{x_k\}_{k\in\mathbb{N}} \in \ell_2^p$ to the output $y = \{y_k\}_{k\in\mathbb{N}} = \mathfrak{T}_a x \in \ell_2^q$ given by the infinite matrix equation

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \ddots \\ a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}.$$

The Nth finite section $(N \geq 1)$ of the Toeplitz operator $\mathfrak{T}_a$ is the finite block Toeplitz matrix

$$\Gamma_N^a := \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{N-1} & \cdots & a_1 & a_0 \end{bmatrix} \in \mathbb{C}^{Nq\times Np}. \quad (1)$$

The following recent result gives a lower bound on the speed of convergence with which a certain singular value of the finite section Γ_N^a converges if a is square and N grows to infinity.

Theorem 1 ([8, Th. 7.3+p. 274]).: Let $a : \mathbb{T} \to \mathbb{C}^{q\times q}$ be rational such that $\det[a(e^{i\theta})] \neq 0$ for all $0 \leq \theta < 2\pi$, and let $k_0 := \dim\ker(\mathfrak{T}_a^+) + \dim\ker(\mathfrak{T}_{a\sim}^+)$. Then, there is a $\delta > 0$ such that $d := \min\{\|\mathfrak{T}_a^+\|_{op}^{-1}, \|\mathfrak{T}_{a\sim}^+\|_{op}^{-1}\}$ satisfies

$$|d - \sigma_{k_0+1}(\Gamma_N^a)| = \begin{cases} O(e^{-\delta\sqrt{N}}), & if\ d < \|\mathfrak{T}_{a^{-1}}\|_{op}^{-1} \\ O\left(\frac{\ln N}{N}\right), & otherwise. \end{cases} \quad (2)$$

It is currently unknown if the estimates in Theorem 1 are tight [9, p. 151]. However, note that there are cases where the convergence in (2) is only quadratic. See, e.g., Example 6.16 in [9].

B. Finite Section Method for the Infimum in System Inversion

In this subsection, we recall a recent result on the infimum in the system inversion problem. The system inversion problem is the following. Given some $F \in \mathcal{RH}_\infty^{q\times p}$ we want to find a right inverse $G \in \mathcal{RH}_\infty^{p\times q}$, i.e., $FG = I$, with (close to) infimal norm. The numerical solution of this problem has been established in [1], [2]. In both papers, it consists of two steps where the first step is the (approximate) solution of the following problem.

Problem 1 (Infimum in $\mathcal{H}_\infty$ System Inversion): What is the value of

$$\gamma_{opt}^{SI}(F) := \inf\left(\{\|G\|_\infty : G \in \mathcal{RH}_\infty^{p\times q}, FG = I\} \cup \{\infty\}\right)?$$

The following recent result shows how the finite section method, which was introduced in the previous subsection, can be used to obtain arbitrarily good lower bounds on γ_{opt}^{SI} in Problem 1.

Theorem 2 ([7, Th. 7]+[2, Th. 5.2]).: Let $F \in \mathcal{RH}_\infty^{q\times p}$, $q \leq p$. Then, the reciprocals of the smallest singular values of the finite sections Γ_N^F converge monotonically increasing towards the solution of Problem 1 as N grows to infinity, i.e.

$$\frac{1}{\sigma_{\min}(\Gamma_1^F)} \leq \frac{1}{\sigma_{\min}(\Gamma_2^F)} \leq \cdots \leq \frac{1}{\sigma_{\min}(\Gamma_N^F)} \to \gamma_{opt}^{SI}(F)[N \to \infty]. \quad (3)$$

Let us give some remarks on the actual numerical computation of the bounds.

Remark 1 (Fourier Coefficients): Note that if $F \in \mathcal{RH}_\infty^{q\times p}$ and $F(z) = D + C(zI - A)^{-1}B$ is a state space realization (i.e., A, B, C, D are constant matrices of suitable dimensions) then the Fourier coefficients of F are $F_{-k} = 0_{q\times p}$, $F_0 = D$ and $F_k = CA^{k-1}B$, $k \geq 1$ [10, §2.7].

Remark 2 (Numerical Accuracy): Suppose that some singular value decomposition (SVD) algorithm has been used to compute approximations $\hat\sigma_{\min}(\Gamma_N^F)$ of the $\sigma_{\min}(\Gamma_N^F)$ such that the approximation errors $|\hat\sigma_{\min}(\Gamma_N^F) - \sigma_{\min}(\Gamma_N^F)|$ are upper bounded by $\epsilon_{fp}\sigma_{\max}(\Gamma_N^F)$, where ϵ_{fp} denotes the machine precision. (This is the usual behavior [11].) Also assume that the condition number $\mathrm{cond}(F) := \gamma_{opt}^{SI}(F)\|F\|_\infty$ is finite and $\mathrm{cond}(F) < \epsilon_{fp}^{-1}$. Then, one can show that the asymptotic approximation error $\limsup_{N\to\infty} |\hat\sigma_{\min}^{-1}(\Gamma_N^F) - \gamma_{opt}^{SI}(F)|$ is upper bounded by $\epsilon_{fp}\gamma_{opt}^{SI}(F)\cdot\mathrm{cond}(F)/(1 - \epsilon_{fp}\mathrm{cond}(F))$. We note that the accuracy of our method may be further improved if a Jacobi-type SVD algorithm is used [11], [12].

Remark 3 (Gap Metric): Problem 1 arises in the computation of the so-called gap metric [13, Rem. 1], which measures how close a feedback loop is to loosing stability. Various authors have researched γ-iterations that solve this problem. See, e.g., [13], [14] and the references therein.

III. Speed of Convergence of the Finite Section Method for the Infimum in System Inversion

In this section, we analyze the approximation method for the system inversion infimum γ_{opt}^{SI} that was given in Theorem 2. The method has several nice properties:

1) The approximations are simple to compute.
2) Each approximation is a lower bound on the exact value.
3) The lower bounds are monotonously increasing in the size of the finite sections.

4) Asymptotically, the approximations are of high numerical precision if the inversion problem itself is not ill-conditioned (i.e., $\gamma_{opt}^{SI}(F)\|F\|_\infty$ is not too large; cf. Remark 2).

However, in the end usefulness of the method crucially depends on the speed of convergence in (3) because slow convergence would make good approximations of the infimum too expensive. The goal of this section is to provide estimates on the speed of convergence. We prove that except for some rare cases convergence is at least root-exponential. In other words, we establish the fact that the approximation method in Theorem 2 is good-natured.

Our main tool in the derivation of the convergence order will be Theorem 1. Two major difficulties prevent direct application of Theorem 1. The first difficulty is that F is not square as required by Theorem 1, but this can be mitigated by considering $a = FF^\sim$ instead of F. The second and more severe difficulty is, to quote [8, p. 292], that "the condition $d < \|\mathfrak{T}_{a^{-1}}\|_{op}^{-1}$ is difficult to verify".[1] Therefore, we have to analyze when this condition is true. The following lemma will be the key to this analysis.

Lemma 1: Let $F \in \mathcal{RH}_\infty^{q\times p}$ have full row normal rank and Fourier coefficients $\{F_k\}_{k\in\mathbb{Z}}$. Furthermore, introduce the input observability matrix $\mathcal{M}_F := [F_0^* \; F_1^* \; \ldots \; F_\kappa^*]^* \in \mathbb{C}^{(\kappa+1)q\times p}$, where κ denotes the McMillan degree of F. Then, the para-pseudoinverse $F^\dagger = F^\sim(FF^\sim)^{-1}$ satisfies $F^\dagger \in \mathcal{RH}_\infty^{p\times q}$ if and only if $\gamma_{opt}^{SI}(F) < \infty$ and $\mathrm{rank}(\mathcal{M}_F) \le q$.

Proof: "$\Rightarrow$": Suppose $F^\dagger \in \mathcal{RH}_\infty^{p\times q}$. Obviously then $\gamma_{opt}^{SI}(F) < \infty$. Now, let $F = \mathcal{O}\mathcal{I}_1$ denote a reduced outer-coinner factorization, i.e., $\mathcal{I}_1 \in \mathcal{RH}_\infty^{q\times p}$, $\mathcal{O}, \mathcal{O}^{-1} \in \mathcal{RH}_\infty^{q\times q}$ and $\mathcal{I}_1\mathcal{I}_1^\sim = I$ [15]. Then, $F^\dagger = \mathcal{I}_1^\sim\mathcal{O}^{-1}$, which shows that $\mathcal{I}_1^\sim = F^\dagger\mathcal{O} \in \mathcal{RH}_\infty^{p\times q}$. Since also $\mathcal{I}_1 \in \mathcal{RH}_\infty^{q\times p}$, we see that the inner factor $\mathcal{I}_1$ has to be a constant matrix. Let us set $U_1 := \mathcal{I}_1$ and choose another scalar matrix U_2 such that $U := [U_1^* \; U_2^*]^* \in \mathbb{C}^{p\times p}$ is unitary. Then, we have $F = [\mathcal{O} \; 0_{q\times(p-q)}]U$ and thus $F_kU_2^* = 0_{q\times(p-q)}$ for all $k \in \mathbb{N}$ by Parseval's relation. This shows

$$\begin{aligned}\mathrm{rank}(\mathcal{M}_F) &= \mathrm{rank}\,(\mathcal{M}_F^*) = \mathrm{rank}\,(U\mathcal{M}_F^*)\\ &= \mathrm{rank}\left(\begin{bmatrix} U_1F_0^* & \ldots & U_1F_\kappa^* \\ 0_{(p-q)\times q} & \ldots & 0_{(p-q)\times q}\end{bmatrix}\right) \le q.\end{aligned}$$

"$\Leftarrow$": On the other hand, suppose that $\gamma_{opt}^{SI}(F) < \infty$ and $\mathrm{rank}(\mathcal{M}_F) \le q$. Bring in a minimal state-space realization $F(z) = D + C(zI - A)^{-1}B$, where $A \in \mathbb{C}^{\kappa\times\kappa}$. The Cayley-Hamilton Theorem (cf., e.g., [10, Th. 2.1]) lets us write $A^\kappa = \alpha_0 I + \alpha_1 A + \ldots + \alpha_{\kappa-1}A^{\kappa-1}$ for some scalars $\alpha_0, \ldots, \alpha_{\kappa-1} \in \mathbb{C}$. Thus, $F_{\kappa+1} = CA^\kappa B = \alpha_0 CB + \alpha_1 CAB + \ldots + \alpha_{\kappa-1}CA^{\kappa-1}B = \sum_{k=1}^{\kappa} \alpha_k F_k$ is a linear combination of the $F_1, \ldots, F_\kappa$. One can iterate this argument, which shows that in fact all F_k with $k > \kappa$ are linear combinations of the $F_1, \ldots, F_\kappa$. Thus

$$\begin{aligned}\mathrm{rank}\,([F_0^* \quad F_1^* \quad F_2^* \quad \ldots]) &= \mathrm{rank}\,([F_0^* \quad \ldots \quad F_\kappa^*])\\ &= \mathrm{rank}(\mathcal{M}_F) \le q.\end{aligned}$$

Therefore, a unitary matrix $U = [U_1^* \; U_2^*]^* \in \mathbb{C}^{p\times p}$ exists such that $U_2[F_0^* \; F_1^* \; F_2^* \; \ldots] = 0_{(p-q)\times\infty}$ is satisfied. We define $\mathcal{O} := FU_1^*$, which is an element of $\mathcal{RH}_\infty^{q\times q}$ because $F \in \mathcal{RH}_\infty^{q\times p}$ and $U_1 \in \mathbb{C}^{q\times p}$. The assumption $\gamma_{opt}^{SI}(F) < \infty$ implies that there exists a $G \in \mathcal{RH}_\infty^{p\times q}$ such that $FG = FU^*UG \overset{(FU_2^*=0)}{=} [\mathcal{O} \; 0_{q\times(p-q)}]UG = \mathcal{O}U_1G = I_q$. Thus, $\mathcal{O}^{-1} = U_1G \in \mathcal{RH}_\infty^{q\times q}$. Now, note that with $\mathcal{I}_1 := U_1$ we have obtained a reduced outer-coinner factorization of F with constant inner factor. Thus, $\mathcal{I}_1^\sim \in \mathcal{RH}_\infty^{p\times q}$, and in particular $F^\dagger = \mathcal{I}_1^\sim\mathcal{O}^{-1} \in \mathcal{RH}_\infty^{p\times q}$. ■

With Lemma 1 in place, we can now establish the upper bounds on the speed of convergence for the approximation method in Theorem 2.

[1] We have adapted the notation in this quote.

Theorem 3: Let $F \in \mathcal{RH}_\infty^{q\times p}$ have a right inverse in $\mathcal{RH}_\infty^{p\times q}$ and denote the Fourier coefficients of F by $\{F_k\}_{k\in\mathbb{Z}}$. Furthermore, introduce the input observability matrix $\mathcal{M}_F := [F_0^* \; F_1^* \; \ldots \; F_\kappa^*]^* \in \mathbb{C}^{(\kappa+1)q\times p}$, where κ is the McMillan degree of F. Then, $\mathrm{rank}(\mathcal{M}_F) \ge q$, and the lower bounds $\sigma_{\min}^{-1}(\Gamma_N^F)$ on the solution $\gamma_{opt}^{SI}(F)$ of Problem 1 in Theorem 2 satisfy

$$\left|\sigma_{\min}^{-1}\left(\Gamma_N^F\right) - \gamma_{opt}^{SI}(F)\right| = \begin{cases} O(e^{-\delta\sqrt{N}}), & \text{if } \mathrm{rank}(\mathcal{M}_F) > q \\ O\left(\frac{\ln N}{N}\right), & \text{if } \mathrm{rank}(\mathcal{M}_F) = q\end{cases}$$

for some $\delta > 0$. Moreover, if $\mathrm{rank}(\mathcal{M}_F) = q$, we have $\gamma_{opt}^{SI}(F) = \mathrm{esssup}_{|z|=1}\sigma_{\min}^{-1}[F(z)]$.

Proof: We start with some preparations. First, we prepare the use of Theorem 1. Let us define $a(e^{i\theta}) := F(e^{i\theta})F^\sim(e^{i\theta})$. We have $\det[a(e^{i\theta})] \ne 0$ for all $0 \le \theta < 2\pi$ because the right invertibility of F (i.e., $\gamma_{opt}^{SI}(F) < \infty$) implies that $a(e^{i\theta}) = F(e^{i\theta})F(e^{i\theta})^*$ is invertible for all $0 \le \theta < 2\pi$. Similarly, we have $\dim\ker(\mathfrak{T}_a^+) = 0$ because [9, Prop. 1.3] shows that $\mathfrak{T}_a = \mathfrak{T}_{FF^\sim} = \mathfrak{T}_F\mathfrak{T}_F^*$ and the right invertibility of $\mathfrak{T}_F$ implies that $\mathfrak{T}_F\mathfrak{T}_F^*$ is invertible. (Note that the right invertibility also is a consequence of [9, Prop. 1.3]. We have $\mathfrak{T}_F\mathfrak{T}_G = \mathfrak{T}_{FG} = I$ for any $G \in \mathcal{RH}_\infty^{p\times q}$ with $FG = I$, and such G exists by assumption.) We see that $k_0 = \dim\ker(\mathfrak{T}_a^+) + \dim\ker(\mathfrak{T}_{a^\sim}^+) = 2\dim\ker(\mathfrak{T}_a^+) = 0$ because $a = a^\sim$. Thus, a satisfies the assumptions in Theorem 1, and the singular value in (2) becomes $\sigma_{k_0+1}(\Gamma_N^a) = \sigma_1(\Gamma_N^a) = \sigma_{\min}(\Gamma_N^a)$. Next, we consider the finite sections Γ_N^a given in (1). The Fourier coefficients $\{a_m\}_{m\in\mathbb{Z}}$ of a can be given directly in terms of the Fourier coefficients of F, i.e., $a_m := \sum_{k=m}^\infty F_kF_{k-m}^*$ for $m \ge 0$ and $a_m := a_{-m}^*$ for $m < 0$. When we compare the block entries of $\Gamma_N^a = [a_{i-j}]_{i,j=0,\ldots,N-1}$ with those of $\Gamma_N^F(\Gamma_N^F)^* =: [b_{i,j}]_{i,j=0,\ldots,N-1}$, which are given by $b_{i,j} = \sum_{k=i-j}^{i} F_kF_{k-(i-j)}^*$, we see that

$$\Gamma_N^a - \Gamma_N^F\left(\Gamma_N^F\right)^* = \Phi_N^F\left(\Phi_N^F\right)^* \ge 0 \qquad (N \ge 1) \tag{4}$$

where $\Phi_N^F := [F_{i+j+1}]_{i=0,\ldots,N-1;j=0,1,\ldots}$ is a $Nq\times\infty$ block Hankel matrix. Finally, note that

$$\begin{aligned} d &= \min\left(\left\|\mathfrak{T}_a^+\right\|_{op}^{-1}, \left\|\mathfrak{T}_{a^\sim}^+\right\|_{op}^{-1}\right)\\ &\overset{(a=a^\sim)}{=} \left\|\mathfrak{T}_{a^\sim}^+\right\|_{op}^{-1}\\ &\overset{(a=FF^\sim)}{=} \left\|\mathfrak{T}_{F^\sim}^+\right\|_{op}^{-2}\\ &\overset{([16,\text{ Th. }9.2.1])}{=} \left[\gamma_{opt}^{SI}(F)\right]^{-2}\end{aligned} \tag{5}$$

and

$$\begin{aligned}\|\mathfrak{T}_{a^{-1}}\|^{-1} &\overset{([8,\text{ p. }274])}{=} \|a^{-1}\|_\infty^{-1}\\ &= \left\|(FF^\sim)^\dagger\right\|_\infty^{-1}\\ &= \left\|(F^\dagger)^\sim F^\dagger\right\|_\infty^{-1}\\ &= \|F^\dagger\|_\infty^{-2}.\end{aligned}$$

We are now finished with the preparations. In the following, let us distinguish two cases.

The first case is $\mathrm{rank}(\mathcal{M}_F) > q$. We then have $d < \|\mathfrak{T}_{a^{-1}}\|_{op}^{-1}$ because Lemma 1 implies $\gamma_{opt}^{SI}(F) > \|F^\dagger\|_\infty$. Application of Theorem 1 shows that there exists some $\delta > 0$ such that $|d - \sigma_{\min}(\Gamma_N^a)| = O(e^{-\delta\sqrt{N}})$. Since

$$\begin{aligned} d &\overset{(5)}{=} \left[\gamma_{opt}^{SI}(F)\right]^{-2} \overset{(\text{Th. }2)}{\le} \sigma_{\min}^2\left(\Gamma_N^F\right)\\ &= \sigma_{\min}\left[\Gamma_N^F\left(\Gamma_N^F\right)^*\right] \overset{(4)}{\le} \sigma_{\min}^2\left(\Gamma_N^a\right)\end{aligned}$$

we obtain $|d - \sigma_{\min}^2(\Gamma_N^F)| = O(e^{-\delta\sqrt{N}})$. It remains to show that also $|d^{-1/2} - \sigma_{\min}^{-1}(\Gamma_N^F)| = O(e^{-\delta\sqrt{N}})$. Define $C_N := [\sigma_{\min}(\Gamma_N^F) +$

$d^{1/2}]\sigma_{\min}(\Gamma_N^F)d^{1/2}$. Then, $\lim_{N\to\infty} C_N = 2d^{3/2} > 0$ and there exists $N_0 \in \mathbb{N}$ such that $d^{3/2} < C_N$ for all $N > N_0$. Hence

$$\begin{aligned} d^{\frac{3}{2}} \left| d^{-\frac{1}{2}} - \sigma_{\min}^{-1}\left(\Gamma_N^F\right)\right| &\le C_N \left| d^{-\frac{1}{2}} - \sigma_{\min}^{-1}\left(\Gamma_N^F\right)\right| \\ &= C_N \left[d^{-\frac{1}{2}} - \sigma_{\min}^{-1}\left(\Gamma_N^F\right)\right] \\ &= \sigma_{\min}^2\left(\Gamma_N^F\right) - d \\ &= \left|\sigma_{\min}^2\left(\Gamma_N^F\right) - d\right| \end{aligned}$$

for all $N > N_0$.

In the second case, $\operatorname{rank}(\mathcal{M}_F) \le q$, Theorem 1 only gives $O(\ln N/N)$ but Lemma 1 implies $\gamma_{opt}^{SI}(F) = \|F^\dagger\|_\infty = \operatorname{esssup}_{|z|=1}\sigma_{\min}^{-1}[F(z)]$. Note that $\gamma_{opt}^{SI}(F) < \infty$ implies $\operatorname{rank}(F_0) = q$, thus $\operatorname{rank}(\mathcal{M}_F) \ge q$, and it must hold $\operatorname{rank}(\mathcal{M}_F) = q$. Hence, $\operatorname{rank}(\mathcal{M}_F) < q$ cannot occur. ■

IV. Finite Section Method for the Infimum in Model Matching

We have already observed that numerical solution of the system inversion problem requires computation of the infimal norm among all right inverses. Something similar can be observed in the model-matching problem that we will consider in this section. Let us give an exact problem statement. The model matching problem is to

$$\begin{aligned} &\text{find } Q \in \mathcal{RH}_\infty^{p_2\times p_1} \text{ such that } M + NQ \in \mathcal{RH}_\infty^{q\times p_1} \\ &\text{and} \quad \|M+NQ\|_\infty \text{ is (close to) infimal} \end{aligned} \tag{6}$$

where $M \in \mathcal{R}^{q\times p_1}$ and $N \in \mathcal{R}^{q\times p_2}$ are arbitrary proper rational matrices. Numerical solution of the model matching problem decomposes into two subproblems [3], [4]. We are interested in the first subproblem, which is the computation of the following infimal norm.

Problem 2 (Infimum in $\mathcal{H}_\infty$ Model Matching): What is the value of

$$\gamma_{opt}^{MM}(M,N) := \inf\left(\left\{\|M+NQ\|_\infty : Q \in \mathcal{RH}_\infty^{p_2\times p_1},\ M+NQ \in \mathcal{RH}_\infty^{q\times p_1}\right\} \cup \{\infty\}\right)?$$

The goal of this section is to show how the computation of the model matching infimum γ_{opt}^{MM} in Problem 2 can be reduced to computation of a system inversion infimum as in Problem 1. This will enable us to use the approximation method in Theorem 2 in order to obtain arbitrarily good lower bounds on γ_{opt}^{MM} given in Problem 2.

Next, we have to discuss some technicalities. Let us assume that N has full column normal rank and no zeros on the unit circle $|z| = 1$. Then, we can compute the generalized inner-outer factorization (cf., e.g., [15], [16])

$$\mathcal{I}(z)\begin{bmatrix}\mathcal{O}(z)\\ 0\end{bmatrix} = \begin{bmatrix} M(z) & N(z) \\ I & 0\end{bmatrix} \tag{7}$$

i.e., $\mathcal{I} \in \mathcal{RH}_\infty^{(q+p_1)\times(q+p_1)}$, $\mathcal{I}^\sim\mathcal{I} = I$, and $\mathcal{O}^{-1} \in \mathcal{RH}_\infty^{(p_1+p_2)\times(p_1+p_2)}$. Without loss of generality we can assume that every unstable pole of $\mathcal{O}$ is a pole of the factorized function, i.e.

$$\begin{aligned} \forall |z_0| \ge 1, u \in \mathcal{RH}_\infty^{(p_1+p_2)\times 1} : \lim_{z\to z_0} \|[M(z) \quad N(z)]\, u(z)\| \\ < \infty \Rightarrow \lim_{z\to z_0}\|\mathcal{O}(z)u(z)\| < \infty. \end{aligned} \tag{8}$$

(The construction of a suitable factorization is discussed in the Appendix.) Our first auxiliary result characterizes the stabilizing parameters in Problem 2 in terms of the outer factor $\mathcal{O}$.

Lemma 2 (Stabilizing Parameters): We consider Problem 2. Assume that N has full column normal rank and no zeros on the unit circle $|z| = 1$, and introduce a generalized inner-outer factorization (7) that satisfies (8). Let $\mathcal{O}_1 \in \mathcal{R}^{(p_1+p_2)\times p_1}$ denote the first p_1 and $\mathcal{O}_2 \in \mathcal{R}^{(p_1+p_2)\times p_2}$ the last p_2 columns of $\mathcal{O}$. Then,

$$\begin{aligned} &\left\{Q \in \mathcal{RH}_\infty^{p_2\times p_1} : M + NQ \in \mathcal{RH}_\infty^{q\times p_1}\right\} \\ &\quad = \left\{Q \in \mathcal{RH}_\infty^{p_2\times p_1} : \mathcal{O}_1 + \mathcal{O}_2 Q \in \mathcal{RH}_\infty^{(p_1+p_2)\times p_1}\right\}. \end{aligned} \tag{9}$$

Proof: Let us denote the right-hand side of (7) by P, and introduce the family of auxiliary systems $Q_I := [I\ Q^T]^T$, where $Q \in \mathcal{RH}_\infty^{p_2\times p_1}$. We also introduce the sets

$$\mathcal{A} := \left\{Q \in \mathcal{RH}_\infty^{p_2\times p_1} : PQ_I \in \mathcal{RH}_\infty^{(q+p_1)\times p_1}\right\}$$

and

$$\mathcal{B} := \left\{Q \in \mathcal{RH}_\infty^{p_2\times p_1} : \mathcal{O}Q_I \in \mathcal{RH}_\infty^{(p_1+p_2)\times p_1}\right\}.$$

Then, the claim (9) is equivalent to $\mathcal{A} = \mathcal{B}$. Before we show the latter, note that the definition of a pole implies that a proper rational matrix $H \in \mathcal{R}_\infty^{q\times p}$ satisfies $H \in \mathcal{RH}_\infty^{q\times p}$ if and only if $\lim_{z\to z_0}\|H(z)\| < \infty$ for all $|z_0| \ge 1$. See, e.g., [6, Def. 3.13].

"$\mathcal{A} \subseteq \mathcal{B}$": Assume that $PQ_I \in \mathcal{RH}_\infty^{(q+p_1)\times p_1}$ for some $Q \in \mathcal{RH}_\infty^{p_2\times p_1}$, and denote the columns of Q_I by $u_1,\ldots,u_{p_1}$. Then, $\lim_{z\to z_0}\|P(z)u_k(z)\| < \infty$ for all $k = 1,\ldots,p_1$ and $|z_0| \ge 1$. The minimality assumption (8) implies that also $\lim_{z\to z_0}\|\mathcal{O}(z)u_k(z)\| < \infty$ for all $k = 1,\ldots,p_1$ and $|z_0| \ge 1$, which in turn shows that

$$\lim_{z\to z_0}\|\mathcal{O}(z)\,[u_1(z) \quad \ldots \quad u_{p_1}(z)]\| = \lim_{z\to z_0}\|\mathcal{O}(z)Q_I\| < \infty$$

for all $|z_0| \ge 1$. Thus, $\mathcal{O}Q_I \in \mathcal{RH}_\infty^{(p_1+p_2)\times p_1}$.

"$\mathcal{A} \supseteq \mathcal{B}$": Assume that $\mathcal{O}_1 + \mathcal{O}_2 Q \in \mathcal{RH}_\infty^{q\times p_1}$ for some $Q \in \mathcal{RH}_\infty^{p_2\times p_1}$. Then, $PQ_I = \mathcal{I}[I\ 0]^T\mathcal{O}Q_I \in \mathcal{RH}_\infty^{(q+p_1)\times p_1}$ because $\mathcal{I} \in \mathcal{RH}_\infty^{(q+p_1)\times(q+p_1)}$ and $\mathcal{O}Q_I \in \mathcal{RH}_\infty^{(q+p_1)\times p_1}$. ■

The main result of this section is as follows.

Theorem 4: Let $M \in \mathcal{R}^{q\times p_1}$ and $N \in \mathcal{R}^{q\times p_2}$ denote proper rational matrices and assume that N has full column normal rank and no zeros on the unit circle $|z| = 1$. Then, we can find a generalized inner-outer factorization (7) that satisfies the minimality assumption (8). Furthermore, the solution of Problem 2 is given by

$$\gamma_{opt}^{MM}(M,N) = \sqrt{\left[\gamma_{opt}^{SI}\left([I_{p_1} \quad 0]\,\mathcal{O}^{-1}\right)\right]^2 - 1}. \tag{10}$$

Proof: Let, like in the proof of Lemma 2, P denote the right-hand side of (7) and set $Q_I := [I\ Q^T]^T$ for any $Q \in \mathcal{RH}_\infty^{p_2\times p_1}$. We also partition $\mathcal{O} = [\mathcal{O}_1\ \mathcal{O}_2]$ where $\mathcal{O}_1 \in \mathcal{R}^{(p_1+p_2)\times p_1}$. Then, we have

$$\begin{aligned} \|M+NQ\|_\infty^2 &= \|PQ_I\|_\infty^2 - 1 \\ &= \left\|\mathcal{I}[I \quad 0]^T\mathcal{O}Q_I\right\|_\infty^2 - 1 \\ &= \|\mathcal{O}_1 + \mathcal{O}_2 Q\|_\infty^2 - 1 \end{aligned}$$

for all $Q \in \mathcal{RH}_\infty^{p_2\times p_1}$ because $\mathcal{I}$ is inner. Lemma 2 shows that (9) is satisfied. Thus

$$\begin{aligned} &\left(\gamma_{opt}^{MM}(M,N)\right)^2 \\ &\quad = \inf_{Q\in\mathcal{RH}_\infty^{p_2\times p_1}, M+NQ\in\mathcal{RH}_\infty^{q\times p_1}} \|M+NQ\|_\infty^2 \\ &\quad = \inf_{Q\in\mathcal{RH}_\infty^{p_2\times p_1}, \mathcal{O}_1+\mathcal{O}_2Q\in\mathcal{RH}_\infty^{(p_1+p_2)\times p_1}} \left(\|\mathcal{O}_1+\mathcal{O}_2Q\|_\infty^2 - 1\right). \end{aligned}$$

Now, let $\mathcal{O}_1^+ \in \mathcal{RH}_\infty^{p_1\times(p_1+p_2)}$ denote the first p_1 rows and $\mathcal{O}_2^+ \in \mathcal{RH}_\infty^{p_2\times(p_1+p_2)}$ the last p_2 rows of $\mathcal{O}^{-1}$, respectively, and note that we have

$$\begin{aligned} &\left\{\mathcal{O}_1 + \mathcal{O}_2 Q : Q \in \mathcal{RH}_\infty^{p_2\times p_1}, \mathcal{O}_1 + \mathcal{O}_2 Q \in \mathcal{RH}_\infty^{(p_1+p_2)\times p_1}\right\} \\ &\quad = \left\{X \in \mathcal{RH}_\infty^{(p_1+p_2)\times p_1} : \mathcal{O}_1^+ X = I\right\} \end{aligned}$$

because, for any $Q \in \mathcal{RH}_\infty^{p_2\times p_1}$, we have

$$\begin{bmatrix}\mathcal{O}_1^+\\ \mathcal{O}_2^+\end{bmatrix}[\mathcal{O}_1 \quad \mathcal{O}_2] = \begin{bmatrix} I & 0\\ 0 & I\end{bmatrix} \Rightarrow \mathcal{O}_1^+(\mathcal{O}_1 + \mathcal{O}_2 Q) = I + 0$$

and, for any $X \in \mathcal{RH}_\infty^{(p_1+p_2)\times p_1}$ with $\mathcal{O}_1^+ X = I$, the parameter $Q := \mathcal{O}_2^+ X \in \mathcal{RH}_\infty^{p_2\times p_1}$ satisfies

$$\begin{bmatrix} I\\ Q\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1^+\\ \mathcal{O}_2^+\end{bmatrix} X \Rightarrow X = [\mathcal{O}_1 \quad \mathcal{O}_2]\begin{bmatrix} I\\ Q\end{bmatrix}.$$

Thus

$$\inf_{Q\in\mathcal{RH}_\infty^{p_2\times p_1},\mathcal{O}_1+\mathcal{O}_2Q\in\mathcal{RH}_\infty^{(p_1+p_2)\times p_1}} \|\mathcal{O}_1+\mathcal{O}_2Q\|_\infty^2 = \inf_{X\in\mathcal{RH}_\infty^{(p_1+p_2)\times p_1},\mathcal{O}_1^+X=I} \|X\|_\infty^2.$$

∎

V. Finite Section Method for the Infimum in Full Information Control

In this section, we make an observation similar to the previous section. We observe that also the numerical solution of the full information control problem decomposes into two steps where the first step is the computation of some infimal norm, and that the computation of this infimal norm can be reduced to the computation of some system inversion infimum. Let us start with a statement of the full information control problem. Consider the state space system

$$\Sigma: \begin{cases} x(t+1) = Ax(t) + B_1w(t) + B_2u(t) \\ z(t) = Cx(t) + D_1w(t) + D_2u(t) \end{cases} \quad (t\in\mathbb{N}) \qquad (11)$$

where $x(t)\in\mathbb{C}^n$, $z(t)\in\mathbb{C}^q$, $w(t)\in\mathbb{C}^{p_1}$, and $u(t)\in\mathbb{C}^{p_2}$ for all $t\in\mathbb{N}$. The A, B_1, B_2, C, D_1, D_2 are complex matrices of suitable dimensions. We want to find an internally stabilizing controller

$$K: \begin{cases} x_K(t+1) = A_Kx_K(t) + B_{K1}x(t) \\ \quad +B_{K2}w(t) \\ u(t) = C_Kx_K(t) + D_{K1}x(t) + D_{K2}w(t) \end{cases} \quad (t\in\mathbb{N})$$

where $x_K(t)\in\mathbb{C}^{n_K}$ and the $A_K, B_{K1}, B_{K2}, C_K, D_{K1}, D_{K2}$ are again complex matrices of suitable dimensions, such that the transfer function

$$T_{zw}(z) = (D_1 + D_2D_{K2}) + [\,C + D_2D_{K1} \quad D_2C_K\,] \times \left(zI - \begin{bmatrix} A+B_2D_{K1} & B_2C_K \\ B_{K1} & A_K \end{bmatrix}\right)^{-1} \begin{bmatrix} B_1+B_2D_{K2} \\ B_{K2} \end{bmatrix}$$

of the resulting closed-loop system

$$\mathcal{F}_l(\Sigma,K): \begin{cases} x(t+1) = (A+B_2D_{K1})x(t) + B_2C_Kx_K(t) \\ \quad +(B_1+B_2D_{K2})w(t) \\ x_K(t+1) = B_{K1}x(t) + A_Kx_K(t) + B_{K2}w(t) \quad (t\in\mathbb{N}) \\ z(t) = (C+D_2D_{K1})x(t) + D_2C_Kx_K(t) \\ \quad +(D_1+D_2D_{K2})w(t) \end{cases}$$

has (close to) infimal norm [5], [6]. Here, internal stability means that $x(t)\to 0$ and $x_K(t)\to 0$ for all $x(0)\in\mathbb{C}^n$ and $x_K(0)\in\mathbb{C}^{n_K}$ if $w(t)=0$ for all $t\in\mathbb{N}$.

As noted above does numerical solution of this problem decompose into two subproblems [5], [6]. The first of these two subproblems is as follows.

Problem 3 (Infimum in $\mathcal{H}_\infty$ Full Information Control): What is the value of

$$\gamma_{opt}^{FI}(\Sigma) := \inf\left(\left\{\left\|T_{\mathcal{F}_l(\Sigma,K)}\right\|_\infty : K \text{ internally stabilizing}\right\} \cup \{\infty\}\right)?$$

The goal of this section is to reduce the computation of γ_{opt}^{FI} in Problem 3 to the computation of a system inversion infimum as in Problem 1. This will enable us to use the approximation method in Theorem 2 in order to obtain arbitrarily good lower bounds on γ_{opt}^{FI}.

The main result of this section is the following.

Theorem 5: We consider the plant Σ given in (11) and introduce two proper rational matrices, $M(z) := D_1 + C(zI-A)^{-1}B_1$ and $N(z) := D_2 + C(zI-A)^{-1}B_2$. Assume that $[A - zI\ B_2]$ has full rank for all $|z|\geq 1$, and that N has full column normal rank and no zeros on the unit circle $|z|=1$. Then, we can find a generalized inner-outer factorization (7) such that that the minimality assumption (8) is satisfied. Furthermore, the solution to Problem 3 is given by

$$\gamma_{opt}^{FI}(\Sigma) = \sqrt{\gamma_{opt}^{SI}\left([I_{p_1} \quad 0]\,\mathcal{O}^{-1}\right)^2 - 1}. \qquad (12)$$

Proof: In the following, we show $\gamma_{opt}^{FI}(\Sigma) = \gamma_{opt}^{MM}(M,N)$. The claim then follows directly from Theorem 4. Our assumption that $[A - zI\ B_2]$ has full rank in $|z|\geq 1$ ensures that we can find a matrix H such that the eigenvalues of $A + B_2H$ are contained in $|z| < 1$ [6, Th. 3.2]. We define two proper rational matrices $T_i(z) := D_i + (C + D_2H)(A + B_2H)^{-1}B_i$ where $i\in\{1,2\}$. The proof of Lemma 1 in [17] shows that $\gamma_{opt}^{FI}(\Sigma) = \gamma_{opt}^{MM}(T_1,T_2)$. A dual version of Lemma 2.5 in [18] finally shows that $\gamma_{opt}^{MM}(T_1,T_2) = \gamma_{opt}^{MM}(M,N)$. ∎

Remark 4 (Non-iterative Computation of γ_{opt}^{FI}): Non-iterative algorithms that compute the infimum γ_{opt}^{FI} exist for certain special cases of the full information control problem [19], [20]. These algorithms are guaranteed to work only if the normal ranks of N and $[M\ N]$ coincide [20, Cor. 5]. In the setting of Theorem 5, this condition may or may not be fulfilled.

VI. Numerical Example

Finally, we present a numerical example. The MATLAB[2] source code used for the generation of this example is available via IEEE Xplore (http://ieeexplore.ieee.org) as additional multimedia content to this technical note. We consider the model matching problem with plant

$$[M(z) \quad N(z)] := \begin{bmatrix} \frac{-3z^3+3z^2-z-7}{6z^3-18z^2+26z-14} & \frac{9z^3-21z^2+27z-7}{6z^3-18z^2+26z-14} \\ \frac{9z^3-25z^2+19z-11}{6z^3-18z^2+26z-14} & \frac{3z^3+5z^2+z-1}{6z^3-18z^2+26z-14} \end{bmatrix}.$$

The infimum in this problem is $\gamma_{opt}^{MM}(M,N) = \sqrt{5}$.[3] We want to approximate $\gamma_{opt}^{MM}(M,N)$. Theorem 4 shows that $\gamma_{opt}^{MM}(M,N) = \sqrt{\gamma_{opt}^{SI}(\mathcal{O}_1^+)^2 - 1}$, where we have obtained

$$\mathcal{O}_1^+(z) := [1 \quad 0]\mathcal{O}^{-1}(z) \approx \left[\frac{0.458z^3 - 0.6902z^2 + 0.65z - 0.1569}{z^3 - 1.107z^2 + 0.7146z - 0.1522} \quad \frac{-0.01719z^3 - 0.4833z^2 + 0.2719z - 0.03631}{z^3 - 1.107z^2 + 0.7146z - 0.1522}\right]$$

by solving (7). Furthermore, Theorem 2 shows that we can approximate $\gamma_{opt}^{SI}(\mathcal{O}_1^+)$ by its lower bounds $\sigma_{\min}(\Gamma_N^{\mathcal{O}_1^+})^{-1}$. Theorem 3 predicts that the approximation error

$$\left|\gamma_{opt}^{MM}(M,N) - \sqrt{\sigma_{\min}\left(\Gamma_N^{\mathcal{O}_1^+}\right)^{-1} - 1}\right| \overset{(\dots)}{=} O\left(\left|\gamma_{opt}^{SI}\left(\mathcal{O}_1^+\right) - \sigma_{\min}\left(\Gamma_N^{\mathcal{O}_1^+}\right)^{-1}\right|\right)$$

decreases at least root-exponentially because the input observability matrix

$$\mathcal{M}_{\mathcal{O}_1^+} \approx \begin{bmatrix} 0.4580 & -0.1832 & 0.1200 & 0.1765 \\ -0.0172 & -0.5023 & -0.2719 & 0.0190 \end{bmatrix}^*$$

has full rank $2 > 1$. Fig. 1 depicts the approximation error $|\sqrt{5} - \sqrt{\sigma_{\min}(\Gamma_N^{\mathcal{O}_1^+})^{-2} - 1}|$ for increasing N (red curve). Indeed, we can even observe exponentially fast convergence until the asymptotic

[2]MATLAB is a registered trademark of The Mathworks, Inc. in Natick, MA. See http://www.mathworks.com. We used version 7.11 of MATLAB with version 9.0 of the Control System Toolbox and version 3.5 of the Robust Control Toolbox.

[3]The plant has been obtained by application of a bilinear transform to the continuous-time plant in [3, Ex. 7.1]. As this does not change the infimum, $\gamma_{opt}^{MM}(M,N)$ equals the infimum given in [3, Ex. 7.1]

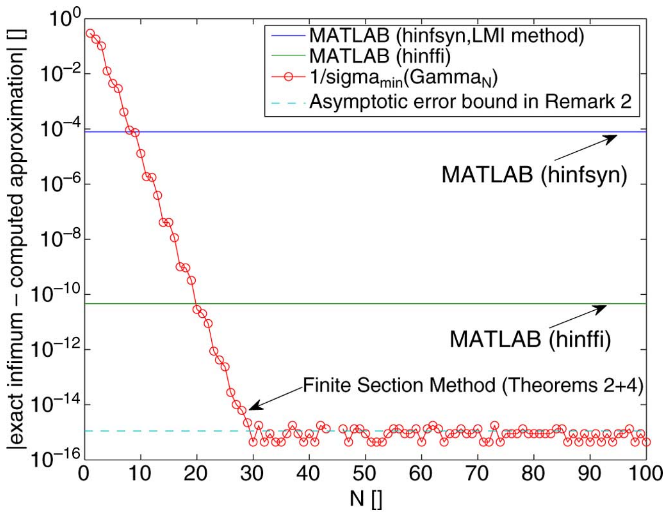

Fig. 1. Numerical example.

error bound in Remark 2 (applied to $\mathcal{O}_1^+$) is hit.[4] In this example, the error bound in Remark 2 is approximately 10^{-15} (dashed line). It is interesting to compare this result to standard γ-iteration algorithms. A comparison of the Theorems 4 and 5 shows that we can equivalently consider the model matching problem as a full information control problem. We have applied MATLAB's `hinfsyn` routine, configured to use a linear matrix inequality method, as well as MATLAB's `hinffi` routine, which implements a Riccati-based approach, to the equivalent full information control problem [3, Eq. (7.24)] in order to obtain two other estimates on $\gamma_{opt}^{MM}(M,N)$. The errors in the obtained estimates are approximately $8 \cdot 10^{-5}$ and $4 \cdot 10^{-11}$, respectively. See Fig. 1 (blue and green line). We note that our method is much more precise. This is not surprising as standard γ-iterations have well-known numerical problems [21], [22].

VII. Conclusion

In this technical note, we have extended an approach to compute arbitrarily good lower bounds on the infimum in the $\mathcal{H}_\infty$ system inversion problem to the $\mathcal{H}_\infty$ model matching problem and the $\mathcal{H}_\infty$ full information control problem, respectively. Thus, we have derived an unified approach to obtain lower bounds on the infima in all three problems. The bounds can e.g. be used to initialize γ-iterations or also for direct approximation of the infima. The effectiveness of our approach was established by showing that the bounds converge at least root-exponentially in most cases. A numerical example illustrated the fast convergence and possible gains in numerical accuracy when compared to standard γ-iteration algorithms.

Appendix
Generalized Inner-Outer Factorization That Satisfies (8)

We show that a suitable factorization can be found from Theorem 2 in [15]. Our assumption that N has full column normal rank and no zeros on the unit circle $|z| = 1$ ensures that the right hand side of (7), which we will denote by P, satisfies the assumption [15, (1)]. Let $x(t+1) = Ax(t) + Bw(t)$, $y(t) = Cx(t) + Du(t)$ denote a stabilizable and detectable (i.e., $[zI - A\ B]$ and $[zI - A^T\ C^T]$ have full rank in $|z| \geq 1$) state-space realization of P. Then, we can apply [15, Th. 2] to this realization. The resulting outer factor given in [15, Th. 2] has the state-space realization $x_O(t+1) = Ax_O(t) + Bu(t)$, $y_O(t) = -HGx_O(t) + Hu(t)$, where H and G are matrices which result from the solution of a Riccati system. Since it is not stated explicitly in [15, Th. 2], let us point out that H is invertible due to our assumption on $N(z)$, and that the inverse of the transfer function of the outer factor, which is given by $\mathcal{O}^{-1}(z) = H^{-1} - G(zI - A - BG)^{-1}BH^{-1}$ (cf. [6, Lem. 3.15]), satisfies $\mathcal{O}^{-1} \in \mathcal{RH}_\infty^{(p_1+p_2)\times(p_1+p_2)}$ because by construction all eigenvalues of $A + BG$ are contained inside the open unit disc $|z| < 1$ (cf. [6, Lem. 3.35]). Now, assume that $U \in \mathcal{RH}_\infty^{p\times 1}$ is given by $U(z) = \sum_{k=0}^\infty u(t)z^{-t}$ on $|z| \geq 1$ and that $\lim_{z\to z_0} \|P(z)U(z)\| < \infty$ for some $|z_0| \geq 1$. Remember that the realization of P is detectable. Thus, there exists a constant matrix M such that all eigenvalues of $A + MC$ are contained in $|z| < 1$ [6, Th. 21.4]. Therewith, we have $x(t+1) = Ax(t) + Bu(t) + M[Cx(t) + Du(t) - y(t)]$ because $Cx(t) + Du(t) - y(t) = 0$ for all $t \in \mathbb{N}$. Equivalently, we can write $X(z) = (zI - A - MC)^{-1}[(B + MD) - MP(z)]U(z)$ in the z-domain, where $X(z) := \sum_{t=0}^\infty x(t)z^{-t}$ is the z-transform of $\{x(t)\}_{t\in\mathbb{N}}$. This shows $\lim_{z\to z_0} \|X(z)\| < \infty$. But then, also $\lim_{z\to z_0} \|\mathcal{O}(z)U(z)\| = \lim_{z\to z_0} \| -HGX_O(z) + HU(z)\| < \infty$, where $X_O(z) := \sum_{t=0}^\infty x_O(t)z^{-t}$, because $x(t) = x_O(t)$ for all $t \in \mathbb{N}$ by construction.

[4]Note that the error bound is not completely exact because finite precision effects arising in the computation of the generalized inner-outer factorization have slightly changed the infimum.

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