Robust Analog Function Computation via Wireless Multiple-Access Channels

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***Abstract*—Wireless sensor network applications often involve the computation of pre-defined functions of the measurements such as for example the arithmetic mean or maximum value. Standard approaches to this problem separate communication from computation: digitized sensor readings are transmitted interference-free to a fusion center that reconstructs each sensor reading and subsequently computes the sought function value. Such separation-based computation schemes are generally highly inefficient as a complete reconstruction of individual sensor readings at the fusion center is not necessary to compute a function of them. In particular, if the mathematical structure of the channel is suitably matched (in some sense) to the function of interest, then channel collisions induced by concurrent transmissions of different nodes can be beneficially exploited for computation purposes. This paper proposes an analog computation scheme that allows for an efficient estimate of linear and nonlinear functions over the wireless multiple-access channel. A match between the channel and the function being evaluated is thereby achieved via some pre-processing on the sensor readings and post-processing on the superimposed signals observed by the fusion center. After analyzing the estimation error for two function examples, simulations are presented to show the potential for huge performance gains over time- and code-division multiple-access based computation schemes.**

***Index Terms*—Computation over multiple-access channels, in-network processing, function estimation, distributed computation, wireless sensor networks.**

I. Introduction

In contrast to traditional wireless networks, wireless sensor networks are deployed to perform various application tasks such as environmental monitoring or disaster alarm. Indeed, rather than transmitting and reconstructing the data of each individual sensor node, wireless sensor network applications often involve the computation of some pre-defined function of these data (called sensor readings), which includes the arithmetic mean, the maximum or minimum value, and different polynomials [3]. In this paper, we address the problem of computing functions over a Wireless Multiple-Access Channel (W-MAC) with a fixed number of sensor nodes and a single receiver that is referred to as the fusion center. A standard approach to such a computation problem widely used in contemporary sensor networks is to let each node transmit a digitized version of its sensor reading to the fusion center as a stream of modulated bits. In order to allow the perfect reconstruction of each (digitized) sensor reading, medium-access protocols such as Time-Division Multiple-Access (TDMA) [4] and Orthogonal Frequency-Division Multiple-Access (OFDMA) [5], [6] are typically used to avoid concurrent transmissions in the same frequency band and with it to establish interference-free connections between each node and the receiver.[1] Once the fusion center is aware of all sensor readings it computes the function of interest. The data transmission and the function computation are therefore completely disjoint processes.

Separation-based medium-access protocols (e.g., TDMA, OFDMA) are in general highly suboptimal when for instance maximizing the computation throughput defined as the rate at which function values are reconstructed at the fusion center subject to some communication constraints. In particular, the information-theoretic results in [8] suggest that the superposition property of the wireless channel can be beneficially exploited if the W-MAC is *matched* in some mathematical sense to the function of interest. The approach, which is known as *Computation over MAC* (CoMAC), can be seen as a method for merging the processes of data transmission and function computation by exploiting channel collisions induced by a concurrent access of different nodes to the common channel. An immediate consequence of this approach can be a higher computation throughput, and with it a reduced latency or lower bandwidth requirements.

The analysis in [8] also shows that in CoMAC scenarios, codes with a certain algebraic structure may outperform random codes. One such an example can be found in [9] (see also [8]) where two correlated sources aim at distributively compressing the parity of two binary messages. The code design in this case is driven by an application which is the modulo-two sum computation, and therefore the example lifts a strict separation between computation and communication. The research on practical structured codes for computation purposes is however still in its infancy [10]. Note that due to the superposition property, the W-MAC can be seen as a summation-type linear operator. Hence, functions naturally matched to this channel are linear, which constitute only one

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[1] In general, more sophisticated protocols for interference avoidance are necessary to achieve throughput-optimality [7].

class of functions with practical relevance.

In light of practical constraints, a serious drawback of the information-theoretic approach in [8] and other related results (see Section I-A) is the implicit assumption that if two symbols are put on the channel input, then the corresponding decoder observes the sum of these inputs. Obviously, this is satisfied in additive white Gaussian noise channels with users perfectly synchronized on the symbol and phase level. In practical wireless sensor networks, however, it may be unreasonably difficult and expensive in terms of resources to ensure such a perfect synchronization [11]. Hence, even if structured codes were available, the question remained how to exploit the superposition property of the wireless channel in the presence of practical impairments.

In this paper, we propose and analyze a novel CoMAC scheme for wireless sensor network applications. The scheme requires only a *coarse block-synchronization* and is therefore robust against synchronization errors. It is a simple *analog* computation scheme in which

1) each sensor node encodes its real-valued message (sensor reading) in the transmit power of a series of random signal pulses, and
2) the receiver estimates the function value from the corresponding received sum-power.

Another crucial advantage of the proposed analog computation scheme is its ability to reliably and efficiently compute nonlinear functions of sensor readings. We achieve such *nonlinear computational capabilities* by letting each sensor node pre-process its sensor readings prior to transmission, followed by a receiver-side post-processing of the received power, which is a noise-corrupted weighted sum of the pre-processed sensor readings from different sensor nodes. The pre-processing functions and the post-processing function are to be chosen such as to match the wireless channel with its superposition property to a function that we intend to evaluate at the sensor readings. The weights are due to the impact of the fading channel, which needs to be compensated in practical systems.

A. Related Work

In the context of sensor networks viewed as collections of distributed computation devices, Giridhar and Kumar took the first steps towards a theory-based framework for *in-network computation* with the aim of characterizing efficient application-specific computation strategies [3]. The work, however, is focused on complexity and protocol aspects and does not explicitly take into account the properties of wireless communication channels. A similar holds true for [12], [13], the information theoretical considerations in [14] and [15] as well as for [16], which is mainly devoted to wired networks. In contrast, harnessing the explicit structure of the channel for reliable function computations is first thoroughly analyzed in [8] with an emphasis on information theoretical insights, whereas computations over noiseless linear channels are considered in [17]. To the best of our knowledge, a first experimental validation that concurrent transmissions of nodes can increase the efficiency in some sensor network computation problems is given in [18].

Function computation in sensor networks is a fundamental building block of gossip and consensus algorithms, a form of distributed in-network data processing aiming at achieving some network-wide objectives based on local computations. Such algorithms, which compute a global function of sensor readings and distribute the function values among the nodes, have attracted a great deal of attention (see [19]–[21] and references therein). Most gossip and consensus protocols, however, require interference-free transmissions between nearby nodes, except for the recent work in [22]–[25], where it is shown that the superposition property of the wireless channel can be advantageously exploited to accelerate convergence speeds.

In [26], an analog joint source-channel communication scheme was proposed to exploit the superposition property of a Gaussian multiple-access channel for the optimal estimation of some desired parameter from a collection of noisy sensor readings. The approach outperforms comparable digital approaches that are based on the standard separation design principle between source and channel coding, as proposed by Shannon in his landmark paper [27]. Extensions of the analog joint source-channel scheme to more general estimation problems in wireless networks can be found in [28]–[31], whereas References [32]–[37] are devoted to the detection counterparts.

Finally, we point out that the basic idea of *physical-layer network coding* is to exploit the superposition property of the wireless channel as well. Indeed, in contrast to the traditional network coding principle applied across the packets on the network layer, the physical-layer network coding generates linear codewords immediately on the wireless channel by letting perfectly synchronized nodes transmit concurrently [38]–[40].

B. Paper Organization

Section II introduces the system model, formulates the problem and provides definitions used throughout the paper. In Section III, we present our novel analog CoMAC scheme for estimating linear and nonlinear functions of sensor readings and study the estimation error and outage behavior under the proposed scheme in Section IV. This analysis is used to define appropriate estimators for two canonical function examples of great practical importance: the arithmetic mean and the geometric mean. Numerical examples in Section V illustrate the performance of the proposed CoMAC scheme and compares it with a TDMA and a Code-Divison Multiple-Access (CDMA) based computation scheme to show the potential for huge performance gains under different network parameters. Finally, Section VI concludes the paper.

C. Notational Remarks

Random variables are denoted with uppercase letters, random vectors by bold uppercase letters, realizations by lowercase letters and vector valued realizations by bold lowercase letters, respectively. The sets of natural, nonnegative integer, real, nonnegative real, positive real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{C}$. The distributions of normally distributed real and proper complex random elements

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are denoted by $\mathcal{N}_{\mathbb{R}}(\cdot,\cdot)$ and $\mathcal{N}_{\mathbb{C}}(\cdot,\cdot)$. $\mathcal{LN}(\cdot,\cdot)$ denotes the log-normal distribution and χ^2_n the Chi-square distribution with n degrees of freedom. The error function and error function compliment are described by $\operatorname{erf}(\cdot)$ and $\operatorname{erfc}(\cdot)$. $\mathbb{1}_{\mathcal{B}}(x)$ is the indicator function on set $\mathcal{B}$ and the imaginary unit is denoted by i.

II. Definitions, System Model and Problem Statement

Throughout the paper, all random elements are defined over an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with sample space Ω, σ-Algebra $\mathcal{A}$ of subsets of Ω and probability measure $\mathbb{P}$ on $\mathcal{A}$. It is assumed that all functions of random variables and stochastic processes are Borel functions to ensure that all resulting random elements are well defined.

We consider a wireless sensor network consisting of $K \in \mathbb{N}$ spatially distributed single-antenna sensor nodes and one designated single-antenna Fusion Center (FC). Without loss of generality, it is assumed that the K nodes are identical and we use $\mathcal{K} := \{1, \dots, K\}$ to denote the set of all sensor nodes (numbered in an arbitrary order). Basically, the sensor nodes have the task to jointly observe a certain physical phenomenon (e.g., temperature, pressure, humidity) and subsequently transmit their suitably encoded sensor readings to the FC. We model the sensor readings as time-discrete $\mathcal{X}$-valued stochastic processes $X_k : \Omega \times \mathcal{T} \to \mathcal{X}$, $(\omega, t) \mapsto X_k[\omega, t]$, $k \in \mathcal{K}$, where $\mathcal{X} := [x_{\min}, x_{\max}]$, for some given $x_{\min} < x_{\max}$, is the underlying compact state space and $\mathcal{T}$ is an at most countable set of ordered measurement times.[2] Without loss of generality, let us assume that $\mathcal{X} \subseteq \mathcal{S} \subset \mathbb{R}$, where $\mathcal{S} := [s_{\min}, s_{\max}]$, $s_{\min} < s_{\max}$, is called the *sensing range*, which is the hardware-dependent range in which the sensors are able to quantify values. Finally, it is assumed that the joint probability density $p_{\boldsymbol{X}} : \mathcal{X}^K \times \mathcal{T} \to \mathbb{R}_+$, $p_{\boldsymbol{X}}(\boldsymbol{x}; t) := p_{X_1,\dots,X_K}(x_1, \dots, x_K; t) \in C_0(\mathcal{X}^K)$ of sensor readings $\boldsymbol{X}[t] := (X_1[t], \dots, X_K[t])^\mathsf{T}$ exists, with $C_0(\mathcal{B})$, $\mathcal{B} \subset \mathbb{R}^n$, being the space of real-valued compactly supported continuous functions over $\mathcal{B}$.

A. Wireless Multiple-Access Channel

The main contribution of this paper lies in a novel transmission scheme that exploits the superposition property of the Wireless Multiple-Access Channel (W-MAC) to efficiently compute functions of sensor readings. The W-MAC is defined as follows.

Definition 1 (W-MAC). For any transmission time $\tau \in \mathbb{Z}_+$, the W-MAC is a map from $\mathbb{C}^K$ into $\mathbb{C}$ defined to be

$$\big(W_1[\tau], \dots, W_K[\tau]\big) \longmapsto \sum_{k=1}^{K} H_k[\tau] W_k[\tau] + N[\tau] =: Y[\tau] \, . \tag{1}$$

Here and hereafter

- $W_k[\tau] \in \mathbb{C}$, $k \in \mathcal{K}$, is the transmit signal of node k with $|W_k[\tau]|^2 \le P_{\max}$, for all $\tau \in \mathbb{Z}_+$, where $P_{\max} > 0$ is the peak power constraint on each node,
- $H_k[\tau]$, $k \in \mathcal{K}$, is an independent complex-valued flat fading process between the k^{th} sensor node and the FC, and
- $N[\tau]$ is an independent proper complex receiver noise process.

Note that $W_k[\tau]$ depends on the k^{th} sensor reading $X_k[t] \in \mathcal{X}$ at any measurement time $t \in \mathcal{T}$. If $H_k[\tau] \equiv 1$ and $N[\tau] \equiv 0$, the W-MAC takes the form

$$\big(W_1[\tau], \dots, W_K[\tau]\big) \longmapsto \sum_{k=1}^{K} W_k[\tau] \, , \tag{2}$$

which is referred to as the *ideal W-MAC*.

Remark 1. The W-MAC is a symbol-synchronous channel similar to the standard synchronous CDMA channel studied for instance in [41], [4]. We would like to emphasize, however, that the computation scheme proposed in this paper does not require such a synchronous channel and the only reason for assuming perfect synchronization is to simplify the error analysis in Section IV and the notation throughout the paper.

B. Pre-processing and Post-processing Functions

As already mentioned, the objective is not to transmit each sensor reading via the W-MAC to the FC but rather to compute a function thereof. Throughout the paper, we use f to denote the function of interest and refer to this function as the *desired function*. Obviously, given a realization of $\boldsymbol{X}[t]$ at some measurement time instant $t \in \mathcal{T}$, we have $f : \mathcal{X}^K \to \mathbb{R}$ with $f\big(x_1[t], \dots, x_K[t]\big) =: (f \circ \boldsymbol{x})[t] = f\big(\boldsymbol{x}[t]\big)$, where $f\big(\boldsymbol{x}[t]\big)$ is the function value which the FC attempts to extract from the corresponding observed receive signal.

The basic idea behind the scheme for an efficient computation of desired functions proposed in this paper is to exploit the broadcast nature of the wireless channel to enable the FC to observe a superposition of signals transmitted by the sensors. A look at Eqs. (1) and (2) shows that the basic mathematical operation which can be naturally performed by the W-MAC on the transmit signals is *addition*. In other words, if all sensors send their readings simultaneously over the same frequency band, then the FC would receive a weighted sum of the sensor readings corrupted by background noise.[3] Now, the reader may be inclined to think that such an approach is inherently restricted to the computation of affine functions, which is in fact true if no additional signal processing is carried out at both the transmitters and the receiver. To overcome this limitation, we propose to perform some pre-processing and post-processing at the sensor nodes and the FC, respectively. To this end, we introduce the following two definitions.

Definition 2 (Pre-processing Function). We define $\varphi_k : \mathcal{X} \to \mathbb{R}$, $\varphi_k \in C_0(\mathcal{X})$, with $\varphi_k(x_k[t]) = (\varphi_k \circ x_k)[t]$, to be a pre-processing function of node $k \in \mathcal{K}$.

Definition 3 (Post-processing Function). The continuous injective function $\psi : \mathbb{C} \to \mathbb{R}$ with $\psi(y[\tau]) = (\psi \circ y)[\tau]$ and $y[\tau]$ given by (1) is said to be a post-processing function.

[2] Throughout the paper, we skip the explicit designation of elementary events $\omega \in \Omega$ in the formulation of stochastic processes and write for example $X_k[t]$ instead of $X_k[\omega, t]$.

[3] In the case of an ideal W-MAC, the FC would observe the sum of the transmit signals.

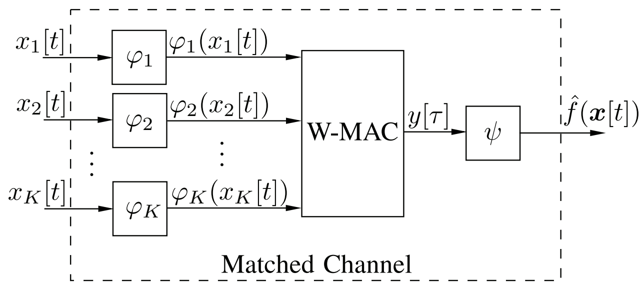

Fig. 1. Block diagram of the overall channel, which is matched to the desired function f. The match results from the transformation of the W-MAC by the corresponding pre-processing functions $\varphi_1, \dots, \varphi_K$ and the post-processing function ψ.

In order to illustrate the above definitions, it is reasonable to consider an ideal W-MAC, in which case the objective of the pre- and post-processing functions is to transform the ideal W-MAC in such a way that the resulting overall channel mapping from $\mathcal{X}^K$ into $\mathbb{R}$ is the desired function. Therefore, $\{\varphi_k\}_{k\in\mathcal{K}}$ and ψ are to be chosen so that

$$f\big(x_1[t], \dots, x_K[t]\big) = \psi\Big(\sum\nolimits_{k\in\mathcal{K}} \varphi_k\big(x_k[t]\big)\Big)\,, \tag{3}$$

where $w_k[\tau] = \varphi_k(x_k[t]) \in \mathbb{R}$ is the transmit signal of node $k \in \mathcal{K}$ at time $\tau \in \mathbb{Z}_+$.

C. Functions Computable via Wireless Multiple-Access Channels

Figure 1 illustrates the functional principle of the *analog computation* scheme proposed in Section III, which is referred to as the CoMAC scheme in what follows. Consequently, the space of all functions $\mathcal{F}(\mathcal{X}^K) \subset C_0(\mathcal{X}^K)$ that can be computed using the CoMAC scheme under the assumption of an ideal W-MAC is given by

$$\mathcal{F}(\mathcal{X}^K) := \Big\{ f : \mathcal{X}^K \to \mathbb{R} \,\Big|\, f(\boldsymbol{x}) = \psi\Big(\sum\nolimits_{k\in\mathcal{K}} \varphi_k(x_k)\Big)\Big\}\,.$$

The space of all affine functions is clearly a subset of $\mathcal{F}(\mathcal{X}^K)$, since any affine function can be computed with $\varphi_k(x) = \nu_k x$, $k \in \mathcal{K}$, and $\psi(y) = ay + b$, for some suitably chosen $(\nu_1, \dots, \nu_K)^\mathsf{T} \in \mathbb{R}^K$ and $a, b \in \mathbb{R}$. It is therefore possible to compute any weighted sum and, in particular, the *arithmetic mean*, which is of great interest in practice. Moreover, we can easily determine the number of active nodes in a network by letting them simultaneously transmit some constant value $c > 0$ and then post-process the received signal by means of $\psi(y) = \frac{1}{c}y$.

Now the following two questions arise immediately:

i) Is the set of all affine functions a *proper* subset of $\mathcal{F}(\mathcal{X}^K)$?
ii) What is exactly the function space $\mathcal{F}(\mathcal{X}^K)$ and how can its elements be computed?

In other words, the first question is one of whether functions other than affine ones are members of $\mathcal{F}(\mathcal{X}^K)$ and therefore are computable using a CoMAC scheme? The answer is obviously positive as we can easily compute the *geometric mean* of some positive sensor readings by choosing $\varphi_k(x) = \log_a(x)$, $a > 1$, $x > 0$, for each $k \in \mathcal{K}$ and $\psi(y) = a^{\frac{1}{K}y}$. Indeed, with this choice of pre- and post-processing functions, we have $f(\boldsymbol{x}) = \psi\big(\sum_{k\in\mathcal{K}} \varphi_k(x_k)\big) = \big(\prod_{k=1}^K x_k\big)^{\frac{1}{K}}$, where the sensor readings are positive so that $0 < x_{\min} \le x_k$ for each $k \in \mathcal{K}$. The second question in contrast is not so easy to answer. Widely considered in wireless sensor network applications is for instance the *maximum* of sensor readings $f(\boldsymbol{x}) = \max_{k\in\mathcal{K}} x_k$. It is, however, not clear how to compute the maximum function using a CoMAC scheme. On the positive side, the maximum function can be arbitrarily closely approximated by a sequence of functions in $\mathcal{F}(\mathcal{X}^K)$. Indeed, it is well-known that $\lim_{q\to\infty} \|x\|_q = f(\boldsymbol{x}) = \max_{k\in\mathcal{K}} x_k$, where $\|x\|_q = \big(\sum_{k=1}^K x_k^q\big)^{\frac{1}{q}} \in \mathcal{F}(\mathcal{X}^K)$, $x_k \ge 0$, $k \in \mathcal{K}$, and the norms can be computed when $\varphi_k(x) = x^q$, for all $k \in \mathcal{K}$, and $\psi(y) = y^{\frac{1}{q}}$.

Recently, it was shown in [42] that essentially *every* function can be computed over the W-MAC by exploiting its superposition property since every multivariate function has a representation of the form (3). It turns out that the pre-processing functions can be chosen to be monotonically increasing and universal, where universality refers to the ability to compute any desired function without the need for adapting the pre-processing functions [43]. The main difficulty, however, lies in a constructive characterization of $\mathcal{F}(\mathcal{X}^K)$ to determine the pre- and post-processing functions for computing some arbitrary members of this function space. Since such a complete constructive characterization is beyond the scope of this paper, we confine our attention in the upcoming sections to the problem of computing some functions of $\mathcal{F}(\mathcal{X}^K)$ that are of high practical relevance.

III. Analog Function Computation via Wireless Multiple-Access Channels

Computing desired functions of the form (3) over the W-MAC (1) seemingly requires a receiver-side constructive superposition of the transmit signals from different sensor nodes. However, such a perfect synchronization at the symbol and phase level is notoriously difficult to realize in wireless networks and in particular in large-scale wireless sensor networks [11]. Therefore, we propose in this section an analog computation scheme, consisting of a computation-transmitter as well as a computation receiver, that tolerates a *coarse block-synchronization* at the FC, which is by far easier to establish and maintain than the perfect synchronization required by traditional approaches.

The basic idea of the scheme consists in letting each sensor node transmit a distinct complex-valued sequence of length $M \in \mathbb{N}$ at a *transmit power* that depends on the pre-processed sensor reading. Under some conditions and a suitable pre-processing strategy, the received energy at the FC equals the sum of all the transmit energies corrupted by background noise. The application of an appropriately chosen post-processing function results then together with some simple arithmetic calculations (to ensure certain estimation properties) in an immediate estimate of the desired function of the sensor readings. The coarse block-synchronization is needed to ensure a sufficiently large overlap of the different signal frames, and with it of the different transmit energies, as illustrated in Fig. 2.

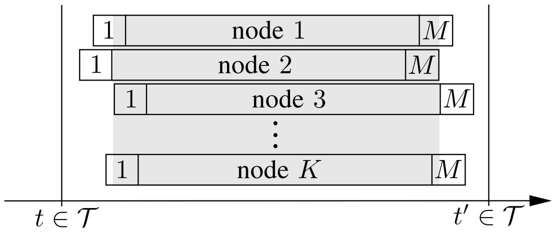

Fig. 2. Transmit sequences of nodes sent between measurement times t and t', respectively, without precise symbol- and phase-synchronization. The gray rectangle emphasizes the maximum overlapping area.

A. Computation Transmitter

1) Data Pre-processing: As each pre-processed sensor reading is to be encoded in transmit power only, it is necessary to apply a suitable bijective continuous mapping $g_\varphi : [\varphi_{\min}, \varphi_{\max}] \to [0, P_{\max}]$ from the set of all pre-processed sensor readings onto the set of all feasible transmit powers, with $\varphi_{\min} := \min_{k\in\mathcal{K}} \inf_{x\in\mathcal{S}} \varphi_k(x)$, $\varphi_{\max} := \max_{k\in\mathcal{K}} \sup_{x\in\mathcal{S}} \varphi_k(x)$ and $P_{\max}$ being the transmit power constraint on each node (see Definition 1). Note that the mapping depends on the pre-processing functions and the sensing range and is independent of k, as the FC does not have access to individual transmit signals but only to the W-MAC output given by (1). We call the quantity

$$P_k[t] := g_\varphi\big(\varphi_k(X_k[t])\big) \tag{4}$$

the transmit power of node k and point out that it is a random variable whenever $X_k[t]$ is random. Moreover, we have $P_k[t] \le P_{\max}$, for all $t \in \mathcal{T}$, and thus the information to be conveyed to the FC is for each $k \in \mathcal{K}$ and each $t \in \mathcal{T}$ encoded in $P_k[t]$.

2) Random Sequences: The transmit power (4) modulates a sequence of random symbols. In what follows, we use $\boldsymbol{S}_k[t] := \big(S_k[1], \dots, S_k[M]\big)^\mathsf{T} \in \mathbb{C}^M$ to denote a sequence of random transmit symbols independently generated by node $k \in \mathcal{K}$ at any measurement time $t \in \mathcal{T}$. The symbols of the sequence are assumed to be of the form $S_k[m] = \mathrm{e}^{i\Theta_k[m]}$, $m = 1, \dots, M$, where $\{\Theta_k[m]\}_{k,m}$ are continuous random phases that are independent identically and uniformly distributed on $[0, 2\pi)$. This implies $\|\boldsymbol{S}_k[t]\|_2^2 = M$ for all $k \in \mathcal{K}$ and a constant envelope of the transmit signals (i.e., $|S_k[m]|^2 = 1$, for all m, k), which is a vital practical constraint. We have two remarks.

Remark 2. Instead of optimizing the sequences assigned to different nodes, we consider sequences $\boldsymbol{S}_k[t] = (\mathrm{e}^{i\Theta_k[1]}, \dots, \mathrm{e}^{i\Theta_k[M]})^\mathsf{T}$, $k \in \mathcal{K}$, with *random phases* and constant envelope to reduce the overhead for coordination and to improve scalability when compared to systems with optimized sequences. It is worth pointing out that the sequence design should be different from that for traditional asynchronous CDMA systems [4], [41], [44], where the objective is to eliminate or mitigate the detrimental impact of interuser interference. Sequences for CoMAC schemes, in contrast, should be designed to harness interference for a common goal, which is the computation of functions of sensor readings.

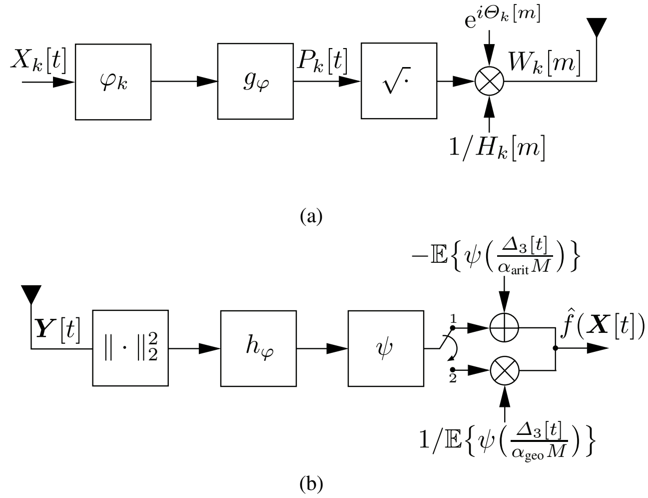

Fig. 3. (a) Block diagram of the CoMAC computation-transmitter of sensor node $k \in \mathcal{K}$. (b) Block diagram of the CoMAC computation-receiver for computing the arithmetic mean (switch position 1) and the geometric mean (switch position 2). Functions h_φ and ψ depend on the choice of the desired function and should be chosen according to the discussion in Section II-C and Definition 5 or Definition 6. For brevity, standard radio components (e.g., modulator, demodulator, filters) are not depicted.

Remark 3. Note that the assumption of continuous random phases is not necessary for our CoMAC scheme to be implemented. Without loss of performance, the phases can take on values on any discrete subset of $[0, 2\pi)$ provided that the resulting sequences are zero-mean.

3) Transmitter-side Channel Inversion: According to (1) is the W-MAC output, and thus the computation result, contaminated by fading. Since a receiver-side elimination of this is generally infeasible, we suggest that each transmitter corrects the fading impact by inverting its own channel. To this end, channel state information is necessary at each transmitter, which can be estimated from a known pilot signal transmitted by the FC. In practical systems, the pilot signal can also be used to wake up sensor nodes and initiate the computation process. With the channel state information available at nodes and the transmit powers given by (4), the W-MAC input of node $k \in \mathcal{K}$ at transmit symbol m (see (1)) takes the form

$$W_k[m] = \frac{\sqrt{P_k[t]}}{H_k[m]} S_k[m] = \frac{\sqrt{g_\varphi\big(\varphi_k(X_k[t])\big)}}{H_k[m]} \mathrm{e}^{i\Theta_k[m]} . \tag{5}$$

In [45] it is shown that dividing by the channel magnitude $|H_k[m]|$ is sufficient so that estimating the channel phase is generally not necessary.

The resulting computation-transmitter structure is depicted in Fig. 3(a).

Remark 4. Notice that any node k with $P_k[t]/|H_k[m]|^2 > P_{\max}$ for some m cannot invert its channel under the power constraint and must therefore be excluded from transmissions associated with measurement time $t \in \mathcal{T}$. One possibility to mitigate the problem is to scale down all transmit powers by the same constant so that the power constraint is satisfied. Of course, this impacts the performance in noisy channels and requires some degree of coordination. We are not going to dwell on this point and assume in the following that the set

$\mathcal{K}$ is chosen such that each node can invert its own channel without violating the power constraint.

B. Computation Receiver

As mentioned before (see Remark 1), in order to avoid cumbersome notation and to simplify the error analysis in the next section, we assume a perfect synchronization of signals from different nodes at the FC. The reader, however, may easily verify that the proposed CoMAC scheme based on a simple energy estimator is insensitive to the lack of synchronization provided that a significant overlap of different signal frames is ensured as illustrated in Fig. 2 (i.e., a coarse frame-synchronization). We also point out that the assumption of perfect synchronization has been widely used when analyzing asynchronous CDMA systems [4].

1) Received Signal: With this assumption in hand, the W-MAC is a memoryless channel and its output follows from (1) with (5) to $Y[m] = \sum_{k=1}^{K} \sqrt{P_k[t]} S_k[m] + N[m]$, $m = 1, \dots, M$. For any given $t \in \mathcal{T}$, we arrange these symbols in a vector $\boldsymbol{Y}[t] := (Y[1], \dots, Y[M])^{\mathsf{T}} \in \mathbb{C}^M$ to obtain the vector notation

$$\boldsymbol{Y}[t] = \sum_{k=1}^{K} \sqrt{P_k[t]}\, \boldsymbol{S}_k[t] + \boldsymbol{N}[t] \,, \tag{6}$$

where $\boldsymbol{N}[t] := (N[1], \dots, N[M])^{\mathsf{T}} \in \mathbb{C}^M$ denotes an independent stationary proper complex Gaussian noise process, that is $\boldsymbol{N}[t] \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \sigma_N^2 \boldsymbol{I}_M)$, $\sigma_N^2 \in (0, \infty)$.

2) Equivalent W-MAC: The observation vector in (6) is a basis for estimating the desired function value $f(X_1[t], \dots, X_K[t])$. Note, however, that the information about the pre-processed sensor readings is contained in the transmit powers (4) such that on the basis of observation $\boldsymbol{Y}[t]$, the sum-energy given by

$$\|\boldsymbol{Y}[t]\|_2^2 = M \sum_{k=1}^{K} P_k[t] + \underbrace{\sum_{k=1}^{K} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{K} \sqrt{P_k[t] P_\ell[t]} \boldsymbol{S}_k[t]^{\mathsf{H}} \boldsymbol{S}_\ell[t]}_{=: \Delta_1[t] \in \mathbb{R}} + \underbrace{2 \sum_{k=1}^{K} \sqrt{P_k[t]}\, \mathrm{Re}\{\boldsymbol{S}_k[t]^{\mathsf{H}} \boldsymbol{N}[t]\}}_{=: \Delta_2[t] \in \mathbb{R}} + \underbrace{\boldsymbol{N}[t]^{\mathsf{H}} \boldsymbol{N}[t]}_{=: \Delta_3[t] \in \mathbb{R}_+} \,, \tag{7}$$

is a sufficient statistic for the sum $\sum_{k=1}^{K} \varphi_k(x_k[t])$ of pre-processed sensor readings. As a consequence, instead of (1), we can equivalently consider the additive noise channel (7), which is expressed in a compact way as

$$\|\boldsymbol{Y}[t]\|_2^2 = M \sum_{k=1}^{K} P_k[t] + \Delta[t] \,, \tag{8}$$

where $\Delta[t] := \Delta_1[t] + \Delta_2[t] + \Delta_3[t] \in \mathbb{R}$. Note that in all that follows, we use the channel in (8), which is a map from $\mathbb{R}_+^K$ into $\mathbb{R}$.

3) Signal Post-processing: Before applying the post-processing function to the equivalent channel output (8), the receiver has to remove the influence of the function g_φ, which was used at the transmitting nodes to map the sensing range onto the set of feasible transmit powers. In other words, if $\Delta[t] \equiv 0$, then an application of the post-processing function must perfectly reconstruct the sought function value, which is expected from any computation or transmission scheme. Now, an examination of (8) with (4) shows that given g_φ, ψ and $\{\varphi_k\}_{k \in \mathcal{K}}$, we need to apply a function $h_\varphi : \mathbb{R} \to \mathbb{R}$ to (8) such that

$$\psi\left(h_\varphi\Big(M \sum_{k \in \mathcal{K}} g_\varphi\big(\varphi_k(x_k[t])\big)\Big)\right) \equiv \psi\left(\sum_{k \in \mathcal{K}} \varphi_k\big(x_k[t]\big)\right) . \tag{9}$$

Thus, given some pre- and post-processing functions, we can compute any desired function of the form (3) provided that $\Delta[t] \equiv 0$ and the pair (g_φ, h_φ) satisfies (9). The following proposition provides a necessary and sufficient condition for the functions to fulfill (9).

Proposition 1. Let $K \geq 2$ be arbitrary. Then, (9) holds with f defined by (3) for some given $\psi, \varphi_1, \dots, \varphi_K$, if and only if g_φ and h_φ are affine functions with $h_\varphi \equiv g_\varphi^{-1} - c$, where the constant $c \in \mathbb{R}$ depends on g_φ.

Proof: The proof is deferred to Appendix A. ■

In addition to the components mentioned above, the signal post-processing requires a further instance that takes into account the statistics of the transformed overall noise $\Delta[t]$ (transformed by h_φ and ψ), as it is illustrated in Fig. 3(b). A more detailed explanation of this part of the computation-receiver as well as examples of the data pre-processing and the signal post-processing functions for the arithmetic mean and the geometric mean are given in Sections IV-B and IV-C.

C. Performance Metric

The performance of the CoMAC scheme is determined in terms of the *function estimation error* defined as follows.

Definition 4 (Function Estimation Error). Let $f \in \mathcal{F}(\mathcal{X}^K)$ be the desired function continuously extended onto $\mathcal{S}'$, where $\mathcal{S}' \subseteq \mathcal{S}$ is an appropriate subset of $\mathcal{S}$.[4] Furthermore, let $\hat{f}$ be a corresponding estimate at the FC, $f_{\max} := \sup_{\boldsymbol{x} \in \mathcal{S}'^K} f(\boldsymbol{x})$ and $f_{\min} := \inf_{\boldsymbol{x} \in \mathcal{S}'^K} f(\boldsymbol{x})$. Then, $E := (\hat{f}(\boldsymbol{X}) - f(\boldsymbol{X}))/(f_{\max} - f_{\min})$ is said to be the *function estimation error* (relative to $\mathcal{S}'$).

Practical systems tolerate estimation errors provided that they are small enough. This means that $|E| < \epsilon$ must be satisfied for some given application-dependent constant $\epsilon > 0$. However, in many applications, the requirement cannot be met permanently due to, for instance, some random influences. In such cases, the main figure of merit is the *outage probability* $\mathbb{P}(|E| \geq \epsilon)$, which is the probability that the function estimation error is larger than or equal to $\epsilon > 0$. It is clear that the smaller the outage probability, the higher the computation accuracy.

[4] $\mathcal{S}'$ is introduced since it may be impossible to continuously extend f onto the entire sensing range $\mathcal{S}$.

IV. Error Analysis

This section is devoted to the performance analysis of the proposed CoMAC scheme in the presence of noise. First, we show that for sufficiently large values of M, the distribution of the computation noise $\Delta[t]$ can be approximated by a normal distribution. Since the function estimation error is strongly influenced by the post-processing function ψ, and with it on the choice of the desired function f, we confine our attention in subsequent subsections to two special cases of great practical importance: *arithmetic mean* and *geometric mean*. Note that these two functions are canonical representatives of the basic arithmetic operations *summation* and *multiplication*. For both cases, we define appropriate estimators $\hat{f}$ by taking into account statistical properties of the transformed overall noise $\Delta[t]$ (transformed by h_φ and ψ), prove some properties and provide accurate approximations of the corresponding outage probabilities. Without loss of generality, we focus on an arbitrary measurement time instant $t \in \mathcal{T}$ and therefore drop the time index for brevity.

A. Approximation of the Overall Error Distribution

The statistics of the overall noise in (8) play a key role when defining an estimator $\hat{f}$ for any desired function $f \in \mathcal{F}(\mathcal{X}^K)$ as well as when evaluating the performance of the proposed CoMAC scheme. Since an exact distribution of $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ conditioned on the sensor readings $\boldsymbol{X} = \boldsymbol{x}$ is difficult to determine, we focus on suitable asymptotic approximations.

To this end, let us first compute the first and second order statistical moments of Δ_1, Δ_2 and Δ_3. As far as Δ_1 is concerned, we have

$$\begin{aligned}\Delta_1 &= \sum_{k=1}^{K} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{K} \sum_{m=1}^{M} \sqrt{P_k P_\ell}\, S_k^*[m] S_\ell[m] \\ &= 2 \sum_{n=1}^{L} \sum_{m=1}^{M} \sqrt{\tilde{P}_n}\, \underbrace{\cos(\Theta'_n[m])}_{=:Z_n[m]} , \end{aligned} \tag{10}$$

where $L := K(K-1)/2$, $\tilde{P}_n := P_k P_\ell$ and $\Theta'_n[m] := (\Theta_\ell[m] - \Theta_k[m]) \bmod 2\pi$ the random phase difference between nodes k and ℓ at sequence symbol m. The mapping $(k, \ell) \mapsto n$ is obtained by $n = n(k, \ell) = \ell + (k-1)K - k(k+1)/2$, $k = 1, \dots, K-1$ and $\ell = k+1, \dots, K$, respectively.

By convolution of the densities of $\Theta_\ell[m]$ and $\Theta_k[m]$, $\Theta'_n[m]$ is independent uniformly distributed over $[0, 2\pi)$, for all n, m. Hence, the probability density of each $Z_n[m]$ in (10) is[5]

$$p_Z(z) = \frac{1}{\pi\sqrt{1-z^2}} \mathbb{1}_{(-1,1)}(z) , \tag{11}$$

which is symmetric around zero. So $\forall n, m : \mathbb{E}\{Z_n[m]\} = 0$ and

$$\mathbb{E}\{\Delta_1\} = 2 \sum_{n=1}^{L} \sum_{m=1}^{M} \mathbb{E}\{\tilde{P}_n^{\frac{1}{2}}\} \mathbb{E}\{Z_n[m]\} = 0 ,$$

for all $p_{\boldsymbol{X}} \in C_0(\mathcal{X}^K)$. Furthermore,

$$\begin{aligned}\mathbb{V}\mathrm{ar}\{\Delta_1\} &= 4 \sum_{n=1}^{L} \sum_{m=1}^{M} \mathbb{E}\{\tilde{P}_n\}\, \mathbb{V}\mathrm{ar}\{Z_n[m]\} \\ &= 2M \sum_{n=1}^{L} \mathbb{E}\{\tilde{P}_n\} \end{aligned} \tag{12}$$

since $\forall m, n \neq n' : \mathbb{C}\mathrm{ov}\{Z_n[m], Z_{n'}[m]\} = 0$ and $\forall m, n : \mathbb{V}\mathrm{ar}\{Z_n[m]\} = 1/2$, where the latter can be concluded by considering (11). As for the second error term Δ_2, we have

$$\Delta_2 = 2 \sum_{k=1}^{K} \sqrt{P_k}\, \mathrm{Re}\{\boldsymbol{S}_k^{\mathsf{H}} \boldsymbol{N}\} = 2 \sum_{k=1}^{K} \sum_{\ell=1}^{2M} \sqrt{P_k}\, U_{k\ell} N'_\ell ,$$

where for any odd ℓ, $U_{k\ell} := \cos(\Theta_k[m])$, $N'_\ell := \mathrm{Re}\{N[m]\}$ and $U_{k\ell} := \sin(\Theta_k[m])$, $N'_\ell := \mathrm{Im}\{N[m]\}$, for any even ℓ. Notice that $\forall \ell : N'_\ell \sim \mathcal{N}_{\mathbb{R}}(0, \frac{1}{2}\sigma_N^2)$ and the probability density function of $U_{k\ell}$ is given by (11). Because N'_ℓ and $U_{k\ell}$ are zero mean and independent for all k, ℓ, it follows for the expectation value

$$\mathbb{E}\{\Delta_2\} = 2 \sum_{k=1}^{K} \sum_{\ell=1}^{2M} \mathbb{E}\{\sqrt{P_k}\}\, \mathbb{E}\{U_{k\ell}\} \mathbb{E}\{N'_\ell\} = 0 ,$$

for all $p_{\boldsymbol{X}} \in C_0(\mathcal{X}^K)$. Arguing along similar lines as in the case of Δ_1, the variance of Δ_2 can be easily shown to be

$$\begin{aligned}\mathbb{V}\mathrm{ar}\{\Delta_2\} &= 4 \sum_{k=1}^{K} \sum_{\ell=1}^{2M} \mathbb{E}\{P_k\}\, \mathbb{V}\mathrm{ar}\{U_{k\ell}\}\, \mathbb{V}\mathrm{ar}\{N'_\ell\} \\ &= 2M\sigma_N^2 \sum_{k=1}^{K} \mathbb{E}\{P_k\} . \end{aligned} \tag{13}$$

Since $\Delta_3 = \sum_{m=1}^{M} |N[m]|^2 \sim \chi^2_{2M}$, we finally conclude $\mathbb{E}\{\Delta_3\} = M\sigma_N^2$ and

$$\mathbb{V}\mathrm{ar}\{\Delta_3\} = M\sigma_N^4 . \tag{14}$$

Lemma 1. Δ_1, Δ_2 and Δ_3 are mutually orthogonal (in the Hilbert space of random variables with the inner product defined to be $\langle \Delta_j, \Delta_{j'} \rangle \equiv \mathbb{E}\{\Delta_j \Delta_{j'}\}$) for all $p_{\boldsymbol{X}} \in C_0(\mathcal{X}^K)$.

Proof: Since the sensor readings, the sequence symbols and the noise are mutually independent random variables with $\mathbb{E}\{N[m]\} = 0$, for all m, a straightforward calculation of the covariances between Δ_1 and Δ_2 as well as between Δ_2 and Δ_3 proves the lemma. ■

The above derivations show that $\forall p_{\boldsymbol{X}} \in C_0(\mathcal{X}^K) : \mathbb{E}\{\Delta\} = M\sigma_N^2$, whereas, by Lemma 1, the variance of Δ is the sum of the variances (12), (13) and (14). Thus,

$$\begin{aligned}\sigma_\Delta^2 &:= \mathbb{V}\mathrm{ar}\{\Delta\} \\ &= 2M \sum_{n=1}^{L} \mathbb{E}\{\tilde{P}_n\} + 2M\sigma_N^2 \sum_{k=1}^{K} \mathbb{E}\{P_k\} + M\sigma_N^4 \end{aligned} \tag{15}$$

and note that when conditioned on $\boldsymbol{X} = \boldsymbol{x}$, the variance in (15) yields

$$\begin{aligned}\sigma_{\Delta|\boldsymbol{x}}^2 &:= \mathbb{E}\{(\Delta - M\sigma_N^2)^2 | \boldsymbol{X} = \boldsymbol{x}\} \\ &= 2M \sum_{n=1}^{L} \tilde{p}_n + 2M\sigma_N^2 \sum_{k=1}^{K} p_k + M\sigma_N^4 . \end{aligned} \tag{16}$$

[5] Note that by the definitions, all the probability density functions and expected values in this section exist.

As mentioned in the introduction to this section, we were not able to find out the exact distribution of the overall noise $\varDelta$, which includes various terms with different distributions. However, since the number of summands $J := K(K-1)M/2 + 2KM + 2M$ in the definition of $\varDelta$ is already for small values of K and M relatively large, we argue that it is well-founded to invoke the central limit theorem so as to approximate the conditional distribution by a normal distribution. The following proposition proves the corresponding convergence as $M \to \infty$.

Proposition 2. Let $\varDelta|\boldsymbol{x}$ be the overall noise according to (7) and (8) conditioned on the sensor readings $\boldsymbol{X} = \boldsymbol{x}$ with $\mathbb{E}\{\varDelta \,|\, \boldsymbol{X} = \boldsymbol{x}\} = M\sigma_N^2$, $0 < \sigma_N^2 < \infty$, and $\sigma^2_{\varDelta|\boldsymbol{x}}$ as defined in (16). Then, for any fixed $K, P_{\max} < \infty$ and a compact set $\mathcal{X}$, we have

$$\forall \boldsymbol{x} \in \mathcal{X}^K : \frac{\varDelta|\boldsymbol{x} - M\sigma_N^2}{\sigma_{\varDelta|\boldsymbol{x}}} \xrightarrow{\text{d}} \mathcal{N}_{\mathbb{R}}(0,1)$$

as $M \to \infty$, where $\xrightarrow{\text{d}}$ denotes the convergence in distribution.

Proof: Since the sum terms of $\varDelta|\boldsymbol{x}$ are neither identically distributed nor independent, the convergence to a normal distribution is not clear. Let us therefore rearrange the sum to obtain:

$$\begin{aligned}
\varDelta|\boldsymbol{x} &= \varDelta_1|\boldsymbol{x} + \varDelta_2|\boldsymbol{x} + \varDelta_3 \\
&= \sum_{n=1}^{L}\sum_{m=1}^{M} \sqrt{\tilde{p}_n}\cos(\varTheta_n'[m]) \\
&\quad + 2\sum_{k=1}^{K}\sum_{\ell=1}^{2M} \sqrt{p_k}U_{k\ell}N_\ell' + \sum_{m=1}^{M} |N[m]|^2 \\
&= \sum_{m=1}^{M}\Bigg[\sum_{n=1}^{L} \sqrt{\tilde{p}_n}\cos(\varTheta_n'[m]) \\
&\quad + \sum_{k=1}^{K}\Big(\mathrm{Re}\{N[m]\}\cos(\varTheta_k[m]) \\
&\quad + \mathrm{Im}\{N[m]\}\sin(\varTheta_k[m])\Big) + |N[m]|^2\Bigg] = \sum_{m=1}^{M} \varLambda_m \,.
\end{aligned}$$

This makes clear that $\varLambda_m$, $m = 1,\ldots,M$, are independent and identically distributed (i.i.d.) non-degenerate (i.e., $\mathbb{V}\mathrm{ar}\{\varLambda_1\} \neq 0$) random variables. Moreover, for any $K, P_{\max}, \sigma_N^2 < \infty$ and a compact set $\mathcal{X}$, $\mathbb{E}\{\varLambda_1^2 \,|\, \boldsymbol{X} = \boldsymbol{x}\} = 2\left(\sum_{n=1}^{L}\tilde{p}_n + \sigma_N^2\sum_{k=1}^{K}p_k + \sigma_N^4\right)$ is finite. Hence, the proposition follows from Theorem 3 in [46, p. 326] with (16) and $\mathbb{E}\{\varDelta \,|\, \boldsymbol{X} = \boldsymbol{x}\} = M\sigma_N^2$. ■

Since Proposition 2 implies the uniform convergence of the sequence of distribution functions associated with $\{\varDelta|\boldsymbol{x}\}_{M\in\mathbb{N}}$, we can conclude that the distribution of $\varDelta|\boldsymbol{x}$ can be approximated by a normal distribution provided that M is sufficiently large. This is summarized in a corollary.

Corollary. If M is sufficiently large, $\varDelta|\boldsymbol{x}$ is close to $\tilde{\varDelta}|\boldsymbol{x} \sim \mathcal{N}_{\mathbb{R}}(M\sigma_N^2, \sigma^2_{\varDelta|\boldsymbol{x}})$ in distribution.

We point out that determining the convergence rate is beyond the scope of this paper but extensive numerical experiments (see Section V) suggest that the approximation stated in the Corollary is justified already for small values of M and most cases of practical interest.

B. Arithmetic Mean Analysis

First, we define a suitable *arithmetic mean* estimator based on the observation of the channel output energy $\|\boldsymbol{Y}\|_2^2$ given by (8). Subsequently, we analyze the outage probability under the proposed estimator.

Definition 5 (Arithmetic Mean Estimate). Let f be the desired function "arithmetic mean" and let the expected value $\mathbb{E}\{\psi(\varDelta_3/(\alpha_{\text{arit}}M))\} = \sigma_N^2/(\alpha_{\text{arit}}K)$ be known to the FC with $\alpha_{\text{arit}} := \frac{P_{\max}}{s_{\max}-s_{\min}}$. Then, given $M \in \mathbb{N}$, the estimate $\hat{f}_M(\boldsymbol{X})$ of $f(\boldsymbol{X})$ is defined to be

$$\hat{f}_M(\boldsymbol{X}) := \psi\big(h_\varphi(\|\boldsymbol{Y}\|_2^2)\big) - \mathbb{E}\left\{\psi\big(\varDelta_3/(\alpha_{\text{arit}}M)\big)\right\} \,. \quad (17)$$

Assuming $M\sum_k g_\varphi(\varphi_k(x_k)) = M\sum_k p_k =: z$, we have

- *Data pre-processing*: $\forall k \in \mathcal{K} : \varphi_k(x) = x$, $g_\varphi(x) = \alpha_{\text{arit}}(x - s_{\min})$, $\varphi_{\min} = s_{\min}$, $\varphi_{\max} = s_{\max}$,
- *Signal post-processing*: $\psi(y) = y/K$ and $h_\varphi(z) = \frac{1}{M\alpha_{\text{arit}}}z + Ks_{\min}$.

The resulting computation-receiver is depicted in Fig. 3(b) with the switch in position 1.

Now, we prove two propositions to show that the arithmetic mean estimator of Definition 5 provides the two most desired properties: unbiasedness and consistency.

Proposition 3. The function value estimator of Definition 5 is unbiased, *that is, we have $\forall \boldsymbol{x} \in \mathcal{X}^K : \mathbb{E}\{\hat{f}_M(\boldsymbol{X}) \,|\, \boldsymbol{X} = \boldsymbol{x}\} = f(\boldsymbol{x})$.*

Proof: With the definitions introduced in Section II-C and Definition 5 in mind, we can write (17) as $\hat{f}_M(\boldsymbol{X}) = f(\boldsymbol{X}) + \frac{1}{\alpha_{\text{arit}}KM}(\varDelta - M\sigma_N^2)$. From this, it follows that $\mathbb{E}\{\hat{f}_M(\boldsymbol{X}) \,|\, \boldsymbol{X} = \boldsymbol{x}\} = f(\boldsymbol{x}) + \frac{1}{\alpha_{\text{arit}}KM}(\mathbb{E}\{\varDelta \,|\, \boldsymbol{X} = \boldsymbol{x}\} - M\sigma_N^2)$. So the proposition follows since $\forall \boldsymbol{x} \in \mathcal{X}^K :$ $\mathbb{E}\{\varDelta \,|\, \boldsymbol{X} = \boldsymbol{x}\} = \mathbb{E}\{\varDelta_3\} = M\sigma_N^2$. ■

Proposition 4. Let $K, P_{\max}, \sigma_N^2 < \infty$ be arbitrary but fixed and let $\{\hat{f}_M\}_{M\in\mathbb{N}}$ be the sequence of estimators (17). Then, the arithmetic mean estimator of Definition 5 is consistent, *that is $\forall \epsilon > 0 : \lim_{M\to\infty} \mathbb{P}(|\hat{f}_M - f| \geq \epsilon) = 0$.*

Proof: Let $c := 1/(\alpha_{\text{arit}}K) > 0$ and $\epsilon > 0$ be arbitrary but fixed. By the preceding proof we know that $E_M := \hat{f}_M - f = \frac{c}{M}(\varDelta - \mathbb{E}\{\varDelta_3\})$. Hence, as $\mathbb{E}\{\varDelta\} = \mathbb{E}\{\varDelta_3\}$, we obtain

$$\begin{aligned}
\mathbb{P}(|E_M| \geq \epsilon) &= \mathbb{P}(E_M^2 \geq \epsilon^2) \\
&= \mathbb{P}\big(c^2M^{-2}(\varDelta - \mathbb{E}\{\varDelta_3\})^2 \geq \epsilon^2\big) \\
&\leq c^2(M\epsilon)^{-2}\mathbb{E}\big\{(\varDelta - \mathbb{E}\{\varDelta\})^2\big\} \\
&= c^2(M\epsilon)^{-2}\sigma_\varDelta^2 \,, \quad (18)
\end{aligned}$$

where we used Markov's inequality [46, p. 47] (also called Chebyshev's inequality). By (15), we have for $K, P_{\max}, \sigma_N^2 < \infty$ that $\sigma_\varDelta^2 \in \mathcal{O}(M)$ so that the right-hand side of the above inequality goes to zero as M tends to infinity. Since $\epsilon > 0$ is arbitrary, this completes the proof. ■

Since the upper bound in (18) typically provides rather loose bounds for finite values of M, we cannot use it to approximate the outage probability. It turns out that a better approach is to invoke Proposition 2 and approximate $\mathbb{P}(|E| \geq \epsilon)$ by using a transformed normal distribution. Note that as $f_{\max} = \sup_{\boldsymbol{x}\in\mathcal{S}^K} f(\boldsymbol{x}) = s_{\max}$ and $f_{\min} = \sup_{\boldsymbol{x}\in\mathcal{S}^K} f(\boldsymbol{x}) = s_{\min}$ with f being continuously extended onto $\mathcal{S}$, the function estimation error of the arithmetic mean conditioned on the sensor readings $\boldsymbol{X} = \boldsymbol{x}$ has the form

$$E|\boldsymbol{x} = \big(\hat{f}_M(\boldsymbol{x}) - f(\boldsymbol{x})\big)/(s_{\max} - s_{\min}) = (\Delta|\boldsymbol{x} - M\sigma_N^2)/\alpha'_{\text{arit}}\,, \tag{19}$$

where $\alpha'_{\text{arit}} := MKP_{\max}$.

The Mann-Wald theorem [46, p. 356] guarantees that for any real continuous mapping $h = h(x)$, one has $h(X_n) \xrightarrow{\text{d}} h(X)$ whenever $X_n \xrightarrow{\text{d}} X$. We can therefore conclude from the corollary to Proposition 2 that for sufficiently large values of M, $E|\boldsymbol{x}$ in (19) can be approximated by a random variable $\tilde{E}|\boldsymbol{x} \sim \mathcal{N}_{\mathbb{R}}(0, \frac{\sigma^2_{\Delta|\boldsymbol{x}}}{\alpha'^2_{\text{arit}}})$ with conditional distribution function $P_{\tilde{E}}(e|\boldsymbol{x}) := \mathbb{P}(\tilde{E} \leq e|\boldsymbol{X} = \boldsymbol{x}) = \frac{1}{2}[1 + \operatorname{erf}(\frac{\alpha'_{\text{arit}} e}{\sigma_{\Delta|\boldsymbol{x}}\sqrt{2}})]$, $e \in \mathbb{R}$.

Since the absolute value is also continuous and $\mathbb{P}(|\tilde{E}| \geq \epsilon|\boldsymbol{X} = \boldsymbol{x}) = 1 - P_{\tilde{E}}(\epsilon|\boldsymbol{x}) + P_{\tilde{E}}(-\epsilon|\boldsymbol{x})$ for any $\epsilon > 0$, we obtain for sufficiently large M

$$\begin{aligned}\mathbb{P}(|E| \geq \epsilon) &\approx \mathbb{P}(|\tilde{E}| \geq \epsilon)\\ &= \int_{\mathcal{X}^K} \mathbb{P}(|\tilde{E}| \geq \epsilon \,|\, \boldsymbol{X} = \boldsymbol{x}) p_{\boldsymbol{X}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\\ &= \int_{\mathcal{X}^K} \operatorname{erfc}\big(\alpha'_{\text{arit}}\epsilon/(2\sigma^2_{\Delta|\boldsymbol{x}})^{\frac{1}{2}}\big) p_{\boldsymbol{X}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\end{aligned} \tag{20}$$

in distribution, where we used the fact that $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ for all $x \in \mathbb{R}$.

C. Geometric Mean Analysis

As in the preceding subsection, we first define an estimator for the desired function *geometric mean* including the required data pre-processing and signal post-processing functions.

Definition 6 (Geometric Mean Estimate). Let f be the desired function "geometric mean" as defined in Section II-C, and let the expected value $\mathbb{E}\{\psi(\Delta_3/(\alpha_{\text{geo}}M))\}$ be known to the FC (see Lemma 2 below) with $\alpha_{\text{geo}} := \frac{P_{\max}}{\log_a(s_{\max}) - \log_a(s')}$. Then, given $M \in \mathbb{N}$, the estimate $\hat{f}_M(\boldsymbol{X})$ of $f(\boldsymbol{X})$ is defined to be

$$\begin{aligned}\hat{f}_M(\boldsymbol{X}) &:= \frac{\psi\big(h_\varphi(\|\boldsymbol{Y}\|_2^2)\big)}{\mathbb{E}\left\{\psi\big(\Delta_3/(\alpha_{\text{geo}}M)\big)\right\}}\\ &= f(\boldsymbol{X})\frac{\psi\big(\Delta/(\alpha_{\text{geo}}M)\big)}{\mathbb{E}\left\{\psi\big(\Delta_3/(\alpha_{\text{geo}}M)\big)\right\}}\,.\end{aligned} \tag{21}$$

Assuming $M\sum_k g_\varphi(\varphi_k(x_k)) = M\sum_k p_k =: z$, we have

- *Data pre-processing*: If $s_{\min} \leq 0$, choose an arbitrary but fixed s' such that $0 < s' \leq x_{\min} < s_{\max}$ (i.e., $\mathcal{S}' := [s', s_{\max}] \subset \mathcal{S}$) and otherwise $s' = s_{\min}$. Then, $\forall k \in \mathcal{K}: \varphi_k(x) = \log_a(x)$, $a > 1$, $\varphi_{\min} = \log_a(s')$, $\varphi_{\max} = \log_a(s_{\max})$, and $g_\varphi(\log_a(x)) = \alpha_{\text{geo}}(\log_a(x) - \log_a(s'))$.
- *Signal post-processing*: $\psi(y) = a^{y/K}$ and $h_\varphi(z) = \frac{1}{M\alpha_{\text{geo}}}z + K\log_a(s')$.

The resulting computation-receiver is shown in Fig. 3(b) with the switch in position 2.

As mentioned in the definition, our computation-receiver requires the knowledge of the expected value $\mathbb{E}\{\psi(\Delta_3/(\alpha_{\text{geo}}M))\}$, which is explicitly given in part (i) of the following lemma. Part (ii) is used in the proof of Proposition 5.

Lemma 2. Let $a > 1$ be given and let α_{geo} be as in Definition 6. Suppose that $\sigma_N^2 \log_{\mathrm{e}}(a) < \alpha_{\text{geo}}KM$. Then

(i) $\lambda_M := \mathbb{E}\left\{\psi\big(\Delta_3/(\alpha_{\text{geo}}M)\big)\right\} = \left(\frac{\alpha_{\text{geo}}KM}{\alpha_{\text{geo}}KM - \sigma_N^2\log_{\mathrm{e}}(a)}\right)^M$

(ii) $\lim_{M\to\infty} \lambda_M = \mathrm{e}^{\frac{\sigma_N^2 \log_{\mathrm{e}}(a)}{\alpha_{\text{geo}}K}}$.

Proof: The proof is deferred to Appendix B. ■

We point out that the expected value λ_M exists if $\sigma_N^2\log_{\mathrm{e}}(a) < \alpha_{\text{geo}}KM$ holds, which is usually fulfilled in practical situations and therefore assumed in what follows.

With the estimator of Definition 6, the function estimation error conditioned on the sensor readings $\boldsymbol{X} = \boldsymbol{x}$ becomes

$$E|\boldsymbol{x} = \frac{1}{\gamma(\boldsymbol{x})}\Xi|\boldsymbol{x} - \beta(\boldsymbol{x}) = \beta(\boldsymbol{x})\left(\frac{\Xi|\boldsymbol{x}}{\lambda_M} - 1\right)\,, \tag{22}$$

where we used the following notation: $f_{\max} = s_{\max}$, $f_{\min} = s'$, $\beta(\boldsymbol{x}) := f(\boldsymbol{x})/(s_{\max} - s')$, $\gamma(\boldsymbol{x}) := \lambda_M/\beta(\boldsymbol{x})$ and $\Xi|\boldsymbol{x} := \psi(\Delta|\boldsymbol{x}/(\alpha_{\text{geo}}M))$.

Note that the estimator of Definition 6 is not necessarily unbiased but it offers the advantage of a simple implementation in practical systems. In contrast, the estimator

$$\hat{f}_M(\boldsymbol{X}) = \frac{\psi\big(h_\varphi(\|\boldsymbol{Y}\|_2^2)\big)}{\mathbb{E}\left\{\psi\big(\Delta/(\alpha_{\text{geo}}M)\big)\right\}} \tag{23}$$

is unbiased but not applicable in practice, because in opposite to the expected value in (21) depends $\mathbb{E}\left\{\psi\big(\Delta/(\alpha_{\text{geo}}M)\big)\right\}$ on the overall noise and thus on the distribution of the sensor readings, which is usually unknown at the FC. Although (21) is not unbiased, the following proposition shows that it is consistent.

Proposition 5. For any fixed $K, P_{\max}, \sigma_N^2 < \infty$, the geometric mean estimator proposed in Definition 6 is consistent.

Proof: Let $K, P_{\max}, \sigma_N^2 < \infty$ and $\epsilon > 0$ be arbitrary but fixed. Let $\{\hat{f}_M\}_{M\in\mathbb{N}}$ be the sequence of estimators given by (21). We show that the outage probability $\mathbb{P}(|E| \geq \epsilon) \to 0$ as $M \to \infty$. To this end, consider $\mathbb{P}(|E|\boldsymbol{x}| \geq \epsilon) := \mathbb{P}(|E| \geq \epsilon|\boldsymbol{X} = \boldsymbol{x})$ for any $\boldsymbol{x} \in \mathcal{X}^K$, and note that $f(\boldsymbol{x}) > 0, \beta(\boldsymbol{x}) > 0, \lambda_M > 0$ and $\Xi_{\boldsymbol{x}} := \Xi|\boldsymbol{x} > 0$. By (22), we have $\mathbb{P}(|E|\boldsymbol{x}| \geq \epsilon) = \mathbb{P}(\Xi_{\boldsymbol{x}}/\lambda_M \geq 1 + \epsilon/\beta(\boldsymbol{x})) + \mathbb{P}(1 - \Xi_{\boldsymbol{x}}/\lambda_M \geq \epsilon/\beta(\boldsymbol{x}))$. An application of Markov's inequality [46, p. 47] yields an upper bound on the first sum term:

$$\begin{aligned}\mathbb{P}\left(\tfrac{\Xi_{\boldsymbol{x}}}{\lambda_M} \geq 1 + \tfrac{\epsilon}{\beta(\boldsymbol{x})}\right) &= \mathbb{P}\left(\log_{\mathrm{e}}\left(\tfrac{\Xi_{\boldsymbol{x}}}{\lambda_M}\right) \geq \log_{\mathrm{e}}\left(1 + \tfrac{\epsilon}{\beta(\boldsymbol{x})}\right)\right)\\ &\leq \frac{\mathbb{E}\{\log_{\mathrm{e}}(\Xi_{\boldsymbol{x}})\} - \log_{\mathrm{e}}(\lambda_M)}{\log_{\mathrm{e}}\big(1 + \epsilon/\beta(\boldsymbol{x})\big)}\,.\end{aligned} \tag{24}$$

By (ii) of Lemma 2, we have $\lim_{M\to\infty}\log_{\mathrm{e}}(\lambda_M) = \log_{\mathrm{e}}(\lim_{M\to\infty}\lambda_M) = \frac{\sigma_N^2\log_{\mathrm{e}}(a)}{\alpha_{\text{geo}}K}$. Due to the results on the distribution functions of random variables that are functions of other random variables [46, pp. 239–240], we obtain $\mathbb{E}\{\log_{\mathrm{e}}(\Xi_{\boldsymbol{x}})\} = \frac{\log_{\mathrm{e}}(a)}{K}\mathbb{E}\{\frac{\Delta|\boldsymbol{x}}{\alpha_{\text{geo}}M}\} = \frac{\sigma_N^2\log_{\mathrm{e}}(a)}{\alpha_{\text{geo}}K}$, where we used $\mathbb{E}\{\Delta\,|\,\boldsymbol{X} = \boldsymbol{x}\} = M\sigma_N^2$ in the last step. Combining

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the results shows that the upper bound in (24) tends to zero as $M \to \infty$. As for $\mathbb{P}(1 - \varXi_{\boldsymbol{x}}/\lambda_M \geq \epsilon/\beta(\boldsymbol{x}))$, note that we can focus on $\epsilon/\beta(\boldsymbol{x}) < 1$ since $\varXi_{\boldsymbol{x}}/\lambda_M > 0$. With this in mind, we have $\mathbb{P}(1 - \varXi_{\boldsymbol{x}}/\lambda_M \geq \epsilon/\beta(\boldsymbol{x})) = \mathbb{P}(\lambda_M/\varXi_{\boldsymbol{x}} \geq 1/(1 - \epsilon/\beta(\boldsymbol{x})))$. Proceeding essentially along the same lines as above shows that this probability goes to zero with $M \to \infty$. Now, by compactness of $\mathcal{X}$ and Theorem 3 or Theorem 4 of [46, p. 188], we have $\lim_{M\to\infty} \mathbb{P}(|E| \geq \epsilon) = \lim_{M\to\infty} \mathbb{E}\{\mathbb{P}(|E| \geq \epsilon \,|\, \boldsymbol{X} = \boldsymbol{x})\} = \mathbb{E}\{\lim_{M\to\infty} \mathbb{P}(|E| \geq \epsilon \,|\, \boldsymbol{X} = \boldsymbol{x})\} = 0$. ■

Remark 5. Notice that the proposition implies that the proposed geometric mean estimator (21) is asymptotically unbiased, that is, we have $\lim_{M\to\infty} \mathbb{E}\{\hat{f}(\boldsymbol{X}) \,|\, \boldsymbol{X} = \boldsymbol{x}\} = f(\boldsymbol{x})$. As a consequence, (21) is asymptotically equivalent to (23).

Unfortunately, $\mathbb{P}(|E| \geq \epsilon)$ cannot be exactly evaluated because we are not able to determine the distribution function of $|E| = \left|\frac{1}{\gamma(\boldsymbol{X})}\varXi - \beta(\boldsymbol{X})\right|$. For this reason, as in the preceding subsection, we approximate the distribution of $\varXi|\boldsymbol{x}$ by a transformed normal distribution since in contrast to the arithmetic mean case depends $\varXi|\boldsymbol{x}$ nonlinearly on the conditioned overall noise $\varDelta|\boldsymbol{x}$.

Lemma 3. Let $K < \infty$, $\mathcal{X}$ be compact and M sufficiently large. Then, $\varXi|\boldsymbol{x}$ can be approximated by a random variable $\tilde{\varXi}|\boldsymbol{x} \sim \mathcal{LN}(\mu_\varXi, \sigma^2_{\varXi|\boldsymbol{x}})$, where $\mu_\varXi = \sigma_N^2 \log_{\mathrm{e}}(a)/(\alpha_{\mathrm{geo}}K)$ and $\sigma^2_{\varXi|\boldsymbol{x}} = \sigma^2_{\varDelta|\boldsymbol{x}}(\log_{\mathrm{e}}(a))^2/(\alpha_{\mathrm{geo}}KM)^2$, respectively.

Proof: The proof can be found in Appendix C. ■

With Lemma 3 in hand, we are now in a position to prove the main result of this section.

Proposition 6. Consider the proposed geometric mean estimator (21) and suppose that E is the corresponding function estimation error. Let $\mu_\varXi$ and $\sigma^2_{\varXi|\boldsymbol{x}}$ be given by Lemma 3, and let $\beta(\boldsymbol{x}), \gamma(\boldsymbol{x}) > 0$ be as defined in (22). Then, for M sufficiently large, the outage probability $\mathbb{P}(|E| \geq \epsilon)$ can be approximated (in distribution) for arbitrary $\epsilon > 0$ by

$$\begin{aligned}\mathbb{P}(|E| \geq \epsilon) &\approx \mathbb{P}(|\tilde{E}| \geq \epsilon)\\ &= \int_{\mathcal{X}^K} \mathbb{P}(|\tilde{E}| \geq \epsilon | \boldsymbol{X} = \boldsymbol{x}) p_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \end{aligned} \quad (25)$$

with $\mathbb{P}(|\tilde{E}| \geq \epsilon | \boldsymbol{X} = \boldsymbol{x})$ equal to

$$\frac{1}{2}\left[2 + \operatorname{erf}\left(\frac{\log_{\mathrm{e}}(\rho^-(\boldsymbol{x},\epsilon)) - \mu_\varXi}{\sqrt{2}\,\sigma_{\varXi|\boldsymbol{x}}}\right) - \operatorname{erf}\left(\frac{\log_{\mathrm{e}}(\rho^+(\boldsymbol{x},\epsilon)) - \mu_\varXi}{\sqrt{2}\,\sigma_{\varXi|\boldsymbol{x}}}\right)\right] \quad (26)$$

for $0 < \epsilon < \beta(\boldsymbol{x})$ and equal to

$$\frac{1}{2}\operatorname{erfc}\left(\frac{\log_{\mathrm{e}}(\rho^+(\boldsymbol{x},\epsilon)) - \mu_\varXi}{\sqrt{2}\,\sigma_{\varXi|\boldsymbol{x}}}\right) \quad (27)$$

for $\beta(\boldsymbol{x}) \leq \epsilon < \infty$, respectively, where $\rho^-(\boldsymbol{x}, \epsilon) := \gamma(\boldsymbol{x})(\beta(\boldsymbol{x}) - \epsilon)$ and $\rho^+(\boldsymbol{x}, \epsilon) := \gamma(\boldsymbol{x})(\beta(\boldsymbol{x}) + \epsilon)$.

Proof: The proof is deferred to Appendix D. ■

In Section V-A, we choose a particular density $p_{\boldsymbol{X}}(\boldsymbol{x})$ and evaluate (25) numerically to indicate the accuracy of the approximation for different network parameters.

V. Numerical Examples

The objective of this section is twofold. First, we show in Section V-A that the approximations of Section IV are highly accurate, and second, we compare in Section V-B the proposed analog CoMAC scheme with a TDMA-based and a CDMA-based scheme to show the huge potential for performance gains in typical sensor network operating points.

As a basis, we consider a classical environmental monitoring scenario in which the FC is interested in the arithmetic mean or geometric mean of temperature measurements carried out by a number of sensor nodes that are distributed over some geographical area. We assume that all nodes are equipped with a low-power temperature sensor operating in a typical sensing range $\mathcal{S} = [-55\,^\circ\mathrm{C}, 130\,^\circ\mathrm{C}]$ [47].

A. Approximation Accuracy

To assess the accuracy of the approximated outage probabilities, we consider two scenarios: one in which the FC estimates the arithmetic mean, and another where the geometric mean is desired. Accordingly, we compare the approximations (20) and (25) with Monte Carlo evaluations of the true outage probability $\mathbb{P}(|E| \geq \epsilon)$ based on $10 \cdot 10^3$ realizations. Note that for both simulation examples, $P_{\max}$ and σ_N^2 have been chosen in agreement with commercial IEEE 802.15.4 compliant sensor platforms [48].

Example 1 (Arithmetic Mean). Let $M = 25, 50, 150, 250$, the number of nodes $K = M$ and the sensor readings uniformly and i.i.d. in $\mathcal{X} = [1\,^\circ\mathrm{C}, 30\,^\circ\mathrm{C}] \subset \mathcal{S}$. The resulting experimental data is depicted in Fig. 4(a).

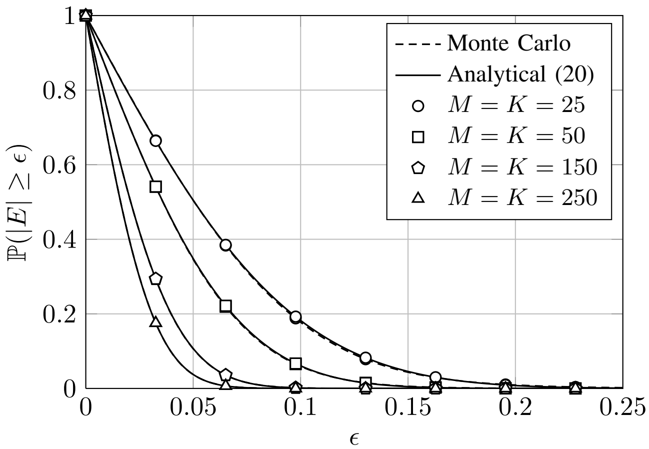
The plots in Fig. 4(a) show that approximation (20) closely matches the true outage probability $\mathbb{P}(|E| \geq \epsilon)$ for all $\epsilon > 0$. Notice that already for relatively short sequence lengths M, differences between the analytical expression and the Monte Carlo simulations are negligible. Furthermore, the plots confirm the consistency statement of Proposition 4, because the probability curves tend to the ordinate axis with growing M.

Example 2 (Geometric Mean). Let $\mathcal{S}' = [s', s_{\max}] = [0.5\,^\circ\mathrm{C}, 130\,^\circ\mathrm{C}] \subset \mathcal{S}$, $\mathcal{X} = [1\,^\circ\mathrm{C}, 30\,^\circ\mathrm{C}] \subset \mathcal{S}'$, $a = 2$, and all other simulation parameters as in Example 1.[6] The resulting experimental data is depicted in Fig. 4(b).

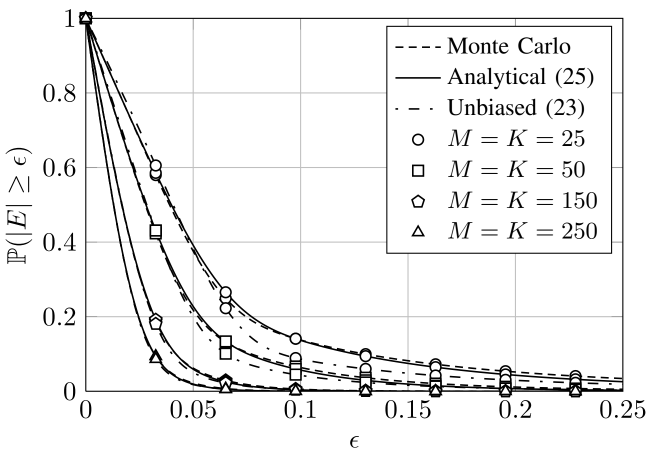
Similar as for Example 1, the plots in Fig. 4(b) show that (25) approximates with (26) and (27) the true outage probability sufficiently accurate, with a negligible deviation for short sequence lengths. Even though the geometric mean estimator (21) is applicable in practice, it has the drawback that unbiasedness is achieved only asymptotically as $M \to \infty$. To quantify this drawback, Fig. 4(b) also depicts Monte Carlo evaluations of $\mathbb{P}(|E| \geq \epsilon)$ using the unbiased (but impractical) estimator (23). Note that the drawback vanishes quickly with increasing M, which therefore confirms Proposition 5 and Remark 5.

Remark 6. Propositions 4 and 5 as well as Examples 1 and 2 demonstrate that the sequence length M is the crucial design parameter that determines the trade-off between computation accuracy and computation throughput.

[6] Notice that the corresponding function estimation error relies on $\mathcal{S}'$ since the geometric mean cannot be continuously extended onto the entire sensing range $\mathcal{S} = [-55\,^\circ\mathrm{C}, 130\,^\circ\mathrm{C}]$.



(a) Arithmetic Mean



(b) Geometric Mean

Fig. 4. Monte Carlo evaluation of the outage probabilities ($10 \cdot 10^3$ realizations) vs. analytical results for different $M = K$.

B. Comparisons with TDMA and CDMA

The numerical examples in the preceding subsection indicate the general behavior of the proposed analog computation architecture without concrete evidence regarding the computation performance compared to standard multiple-access schemes. Therefore, we demonstrate in this subsection the advantage of the proposed CoMAC architecture over idealized uncoded TDMA and CDMA schemes. In both cases, the individual nodes quantize their sensor readings uniformly over the sensing range $\mathcal{S}$ with $Q \in \mathbb{N}$ bit and transmit them as bipolar symbol streams to the FC, which reconstructs all readings to subsequently compute the sought function value.

1) TDMA: To ensure fairness between CoMAC and TDMA, with fixed degrees of freedom (e.g., bandwidth, symbol duration), both schemes should induce the same costs per function value computation with respect to transmit energy and transmit time. Therefore, let $T \in \mathbb{R}_{++}$ be the common symbol duration and let $p_{\text{TDMA},k} \in \mathbb{R}_{++}$ denote the instantaneous TDMA transmit power on node $k \in \mathcal{K}$. Then, the transmit times per function value are $T_{\text{CoMAC}} = MT$ and $T_{\text{TDMA}} = QKT$, whereas the transmit energies can be written as $\mathsf{E}_{\text{CoMAC},k} = Mp_k T$ and $\mathsf{E}_{\text{TDMA},k} = Qp_{\text{TDMA},k}T$, respectively. Now, from the fairness conditions $T_{\text{CoMAC}} = T_{\text{TDMA}}$ and $\mathsf{E}_{\text{CoMAC},k} = \mathsf{E}_{\text{TDMA},k}$, $k \in \mathcal{K}$, it follows $M = QK$ for the CoMAC sequence length and $p_{\text{TDMA},k} = \frac{p_k M}{Q} = \frac{g_\varphi(\varphi_k(x_k))M}{Q}$, $k \in \mathcal{K}$, for the required instantaneous TDMA transmit powers.

In addition to the fairness aspect requires an adequate comparison the determination of a common system operating point, which can be done in terms of an average Signal-to-Noise Ratio (SNR). Assume for simplicity that the sensor readings X_k are i.i.d. in $\mathcal{X}$, for all $k \in \mathcal{K}$, such that the average received TDMA-SNR per node can be defined as[7]

$$\text{SNR}_{\text{TDMA}} := \frac{2M\mathbb{E}\{P_1\}}{\sigma_N^2 Q} . \tag{28}$$

Example 3 (Small Network Size). Let $K = 25$, $Q = 10\,\text{bit}$, the sequence length $M = QK$, and let $P_{\max}$ and σ_N^2 be chosen such that $\text{SNR}_{\text{TDMA}}^{\text{dB}} := 10\log_{10}(\text{SNR}_{\text{TDMA}}) \in \{0, 2, 4, 6, 8, 10\}$. Furthermore, let the sensor readings be uniformly and i.i.d. in $\mathcal{X} = [5\,°\text{C}, 30\,°\text{C}] \subset \mathcal{S}$ and let the desired function be "arithmetic mean". The corresponding simulation data is depicted in Fig. 5(a).

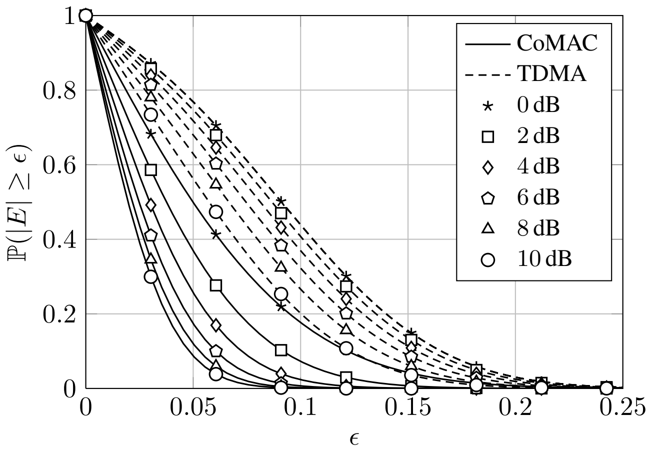
Example 4 (Medium Network Size). Let $K = 250$, the desired function be "geometric mean" with $\mathcal{S}' = [1\,°\text{C}, 130\,°\text{C}] \subset \mathcal{S}$, $a = 2$, and let all other simulation parameters as in Example 4. The corresponding simulation data is shown in Fig. 5(b).

Figures 5(a) and 5(b) indicate the huge potential of the proposed analog CoMAC scheme for efficiently computing linear and nonlinear functions over the wireless channel. In both examples, CoMAC entirely outperforms TDMA with respect to the computation accuracy for different network parameters. It should be clear that the shown performance gains can be trade off for better computation throughput or higher energy efficiency.

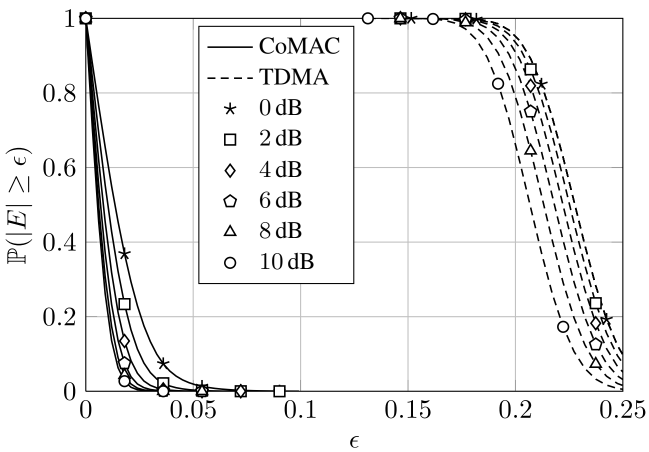
Remark 7. It is important to emphasize that the shown performance gains are quite conservative since the simulated TDMA scheme was idealized in many ways. For example, a realistic TDMA would require an established protocol stack with considerable amount of overhead per frame (e.g., header, synchronization information, check sum) such that the overall TDMA transmission time would extend to $T_{\text{TDMA}} = (Q + R)KT$ with some $R \in \mathbb{N}$.

2) CDMA: In contrast to the TDMA-based approach in which the sensor readings are transmitted in an interference-free manner to the FC, the nodes can employ the direct-sequence CDMA principle to also transmit concurrently in the same frequency band. Therefore, to compare a CDMA-based approach with our CoMAC scheme, let for CDMA each node spread its quantized sensor reading prior to transmission with an individual fixed bipolar Welch-Bound-Equality (WBE) sequence of length $\tilde{M} \in \mathbb{N}$, constructed according to [49] from some m-sequence. The FC estimates all sensor readings from the received signal using the standard CDMA matched-filter receiver [4], which becomes in conjunction with WBE sequences an optimal linear minimum mean square error receiver.

[7] The factor of 2 in (28) and (29) results from the fact that the considered TDMA- and CDMA-based approaches use in contrast to CoMAC only one real dimension per transmission.



(a) Arithmetic Mean, $K = 25$ nodes



(b) Geometric Mean, $K = 250$ nodes

Fig. 5. CoMAC vs. TDMA: outage probabilities for quantization with $Q = 10$ bit (in the case of TDMA), sequence length $M = QK$, and $\text{SNR}_{\text{TDMA}}^{\text{dB}} = 0, 2, 4, 6, 8, 10\,\text{dB}$.

To make the comparison fair, we assume again that both schemes induce the same costs per function computation, from which it follows that the CoMAC sequence length M has to be chosen such that it equals $Q\tilde{M}$. This in conjunction with equally set transmit powers (i.e., $P_{\text{CDMA},k} = P_k = g_\varphi(\varphi_k(X_k))$, $k \in \mathcal{K}$) and X_k uniformly i.i.d. over $\mathcal{X} = [5\,°\text{C}, 30\,°\text{C}]$ results in equal transmit energy consumptions per transmitted symbol and in the average CDMA-SNR (per node)

$$\text{SNR}_{\text{CDMA}} := \frac{2\mathbb{E}\{P_1\}}{\sigma_N^2} \tag{29}$$

at the output of the matched filter.

Example 5. Let $K = 256$, $Q = 10\,\text{bit}$, $\tilde{M} = 15$ ($\Rightarrow M = 150$), the desired function be "geometric mean", $P_{\max}$ and σ_N^2 chosen such that $\text{SNR}_{\text{CDMA}}^{\text{dB}} := 10\log_{10}(\text{SNR}_{\text{CDMA}}) \in \{0, 2, 4, 6, 8, 10\}$, and let all other simulation parameters as in Example 4. The corresponding achievable outage probabilities are depicted in Fig. 6.

The plots show that in the considered range of typical sensor network operating points, our analog CoMAC scheme also entirely outperforms a computation approach that is based on the CDMA principle.[8] The reader should notice that even if the proposed CoMAC scheme seems to be quite similar to the CDMA concept from a transmitter-side perspective, the huge performance gains result from estimating the desired function *immediately* from the W-MAC output instead of trying to reconstruct all individual sensor readings by multiuser detection.

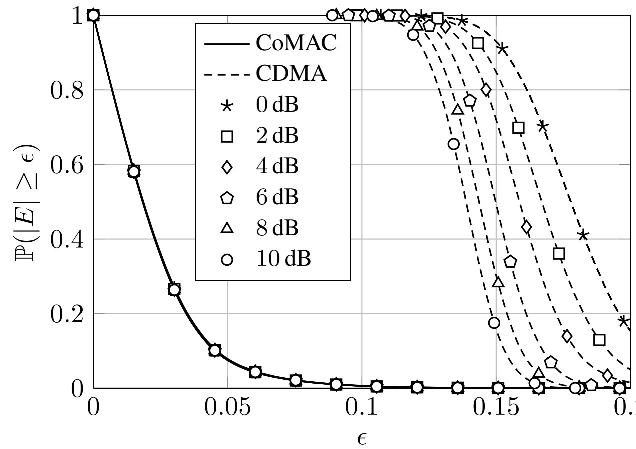


Fig. 6. CoMAC vs. CDMA: outage probabilities for quantization with $Q = 10$ bit (in the case of CDMA), $K = 256$ nodes, sequence length $M = 150$, $\text{SNR}_{\text{CDMA}}^{\text{dB}} = 0, 2, 4, 6, 8, 10\,\text{dB}$, and desired function "geometric mean".

Remark 8. In the terminology of CDMA represents Example 5 an overloaded case (i.e., $M < K$). This was considered because signature waveforms can always be chosen to be mutually orthogonal whenever $M \geq K$ such that the resulting outage performance would at best be the same as for TDMA (see Fig. 5(a) and 5(b)), and would therefore not provide new insights. Note that the overloaded case was the reason for letting the nodes spread their quantized sensor readings with WBE sequences since for $M < K$ they are optimal in the sense that they can achieve the lower bound on the sum of cross-correlations between spreading sequences [49].

VI. Conclusion

In this paper, we proposed a simple analog scheme for efficiently computing functions of the measurements in wireless sensor networks. The main idea of the approach is to exploit the natural superposition property of the wireless channel by letting nodes transmit simultaneously to a fusion center. Applying an appropriate pre-processing function to each sensor reading prior to transmission and a post-processing function to the signal received by the fusion center, which is the superposition of the signals transmitted by the individual nodes, the approach allows the analog computation of a huge set of linear and nonlinear functions over the channel. To relax corresponding synchronization requirements, the nodes transmit some random sequences at a transmit power that is proportional to the respective pre-processed sensor information. As a consequence, only a coarse frame synchronization is required such that the scheme is robust against synchronization

[8] The CoMAC plots in Fig. 6 coincide with each other due to the fact that the CoMAC outage performance is in the considered parameter range mainly determined by the cross-correlations between the random transmit sequences and less by the receiver noise.

errors on the symbol and phase level. The second essential part of the scheme consists of an analog computation-receiver that is designed to appropriately estimate desired function values from the post-processed received sum of transmit energies. Since the estimator has to be matched to the desired function, we considered two canonical function examples and proposed corresponding estimators with good statistical properties.

Numerical comparisons with a TDMA- and a CDMA-based approach have shown that the proposed analog computation scheme has the potential to achieve huge performance gains in terms of computation accuracy or computation throughput over medium-access strategies that separate the transmissions of nodes in time or code space, why the gains remain the same when the scheme would be compared with OFDMA. In addition to the weaker requirements regarding the synchronization of sequences, the scheme needs no explicit protocol structure, which significantly reduces overhead. Computation schemes following the described design rule are therefore energy and complexity efficient and can be easily implemented in practice. Finally, the hardware-effort is reduced as well since energy consuming digital components (e.g., analog-to-digital converters, registers) are not required.

Note that the proposed computation scheme can be used as a building block of complex in-network processing schemes and wireless networking protocols. For example, if sensor nodes are equipped with both the computation-transmitter and the computation-receiver described in Sections III-A and III-B, respectively, our scheme can be combined with adaptive clustering methods as those proposed in [50].

APPENDIX

A. Proof of Proposition 1

Let $g'_\varphi := Mg_\varphi$ and $h'_\varphi := h_\varphi/M$ so that we have to show that $h'_\varphi(\sum_k g'_\varphi(\xi_k)) = \sum_k \xi_k$ with $\xi_k \in [\varphi_{\min}, \varphi_{\max}]$, $k \in \mathcal{K}$, holds if and only if g_φ and h_φ are affine functions. The "$\Leftarrow$" direction is trivial, while the other direction is shown by contradiction. Suppose g'_φ is bijective and continuous but not affine. Then, there exist two points $(\xi_1, \ldots, \xi_K)$ and $(\tilde{\xi}_1, \ldots, \tilde{\xi}_K)$ in $[\varphi_{\min}, \varphi_{\max}]^K$ with $\sum_k \xi_k \neq \sum_k \tilde{\xi}_k$ but $\sum_k g'_\varphi(\xi_k) = \sum_k g'_\varphi(\tilde{\xi}_k)$. By the last equation, we have

$$\textstyle\sum_k \xi_k = h'_\varphi\big(\sum_k g'_\varphi(\xi_k)\big) = h'_\varphi\big(\sum_k g'_\varphi(\tilde{\xi}_k)\big) = \sum_k \tilde{\xi}_k\ ,$$

which however contradicts $\sum_k \xi_k \neq \sum_k \tilde{\xi}_k$. Hence, g'_φ is affine and so is g_φ. Moreover, we have $h_\varphi(\sum_k g'_\varphi(\xi_k)) = h_\varphi(Mg_\varphi(\sum_k \xi_k) + \tilde{c})$ for some $\tilde{c} \in \mathbb{R}$, from which we conclude that h_φ is an affine function as well with $h_\varphi \equiv g_\varphi^{-1} - c$ and some constant $c \in \mathbb{R}$ that depends on g_φ.

B. Proof of Lemma 2

Since $\Delta_3 \sim \chi^2_{2M}$, the probability density of Δ_3 is $p_{\Delta_3}(x) = \frac{1}{\sigma_N^{2M}\,\Gamma(M)} x^{M-1}\,\mathrm{e}^{-x/\sigma_N^2}\,\mathbb{1}_{[0,\infty)}(x)$, where $\Gamma(z)$ with $\mathrm{Re}\{z\} > 0$ is used to denote the Gamma function. Hence, one obtains

$$\begin{aligned}&\mathbb{E}\{\psi(\tfrac{\Delta_3}{\alpha_{\text{geo}}M})\}\\&= \tfrac{1}{\sigma_N^{2M}\,\Gamma(M)} \int_0^\infty x^{M-1} \exp\Big(-\Big(\tfrac{\alpha_{\text{geo}}KM - \sigma_N^2 \log_{\mathrm{e}}(a)}{\sigma_N^2 \alpha_{\text{geo}}KM}\Big)x\Big)\,\mathrm{d}x.\end{aligned} \tag{30}$$

Now, assume $\sigma_N^2 \log_{\mathrm{e}}(a) < \alpha_{\text{geo}}KM$ and note that $\Gamma(z) = \int_0^\infty x^{z-1}\,\mathrm{e}^{-x}\,\mathrm{d}x = k^z \int_0^\infty x^{z-1}\,\mathrm{e}^{-kx}\,\mathrm{d}x$, $\mathrm{Re}\{z\} > 0$, which holds for any $\mathrm{Re}\{k\} > 0$ [51]. So substituting this into (30) with an appropriately chosen k proves (i). As for (ii), if $\sigma_N^2 \log_{\mathrm{e}}(a) < \alpha_{\text{geo}}KM$, then it follows from (i) that $\lim_{M\to\infty}\Big(\frac{\alpha_{\text{geo}}KM}{\alpha_{\text{geo}}KM - \sigma_N^2 \log_{\mathrm{e}}(a)}\Big)^M = \lim_{M\to\infty}(1 + \frac{u}{M})^{-M} = \mathrm{e}^{-u}$, where $u := -\frac{\sigma_N^2 \log_{\mathrm{e}}(a)}{\alpha_{\text{geo}}K}$.

C. Proof of Lemma 3

Let $\mathcal{X}$ be an arbitrary compact set and $K < \infty$ any fixed natural number. By Section II-C and Definition 6, we know that $\Xi|\boldsymbol{x} = \psi(\frac{\Delta|\boldsymbol{x}}{\alpha_{\text{geo}}M}) = a^{\frac{1}{\alpha_{\text{geo}}KM}\Delta|\boldsymbol{x}}$, $K, \alpha_{\text{geo}} > 0$, $a > 1$. Since ψ is continuous and strictly increasing, $P_\Xi(\xi|\boldsymbol{x}) = \mathbb{P}(\Xi \le \xi|\boldsymbol{X} = \boldsymbol{x}) = \mathbb{P}(\Delta \le \alpha_{\text{geo}}KM\log_a(\xi)|\boldsymbol{X} = \boldsymbol{x}) = P_\Delta(\alpha_{\text{geo}}KM\log_a(\xi)|\boldsymbol{x})$, $\xi > 0$. Thus, with the corollary to Proposition 2, we can conclude that $\Delta|\boldsymbol{x}$ can be approximated by a random variable $\tilde{\Delta}|\boldsymbol{x} \sim \mathcal{N}_{\mathbb{R}}(M\sigma_N^2, \sigma^2_{\Delta|\boldsymbol{x}})$ for M sufficiently large. An immediate consequence of this is that for sufficiently large values of M, the distribution function of $\Delta|\boldsymbol{x}$ can be approximated by $P_{\tilde{\Delta}}(\delta|\boldsymbol{x}) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\big(\frac{\delta - M\sigma_N^2}{\sqrt{2}\sigma_{\Delta|\boldsymbol{x}}}\big)$ (i.e., $P_\Delta(\delta|\boldsymbol{x}) \approx P_{\tilde{\Delta}}(\delta|\boldsymbol{x})$). Moreover, for M large enough, the Mann-Wald theorem [46, p. 356] implies $P_\Xi(\xi|\boldsymbol{x}) \approx P_{\tilde{\Xi}}(\xi|\boldsymbol{x}) = P_{\tilde{\Delta}}(\alpha_{\text{geo}}KM\log_a(\xi)|\boldsymbol{x})$, where for $\xi \in \mathbb{R}_{++}$

$$\begin{aligned}&P_{\tilde{\Delta}}\big(\alpha_{\text{geo}}KM\log_a(\xi)|\boldsymbol{x}\big)\\&= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\alpha_{\text{geo}}KM\log_{\mathrm{e}}(\xi) - \sigma_N^2 M \log_{\mathrm{e}}(a)}{\sqrt{2}\,\log_{\mathrm{e}}(a)\sigma_{\Delta|\boldsymbol{x}}}\right).\end{aligned} \tag{31}$$

Note that (31) describes the distribution function of a log-normally distributed random variable with parameters $\frac{\sigma_N^2 \log_{\mathrm{e}}(a)}{\alpha_{\text{geo}}K} =: \mu_\Xi$ and $\big(\frac{\log_{\mathrm{e}}(a)}{\alpha_{\text{geo}}KM}\sigma_{\Delta|\boldsymbol{x}}\big)^2 =: \sigma^2_{\Xi|\boldsymbol{x}}$. Thus, $\Xi|\boldsymbol{x}$ can be approximated by $\tilde{\Xi}|\boldsymbol{x} \sim \mathcal{LN}(\mu_\Xi, \sigma^2_{\Xi|\boldsymbol{x}})$.

D. Proof of Proposition 6

Note that it is sufficient to show (26) and (27). Because $|E|\boldsymbol{x}| = |\gamma(\boldsymbol{x})^{-1}\Xi|\boldsymbol{x} - \beta(\boldsymbol{x})|$ is continuous in $\Xi|\boldsymbol{x}$, Lemma 3 and the Mann-Wald theorem allow for the approximation of $|\Xi|\boldsymbol{x}|$ by $|\tilde{E}|\boldsymbol{x}| = |\gamma(\boldsymbol{x})^{-1}\tilde{\Xi}|\boldsymbol{x} - \beta(\boldsymbol{x})|$, where the probability distribution function of $\tilde{\Xi}|\boldsymbol{x} \sim \mathcal{LN}(\mu_\Xi, \sigma^2_{\Xi|\boldsymbol{x}})$ is given by (31). Since $0 < \beta(\boldsymbol{x}), \gamma(\boldsymbol{x}) < \infty$, we have $\mathbb{P}(|E| \ge \epsilon|\boldsymbol{X} = \boldsymbol{x}) \approx \mathbb{P}(|\tilde{E}| \ge \epsilon|\boldsymbol{X} = \boldsymbol{x}) = 1 - \mathbb{P}(-\epsilon < \tilde{E} < \epsilon|\boldsymbol{X} = \boldsymbol{x}) = 1 - \mathbb{P}(-\epsilon < \gamma(\boldsymbol{X})^{-1}\tilde{\Xi} - \beta(\boldsymbol{X}) < \epsilon|\boldsymbol{X} = \boldsymbol{x})$, which leads to

$$\begin{aligned}&\mathbb{P}(|\tilde{E}| \ge \epsilon|\boldsymbol{X} = \boldsymbol{x})\\&= \begin{cases}1 - P_{\tilde{\Xi}}\big(\rho^+(\boldsymbol{x},\epsilon)|\boldsymbol{x}\big) + P_{\tilde{\Xi}}\big(\rho^-(\boldsymbol{x},\epsilon)|\boldsymbol{x}\big), & 0 < \epsilon < \beta(\boldsymbol{x})\\ 1 - P_{\tilde{\Xi}}\big(\rho^+(\boldsymbol{x},\epsilon)|\boldsymbol{x}\big), & \beta(\boldsymbol{x}) \le \epsilon < \infty\end{cases}\end{aligned} \tag{32}$$

with $\rho^+(\boldsymbol{x},\epsilon) := \gamma(\boldsymbol{x})(\beta(\boldsymbol{x}) + \epsilon)$ and $\rho^-(\boldsymbol{x},\epsilon) := \gamma(\boldsymbol{x})(\beta(\boldsymbol{x}) - \epsilon)$. Inserting the right-hand side of (31) into expression (32) and using $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$, for all $x \in \mathbb{R}$, shows (26) and (27) and thus completes the proof.

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