COMPLETE CHARACTERIZATION OF BANDLIMITED SIGNALS WITH BOUNDED HILBERT TRANSFORM

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ABSTRACT

For the space of bounded bandlimited signals a definition of the Hilbert transform by the usual Hilbert transform integral is not possible, because the integral diverges for certain bounded bandlimited signals. There are other ways to define the Hilbert transform meaningfully. Recently, it was shown that, for bounded bandlimited signals, a simple formula can be used to calculate the Hilbert transform. However, the Hilbert transform of a bounded bandlimited signal is not necessarily bounded again. In this paper, we completely characterize the bounded bandlimited signals that have a bounded Hilbert transform by giving a necessary and sufficient condition for the boundedness. Further, we use this condition to prove that there exist bounded bandlimited signals that even vanish at infinity, the Hilbert transform of which is unbounded.

Index Terms— Hilbert transform, bounded bandlimited signal, peak value, analytical signal

1. INTRODUCTION AND NOTATION

The Hilbert transform is an important tool in communication theory and signal processing. For example, the "analytical signal" [1], which was introduced in Dennis Gabor's "Theory of Communication" [2], and the definition of the instantaneous amplitude, frequency, and phase of a signal [3, 4, 1] are based on the Hilbert transform. Further, the Hilbert transform is used in the theory of modulation [5, 1]. Classically, the Hilbert transform of a smooth signal f with compact support is defined as the principal value integral

$$(Hf)(t) = \frac{1}{\pi} \text{V.P.}\int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau}\mathrm{d}\tau = \frac{1}{\pi}\lim_{\epsilon\to 0} \int\limits_{\epsilon \le |t-\tau| \le \frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau}\mathrm{d}\tau$$

$$= \frac{1}{\pi}\lim_{\epsilon\to 0}\left(\int_{t-\frac{1}{\epsilon}}^{t-\epsilon} \frac{f(\tau)}{t-\tau}\mathrm{d}\tau + \int_{t+\epsilon}^{t+\frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau}\mathrm{d}\tau\right). \quad (1)$$

In general, the principal value integral (1) cannot be used to define the Hilbert transform for bounded signals in $L^\infty(\mathbb{R})$, because there are signals in $L^\infty(\mathbb{R})$ such that (1) diverges for all $t \in \mathbb{R}$. For example, for the bounded bandlimited signal

$$f_{\mathrm{L}}(t) = -\frac{2}{\pi}\int_0^\pi \frac{\sin(\omega t)}{\omega}\mathrm{d}\omega,$$

which is closely related to the sine integral, we have for all $t \in \mathbb{R}$

$$\frac{1}{\pi}\lim_{\epsilon\to 0} \int\limits_{\epsilon \le |t-\tau| \le \frac{1}{\epsilon}} \frac{f_{\mathrm{L}}(\tau)}{t-\tau}\mathrm{d}\tau = \infty.$$

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Note that this result does not imply that the Hilbert transform cannot be defined for the space of bounded bandlimited signals. There may be other definitions, apart from the principal value integral definition (1) that are meaningful.

A class of signals for which the Hilbert transform as principal value integral (1) exists and is bounded, are bounded bandpass signals. If f is a bandpass signals, the distributional Fourier transform of which vanishes outside $[-\pi, -\epsilon\pi] \cup [\epsilon\pi, \pi]$, $0 < \epsilon < 1$, then f has a bounded Hilbert transform satisfying

$$\|Hf\|_\infty \le \left(A + \frac{2}{\pi}\log\left(\frac{1}{\epsilon}\right)\right)\|f\|_\infty,$$

where $A < 4/\pi$ is a constant [6, 5]. That is, the upper bound on the peak value of the Hilbert transform diverges as ϵ tends to zero. Probably, observations of this kind led to the misbelief "that an arbitrary bounded bandlimited function does not have a Hilbert transform…" [5, p. 502].

Regardless of the convergence problems of the principal value integral (1), the Hilbert transform can be meaningfully defined for signals in $L^\infty(\mathbb{R})$, by using Fefferman's duality theorem [7]. Unfortunately, this rather abstract definition does not provide a constructive procedure for the calculation of the Hilbert transform. We do not go further into this definition, because it was shown in [8] that for the subspace of bounded bandlimited signals $\mathcal{B}_\pi^\infty$ a much simpler, constructive approach can be taken for the calculation of the Hilbert transform. This approach will be presented in Section 2.

A main result of this new theory is that the Hilbert transform of a bounded bandlimited signal is again bandlimited but not necessarily bounded. That is, the peak value of the Hilbert transform can be arbitrarily large. The peak value is a basic characteristic of signals. In many applications it is crucial to control the peak value. For example, in wireless communication systems high peak-to-average power ratios (PAPRs) are problematic because high peak values can overload the power amplifiers, which in turn leads to undesired out-of-band radiation. For this reason it is important to characterize the signals that have a bounded Hilbert transform.

In this paper we study the Hilbert transform for bounded bandlimited signals, and provide a simple test for the boundedness of the Hilbert transform. Further, we completely characterize the signals for which the integral (1) diverges unboundedly and identify a large class of signals for which the integral (1) converges.

We need some definitions and notation. By $L^p(\mathbb{R})$, $1 \le p < \infty$, we denote the space of all pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and by $L^\infty(\mathbb{R})$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. For $0 < \sigma < \infty$ let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \le C(\epsilon)\exp((\sigma+\epsilon)|z|)$ for all $z \in \mathbb{C}$. The Bernstein space $\mathcal{B}_\sigma^p$, $1 \le$

$p \leq \infty$, consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line, i.e., $\|\cdot\|_{\mathcal{B}_\sigma^p} = \|\cdot\|_p$. A signal in $\mathcal{B}_\sigma^p$ is called bandlimited to σ, and $\mathcal{B}_\sigma^\infty$ is the space of bandlimited signals that are bounded on the real axis. We call a signal in $\mathcal{B}_\pi^\infty$ bounded bandlimited signal.

2. THE HILBERT TRANSFORM FOR $\mathcal{B}_\pi^\infty$

The Hilbert transform of a general bounded signal can be meaningfully defined by using Fefferman's duality theorem, which states that the dual space of $\mathcal{H}^1$ is BMO($\mathbb{R}$) [1]. In this definition, the Hilbert transform $\mathfrak{H}f$ of $f \in L^\infty(\mathbb{R})$ is a function in BMO($\mathbb{R}$), which is only unique up to an arbitrary additive constant C_{BMO} [2].

Next, we present a constructive approach for the calculation of the Hilbert transform $\mathfrak{H}f$ for signals in $\mathcal{B}_\pi^\infty$. A key ingredient is the operator $Q^{\text{E}} : \mathcal{B}_\pi^\infty \to \mathcal{B}_\pi^\infty$, defined by

$$(Q^{\text{E}} f)(z) = \sum_{k=-\infty}^{\infty} a_k f(z-k), \quad z \in \mathbb{C}, \tag{2}$$

where the coefficients a_k, $k \in \mathbb{Z}$, are given by

$$a_k = \begin{cases} \frac{\pi}{2}, & k = 0, \\ \frac{(-1)^k - 1}{\pi k^2}, & k \neq 0. \end{cases}$$

It can be shown [8] that $Q^{\text{E}} : \mathcal{B}_\pi^\infty \to \mathcal{B}_\pi^\infty$ is a bounded linear operator with norm $\|Q^{\text{E}}\| = \pi$. Further, since $Q^{\text{E}} : \mathcal{B}_\pi^\infty \to \mathcal{B}_\pi^\infty$ is a bounded linear operator, the operator $\mathfrak{I}$ given by

$$(\mathfrak{I}f)(z) = \int_{\Gamma(0,z)} (Q^{\text{E}} f)(\xi) \mathrm{d}\xi, \quad z \in \mathbb{C}, \tag{3}$$

where $\Gamma(0, z)$ is an arbitrary piecewise smooth curve in the complex plane from 0 to z, is well defined for every $f \in \mathcal{B}_\pi^\infty$.

For bandlimited signals with finite energy, i.e., signals in $\mathcal{B}_\pi^2$, the operator Q^{E} is nothing else than the concatenation of the Hilbert transform H and the differential operator D, i.e., $Q^{\text{E}} = DH$. Thus, for $g \in \mathcal{B}_\pi^2$, the integral of $Q^{\text{E}} g$ as in (3) gives—up to a constant—the Hilbert transform Hg of g. More precisely, for $g \in \mathcal{B}_\pi^2$ we have

$$\begin{aligned}(\mathfrak{I}g)(t) &= \int_0^t (Q^{\text{E}} g)(\tau) \mathrm{d}\tau = \int_0^t (Qg)(\tau) \mathrm{d}\tau \\ &= \int_0^t (DHg)(\tau) \mathrm{d}\tau = (Hg)(t) - (Hg)(0). \end{aligned} \tag{4}$$

That is, for every signal $g \in \mathcal{B}_\pi^2$, we have $(Hg)(t) = (\mathfrak{I}g)(t) + C_1(g)$, $t \in \mathbb{R}$, where $C_1(g)$ is a constant that depends on g.

Based on this observation it is natural to assume that, for signals $f \in \mathcal{B}_\pi^\infty$, the integral $\mathfrak{I}f$ is somehow connected to the Hilbert $\mathfrak{H}f$ transform of f. In [8], it was shown that such a connection exists in the sense that $\mathfrak{H}f = \mathfrak{I}f + C_{\text{BMO}}$, where C_{BMO} is an arbitrary constant [3].

Theorem 1. *Let $f \in \mathcal{B}_\pi^\infty$. Then we have $\mathfrak{H}f = \mathfrak{I}f + C_{BMO}$. Further, the Hilbert transform is again bandlimited, because $\mathfrak{I}f \in \mathcal{B}_\pi$.*

Theorem 1 is very useful, because it enables us to compute the Hilbert transform of a bounded bandlimited signals by using the constructive formula (3), instead of using the abstract definition which is based on the $\mathcal{H}^1$-BMO($\mathbb{R}$) duality. Note that $\mathfrak{I}f$ is well defined for all signals $f \in \mathcal{B}_\pi^\infty$, which also means that $(\mathfrak{I}f)(t)$ can be computed and is finite for all $t \in \mathbb{R}$.

3. CONDITION FOR THE BOUNDEDNESS OF THE HILBERT TRANSFORM

Thanks to Theorem 1, we can use the simple formula (3) to compute the Hilbert transform of bounded bandlimited signal. In [8, 10] the properties of $\mathfrak{I}f$, i.e., of the Hilbert transform, were studied for signals $f \in \mathcal{B}_\pi^\infty$. It was found that there exists a signal $f_1 \in \mathcal{B}_\pi^\infty$ such that $\mathfrak{I}f_1$ is unbounded on the real axis. Thus, the Hilbert transform of a bounded bandlimited signal is again a bandlimited (Theorem 1) but not necessarily a bounded signal.

For practical applications is important to know when the Hilbert transform is bounded. Theorem 2 gives a necessary and sufficient condition for the boundedness of the Hilbert transform.

Theorem 2. *Let $f \in \mathcal{B}_\pi^\infty$ be real-valued. We have $\mathfrak{I}f \in \mathcal{B}_\pi^\infty$ if and only if there exists a constant C_2 such that*

$$\left| \frac{1}{\pi} \int_{\epsilon \leq |t-\tau| \leq \frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau \right| \leq C_2 \tag{5}$$

for all $0 < \epsilon < 1$ and all $t \in \mathbb{R}$.

Remark 1. By Theorem 2 we have a complete characterization of the signals in $\mathcal{B}_\pi^\infty$ that have a bounded Hilbert transform. Theorem 2 further shows that the unbounded divergence of the principal value integral is connected to the unboundedness of the Hilbert transform.

For the proof of Theorem 2 we need Lemma 1.

Lemma 1. *Let $f \in \mathcal{B}_\pi^\infty$ and $\mathfrak{I}f \in \mathcal{B}_\pi^\infty$. Then, for $F = f + i\mathfrak{I}f$, we have $|F(t+iy)| \leq \|F\|_\infty$ for all $t \in \mathbb{R}$ and $y \geq 0$.*

Proof. Let $t \in \mathbb{R}$ and $y > 0$ be arbitrary but fixed. Since $f \in \mathcal{B}_\pi^\infty$ and $\mathfrak{I}f \in \mathcal{B}_\pi^\infty$, we have $F \in \mathcal{B}_\pi^\infty$ and therefore the integral

$$\frac{1}{\pi} \int_{-\infty}^{\infty} F(\tau) \frac{y}{y^2 + (t-\tau)^2} \mathrm{d}\tau$$

is absolutely convergent. It can be shown that

$$F(t+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\tau) \frac{y}{y^2 + (t-\tau)^2} \mathrm{d}\tau.$$

Then, it follows that

$$\begin{aligned} |F(t+iy)| &\leq \frac{1}{\pi} \int_{-\infty}^{\infty} |F(\tau)| \frac{y}{y^2 + (t-\tau)^2} \mathrm{d}\tau \\ &\leq \|F\|_\infty \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (t-\tau)^2} \mathrm{d}\tau = \|F\|_\infty. \quad \square \end{aligned}$$

Proof of Theorem 2. Since the "$\Leftarrow$" direction is not needed in the rest of the paper, it is omitted, due to space constraints. It follows the proof of the "$\Rightarrow$" direction. Let $f \in \mathcal{B}_\pi^\infty$ be real-valued, such that $\mathfrak{I}f \in \mathcal{B}_\pi^\infty$. Further, let ϵ with $0 < \epsilon < 1$ and $t \in \mathbb{R}$ be arbitrary but fixed, and consider the complex contour that is depicted in Fig. 1. Since $F = f + i\mathfrak{I}f$ is an entire function, we have according to Cauchy's integral theorem that

$$\int_{P_{\epsilon,t}} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi = 0.$$

[1] For details and a definition of the spaces $\mathcal{H}^1$ and BMO($\mathbb{R}$), see [9].

[2] In a strict mathematical sense, the Hilbert transform in this definition is not a function but an equivalence class that contains all functions that differ only by a constant. (Hence, we use a different notation $\mathfrak{H}f$.) For technical details see [8, 10].

[3] More precisely, $\mathfrak{I}f$ is a representative of the equivalence class $\mathfrak{H}f$.

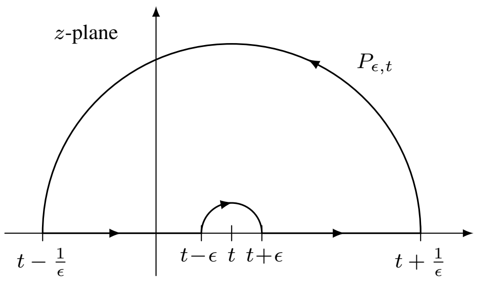

Fig. 1. Integration path $P_{\epsilon,t}$ in the complex plane.

Further, we have

$$\int\limits_{P_{\epsilon,t}} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi = \int\limits_{\curvearrowright} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi + \int\limits_{\frown} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi + \int\limits_{\rightarrow\,\rightarrow} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi.$$

Thus, it follows that

$$\int\limits_{\rightarrow\,\rightarrow} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi = -\int\limits_{\curvearrowright} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi - \int\limits_{\frown} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi. \tag{6}$$

Next, we analyze the two integrals on the right hand side of (6). For the first integral we have

$$\begin{aligned}\int\limits_{\curvearrowright} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi &= \int_{-\pi}^{0} \frac{F(t+\epsilon\,\mathrm{e}^{i\phi})}{\epsilon\,\mathrm{e}^{i\phi}} i\epsilon\,\mathrm{e}^{i\phi}\,\mathrm{d}\phi \\ &= i\int_{-\pi}^{0} F(t+\epsilon\,\mathrm{e}^{i\phi})\mathrm{d}\phi, \end{aligned} \tag{7}$$

and consequently

$$\left|\int\limits_{\curvearrowright} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi\right| \leq \pi \sup_{\mathrm{Im}(z)\geq 0} F(z) \leq \pi\|F\|_\infty, \tag{8}$$

where we used Lemma 1 in the last inequality. For the second integral, a similar calculation yields

$$\left|\int\limits_{\frown} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi\right| \leq \pi\|F\|_\infty. \tag{9}$$

Combining (6), (8), and (9), we obtain $\left|\frac{1}{\pi}\int\limits_{\rightarrow\,\rightarrow} \frac{F(\xi)}{t-\xi}\mathrm{d}\xi\right| \leq 2\|F\|_\infty$. Since $|\mathrm{Re}\, z| \leq |z|$ for all $z \in \mathbb{C}$ and f is real-valued, this implies that $\left|\frac{1}{\pi}\int\limits_{\rightarrow\,\rightarrow} \frac{f(\xi)}{t-\xi}\mathrm{d}\xi\right| \leq 2\|F\|_\infty$, which completes the proof of the "⇒" direction. □

4. SIGNAL WITH UNBOUNDED HILBERT TRANSFORM

We can use Theorem 2 to show that the Hilbert transform of the signal

$$f_1(t) = -\frac{1}{2}\int_0^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \frac{\sin(\omega t)}{\omega}\mathrm{d}\omega,$$

which is plotted in Fig. 2, is unbounded. f_1 is a bounded bandlimited signal that satisfies $\lim_{|t|\to\infty} f_1(t) = 0$, i.e., vanishes on the real axis at infinity.

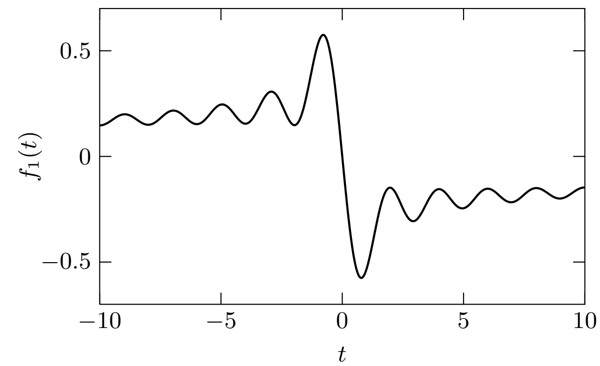

Fig. 2. Plot of the signal f_1.

Theorem 3. *We have* $\|\mathfrak{H}f_1\|_\infty = \infty$.

Proof. According to Theorem 1, it suffices to show that $\|\mathfrak{I}f_1\|_\infty = \infty$. We use an indirect proof and show that the assumption $\|\mathfrak{I}f_1\|_\infty < \infty$ leads to a contradiction.

Assume that $\|\mathfrak{I}f_1\|_\infty < \infty$. Since $f_1 \in \mathcal{B}_\pi^\infty$, we have $\mathfrak{I}f_1 \in \mathcal{B}_\pi^\infty$, due to Theorem 1. Therefore, Theorem 2 implies that there exists a constant C_2 such that

$$\left|\frac{1}{\pi}\int\limits_{\epsilon\leq|\tau|\leq\frac{1}{\epsilon}} \frac{f_1(\tau)}{-\tau}\mathrm{d}\tau\right| \leq C_2 \tag{10}$$

for all $0 < \epsilon < 1$. Next, we analyze the integral in (10). Let $0 < \epsilon < 1$ be arbitrary but fixed. Since $f(-t) = -f(t)$, $t \in \mathbb{R}$, we have

$$\int\limits_{\epsilon\leq|\tau|\leq\frac{1}{\epsilon}} \frac{f_1(\tau)}{-\tau}\mathrm{d}\tau = -2\int_\epsilon^{1/\epsilon} \frac{f_1(\tau)}{\tau}\mathrm{d}\tau, \tag{11}$$

and further

$$\begin{aligned} -2\int_\epsilon^{1/\epsilon} \frac{f_1(\tau)}{\tau}\mathrm{d}\tau &= \int_\epsilon^{1/\epsilon} \frac{1}{\tau}\int_0^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \frac{\sin(\omega\tau)}{\omega}\mathrm{d}\omega\mathrm{d}\tau \\ &= \int_0^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \int_\epsilon^{1/\epsilon} \frac{\sin(\omega\tau)}{\omega\tau}\mathrm{d}\tau\mathrm{d}\omega. \end{aligned} \tag{12}$$

The order of integration was exchanged according to Fubini's theorem, which can be applied because

$$\begin{aligned} \int_0^\pi \int_\epsilon^{1/\epsilon} \left|\frac{\sin(\omega\tau)}{\omega\tau\log\left(\frac{2\pi}{\omega}\right)}\right| \mathrm{d}\tau\mathrm{d}\omega &\leq \int_0^\pi \int_\epsilon^{1/\epsilon} \left|\frac{1}{\log\left(\frac{2\pi}{\omega}\right)}\right| \mathrm{d}\tau\mathrm{d}\omega \\ &\leq \left(\frac{1}{\epsilon}-\epsilon\right)\pi\frac{1}{\log(2)} < \infty, \end{aligned}$$

were we used $|\sin(t)/t| \leq 1$, for all $t \in \mathbb{R}$. Moreover, we have

$$\begin{aligned} &\int_0^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \int_\epsilon^{1/\epsilon} \frac{\sin(\omega\tau)}{\omega\tau}\mathrm{d}\tau\mathrm{d}\omega \\ &= \int_0^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \left(\int_0^{1/\epsilon} \frac{\sin(\omega\tau)}{\omega\tau}\mathrm{d}\tau - \int_0^\epsilon \frac{\sin(\omega\tau)}{\omega\tau}\mathrm{d}\tau\right)\mathrm{d}\omega \\ &\geq \int_0^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \int_0^{1/\epsilon} \frac{\sin(\omega\tau)}{\omega\tau}\mathrm{d}\tau\mathrm{d}\omega - C_3, \end{aligned} \tag{13}$$

because

$$\int_0^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \int_0^\epsilon \frac{\sin(\omega\tau)}{\omega\tau} \mathrm{d}\tau \mathrm{d}\omega \leq \frac{\epsilon\pi}{\log(2)} =: C_3,$$

which follows from

$$\int_0^\epsilon \frac{\sin(\omega\tau)}{\omega\tau} \mathrm{d}\tau \leq \epsilon, \quad \omega \in [0,\pi],$$

and the fact that $1/\log(2\pi/\omega)$ is monotone increasing and non-negative for $\omega \in [0,\pi]$. Combining (11),(12), and (13), we therefore obtain

$$\int\limits_{\epsilon\leq|\tau|\leq\frac{1}{\epsilon}} \frac{f_1(\tau)}{-\tau} \mathrm{d}\tau \geq \int_0^\pi \underbrace{\frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \int_0^{1/\epsilon} \frac{\sin(\omega\tau)}{\omega\tau} \mathrm{d}\tau\, \mathrm{d}\omega}_{=:g(\omega,\epsilon)} - C_3,$$

which holds for all $0 < \epsilon < 1$, because $0 < \epsilon < 1$ was arbitrary. Let $0 < a < 1$ be arbitrary but fixed. Since $g(\omega,\epsilon) \geq 0$ for all $\omega \in [0,\pi]$ and $0 < \epsilon < 1$, we can apply Fatou's Lemma to obtain

$$\begin{aligned}
&\liminf_{\epsilon\to0} \int_0^\pi g(\omega,\epsilon)\mathrm{d}\omega \geq \liminf_{\epsilon\to0} \int_a^\pi g(\omega,\epsilon)\mathrm{d}\omega \\
&\geq \int_a^\pi \liminf_{\epsilon\to0} g(\omega,\epsilon)\mathrm{d}\omega = \int_a^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \frac{\pi}{2\omega}\mathrm{d}\omega \\
&= \frac{\pi}{2} \log\left(\frac{\log\left(\frac{2\pi}{a}\right)}{\log(2)}\right), \qquad (14)
\end{aligned}$$

where we used in the first equality that

$$\lim_{\epsilon\to0} \int_0^{1/\epsilon} \frac{\sin(\omega\tau)}{\omega\tau} \mathrm{d}\tau = \frac{1}{\omega} \lim_{\epsilon\to0} \int_0^{\omega/\epsilon} \frac{\sin(\xi)}{\xi} \mathrm{d}\xi = \frac{\pi}{2\omega}$$

for all $\omega \in [a,\pi]$. Since (14) is valid for all $0 < a < 1$ and

$$\lim_{a\to0} \frac{\pi}{2} \log\left(\frac{\log\left(\frac{2\pi}{a}\right)}{\log(2)}\right) = \infty,$$

it follows that $\liminf_{\epsilon\to0} \int_0^\pi g(\omega,\epsilon)\mathrm{d}\omega = \infty$, and consequently that

$$\lim_{\epsilon\to0} \int\limits_{\epsilon\leq|\tau|\leq\frac{1}{\epsilon}} \frac{f_1(\tau)}{-\tau} \mathrm{d}\tau = \infty,$$

which is a contradiction to (10). □

In the proof of Theorem 3 we have seen that the Hilbert transform integral (1) diverges unboundedly for the signal f_1 and $t = 0$. However, this divergence is not restricted to $t = 0$. It can be shown that the divergence occurs for all $t \in \mathbb{R}$.

5. CONVERGENCE OF THE HILBERT TRANSFORM INTEGRAL

Theorem 2 characterizes when $\mathfrak{I}f$ is bounded. It links the boundedness of $\mathfrak{I}f$ to the boundedness of the principal value integral (1). However, it makes no statement about the convergence of the principal value integral (1). This convergence is treated in the next theorem. In Theorem 4 we characterize a subset of the bounded bandlimited signals, for which the integral (1) converges, and thus give a sufficient condition for being able to calculate the Hilbert transformation (modulo an additive constant) by the integral (1).

Theorem 4. *Let $f \in \mathcal{B}_{\pi,0}^\infty$ be real-valued. If $\mathfrak{I}f - C_I \in \mathcal{B}_{\pi,0}^\infty$ for some constant C_I, then we have*

$$\lim_{\epsilon\to0} \frac{1}{\pi} \int\limits_{\epsilon\leq|t-\tau|\leq\frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau = (\mathfrak{I}f)(t) - C_I$$

and

$$\lim_{\epsilon\to0} \frac{1}{\pi} \int\limits_{\epsilon\leq|t-\tau|\leq\frac{1}{\epsilon}} \frac{(\mathfrak{I}f)(\tau)}{t-\tau} \mathrm{d}\tau = f(t)$$

for all $t \in \mathbb{R}$.

For the proof of Theorem 4 we need the following lemma.

Lemma 2. *Let $f \in \mathcal{B}_{\pi,0}^\infty$ such that $\mathfrak{I}f - C_I \in \mathcal{B}_{\pi,0}^\infty$ for some constant C_I, and let $F^{C_I}(t+iy) = f(t+iy) + i((\mathfrak{I}f)(t+iy) - C_I)$. Then, for all $\epsilon > 0$ there exists a natural number $R_0 = R_0(\epsilon)$ such that $|F^{C_I}(t+iy)| < \epsilon$ for all $t \in \mathbb{R}$ and $y \geq 0$, satisfying $\sqrt{t^2+y^2} \geq R_0$.*

Proof. Consider the Möbius transformation $\phi(z) = (z-i)/(z+i)$, which maps the upper half plane to the unit disk. The inverse mapping is given by $\phi^{-1}(z) = i(1+z)/(1-z)$. Since F^{C_I} is analytic in $\mathbb{C}$ and $|F^{C_\mathrm{I}}(t+iy)| \leq \|F^{C_\mathrm{I}}\|_\infty$ for all $t \in \mathbb{R}$ and $y \geq 0$, according to Lemma 1, it follows that

$$G(z) = F^{C_\mathrm{I}}(\phi^{-1}(z)) = F^{C_\mathrm{I}}\left(i\frac{1+z}{1-z}\right)$$

is analytic for $|z| < 1$ and that $\sup_{|z|<1}|G(z)| < \infty$. Further, G is continuous on the unit circle, because F^{C_I} is continuous on the real axis,

$$\lim_{\omega\searrow0} G(\mathrm{e}^{i\omega}) = \lim_{t\to-\infty} F^{C_\mathrm{I}}(t) = 0, \qquad (15)$$

and

$$\lim_{\omega\nearrow0} G(\mathrm{e}^{i\omega}) = \lim_{t\to\infty} F^{C_\mathrm{I}}(t) = 0. \qquad (16)$$

Hence, by [11, p. 340, Theorem 17.11], we have

$$G(\rho\,\mathrm{e}^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^\pi G(\mathrm{e}^{i\omega}) \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2} \mathrm{d}\omega$$

for all $0 \leq \rho < 1$ and $-\pi < \theta < \pi$.

Let $\epsilon > 0$ be arbitrary but fixed. Equations (15) and (16) imply that there exits a $\omega_0 = \omega_0(\epsilon)$, $0 < \omega_0 < \pi$, such that

$$|G(\mathrm{e}^{i\omega})| < \frac{\epsilon}{2} \qquad (17)$$

for all $|\omega| \leq \omega_0$. Further, there exists a $\rho_0 = \rho_0(\epsilon)$, $0 < \rho_0 < 1$, such that

$$\frac{\|F^{C_\mathrm{I}}\|_\infty (1-\rho)}{\rho\left(1-\cos\left(\frac{\omega_0}{2}\right)\right)} < \frac{\epsilon}{2} \qquad (18)$$

for all $\rho_0 \leq \rho < 1$.

Next, let ρ satisfying $\rho_0 \leq \rho < 1$, and θ satisfying $-\omega_0/2 \leq \theta \leq \omega_0/2$, be arbitrary but fixed. Then, we have

$$\begin{aligned}
|G(\rho\,\mathrm{e}^{i\theta})| &\leq \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} |G(\mathrm{e}^{i\omega})| \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2} \mathrm{d}\omega \\
&\quad + \frac{1}{2\pi} \int\limits_{\omega_0\leq|\omega|\leq\pi} |G(\mathrm{e}^{i\omega})| \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2} \mathrm{d}\omega \\
&< \frac{\epsilon}{2} + \frac{\|F^{C_\mathrm{I}}\|_\infty}{2\pi} \int\limits_{\omega_0\leq|\omega|\leq\pi} \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2} \mathrm{d}\omega,
\end{aligned}$$

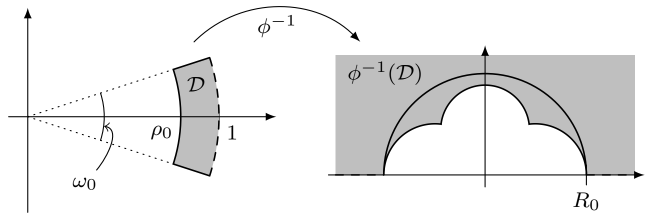

Fig. 3. Visualization of the set $\phi^{-1}(\mathcal{D})$.

where we used (17) and the fact [11, p. 233] that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2}\mathrm{d}\omega = 1.$$

Further, we have

$$\begin{aligned}&\frac{\|F^{C_{\mathrm{I}}}\|_\infty}{2\pi}\int\limits_{\omega_0\le|\omega|\le\pi}\frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2}\mathrm{d}\omega\\ &\le \frac{\|F^{C_{\mathrm{I}}}\|_\infty}{2\pi}\int\limits_{\omega_0\le|\omega|\le\pi}\frac{1-\rho^2}{1-2\rho\cos\left(\frac{\omega_0}{2}\right)+\rho^2}\mathrm{d}\omega\\ &< \frac{\|F^{C_{\mathrm{I}}}\|_\infty(1-\rho^2)}{1-2\rho\cos\left(\frac{\omega_0}{2}\right)+\rho^2} = \frac{\|F^{C_{\mathrm{I}}}\|_\infty(1-\rho)(1+\rho)}{(1-\rho)^2+2\rho\left(1-\cos\left(\frac{\omega_0}{2}\right)\right)}\\ &< \frac{\|F^{C_{\mathrm{I}}}\|_\infty(1-\rho)}{\rho\left(1-\cos\left(\frac{\omega_0}{2}\right)\right)} < \frac{\epsilon}{2},\end{aligned}$$

where we used (18) in the last inequality. Hence, it follows that $|G(\rho\,\mathrm{e}^{i\theta})| < \epsilon$ for all $\rho_0 \le \rho < 1$ and $-\omega_0/2 \le \theta \le \omega_0/2$. Let $\mathcal{D} = \{\rho\,\mathrm{e}^{i\theta} : \rho_0 \le \rho < 1, -\omega_0/2 \le \theta \le \omega_0/2\}$. Thus, for $z \in \phi^{-1}(\mathcal{D})$, we have $F^{C_{\mathrm{I}}}(z) < \epsilon$. The image of $\mathcal{D}$ under the mapping ϕ^{-1} is depicted in Figure 3. Finally, let R_0 be the radius of the smallest circle around the origin, whose restriction to the upper half plane lies completely in $\phi^{-1}(\mathcal{D})$. Then, we have $|F^{C_{\mathrm{I}}}(t+iy)| < \epsilon$ for all $t \in \mathbb{R}$ and $y \ge 0$, satisfying $\sqrt{t^2+x^2} \ge R_0$. □

Now we are in the position to prove Theorem 4

Proof of Theorem 4. Let $f \in \mathcal{B}_{\pi,0}^\infty$ be real-valued, such that $\mathfrak{I}f - C_{\mathrm{I}} \in \mathcal{B}_{\pi,0}^\infty$ for some constant C_{I}. Further, let $t \in \mathbb{R}$ be arbitrary but fixed. Since $F^{C_{\mathrm{I}}} = f + i(\mathfrak{I}f - C_{\mathrm{I}}) \in \mathcal{B}_\pi$ is an entire function, we can use the same argumentation as in the proof of Theorem 2 to obtain

$$\underset{\rightarrow\;\rightarrow}{\int}\frac{F^{C_{\mathrm{I}}}(\xi)}{t-\xi}\mathrm{d}\xi = -\underset{\frown}{\int}\frac{F^{C_{\mathrm{I}}}(\xi)}{t-\xi}\mathrm{d}\xi - \underset{\frown}{\int}\frac{F^{C_{\mathrm{I}}}(\xi)}{t-\xi}\mathrm{d}\xi. \tag{19}$$

From (7) we see that $\lim_{\epsilon\to 0}\underset{\frown}{\int}\frac{F^{C_{\mathrm{I}}}(\xi)}{t-\xi}\mathrm{d}\xi = \pi i F^{C_{\mathrm{I}}}(t)$. Let $\delta > 0$ be arbitrary but fixed. Then, according to Lemma 2, there exists a natural number $R_0 = R_0(\delta)$ such that $|F^{C_{\mathrm{I}}}(t+iy)| < \delta$ for all $t \in \mathbb{R}$ and $y \ge 0$, satisfying $\sqrt{t^2+y^2} \ge R_0$. Let $\epsilon_0 = 1/(R_0 + |t|)$. Then it follows that $\left|t + \frac{1}{\epsilon}\mathrm{e}^{i\phi}\right| \ge R_0$ for all $0 < \epsilon \le \epsilon_0$ and consequently that $\left|F^{C_{\mathrm{I}}}\left(t+\frac{1}{\epsilon}\mathrm{e}^{i\phi}\right)\right| < \delta$ for all $0 < \epsilon \le \epsilon_0$ and $0 \le \phi \le \pi$. It follows that

$$\left|\underset{\frown}{\int}\frac{F^{C_{\mathrm{I}}}(\xi)}{t-\xi}\mathrm{d}\xi\right| \le \int_0^\pi\left|F^{C_{\mathrm{I}}}\left(t+\frac{1}{\epsilon}\mathrm{e}^{i\phi}\right)\right|\mathrm{d}\phi \le \pi\delta$$

for all $0 < \epsilon \le \epsilon_0$, which shows that

$$\lim_{\epsilon\to 0}\underset{\frown}{\int}\frac{F^{C_{\mathrm{I}}}(\xi)}{t-\xi}\mathrm{d}\xi = 0.$$

Hence, it follows that

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\underset{\rightarrow\;\rightarrow}{\int}\frac{F^{C_{\mathrm{I}}}(\xi)}{t-\xi}\mathrm{d}\xi = -iF^{C_{\mathrm{I}}}(t), \tag{20}$$

which in turn implies that the real part of the left hand side of (20) converges to the real part of the right hand side of (20), i.e., that

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\underset{\rightarrow\;\rightarrow}{\int}\frac{f(\xi)}{t-\xi}\mathrm{d}\xi = (\mathfrak{I}f)(t),$$

and that the imaginary part of the left hand side of (20) converges to the imaginary part of the right hand side of (20), i.e., that

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\underset{\rightarrow\;\rightarrow}{\int}\frac{(\mathfrak{I}f)(\xi)-C_{\mathrm{I}}}{t-\xi}\mathrm{d}\xi = \lim_{\epsilon\to 0}\frac{1}{\pi}\underset{\rightarrow\;\rightarrow}{\int}\frac{(\mathfrak{I}f)(\xi)}{t-\xi}\mathrm{d}\xi = f(t).\quad\square$$

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