The Geometry of Brackets and the Area Principle

Susanne Apel

Fakultät für Mathematik
Technische Universität München
D-85747 Garching
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Technische Universität München
Fakultät für Mathematik

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Susanne Jasmin Apel

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Vorsitzende:		Univ.-Prof. Dr. Caroline Lasser
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Zusammenfassung

Diese Arbeit beschäftigt sich mit dem Zusammenspiel von Geometrie und Algebra. Als Geometrie behandeln wir projektive Geometrie sowie euklidische Spezialisierungen. Die Algebra, um die es hier gehen soll, baut auf Determinanten auf deren symbolische Darstellung im Englischen "Brackets" heißen. Das von Grünbaum und Shephard wiederentdeckte "Flächenprizip" macht es möglich, Brüche von Brackets als orientierte Längenverhältnisse zu interpretieren. Diese Interpretation wird in zwei Teilen der Arbeit verwendet:

Zunächst werden sogenannte Γ-Kreise untersucht. Diese ergeben sich auf natürliche Weise bei der Interpretation von *binomischen* oder *biquadratischen Beweisen* von Schließungssätzen mittels des Flächenprinzips. Diese kann angewendet werden, da solche Beweise, wie von Richter-Gebert eingeführt, in der Sprache der Brackets formuliert sind. Γ-Kreise können als grundlegende Bausteine in dieser Theorie betrachtet werden. Man kann die Γ-Kreise zusammen mit *biquadratischen Brüchen*, die den biquadratischen Beweisen entspringen, auch nutzen, um daraus selbst Schließungssätze zu konstruieren. Dabei ermöglicht die Interpretation durch das Flächenprinzip geometrische Kontrolle über die Kürzungsmuster. Die formale Behandlung der Bausteine orientiert sich an der Sprache von Dress und Wenzel. Die in unserer Behandlung nicht vollständig vorhandene Information über Kollinearitäten impliziert, dass sich nicht automatisch alle Γ-Kreise auf einige wenige ihrer Art zurückführen lassen. Daher werden sogenannte *irreduzible* Γ-Kreise genauer untersucht, was es zudem ermöglicht sie zu zählen. Wendet man das Flächenprinzip auf (Kombinationen von) Γ-Kreisen an, so erhält man Resultate, die die von Grünbaum und Shephard verallgemeinern.

Anschließend wird das Problem der Cayley-Faktorisierung behandelt. Hierbei möchte man mit einer Linealkonstruktion überprüfen, ob sich endlich viele Punkte der projektiven Ebene in spezieller Lage befinden. Die spezielle Lage der Punkte wird durch das Verschwinden eines Polynoms in Brackets beschrieben, das wir mit B bezeichnen wollen. Es ist durch Sturmfels und Whiteley bekannt, dass dies nicht immer möglich ist – stattdessen kann man immer $M \cdot B$ durch eine Linealkonstruktion charakterisieren, wobei M ein *Monom* in Brackets ist. Wir stellen einen neuen Algorithmus für das gleiche Resultat vor, der einen geringeren Grad von M aufweist. Der konstante Faktor in der Gradschranke wird von 105 auf 9 reduziert. Dies wird durch das implizite Einführen von *lokalen* Koordinatensystemen mittels des Flächenprinzips erreicht. Die gefundenen orientierten Längenverhältnisse können dann mit den Konstruktionen von Ceva und Menelaus sowie der von-Staudt'schen projektiven Arithmetik kombiniert werden. Der Algorithmus kann den Satz von Pascal erklären und ist prägnant genug, um auf dem Computer implementiert zu werden. Zwei beispielhafte Ausgaben werden angegeben.

Abstract

The present thesis is on the interplay of geometry and algebra. As geometry we consider projective geometry as well as Euclidean specializations. The algebra in question is based on determinants whose symbolic representation is called *bracket*. The *area principle*, which was rediscovered by Grünbaum and Shephard, allows for interpreting ratios of brackets as oriented length ratios. This interpretation is made use of in two parts of the thesis:

At first, so-called Γ-cycles are investigated. They emerge in a natural way when interpreting *binomial* or *biquadratic proofs* of incidence theorems via the area principle. It can be applied, since such proofs, as introduced by Richter-Gebert, are formulated in the language of brackets. Γ-cycles can be understood as basic building blocks within this theory. One can use Γ-cycles together with biquadratic fractions, which arise from biquadratic proofs, also in order to construct incidence theorems. In doing so, the interpretation via the area principle provides geometric control over the cancellation patterns. The formal treatment of these building blocks follows the language of Dress and Wenzel. Since the information about collinearities is not complete in our setup, we cannot automatically reduce the number of relevant Γ-cycles to only a few. Therefore, so-called *irreducible* Γ-cycles are analyzed in more detail, which allows for also counting them. Applying the area principle to (a combination of) Γ-cycles results in statements that generalize those of Grünbaum and Shephard.

Afterwards, the problem of Cayley factorization is treated. Here one wishes to check with a ruler construction whether finitely many points in the projective plane are located in special position. The special position of the points is described by the vanishing of a polynomial in brackets which is denoted by B. It is known due to a result of Sturmfels and Whiteley that this is not always possible—instead one can characterize always $M \cdot B$ by a ruler construction, where M is a *monomial* in brackets. We present a new algorithm for the same result and show that it implies a smaller degree of M. The constant factor in the bound on the degree is reduced from 105 to 9. This is achieved by implicitly introducing *local* coordinate systems via the area principle. The obtained oriented length ratios can be combined with the constructions of Ceva and Menelaus as well as with the projective arithmetic given by von Staudt. The algorithm is able to explain Pascal's theorem and is concise enough to be implemented on a computer. Two exemplary outputs are given.

Acknowledgment

First of all, I wish to thank my advisor Jürgen Richter-Gebert for giving me the opportunity to do this thesis at his chair and within the program of TopMath. His style of supervision includes an amount of freedom that made it possible to find topics that suited me well. He taught me many things including of course mathematics, reasoning, didactics, programming, using Cinderella and enjoying what one is doing. I also learned my geometry from him and he showed me many tools that are used within this thesis. Through him, I got to know problems and results which were the starting points for all parts of the present text. Together, we got interested and started to analyze Γ-cycles which are one of the central topics treated here. He always encouraged me to go on with this topic.

Next, I want to thank Walter Whiteley for examining this thesis. Furthermore I am thankful for the discussions I had with him during my stay at the Fields Institute in Toronto. The thematic program there was a great opportunity for a stay abroad and meeting many interesting people. Especially the discussions with him about bracket polynomials and weights as well as about his results obtained together with Bernd Sturmfels on Cayley factorization enabled me to come up with and pursuing the ideas for improving their result.

Moreover, I am very grateful to Eva Maria Feichtner for kindly agreeing to act as examiner of this thesis.

The graduate program "TopMath" within the Elite Network of Bavaria allowed me to start doing research at a very early level. I am thankful for covering expenses of my scientific travels and for the opportunity to participate in the program of the Elite Network of Bavaria.

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Contents

1. Introduction

> “ Ich bemerke nur noch, dass ich das [...] Verfahren, nach welchem der positive oder negative Werth einer Linie durch die verschiedene Nebeneinanderstellung der die Endpuncte der Linie bezeichnenden Buchstaben ausgedrückt wird, durchgehends angewendet und auch auf die Bezeichnung des Inhaltes von Dreiecken [...] und dreiseitigen Pyramiden erweitert habe. Es wird hierdurch, so wie auch zum Theil durch den barycentrischen Calcul selbst, die Anschaulichkeit der synthetischen Methode mit der Allgemeinheit der analytischen in möglichst nahe Verbindung gebracht, indem man mit Anwendung rein geometrischer Zeichen, dergleichen die für die Puncte einer Figur gewählten Buchstaben sind, die arithmetischen Beziehungen zwischen den Theilen der Figur durch Formeln darstellt, welche für alle möglichen Lagen der Theile Gültigkeit haben.”
>
> August Ferdinand Möbius, Der barycentrische Calcul, 1827

The topic of this thesis in its most general sense is the interplay of geometry and algebra which both have a very long history. In the following it shall become clear why we feel connected to the above statement made by Möbius which in a loose translation and with a language adjusted to the present thesis can be summarized as follows: The method of considering oriented lengths and volumes, indicated by the order of points naming them, is able to connect the imagination of the synthetic method and the generality of the analytic one by naming geometric points with letters and representing geometric relations with formulas such that they are valid in all possible concrete geometric situations. The so-called area principle is a special case of considering *ratios* of oriented lengths and volumes. It is used within this thesis to address several problems. However, as we will show in Section 3.9, it can often be translated into considerations regarding weights as it was Möbius's starting point for introducing barycentric coordinates. To begin with, we now describe the synthetic and algebraic methods and language used in the present text.

Our setup is projective geometry (for a very brief introduction see Chapter 2). Here, three points are collinear if and only if the matrix whose columns are the 3-vectors of the homogenous coordinates of the points has a vanishing determinant. Naming the homogenous coordinates of the three points p, q and r, this can be written as

$$|p, q, r| := \det(p, q, r) = 0.$$

This notion does not refer to the concrete coordinates of the points but only to its names. This motivates the notion of a symbolized version of a determinant using only *names of points*:

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}]$$

which is called a *bracket*. The names $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ themselves have a symbolic nature and can be considered to be taken from an alphabet $\mathbf{P}$ describing a point set. This language emphasizes the symbolic and combinatoric way points will be treated. The notion of brackets is useful since many geometric properties can be represented by combinations of brackets, more precisely with bracket polynomials. E.g. three lines spanned by the pairs of points $\{a, b\}$, $\{c, d\}$ and $\{e, f\}$ are concurrent if and only if

$$|a, b, c||d, e, f| - |a, b, d||c, e, f| \tag{1.1}$$

vanishes. Expressing the same property on the level of concrete homogenous coordinates yields much longer expressions (see Section 4.6). This is one reason why geometers since Cayley have been interested in brackets.

Another approach to brackets is invariant theory (see Section 3.3 for more details). According to Klein's Erlangen program, geometry can always be considered as being invariant theory. This seems reasonable when recalling that in Euclidean geometry the property of a triangle being equilateral does not depend on its position relative to the origin. Therefore this property is at least invariant under translations. It turns out that all Euclidean properties are invariant under the group of Euclidean moves.

In projective geometry one considers those transformations that maintain collinearity of points. It is the first fundamental theorem of (real) projective geometry that each such transformation is induced by multiplying the homogenous coordinates of a point with an invertible matrix. Now (1.1) vanishing is a projectively invariant property, since

$$\begin{aligned}|M \cdot a, M \cdot b, M \cdot c||M \cdot d, M \cdot e, M \cdot f| - |M \cdot a, M \cdot b, M \cdot d||M \cdot c, M \cdot e, M \cdot f| \\ = \det(M)^2\big(|a, b, c||d, e, f| - |a, b, d||c, e, f|\big)\end{aligned}$$

for a real invertible 3×3 matrix M and due to the multiplicativity of the determinant. For technical reasons we restrict ourselves to polynomial invariants with respect to the multiplication of matrices M with determinant equal to 1. These invariants can be classified by the bracket polynomials and the corresponding fundamental theorems of invariant theory (which is described in Section 4.5 in more detail). Projectively invariant properties can be easily constructed by their means. They are considered to be elements of the *bracket ring* as introduced by White in [120]. Here the names of the points are treated as symbols, not as vectors, which allows for generalizing the concept to more general combinatorial geometries.

The usage of bracket polynomial and invariant language turned out to be useful for many geometers. There are also applications that use the algebra of brackets to determine special positions of points. These applications include robotics, statics, rigidity of frameworks and scene analysis (see e.g. [125, 127, 33, 129, 28, 27, 81, 32]).

Until now this introduction was very algebraic. In the following we will present a more geometric point of view: there are natural geometric constructions in the projective plane. The most fundamental ones are the operations *join* and *meet* whose existence is guaranteed by the axioms of projective planes: two points can be joined to a unique line and two lines meet in a unique point. We will focus on these two constructions. An exemplary computation in the more general Grassmann-Cayley algebra shows that the intersection of the lines spanned by the pairs of points $\{c, d\}$ and $\{a, b\}$ can be given by

$$x := |a,\ b,\ d| \cdot c - |a,\ b,\ c| \cdot d. \tag{1.2}$$

This point being collinear also with e and f is characterized by (1.1) vanishing. The previous was supposed to give a first glimpse on the constructions possible in Grassmann-Cayley algebra. In its general form it allows for transforming (closed) linear geometric constructions into bracket polynomials. Calculations will be done in the language of brackets and in an affine version also in terms of oriented length ratios. The latter version is obtained by the very powerful *area principle* which in this form is was rediscovered by Grünbaum and Shephard. However, without considering ratios of oriented lengths and volumes but absolute oriented values is used by and generalized from lines to triangles and tetrahedra by Möbius in [86]. For more historical background see Section 3.1. The principle says that the ratio of the determinants given in (1.2) equals an oriented length ratio

$$-\frac{|a,\ c, d|}{|b,\ c, d|} = \frac{\overline{a\,x}}{\overline{x\,b}}$$

in a Euclidean interpretation (see also Figure 1.1). The principle is called "area principle", since ratios of determinants can be considered as ratios of areas of triangles in the right context. This interpretation in terms of oriented length ratios has the advantage that they provide much more intuition. It will be the key intuition for the considerations done in Part II and in Part III. Furthermore, as it is pointed out in Section 3.9, it is closely related with the theory of weights and centers of mass referred to by Möbius in the introductory quotation.

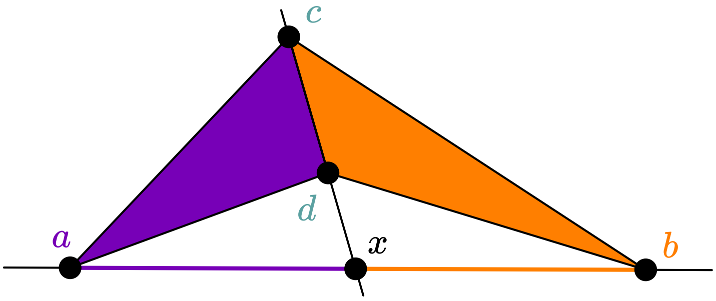


Figure 1.1.: The length ratio $\overline{ax}/\overline{xb}$ equals $\frac{|a,d,c|}{|b,c,d|}$, a ratio of determinants.

Part I: Basics

Part I introduces many of the previously mentioned concepts in more detail. It summarizes the basic definition and computing techniques (projective geometry, bracket algebra, Grassmann-Cayley algebra, tensor diagrams and evaluation) as well as some tools or basic building blocks (the area principle, the theorems of Ceva and Menelaus and von Staudt constructions) used in both Parts II and III. First, this part will focus on the plane case. For most of the results in Parts II and III, this is sufficient. Since the theory also extends to arbitrary dimensions results in Parts II and III can sometimes also be applied to general dimensions. This will not be treated explicitly but should be mentioned. We remark that we give a *symbolic* version of the definitions for the Grassmann-Cayley algebra which is not the standard one (see Chapter 5). The symbolic definition is very close to the formulation of Möbius who is amazed by the possibility to calculate only with the *names* of the points. This algebra can be thought of a tool for computing with subspaces both of $\mathbb{R}^n$ and also with projective subspaces. We choose the symbolic version for the following reasons: as before when introducing brackets, it is natural to work only with abstract *names* of points not with vectors representing homogenous coordinates as it is done in classical introductions which mostly go back to [37] by Doubilet, Rota and Stein in which the ideas go back to Grassmann's "Ausdehnungslehre" (see [48, 49]). This approach fits better with the symbolic treatment in the bracket ring. This was already done by the school of Rota and is called the "White module" there. It is located at the transition to a more general, supersymmetric theory and the very detailed geometric ideas presented in [37] are less present. We want to connect both in the introductive treatment of Grassmann-Cayley algebra.

It is an important basic fact that in Grassmann-Cayley algebra, any "closed" construction done by the means of *join* and *meet* with respect to some (abstract) base points $\mathbf{a}$, $\mathbf{b}$, ..., $\mathbf{e}$ can be evaluated to a (multihomogenous) bracket polynomial B in the points $\mathbf{a}$, $\mathbf{b}$, ..., $\mathbf{e}$ which has integer coefficients. This serves as motivation for the problem treated in Part III. A closed construction can be thought of a construction that asks for the collinearity of three points that are constructed by the means of *join* and *meet* depending on the points $\mathbf{a}$, $\mathbf{b}$, ..., $\mathbf{e}$. The resulting bracket polynomial is unique in the previously mentioned bracket ring. Nevertheless, the bracket ring is a factor ring, say R/I with a non-trivial modulo I. Therefore, it makes sense to consider several methods for evaluation since their output might look quite different and be identical only modulo I (see Chapter 6). We also give an introduction to the background of incidence theorems (see Chapter 7). Theorems of this kind are the starting point for the considerations in Part II. The techniques used in this part are in turn used to attack the problems in Part III. Incidence theorems are theorems using only incidences in their formulation of hypotheses and conclusions. The most famous examples are the theorems of Pappos and of Desargues.

Part II: Γ-Cycles

Part II is about so-called Γ-cycles, which is a new concept developed together with Jürgen Richter-Gebert and describing a special class of binomial identities in the bracket algebra. In general, binomial identities in the bracket algebra are obvious. However, in the case of Γ-cycles its only at first sight and without knowing any context that they could be considered being trivial. At second sight, any Γ-cycle can be interpreted as a theorem about oriented length ratios in affine geometry. Regarding them on their own, these Euclidean theorems are surprising and do not look trivial any longer. An example seems to be appropriate: In Figure 1.2 on the left-hand side, a Γ-cycle in its graphical version is shown. It induces the identity

$$\frac{[\mathbf{a},\mathbf{f},\mathbf{g}]}{[\mathbf{a},\mathbf{b},\mathbf{f}]}\cdot\frac{[\mathbf{a},\mathbf{b},\mathbf{f}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}\cdot\frac{[\mathbf{a},\mathbf{b},\mathbf{c}]}{[\mathbf{a},\mathbf{c},\mathbf{d}]}\cdot\frac{[\mathbf{a},\mathbf{c},\mathbf{d}]}{[\mathbf{a},\mathbf{d},\mathbf{e}]}\cdot\frac{[\mathbf{a},\mathbf{d},\mathbf{e}]}{[\mathbf{d},\mathbf{e},\mathbf{f}]}\cdot\frac{[\mathbf{d},\mathbf{e},\mathbf{f}]}{[\mathbf{e},\mathbf{f},\mathbf{g}]}\cdot\frac{[\mathbf{e},\mathbf{f},\mathbf{g}]}{[\mathbf{a},\mathbf{f},\mathbf{g}]}=1,$$

which can also be written as a vanishing binomial in the bracket ring, since all brackets occurring somewhere in a numerator also occur somewhere in a denominator. In the concrete form given, every fraction can be interpreted via the area principle yielding that the product of oriented length ratios in the right-hand side of Figure 1.2 equals 1 (for more details see Chapter 10).

Γ-cycles are exactly those binomial identities, for which such an interpretation in terms of oriented lengths is possible. This is due to the fact that they can be interpreted as cycles in a Γ-graph known from matroid theory (see [84]).

Furthermore, Γ-cycles do appear naturally when one tries to interpret so-called binomial or biquadratic proofs (see [92, 93, 9, 30]) in terms of length ratios in an

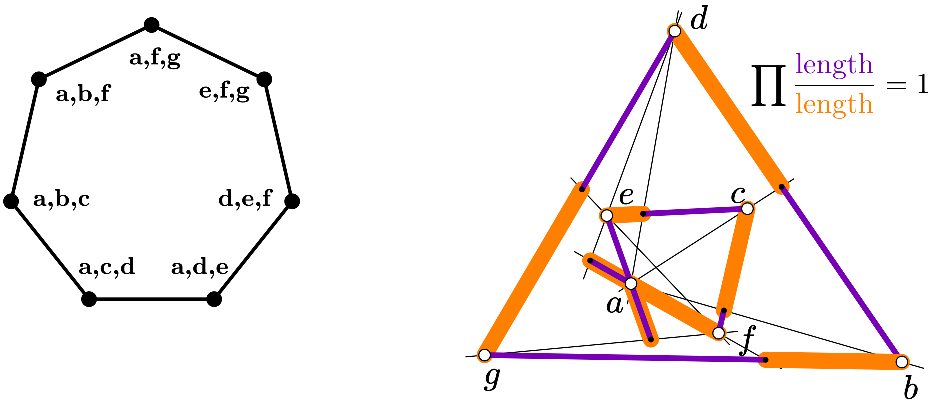


Figure 1.2.: A Γ-cycle and a Euclidean interpretation.

affine setup. In fact, this [94, 2] was the starting point of our considerations. So the exposition in Part II starts with an exemplary interpretation of a binomial proof in terms of length ratios (see Chapter 9). Of special interest are *irreducible* Γ-cycles, which cannot be understood as *products* of smaller Γ-cycles. Here the product is considered in an appropriate group defined in Chapter 10. It seems to be natural to consider this group as it was similarly done by Dress and Wenzel in a series of papers, e.g. in [38, 40, 41, 39]. In our context, the formulation of groups is useful to define (different versions of) irreducibility. Dress and Wenzel use the groups for encoding information about very general matroids. Γ-cycles can be understood as mimicking these concepts in a setup where only partial information is present and the setup is more concrete, namely the projective plane. For knowing all of the possible building blocks when interpreting binomial proofs, a counting of irreducible Γ-cycles is given in Chapter 11. Furthermore, the structure of the supporting lines of the induced oriented length ratios is analyzed and shortcuts for applying the area principle are established (see Chapter 12).

As it turns out, Γ-cycles are also related to phenomena described by Grünbaum and Shephard (see [53, 52, 54]) who rediscovered the area principle. They enumerate cyclic theorems about oriented length ratios in polygons. We find that many of the calculations can be restated with Γ-cycles alone (see Chapter 13). For the results in [54] another ingredient is needed to prove the theorems in the language of brackets: biquadratic fractions, which are the basic building blocks in binomial proofs, which in turn were the starting point of our consideration in Part II. Now all results in [54] can be restated within the concise language of Γ-cycles and biquadratic fractions.

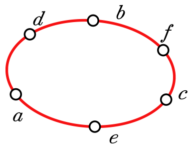
Biquadratic fractions can be also used in a more general setup in order to impose incidence relations on the points contained in a Γ-cycle. They are also included in the formulations via groups done in Chapter 10. The tools of Γ-cycles and biquadratic fractions can be used in at least two setups: in the style of Grünbaum and Shephard, new theorems about oriented length ratios can be constructed. They generalize their concept in the sense that no symmetry is required in the constructed theorems. However, one can also come up with theorems having a symmetry different from the cyclic one considered by Grünbaum and Shephard (see again Chapter 13).

On the other hand, e.g. the theorems of Ceva and Menelaus (introduced in Part I) allow for reducing the number of oriented length ratios involved. Doing this often enough one ends up with an incidence theorem. Therefore one also needs information on how the length ratios are arranged. It turns out that they arrange in cycles which we call orbits to avoid confusion. Chapter 12 gives shortcuts on how these orbits are obtained and that irreducible Γ-cycles do have at most three orbits. All in all we have a construction method for incidence theorems which was inspired by [94]. Brackets cancel as in binomial proofs but the interpretation via oriented length ratios provides (geometric) control over the cancellation process.

Part III: Cayley Factorization

In the most general point of view, the problem of Cayley factorization asks for a translation from algebra to geometry. It treats the question whether a construction given previously in the description of Part I can be reversed. There we pointed out that a closed construction done by the means of join and meet yields a (multihomogeneous) bracket polynomial with coefficients in $\mathbb{Z}$. Reversing this process means starting with a multihomogenous bracket polynomial B with integer coefficients and searching for a *join-meet*-construction whose evaluation results in B. This is a reasonable question, since in the above mentioned applications of bracket polynomials those are often obtained algebraically and need to be interpreted. Therefore the problem of Cayley factorization is formulated since a join-meet-construction results in a bracket polynomial and has a chance to interpret it. A very short first example would be to search for an interpretation of (1.1) vanishing. We saw before that this interpretation can be given by the concurrence of three lines. The most famous example is surely induced by Pascal's theorem characterizing six points lying on a common conic by a construction. Its statement is summarized in Figure 1.3. It is not too hard to derive the determinantal equation given in the middle of the figure by geometric and algebraic considerations (see e.g. [95]). A Cayley factorization can be considered as "automatically deriving" Pascal's theorem.

For attacking the problem there is an algorithm due to White for deciding the *multilinear* case (see [124]). For the general case there are methods for bracket polynomials of small degree given in [80]. There are also results in more special cases which are given in Chapter 14. It was shown in [104] by Sturmfels and Whiteley



$$\Leftrightarrow \quad \begin{array}{c} |a,c,e| \cdot |a,d,f| \cdot |b,c,f| \cdot |b,d,e| \\ -|a,c,f| \cdot |a,d,e| \cdot |b,c,e| \cdot |b,d,f| = 0 \end{array} \quad \Leftrightarrow$$

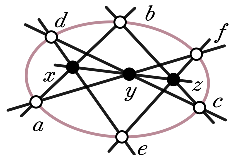


Figure 1.3.: Pascal's theorem: six points a, b, c, d, e and f lie on a common conic if and only if the depending points x, y, z lie on a common line.

that a factorization is not always possible. However, they also show that for any bracket polynomial B (homogenous, with integer coefficients), there always exists a bracket *monomial* M such that $M \cdot B$ can be factored. This is the reformulation of the problem which we focus on. In the present thesis, the degree of the monomial M and therefore also the complexity of the construction can be reduced to an extent that makes an implementation possible. The constant factor in the bound on the degree of M can be reduced from 105 (in the algorithm used in [104]) to 9.

The algorithm presented here relies on the (intuition of the) area principle and in its first version interprets only bracket *binomials*. The configurations of Ceva and Menelaus as well as von Staudt constructions are used to perform multiplication on oriented length ratios and cross-ratios. In a next step, the approach is generalized to the case of an arbitrary number of summands. Von Staudt's construction for projective addition is used to combine several summands. Some optimizations are done in order to keep the degree of the leading monomial M low. The most relevant optimizations done are factoring out common factors and reusing existing points in the constructions.

The algorithm is able to explain Pascal's theorem: The algorithm has some freedom of choice during its execution. We will show in Section 16.7 that *local optimizations* within the algorithm yield Pascal's theorem. In a more general and less symmetric setup this cannot be expected. However we think that this derivation of Pascal's theorem shows that our algorithm is very close to geometry. In addition, as previously mentioned, a concrete implementation in *Mathematica* ([136]) together with the *Combinatorica* package was possible. With its help we include two examples of the output of the implementation (see Chapter 18). One of its examples can be considered to be a generalization of Pascal's theorem since it factors a polynomial describing the property of ten points lying on a common cubic. The concrete representation of the bracket polynomial that is factored has 20 summands. It is due to Richter-Gebert and it is the shortest representation known to our group.

Part I.

Basics, Language and Fundamental Ideas

2. Fundamental Definitions in Projective Geometry

This and the following chapter give a very short reference for basic concepts in projective geometry which are used later on in this text. We will mostly focus on the *real* projective *plane*. The exposition given here is by no means exhaustive. For a more extensive treatise, there are a lot of monographs on the topic, e.g. [24, 25, 43, 110, 65] to mention only a few of them. They provide most of the concepts used here. We follow closely the exposition in [95]. E.g. [24, 95] also provide comments on the historical development.

2.1. Homogenous Coordinates of Points

In the real projective plane $\mathbb{RP}^2$, a *point* is a one-dimensional subspace of $\mathbb{R}^3$. This motivates why the two-dimensional real projective plane is also called projective geometry of *rank* 3. We define an equivalence relation on the vectors spanning the one-dimensional subspaces. For $v \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ let

$$[v] := \big\{ v' \in \mathbb{R}^3 \,|\, v' = \lambda \cdot v \text{ for } \lambda \in \mathbb{R} \setminus \{0\} \big\}.$$

The points $\mathcal{P}$ of $\mathbb{RP}^2$ are all of these equivalence classes:

$$\mathcal{P} := \frac{\mathbb{R}^3 \setminus \{(0,0,0)\}}{\mathbb{R} \setminus \{0\}} \;:=\; \big\{ [v] \mid v \in \mathbb{R}^3 \setminus \{(0,0,0)\} \big\}.$$

Topologically, $\mathcal{P}$ is in fact a two-dimensional structure. The usual affine plane $\mathbb{R}^2$ can be embedded in $\mathbb{RP}^2$. One possible embedding is shown in Figure 2.1. This embedding, where the last coordinate equals 1, is called the *standard embedding.* It provides an intuition for most of the points in $\mathbb{RP}^2$ except for points of the shape $(x, y, 0)$, where $x, y \in \mathbb{R}$. Those can be thought as being points *infinitely far away pointing in direction* (x, y) in the corresponding affine picture in $\mathbb{R}^2$. All lines with direction (x, y) have this point in common. This brings us to the next ingredient of the projective plane: lines.

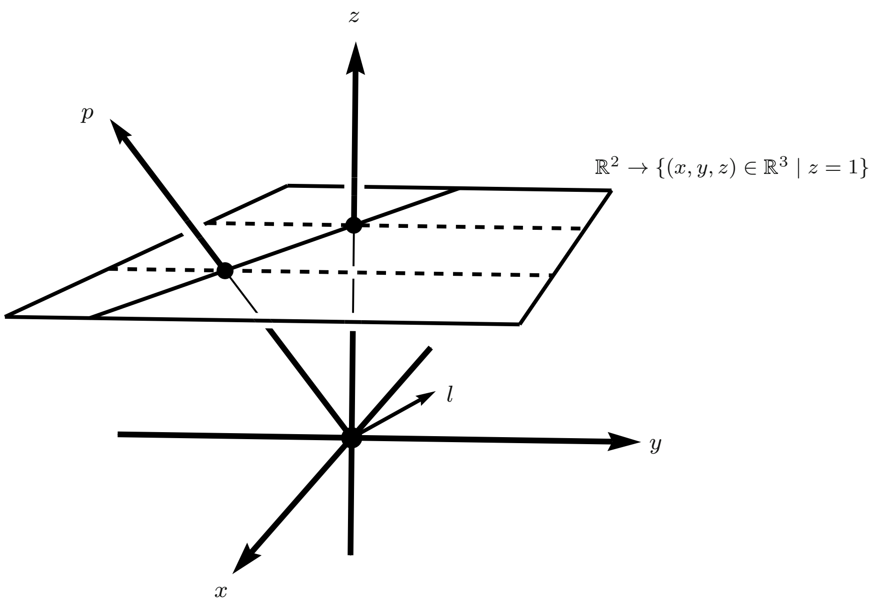


Figure 2.1.: Embedding the Euclidean plane in $\mathbb{R}^3$. The figure is due to Richter-Gebert and e.g. used in [95].

2.2. Homogenous Coordinates of Lines

Lines in $\mathbb{RP}^2$ should be defined in such a way that points, that lie on a common line in $\mathbb{R}^2$, should be collinear in their embedding in $\mathbb{RP}^2$. The embedded points of a line in $\mathbb{R}^2$ form planes that pass through the (three-dimensional) origin. The plane can be identified by its normal vector, which is unique up to non-zero scalar multiples. $\mathcal{L}$ shall denote the set of lines. We can set

$$\mathcal{P} \;=\; \frac{\mathbb{R}^3 \setminus \{(0,0,0)\}}{\mathbb{R} \setminus \{0\}} \;=\; \mathcal{L} \tag{2.1}$$

and for $[p] \in \mathcal{P}$ and $[l] \in \mathcal{L}$

$$[p] \text{ incident to } [l] \quad :\Longleftrightarrow \quad \langle p, l \rangle = 0, \tag{2.2}$$

where $\langle *, * \rangle$ denotes the usual scalar product in $\mathbb{R}^3$. One can check that the resulting plane $\mathbb{RP}^2$ characterized by $\mathcal{P}$, $\mathcal{L}$ and the incidence relation just described, fulfills the axioms for (general) projective planes given e.g. in [24]. These axioms as well as the equations (2.1) and (2.2) indicate that there is a dual relationship between points and lines in this geometry.

2.3. Join, Meet and Collinearity in Homogenous Coordinates

Even though non-zero scalar-multiples of a vector in $\mathbb{R}^3$ represent the same point or line, when doing concrete calculations, the actual vector does matter. This will be the case in large parts of the present thesis. We will refer to the concrete vector by referring to the *homogeneous coordinates* of a point. The approach of taking into account the concrete homogeneous coordinates can be considered to be the *structural geometry* described in [131, p. 16, p. 136].

Suppose we are given the homogeneous coordinates p and q of two distinct points in $\mathbb{RP}^2$. From (2.2) one can conclude how to compute homogeneous coordinates l of the line joining the points $[p]$ and $[q]$. The line $[l]$ is orthogonal to p and q. Therefore

$$l = p \times q$$

where $\times$ denotes the usual cross product or vector product from linear algebra. Furthermore we have $l \neq 0$, since p and q are linearly independent. By duality we can summarize

$$\mathbf{join}(p, q) := p \times q \quad \text{and} \quad \mathbf{meet}(l, m) := l \times m \tag{2.3}$$

for homogeneous coordinates of the line joining the two points $[p]$ and $[q]$ as well as homogeneous coordinates of the point of intersection of the lines $[l]$ and $[m]$. The definition of **join** and **meet** can be extended to the case where p, q, l or m are arbitrary elements in $\mathbb{R}^3$. We have

$$\mathbf{join}(p, q) = 0$$

if and only if $p = (0, 0, 0)$ or $q = (0, 0, 0)$ or $[p] = [q]$ which describes degenerate cases. With the help of (2.3) (or with the intuition provided by Figure 2.1) one can deduce a condition for three points $[p]$, $[q]$ and $[r]$ being collinear:

$$\begin{aligned} [p],\ [q],\ [r] \text{ collinear} \quad &\iff \quad \langle \mathbf{join}(p, q), r\rangle = 0 \\ &\iff \quad \langle p \times q, r\rangle = 0 \quad \iff \quad |p, q, r| = 0 \end{aligned} \tag{2.4}$$

where $|p, q, r|$ shall be a shortcut for $\det(p, q, r)$. Furthermore, in the case that p, q and r are induced by the standard embedding, we have

$$|p, q, r| \;=\; \begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ 1 & 1 & 1 \end{vmatrix} \;=\; 2 \cdot \triangle(p, q, r) \tag{2.5}$$

where $\triangle(p, q, r)$ denotes the oriented area of the triangle that corresponds to p, q and r in the Euclidean plane. $\triangle(p, q, r)$ is positive if and only if a counterclockwise triangle is induced.

2.4. The Intersection of Two Lines Given as Spans of Points via Plücker's μ

Let a, b, c and d be the homogeneous coordinates of points in $\mathbb{RP}^2$. We seek for an easy representation of $\mathbf{meet}(\mathbf{join}(a, b), \mathbf{join}(c, d))$. We assume non-degenerate situations for the moment. By (2.4) we search homogeneous coordinates x with $|x, a, b| = 0$ as well as $|x, c, d| = 0$. For any $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we have

$$|\mu \cdot b + \lambda \cdot a,\ a,\ b| = 0 \tag{2.6}$$

and therefore $[\mu \cdot b + \lambda \cdot a]$ lies on the line spanned by $[a]$ and $[b]$. Testing the collinearity with $[\,\mathbf{join}(c, d)]$ is testing

$$|\mu \cdot b + \lambda \cdot a,\ c,\ d| = \mu \cdot |b,\ c,\ d| + \lambda \cdot |a,\ c,\ d| \quad = \quad 0.$$

Therefore, the choice $\mu = |a,\ c,\ d|$ and $\lambda = -|b,\ c,\ d|$ are valid parameters for describing x and it holds:

$$\big[\mathbf{meet}(\mathbf{join}(a, b), \mathbf{join}(c, d))\big] = \big[\ |a,\ c,\ d| \cdot b - |b,\ c,\ d| \cdot a\big]. \tag{2.7}$$

Furthermore, one can convince oneself that this formula also holds on the level of homogeneous coordinates and due to the commutation rules for the vector product it holds:

$$\begin{aligned} \mathbf{meet}(\mathbf{join}(a, b), \mathbf{join}(c, d)) &= -|a,\ b,\ c| \cdot d\ + |a,\ b,\ d| \cdot c \\ &= \quad |c,\ d,\ a| \cdot b\ - |c,\ d,\ b| \cdot a. \end{aligned} \tag{2.8}$$

Though this equation was not too hard to find, in rank 3 it motivates definitions in the bracket ring and in the Grassmann-Cayley algebra (see Chapter 4 and Chapter 5). Grassmann-Cayley algebra in general rank can be considered as the symbolic computation of spans and intersections. Assigning concrete homogeneous coordinates to the symbols results in *exact* calculations with coordinates.

2.5. Points in Projective Space of Rank n

Analogously to Section 2.1, one can define a real projective geometry $\mathbb{RP}^{n-1}$ of rank n. Its points are given by imposing an equivalence relation on non-zero vectors in

$\mathbb{R}^n$: For $v \in \mathbb{R}^n \setminus \{0\}$ let

$$[v] := \{v' \in \mathbb{R}^n \mid v' = \lambda \cdot v \text{ for } \lambda \in \mathbb{R} \setminus \{0\}\}.$$

The points $\mathcal{P}^{(n-1)}$ of $\mathbb{RP}^{n-1}$ are all of these equivalence classes:

$$\mathcal{P}^{(n-1)} := \frac{\mathbb{R}^n \setminus \{(0, \ldots, 0)\}}{\mathbb{R} \setminus \{0\}} \;:=\; \big\{[v] \mid v \in \mathbb{R}^n \setminus \{0\}\big\}.$$

In rank 3 we proceeded by giving coordinates of lines and incidence relations between points and lines (see Section 2.2). In general rank n, it does make sense not only to talk about lines but also about planes, hyperplanes and subspaces of a rank between 0 and n. A detailed treatment of these properties very soon leads to the introduction of Grassmann coordinates (see e.g. [95, 65] also for more information about projective *spaces*). In principle, the Grassmann-Cayley algebra introduced in Chapter 5 is a symbolic version of these coordinates. For future reference, we observe: The $(n-1)$-dimensional Euclidean space $\mathbb{R}^{n-1}$ can be embedded in $\mathbb{RP}^{n-1}$ by the same standard embedding as before, i.e. by homogeneous coordinates whose last coordinate equals 1. As before, the homogeneous coordinates of points spanning subspaces in $\mathbb{RP}^{n-1}$ span *linear* subspaces in $\mathbb{R}^n$. Therefore, the foundations of an algebraic treatment of projective $(n-1)$-space is linear algebra. Some general relations between vectors in $\mathbb{R}^n$ will be given in Section 4.4. They are used to formulate (symbolic versions of) incidence relations of points in $\mathbb{R}^n$ and in $\mathbb{RP}^{n-1}$ in Chapter 5.

3. Geometric Tools and Constructions

This chapter is to be understood as loose collection of geometric tools and techniques used throughout the thesis.

3.1. The Area Principle

Assuming the standard embedding and using triangle areas leads to an interpretation of ratios of determinants: let a, b, c and d be homogeneous coordinates of points in $\mathbb{RP}^2$ with last coordinates equal to 1 and located in such a way that

$$x = \mathbf{meet}(\mathbf{join}(a, b), \mathbf{join}(c, d)) = |a, c, d|\, b - |b, c, d|\, a \tag{3.1}$$

is a well defined point in the embedded version of $\mathbb{R}^2$ (i.e. the last coordinate of x is not 0) and $[x]$ is distinct from $[a]$ and $[b]$. Due to (2.5) it holds:

$$\frac{|a,\, d, c|}{|b,\, c, d\,|} = \frac{\triangle(a, d, c)}{\triangle(b, c, d)} = \frac{\overline{a\, x}}{\overline{x\, b}}, \tag{3.2}$$

where and the notation $\frac{\overline{*\,*}}{\overline{*\,*}}$ denotes an oriented length ratio in the corresponding Euclidean setup (see also Figure 3.1). This identity can easily be checked by means of elementary geometry. We will use this principle very often. It is able to give

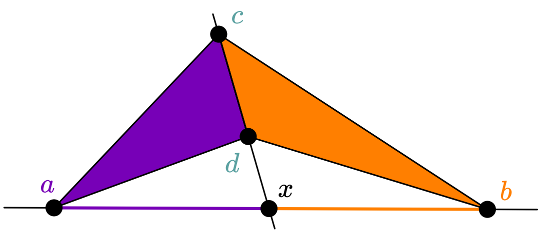


Figure 3.1.: The area principle: $\frac{|a,d,c|}{|b,c,d|} = \overline{a\,x}\,/\,\overline{x\,b}$.

a *geometric* interpretation of determinantal expressions. The area principle will be the key ingredient and starting point for Γ-cycles in Part II as well as the key intuition for Cayley factorization in Part III. Observe that formula for x given in (3.1) uses the same determinants as the area principle in (3.2). One can say, that starting with the determinants, one can *generate* them as coefficients in a linear combination expressing x. This observation will be used later on as well. It is not new and was recovered e.g. by Grünbaum and Shephard in [53]. On page 182 in a subsequent publication [54] the same authors give some historical remarks. The oldest emergence of the principle given there is [34] from 1816. They also give the reference [21] which is closely related to incidence theorems. They are the origin from which Part II emerged. As mentioned earlier, the principle is closely related with the theory of weights and centers of mass used by Möbius in [86]. Generating the determinants in linear combinations by the above construction was also used in [104] which will get more important in Part III.

When doing calculations with oriented length ratios, there is always a formulation via brackets nearby and no explicit specialization to Euclidean space is needed for formal calculations. First demonstrations of this fact will be done in Section 3.2 and Section 3.6.

In general, the area principle can be used in two directions: the first one is to express oriented length ratios via determinants. This will be used e.g. in the next section and allows for simple proofs of theorems about oriented length ratios as well as incidence theorems (see Part II).

When we take the opposite direction and we use the words "applying the area principle": assume that we have two determinants (or entities closely related to determinants, see Chapter 4 and Chapter 10) which differ by exactly one entry. This fits with the combinatorics of (3.2) and the ratio of determinants can be interpreted as oriented length ratio. The combinatorics of (3.2) are highlighted in a colored version below:

$$\frac{|a,\ d, c|}{|b,\ c, d\,|}$$

In fact, for applying the area principle to generalized determinants, it will suffice that they differ by exactly one element for permutations of the entries can by accounted by sign changes of the result. We emphasize the combinatorics, since we want to show how they are interpreted when "applying the area principle" and to introduce some terminology. The blue-grey entries are the common elements of the determinants, the blue-violet entry only occurs in the numerator, the orange entry only in the denominator. The blue-grey entries span the *transversal*, the blue-violet and the orange entry span the *reference line*. x is the intersection of the transversal and the reference line. The length ratio is taken along the reference line with respect to x. Sometimes a and b are called *endpoints* and x is called *intermediate point*. The complete picture including the newly introduced terms is shown in Figure 3.2.

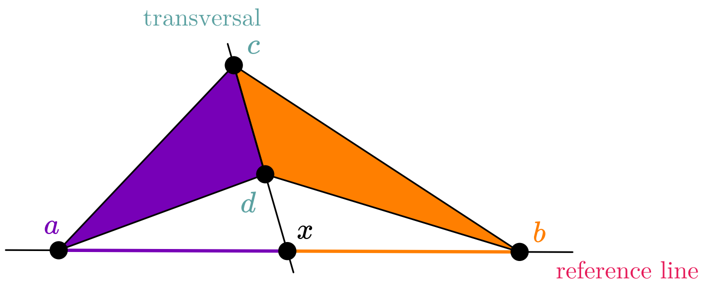
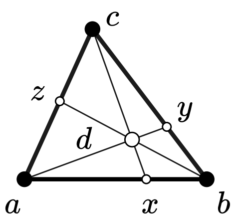


Figure 3.2.: The length ratio $\overline{a\,x}\,/\,\overline{x\,b}$ is induced by the combinatorics of $\frac{|a,d,c|}{|b,c,d|}$.

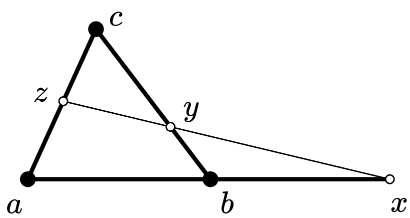
3.2. The Theorems of Ceva and Menelaus

The area principle can be used to give very short proofs of the classical theorems of Giovanni Ceva (1647–1734) and Menelaus of Alexandria (c. 70–140) in Euclidean geometry (see also Figure 3.3):

- Ceva's theorem states that if in a triangle the sides are cut by three lines concurring in a single point (the *Ceva center* or *centroid*) and passing through the corresponding opposite vertex, then the product of the three (oriented) length ratios along each side equals 1.
- Menelaus's theorem states that this product is -1 if the cuts along the sides are induced by a single line, called *Menelaus line*.



Ceva's Thm: $\dfrac{\overline{a\,x}}{\overline{x\,b}} \cdot \dfrac{\overline{b\,y}}{\overline{y\,c}} \cdot \dfrac{\overline{c\,z}}{\overline{z\,a}} = 1$



Menelaus's Thm: $\dfrac{\overline{a\,x}}{\overline{x\,b}} \cdot \dfrac{\overline{b\,y}}{\overline{y\,c}} \cdot \dfrac{\overline{c\,z}}{\overline{z\,a}} = -1$

Figure 3.3.: Theorems of Ceva and Menelaus.

A reformulation of the claims of both theorems via the area principle (see also [53]) and assuming homogenous coordinates in the standard embedding makes the statement of the theorems almost trivial: in the Ceva case we have

$$\frac{\overline{a\,x}}{\overline{x\,b}} \cdot \frac{\overline{b\,y}}{\overline{y\,c}} \cdot \frac{\overline{c\,z}}{\overline{z\,a}} = \frac{|a,c,d|}{|b,d,c|} \cdot \frac{|b,a,d|}{|c,d,a|} \cdot \frac{|c,b,d|}{|a,d,b|} = 1 \tag{3.3}$$

and in the Menelaus case we have

$$\frac{\overline{a\,x}}{\overline{x\,b}} \cdot \frac{\overline{b\,y}}{\overline{y\,c}} \cdot \frac{\overline{c\,z}}{\overline{z\,a}} = \frac{|a,x,y|}{|b,y,x|} \cdot \frac{|b,x,y|}{|c,y,x|} \cdot \frac{|c,x,y|}{|a,y,x|} = -1. \tag{3.4}$$

Observe that for each $\alpha \in \mathbb{R} \setminus \{0\}$, there is exactly one vector w of homogeneous coordinates (with last coordinate 1) such that $\overline{c\,w}/\overline{w\,a} = \alpha$. This implies that also the converse of the above formulation is true: from $\overline{a\,x}/\overline{x\,b} \cdot \overline{b\,y}/\overline{y\,c} \cdot \overline{c\,w}/\overline{w\,a} = 1$ or $\overline{a\,x}/\overline{x\,b} \cdot \overline{b\,y}/\overline{y\,c} \cdot \overline{c\,w}/\overline{w\,a} = -1$ we can deduce the existence of a centroid or a Menelaus line, resp. This can be seen by the fact that x and y already determine the centroid of the Menelaus line. Therefore there is a point z' fulfilling the adapted versions of (3.3) and (3.4). It has to coincide with z since $\overline{c\,z}/\overline{z\,a} = \overline{c\,z'}/\overline{z'\,a}$. In the following, we will consider the theorems of Ceva and Menelaus as containing both directions of implication.

3.3. Projective Transformations and Invariants

This section will be formulated for general rank n in order to establish the necessary concepts needed in Section 4 which treats the bracket ring. Consider an invertible $n \times n$ matrix M. The left multiplication of M with the homogeneous coordinates of a point x in $\mathbb{RP}^{n-1}$ induces a map $\mathcal{P}^{(n-1)} \to \mathcal{P}^{(n-1)}$. Any such map is called a *projective transformation.* The class of projective transformations also covers the better known (embedded versions of) affine transformations such as rotations, scaling and shearing. A projective transformation is intrinsic to projective geometry in the sense that a projective transformation preserves collinearity, i.e.

$$[a],\ [b],\ [c] \in \mathcal{P}^{n-1} \text{ are collinear } \implies [M \cdot a],\ [M \cdot b],\ [M \cdot c] \text{ are collinear.}$$

In rank 3 this can be easily seen by assuming that l provides the coordinates of the line incident with $[a]$, $[b]$ and $[c]$. Now, the line with coordinates $M^{-T}l$ is incident with $[M \cdot a]$, $[M \cdot b]$ and $[M \cdot c]$. It is a classical result traditionally formulated for the plane that in the *real* projective plane, also the converse is true.

Theorem 3.1 (Fundamental Theorem of Projective Geometry)**.** *If $\Phi : \mathcal{P} \to \mathcal{P}$ is any bijective map on the points of the real projective plane $\mathbb{RP}^2$ that preserves the collinearity of points, then Φ can be expressed as multiplication by a invertible 3×3 matrix.*

This classical result is known as *fundamental theorem of projective geometry.* One can find proofs in many textbooks on projective geometry. A detailed description of the history of the theorem is given in [111]. Its origins are attributed to Möbius, Chasles and Steiner whereas the real treatment of the questions starts with von Staudt [112]. The constructions he uses will be restated in Section 3.7. Under a general projective transformation, lengths and ratios of lengths are not preserved. This leads to the question, which properties are preserved and to the definition of a projective invariant. This kind of question can be regarded as being fundamental to all kind of geometries and is closely related to Klein's Erlangen program.

Note that the proof of Theorem 3.1 actually works with properties on projective lines and incorporates each additional dimension by introducing a new projective line as reference system. Therefore Theorem 3.1 also holds in all higher dimensions (see e.g. [3] for a verification of this fact). Observe that each element of $\mathbb{R}^{n\cdot k}$ can be interpreted as giving a point configuration: we can consider it as a matrix whose columns are coordinates of points.

Definition 3.2. *Let S be an arbitrary set. A* projective invariant *of rank n on k points in the real projective space $\mathbb{RP}^{n-1}$ is a map f from a dense subset of $\mathbb{R}^{n\cdot k}$ to S such that for all invertible real $n\times n$ matrices $M\in\mathrm{GL}(\mathbb{R},n)$ and $k\times k$ invertible real diagonal matrices $D\in\mathrm{diag}(\mathbb{R}\setminus\{0\},k)$ and for any configuration P, the map f is defined on $M\cdot P\cdot D$ and we have*

$$f(P)=f(M\cdot P\cdot D).$$

In the case that S is a binary set, f is called a projectively invariant property*.*

Each projective invariant is also an SL-invariant as defined below. We consider SL-invariants since the polynomial SL-invariants have a nice structure as we will see in Chapter 4. Therefore all objects built on this structure, as the Grassmann-Cayley algebra introduced in Chapter 5, are closely related with SL-invariants.

Definition 3.3. *Let S be an arbitrary set. A* SL-invariant *of rank n on k points in $\mathbb{R}^n$ is a map f from a dense subset of $\mathbb{R}^{n\cdot k}$ to S such that for all real $n\times n$ unimodular matrices M, i.e. $M\in\mathrm{SL}(\mathbb{R},n)$, and for any configuration P, the map f is defined on $M\cdot P$ and we have*

$$f(P)=f(M\cdot P).$$

When no rank is explicitly specified we assume rank 3. In Section 3.4, we will get to know the prototype of a projective invariant, whereas Chapter 4 will provide a complete description of (polynomial) SL-invariant functions. It turns out that some of them induce *projectively* invariant properties. In fact, in the context of projective geometry, there is a whole zoo of different definitions of invariants. There is also a definition of a relative invariant or semi-invariant. In [37] (used as foundation of Chapter 5) the relative invariant is called an invariant. So the reader should be careful in this context. In the present text, we decided to chose the projective invariants and the SL-invariants in order to have a clear setup. We are guided by [95, 102].

3.4. The Cross-Ratio

Another fundamental ingredient to projective geometry used extensively in Part III is the cross-ratio. It forms the smallest projective invariant and is in fact an invariant involving four points on a projective *line*. The origins of the concept even dates back to the ancient greeks. In order to keep the exposition short, we embed the definition in $\mathbb{RP}^2$.

Definition 3.4. *The* cross-ratio *of the points* $[a]$, $[b]$, $[x]$, $[y] \in \mathcal{P}$ *(in this order) seen from* $[o] \in \mathcal{P}$ *is defined as the magnitude*

$$([a],[b];[x],[y])_{[o]} \;:=\; \frac{|o,\,a,x|}{|o,\,x,b\,|}\cdot\frac{|o,\,b,y\,|}{|o,\,y,a|} \tag{3.5}$$

whenever the magnitude can be considered as a value in $\mathbb{R}\cup\{\infty\}$*, i.e. whenever* $[o]$, $[a]$, $[b]$, $[x]$, $[y]$ *are such that the right-hand side of* (3.5) *is in* $\mathbb{R}$ *or they are such that the denominator vanishes but the numerator does not vanish.*

Due to the linearity of the determinant, the cross-ratio is well defined for points in $\mathcal{P}$. The multiplicativity and multilinearity of the determinant imply that the cross-ratio seen form $[o]$ is a projective invariant. Furthermore, in non-degenerate situations, replacing e.g. a by $o+\lambda a$ (for $\lambda\in\mathbb{R}\setminus\{0\}$) results in the same value for the cross-ratio seen from $[o]$. So the cross-ratio seen from $[o]$ in fact depends on the *lines* through $[o]$ and $[a]$, $[b]$, $[x]$, $[y]$, resp. (see also Figure 3.4 where the square brackets indicating equivalence classes are omitted in order to obtain a clear arrangement).

On the other hand, assuming that $[a]$, $[b]$, $[x]$ and $[y]$ are on a common line not incident to $[o]$, the expression does not depend on the specific choice of the point $[o]$: Since $([a],[b];[x],[y])_{[o]}$ is invariant under projective transformations and rescaling, we can assume that a, b, x, y and o are coordinates of the standard embedding: if the last coordinate of a, b, x, y or o equals 0, one can find an invertible matrix such that the transformed coordinates can be rescaled to coordinates of points in the standard embedding. We will not give such a matrix explicitly. However almost all matrices are appropriate. In a non-degenerate situation, we can apply the area

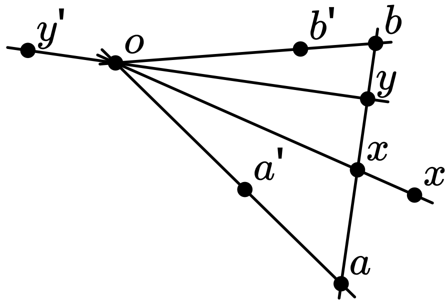


Figure 3.4.: In $([a],[b];[x],[y])_{[o]}$ we can replace a by a', b by b', x by x' and y by y' and we obtain the same value.

principle from 3.1 to the defining equation of the cross-ratio (see also Figure 3.5). This implies that we have

$$([a],[b];[x],[y])_{[o]} = \frac{\overline{a\,x}}{\overline{x\,b}} \cdot \frac{\overline{b\,y}}{\overline{y\,a}} \tag{3.6}$$

which does not depend on $[o]$. This also holds in degenerate situations and we can define:

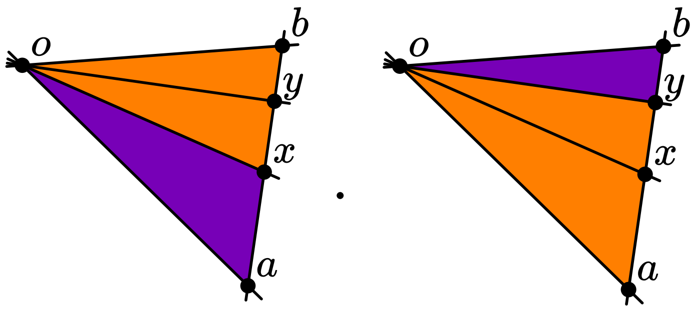


Figure 3.5.: Applying the area principle to the defining equation of the cross-ratio.

Definition 3.5. *The* cross-ratio *of four points* $[a]$, $[b]$, $[x]$, $[y] \in \mathcal{P}$ *(in this order) incident with a line* $[l] \in \mathcal{L}$ *is defined as*

$$([a],[b];[x],[y]) := \frac{|o,\,a,x|}{|o,\,x,b|} \cdot \frac{|o,\,b,y|}{|o,\,y,a|}$$

with arbitrary $[o] \in \mathcal{P}$ *not on the line* $[l]$.

Due to the possible replacement of e.g. a by $o + \lambda a$ in $([a],[b];[x],[y])_{[o]}$ we have:

Proposition 3.6 (Invariance of Cross-Ratios under Projections)**.** *Let* $[o]$ *be a point and let* $[l]$ *and* $[l']$ *be two lines not passing through* $[o]$. *If four points* $[a]$, $[b]$, $[c]$, $[d]$ *on* $[l]$ *are projected by the viewpoint* $[o]$ *to four points* $[a']$, $[b']$, $[c']$, $[d']$ *on* $[l']$, *then the cross-ratios satisfy* $([a],[b];[c],[d]) = ([a'],[b'];[c'],[d'])$.

3.5. Projective Number Lines

Figure a line in $\mathbb{RP}^2$ and three distinct points $[p_0]$, $[p_1]$ and $[p_\infty]$ on this line. W.l.o.g. we can assume that they are scaled in such a way that $p_1 = p_0 + p_\infty$. This fixes the relative scaling between the three given points. Any other point $[p_x]$ on this line, that different from $[p_0]$, is identical to $[p_0 + x \cdot p_\infty]$ for some $x \in \mathbb{R}$. One can easily check that it holds

$$([p_0]\,,\,[p_\infty]\,;[p_x]\ \ ,\,[p_1]) = x$$

for any such x and also

$$\begin{aligned}
([p_0]\,,\,[p_\infty]\,;[p_\infty]\,,\,[p_1]) &= \infty,\\
([p_0]\,,\,[p_\infty]\,;[p_0]\ \ ,\,[p_1]) &= 0,\\
([p_0]\,,\,[p_\infty]\,;[p_1]\ \ ,\,[p_1]) &= 1.
\end{aligned}$$

Thus by its projective invariance, the cross-ratio can be interpreted as a number obtained by rescaling three of the points and expressing a fourth point with respect to the other three ones.

Furthermore, for *any* point $[p]$ on the line spanned by distinct points $[p_0]$ and $[p_\infty]$, there is the tuple of real numbers (μ_p, λ_p) different from $(0,0)$ such that $[p] = [\mu_p\, p_0 + \lambda_p\, p_\infty]$. Any rescaling of the pair (λ_p, μ_p) gives the same point. This induces homogenous coordinates of the *line* with respect to the concrete homogeneous coordinates p_0 and p_∞. In a situation analogue to Section 2.1 these coordinates have the structure of $\frac{\mathbb{R}^2\setminus\{(0,0)\}}{\mathbb{R}\setminus\{0\}}$. So if p_0 and p_∞ are scaled in such a way that $p_1 = p_0 + p_\infty$ and $[q]$ is a point on the line with $q = \mu_q\, p_0 + \lambda_q\, p_\infty$ we have that

$$([p_0]\,,\,[p_\infty]\,;\,[q]\,,\,[p_1]) \;=\; \frac{\lambda_q}{\mu_q}.$$

Alltogether, the cross-ratio can be considered as line coordinates of any point on a line with respect to three given base points $[p_0]$, $[p_\infty]$ and $[p_1]$. Since the properties describing a basis do *not* depend on the concrete homogeneous coordinates of the points, it is possible to talk about three points in $\mathcal{P}$ being a basis.

3.6. Quadrilateral Sets

Quadrilateral sets are the foundation for doing *constructive calculations* on projective number lines in the next section. In the (old) literature they are often called "points of an involution". Consider the situation in Figure 3.6: there it is given a complete quadrilateral $[z]$, $[z_1]$, $[z_2]$, $[z_3]$ which sides are cut with another line resulting in points $[a_1]$, $[a_2]$, $[a_3]$, $[b_1]$, $[b_2]$ and $[b_3]$. Again, we omit the square brackets indicating equivalence classes in order to obtain a clear arrangement.

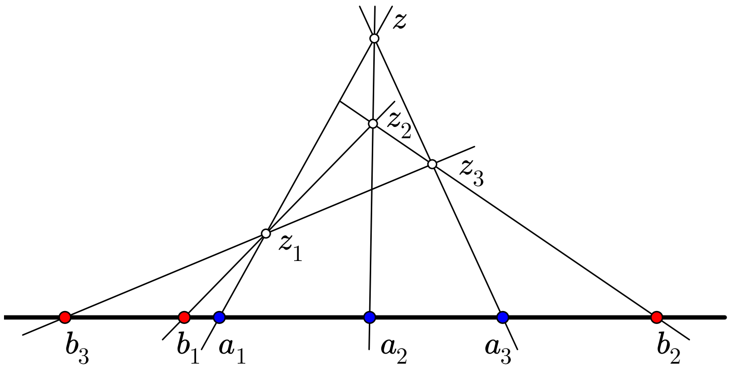


Figure 3.6.: The incidence configuration shows that (3.7) holds and that $\{\{[a_1],[b_2]\},\{[a_2],[b_3]\},\{[a_3],[b_1]\}\}$ is a quadrilateral set.

We will show that in this case for any point $[o]$ it holds that

$$|o,\, a_1, b_1| \cdot |o,\, a_2, b_2| \cdot |o,\, a_3, b_3| = |o,\, a_1, b_3| \cdot |o,\, a_2, b_1| \cdot |o,\, a_3, b_2|. \tag{3.7}$$

In order to show this and assuming a non-degenerate situation, where all the determinants do not vanish, we can rewrite the identity as

$$\frac{|o,\, a_1, b_1|}{|o,\, a_2, b_1|} \cdot \frac{|o,\, a_2, b_2|}{|o,\, a_3, b_2|} \cdot \frac{|o,\, a_3, b_3|}{|o,\, a_1, b_3|} = 1. \tag{3.8}$$

Since the left-hand side of (3.8) is again invariant under projective transformations and rescaling of the points, we can w.l.o.g. assume homogeneous coordinates with last coordinate equaling 1. This allows for an reformulation of (3.8) equation via the area principle:

$$\frac{\overline{a_1\,b_1}}{\overline{b_1\,a_2}} \cdot \frac{\overline{a_2\,b_2}}{\overline{b_2\,a_3}} \cdot \frac{\overline{a_3\,b_3}}{\overline{b_3\,a_1}} = -1.$$

This identity can be proved by applying Menelaus's theorem to the three triangles (a_1, a_2, z), (a_2, a_3, z) and (a_3, a_1, z). This allows us to rewrite all *parts* of the above equation in a way such that their overall product equals -1:

$$\frac{\overline{a_1\,b_1}}{\overline{b_1\,a_2}} = -\frac{\overline{a_1\,z_1}}{\overline{z_1\,z}} \cdot \frac{\overline{z\,z_2}}{\overline{z_2\,a_2}},$$

$$\frac{\overline{a_2\,b_2}}{\overline{b_2\,a_3}} = -\frac{\overline{a_2\,z_2}}{\overline{z_2\,z}} \cdot \frac{\overline{z\,z_3}}{\overline{z_3\,a_3}},$$

$$\frac{\overline{a_3\,b_3}}{\overline{b_3\,a_1}} = -\frac{\overline{a_3\,z_3}}{\overline{z_3\,z}} \cdot \frac{\overline{z\,z_1}}{\overline{z_1\,a_1}}.$$

As announced in Section 3.1, the advantage of the area method is that it comes with a less Euclidean and more projective formulation of this derivation, which is a little harder to read but does not make assumptions on the shape of homogeneous coordinates of the points involved:

$$\frac{|o,\,a_1,b_1|}{|o,\,a_2,b_1|} = \frac{|z_1,z_2,\,a_1|}{|z_1,z_2,\,a_2|} = \frac{|z_1,z_2,\,a_1|}{|z_1,z_2,\,z\,|} \cdot \frac{|z_1,z_2,\,z\,|}{|z_1,z_2,\,a_2|} = \frac{|z_1,z_2,\,a_1|}{|z_1,z_2,\,z\,|} \cdot \frac{|z_2,z_3,\,z\,|}{|z_2,z_3,\,a_2|},$$

$$\frac{|o,\,a_2,b_2|}{|o,\,a_3,b_2|} = \frac{|z_2,z_3,\,a_2|}{|z_2,z_3,\,a_3|} = \frac{|z_2,z_3,\,a_2|}{|z_2,z_3,\,z\,|} \cdot \frac{|z_2,z_3,\,z\,|}{|z_2,z_3,\,a_3|} = \frac{|z_2,z_3,\,a_2|}{|z_2,z_3,\,z\,|} \cdot \frac{|z_1,z_3,\,z\,|}{|z_1,z_3,\,a_3|},$$

$$\frac{|o,\,a_3,b_3|}{|o,\,a_1,b_3|} = \frac{|z_1,z_3,\,a_3|}{|z_1,z_3,\,a_1|} = \frac{|z_1,z_3,\,a_3|}{|z_1,z_3,\,z\,|} \cdot \frac{|z_1,z_3,\,z\,|}{|z_1,z_3,\,a_1|} = \frac{|z_1,z_3,\,a_3|}{|z_1,z_3,\,z\,|} \cdot \frac{|z_1,z_2,\,z\,|}{|z_1,z_2,\,a_1|}.$$

In a Euclidean setup, the identities can be considered to be true since they can be derived by applying the area principle to the Menelaus identities given above.

By a more careful investigation one can find two basic kind of operations in the detailed calculation:

- In each row, the first and the last equality can be considered as changing the points involved in the area principle. Therefore one needs the existence of some collinearities.
- The second equality explains itself by canceling on the right-hand side.

Both tools will be introduced more formally in Chapter 10 and are called *biquadratic fractions* and Γ-*cycles*. These calculations have the advantage that they are more projective in nature. However, they are less intuitive and more algebraic and non-geometric in nature, unless the area principle can be applied.

In fact, (3.7) has some symmetries which can be captured by saying that

$$\big\{\{[a_1],[b_2]\},\{[a_2],[b_3]\},\{[a_3],[b_1]\}\big\} \tag{3.9}$$

forms a *quadrilateral set* whenever (3.7) is fulfilled. We use the notation of sets which shall be allowed to be multisets. Furthermore, this condition does not depend on the order of elements of the sets in (3.9). So the construction in Figure 3.6 witnesses the fact, that $\{\{[a_1],[b_2]\},\{[a_2],[b_3]\},\{[a_3],[b_1]\}\}$ forms a quadrilateral set. However, due to the symmetry just described it is not the only construction for witnessing (3.9). For our purposes we conclude that the combinatorics of a quadrilateral set given by the construction in Figure 3.6 can be *identified* by following pairs of lines that do not share a common point in the construction.

3.7. Von Staudt's Constructions: Arithmetic on Projective Number Lines

In this section, no explicit reference to the concrete homogeneous coordinates is needed and we can do our considerations without square brackets denoting equivalence classes. Let p_0, p_1 and p_∞ be in $\mathcal{P}$ such that they form a projective basis of a number line in $\mathbb{RP}^2$. Let p_x and p_y two other points on this line with $(p_0\,,\,p_\infty\,;\,p_x\,,\,p_1)=x$ and $(p_0\,,\,p_\infty\,;\,p_y\,,\,p_1)=y$. We are interested in points p_{x+y} with $(p_0\,,\,p_\infty\,;\,p_z\,,\,p_1)=x+y$ and $p_{x\cdot y}$ with $(p_0\,,\,p_\infty\,;\,p_z\,,\,p_1)=x\cdot y$. One can easily check by hand calculation that

$$\big\{\{p_0\,,\,p_{x+y}\}\,;\,\{p_x\,,\,p_y\}\,;\,\{p_\infty\,,\,p_\infty\}\big\} \quad \text{and} \quad \big\{\{p_0\,,\,p_\infty\}\,;\,\{p_x\,,\,p_y\}\,;\,\{p_1\,,\,p_{x\cdot y}\}\big\}$$

form quadrilateral sets. Figure 3.7 induces a construction of the points p_{x+y} and $p_{x\cdot y}$. Observe that the symmetry of quadrilateral sets implies the commutativity of projective addition and multiplication.

The tools of projective addition and multiplication where originally introduced by von Staudt in [112]. He introduces the language of *Würfe* (which is translated to *throws* e.g. in [110]). This concept makes it possible to avoid the assignment of coordinates. Von Staudt uses throws for his version of the fundamental theorem of projective geometry (see Theorem 3.1 in Section 3.3). Those constructions are not contained in every textbook on projective geometry. E.g. in [25] as well as in [110] they are introduced. The latter has the advantage that it gives a translation from the old notions of von Staudt into more modern terms of language.

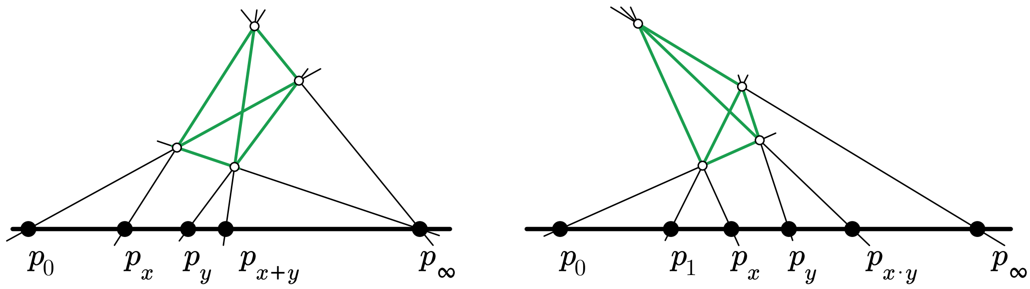


Figure 3.7.: Von Staudt constructions for projective addition and multiplication.

3.8. Harmonic Sets

In Section 16.4, we will once use the classical construction for a harmonic set. In fact, this concept is much more fundamental than the arithmetic constructions just seen. However it can be easily explained with their help: We say that

$$\big\{\{a\,,\,b\}\,;\,\{c\,,\,d\}\big\} \tag{3.10}$$

$(a, b, c, d \in \mathcal{P})$ is a harmonic set whenever

$$(a, b; c, d) = -1.$$

Observe that after canceling this is equivalent to

$$\big\{\{a, a\}, \{b, b\}, \{c, d\}\big\}$$

being a quadrilateral set. This implies that the order of the elements in the sets in in (3.10) does not affect the property of being a harmonic set. In particular, the considerations done in Section 3.6 imply that the $\big\{\{a\,,\,b\}\,;\,\{c\,,\,d\}\big\}$ is a harmonic set if the construction shown in Figure 3.8 is possible.

3.9. Configurations of Ceva and Menelaus Revisited

Most of the considerations involving oriented length ratios can be restated in terms of *weights*. Therefore we will use physical properties and intuition which we will not prove. If one wishes, one can translate large parts of the thesis into terms of weight ratios. A overall illustration of the phenomena described is shown in Figure 3.9. Consider homogeneous coordinates a, b and c of three non-collinear points in the standard embedding, i.e. the last coordinate a, b and c equals 1. Assume w_a, x_b in

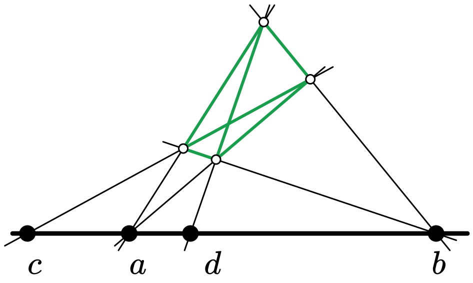


Figure 3.8.: Construction for witnessing $\{\{a,b\};\{c,d\}\}$ being a harmonic set.

$\mathbb{R} \setminus \{0\}$ are given such that the linear combination

$$w_a\, a + w_b\, b$$

is a finite point, i.e. its last coordinate is distinct from 0. The points Euclidean interpretations of a, b and c form a triangle. Let x be the rescaled version of the above linear combination that allows us to consider oriented length ratios in the Euclidean plane. Due to the area principle it holds

$$\frac{\overline{a\,x}}{\overline{x\,b}} = \frac{w_b}{w_a}. \tag{3.11}$$

This property establishes the connection between lengths and weights: assume there are weights w_a, w_b and w_c in $\mathbb{R} \setminus \{0\}$ attached to the points a, b and c. In homogeneous coordinates, the center of mass of the segment with endpoints a and b is given by $z_{ab} := w_a\, a + w_b\, b$ and similarly for the other two segments. This is the lever rule from physics stated in homogenous coordinates. By physics we know that there exists a center of mass of the whole triangle. Observe that any point on the line spanned by $[a]$ and $[b]$ can be written as linear combination of a and b. Therefore *any* point of the line can be interpreted as a center of mass where the weights are the coefficients of the linear combination. Now Ceva's theorem can be rephrased as:

"Consider homogeneous coordinates of points x_{ab}, x_{bc} and x_{ca} on the lines supporting the corresponding triangle edges. The lines

$$[\mathbf{join}(a, x_{bc})],\ [\mathbf{join}(b, x_{ca})] \ \text{ and } \ [\mathbf{join}(c, x_{ab})]$$

meet in a single point if and only if we can find weights w_a, w_b and w_c such that $[x_{ab}]$, $[x_{bc}]$ and $[x_{ca}]$ are the same points as the induced centers of mass."

A centroid implying weights is the existence of barycentric coordinates introduced by Möbius who starts his investigations by studying the center of mass (see [86]). He also relates length ratios together with an orientation to the ratio of masses. Barycentric coordinates can be understood as homogenous coordinates of an embedding of the Euclidean plane into projective space which is different from the standard embedding introduced in Section 2.1. This shows the deep connection between Ceva's theorem and weights.

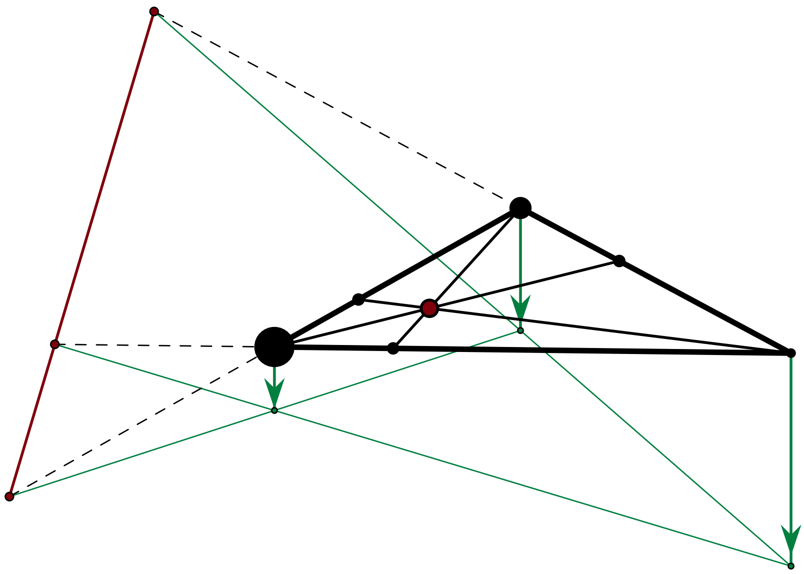


Figure 3.9.: Weights are given for the vertices of the triangle. They are indicated by the size of the vertices. Its inverses are indicated by green arrows which are considered to be forces located at the vertices. The centroid and the Menelaus lines are given. The latter contains the centers of motions of the edges which are induced by the green forces.

Also for the theorem of Menelaus it can be given an interpretation which depends on *the same weights* w_a, w_b and w_c: consider again the segment with endpoints a and b and forces induced by ${}^1\!/w_a$ and ${}^1\!/w_b$ at the corresponding vertices of the segment. Due to (3.11), the ratio of these values is the same as $\frac{\overline{a\,x}}{\overline{x\,b}}$. There is an associated rotation of the segment with center

$$y_{ab} := w_a\, a - w_b\, b.$$

Observe that y_{ab} is related to z_{ab} by $\{\{[a], [b]\}; \{[z_{ab}], [y_{ab}]\}\}$ being a harmonic point set. Assume y_{bc} and y_{ca} are defined analogously. Now y_{ab} and y_{bc} suffice to induce a rotation of the whole triangle. This motion has an *axis* which is spanned by y_{ab} and y_{bc}. The point y_{ca} has to lie on this axis as well. This fact can be understood as Menelaus's theorem. It can be paraphrased as: "Given instantaneous motions on the three vertices of a triangle, the induces centers of motion on the edges lie on a common line."

A generalization of this fact, which uses weights on the vertices of a tetrahedron in 3-space and its the inverses of the weights as instantaneous motions, induces the well known incidence theorem of Desargues. For more information about incidence theorems see Chapter 7. A generalization to 4-space should be possible as well. A related reasoning, which uses three vertices of a triangle as a basis for further constructions, is given in [139]. The three base points remain fixed during the calculations. The method can be interpreted as using barycentric coordinates with respect to the three base points chosen.

3.10. Conics and Pascal's Theorem

For future reference, we also state another classical theorem about conics. It is due to Pascal who proved it in 1640 (see [22] for an english translation). A *conic* $\mathcal{C}$ consists of those points $[(x, y, z)]$ in $\mathbb{RP}^2$ that fulfill

$$\alpha \cdot x^2 + \beta \cdot y^2 + \gamma \cdot xy + \delta \cdot xz + \epsilon \cdot yz + \zeta \cdot z^2 = 0$$

for some real α, β, γ, δ, ϵ and ζ depending on $\mathcal{C}$. By counting degrees of freedom one finds that five points in general position supply enough equations to determine these values. Therefore, six points, that lie on the same conic, are in a *special position.* It is not hard to derive (see [95, p. 169]) that the six points $[a]$, $[b]$, $[c]$, $[d]$, $[e]$ and $[f]$ lie on a common conic if and only if

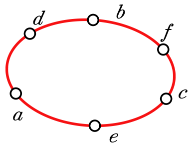
$$|a,c,e| \cdot |a,d,f| \cdot |b,c,f| \cdot |b,d,e| - |a,c,f| \cdot |a,d,e| \cdot |b,c,e| \cdot |b,d,f| = 0. \tag{3.12}$$

It is the statement of Pascal's theorem that both is also equivalent to the fact that the points $[x]$, $[y]$ and $[z]$, whose construction is shown in Figure 3.10 on the right, are collinear. We will prove the equivalence of (3.12) and the collinearity of $[x]$, $[y]$ and $[z]$ in Chapter 6. Observe that a and b never occur in the same determinant. The defining equation can be rewritten in terms of cross-ratios of line bundles:

$$\frac{|a,c,e| \cdot |a,d,f|}{|a,c,f| \cdot |a,d,e|} = \frac{|b,c,e| \cdot |b,d,f|}{|b,c,f| \cdot |b,d,e|}$$

which is equivalent to

$$([c],[d];[e],[f])_{[a]} = ([c],[d];[e],[f])_{[b]}.$$



$$\Leftrightarrow \quad \begin{array}{c} |a,c,e| \cdot |a,d,f| \cdot |b,c,f| \cdot |b,d,e| \\ -|a,c,f| \cdot |a,d,e| \cdot |b,c,e| \cdot |b,d,f| = 0 \end{array} \quad \Leftrightarrow$$

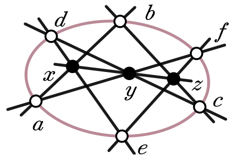


Figure 3.10.: Pascal's theorem: six points a, b, c, d, e and $f \in \mathcal{P}$ lie on a common conic if and only if the depending points x, y, z lie on a common line.

4. The Bracket Ring and the Fundamental Theorems of Invariant Theory

A lot of properties in projective geometry can be stated with the help of determinants. The cross-ratio (Section 3.4) and the condition for six points lying on a common conic (Section 3.10) are only a few examples. Theorem 4.9 will show that in a certain sense, *any* projectively invariant (polynomial) property can be stated with the help of determinants. To formulate this result, we introduce some notation in the current chapter. In the literature (see below), there are many slightly different versions of these results. We follow the approach given by Sturmfels in [102]. The concept, that all geometry is in fact invariant theory, motivates many of the considerations in the following Chapters and is subject of Klein's Erlangen program.

In the present chapter, we give an algebraic structure that describes determinants *symbolically*. We dissociate the determinants from their definition via vector coordinates and instead view them as formal symbols only depending on the *names* of the points without referring to concrete coordinates. Therefore the notion of determinants is simplified and square brackets, e.g. $[*,*,*]$ in rank 3, are used. This explains the name "bracket ring". We briefly describe the main ideas in the example of rank 3 and then give some additional informations about the origin of the bracket ring. Let $\mathbf{P}$ be a finite set of names of points and let $\boldsymbol{\Delta}(\mathbf{P}) = \{\, [\mathbf{i},\mathbf{j},\mathbf{k}] \mid \mathbf{i},\mathbf{j},\mathbf{k} \in \mathbf{P}\,\}$ be the set of formal determinants. The points in $\mathbf{P}$ are thought to be points in $\mathbb{RP}^2$ or non-zero points in $\mathbb{R}^3$. $\mathbb{R}[\Delta(\mathbf{P})]$ is the ring over $\mathbb{R}$ which is generated by these symbols. We aim to model the behavior of true determinants as well as possible. This is in fact achieved quite well as, we will see in Section 4.5. We will ensure that e.g. $[\mathbf{a},\mathbf{b},\mathbf{a}] = 0$ and $[\mathbf{a},\mathbf{b},\mathbf{c}] = -[\mathbf{a},\mathbf{c},\mathbf{b}]$. Apart from that, there are other phenomena which we have to take into account. There are relations between determinants which are called Grassmann-Plücker relations. We will provide an interpretation of them. In Section 4.4, we will generalize the concept of the bracket ring to rank n, i.e. for $\mathbb{RP}^{n-1}$ or $\mathbb{R}^n$.

Another approach to the bracket ring is to consider SL-invariants. The determinant is an SL-invariant. The first fundamental theorem of invariant theory can be summarized as stating that combinations of determinants—or bracket polynomials, which are their symbolic counterparts—are enough to describe *all* (polynomial) SL-invariants. The second fundamental theorem describes the relations holding between different bracket polynomials. These relations will match the ones given in the definition of the bracket ring. They also imply that a representation of a bracket polynomial is not unique. Therefore we will briefly describe the straightening algorithm which is able to compute a normal form of bracket polynomials.

The definitions given here are taken from [95] and [102]. First, a definition for rank 3 is given. In the context of Grassmann coordinates, a symbolic treatment of determinants is already used in [65] whereas a systematic symbolic treatment is done by White in [120] in the more general setup of combinatorial geometries (see e.g. [31]). This point of view is fundamental for the definitions as they are done in the present chapter. Great insight of the subject is also shown in [109] and [107]. However, the treatment there is not symbolic in nature. This branch of invariant theory is (in this context) often referred to as "classical invariant theory". A linkage between new and old invariant theory is given in [16] by Carrell and Dieudonné. It is based on Weyl's classical groups [119]. Also classical invariant theory considers its computations as "symbolic method". However the terms and definitions used rely on coordinates. A formal language, which uses names of points, is also given by Whiteley in [135, 130, 134]. Here first order logic is considered. This means that relations under investigation include quantifiers. There, there are also given fundamental theorems of invariant theory. In particular, an invariant first order formula in coordinates can be translated into a first order formula in brackets. Within these notions it is natural to talk about models which are distinct from the language. In [51, 68, 99] Rota and his collaborators use a symbolic method which is generalized to model symmetric and antisymmetric tensors. This is done by generalizing the straightening algorithm for comparing bracket polynomials (see Section 4.7). In the case that the signed alphabet used contains only negative letters, classical invariant theory is obtained. This case is treated in more detail in [68, Chap. 8]. It is pointed out that the bracket ring is a special case of the supersymmetric bracket algebra introduced there. A point configuration assigning concrete homogeneous coordinates to the point names is considered being a *model*. Again the word model is borrowed from logic. This distinction will also be done in the present and the next chapter. More comments on this development, which is also related to the following chapter, can be found in [50].

Proofs for (versions of) the crucial results given in Section 4.5 can be found e.g. in [102, 37, 68, 65, 71, 119, 16]. A detailed summary of related topics including the definitions and relations given in the next chapters is given in the recent book [77] by Li. Again observe that the following treatment is symbolic in nature and that we never have to refer to concrete coordinates.

4.1. Grassmann-Plücker Relations in Rank 3

We start with a definition of the bracket ring in rank 3. In this case, we can show some connections to geometric and algebraic facts derived in Chapters 2 and Chapter 3. On the level of true determinants, (2.8) implies that for any homogeneous coordinates a, b, c, d, e and f of points in $\mathbb{RP}^2$, there are two ways of expressing $\big|a,\, b,\, \mathbf{meet}(\mathbf{join}(c,d),\mathbf{join}(e,f))\big|$ and it holds

$$\begin{aligned} -\,|c,\, d,\, e|\cdot|a,\, b,\, f| \,+\, |c,\, d,\, f|\cdot|a,\, b,\, e| \,=\, |e,\, f,\, c|\cdot|a,\, b,\, d| \\ -\,|e,\, f,\, d|\cdot|a,\, b,\, c|. \quad (4.1) \end{aligned}$$

This motivates the *4-term Grassmann-Plücker relation*

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}][\mathbf{d},\,\mathbf{e},\,\mathbf{f}]-[\mathbf{a},\,\mathbf{b},\,\mathbf{d}][\mathbf{c},\,\mathbf{e},\,\mathbf{f}]+[\mathbf{a},\,\mathbf{b},\,\mathbf{e}][\mathbf{c},\,\mathbf{d},\,\mathbf{f}]-[\mathbf{a},\,\mathbf{b},\,\mathbf{f}][\mathbf{c},\,\mathbf{d},\,\mathbf{e}]=0$$

$$\text{for } \mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\mathbf{e},\,\mathbf{f}\in\mathbf{P} \quad (4.2)$$

for the symbolized versions of determinants. The equation will be required to hold in the following definition of the bracket ring.

Remark 4.1. Assuming $\mathbf{f} = \mathbf{a}$ and the rules for permutations of the point names given in the next definition (and in particular implying that $[\mathbf{a},\,\mathbf{b},\,\mathbf{a}] = 0$) we can deduce the *3-term Grassmann-Plücker relation* for $\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\mathbf{e}\in\mathbf{P}$:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}][\mathbf{a},\,\mathbf{d},\,\mathbf{e}]-[\mathbf{a},\,\mathbf{b},\,\mathbf{d}][\mathbf{a},\,\mathbf{c},\,\mathbf{e}]+[\mathbf{a},\,\mathbf{b},\,\mathbf{e}][\mathbf{a},\,\mathbf{c},\,\mathbf{d}]=0. \quad (4.3)$$

In fact, [77, Prop. 2.16] tells us, that in a specific sense, the 3-term Grassmann-Plücker relations can also generate 4-termed Grassmann-Plücker relations. This is also true in higher rank.

4.2. The Bracket Ring $\mathcal{B}_\mathbf{P}$ in Rank 3

The previous considerations give rise to the following definition:

Definition 4.2 (Bracket Ring of Rank 3). *Let* $\mathbf{P}$ *be a finite set of point names.* $\Delta(\mathbf{P}) := \{\, [\mathbf{i},\mathbf{j},\mathbf{k}] \,|\, \mathbf{i},\mathbf{j},\mathbf{k}\in\mathbf{P}\,\}$ *defines a set of indeterminantes in the ring* $\mathbb{R}[\Delta(\mathbf{P})]$. *With* $\mathcal{S}_3$ *the symmetric group on three elements,* $\mathbb{R}[\Delta(\mathbf{P})]$ *contains the ideals*

$$\mathbf{I}_{repeat} := \Big\langle\, \big\{[\mathbf{i},\mathbf{j},\mathbf{k}]\in\Delta(\mathbf{P}) \,\big|\, \mathbf{i}=\mathbf{j} \textit{ or } \mathbf{i}=\mathbf{k} \textit{ or } \mathbf{j}=\mathbf{k}\big\}\,\Big\rangle$$

$$\mathbf{I}_{altern} := \Big\langle \left\{ [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3] + \operatorname{sgn}(\sigma)\, [\mathbf{p}_{\sigma(1)}, \mathbf{p}_{\sigma(2)}, \mathbf{p}_{\sigma(3)}] \,\middle|\, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbf{P}, \sigma \in \mathcal{S}_3 \right\} \Big\rangle$$

$$\begin{aligned}\mathbf{I}_{GP} \quad := \Big\langle \Big\{ &[\mathbf{a},\, \mathbf{b},\, \mathbf{c}][\mathbf{d},\, \mathbf{e},\, \mathbf{f}] - [\mathbf{a},\, \mathbf{b},\, \mathbf{d}][\mathbf{c},\, \mathbf{e},\, \mathbf{f}] + [\mathbf{a},\, \mathbf{b},\, \mathbf{e}][\mathbf{c},\, \mathbf{d},\, \mathbf{f}] \\ &- [\mathbf{a},\, \mathbf{b},\, \mathbf{f}][\mathbf{c},\, \mathbf{d},\, \mathbf{e}] \,\Big|\, \mathbf{a},\, \mathbf{b},\, \mathbf{c},\, \mathbf{d},\, \mathbf{e},\, \mathbf{f} \in \mathbf{P} \Big\} \Big\rangle.\end{aligned}$$

The bracket ring is defined as

$$\mathcal{B}_{\mathbf{P}} := \mathbb{R}[\Delta(\mathbf{P})] \Big/ \langle \mathbf{I}_{repeat} \cup \mathbf{I}_{altern} \cup \mathbf{I}_{GP} \rangle.$$

Remark 4.3. Observe that for $\mathbf{Q}$ and $\mathbf{P}$ disjoint we have

$$\mathcal{B}_{\mathbf{P}} \subset \mathcal{B}_{\mathbf{P} \dot{\cup} \mathbf{Q}}\ .$$

Following [102], consider an abstract point configuration X consisting of $|\mathbf{P}|$ homogeneous coordinates in $\mathbb{RP}^2$: let $X = (x_{ij})$ be an $3 \times |\mathbf{P}|$-matrix whose entries are indeterminates and whose columns correspond to elements of $\mathbf{P}$. $\mathbb{R}[x_{ij}]$ is the corresponding ring in $3 \cdot |\mathbf{P}|$ variables. To make the connection between the abstract bracket ring and the projective geometry introduced in Chapter 2, we define an algebra homomorphism from the bracket ring $\mathcal{B}_{\mathbf{P}}$ to $\mathbb{R}[x_{ij}]$. Therefore, fix an order on $\mathbf{P}$ and let $\mathbf{i}$ be the element of $\mathbf{P}$ at position i. We allow ourselves to write $x_{\mathbf{i}\_}$ instead of $x_{\_i}$.

Definition 4.4. *The map*

$$\phi : \begin{cases} \mathbb{R}[\Delta(\mathbf{P})] & \to \mathbb{R}[x_{ij}] \\ [\mathbf{i}, \mathbf{j}, \mathbf{k}] & \mapsto \det \begin{pmatrix} x_{\mathbf{i}1} & x_{\mathbf{j}1} & x_{\mathbf{k}1} \\ x_{\mathbf{i}2} & x_{\mathbf{j}2} & x_{\mathbf{k}2} \\ x_{\mathbf{i}3} & x_{\mathbf{j}3} & x_{\mathbf{k}3} \end{pmatrix} \end{cases}$$

vanishes on $\mathbf{I}_{repeat}$*,* $\mathbf{I}_{altern}$ *and* $\mathbf{I}_{GP}$*,* ϕ*. Therefore it induces a map*

$$\Phi : \mathcal{B}_{\mathbf{P}} \to \mathbb{R}[x_{ij}]$$

which is called the generic coordinatization*.*

Where identities induces by Φ hold for *all* point configurations, the following definition of Φ^P can check for relations between concrete homogeneous coordinates of points:

Definition 4.5. *Let P be a (concrete) point configuration associated to $|\mathbf{P}|$ many points in $\mathbb{RP}^2$: let $P = (p_{ij})$ be an $3 \times |\mathbf{P}|$-matrix, whose columns are homogeneous coordinates. The coordinatization map*

$$\Phi^P : \mathbb{R}[\Delta(\mathbf{P})] \to \mathbb{R}$$

is the composition of Φ and the substitution homomorphism given by the values of P.

4.3. Grassmann-Plücker Relations in Rank n

The bracket ring can be defined not only for rank 3. The Grassmann-Plücker relations are induced essentially by Cramer's rule from linear algebra. It can be restated as follows: Searching for solutions of the linear equation $(b_1, \dots, b_n)\, x = b_{n+1}$ in $\mathbb{R}^n$ gives an identity on the level of vectors: assuming the solvability of the equation implies

$$b_{n+1} = (b_1, \dots, b_n)\, x = \sum_{i=1}^{n} x_i\, b_i$$

with x_i being the entries of a solution x (for $1 \le i \le n$). Furthermore, the multilinearity of the determinant gives

$$|b_1, \dots, b_{i-1}, b_{n+1}, b_{i+1}, \dots, b_n| = x_i\, |b_1, \dots, b_n| \quad \text{for } 1 \le j \le n$$

and therefore

$$\sum_{i=1}^{n} |b_1, \dots, b_{i-1}, b_{n+1}, b_{i+1}, \dots, b_n|\, b_i \quad = \quad |b_1, \dots, b_n| \sum_{i=1}^{n} x_i\, b_i$$

which implies

$$\sum_{i=1}^{n} |b_1, \dots, b_{i-1}, b_{n+1}, b_{i+1}, \dots, b_n|\, b_i \quad = \quad |b_1, \dots, b_n|\, b_{n+1}. \tag{4.4}$$

In the case that $(b_1, \dots, b_n)\, x = b_{n+1}$ is not solvable in $\mathbb{R}^n$, assume there is another n-element subset of $b_1, \dots, b_{n+1}$, such that the corresponding determinant does not vanish (otherwise, (4.4) already holds true). Letting play these vector the roles of the $b_1, \dots, b_n$ gives the same result. Thus (4.4) holds in general. Plugging it in the last column of $\det(a_1, \dots, a_{n-1}, *)$ gives the coordinate version of a general Grassmann-Plücker relation holding for all vectors in $\mathbb{R}^n$ and for all homogeneous coordinates of points in $\mathbb{RP}^{n-1}$ and motivates Definition 4.6.

4.3.1. Interpreting Grassmann-Plücker Relations: Intersection of Subspaces

(4.4) has a similar interpretation as (4.1): for any $2 \leq k \leq n-1$ (4.4), can be rewritten as

$$\sum_{i=1}^{k}(-1)^i \,|b_1,\,\ldots,b_{i-1},\,b_{i+1},\,\ldots,b_{n+1}|\,b_i = \sum_{i=k+1}^{n+1}(-1)^{i+1}\,|b_1,\,\ldots,b_{i-1},\,b_{i+1},\,\ldots,b_{n+1}|\,b_i. \quad (4.5)$$

Both sides of the equation give homogeneous coordinates of points in $\mathbb{RP}^{n-1}$. The left-hand side of the equation indicates that the point lies in the subspace spanned by the points $b_1,\,\ldots,b_k$ and the right-hand side of the equation indicates that it also lies in the space spanned by the points $b_{k+1},\,\ldots,b_{n+1}$. Thus, (4.5) allows for expressing the intersection of two subspaces of suitable dimensions in two different ways. This phenomenon will be generalized in Chapter 5.

4.4. The Bracket Ring $\mathcal{B}_{\mathbf{P}}$ in Rank n

Definition 4.6 (Bracket Ring of Rank n)**.** *Let $\mathbf{P}$ be a finite set of point names and let n be a natural number. $\Delta(\mathbf{P}) := \{\, [\mathbf{p}_1, \mathbf{p}_2,\,\ldots,\mathbf{p}_n] \,|\, \mathbf{p}_1, \mathbf{p}_2,\,\ldots,\mathbf{p}_n \in \mathbf{P}\,\}$ defines a set of indeterminantes for the ring $\mathbb{R}[\Delta(\mathbf{P})]$. With $\mathcal{S}_n$ the symmetric group on n elements, it contains the ideals*

$$\mathbf{I}_{altern} := \Big\langle \big\{[\mathbf{p}_1,\mathbf{p}_2,\,\ldots,\mathbf{p}_n] + \operatorname{sgn}(\sigma)\,[\mathbf{p}_{\sigma(1)},\mathbf{p}_{\sigma(2)},\,\ldots,\mathbf{p}_{\sigma(n)}] \;\big|\; \mathbf{p}_1,\mathbf{p}_2,\,\ldots,\mathbf{p}_n \in \mathbf{P}, \sigma \in \mathcal{S}_n\big\}\Big\rangle,$$

$$\mathbf{I}_{repeat} := \Big\langle \big\{[\mathbf{p}_1,\mathbf{p}_2,\,\ldots,\mathbf{p}_n] \,\big|\, \mathbf{p}_1 = \mathbf{p}_2,\, \mathbf{p}_1,\mathbf{p}_2,\,\ldots,\mathbf{p}_n \in \mathbf{P}\big\}\Big\rangle,$$

$$\mathbf{I}_{GP} := \Big\langle \big\{\sum_{i=1}^{n+1}(-1)^i[\mathbf{a}_1,\,\ldots,\mathbf{a}_{n-1},\mathbf{b}_i][\mathbf{b}_1,\,\ldots,\mathbf{b}_{i-1},\mathbf{b}_{i+1},\,\ldots,\mathbf{b}_{n+1}] \;\big|\; \mathbf{a}_1,\,\ldots,\mathbf{a}_{n-1},\,\mathbf{b}_1,\,\ldots,\mathbf{b}_{n+1} \in \mathbf{P}\big\}\Big\rangle.$$

The bracket ring of rank n is defined as

$$\mathcal{B}^{(n)}_{\mathbf{P}} := \mathcal{B}_{\mathbf{P}} := \mathbb{R}[\Delta(\mathbf{P})]\Big/\big\langle \mathbf{I}_{repeat} \cup \mathbf{I}_{altern} \cup \mathbf{I}_{GP}\big\rangle.$$

ϕ, Φ and Φ^P can be defined in analogy to the rank 3 case in Definitions 4.4 and 4.5.

4.5. Fundamental Theorems of Invariant Theory

ϕ and Φ make it easier to formulate the fundamental theorems of invariant theory which emphasize the fundamental role of the bracket ring. Again, all statements involving Φ are statements which hold for *all* possible point configurations (in contrast to Φ^P). No proofs will be given. Proofs can be found e.g. in [102] or other references given in the beginning of the chapter.

As it turns out, the ideals $\mathbf{I}_{\text{repeat}}$, $\mathbf{I}_{\text{altern}}$ and $\mathbf{I}_{\text{GP}}$ generate *all* dependencies along determinants and the bracket polynomials in $\mathbb{R}[\Delta(\mathbf{P})]$ vanishing in any coordinatization and for arbitrary rank:

Theorem 4.7 (Second Fundamental Theorem of Invariant Theory)**.** *The homomorphism Φ is injective. In other words,* $\ker\phi$ *(which is a subset of* $\mathbb{R}[\Delta(\mathbf{P})]$*) equals* $\left\langle \mathbf{I}_{repeat} \cup \mathbf{I}_{altern} \cup \mathbf{I}_{GP} \right\rangle$*.*

Any element of $\ker \Phi_{\mathbf{P}} = \left\langle \mathbf{I}_{\text{repeat}} \cup \mathbf{I}_{\text{altern}} \cup \mathbf{I}_{\text{GP}} \right\rangle$ is called a *syzygy*. The following first fundamental theorem of invariant theory states in essence that *all* (polynomial) invariants are induced by bracket polynomials. For the correct formulation we define

Definition 4.8. *Let* $\mathbb{R}[x_{ij}]^{\text{SL}}$ *be the ring of polynomials in* $\mathbb{R}[x_{ij}]$ *that are* SL-*invariants when considered as functions.*

Theorem 4.9 (First Fundamental Theorem of Invariant Theory)**.** Φ *is surjective, i.e. it holds*

$$\operatorname{im}(\phi) = \phi(\, \mathbb{R}[\Delta(\mathbf{P})]\,) = \mathbb{R}[x_{ij}]^{\text{SL}}.$$

A concrete algorithm for finding a bracket representation of an element in $\mathbb{R}[x_{ij}]^{\text{SL}}$ is provided by [102, Alg. 3.2.8]. First and second fundamental theorem together imply:

Corollary 4.10. *The Bracket ring is isomorphic to the invariant ring:*

$$\mathcal{B}_{\mathbf{P}} \cong \mathbb{R}[x_{ij}]^{\text{SL}}.$$

One can also derive a "first fundamental theorem" for rational functions. However, it is needed precisely once in this thesis so it is implicitly given in the last part of the proof of Theorem 5.15.

4.6. Applications to Projective Geometry

So for any SL-invariant, we can chose whether we want to give it as an element of $\mathcal{B}_{\mathbf{P}}$ or as an element of $\mathbb{R}[x_{ij}]^{\text{SL}}$. It is a deep believe of many geometers, that the representation within the bracket ring is "better". We give two reasons for that: First of all, this formulation is independent on the choice of coordinates. This is a basic property of properties in geometry. Consider e.g. the statement of Pascal's theorem (see Section 3.10): the fact that six points lie on a common conic should not depend on the position of the six points relative to the origin. Secondly, the representation of the invariants of interest is usually much shorter in the language of brackets than in the language of homogeneous coordinates. We borrow an example from [102] to demonstrate this: with $\mathbf{P} = \{\mathbf{1}, \ldots, \mathbf{6}\}$ we have that $\Phi\big([\mathbf{1},\mathbf{2},\mathbf{3}][\mathbf{4},\mathbf{5},\mathbf{6}] - [\mathbf{1},\mathbf{2},\mathbf{4}][\mathbf{3},\mathbf{5},\mathbf{6}]\big)$ equals

$$
\begin{aligned}
&-x_{\mathbf{1},3}x_{\mathbf{2},2}x_{\mathbf{3},3}x_{\mathbf{4},1}x_{\mathbf{5},2}x_{\mathbf{6},1} + x_{\mathbf{1},2}x_{\mathbf{2},3}x_{\mathbf{3},3}x_{\mathbf{4},1}x_{\mathbf{5},2}x_{\mathbf{6},1} + x_{\mathbf{1},3}x_{\mathbf{2},1}x_{\mathbf{3},3}x_{\mathbf{4},2}x_{\mathbf{5},2}x_{\mathbf{6},1}\\
&-x_{\mathbf{1},1}x_{\mathbf{2},3}x_{\mathbf{3},3}x_{\mathbf{4},2}x_{\mathbf{5},2}x_{\mathbf{6},1} + x_{\mathbf{1},3}x_{\mathbf{2},2}x_{\mathbf{3},1}x_{\mathbf{4},3}x_{\mathbf{5},2}x_{\mathbf{6},1} - x_{\mathbf{1},2}x_{\mathbf{2},3}x_{\mathbf{3},1}x_{\mathbf{4},3}x_{\mathbf{5},2}x_{\mathbf{6},1}\\
&-x_{\mathbf{1},3}x_{\mathbf{2},1}x_{\mathbf{3},2}x_{\mathbf{4},3}x_{\mathbf{5},2}x_{\mathbf{6},1} + x_{\mathbf{1},1}x_{\mathbf{2},3}x_{\mathbf{3},2}x_{\mathbf{4},3}x_{\mathbf{5},2}x_{\mathbf{6},1} + x_{\mathbf{1},3}x_{\mathbf{2},2}x_{\mathbf{3},2}x_{\mathbf{4},1}x_{\mathbf{5},3}x_{\mathbf{6},1}\\
&-x_{\mathbf{1},2}x_{\mathbf{2},3}x_{\mathbf{3},2}x_{\mathbf{4},1}x_{\mathbf{5},3}x_{\mathbf{6},1} - x_{\mathbf{1},3}x_{\mathbf{2},2}x_{\mathbf{3},1}x_{\mathbf{4},2}x_{\mathbf{5},3}x_{\mathbf{6},1} + x_{\mathbf{1},2}x_{\mathbf{2},3}x_{\mathbf{3},1}x_{\mathbf{4},2}x_{\mathbf{5},3}x_{\mathbf{6},1}\\
&-x_{\mathbf{1},2}x_{\mathbf{2},1}x_{\mathbf{3},3}x_{\mathbf{4},2}x_{\mathbf{5},3}x_{\mathbf{6},1} + x_{\mathbf{1},1}x_{\mathbf{2},2}x_{\mathbf{3},3}x_{\mathbf{4},2}x_{\mathbf{5},3}x_{\mathbf{6},1} + x_{\mathbf{1},2}x_{\mathbf{2},1}x_{\mathbf{3},2}x_{\mathbf{4},3}x_{\mathbf{5},3}x_{\mathbf{6},1}\\
&-x_{\mathbf{1},1}x_{\mathbf{2},2}x_{\mathbf{3},2}x_{\mathbf{4},3}x_{\mathbf{5},3}x_{\mathbf{6},1} + x_{\mathbf{1},3}x_{\mathbf{2},2}x_{\mathbf{3},3}x_{\mathbf{4},1}x_{\mathbf{5},1}x_{\mathbf{6},2} - x_{\mathbf{1},2}x_{\mathbf{2},3}x_{\mathbf{3},3}x_{\mathbf{4},1}x_{\mathbf{5},1}x_{\mathbf{6},2}\\
&-x_{\mathbf{1},3}x_{\mathbf{2},1}x_{\mathbf{3},3}x_{\mathbf{4},2}x_{\mathbf{5},1}x_{\mathbf{6},2} + x_{\mathbf{1},1}x_{\mathbf{2},3}x_{\mathbf{3},3}x_{\mathbf{4},2}x_{\mathbf{5},1}x_{\mathbf{6},2} - x_{\mathbf{1},3}x_{\mathbf{2},2}x_{\mathbf{3},1}x_{\mathbf{4},3}x_{\mathbf{5},1}x_{\mathbf{6},2}\\
&+x_{\mathbf{1},2}x_{\mathbf{2},3}x_{\mathbf{3},1}x_{\mathbf{4},3}x_{\mathbf{5},1}x_{\mathbf{6},2} + x_{\mathbf{1},3}x_{\mathbf{2},1}x_{\mathbf{3},2}x_{\mathbf{4},3}x_{\mathbf{5},1}x_{\mathbf{6},2} - x_{\mathbf{1},1}x_{\mathbf{2},3}x_{\mathbf{3},2}x_{\mathbf{4},3}x_{\mathbf{5},1}x_{\mathbf{6},2}\\
&-x_{\mathbf{1},3}x_{\mathbf{2},1}x_{\mathbf{3},2}x_{\mathbf{4},1}x_{\mathbf{5},3}x_{\mathbf{6},2} + x_{\mathbf{1},1}x_{\mathbf{2},3}x_{\mathbf{3},2}x_{\mathbf{4},1}x_{\mathbf{5},3}x_{\mathbf{6},2} + x_{\mathbf{1},2}x_{\mathbf{2},1}x_{\mathbf{3},3}x_{\mathbf{4},1}x_{\mathbf{5},3}x_{\mathbf{6},2}\\
&-x_{\mathbf{1},1}x_{\mathbf{2},2}x_{\mathbf{3},3}x_{\mathbf{4},1}x_{\mathbf{5},3}x_{\mathbf{6},2} + x_{\mathbf{1},3}x_{\mathbf{2},1}x_{\mathbf{3},1}x_{\mathbf{4},2}x_{\mathbf{5},3}x_{\mathbf{6},2} - x_{\mathbf{1},1}x_{\mathbf{2},3}x_{\mathbf{3},1}x_{\mathbf{4},2}x_{\mathbf{5},3}x_{\mathbf{6},2}\\
&-x_{\mathbf{1},2}x_{\mathbf{2},1}x_{\mathbf{3},1}x_{\mathbf{4},3}x_{\mathbf{5},3}x_{\mathbf{6},2} + x_{\mathbf{1},1}x_{\mathbf{2},2}x_{\mathbf{3},1}x_{\mathbf{4},3}x_{\mathbf{5},3}x_{\mathbf{6},2} - x_{\mathbf{1},3}x_{\mathbf{2},2}x_{\mathbf{3},2}x_{\mathbf{4},1}x_{\mathbf{5},1}x_{\mathbf{6},3}\\
&+x_{\mathbf{1},2}x_{\mathbf{2},3}x_{\mathbf{3},2}x_{\mathbf{4},1}x_{\mathbf{5},1}x_{\mathbf{6},3} + x_{\mathbf{1},3}x_{\mathbf{2},2}x_{\mathbf{3},1}x_{\mathbf{4},2}x_{\mathbf{5},1}x_{\mathbf{6},3} - x_{\mathbf{1},2}x_{\mathbf{2},3}x_{\mathbf{3},1}x_{\mathbf{4},2}x_{\mathbf{5},1}x_{\mathbf{6},3}\\
&+x_{\mathbf{1},2}x_{\mathbf{2},1}x_{\mathbf{3},3}x_{\mathbf{4},2}x_{\mathbf{5},1}x_{\mathbf{6},3} - x_{\mathbf{1},1}x_{\mathbf{2},2}x_{\mathbf{3},3}x_{\mathbf{4},2}x_{\mathbf{5},1}x_{\mathbf{6},3} - x_{\mathbf{1},2}x_{\mathbf{2},1}x_{\mathbf{3},2}x_{\mathbf{4},3}x_{\mathbf{5},1}x_{\mathbf{6},3}\\
&+x_{\mathbf{1},1}x_{\mathbf{2},2}x_{\mathbf{3},2}x_{\mathbf{4},3}x_{\mathbf{5},1}x_{\mathbf{6},3} + x_{\mathbf{1},3}x_{\mathbf{2},1}x_{\mathbf{3},2}x_{\mathbf{4},1}x_{\mathbf{5},2}x_{\mathbf{6},3} - x_{\mathbf{1},1}x_{\mathbf{2},3}x_{\mathbf{3},2}x_{\mathbf{4},1}x_{\mathbf{5},2}x_{\mathbf{6},3}\\
&-x_{\mathbf{1},2}x_{\mathbf{2},1}x_{\mathbf{3},3}x_{\mathbf{4},1}x_{\mathbf{5},2}x_{\mathbf{6},3} + x_{\mathbf{1},1}x_{\mathbf{2},2}x_{\mathbf{3},3}x_{\mathbf{4},1}x_{\mathbf{5},2}x_{\mathbf{6},3} - x_{\mathbf{1},3}x_{\mathbf{2},1}x_{\mathbf{3},1}x_{\mathbf{4},2}x_{\mathbf{5},2}x_{\mathbf{6},3}\\
&+x_{\mathbf{1},1}x_{\mathbf{2},3}x_{\mathbf{3},1}x_{\mathbf{4},2}x_{\mathbf{5},2}x_{\mathbf{6},3} + x_{\mathbf{1},2}x_{\mathbf{2},1}x_{\mathbf{3},1}x_{\mathbf{4},3}x_{\mathbf{5},2}x_{\mathbf{6},3} - x_{\mathbf{1},1}x_{\mathbf{2},2}x_{\mathbf{3},1}x_{\mathbf{4},3}x_{\mathbf{5},2}x_{\mathbf{6},3}.
\end{aligned}
$$

Now, one could ask whether the SL-invariants are of any interest for a geometer who is expected to be interested in projective invariants. In fact, for any point configuration P, the vanishing of $\Phi^P\big([\mathbf{1},\mathbf{2},\mathbf{3}][\mathbf{4},\mathbf{5},\mathbf{6}] - [\mathbf{1},\mathbf{2},\mathbf{4}][\mathbf{3},\mathbf{5},\mathbf{6}]\big)$ expresses the fact that the lines with coordinates $\mathbf{join}\big(\Phi^P(\mathbf{1}), \Phi^P(\mathbf{2})\big)$, $\mathbf{join}\big(\Phi^P(\mathbf{3}), \Phi^P(\mathbf{4})\big)$ and $\mathbf{join}\big(\Phi^P(\mathbf{5}), \Phi^P(\mathbf{6})\big)$ meet in a single point. Here $\Phi^P(\mathbf{p})$ is to be understood as the point in P corresponding to $\mathbf{p} \in \mathbf{P}$ and the geometric interpretation follows from Section 2.4. The following Definition 4.11 is needed in order to see which of the SL-invariants also lead to more general invariants, e.g. projectively invariant properties (see Section 3.3). Before that, we observe that giving SL-invariants as elements of the bracket ring $\mathcal{B}_{\mathbf{P}}$ has the disadvantage that there are many representations of the same invariant. Section 4.7 will deal with this problem in more detail.

Definition 4.11. *Let* $\mathbf{P}$ *be a finite set. For a bracket monomial* m *in* $\mathbb{R}[\Delta(\mathbf{P})]$ *and for* $\mathbf{p} \in \mathbf{P}$ *we define the relative degree* $\deg(m, \mathbf{p})$ *as the total number of occurrences of* $\mathbf{p}$ *in* m*. Let* $m_1 + \cdots + m_k$ *be a polynomial in* $\mathbb{R}[\Delta(\mathbf{P})]$ *given as a sum of* *monomials. We say that* $m_1 + \cdots + m_k$ *is* multihomogenous *if*

$$\deg(m_1, \mathbf{p}) \;=\; \ldots \;=\; \deg(m_k, \mathbf{p}) \qquad \textit{for all } \; \mathbf{p} \in \mathbf{P}.$$

If $m_1 + \cdots + m_k$ *is also in* $\mathbb{Z}[\Delta(\mathbf{P})]$ *then* $m_1 + \cdots + m_k$ *is said to have* integer coefficients. *For* $k = 1, 2, 3, \ldots,$ *the expression* $m_1 + \cdots + m_k$ *is called* monomial, binomial, trinomial, ..., *resp. as long as* $m_1, \ldots, m_k \notin \langle \mathbf{I}_{repeat} \rangle$.

A bracket polynomial B in $\mathcal{B}_{\mathbf{P}}$ *has a subset of the properties just defined whenever it has a representation in* $\mathbb{R}[\Delta(\mathbf{P})]$ *satisfying all properties in the subset at once.*

By the multiplicativity of the determinant, every multihomogenous bracket polynomial induces a projectively invariant property: Interpreting Φ as function and following that up with applying a function that decides whether the result equals zero, induces the desired invariant. To see that the resulting functions are projectively invariant property, observe that is holds since e.g. in rank 3:

$$|M \cdot \lambda_a \cdot a, M \cdot \lambda_b \cdot b, M \cdot \lambda_c \cdot c| = \det(M) \cdot \lambda_a \, \lambda_b \, \lambda_c \cdot |a, b, c|$$

for $\lambda_a, \lambda_b, \lambda_c \in \mathbb{R} \setminus \{0\}$, $M \in \mathrm{GL}(\mathbb{R}, 3)$ and $a, b, c \in \mathbb{R}^3$. Thus in a (representation of a) multihomogeneous bracket polynomial, each summand induces the same non-zero scalar factors when testing the function being a projectively invariant property. We summarize:

Proposition 4.12. *Multihomogeneous bracket polynomials induce projectively invariant properties.*

4.7. Comparing Expressions in $\mathcal{B}_{\mathbf{P}}$ and the Straightening Algorithm

The advantage of short expressions for geometric properties comes with a big disadvantage: especially $\mathbf{I}_{\mathrm{GP}}$ makes it hard to tell whether two bracket polynomial equal each other or not (or equivalently to tell whether a bracket polynomial is in fact the zero polynomial). The standard answer for this problem is the classical straightening algorithm which goes back to Young (see [138]). It is also treated in [37]. [103, 102] show the connection to Gröbner bases as introduced by Buchberger in [14, 15] and by Hironaka in [64]. Another introduction on Gröbner bases can be found e.g. in [113]. More precisely, [103, 102] show that the straightening algorithm *is* as a Gröbner bases algorithm where in addition the shape of the Gröbner basis is known beforehand. It is able to transform any bracket polynomial into a normal form. After reducing two bracket polynomials to normal form, they can be compared. Nevertheless, in practice, the normal forms tend to have much more summands than the polynomials one started with. Due to [93], the straightening algorithm requires overexponential CPU time which is not surprising, since it is a Gröbner bases algorithm.

We will give a very rough sketch of the algorithm in order to get an impression on how it works. The normal form is characterized by the fact that all monomials of the polynomial are *standard*. A monomial is standard, whenever all of its sorted brackets (sorted with respect to the some fixed order imposed on $\mathbf{P}$) can be arranged as rows of a matrix in such a way that the columns of the matrix are non-decreasing. The ordering of $\mathbf{P}$ also induces a *monomial order* which orders the set of monomials.[1] Whenever there is a biggest monomial of the current representation of the bracket polynomial that is non-standard, it can be replaced by a bracket polynomial (taken from the Göbner bases) whose summands are all smaller in the monomial order. Monomial orders have the property that they cannot have infinite decreasing chains. Therefore, this process has to terminate. This is the required normal form. It can be shown that the set of so-called van der Waerden syzygies is a Gröbner basis for $\langle \mathbf{I}_{\text{repeat}} \cup \mathbf{I}_{\text{altern}} \cup \mathbf{I}_{\text{GP}} \rangle$ (see again [103, 102]). One can use not only van der Waerden syzygies but also other additional syzygies in order to eliminate non-standard monomials as long as the monomial order is decreased. One can hope for better running times of the straightening algorithm. This is done in [122] by White including a comparison of the performance of the five variations given. Furthermore, if the bracket polynomial is antisymmetric in some points of $\mathbf{P}$, an optimized version of the algorithm is given in [85], called the dotted straightening algorithm. According to [102, p. 117] it can be understood as a special case of the straightening law in the superalgebra [51]. Considering antisymmetry is closely related with Grassmann-Cayley algebra treated in the next chapter and with multilinear Cayley factorization mentioned in Part III. Here in this context, the straightening algorithm is treated in sake of completeness. It will not be used within this thesis.

[1] In rank 3, it suffices to order the indeterminantes $\{ [\mathbf{i}, \mathbf{j}, \mathbf{k}] \mid \mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{P} \text{ and } \mathbf{i} < \mathbf{j} < \mathbf{k} \}$. One choses the lexicographical order on these variables and the induced *degree reverse lexicographic order* is the monomial order in question.

5. A Symbolic Version of the Grassmann-Cayley Algebra

Grassmann-Cayley algebra is used to calculate with subspaces of $\mathbb{R}^n$ and therefore also with subspaces of $\mathbb{RP}^{n-1}$ (see also Section 2.5). In the classical literature, this is done by referring to the concrete coordinates of points in $\mathbb{R}^n$. We will again use the names of points to introduce the elements of a symbolic version of Grassmann-Cayley algebra. We are interested in those elements of the algebra that *can* be interpreted as being subspaces of $\mathbb{R}^n$. The *join* operation allows for spanning a subspace with some points. The *meet* operation can be interpreted as an intersection of subspaces. We give formal definitions and afterwards interpretations in order to show that the definitions meet the intuition of join and meet just described. To do so on a symbolic level, relations are established in Theorem 5.13. Theorem 5.15 is a symbolic version of the fact that in classical Grassmann-Cayley algebra "identical subspaces are scalar multiples of each other". This will be used as a motivation for different evaluations in Chapter 6.

Classical introductions can be found e.g. in [126, 127, 4, 102] and also in [77, Chap. 2.3]. In [77] and in [73] many additional references can be found. [4, 36] provide many historical comments and give many credits to Grassmann for his *Ausdehnungslehre* (see [48, 49]). There are also many articles honoring Grassmann in [90] celebrating the bicentennial birthday of Grassmann. Within this community it is also provided a software package for Grassmann algebra calculations. It is distributed with an introductory book (see [13]). Forder [44] can be understood as giving an early restatement of Grassmann's original ideas in modern language. Furthermore, many examples are given within this book. In [77], in addition to the conventional meet, also a *total meet product* is given by Li which is inspired by the coproduct given in [51]. It is also given a more recent development to connect Grassmann-Cayley algebra with Clifford algebra or geometric algebra. For an introduction to the latter we refer to [89]. On page 49 it is stated that "Grassmann-Cayley algebra and Clifford algebra are equivalent in the sense that anything expressed in one of them can also be expressed in the other. Which one you prefer is probably a matter of taste." The foundations of geometric algebra are given by Hestenes in [59] and first applications to projective geometry are given in [61]. Another overview over related algebras is given in [60]. As one investigates literature using Grassmann-Cayley algebra, one finds that one often calculates symbolically.

Therefore we will use a symbolized version of this algebra and refer to it as "Grassmann-Cayley algebra" as well when no confusion can arise. We saw another symbolic treatment before in the case of bracket polynomials: here in every expression, the names of points in $\mathbf{P}$ are unbreakable symbols in the sense that it is never referred to coordinates of the points. In the context of Grassmann-Cayley algebra, this has the advantage that no coordinates for subspaces have to be defined. This definition is a little cumbersome and needs an interpretation as determinants of points given by coordinates with respect to a specific basis. This does not provide geometric interpretation in the first place. The symbolic version captures geometric meaning and emphasizes the nature of the computations done in this thesis. Furthermore, we do not need to specify a certain basis. Of course, when interpreting a formula, some points can be considered constituting a basis. However, interpreting another formula, other points can be considered yielding a basis.

Another motivation for the symbolic treatment is that in the literature, the bracket ring is considered to be a substructure of Grassmann-Cayley algebra. However, in a formal way of thinking, i.e. without using isomorphisms, this is not the case when we use the symbolic version of the bracket ring given by White in [120] and the classical definition of the Grassmann-Cayley algebra based on coordinates.

The symbolic version $\Lambda(\mathbf{P})$ of the Grassmann-Caylay algebra given below allows for this and is completely coordinate-free. It is called White module in [18]. The elements of $\mathbf{P}$ play again the role of *generic* points in $\mathbb{R}^n$. In order to see the beauty and structure of the algebra, we will introduce it for general geometric rank n. In rank 3, a lot of phenomena cannot be seen. The definition of $\Lambda(\mathbf{P})$ depends on the geometric rank n as well as on (the cardinality of) the set $\mathbf{P}$. It uses the definition of the bracket ring in rank n on $\mathbf{P}$. It is used to ensure that any $n+1$ points are dependent. Furthermore, we will also give an interpretation of the elements of $\Lambda(\mathbf{P})$ for a concrete instance of points. This is done by inserting concrete homogenous coordinates in brackets. In principle, this interpretation can be slightly extended in order to achieve the classical version of Grassmann-Cayley algebra including Grassmann coordinates as defined e.g. in [65]. In essence, this follows from letting $\mathbf{P}$ be a n-elements set which is considered to be a basis of the vector space. This approach is taken in [28].

The exposition given in the current chapter follows [37] by Doubilet, Rota and Stein. The following references can be considered to emerge from a school which goes back to Rota. The definitions in [37] are closely related to concrete coordinates. However, almost every definition given there can be literally translated to names of points instead of points in $\mathbb{R}^n$. This makes the presentation symbolic in nature. With their vocabulary, it is in essence (a symbolized version of) a *non-standard Cayley or Peano space* (see also [4, 28]). This is a vector space together with a so-called bracket, which is in fact a determinant there. The Peano space is called non-standard, if the length of the bracket is not the same as the dimension of the vector space. This case is neglected in the exposition in [37]. In our case, the vector space is replaced by a module with coefficients in $\mathcal{B}_{\mathbf{P}}$. As it was mentioned in the previous chapter, this symbolic approach using only *point names* or an *alphabet* was already taken in [51, 68, 99] in order to treat generalizations of the bracket ring. In [37], it is pointed out that it is the first time since the period between Grassmann and Bourbaki that the so-called meet product is defined without referring to duality of vector spaces. This feature is essential for the symbolized treatment given here since dualizing refers explicitly to single coordinates of points (or to a fixed basis). This makes it hard to treat points in a whole. [4] treats classical Grassmann-Cayley algebra which is called double algebra there. It also provides much historical background.

As mentioned before, the symbolic version of Grassmann-Cayley algebra given here is in essence the White module considered in [18, 12] and also [11]. [18] introduces the White module as a motivation for the more general superalgebras. In [12] the White module is called $G_n^-(\mathbf{P})$ on a negative alphabet $\mathbf{P}$. Based on a prejoin and a premeet a join and a meet are defined. The fact that the premeet is well defined follows from the existence of a Hopf algebra structure. This approach, which is inspired by the connection of the exterior product and Hopf algebras, is also mentioned in [4]. We think that the White module can be considered as an intermediate step in the passage from the classical version of Grassmann-Cayley algebra and superalgebras. Another introduction in this context is given in [57, 29]. We consider the passage to the White module as being a natural concept in the same way as the symbolic treatment of the bracket ring is natural. It is less general than the supersymmetric theory but it is more in tradition with the very geometric ideas given in [37]. Furthermore, in our definition we will not need to refer to a Hopf algebra. Instead we refer to the bracket ring.

Observe that while trying to restate known properties of Grassmann-Cayley algebra in a purely symbolic manner, one can also see the limitations of symbolic computing in general. Or to state it differently, one sees in which parts of the theory coordinates *are* involved. For example, as mentioned before, the definition of the Hodge star operator for dualization and therefore presumably the concept of duality relies on coordinates. Furthermore, in the symbolic formulation, one can always see which parts of a formula (may) depend on which points in $\mathbf{P}$. When in classical theory it is easy to argue that entities representing the same subspace in $\mathbb{R}^n$ differ only by a scalar factor, this gets more difficult in a symbolic version (see Theorem 5.15). However, one gains information about this scalar factor and sees that it can be given as (the quotient of) two SL-invariants on the points in $\mathbf{P}$.

Doing this, it turns out that in Grassmann-Cayley algebra calculations with projective objects are done exactly, i.e. taking scalar factors into account. Thus these calculations can be regarded as being also valid for points in $\mathbb{R}^n$. Nevertheless, the interpretation of the elements as subspaces of $\mathbb{R}^n$ (or $\mathbb{RP}^{n-1}$) does not take scalar multiples into account.

5.1. The Elements

The Grassmann-Cayley algebra has two operations: the join $\vee$ and the meet $\wedge$. We will first define the elements of the algebra as elements of a quotient space and treat the two (compatible) operations later on.

Definition 5.1 (Elements of the Grassmann-Cayley Algebra of Rank n)**.** *Let* $\mathbf{P}$ *be a finite set of names of points and let* n *be a natural number. The set of all strings over* $\mathbf{P}$ *of length* j *is denoted* $\mathbf{P}^j$ *with* $\mathbf{P}^0 := \{1\}$*. Let* $\mathcal{B}_\mathbf{P} = \mathcal{B}_\mathbf{P}^{(n)}$ *be the bracket ring of rank* n *on* $\mathbf{P}$*. For* $1 \leq k \leq n$ *let* $\Lambda^k(\mathbf{P})$ *be the quotient of the free* $\mathcal{B}_\mathbf{P}$*-module on* $\mathbf{P}^k$ *and the submodule which contains all elements* $\sum_{\mathbf{p}_1\cdots\mathbf{p}_k\in\mathbf{P}^k} \alpha_{\mathbf{p}_1\cdots\mathbf{p}_k}\, \mathbf{p}_1\cdots\mathbf{p}_k$ *with* $\alpha_{\mathbf{p}_1\cdots\mathbf{p}_k} \in \mathcal{B}_\mathbf{P}$ *such that*

$$\sum_{\mathbf{p}_1\cdots\mathbf{p}_k\in\mathbf{P}^k} \alpha_{\mathbf{p}_1\cdots\mathbf{p}_k}\, [\mathbf{p}_1, \ldots, \mathbf{p}_k,\, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}] \quad \in \mathcal{B}_{\mathbf{P}\dot\cup\{\mathbf{z}_1,\ldots,\mathbf{z}_{n-k}\}} \tag{5.1}$$

is a representation of 0 *in* $\mathcal{B}_{\mathbf{P}\dot\cup\{\mathbf{z}_1,\ldots,\mathbf{z}_{n-k}\}}$ *(for symbols* $\mathbf{z}_1, \ldots, \mathbf{z}_{n-k}$ *not in* $\mathbf{P}$*).* $\Lambda(\mathbf{P})$ *is the graded direct sum*

$$\Lambda(\mathbf{P}) := \bigoplus_{k=0}^{n} \Lambda^k(\mathbf{P}).$$

Elements of $\Lambda^1(\mathbf{P})$ *are called* points.

The elements of $\Lambda^k(\mathbf{P})$ can be regarded as incomplete brackets. Due to (5.1), its elements equal each other, whenever the completed brackets equal each other for any choice of completion. Thus, $\mathbf{z}_1, \ldots, \mathbf{z}_{n-k}$ are regarded as generic points with respect to the elements in $\mathbf{P}$.

Remark 5.2. Observe that whenever (5.1) equals 0, it equals 0 as well for letting $\mathbf{z}_1, \ldots, \mathbf{z}_{n-k}$ be arbitrary names of points in $\mathbf{P}\,\dot\cup\,\{\mathbf{z}_1, \ldots, \mathbf{z}_{n-k}\}$ due to the second fundamental theorem of invariant theory.

The points, i.e. the elements of $\Lambda^1(\mathbf{P})$, can be easily interpreted, since they are linear combination of the points in $\mathbf{P}$: let P be a point configuration describing $\mathbf{P}$ in $\mathbb{R}^n$. We write $\Phi^P(\mathbf{p})$ for the column in P that corresponds to $\mathbf{p} \in \mathbf{P}$. There is a natural extension of the coordinatization Φ^P defined on $\mathcal{B}_{\mathbf{P}}$:

$$\sum_{\mathbf{p}\in\mathbf{P}} \alpha_{\mathbf{p}}\,\mathbf{p} \ \in\ \Lambda^1(\mathbf{P}) \text{ is mapped to } \quad \sum_{\mathbf{p}\in\mathbf{P}} \Phi^P(\alpha_{\mathbf{p}})\,\Phi^P(\mathbf{p})$$

which is the same linear combination with coordinates substituted in the coefficients (which are determinants) as well as in the points themselves. Φ^P is well defined: from $\sum_{\mathbf{p}\in\mathbf{P}^k} \alpha_{\mathbf{p}}\,\mathbf{p} = 0$ it follows that

$$\left| \sum_{\mathbf{p}\in\mathbf{P}} \Phi^P(\alpha_{\mathbf{p}})\,\Phi^P(\mathbf{p}),\ \ z_1,\,\ldots,z_{n-1} \right| = 0 \text{ for all } z_1,\,\ldots,z_{n-1}\in\mathbb{R}^n.$$

This implies that $\sum_{\mathbf{p}\in\mathbf{P}} \Phi^P(\alpha_{\mathbf{p}})\,\Phi^P(\mathbf{p}) = 0$ and Φ^P is well defined. For future reference, we remark that the same construction can be also done for the *generic* coordinatization Φ.

The interpretation of $\Lambda^0(\mathbf{P})$ is easy, since it is the bracket ring $\mathcal{B}_{\mathbf{P}}$. Furthermore, $\Lambda^n(\mathbf{P})$ can also be identified with the bracket ring (see also [102, p. 96]): for $\mathbf{p}_1,\ldots,\mathbf{p}_n\in\mathbf{P}$ we map $\mathbf{p}_1\cdots\mathbf{p}_n$ to $[\mathbf{p}_1,\ldots,\mathbf{p}_n]$. The multilinear extension of this map is a well-defined injective map. Parts of Φ^P can be generalized to more general $\Lambda^k(\mathbf{P})$:

Definition 5.3. *For $\mathbf{P}$, P and n as before, and for $\sum_{\mathbf{p}_1\cdots\mathbf{p}_k\in\mathbf{P}^k} \alpha_{\mathbf{p}_1\cdots\mathbf{p}_k}\ \mathbf{p}_1\cdots\mathbf{p}_k\in\Lambda^k(\mathbf{P})$ we say that*

$$\Phi^P\left(\sum_{\mathbf{p}_1\cdots\mathbf{p}_k\in\mathbf{P}^k} \alpha_{\mathbf{p}_1\cdots\mathbf{p}_k}\ \mathbf{p}_1\cdots\mathbf{p}_k\right) = 0$$

if and only if

$$\sum_{\mathbf{p}_1\cdots\mathbf{p}_k\in\mathbf{P}^k} \Phi^P(\alpha_{\mathbf{p}_1\cdots\mathbf{p}_k})\cdot\left|\Phi^P(\mathbf{p}_1),\,\ldots,\Phi^P(\mathbf{p}_k),\,z_1,\,\ldots,z_{n-k}\right| = 0$$

holds for all $z_1,\,\ldots,z_{n-k}\in\mathbb{R}^n$.

The fact that Definition 5.3 is well-defined follows from the following remark:

Remark 5.4. Observe that with this terminology and $\mathbf{e}\in\Lambda^k(\mathbf{P})$ we have:

$$\mathbf{e} = 0 \text{ in } \Lambda^k(\mathbf{P}) \quad\Longleftrightarrow\quad \Phi^P(\mathbf{e}) = 0 \text{ for all } P$$

due to Definition 5.1 and the second fundamental theorem of invariant theory (Theorem 4.7).

For an interpretation of $\Phi^P(\mathbf{e}) = 0$, for the time being we analyze

$$\Phi^P(\mathbf{p}_1 \cdots \mathbf{p}_k) = 0 \quad \text{with} \quad \mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbf{P}. \tag{5.2}$$

This equation must hold true also in the case that the coordinates of $z_1, \ldots, z_{n-k}$ are chosen in such a way that they are linearly independent among themselves and not in the space spanned by $\Phi^P(\mathbf{p}_1), \ldots, \Phi^P(\mathbf{p}_k)$. Thus $\Phi^P(\mathbf{p}_1 \cdots \mathbf{p}_k) = 0$ if and only if $\Phi^P(\mathbf{p}_1), \ldots, \Phi^P(\mathbf{p}_k)$ are linearly dependent. In particular, if $\Phi^P(\mathbf{p}_1), \ldots, \Phi^P(\mathbf{p}_{k-1})$ are linearly independent, it describes the subspace spanned by them. The condition in (5.2) checks for $\Phi^P(\mathbf{p}_k)$ lying in this subspace. Therefore, $\mathbf{p}_1 \cdots \mathbf{p}_{k-1}$ is meant to describe the space spanned by the points $\mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$ on a symbolic level. The following definition of the join allows for extending this spanning property also to points in $\Lambda^1(\mathbf{P})$. It is the linear extension of the juxtaposition as defined in the following section.

5.2. The Join $\vee$

The join $\vee$ is a symbolic version of the exterior product. In any coordinatization, it will be the exterior product. Though the traditional symbol for the exterior product is $\wedge$, most authors in the context of Grassmann-Cayley algebra use the symbol $\vee$. This is due to geometric reasons, since it resembles more the symbol $\cup$.

Definition 5.5 (Join). *For $A = \mathbf{a}_1 \cdots \mathbf{a}_k$ and $B = \mathbf{b}_1 \cdots \mathbf{b}_l$ in $\bigcup_{j=1}^n \mathbf{P}^j$ the join $\vee$ is defined as the juxtaposition*

$$AB := A \vee B := \begin{cases} \mathbf{a}_1 \cdots \mathbf{a}_k \mathbf{b}_1 \cdots \mathbf{b}_l & \text{if } k + l \leq n. \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore,

$$1\,A := 1 \vee A := \ A \ =: A \vee 1 =: A\,1.$$

The general join $\vee$ is this associative map extended linearly to a binary operation on $\Lambda(\mathbf{P})$ and can also be written as juxtaposition.

The join is well defined since it vanishes on the defining submodules due to Remark 5.2. By multilinearity of the join and the determinant, the following holds also for general *points* $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \Lambda^1(\mathbf{P})$:

$$\Phi^P(\mathbf{a}_1 \cdots \mathbf{a}_k) = 0 \iff \left| \, \Phi^P(\mathbf{a}_1), \ldots, \Phi^P(\mathbf{a}_k), z_1, \ldots, z_{n-k} \, \right| = 0$$
$$\text{for all } z_1, \ldots, z_{n-k} \in \mathbb{R}^n \tag{5.3}$$

and similarly for $k = n$ it holds

$$\Phi^P(\mathbf{a}_1 \cdots \mathbf{a}_n) = \Big| \Phi^P(\mathbf{a}_1), \ldots, \Phi^P(\mathbf{a}_n) \Big|.$$

Remark 5.6 (The join is antisymmetric). Due to Definition 5.1 of the elements of $\Lambda(\mathbf{P})$ and due to $\mathbf{I}_{\text{altern}}$ and $\mathbf{I}_{\text{repeat}}$, the juxtaposition of elements, i.e. the join, is antisymmetric.

5.3. The Meet $\wedge$

Definition 5.7. *For $A = \mathbf{a}_1 \cdots \mathbf{a}_k$ and $B = \mathbf{b}_1 \cdots \mathbf{b}_l$ in $\bigcup_{j=0}^{n} \mathbf{P}^j$ with $k + l \geq n$ the meet $\wedge$ is defined as*

$$A \wedge B = \sum_{\sigma} \operatorname{sgn}(\sigma)[\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(n-l)}, \mathbf{b}_1, \ldots, \mathbf{b}_l]\, \mathbf{a}_{\sigma(n-l+1)} \cdots \mathbf{a}_{\sigma(k)} \tag{5.4}$$

where the sum is taken over all permutations σ of $\{1, \ldots, k\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n-l)$ and $\sigma(n-l+1) < \sigma(n-l+2) < \cdots < \sigma(k)$. Such permutations are called shuffles *of the $(n-l, k-(n-l))$ split of A. For $k + l < n$ we define $A \wedge B := 0$. The general meet $\wedge$ is this map extended linearly to a binary operation on $\Lambda(\mathbf{P})$. The join has higher precedence than the meet.*

The fact, that the meet is well-defined, follows from Theorem 5.9 and Remark 5.2. Before stating it, we remark

Remark 5.8. There is a natural notion for brackets also for $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \Lambda^1(\mathbf{P})$. It is the multilinear extension of the bracket. This is consistent with identifying $\Lambda^0(\mathbf{P})$ with $\Lambda^n(\mathbf{P})$. With this notion and by multilinearity, (5.4) is also valid for $\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{b}_1, \ldots, \mathbf{b}_l \in \Lambda^1(\mathbf{P})$ and

$$\Phi^P\big([\mathbf{a}_1, \ldots, \mathbf{a}_n]\big) = \Big| \Phi^P(\mathbf{a}_1), \ldots, \Phi^P(\mathbf{a}_n) \Big|$$

for any configuration P and for $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \Lambda^1(\mathbf{P})$. For future reference we also state that due to the second fundamental theorem of invariant theory, we have

$$\sum_{i=1}^{n+1} (-1)^i [\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_i][\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_{n+1}] = 0$$

in $\mathcal{B}_{\mathbf{P}}^{(n)}$ also for $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_1, \ldots, \mathbf{b}_{n+1} \in \Lambda^1(\mathbf{P})$ (compare Definition 4.6).

Theorem 5.9 (Anticommutativity of the Meet)**.** *For* $A = \mathbf{a}_1 \cdots \mathbf{a}_k$ *and* $B = \mathbf{b}_1 \cdots \mathbf{b}_l$ *in* $\bigcup_{j=0}^{n} \mathbf{P}^j$ *it holds*

$$A \wedge B = (-1)^{(n-k)(n-l)}\ B \wedge A.$$

Proof (sketch). Just as it is done in [102, Thm. 3.2.2.], one can show by induction on $k+l$ that for $\mathbf{z}_1, \ldots, \mathbf{z}_{2n-k-l}$ not in $\mathbf{P}$

$$\sum_{\sigma} \operatorname{sgn}(\sigma)[\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(n-l)}, \mathbf{b}_1, \ldots, \mathbf{b}_l][\mathbf{a}_{\sigma(n-l+1)}, \ldots, \mathbf{a}_{\sigma(k)}, \mathbf{z}_1, \ldots, \mathbf{z}_{2n-k-l}]$$
$$- (-1)^{(n-k)(n-l)} \sum_{\tau} \operatorname{sgn}(\tau)[\mathbf{b}_{\tau(1)}, \ldots, \mathbf{b}_{\tau(n-k)}, \mathbf{a}_1, \ldots, \mathbf{a}_k]$$
$$[\mathbf{b}_{\tau(n-k+1)}, \ldots, \mathbf{b}_{\tau(l)}, \mathbf{z}_1, \ldots, \mathbf{z}_{2n-k-l}]$$

is a syzygy. □

Remark 5.10. Due to the linearity of all operations involved, Theorem 5.9 holds also for $\mathbf{a}_1, \ldots, \mathbf{a}_k,\ \mathbf{b}_1, \ldots, \mathbf{b}_l \in \Lambda^1(\mathbf{P})$.

Corollary 5.11. *The meet is associative.*

Proof. We follow the proof in [102, Thm. 3.2.2.]. Due to the linearity of the join we consider $A = \mathbf{a}_1, \ldots, \mathbf{a}_k$, $B = \mathbf{b}_1, \ldots, \mathbf{b}_l$ and $C = \mathbf{c}_1, \ldots, \mathbf{c}_m$ which are joins of points in $\mathbf{P}$ with $1 \leq k,\ l,\ m \leq n$. By Theorem 5.9, we have

$$\begin{array}{rll} A \wedge (B \wedge C) &= (-1)^{(n-k)(2n-l-m)} & (B \wedge C) \wedge A \quad \text{and} \\ (A \wedge B) \wedge C &= (-1)^{(n-k)(n-l)} & (B \wedge A) \wedge C. \end{array} \tag{5.5}$$

By the iterated application of Definition 5.7, one finds that both expressions are identical: in the non-trivial case that $k+l+m \geq 2n$ the expressions $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$ result both in

$$\sum_{\tau} \widetilde{\operatorname{sgn}}(\tau)[\mathbf{b}_{\tau(1)}, \ldots, \mathbf{b}_{\tau(n-m)},\ \mathbf{c}_1, \ldots, \mathbf{c}_m]$$
$$[\mathbf{b}_{\tau(n-m+1)}, \ldots, \mathbf{b}_{\tau(2n-m-k)},\ \mathbf{a}_1, \ldots, \mathbf{a}_k]\, \mathbf{b}_{\tau(2n-m-k+1)} \cdots \mathbf{b}_{\tau(l)}$$

and the sum is taken over the generalized splits τ, i.e. such that the indices of the $\mathbf{b}$s are ordered inside each bracket and outside. $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$ differ in the value for $\widetilde{\operatorname{sgn}}(\tau)$: the value for the former is induced by sorting $\tau(1), \ldots, \tau(n-m),\ \tau(n-m+1), \ldots, \tau(2n-m-k),\ \tau(2n-m-k+1), \ldots, \tau(l)$ whereas the latter is induced by sorting $\tau(n-m+1), \ldots, \tau(2n-m-k),\ \tau(1), \ldots, \tau(n-m),\ \tau(2n-m-k+1), \ldots, \tau(l)$. Therefore both values differ by $(-1)^{(n-k)(n-m)}$ which shows the claim. □

Remark 5.12. Identifying the elements in $\Lambda^0(\mathbf{P})$ with the ones in $\Lambda^n(\mathbf{P})$ as indicated before, one finds that for $A \in \Lambda^k(\mathbf{P})$ and $B \in \Lambda^l(\mathbf{P})$ holds

$$A \wedge B = A \vee B \quad \text{when} \quad k + l = n. \tag{5.6}$$

In what follows, we want to interpret the meet $A \wedge B$ in the case for $A = \mathbf{a}_1 \cdots \mathbf{a}_k$ and $B = \mathbf{b}_1 \cdots \mathbf{b}_l$ being joins of points in $\Lambda^1(\mathbf{P})$. For $k + l = n + 1$, the meet $\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{b}_1, \ldots, \mathbf{b}_l$ is a point. By Definition 5.7 and Theorem 5.9, these points are linear combinations of the points $\mathbf{a}_1, \ldots, \mathbf{a}_k$ as well as of the points $\mathbf{b}_1, \ldots, \mathbf{b}_l$. Thus their meet can be regarded as lying in the intersection of both objects (compare also Section 4.3.1). For the more general situation of $k + l \geq n + 1$, intuition is provided by the following theorem. Points $\mathbf{c}_1, \ldots, \mathbf{c}_{k+l-n}$ are defined that lie in the intersection of both subspaces represented by A and B. Taking enough of them results in spanning the subspace. Therefore $A \wedge B$ can be considered as representing the *intersection* of A and B. The exact formulation of this reads as:

5.4. Interpretations and Relations

Theorem 5.13. *For $A = \mathbf{a}_1 \cdots \mathbf{a}_k$ and $B = \mathbf{b}_1 \cdots \mathbf{b}_l$ with $\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{b}_1, \ldots, \mathbf{b}_l \in \Lambda^1(\mathbf{P})$ and $k + l \geq n + 1$ let*

$$\begin{array}{lcllcl}
\mathbf{c}_1 & := & \mathbf{a}_1 \cdots \mathbf{a}_{n-l} & \mathbf{a}_{n-l+1} & \wedge B \\
\mathbf{c}_2 & := & \mathbf{a}_1 \cdots \mathbf{a}_{n-l} & \mathbf{a}_{n-l+2} & \wedge B \\
 & \vdots & & & \\
\mathbf{c}_{k+l-n} & := & \mathbf{a}_1 \cdots \mathbf{a}_{n-l} & \mathbf{a}_k & \wedge B.
\end{array}$$

It holds

$$[\mathbf{a}_1, \ldots, \mathbf{a}_{n-l}, \mathbf{b}_1, \ldots, \mathbf{b}_l]^{k+l-n-1} \, A \wedge B \; = \; \mathbf{c}_1 \, \mathbf{c}_2 \, \ldots \, \mathbf{c}_{k+l-n}. \tag{5.7}$$

Proof. We will proceed by induction on the number $l - (n - k)$ of joins on the right-hand side of (5.7), which is also the rank of the resulting space. The base step is the case $k + l = n + 1$ which was treated before. We give a short overview of the (quite lengthy) inductive step: the right-hand side will be expanded which can be shown to be identical with the left-hand side. Therefore the inductive hypothesis will be applied to the first part of the right-hand side of (5.7). Performing the last join will result in a linear combination of elements $\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_{k+l-n}}$ $(i_1, \ldots, i_{k+l-n} \in \{1, \ldots, k\})$. The coefficients of this linear combination can be reduced to the ones emerging from applying Definition 5.7 to the left-hand side of (5.7).

We apply the inductive hypothesis to $\mathbf{a}_1 \cdots \mathbf{a}_{k-1}$ and B in order to give an expression for $\mathbf{c}_1 \, \mathbf{c}_2 \, \ldots \, \mathbf{c}_{k+l-n-1}$ and use Remark 5.8 on the validity of (5.4). This reduces the right-hand side of (5.7) to

$$
\begin{aligned}
&[\mathbf{a}_1,\,\ldots,\mathbf{a}_{n-l},\,B]^{k+l-n-2}\,(\mathbf{a}_1\cdots\mathbf{a}_{k-1}\,\wedge\,B)\vee(\mathbf{a}_1\cdots\mathbf{a}_{n-l}\,\mathbf{a}_k\,\wedge\,B)\\
&=[\mathbf{a}_1,\,\ldots,\mathbf{a}_{n-l},\,B]^{k+l-n-2}\\
&\qquad\sum_{\substack{\sigma\in\mathcal{R}_{\{1,\,\ldots,k-1\}}\\ \tau\in\mathcal{R}_{\{1,\,\ldots,n-l,k\}}}}\Big(\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)[\mathbf{a}_{\sigma(1)},\,\ldots,\mathbf{a}_{\sigma(n-l)},B][\mathbf{a}_{\tau(1)},\,\ldots,\mathbf{a}_{\tau(n-l)},B]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\mathbf{a}_{\sigma(n-l+1)}\cdots\mathbf{a}_{\sigma(k-1)}\,\mathbf{a}_{\tau(k)}\Big)
\end{aligned}
$$

if we use $[*,\ldots,*,B]$ as a shortcut for $[*,\ldots,*,\mathbf{b}_1,\ldots,\mathbf{b}_l]$ and $\mathcal{R}_{\{i_1,,\ldots,i_t\}}$ denotes all permutations of $\{i_1,,\,\ldots,i_t\}$ that indicate $(n-l,t-(n-l))$ splits of $\mathbf{a}_{i_1}\cdots\mathbf{a}_{i_t}$. Since the join is antisymmetric, there are coefficients α_π such that we can rewrite the summation part of the above equation:

$$
\begin{aligned}
&\sum_{\substack{\sigma\in\mathcal{R}_{\{1,\,\ldots,k-1\}}\\ \tau\in\mathcal{R}_{\{1,\,\ldots,n-l,k\}}}}\Big(\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)[\mathbf{a}_{\sigma(1)},\,\ldots,\mathbf{a}_{\sigma(n-l)},B][\mathbf{a}_{\tau(1)},\,\ldots,\mathbf{a}_{\tau(n-l)},B]\\
&\qquad\qquad\qquad\qquad\qquad\mathbf{a}_{\sigma(n-l+1)}\cdots\mathbf{a}_{\sigma(k-1)}\,\mathbf{a}_{\tau(k)}\Big)\\
&\quad=\sum_{\pi\in\mathcal{R}_{\{1,\,\ldots,k\}}}\alpha_\pi\;\mathbf{a}_{\pi(n-l+1)}\cdots\;\mathbf{a}_{\pi(k)}.
\end{aligned}
\tag{5.8}
$$

We fix a $\pi\in\mathcal{R}_{\{1,\ldots,k\}}$ and determine α_π. In order to do so, we have to identify all pairs (σ,τ) with $\sigma\in\mathcal{R}_{\{1,\ldots,k-1\}}$ and $\tau\in\mathcal{R}_{\{1,\ldots,n-l,k\}}$ such that the sets

$$\mathcal{S}_{\sigma,\tau}:=\big\{\sigma(n-l+1),\,\ldots,\sigma(k-1),\tau(k)\big\}\quad\text{and}\quad\mathcal{P}_\pi:=\big\{\pi(n-l+1),\,\ldots,\pi(k)\big\}$$

are identical. From now on let σ, τ and π always denote elements in $\mathcal{R}_{\{1,\ldots,k-1\}}$, $\mathcal{R}_{\{1,\ldots,n-l,k\}}$ and $\mathcal{R}_{\{1,\ldots,k\}}$, resp. Let $s_{\sigma,\tau}$ be the signature of the permutation that sorts

$$\big(\sigma(n-l+1),\,\ldots,\sigma(k-1),\tau(k)\big).\tag{5.9}$$

This sorting results in the order of $(\pi(n-l+1),\,\ldots,\pi(k))$ which is already sorted, since π is a split (see Definition 5.7). With

$$
\begin{aligned}
\beta_\sigma&:=\operatorname{sgn}(\sigma)\,[\mathbf{a}_{\sigma(1)},\,\ldots,\mathbf{a}_{\sigma(n-l)},B]\quad\text{and}\\
\gamma_\tau&:=\operatorname{sgn}(\tau)\,[\mathbf{a}_{\tau(1)},\,\ldots,\mathbf{a}_{\tau(n-l)},B]
\end{aligned}
\tag{5.10}
$$

we have

$$\alpha_\pi=\sum_{\mathcal{S}_{\sigma,\tau}=\mathcal{P}_\pi}s_{\sigma,\tau}\cdot\beta_\sigma\cdot\gamma_\tau.\tag{5.11}$$

In order to determine α_π, we assume that σ and τ are given such that

$$\mathcal{S}_{\sigma,\tau} = \mathcal{P}_\pi. \tag{5.12}$$

We distinguish two cases:
Case 1: Assume $k \in \mathcal{P}_\pi$. Since π is ordered within the parts of the split, we have $\pi(k) = k$. Furthermore, k cannot be in the range of σ and due to (5.12) we have $\tau(k) = k$ and $\tau = \mathrm{id}$. Again (5.12) implies

$$\begin{aligned}&\big\{\sigma(n-l+1), \ldots, \sigma(k-1)\big\} = \{\pi(n-l+1), \ldots, \pi(k-1)\big\} \quad \text{and}\\ &\big\{\sigma(1), \ldots, \sigma(n-l)\big\} = \{\pi(1), \ldots, \pi(n-l)\big\}.\end{aligned}$$

Since π and σ are ordered within the parts of the splits, they coincide on $\{1, \ldots, k-1\}$. Therefore we have

$$\alpha_\pi = 1 \cdot \mathrm{sgn}(\pi) \cdot 1\,[\mathbf{a}_{\pi(1)}, \ldots, \mathbf{a}_{\pi(n-l)}, B]\,[\mathbf{a}_1, \ldots, \mathbf{a}_{n-l}, B]$$

due to (5.11) and (5.10).
Case 2: Assume $k \notin \mathcal{P}_\pi$. Since π is a split, its range can be divided in four parts as given by

$$\underbrace{\pi(1), \ldots, \pi(J)}_{\in\{1,\ldots,n-l\}},\ \underbrace{\pi(J+1), \ldots, \overbrace{\pi(n-l)}^{=k}}_{\in\{n-l+1,\ldots,k\}},\ \underbrace{\pi(n-l+1), \ldots, \pi(j)}_{\in\{1,\ldots,n-l\}},\ \underbrace{\pi(j+1), \ldots, \pi(k)}_{\in\{n-l+1,\ldots,k-1\}}$$

with some $0 \le J \le n-l-1$ and $n-l+1 \le j \le k$. Let

$$I := \big\{\pi(n-l+1), \ldots, \pi(j)\big\}.$$

Observe that I cannot be the empty set since $k \notin \mathcal{P}_\pi$ and $\{\pi(n-l+1), \ldots, \pi(j)\}$ alone does not contain enough elements to constitute $\mathcal{P}_\pi$. Due to (5.12) and due to the range of τ there has to be a

$$x \in I \text{ such that } \tau(k) = x.$$

The range of σ and the fact that σ is ordered within the parts split this imply that the following identities on sets completely determine σ and τ:

$$\begin{aligned}\big\{\pi(j+1), \ldots, \pi(k)\big\} &= \big\{\sigma(j), \ldots, \sigma(k-1)\big\},\\ I \setminus \{x\} &= \big\{\sigma(n-l+1), \ldots, \sigma(j-1)\big\},\\ \big\{\sigma(1), \ldots, \sigma(n-l)\big\} &= \big\{\pi(1), \ldots, \pi(n-l-1)\big\} \cup \{x\},\\ \big\{\tau(1), \ldots, \tau(n-l)\big\} &= \big\{1, \ldots, n-l\big\} \setminus \{x\} \cup \{x\}.\end{aligned}$$

Let $y \in \{n-l+1, \ldots, j\}$ be the preimage of x under π: $\pi(y) = x$. This eases the description of $I \setminus \{x\}$. In order to determine $s_{\sigma,\tau}$ one has to sort the following sequence (see (5.9)):

$$\begin{aligned}&\big(\sigma(n-l+1), \ldots, \sigma(k-1), \tau(k)\big)\\ &= \big(\pi(n-l+1), \ldots, \pi(y-1), \quad \pi(y+1), \ldots, \pi(k), \quad \tau(k) = x = \pi(y)\big).\end{aligned}$$

This implies

$$s_{\sigma,\tau} = (-1)^{k+y}.$$

Comparing the sorting of complete sequences given by σ and π yields

$$\beta_\sigma = \operatorname{sgn}(\pi) \cdot (-1)^{y+k} \, [\mathbf{a}_{\pi(1)}, \ldots, \mathbf{a}_{\pi(J)}, \quad \mathbf{a}_{\pi(J+1)}, \ldots, \mathbf{a}_{\pi(n-l-1)}, \quad \mathbf{a}_{x=\pi(y)}, B].$$

To determine γ_τ one has to sort the sequence

$$\pi(1), \ldots, \pi(J), \;\; \pi(n-l+1), \ldots, \pi(y-1), \;\; \pi(y+1), \ldots, \pi(j), \;\; k, \;\; x = \pi(y) = \tau(k).$$

It can be ordered with a permutation of signature $(-1)^{y+j} \cdot \varepsilon$. Here ε is the signature obtained by sorting

$$\pi(1), \ldots, \pi(J), \quad \pi(n-l+1), \ldots, \pi(j).$$

Therefore, we obtain

$$\begin{aligned}\gamma_\tau &= (-1)^{y+j} \cdot \varepsilon\\ &\qquad [\mathbf{a}_{\pi(1)}, \ldots, \mathbf{a}_{\pi(J)}, \;\; \mathbf{a}_{\pi(n-l+1)}, \ldots, \mathbf{a}_{\pi(y-1)}, \;\; \mathbf{a}_{\pi(y+1)}, \ldots, \mathbf{a}_{\pi(j)}, \;\; \mathbf{a}_k, \, B].\end{aligned}$$

Due to (5.11), we can give the complete coefficient:

$$\begin{aligned}\alpha_\pi &= \operatorname{sgn}(\pi) \cdot (-1)^j \cdot \varepsilon \sum_{n-l+1 \,\le\, y \,\le j} (-1)^y\\
&\qquad [\mathbf{a}_{\pi(1)}, \ldots, \mathbf{a}_{\pi(J)}, \;\; \mathbf{a}_{\pi(J+1)} \;\;, \qquad \ldots, \qquad \mathbf{a}_{\pi(n-l-1)}, \mathbf{a}_{\pi(y)}, \qquad B]\\
&\qquad [\mathbf{a}_{\pi(1)}, \ldots, \mathbf{a}_{\pi(J)}, \;\; \mathbf{a}_{\pi(n-l+1)}, \mathbf{a}_{\pi(y-1)}, \;\; \mathbf{a}_{\pi(y+1)}, \ldots, \mathbf{a}_{\pi(j)}, \qquad \mathbf{a}_k, \qquad B]\\
&= \operatorname{sgn}(\pi) \cdot (-1)^j \cdot \varepsilon \cdot (-1)^j\\
&\qquad [\mathbf{a}_{\pi(1)}, \ldots, \mathbf{a}_{\pi(J)}, \;\; \mathbf{a}_{\pi(J+1)} \;\;, \qquad \ldots, \qquad \mathbf{a}_{\pi(n-l-1)}, \mathbf{a}_{k=\pi(n-l)}, B]\\
&\qquad [\mathbf{a}_{\pi(1)}, \ldots, \mathbf{a}_{\pi(J)}, \;\; \mathbf{a}_{\pi(n-l+1)}, \qquad \ldots, \qquad \mathbf{a}_{\pi(j-1)}, \quad \mathbf{a}_{\pi(j)}, \qquad B]\\
&= \operatorname{sgn}(\pi)\\
&\qquad [\mathbf{a}_{\pi(1)}, \qquad \ldots, \qquad \mathbf{a}_{\pi(n-l)}, \quad B]\\
&\qquad [\mathbf{a}_1, \qquad \ldots, \qquad \mathbf{a}_{n-l}, \quad B]\end{aligned}$$

where the first identity is due to $\mathbf{I}_{\mathrm{GP}}$ in Definition 4.6 and the second identity is due to the definition of ε.
In both cases and due to (5.8), we have that the right-hand side of (5.7) now equals

$$\begin{aligned}
=&[\mathbf{a}_1, \ldots, \mathbf{a}_{n-l}, B]^{k+l-n-1} \\
&\qquad \sum_{\pi \in \mathcal{T}_{\{1,\ldots,k1\}}} \operatorname{sgn}(\pi)[\mathbf{a}_{\pi(1)}, \ldots, \mathbf{a}_{\pi(n-l)}, B]\, \mathbf{a}_{\pi(n-l+1)} \cdots \mathbf{a}_{\pi(k-1)} \\
=&[\mathbf{a}_1, \ldots, \mathbf{a}_{n-l}, B]^{k+l-n-1}\, A \wedge B
\end{aligned}$$

due to the definition of $A \wedge B$. □

Remark 5.14. Suppose $A = \mathbf{a}_1 \cdots \mathbf{a}_k$ and $B = \mathbf{b}_1 \cdots \mathbf{b}_l$ with $\mathbf{a}_1 \cdots \mathbf{a}_k$, $\mathbf{b}_1 \cdots \mathbf{b}_l \in \Lambda^1(\mathbf{P})$. Definition 5.7 and Theorem 5.9 together with Remark 5.8 give representations for $A \wedge B$ that are linear combinations of either $\mathbf{a}$'s or $\mathbf{b}$'s. In [77, p. 68 f.] there are given representations where $\mathbf{a}$'s and $\mathbf{b}$'s can be mixed. This gives new representations starting with rank 4.

In classical Grassmann-Cayley algebra there exists a statement which can be summarized informally as "identical subspaces are scalar multiples of each other". The following theorem gives a symbolic version of this statement.

Theorem 5.15. *Suppose there are given $A, B \in \Lambda(\mathbf{P})$ such that a multiple of them can be given as joins of points: suppose that $\alpha\, A = \mathbf{a}_1 \cdots\ \mathbf{a}_k \neq 0$ and $\beta\, B = \mathbf{b}_1 \cdots \mathbf{b}_k \neq 0$ with α, $\beta \in \mathcal{B}_{\mathbf{P}} \setminus \{0\}$ and $\mathbf{a}_1, \ldots, \mathbf{a}_k$, $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \Lambda^1(\mathbf{P})$ as well as $0 \leq k \leq n$. For $1 \leq k \leq n-1$ assume furthermore that*

$$\begin{aligned}
&\mathbf{a}_1 \cdots \mathbf{a}_k\, \mathbf{b}_i = 0 \quad \textit{for all} \quad 1 \leq i \leq k \qquad \textit{and} \\
&\mathbf{b}_1 \cdots \mathbf{b}_k\, \mathbf{a}_i = 0 \quad \textit{for all} \quad 1 \leq i \leq k.
\end{aligned}$$

There are λ, $\mu \in \mathcal{B}_{\mathbf{P}} \setminus \{0\}$ such that

$$\lambda\, A \;=\; \mu\, B.$$

Remark 5.16. The above theorem should be read as: Whenever A and B represent the same subspace, they are related be multiplicative λ and μ in $\mathcal{B}_{\mathbf{P}}$. Furthermore, λ and μ are indeed elements of $\mathcal{B}_{\mathbf{P}}$, i.e. they SL-invariant on a coordinate level. In addition, it must be possible to obtain $\mu\, B$ from $\lambda\, A$ by essentially only applying Grassmann-Plücker relations.

Proof of Theorem 5.15. The statement is trivial for $k \in \{0, n\}$ and we can assume $1 \leq k \leq n-1$ from now on. The overall strategy for the proof can be described as showing that some suitable λ and μ are SL-invariant polynomials (in the sense of

Definition 4.8) and then to use a fundamental theorem of invariant theory. Therefore, it is necessary to have control over λ and μ. However, the λ' and μ' we are able to give directly are *not* SL-invariant, but their ratio $\frac{\lambda'}{\mu'}$ is. So the first part of the proof is dedicated to giving polynomials in coordinats and showing that their ratio is SL-invariant. The second part of the proof can be considered as giving a rational version of the first fundamental theorem. It provides the existence of $\lambda, \mu \in \mathcal{B}_{\mathbf{P}} \setminus \{0\}$.

Finding λ' and μ' if $\alpha = 1 = \beta$: for now, we are looking for polynomials λ' and μ' only depending on the generic coordinates of $\mathbf{a}_1, \ldots, \mathbf{a}_k$, $\mathbf{b}_1, \ldots, \mathbf{b}_k$ such that

$$\lambda' \cdot \Phi(\, A \vee \mathbf{z}_1 \cdots \mathbf{z}_{n-k}) \;=\; \mu' \cdot \Phi(\, B \vee \mathbf{z}_1 \cdots \mathbf{z}_{n-k}) \tag{5.13}$$

(for $\mathbf{z}_1 \cdots \mathbf{z}_{n-k}$ not in $\mathbf{P}$, compare Definition 5.1) and $\frac{\lambda'}{\mu'}$ is SL-invariant. It trivially holds in $\mathcal{B}_{\mathbf{P} \,\dot\cup\, \{\mathbf{z}_1, \ldots, \mathbf{z}_{n-k}\}}$ that

$$\begin{aligned}&[\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}][\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}] \\ &\qquad = [\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}][\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}].\end{aligned}$$

Our strategy is to show that there are polynomial functions λ' and μ' only depending on the generic coordinates of $\mathbf{a}_1, \ldots, \mathbf{a}_k$, $\mathbf{b}_1, \ldots, \mathbf{b}_k$ such that

$$\frac{\lambda'}{\mu'} = \frac{\Phi\big([\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}]\big)}{\Phi\big([\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}]\big)}. \tag{5.14}$$

In particular, $\frac{\lambda'}{\mu'}$ is SL-invariant since the right-hand side of (5.14) is. Now for the concrete values of λ' and μ': Since $\mathbf{a}_1 \cdots \mathbf{a}_k \neq 0$, there exist row indices $s = \{s_1, \ldots, s_k\} \subset \{1, \ldots, n\}$ such that the subdeterminant $\Phi(\mathbf{a}_1, \ldots, \mathbf{a}_k \,|\, s)$ consisting of the rows s of $\big(\Phi(\mathbf{a}_1), \ldots, \Phi(\mathbf{a}_k)\big)$ is a non-zero polynomial. The notation is borrowed from [51]. We let

$$\lambda' := \Phi(\mathbf{b}_1, \ldots, \mathbf{b}_k \,|\, s) \text{ and } \mu' := \Phi(\mathbf{a}_1, \ldots, \mathbf{a}_k \,|\, s).$$

We show that

$$\lambda' \cdot \Phi\big([\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}]\big) \;=\; \mu' \cdot \Phi\big([\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{n-k}]\big). \tag{5.15}$$

It is a task of linear algebra to show that this equation holds for all values, i.e. to show for any configuration P and for any $z_1, \ldots, z_{n-k} \in \mathbb{R}^n$ that

$$\begin{aligned}&\Big|\Phi^P(\mathbf{b}_1), \ldots, \Phi^P(\mathbf{b}_k) \,|\, s\Big| \cdot \Big|\Phi^P(\mathbf{a}_1), \ldots, \Phi^P(\mathbf{a}_k), z_1, \ldots, z_{n-k}\Big| \\ &\qquad = \Big|\Phi^P(\mathbf{a}_1), \ldots, \Phi^P(\mathbf{a}_k) \,|\, s\Big| \cdot \Big|\Phi^P(\mathbf{b}_1), \ldots, \Phi^P(\mathbf{b}_k), z_1, \ldots, z_{n-k}\Big|\end{aligned} \tag{5.16}$$

where $\left|\Phi^P(*),\, \ldots,\, \Phi^P(*) \,|\, s\right|$ denotes the corresponding subdeterminant. We fix a point configuration P. We can assume that the set $\{\Phi^P(\mathbf{a}_1),\, \ldots,\, \Phi^P(\mathbf{a}_k)\}$ is linearly independent (otherwise (5.16) holds trivially). So due to $\mathbf{a}_1 \cdots \mathbf{a}_k\ \mathbf{b}_i = 0$ (for all $1 \leq i \leq k$) there are scalars $\gamma_{i,j} \in \mathbb{R}$ $(1 \leq i, j \leq k)$ such that

$$\Phi^P(\mathbf{b}_i) = \sum_{j=1}^{k} \gamma_{i,j}\, \Phi^P(\mathbf{a}_j)$$

and we have

$$\left|\Phi^P(\mathbf{b}_1),\, \ldots,\, \Phi^P(\mathbf{b}_k) \,|\, s\right| = \begin{vmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k} \\ \vdots & & \vdots \\ \gamma_{k,1} & \cdots & \gamma_{k,k} \end{vmatrix} \cdot \left|\Phi^P(\mathbf{a}_1),\, \ldots,\, \Phi^P(\mathbf{a}_k) \,|\, s\right|$$

and therefore also

$$\begin{aligned} &\left|\Phi^P(\mathbf{b}_1),\, \ldots, \Phi^P(\mathbf{b}_k),\, z_1,\, \ldots, z_{n-k}\right| \\ &\qquad = \begin{vmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k} \\ \vdots & & \vdots \\ \gamma_{k,1} & \cdots & \gamma_{k,k} \end{vmatrix} \cdot \left|\Phi^P(\mathbf{a}_1),\, \ldots, \Phi^P(\mathbf{a}_k),\, z_1,\, \ldots, z_{n-k}\right| \end{aligned}$$

for all $z_1, \ldots, z_{n-k} \in \mathbb{R}^n$ Substituting both identities in (5.16) and using (5.14) implies (5.13).

Finding λ' and μ' in the general case: in the case that $\alpha \neq 1$ or $\beta \neq 1$ it is not hard to see that

$$\begin{aligned} \Phi(\mathbf{a}_1,\, \ldots,\, \mathbf{a}_k \,|\, s) &= \Phi(\alpha) \cdot \Phi(A \,|\, s) \qquad \text{and} \\ \Phi(\mathbf{b}_1,\, \ldots,\, \mathbf{b}_k \,|\, s) &= \Phi(\beta) \cdot \Phi(B \,|\, s) \end{aligned}$$

where the definition of $\Phi(* \,|\, s)$ is linearly extended and s is determined by the previous case in which we had $\alpha = 1 = \beta$. Substituting this and $\alpha\, A = \mathbf{a}_1 \cdots\ \mathbf{a}_k$ and $\beta\, B = \mathbf{b}_1 \cdots \mathbf{b}_k$ into (5.15) gives

$$\begin{aligned} &\Phi(\beta) \cdot \Phi\big(B \,|\, s\big) \cdot \Phi(\alpha) \cdot \Phi\big(A \vee \mathbf{z}_1,\, \ldots, \mathbf{z}_{n-k}\big) \\ &\qquad\qquad = \Phi(\alpha) \cdot \Phi\big(A \,|\, s\big) \cdot \Phi(\beta) \cdot \Phi\big(B \vee \mathbf{z}_1,\, \ldots, \mathbf{z}_{n-k}\big) \end{aligned}$$

which due to $\alpha \neq 0 \neq \beta$ implies (5.13) for

$$\lambda' := \Phi(B \,|\, s) \text{ and } \mu' := \Phi(A \,|\, s).$$

Furthermore, $\frac{\lambda'}{\mu'}$ is SL-invariant since we can rewrite the SL-invariant

$$\frac{\Phi(\mathbf{b}_1, \ldots, \mathbf{b}_k \mid s)}{\Phi(\mathbf{a}_1, \ldots, \mathbf{a}_k \mid s)} = \frac{\Phi(\beta)}{\Phi(\alpha)} \cdot \frac{\Phi(B \mid s)}{\Phi(A \mid s)}.$$

Since the first factor on the right-hand side is SL-invariant, the second one has to be as well.

Finding λ and μ via a rational version of the first fundamental theorem: we use the fact, that any SL-invariant ratio of polynomials (e.g. $\frac{\lambda'}{\mu'}$) can be written as the ratio of two SL-invariant polynomials. To see this we follow [16, Chap. 1, Prop. 1]. A first verification of this statement was found in [91, Thm. 3.3]. Here the setup is much more general. In the situation given, a short proof can be obtained as follows: since $\mathbb{R}[x_{ij}]$ is factorial, there are λ'' and μ'' in $\mathbb{R}[x_{ij}]$ that are relative prime and

$$\frac{\lambda''}{\mu''} = \frac{\lambda'}{\mu'}.$$

In the following, we introduce a notion and employ that it denotes a group action. Therefore, let $M \in \mathrm{SL}(\mathbb{R}, n)$ and let $\lambda \in \mathbb{R}[x_{ij}]$ and consider the x_{ij}s as entries of a matrix X. Now insert the entry at position (i, j) of $M \cdot X$ in λ instead of x_{ij}. The resulting polynomial is denoted by $M * \lambda$. With this notion and due to the SL-invariance of $\frac{\lambda'}{\mu'}$, we have also SL-invariance for the fully canceled ratio:

$$\frac{\lambda''}{\mu''} = \frac{M * \lambda''}{M * \mu''} \quad \text{which implies} \quad (M * \mu'') \cdot \lambda'' = (M * \lambda'') \cdot \mu''$$

for every $M \in \mathrm{SL}(\mathbb{R}, n)$. Since λ'' does not divide μ'', it divides $M * \lambda''$. Due to the fact that $M * \lambda''$ and λ'' have the same degree, $d(M) := \frac{M * \lambda''}{\lambda''}$ does only depend on M. We know from linear algebra that $M \cdot (M' \cdot X) = (M \cdot M') \cdot X$ for X as before and $M, M' \in \mathrm{SL}(\mathbb{R}, n)$. We obtain

$$(M \cdot M') * \lambda'' = M * (M' * \lambda'') \text{ which implies } d(M \cdot M') = d(M) \cdot d(M'). \quad (5.17)$$

This is the defining equation for d being a *character* (since $\lambda'' \neq 0$ due to the definition of $\lambda' \neq 0$). $d(M) := \frac{M * \lambda''}{\lambda''}$ can be read as giving a rational definition of d in terms of the entries of M. This definition can be also applied to elements of $\mathrm{GL}(\mathbb{R}, n)$. Obviously (5.17) is still valid as an identity of polynomials and $d(M)$ can be understood as an character of $\mathrm{GL}(\mathbb{R}, n)$. Now the sole theorem in [16, Chap. 2, Sec. 4] tells us that every rational abelian character of $\mathrm{GL}(\mathbb{R}, n)$ is an integer exponent of the determinant. Therefore we have $d(M) = \det(M)^g$ for some

$g \in \mathbb{Z}$ whenever $M \in \mathrm{GL}(\mathbb{R}, n)$ and

$$d(M) = 1 \text{ if } M \in \mathrm{SL}(\mathbb{R}, n).$$

Due to the definition of d, the polynomial λ'' is an SL-invariant. Due to symmetry, the same applies for μ''.

Now the (regular) first fundamental theorem of invariant theory applied to λ'' and μ'' implies that there are λ and $\mu \in \mathcal{B}_{\mathbf{P}}$ such that

$$\Phi(\lambda) = \lambda'' \quad \text{and} \quad \Phi(\mu) = \mu''.$$

The existence of these λ and μ together with (5.13) completes the proof. □

5.5. Grassmann-Cayley Algebra Expressions

The following definition is taken from [102, p. 98].

Definition 5.17. *Any element of $\Lambda(\mathbf{P})$ that can be written as an expression only containing the symbols $\vee$, $\wedge$ and points in $\mathbf{P}$ is called a* (simple) Grassmann-Cayley algebra expression*. Whenever this expression is an element of $\Lambda^0(\mathbf{P})$ or $\Lambda^n(\mathbf{P})$, it is called* closed (simple) Grassmann-Cayley algebra expression *or* (closed) synthetic construction*. A formula for a (simple) Grassmann-Cayley algebra expression in $\Lambda^1(\mathbf{P})$ can also be called* construction.

Observe that the simple Grassmann-Cayley algebra expressions are considered to represent subspaces. Theorem 5.13 and Theorem 5.15 imply that whenever A and B are simple Grassmann-Cayley algebra expressions that represent the same subspace, there are $\lambda, \mu \in \mathcal{B}_{\mathbf{P}}$ such that

$$\lambda A \; = \; \mu B.$$

Furthermore, observe that in each step of the evaluation, any summand has the same number of occurrences of any element in $\mathbf{P}$. In addition, this evaluation can only produce coefficients in $\mathbb{Z}$. We summarize:

Theorem 5.18. *Each closed simple Grassmann-Cayley algebra expression is a multihomogenous bracket polynomial with integer coefficients.*

Remark 5.19. The classical version of the Grassmann-Cayley algebra is the one where $|\mathbf{P}| = n$, say $\mathbf{P} = \mathbf{e}_1, \ldots, \mathbf{e}_n$, and the coefficients for linear combinations are allowed to be in the ground field $\mathbb{R}$. In this case, any expression in the resulting algebra can be uniquely written as linear combinations of the *ordered* words on $\mathbf{P}$. This is the way of coordinating objects in the classical Grassmann-Cayley algebra.

The join can easily be performed in the framework of coordinates. For the meet, dualization allows for deriving a formula that is well-structured on the level of coordinates see e.g. [95].

6. Evaluations

In the present chapter we want to emphasize the role of different evaluations of Grassmann-Cayley algebra expressions. We will demonstrate the effects of different strategies for evaluation with the help of the example given by the construction underlying Pascal's theorem. We will promote a graphical evaluation technique which can be considered as the technique of tensor diagrams due to Blinn. In addition, we will introduce a notation which, in the cases considered within this thesis, can encode the choices, which done in the evaluation, in a single graphic. This notation is illustrated with an example taken from Part III.

From now on, we will work in rank 3, we assume $\mathbf{P}$ to be a finite set of names of points and we will identify $\Lambda^0(\mathbf{P})$ with $\Lambda^3(\mathbf{P})$. In principle, in rank 3, a closed synthetic construction can always be evaluated by essentially knowing that

$$\mathbf{ab} \wedge \mathbf{cd} = [\mathbf{a}, \mathbf{c}, \mathbf{d}]\mathbf{b} - [\mathbf{b}, \mathbf{c}, \mathbf{d}]\mathbf{a} = [\mathbf{a}, \mathbf{b}, \mathbf{d}]\mathbf{c} - [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d} \tag{6.1}$$

for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \Lambda^1(\mathbf{P})$ (compare also (2.8)). However, we know from Chapter 5 on Grassmann-Cayley algebra that the elements of $\Lambda(\mathbf{P})$ may have many different representations. When doing calculations in the Grassmann-Cayley algebra, some representations are more useful than others. For example, let $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be *any* (possibly dependent) points in $\Lambda^1(\mathbf{P})$. In the case that $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = (\mathbf{p} \vee \mathbf{q}) \wedge \mathbf{r} = 0$, verifying this identity may be nontrivial and can be done e.g. by using the straightening algorithm. The use of different representations can be illustrated as follows: Theorem 5.15 tells us that if $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = 0$, there are $\lambda, \mu \in \mathcal{B}_{\mathbf{P}} \setminus \{0\}$ such that

$$\lambda\, \mathbf{pq} = \mu\, \mathbf{pr}.$$

So if those λ and μ were known beforehand, $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = 0$ could be seen much more easily since it would hold

$$\lambda\, [\mathbf{p}, \mathbf{q}, \mathbf{r}] = \mu\, [\mathbf{p}, \mathbf{r}, \mathbf{r}] = 0$$

which would imply $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = 0$, since $\lambda \neq 0$ and $\mathcal{B}_{\mathbf{P}}$ has no zero divisors. In general it is not at all obvious how to find suitable representations. However, whenever $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are simple Grassmann-Cayley algebra expressions, we have some freedom in evaluating them. The different evaluations give many different representations of the same object. It is obvious that the anticommutativity of the meet allows for (at least) two different evaluations of the meet given in (6.1). Choosing a suitable one will sometimes ease the whole calculation. And there is even more freedom in choosing a concrete evaluation. We will illustrate different phenomena with the help of three different evaluations of the construction underlying Pascal's theorem (see also Section 3.10 and the non-gray part of Figure 6.1). This will establish a symbolic version of the already diagrammatically notion of tensor diagrams introduced by Blinn in [7, 8, 6] incorporated in [96]. These references follow in their definitions more the lines of non-symbolic tensors as given in [55, 82]. The capacity of tensor diagram evaluations is in principal the same as the basic expansions in [77] given by Li. We will argue that the diagrammatically notion of tensor diagrams is intuitive and symmetric in the sense that no *root* of the calculation tree has to be specified. This is an advantage of the diagram notion which is not present in this clear manner in the formalism of basic expansions given in [77]. There it is also brought up the problem of *binomial evaluations:* The author states that in all of his experiments, his evaluation algorithm achieves binomial evaluations whenever the result is binomial. I.e. in all steps of the algorithm, the expressions stays binomial. In [77, 80] a whole *Cayley expansion theory* is given. The notion of *Cayley bracket* evaluations seems to be similar to the exposition in terms of tensor diagrams given here. However, without a graphical representation, the computations can be considered being less obvious. In the present chapter, we give a notion to indicate evaluations in terms of tensor diagrams. This notion is appropriate especially when the expansions are binomial (or trinomial, in the case that the result is trinomial). It is used in Part III of the thesis in order to give concise certificates for the evaluations stated there. The induced calculations have the same property as the executions of Li's algorithm: the intermediate steps do not have more summands than the result (perhaps except for the computation in Section 17.1 which is a special case since integers are constructed and projective addition is used).

6.1. Evaluations of Pascal's Construction

In Grassmann-Cayley algebra, the collinearity of the three constructed points x, y and z in Pascal's construction (see Section 3.10) in a given point configuration is expressed by

$$[\mathbf{ab} \wedge \mathbf{de}, \mathbf{cd} \wedge \mathbf{fa}, \mathbf{ef} \wedge \mathbf{bc}] = (\mathbf{ab} \wedge \mathbf{de})(\mathbf{cd} \wedge \mathbf{fa})(\mathbf{ef} \wedge \mathbf{bc}) \tag{6.2}$$

vanishing in the coordinatization. Due to (3.12) we expect (6.2) to be (a scalar multiple of)

$$[\mathbf{a}, \mathbf{c}, \mathbf{e}]\, [\mathbf{a}, \mathbf{d}, \mathbf{f}]\, [\mathbf{b}, \mathbf{c}, \mathbf{f}]\, [\mathbf{b}, \mathbf{d}, \mathbf{e}] - [\mathbf{a}, \mathbf{c}, \mathbf{f}]\, [\mathbf{a}, \mathbf{d}, \mathbf{e}]\, [\mathbf{b}, \mathbf{c}, \mathbf{e}]\, [\mathbf{b}, \mathbf{d}, \mathbf{f}]. \tag{6.3}$$

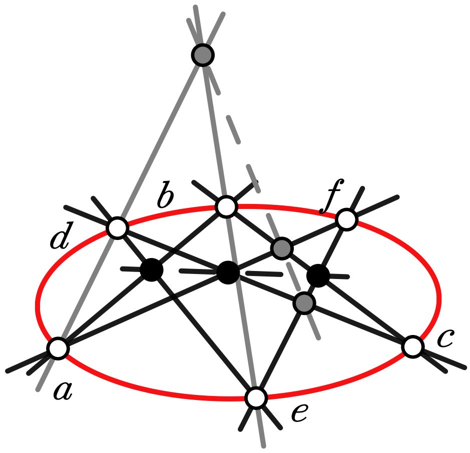


Figure 6.1.: Two Pascal lines for a conic.

6.1.1. First Evaluation

In order to reduce the number of summands after expanding the first two parts of (6.2) it seems to be a good idea to evaluate it like

$$([\mathbf{a},\mathbf{d},\mathbf{e}]\mathbf{b}-[\mathbf{b},\mathbf{d},\mathbf{e}]\mathbf{a})\ ([\mathbf{c},\mathbf{d},\mathbf{a}]\mathbf{f}-[\mathbf{c},\mathbf{d},\mathbf{f}]\mathbf{a})\ (\mathbf{ef}\wedge\mathbf{bc}).$$

This way, some summands will vanish. By evaluating $\mathbf{ef}\wedge\mathbf{bc}$ to $[\mathbf{e},\mathbf{b},\mathbf{c}]\mathbf{f}-[\mathbf{f},\mathbf{b},\mathbf{c}]\mathbf{e}$, (6.2) expands to

$$\begin{aligned}&-[\mathbf{b},\mathbf{f},\mathbf{e}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{c},\mathbf{d},\mathbf{a}][\mathbf{f},\mathbf{b},\mathbf{c}]-[\mathbf{b},\mathbf{a},\mathbf{f}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{c},\mathbf{d},\mathbf{f}][\mathbf{e},\mathbf{b},\mathbf{c}]\\&\qquad+[\mathbf{b},\mathbf{a},\mathbf{e}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{c},\mathbf{d},\mathbf{f}][\mathbf{f},\mathbf{b},\mathbf{c}]+[\mathbf{a},\mathbf{f},\mathbf{e}][\mathbf{b},\mathbf{d},\mathbf{e}][\mathbf{c},\mathbf{d},\mathbf{a}][\mathbf{f},\mathbf{b},\mathbf{c}].\end{aligned}\tag{6.4}$$

This is not a representation we expected. We could straighten (6.3) as well (6.4) and compare both expressions. We will not do this and try another evaluation.

6.1.2. Symmetric (Classical) Evaluation

We chose to evaluate (6.2) as

$$([\mathbf{a},\mathbf{d},\mathbf{e}]\mathbf{b}-[\mathbf{b},\mathbf{d},\mathbf{e}]\mathbf{a})\ ([\mathbf{c},\mathbf{f},\mathbf{a}]\mathbf{d}-[\mathbf{d},\mathbf{f},\mathbf{a}]\mathbf{c})\ ([\mathbf{e},\mathbf{b},\mathbf{c}]\mathbf{f}-[\mathbf{f},\mathbf{b},\mathbf{c}]\mathbf{e}).$$

Here no summand vanishes by repeated points **a**, **b**, **c**, **d**, **e**, **f** in the expansion of the above joins. This evaluation is symmetric in the sense that any point appears exactly once outside the brackets. Expanding this evaluation gives

$$\begin{aligned}
&[\mathbf{b},\mathbf{d},\mathbf{f}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{c},\mathbf{f},\mathbf{a}][\mathbf{e},\mathbf{b},\mathbf{c}] - [\mathbf{b},\mathbf{d},\mathbf{e}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{c},\mathbf{f},\mathbf{a}][\mathbf{f},\mathbf{b},\mathbf{c}]\\
&\qquad - [\mathbf{b},\mathbf{c},\mathbf{f}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{d},\mathbf{f},\mathbf{a}][\mathbf{e},\mathbf{b},\mathbf{c}] + [\mathbf{b},\mathbf{c},\mathbf{e}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{d},\mathbf{f},\mathbf{a}][\mathbf{f},\mathbf{b},\mathbf{c}]\\
&\qquad - [\mathbf{a},\mathbf{d},\mathbf{f}][\mathbf{b},\mathbf{d},\mathbf{e}][\mathbf{c},\mathbf{f},\mathbf{a}][\mathbf{e},\mathbf{b},\mathbf{c}] + [\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{b},\mathbf{d},\mathbf{e}][\mathbf{c},\mathbf{f},\mathbf{a}][\mathbf{f},\mathbf{b},\mathbf{c}]\\
&\qquad\qquad + [\mathbf{a},\mathbf{c},\mathbf{f}][\mathbf{b},\mathbf{d},\mathbf{e}][\mathbf{d},\mathbf{f},\mathbf{a}][\mathbf{e},\mathbf{b},\mathbf{c}] - [\mathbf{a},\mathbf{c},\mathbf{e}][\mathbf{b},\mathbf{d},\mathbf{e}][\mathbf{d},\mathbf{f},\mathbf{a}][\mathbf{f},\mathbf{b},\mathbf{c}]
\end{aligned}$$

which directly simplifies to

$$= [\mathbf{a},\mathbf{c},\mathbf{f}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{b},\mathbf{c},\mathbf{e}][\mathbf{b},\mathbf{d},\mathbf{f}] - [\mathbf{a},\mathbf{c},\mathbf{e}][\mathbf{a},\mathbf{d},\mathbf{f}][\mathbf{b},\mathbf{c},\mathbf{f}][\mathbf{b},\mathbf{d},\mathbf{e}], \tag{6.5}$$

since a lot of identical summands are produced. This is exactly (6.3) with an additional factor of -1. In particular, we know that (6.4) is identical to (6.5) in $\mathcal{B}_{\mathbf{P}}$, i.e. one can be transferred to the other one by adding some Grassmann-Plücker relations. With the evaluation just seen a short representation can be obtained but in between the expression grows exponentially.

6.1.3. Evaluation by Rerooting the Expression

Another possibility to evaluate (6.2) is applying Remark 5.12 which states that for elements $A \vee B \in \Lambda^n(\mathbf{P})$, the join can be replaced by the meet. Furthermore, the meet is associative. This implies:

$$\begin{aligned}
&(\mathbf{ab}\wedge\mathbf{de})(\mathbf{cd}\wedge\mathbf{fa})(\mathbf{ef}\wedge\mathbf{bc}) && (6.6)\\
&= \big((\mathbf{ab}\wedge\mathbf{de})(\mathbf{cd}\wedge\mathbf{fa})\big)\wedge\big(\mathbf{ef}\wedge\mathbf{bc}\big) && (6.7)\\
&= \big(\big((\mathbf{ab}\wedge\mathbf{de})(\mathbf{cd}\wedge\mathbf{fa})\big)\wedge\mathbf{ef}\big)\ \wedge\mathbf{bc} && (6.8)\\
&= \big(\big[(\mathbf{ab}\wedge\mathbf{de}),\mathbf{e},\mathbf{f}\big](\mathbf{cd}\wedge\mathbf{fa}) - \big[(\mathbf{cd}\wedge\mathbf{fa}),\mathbf{e},\mathbf{f}\big](\mathbf{ab}\wedge\mathbf{de})\big)\ \wedge\mathbf{bc}\\
&= \big[(\mathbf{ab}\wedge\mathbf{de}),\mathbf{e},\mathbf{f}\big]\big[(\mathbf{cd}\wedge\mathbf{fa}),\mathbf{b},\mathbf{c}\big] - \big[(\mathbf{cd}\wedge\mathbf{fa}),\mathbf{e},\mathbf{f}\big]\big[(\mathbf{ab}\wedge\mathbf{de}),\mathbf{b},\mathbf{c}\big]\\
&= [\mathbf{a},\mathbf{b},\mathbf{e}][\mathbf{d},\mathbf{e},\mathbf{f}][\mathbf{c},\mathbf{f},\mathbf{a}][\mathbf{d},\mathbf{b},\mathbf{c}] - [\mathbf{c},\mathbf{d},\mathbf{f}][\mathbf{a},\mathbf{e},\mathbf{f}][\mathbf{b},\mathbf{d},\mathbf{e}][\mathbf{a},\mathbf{b},\mathbf{c}]. && (6.9)
\end{aligned}$$

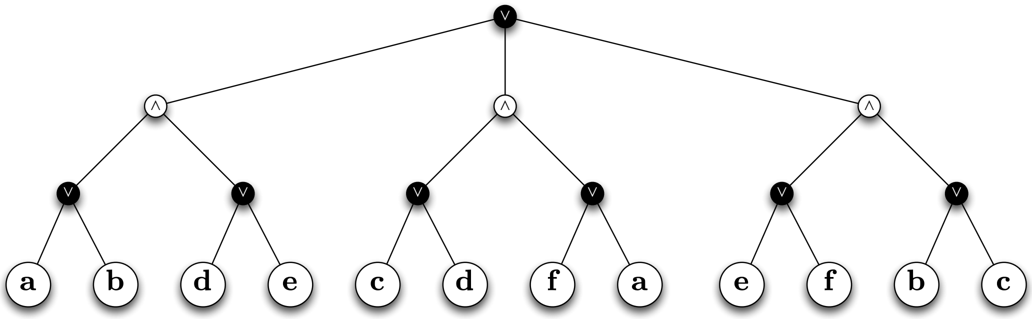
A remarkable property of this evaluation is that it stays binomial during all of the calculation. Furthermore, in contrast to many evaluations in the classical literature, the evaluation starts not in the first part of the underlying construction (which is e.g. evaluating $(\mathbf{ab}\wedge\mathbf{de})$) but leaves points unevaluated. This allows for evaluating it in two different ways in the last step. The resulting bracket polynomial differs in its representation from the one in (6.5). However, exchanging the points $\mathbf{b}$ and $\mathbf{d}$ lets them coincide. This implies that

$$(\mathbf{ab}\wedge\mathbf{de})(\mathbf{cd}\wedge\mathbf{fa})(\mathbf{ef}\wedge\mathbf{bc}) = (\mathbf{ad}\wedge\mathbf{be})(\mathbf{cb}\wedge\mathbf{fa})(\mathbf{ef}\wedge\mathbf{dc})$$

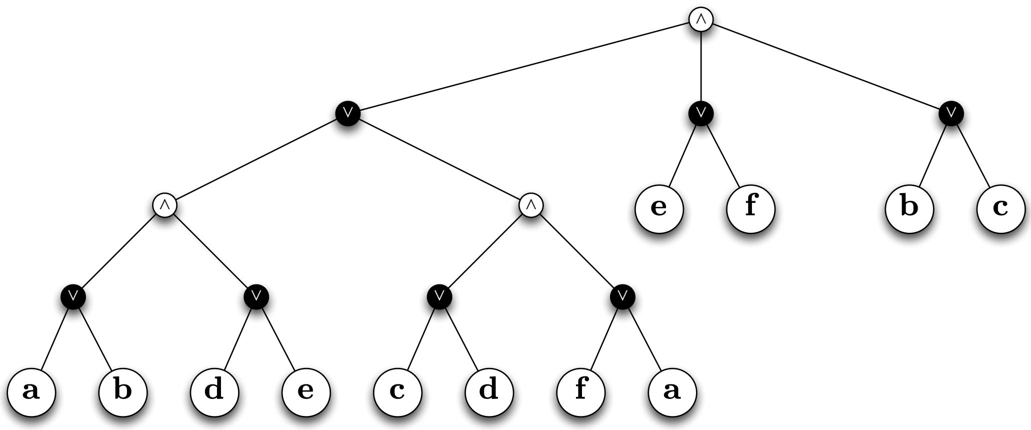
and in any coordinatization, the three gray points are collinear as well (see Figure 6.1). In fact, there is much more underlying symmetry, since the condition for six points lying on a common conic is symmetrical. In each instance, the lines obtained by permuting the points (6.2) are called *Pascal lines*. In fact it is a classical result that there are 60 of them. They are again related an meet in bundles in Steiner points (see [58, 100] and more recently [23]).

6.1.4. Diagram Representations

We illustrate the above calculations by giving expression trees. This helps to also access general evaluations more easily. (6.6) and (6.2) are read as



(6.7) and (6.8) can be read as

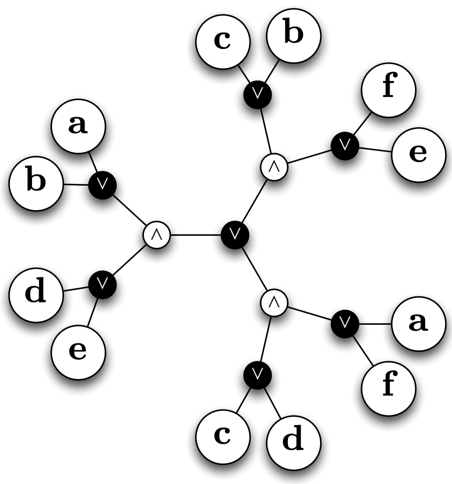


6.2. Rerooting in Diagrams

We see, that the expression tree as a topological structure is not changed. However, at each vertex, the order of the neighboring vertices has a meaning and does matter. One can argue inductively and traveling along edges, that the expression tree can be *rerooted* at any inner vertex $\vee$ or $\wedge$. If the *order* around inner vertices is maintained, this will result in an equivalent Grassmann-Cayley expression: since we are in rank 3 and 3 is an odd number, we have that

$$\mathbf{p} \vee \mathbf{q} \vee \mathbf{r} = \mathbf{q} \vee \mathbf{r} \vee \mathbf{p} \quad \text{and} \quad \mathbf{P} \wedge \mathbf{Q} \wedge \mathbf{R} = \mathbf{Q} \wedge \mathbf{R} \wedge \mathbf{P}$$

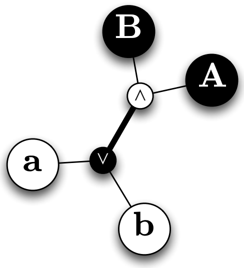
for all $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \Lambda^1(\mathbf{P})$ and $\mathbf{P}, \mathbf{Q}, \mathbf{R} \in \Lambda^2(\mathbf{P})$. This implies that the *rooting* can be completely omitted. What matters, is the topological structure of the evaluation tree including the *cyclic order* around the inner vertices. Therefore, the previous diagram can by symmetrized as



It encodes the same Grassmann-Cayley expression as the diagrams given before. The representation of Grassmann-Cayley expressions as expression trees has the advantage that it is invariant under rerooting in contrast to the underlying geometric construction.

6.3. Splitting Diagrams along Inner Edges and Notation

We call an edge an *inner edge*, whenever it is an edge between inner nodes. Any inner edge has the shape

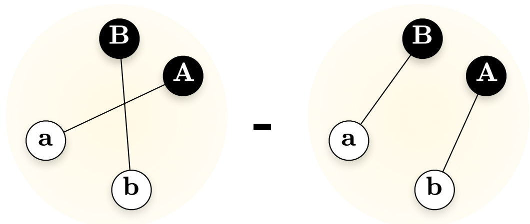


$$= \quad \mathbf{B} \wedge \mathbf{ab} \wedge \mathbf{A} \tag{6.10}$$

for some $\mathbf{a}$, $\mathbf{b} \in \Lambda^1(\mathbf{P})$ and $\mathbf{A}$, $\mathbf{B} \in \Lambda^2(\mathbf{P})$. We know that

$$\mathbf{ab} \wedge \mathbf{A} = [\mathbf{a}, \mathbf{A}]\mathbf{b} - [\mathbf{b}, \mathbf{A}]\mathbf{a}$$

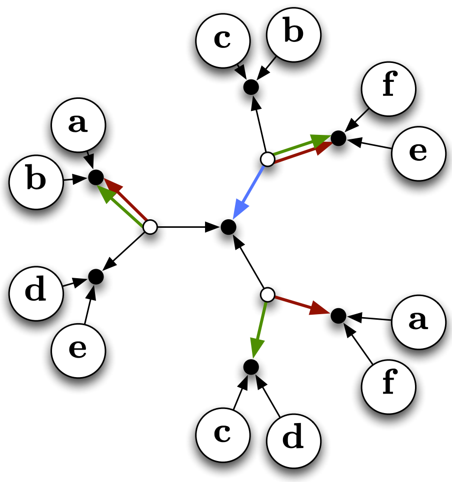
(where $[\mathbf{a}, \mathbf{A}]$ is a shorthand for $\mathbf{a} \wedge \mathbf{A}$ if $\mathbf{a} \in \Lambda^1(\mathbf{P})$ and $\mathbf{A} \in \Lambda^2(\mathbf{P})$) which establishes a general rule for the evaluation of inner edges as given in (6.10) to



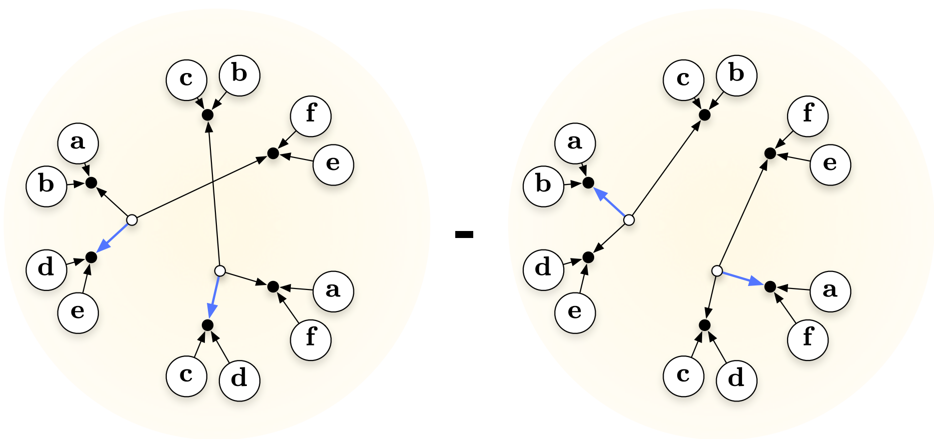
(6.11)

Here different connected components of the same summand are regarded as related by multiplication. The equality of (6.10) and (6.11) is a symbolic versions of the so-called ε-δ rule for graphical calculations with tensors as introduced e.g. in [7, 96]. In this form originating from Blinn. We will refer to it by saying *splitting a diagram* along a given edge. Observe that for correctly splitting along an edge given in (6.10), the endpoints of new edges introduced in (6.11) have to be placed at the position of the endpoints of old edges at the nodes labeled with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{A}$ and $\mathbf{B}$. We have to do this since we want to obtain a correct cyclic order of arrows around the nodes $\mathbf{a}$, $\mathbf{b}$, $\mathbf{A}$ and $\mathbf{B}$. In [7, 96], the nodes  and  are so-called ε-tensors. Both of them are denoted by a small black dot in [7, 96]. The distinction between points and lines in our setup is done by coloring the nodes. In [7, 96] the edges are directed and it is distinguished between *covariant and contravariant indices*. Outgoing arrows (with respect to one node) are called covariant where incoming arrows are called contravariant. So point-like nodes have only covariant arrows (or indices) and line-like nodes have contravariant ones. We adjust to this notation by directing the edges pointing from white nodes to black nodes. This might be convenient for readers familiar with this notion. Nevertheless, we can drop the symbols inside the inner nodes and use $\bullet$ and $\circ$ instead of $\vee$ and $\wedge$.

With these notions, the three evaluations of Pascals construction can be summarized in

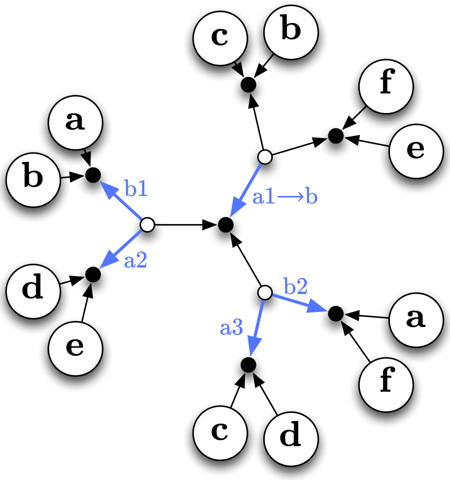


Splitting along the red edges correspond to the first evaluation, splitting along green edges to the second and splitting along the blue edges is the first step of the third evaluation. The next step in the third evaluation is now splitting along the blue edges in the result:

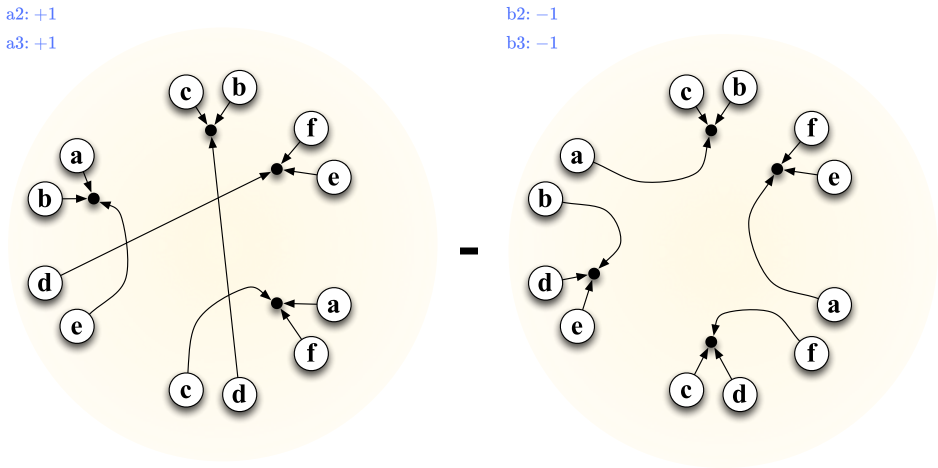
 (6.12)

We say that *Classical evaluations*, like the first and the second evaluations of Pascal's construction, distinguish a single inner node as root and then evaluate the diagram from outside to inside. More precisely, after the specification of a root, it is split only along edges that enter a white node ○ . Here entering is understood as being contained in a directed path from the leave to the distinguished root. Therefore, any classical evaluation can be summarized in a single diagram with highlighted edges. The highlights correspond to the splits and the order of splittings does not really matter. More general, i.e. non-classical evaluations may not be summarized that easily. However, the additional degree of freedom in evaluating may lead to binomial evaluations.

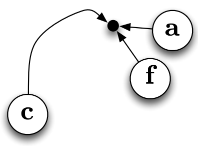
With the help of some conventions we can also summarize both steps of the third evaluation in a single diagram:



It should be read as follows: start splitting along the edge a1. In a generic general case this will result in two summands. This is also true for the actual edge a1 given here. Therefore it is labeled with an arrow ⟶ and a letter b. This means that in the first summand, one should continue splitting edges whose labels start with an a. In the second summand one should start with the newly introduced letter b and split along the edge b1. So in the case of splitting edge a1⟶b in the first step one splits along a2 in the first induced summand and along b1 in the second induced summand. The edges, where the label does not contain an arrow ⟶, have the property that after splitting along them one of the induced summands trivially vanishes. After splitting along the edge a1 the diagram essentially looks like (6.12). After all steps it looks like



The blue label a2: +1 indicates that after splitting along the edge labeled a2, only the first summand survives. It induces a factor +1 for the first summand. Now the result of the evaluation can be easily read of: The small black dots with three adjacent points correspond to joins. In order to read of an expression in $\Lambda(\mathbf{P})$, we recall that they also represent expression trees. Therefore e.g.



can be read as $\quad \mathbf{c} \vee \mathbf{f} \vee \mathbf{a} = [\mathbf{c}, \mathbf{f}, \mathbf{a}]$.

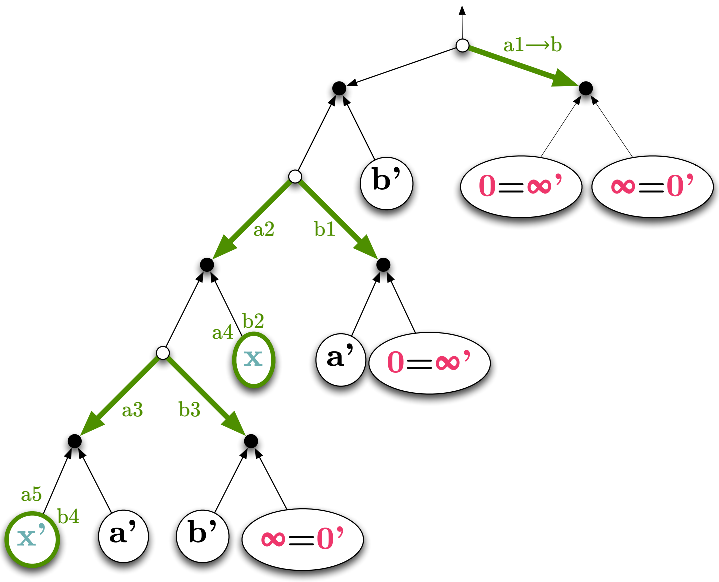
So after splitting any inner edges, the summands consist of products of small diagrams that can be interpreted as brackets when the letters are read in *counterclockwise* order. Therefore the above evaluation can be read as

$$(+1)\cdot(+1)\cdot[\mathbf{a},\mathbf{b},\mathbf{e}][\mathbf{d},\mathbf{e},\mathbf{f}][\mathbf{c},\mathbf{f},\mathbf{a}][\mathbf{d},\mathbf{b},\mathbf{c}]-(-1)\cdot(-1)\cdot[\mathbf{c},\mathbf{d},\mathbf{f}][\mathbf{a},\mathbf{e},\mathbf{f}][\mathbf{b},\mathbf{d},\mathbf{e}][\mathbf{a},\mathbf{b},\mathbf{c}]$$

which is the same as (6.9). This notion is appropriate for evaluations with few summands. In this case, one does not have to introduce to many new letters. Observe that now the order of splittings might be crucial for the property to have few summands during the evaluation. Furthermore, the notion is (not yet) adjusted to the case that in some step of the evaluation one wants to split along an edge that was not present in the very first diagram. We think, in principle, this problem can be resolved. However, we do not need this kind of notion in this exposition and therefore we use the description of calculations as introduced.

6.4. Exemplary Evaluation of an Expression of Part III

In the following, we extend the notion of expansion in order to use not only points in $\mathbf{P}$ as arguments of simple Grassmann-Cayley algebra expressions of diagrams. We also use points that are given by a linear combination. We demonstrate this and some additional phenomena arising in this treatment with an example used in Section 16.5. There it is labeled with Case 2 (ii) and it is given by the diagram



(6.13)

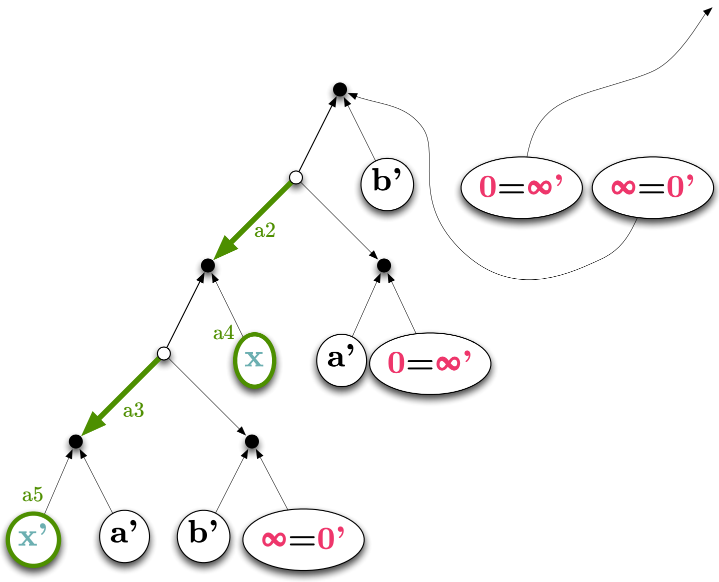
with $\mathbf{0} = \infty'$, $\infty = \mathbf{0}'$, $\mathbf{a}'$ and $\mathbf{b}' \in \mathbf{P}$. x and x' are given as

$$\mathrm{x} = M_{\mathrm{x}}\,(\mu_{\mathrm{x}}\mathbf{0} - \lambda_{\mathrm{x}}\infty) \quad \text{and} \quad \mathrm{x}' = M_{\mathrm{x}'}\,(\mu_{\mathrm{x}'}\mathbf{0}' - \lambda_{\mathrm{x}'}\infty') \tag{6.14}$$

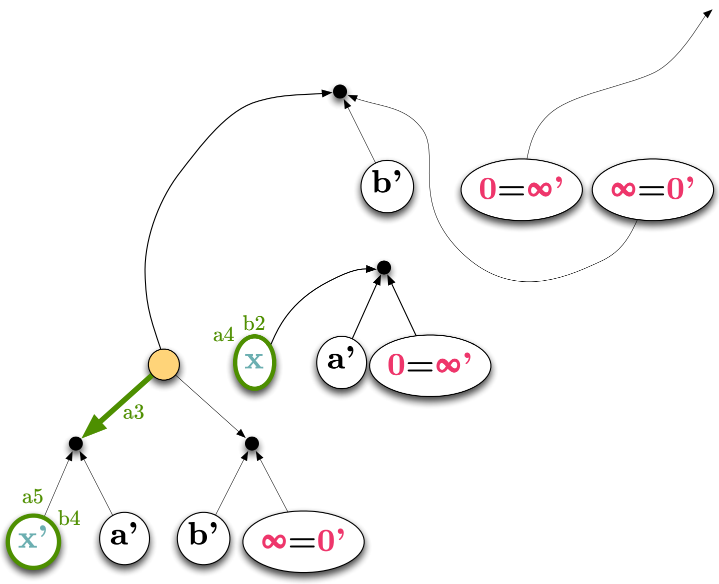
with μ_{x}, λ_{x}, $\mu_{\mathrm{x}'}$ and $\lambda_{\mathrm{x}'} \in \mathcal{B}_{\mathbf{P}}$. Furthermore the shape of $\mu_{\mathbf{1}'}$ and $\lambda_{\mathbf{1}'}$ is known.

$$\mu_{\mathbf{1}'} = [\mathbf{a}', \mathbf{b}', \mathbf{0}'] \quad \text{and} \quad \lambda_{\mathbf{1}'} = [\mathbf{a}', \mathbf{b}', \infty'].$$

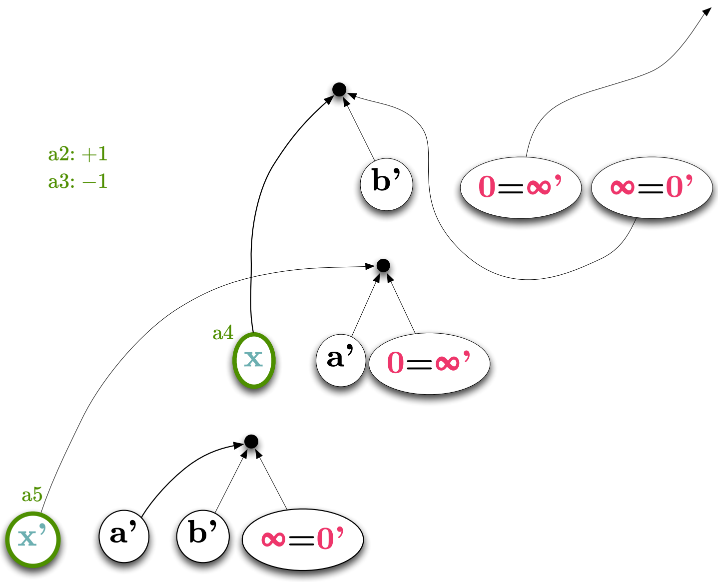
We split along the edge a1 and consider the first summand. It looks like



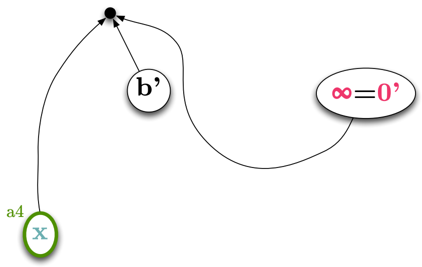
The second summand in the subsequent splitting along edge a2 results in



which vanishes since at the highlighted inner node, two identical subtrees meet. Splitting along edges a2 and a3 in the first summand results in

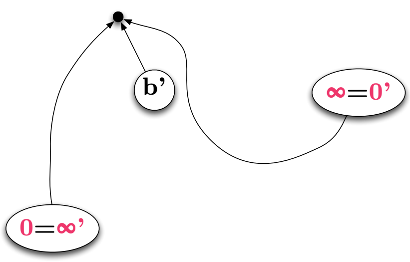


The label a4 indicates that now, we should plug in the value of x given in (6.14). This means that we evaluate

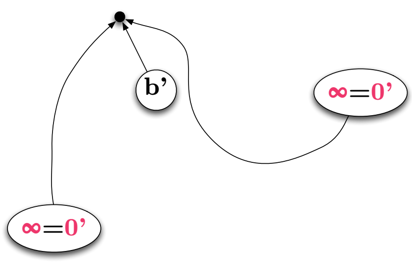


$=$

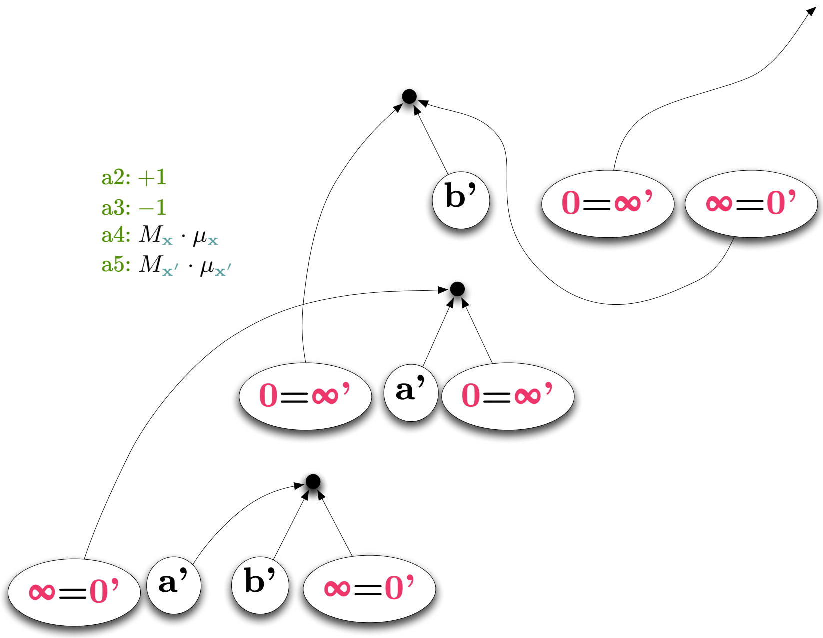
$M_{\mathrm{x}}\cdot\mu_{\mathrm{x}}\cdot$



$-M_{\mathrm{x}}\cdot\lambda_{\mathrm{x}}\cdot$

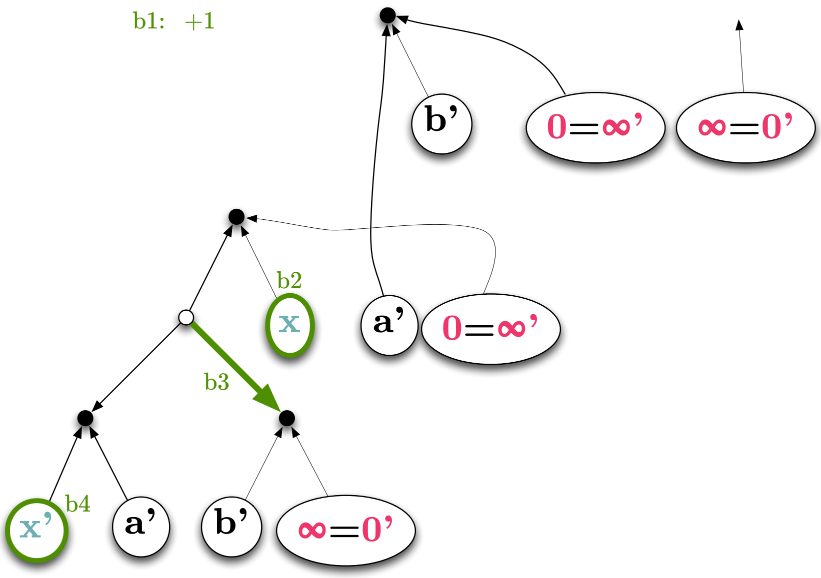


where the second summand vanishes. Plugging in also the value of x' implies that the first summand after splitting (6.13) along the edge a1 can be written as

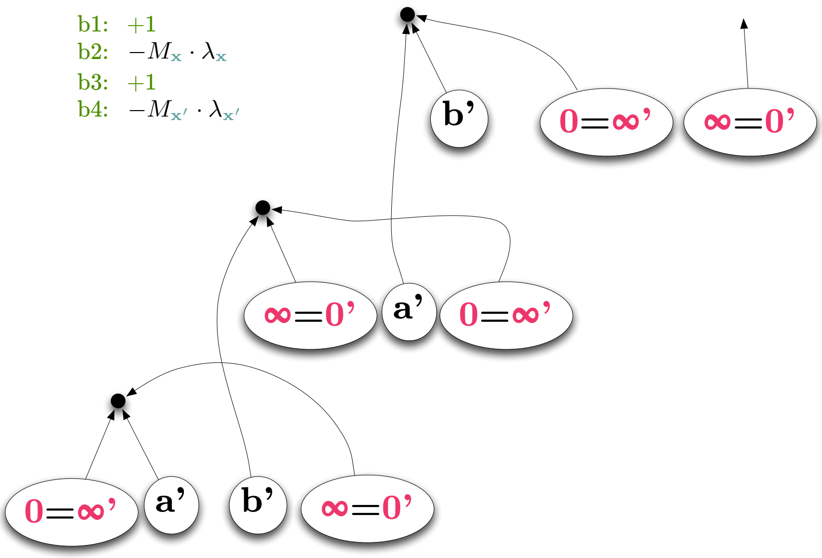


(6.15)

where all values in the top left are considered as factors of the diagram. Now for the second summand. After splitting along the edge b1 the diagram looks like



We can plug in the value of $\mathbf{x}$ by now. In total the second summand evaluates to



(6.16)

The evaluations of both summands (6.15) and (6.16) together give an evaluation of (6.13) as

$$
\begin{aligned}
&(+1)\cdot(-1)\cdot M_{\mathbf{x}}\cdot \mu_{\mathbf{x}}\cdot M_{\mathbf{x}'}\cdot \mu_{\mathbf{x}'}\cdot \\
&\qquad \underbrace{[\mathbf{a}',\mathbf{b}',\infty=\mathbf{0}']}_{\mu_{\mathbf{1}'}}[\infty=\mathbf{0}',\mathbf{a}',\mathbf{0}=\infty'][\mathbf{0}=\infty',\mathbf{b}',\infty=\mathbf{0}']\,\mathbf{0} \\
&-(+1)\cdot(-M_{\mathbf{x}}\cdot\lambda_{\mathbf{x}})\cdot(+1)\cdot(-M_{\mathbf{x}'}\cdot\lambda_{\mathbf{x}'})\cdot \\
&\qquad [\mathbf{0}=\infty',\mathbf{a}',\infty=\mathbf{0}'][\mathbf{b}',\infty=\mathbf{0}',\mathbf{0}=\infty']\,\underbrace{[\mathbf{a}',\mathbf{b}',\mathbf{0}=\infty']}_{\lambda_{\mathbf{1}'}}\,\infty \\
&= [\mathbf{0},\infty,\mathbf{b}'][\mathbf{0},\infty,\mathbf{a}']\cdot M_{\mathbf{x}'}\cdot M_{\mathbf{x}}\,\left(\mu_{\mathbf{1}'}\mu_{\mathbf{x}}\mu_{\mathbf{x}'}\mathbf{0}-\lambda_{\mathbf{1}'}\lambda_{\mathbf{x}}\lambda_{\mathbf{x}'}\infty\right)
\end{aligned}
$$

as claimed in Section 16.5. The example just seen covers all phenomena arising in the evaluation of expressions given in Section 16.5: if one does the evaluation as stated there, one sometimes has to find inner vertices where identical subtrees meet. Doing this, all evaluations are *binomial*. In the last step, one has to identify $\lambda_{\mathbf{1}}$ and $\mu_{\mathbf{1}}$ as just seen in the example. The evaluations of the expressions given in Section 17.2 are all trinomial as it are the results given there. Observe that this is a very concise way of evaluation. Even though the last computations occupied several pages to explain them, the actual results (6.15) and (6.16) can be given in a single diagram. One can do the calculations easily by printing (6.13) twice, considering each copy as a summand, deleting edges and keeping track of factors.

7. Incidence Theorems

There are some reasons to have a separate chapter on incidence theorems. One is that it is a classical part of projective geometry. The defining axioms of projective geometry are essentially about incidence and therefore it is interesting, which kind of theorems can be deduced. Some of the incidence theorems date back to the ancient Greeks. Another reason is that it is closely related to everything presented in this thesis. Although most of the time, this connection is not explicit, it is there. Furthermore, incidence theorems are the starting point for the considerations in Part II. We will only treat incidence theorems in $\mathbb{RP}^2$. We start with describing the general structure of incidence theorems. We will give background information on automated proving of (incidence) theorems. Sometimes incidence theorems can also be *detected* automatically. We will argue that it is important to consider so-called non-degeneracy conditions. Many approaches use construction sequences to prove the theorem. This might restrict the classes of theorems as well as the generality of the statements that are considered. For both, stating and proving incidence relations, it plays a role how theorems are actually given. We will restate our point of view on this subject. It was formalized in [2]. It captures theorems that do not have a construction sequence. Furthermore it can automatically take mild non-degeneracy conditions into account.

7.1. General Structure of Incidence Theorems

In its most general form an incidence theorem may roughly be stated as "Given some objects in the projective plane and some prescribed incidence relations hold between them. Then an additional incidence relation holds."

7.2. Pascal's Theorem Revisited

We already encountered a statement matching this description: Pascal's theorem (see Section 3.10) can be read as:

- Given a conic $\mathcal{C}$, points a, b, c, d, e, f, x, y and z points in the projective plane $\mathbb{RP}^2$ and lines l_1, l_2, l_3, l_4, l_5 and l_6

- such that a, b, c, d, e and f are incident with $\mathcal{C}$ and

$$\begin{array}{l} a,\ b \text{ and } x \text{ are incident with } l_1, \\ e,\ d \text{ and } x \text{ are incident with } l_2, \\ a,\ f \text{ and } y \text{ are incident with } l_3, \\ c,\ d \text{ and } y \text{ are incident with } l_4, \\ e,\ f \text{ and } z \text{ are incident with } l_5 \text{ and} \\ b,\ c \text{ and } z \text{ are incident with } l_6. \end{array}$$

- Then the points x, y and z are incident with some line l.

A different formulation with the incidence constraints more implicitly would be:

- Given points a, b, c, d, e, f, x, y and z in the projective plane
- such that a, b, c, d, e and f are on a common conic and the triples of points (a, b, x), (e, d, x), (a, f, y), (c, d, y), (e, f, z) and (b, c, z) are collinear.
- Then the points x, y and z are collinear as well.

In fact, often in the literature, incidence theorems are stated in this fashion but they are not completely correct in this way. Figure 7.1 gives a configuration in $\mathbb{RP}^2$ fulfilling all requirements of the above formulation of Pascal's theorem where the conclusion does not hold. In fact, Figure 7.1 is a *degenerate* configuration. So

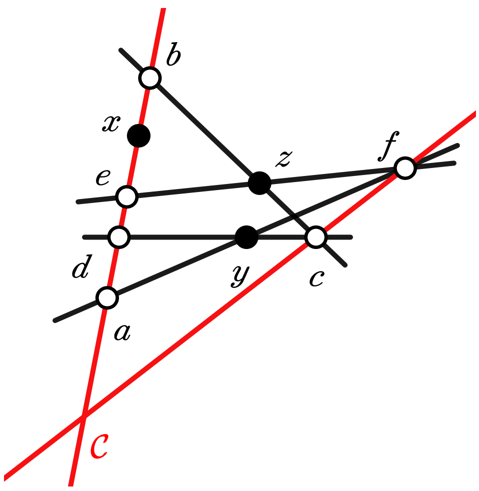


Figure 7.1.: A degenerate configuration where the conic consists of the two red lines. Here a naive formulation of Pascal's theorem does not hold.

in the above formulation of Pascal's theorem, it is not taken care of degenerate configurations. A correct formulation of the theorem that matches the present exhibition could be:

Whenever points a, b, c, d, e and f lie on a common conic it holds that the determinant

$$\Big|\mathbf{meet}\big(\mathbf{join}(a,b),\mathbf{join}(d,e)\big),\mathbf{meet}\big(\mathbf{join}(c,d),\mathbf{join}(f,a)\big),\\ \mathbf{meet}\big(\mathbf{join}(e,f),\mathbf{join}(b,c)\big)\Big| \tag{7.1}$$

vanishes. In the case that $\mathbf{meet}\big(\mathbf{join}(a,b),\mathbf{join}(d,e)\big)$, $\mathbf{meet}\big(\mathbf{join}(c,d),\mathbf{join}(f,a)\big)$ and $\mathbf{meet}\big(\mathbf{join}(e,f),\mathbf{join}(b,c)\big)$ can be interpreted as homogenous coordinates, this fact implies the collinearity of the three points. However, it may happen, that one of the three vectors is zero. This can only happen, if the underlying construction degenerates. In this case, (7.1) holds trivially and one cannot conclude a "collinearity" of three points.

This example is supposed to illustrate the fact, that depending on the formulation of an incidence theorem, non-degenerate conditions have to be formulated. Supposing a correct formulation, Pascal's theorem is surely an incidence theorem. However, in the present text, we will focus mostly on incidence relations and incidence theorems involving only points and lines. So unless mentioned differently, from now on, an incidence theorem will be formulated in terms of points and lines.

7.3. Incidence Theorems of Points and Lines

A first example of a theorem on points and lines is in fact another degenerate situation of Pascal's theorem: consider the case where the conic $\mathcal{C}$ degenerates to two lines which are incident to the points a, c and e resp. b, d and f (see Figure 7.2).

In fact, Pappos's theorem is very special in a lot of senses. For example, it is the smallest of all incidence theorems and also the only one on nine points. In the literature you can find many other special properties and also a variety of different proofs. See [95] for nine proofs and more relations. It is also treated in almost every textbook on projective geometry. We referred to some of them already in Chapter 2. We will present two ideas for proofs. The first one is just the fact that (7.1) holds since points $a, \dots, f$ lie on a common conic. Furthermore, in general, within this specialization, the construction does not degenerate. So this proves, that whenever the construction indicated in Figure 7.2 does not degenerate, the constructed points are collinear. One can see this also without the knowledge that a pair of lines constitutes a degenerate conic: reconsider from Chapter 6 that in

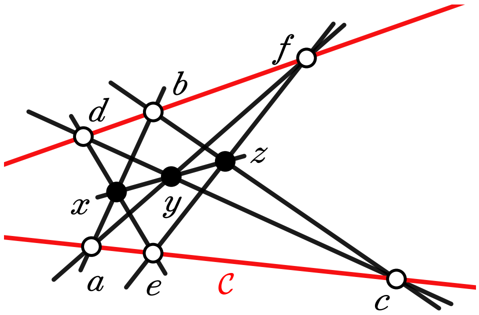


Figure 7.2.: A degenerate configuration of Pascal's theorem gives Pappos's theorem.

Grassmann-Cayley algebra it holds

$$\begin{aligned}&(\mathbf{ab} \wedge \mathbf{de})(\mathbf{cd} \wedge \mathbf{fa})(\mathbf{ef} \wedge \mathbf{bc}) \\ &\qquad = [\mathbf{a}, \mathbf{c}, \mathbf{f}][\mathbf{a}, \mathbf{d}, \mathbf{e}][\mathbf{b}, \mathbf{c}, \mathbf{e}][\mathbf{b}, \mathbf{d}, \mathbf{f}] - [\mathbf{a}, \mathbf{c}, \mathbf{e}][\mathbf{a}, \mathbf{d}, \mathbf{f}][\mathbf{b}, \mathbf{c}, \mathbf{f}][\mathbf{b}, \mathbf{d}, \mathbf{e}].\end{aligned}$$

Plugging in $\mathbf{e} = \mathbf{ac} \wedge \mathbf{uv} = [\mathbf{a}, \mathbf{u}, \mathbf{v}]\mathbf{c} - [\mathbf{c}, \mathbf{u}, \mathbf{v}]\mathbf{a}$ and $\mathbf{f} = \mathbf{bd} \wedge \mathbf{xy} = [\mathbf{b}, \mathbf{x}, \mathbf{y}]\mathbf{d} - [\mathbf{d}, \mathbf{x}, \mathbf{y}]\mathbf{b}$ gives zero (for some $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}$ and $\mathbf{y}$). This can be also read as a proof for a pair of lines being a conic. The above formulation in Grassmann-Cayley algebra implies the existence of an underlying construction. All incidence theorems that are formulated with the help of an *underlying construction* by the ruler alone can in principle be proved with the help of Grassmann-Cayley algebra and a method for showing the vanishing of a resulting bracket polynomial. This method might be the straightening algorithm or substituting symbolic coordinates with the help of Φ.

7.4. (General) Automated Theorem Proving

The method last described can be considered a an automated version of theorem proving. Actually, there is a whole branch of mathematics and computer science dedicated to automated geometric reasoning. Overviews of the development (up to the point of publication) are given in [93, 114] by Richter-Gebert and Wang. The latter contains many references for the development of automated reasoning. The first steps date back to [46]. First theoretical results are given by the fact that arbitrary algebraic equations can be encoded as projective incidence theorems ([83, 10, 121]). Furthermore, realizability is theoretically decidable ([105]).

It was Wu who gave a systematic translation routine from geometry to algebra (see e.g. [137]). He uses Euclidean geometry and translates constructions to polynomials in coordinates. Therefore, he gives a complete restatement of Hilbert's "Foundations of Geometry" (see [62]). He uses a constructive treatment of algebraic geometry introduced previously by Ritt (see e.g. [98]). The method encodes the hypotheses of a theorems in the vanishing of a polynomial (in the coordinates of points). The conclusion is encoded in a polynomial as well. It aims to show that a *multiple* of this polynomial vanishes in behalf of the hypotheses. The theorem is true in a concrete instance, whenever the multiplier monomial not yields zero. The multiplier is said to be a *non-degeneracy condition*. Wu's method tries also to interpret the non-degeneracy conditions derived. To show the vanishing of a multiple of the conclusion polynomial, a construction sequence of the theorem is assumed and used in order to eliminate variables. Using the translation from geometry to algebra done by Wu, in [72], Kutzler and Stifter use a different approach to do the algebraic calculations: they use Buchberger's algorithm using Gröbner bases. Both methods are introduced and compared in [20]. A variant of using Gröbner bases was introduced by Kutzler (see e.g. [70]). For another overview regarding Gröbner bases in geometric theorem proving see also [117]. Here one finds also a section on discovering theorems.

A view words on theorems with a construction sequence (or theorems of constructive type) are appropriate: As stated in [20, p. 19], the choice of a construction sequence is intimately related not only with the selection of parameters and dependent variables but also to the validity of the theorem as well as the non-degeneracy conditions given by an algorithm. Recalling the re-rooting process from the previous chapter, one sees that this approach may not cover the spirit that (some) points have same rights. For example, there might be constructions where one can construct a point a depending on b as well as the point b depending on a.

There are also coordinate-free methods. One of its is the so-called area method summarized in [21]. An external overview over the method is given in [69]. It is coordinate free and expresses relations via areas and length ratios and is therefore closely related with the content of this thesis. The area method also uses a construction sequence in order to make symbolic manipulations. Another coordinate-free method is bracket algebra together with Grassmann-Cayley algebra (see e.g. [30, 74, 78]). A final polynomial exists whenever an (oriented) matroid is non-realizable. Therefore an incidence theorem is proved whenever a counterexample is non-realizable. Final polynomials trace back to Bokowski and Sturmfels ([10]), and independently to Whiteley ([135, 130, 132, 133]). The connection between both theories is emphasized in [134]. The latter also addresses the problem of interpreting non-degenerate multipliers appearing in the methods described above. This is done by considering invariant properties and translating them into invariant language. The special case of biquadratic final polynomials due to Bokowski and Richter-Gebert is easier to compute and works in many cases (see [92, 93, 9, 30]). We will give an example of this method in Section 9. In [101], Sturmfels points out that incidence theorems are special cases of configurations and that also [63] gives comments about the great significance of configurations in former times. In the concrete computations he uses (partial) construction sequences in order to get algebraic expressions. Results of Mnëv and Shor are cited in order to obtain the result that the real realizability problem is NP-hard. Due to [10, p. 5] the existence of a construction sequence is the question of rational realizability.

Closely related to bracket algebra and Grassmann-Cayley algebra is also the method of Clifford algebra (which is closely related with Grassmann-Cayley algebra). This approach is due to Li and Cheng (see e.g. [67, 73, 116]). In these approaches the borders are not sharp and also variants of the previously mentioned approaches are used. See also the work of Li e.g. in [75, 74, 79, 80, 76]. The detailed book [77] can be seen as a summary of the above mentioned. [42] can be read as generalizing the elimination methods seen e.g. in Wu's method to Clifford Algebra.

More developments on the subject are given in several collections e.g. [26] and the proceedings on the workshops of automated deduction in geometry starting with [115, 45, 97]. More literature can be found under `http://www-calfor.lip6.fr/~wang/GRBib/`.

Applications of mechanical theorem proving are due to [93] given in computer-aided geometric reasoning, robotics and robot motion planning, computer vision and scene analysis, rigidity of frameworks, molecular conformation, computer-aided design and computer-aided manufacturing and many other related topics.

7.5. Modeling Incidence Theorems of Points and Lines

The starting point for the investigation of Γ-cycles in Part II is the binomial or biquadratic proving method given in [92, 93, 9, 30]. It provides a construction-free formulation and treatment. Pappos's theorem in this kind of formulation can be stated as: if (a, e, c), (b, d, f), (a, b, x), (e, d, x), (a, f, y), (c, d, y), (e, f, z) and (b, c, z) are triples of collinear points then x, y and z are collinear as well. Also this formulation needs non-degeneracy conditions: the theorem is false in the very degenerate situation where we identify the points $a, \dots, f$ and put x, y and z anywhere in a non-collinear situation in the projective plane. A more general form of a construction-free formulation including non-degeneracy conditions is restated from [2]:

Definition 7.1. *A* real projective incidence assertion $\mathcal{T}$ *on a finite point set* $\mathbf{P}$ *is a triple* $(\mathbf{H}, \mathbf{B}, C)$ *such that:*

- $\mathbf{H}, \mathbf{B} \subset \big\{\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \mid \mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{P}$ *pairwise distinct*$\big\}$
- $C = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *with* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{P}$ *pairwise distinct and* $C \notin \mathbf{H}$.

A point configuration P in $\mathbb{RP}^2$ is called an instance *of* $\mathcal{T}$ *if* $[\mathbf{i}, \mathbf{j}, \mathbf{k}]_P := \Phi^P([\mathbf{i}, \mathbf{j}, \mathbf{k}]) = 0$ *for all* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \in \mathbf{H}$ *and* $[\mathbf{i}, \mathbf{j}, \mathbf{k}]_P \neq 0$ *for all* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \in \mathbf{B}$. *If in addition for* *every instance P of $\mathcal{T}$ also* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]_P = 0$ *holds, then $\mathcal{T}$ is called a* valid assertion *or a* theorem.

Observe that although the formulation in Definition 7.1 is similar to the one of the Grassmann-Cayley algebra, the indices in $\mathbf{P}$ play a different role. In Grassmann-Cayley algebra, they are considered to play the role of *free generic points. Depending* on points in $\mathbf{P}$, new points can be constructed. In contrast to this, Definition 7.1 uses a more symmetric formulation that does not need the distinction between free and dependent points. So it might allow for changing the underlying construction without changing the validity of the theorem.

Non-degeneracy conditions are cumbersome to deal with. The proposed formulation as triples of points that are considered to be non-collinear has the advantage that they can be easily formulated and interpreted. It turns out, that there is a natural choice of $\mathbf{B}$ depending on $\mathbf{H}$ and C. This natural choice ensures essentially that points on the same line may not coincide, as long as this line is relevant to the formulation of the theorem. Furthermore, the dual shall also hold: two (distinct) lines sharing a point in $\mathbf{P}$ may not coincide.

For some considerations in Part II we restate some definitions from [2]. Let $\mathbf{A} := \mathbf{H} \cup \{C\}$. Assume w.l.o.g. that $\mathbf{A}$ is *saturated*, i.e. for

$$\{a, b, c\} \in \mathbf{A} \text{ and } \{b, c, d\} \in \mathbf{A} \implies \{a, b, d\} \in \mathbf{A}.$$

We define

$$f(\{a, b, c\}) := \{i \mid \{a, b, i\} \in \mathbf{A}\}$$

which can be thought as the line supported by a, b and c. f is well defined due to the saturation of $\mathbf{A}$. The set of all *derived* lines is collected in a set

$$\mathcal{G} := \big\{f(\{a, b, c\}) \mid \{a, b, c\} \in \mathbf{A}\big\}.$$

These notions allow us to define the natural choice for $\mathbf{B}$ mentioned above:

$$\mathbf{B}(\mathbf{A}) := \Big\{\{a, b, c\} \;\big|\; \exists G, H \in \mathcal{G} \text{ with } |G \cap H| = 1 \text{ and } a \in G \setminus H,\ b \in H \setminus G,\ c \in H \cup G\Big\}.$$

$\mathbf{B}(\mathbf{A})$ is meant to be a default value for non-degeneracy conditions and depends only on the collinearities and the conclusion of a (generic drawing of) an incidence theorem. As an example, we consider Theorem 9.1 which is treated in more detail later. It is illustrated in Figure 9.1. We have $\{\mathbf{h}, \mathbf{g}, \mathbf{p}\} \in \mathbf{B}(\mathbf{A})$ since $\{\mathbf{0}, \mathbf{g}, \mathbf{h}, \mathbf{k}\}$ and $\{\infty, \mathbf{k}, \mathbf{m}, \mathbf{p}\}$ are in $\mathcal{G}$ and intersect in the single element $\mathbf{k}$. On the other hand, $\{\mathbf{h}, \mathbf{m}, \mathbf{c}\}$ is not an element of $\mathbf{B}(\mathbf{A})$. It also turns out in [2] that this natural choice of $\mathbf{B}$ is well-suited for the purposes of binomial proofs and Ceva-Menelaus proofs investigated there: the equivalence of both proving methods. The idea is already presented in [94]. In Chapter 9, we will exemplify the implication which starts with binomial proofs and ends up with Ceva-Menelaus proofs. This implication is the more interesting one.

The definition of $\mathbf{B}(\mathbf{A})$ is crucial for this translation process. Since we want to derive a geometric reasoning working in any instance, we have to restrict to elements in $\mathbf{B}(\mathbf{A})$. We will see in Section 10.3 that the shape of $\mathbf{B}(\mathbf{A})$ influences the way some *cycles* can be decomposed. So a few comments on why we believe the definition makes sense are appropriate. First of all, the fact that collinear points shall not coincide (and its dual version) seems natural to us. The non-degeneracy conditions may be in some cases a little too restrictive. However, they do not fix all points in the construction and some degeneracies are still allowed. Having in mind that $\mathbf{B}(\mathbf{A})$ can be stated automatically, this seems to be a good tradeoff. Furthermore, as mentioned before, the conditions are strong enough to do the translation process from biquadratic proofs to Ceva-Menelaus proofs.

There is an additional advantage of the construction-free approach for formulating incidence theorems: one can also require to have relations between points that are not ruler-constructible. This happens e.g. if one implicitly needs to have (irrational) roots of real numbers. Von Staudt constructions (see Section 3.7) allow for checking whether we have a root but not to construct one. This happens e.g. in the projective version of a regular pentagon. It can be summarized as follows: consider five points determining a pentagon. Construct all intersections of an edge and the diagonal opposite to it. Whenever four of these intersections lie on a common line, the fifth does lie on the same line (see Figure 7.3). This statement can be made a rigorous one in the sense of Definition 7.1 but no construction sequence can be given. However, there exists a binomial proof for it ([93, Ex. 14] and [47] for another occurrence of the configuration). For other non-rational incidence types, e.g. configurations without construction sequence see [10, Chap. 5].

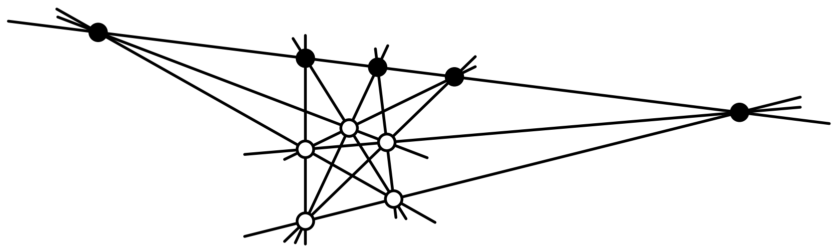


Figure 7.3.: A non-constructible incidence theorem: as soon as 4 of the black points are on a line l, the last black point is on l, too.

Part II.

Γ-Cycles

8. Introduction to Γ-Cycles

The present part of the thesis is dedicated to so-called Γ-cycles. They emerged in the investigation of binomial proofs (see [92, 93, 9, 30]). This kind of investigation can be found in [94, 2]. An example of it will be given in Chapter 9. There, the area principle is used to translate an algebraic proof of an incidence theorem into an argument working with oriented length ratios. These oriented length ratios are structured in such a way that they can be interpreted as being induced by configurations of Ceva and Menelaus. The overall result is a collection of Ceva and Menelaus triangles which are glued together. This resultis in the structure of a triangulated oriented combinatorial 2-manifold. The existence of all but one Ceva or Menelaus triangles implies the existence of the last one. This gives another proof for the incidence theorem in question. It is formulated in the language of length ratios. At some point during the translation process, one is left with a collection of (possibly long) cycles in a base graph Γ (defined in detail in Chapter 10). In [2] the cycles can be decomposed into triangles by adding new points to the overall setup. These triangles indicate the configurations of Ceva and Menelaus. However, the number of additional triangles needed might be big. Therefore it makes sense to consider the cycles on its own. This motivates the treatment of Γ-cycles. All cycles obtained in the translation process are in fact Γ-cycles.

All edges in the overall base graph Γ can be interpreted via the area principle (see Section 3.1). The edges can be considered as unbreakable ratios of determinants and as elements of a group $\mathcal{RG}_{\mathbf{B}}$ defined in Chapter 10. The group is related to a group $\mathcal{BG}_{\mathbf{B}}$ encoding all products of bracket (and its inverses). This relation is also used in the translation process of a binomial proof. It is based on theory and definitions due to Dress and Wenzel (see [38, 40, 41, 39]). Their theory is developed for very general setups of finite or infinite matroids with coefficients. Our more specific setup is the projective plane. However, we are in a situation, where not the full information about triples of points being collinear or not being collinear is present.

As the edges of Γ, each edge of a Γ-cycle can be interpreted as a length ratio in Euclidean geometry (with some additional assumptions on the coordinates). The fact, that the length ratios are induced by cycles yields that the overall product of all length ratios induced by the cycle equals $+1$ or -1. This is an identity in Euclidean geometry called a Γ-*cycle theorem* and is introduced in Section 10.4.1. Γ-cycle theorems can be understood as generalization of Ceva's and Menelaus's theorem.

Similar theorems are also considered by Grünbaum and Shephard (see [53, 52, 54]). The connections of Γ-cycle theorems and their results is given in Chapter 13.

The investigation of Γ-cycles themselves first focuses on irreducibility. The concept emerges from the translation process described before and is also considered by Dress and Wenzel. A Γ-cycle is said to be weakly irreducible whenever it cannot be obtained by combining two shorter Γ-cycles which use only vertices of the first Γ-cycle. For the definition of strongly irreducibility one includes additional information about non-degeneracy conditions as it is possible when treating incidence theorems. Since this information is very specific we focus on weakly irreducibility and omit the word "weakly". These restrictions allow for structuring the irreducible Γ-cycles. We give a table representation which provides a good visualization and allows for counting irreducible Γ-cycles of fixed length (see Chapter 11). The table representation and the suggestion of an underlying structure is due to Richter-Gebert. Furthermore, this coding of Γ-cycles allows for extracting additional information about Γ-cycles. This information (see Section 12.2) is helpful when further *using* the Γ-cycle (see Section 12.1). By *using* Γ-cycles we mean imposing additional incidence relations. This allows for more complicated theorems about oriented length ratios (see Chapter 13) and also for constructing new projective incidence theorems. This construction is possible since one has good *geometric* control of the cancellation process by glueing length ratios. This gluing process is the reason why Chapter 10 also includes definitions of groups capturing informations about triples of points being collinear. These identities are also present in the expositions of Dress and Wenzel.

9. Motivation: Example on How to Transform a Binomial Proof to Γ-Cycles

The topic of Γ-cycles treated in this part of the thesis emerged from the investigation of the relations between *biquadratic proofs* (introduced by Richter-Gebert in [92, 93, 9, 30]) and *Ceva-Menelaus* proofs (see [95, 94, 2]). Since this emergence is essential to the understanding of the relevance and concrete treatment of Γ-cycles, we will give another brief introduction with the help of a concrete example. Other examples can be found in [94, 2]. In this introduction, all considerations are exemplary. They generalize as it was shown in [2].

The example considered here is not the smallest one possible. However, its size may give an impression on how the general case behaves. Furthermore, the construction considered fits quite well into the topics of this thesis and can be extended to show the limitations of the proofing method (see Section 9.4).

Now for the example. Consider Figure 9.1. Assuming all incidences except the one indicated by the red line implies the existence of the red line. This is due to the fact that addition is associative. To see this, recall Section 3.7 and observe:

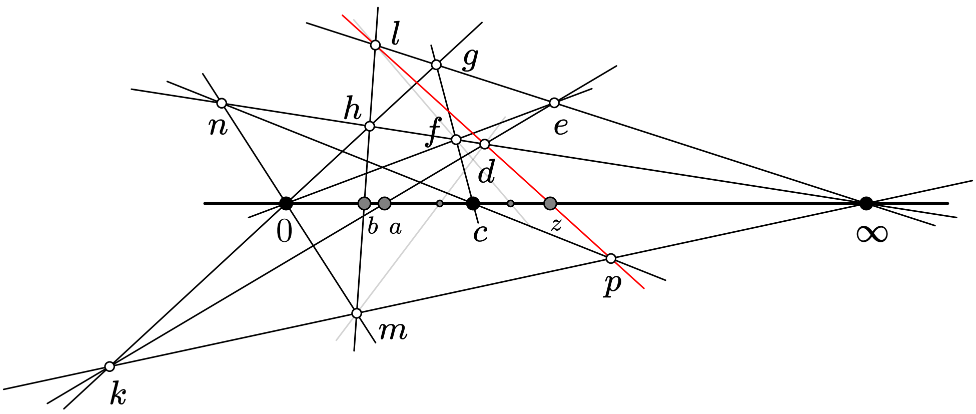


Figure 9.1.: $(a+b)+c = a+(b+c)$ encoded in geometry via von Staudt constructions.

the quadrilateral with vertices d, k, m and h induces a quadrilateral set such that the small gray point on the line spanned by m and d represents the sum of a and b. The quadrilateral set induced by d, m, n and p adds c to it. This means that $\mathbf{meet}(\mathbf{join}(d,p),\mathbf{join}(0,\infty))$ represents the sum $(a+b)+c$. Similarly, the quadrilateral sets induced by f, g, h and l and the one induced by d, e, f and l together compute $a+(b+c)$. The fact that both values are identical is encoded in the fact that the lines $\mathbf{join}(d,p)$ and $\mathbf{join}(d,l)$ meet $\mathbf{join}(0,\infty)$ in the same point. In other words (and in a non-degenerate situation): d, l and p are collinear. All gray points do not play a crucial role in the construction and can be omitted. This results in the following formulation of our toy example:

Theorem 9.1. *Let* $\mathbf{P} = \{\mathbf{0},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f},\mathbf{g},\mathbf{h},\infty,\mathbf{k},\mathbf{l},\mathbf{m},\mathbf{n},\mathbf{p}\}$*, let* $\mathbf{H}$ *consist of all 3-element subsets of the sets*

$$\{\infty,\mathbf{0},\mathbf{c}\},\{\mathbf{0},\mathbf{e},\mathbf{f}\},\{\mathbf{0},\mathbf{m},\mathbf{n}\},\{\mathbf{c},\mathbf{f},\mathbf{g}\},\{\mathbf{c},\mathbf{n},\mathbf{p}\},\{\mathbf{d},\mathbf{e},\mathbf{k}\},\{\mathbf{h},\mathbf{l},\mathbf{m}\},$$
$$\{\mathbf{0},\mathbf{g},\mathbf{h},\mathbf{k}\},\{\infty,\mathbf{e},\mathbf{g},\mathbf{l}\},\{\infty,\mathbf{k},\mathbf{m},\mathbf{p}\},\{\infty,\mathbf{d},\mathbf{f},\mathbf{h},\mathbf{n}\}.$$

Let $C = \{\mathbf{d},\mathbf{l},\mathbf{p}\}$*. In the sense of Definition 7.1,* $\mathcal{A} := (\mathbf{H},\mathbf{B} = \mathbf{B}(\mathbf{A}),C)$ *is an incidence theorem.*

9.1. Formal Language for Stating Biquadratic Proofs

The formalization of biquadratic or binomial proofs done in [2] points out that the domain of its calculations is a multiplicative group generated by formal determinants of (free *and* dependent) points together with a symbol ϵ that plays the role of a formal -1. We will need the following definition as a setup for symbolic calculations, e.g. for binomial proofs:

Definition 9.2. *Let* $(\mathbf{H},\mathbf{B},C)$ *be a real projective incidence theorem on* $\mathbf{P}$*. Let* $\mathcal{BG}_{\mathbf{B}}$ *be the quotient of the free multiplicatively written abelian group generated by*

$$\big\{[\mathbf{a},\mathbf{b},\mathbf{c}] \;\big|\; \{\mathbf{a},\mathbf{b},\mathbf{c}\} \in \mathbf{B}\big\} \cup \big\{\epsilon\big\}$$

and the group generated by

$$\left\{ \frac{[\mathbf{a},\mathbf{b},\mathbf{c}]}{[\mathbf{b},\mathbf{c},\mathbf{a}]} \;\middle|\; \{\mathbf{a},\mathbf{b},\mathbf{c}\} \in \mathbf{B} \right\} \;\cup\; \left\{ \epsilon \cdot \frac{[\mathbf{a},\mathbf{b},\mathbf{c}]}{[\mathbf{b},\mathbf{a},\mathbf{c}]} \;\middle|\; \{\mathbf{a},\mathbf{b},\mathbf{c}\} \in \mathbf{B} \right\} \cup \left\{ \epsilon^2 \right\}.$$

This definition allows for formal multiplicative calculations with determinants that are supposed to be non-zero. The permutational properties of the determinants are implemented. We define a subgroup of $\mathcal{BG}_{\mathbf{B}}$ encoding properties of the collinearities encoded in $\mathbf{H}$:

Definition 9.3. *Let* $(\mathbf{H}, \mathbf{B}, C)$ *be a real incidence projective theorem on* $\mathbf{P}$. *Let* $\mathcal{HG}_{\mathbf{B},\mathbf{H}}$ *be the subgroup of* $\mathcal{BG}_{\mathbf{B}}$ *generated by*

$$\left\{ \frac{[\mathbf{a},\mathbf{b},\mathbf{x}][\mathbf{a},\mathbf{c},\mathbf{y}]}{[\mathbf{a},\mathbf{b},\mathbf{y}][\mathbf{a},\mathbf{c},\mathbf{x}]} \in \mathcal{BG}_{\mathbf{B}} \;\middle|\; \{\mathbf{a},\mathbf{b},\mathbf{c}\} \in \mathbf{H},\ \mathbf{x},\mathbf{y} \in \mathbf{P} \right\}.$$

The generating elements of $\mathcal{HG}_{\mathbf{B},\mathbf{H}}$ *are called* biquadratic fractions.

Proposition 9.4. *Let* P *be a point configuration in* $\mathbb{RP}^2$ *that is an instance of a real projective incidence theorem* $(\mathbf{H}, \mathbf{B}, C)$. *There is a well-defined homomorphism* $\Phi^P : \mathcal{BG}_{\mathbf{B}} \to (\mathbb{R} \setminus \{0\}, \cdot)$ *which is the extension of the map* Φ^P *defined in Definition 4.5 and which maps* ϵ *to* -1. *Furthermore we have*

$$\Phi^P\big(\mathcal{HG}_{\mathbf{B},\mathbf{H}}\big) = \{\,1\,\}.$$

Proof. It suffices to show the last statement. Let $\frac{[\mathbf{a},\mathbf{b},\mathbf{x}][\mathbf{a},\mathbf{c},\mathbf{y}]}{[\mathbf{a},\mathbf{b},\mathbf{y}][\mathbf{a},\mathbf{c},\mathbf{x}]} \in \mathcal{BG}_{\mathbf{B}}$ with $\{\mathbf{a},\mathbf{b},\mathbf{c}\} \in \mathbf{H}$, $\mathbf{x},\mathbf{y} \in \mathbf{P}$. By writing a, b, c, x and y for the points in P corresponding to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$ and $\mathbf{y}$ we have that

$$\Phi^P\left(\frac{[\mathbf{a},\mathbf{b},\mathbf{x}][\mathbf{a},\mathbf{c},\mathbf{y}]}{[\mathbf{a},\mathbf{b},\mathbf{y}][\mathbf{a},\mathbf{c},\mathbf{x}]}\right) = \frac{[\mathbf{a},\mathbf{b},\mathbf{x}]_P[\mathbf{a},\mathbf{c},\mathbf{y}]_P}{[\mathbf{a},\mathbf{b},\mathbf{y}]_P[\mathbf{a},\mathbf{c},\mathbf{x}]_P} = \frac{|a,b,x| \cdot |a,c,y|}{|a,b,y| \cdot |a,c,x|}$$

and it suffices to show that

$$|a,b,x| \cdot |a,c,y| - |a,b,y| \cdot |a,c,x| = 0.$$

This is an easy consequence of the 3-term Grassmann-Plücker relation (4.3) implying that

$$|a,b,x| \cdot |a,c,y| - |a,b,y| \cdot |a,c,x| + |a,b,c| \cdot |a,y,x| = 0.$$

Since $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is in $\mathbf{H}$ and P is an instance, we have $|a,b,c| = 0$ which proofs the claim. $\square$

9.2. A Biquadratic Proof for the Given Example

A binomial proof for $\mathcal{A}$ is formulated within this language is given by:

$$\frac{[\mathbf{0},\infty,\mathbf{g}] \cdot [\infty,\mathbf{c},\mathbf{n}]}{[\infty,\mathbf{c},\mathbf{g}] \cdot [\mathbf{0},\infty,\mathbf{n}]} \quad \{\mathbf{0},\infty,\mathbf{c}\} \qquad \frac{[\infty,\mathbf{d},\mathbf{g}] \cdot [\mathbf{0},\infty,\mathbf{e}]}{[\infty,\mathbf{d},\mathbf{e}] \cdot [\mathbf{0},\infty,\mathbf{g}]} \quad \{\infty,\mathbf{e},\mathbf{g},\mathbf{l}\}$$

$$\frac{[\mathbf{0},\infty,\mathbf{m}] \cdot [\mathbf{0},\mathbf{c},\mathbf{k}]}{[\mathbf{0},\mathbf{c},\mathbf{m}] \cdot [\mathbf{0},\infty,\mathbf{k}]} \quad \{\mathbf{0},\infty,\mathbf{c}\} \qquad \frac{[\infty,\mathbf{n},\mathbf{p}] \cdot [\mathbf{0},\mathbf{c},\mathbf{n}]}{[\infty,\mathbf{c},\mathbf{n}] \cdot [\mathbf{0},\mathbf{n},\mathbf{p}]} \quad \{\mathbf{c},\mathbf{n},\mathbf{p}\}$$

$\frac{[\mathbf{0},\infty,\mathbf{f}]\cdot[\mathbf{0},\mathbf{c},\mathbf{e}]}{[\mathbf{0},\infty,\mathbf{e}]\cdot[\mathbf{0},\mathbf{c},\mathbf{f}]}$	$\{\mathbf{0},\infty,\mathbf{c}\}$	$\frac{[\mathbf{0},\mathbf{d},\mathbf{k}]\cdot[\infty,\mathbf{e},\mathbf{k}]}{[\infty,\mathbf{d},\mathbf{k}]\cdot[\mathbf{0},\mathbf{e},\mathbf{k}]}$	$\{\mathbf{d},\mathbf{e},\mathbf{k}\}$
$\frac{[\infty,\mathbf{d},\mathbf{k}]\cdot[\mathbf{d},\mathbf{h},\mathbf{m}]}{[\mathbf{d},\mathbf{h},\mathbf{k}]\cdot[\infty,\mathbf{d},\mathbf{m}]}$	$\{\infty,\mathbf{d},\mathbf{f},\mathbf{h},\mathbf{n}\}$	$\epsilon\cdot\frac{[\mathbf{0},\infty,\mathbf{k}]\cdot[\mathbf{k},\mathbf{l},\mathbf{m}]}{[\infty,\mathbf{k},\mathbf{l}]\cdot[\mathbf{0},\mathbf{k},\mathbf{m}]}$	$\{\infty,\mathbf{k},\mathbf{m},\mathbf{p}\}$
$\frac{[\mathbf{0},\infty,\mathbf{n}]\cdot[\infty,\mathbf{d},\mathbf{p}]}{[\infty,\mathbf{n},\mathbf{p}]\cdot[\mathbf{0},\infty,\mathbf{d}]}$	$\{\infty,\mathbf{d},\mathbf{f},\mathbf{h},\mathbf{n}\}$	$\epsilon\cdot\frac{[\mathbf{0},\mathbf{m},\mathbf{p}]\cdot[\infty,\mathbf{d},\mathbf{m}]}{[\mathbf{0},\infty,\mathbf{m}]\cdot[\mathbf{d},\mathbf{m},\mathbf{p}]}$	$\{\infty,\mathbf{k},\mathbf{m},\mathbf{p}\}$
$\frac{[\mathbf{0},\infty,\mathbf{d}]\cdot[\infty,\mathbf{f},\mathbf{g}]}{[\infty,\mathbf{d},\mathbf{g}]\cdot[\mathbf{0},\infty,\mathbf{f}]}$	$\{\infty,\mathbf{d},\mathbf{f},\mathbf{h},\mathbf{n}\}$	$\frac{[\mathbf{0},\mathbf{n},\mathbf{p}]\cdot[\mathbf{0},\mathbf{c},\mathbf{m}]}{[\mathbf{0},\mathbf{c},\mathbf{n}]\cdot[\mathbf{0},\mathbf{m},\mathbf{p}]}$	$\{\mathbf{0},\mathbf{m},\mathbf{n}\}$
$\epsilon\cdot\frac{[\mathbf{d},\mathbf{h},\mathbf{k}]\cdot[\mathbf{0},\mathbf{k},\mathbf{m}]}{[\mathbf{0},\mathbf{d},\mathbf{k}]\cdot[\mathbf{h},\mathbf{k},\mathbf{m}]}$	$\{\mathbf{0},\mathbf{g},\mathbf{h},\mathbf{k}\}$	$\epsilon\cdot\frac{[\mathbf{d},\mathbf{l},\mathbf{m}]\cdot[\mathbf{h},\mathbf{k},\mathbf{m}]}{[\mathbf{d},\mathbf{h},\mathbf{m}]\cdot[\mathbf{k},\mathbf{l},\mathbf{m}]}$	$\{\mathbf{h},\mathbf{l},\mathbf{m}\}$
$\frac{[\mathbf{0},\mathbf{c},\mathbf{h}]\cdot[\mathbf{0},\mathbf{e},\mathbf{k}]}{[\mathbf{0},\mathbf{e},\mathbf{h}]\cdot[\mathbf{0},\mathbf{c},\mathbf{k}]}$	$\{\mathbf{0},\mathbf{g},\mathbf{h},\mathbf{k}\}$	$\frac{[\mathbf{0},\mathbf{e},\mathbf{h}]\cdot[\mathbf{0},\mathbf{c},\mathbf{f}]}{[\mathbf{0},\mathbf{f},\mathbf{h}]\cdot[\mathbf{0},\mathbf{c},\mathbf{e}]}$	$\{\mathbf{0},\mathbf{e},\mathbf{f}\}$
$\frac{[\mathbf{0},\mathbf{f},\mathbf{h}]\cdot[\mathbf{0},\mathbf{c},\mathbf{g}]}{[\mathbf{0},\mathbf{c},\mathbf{h}]\cdot[\mathbf{0},\mathbf{f},\mathbf{g}]}$	$\{\mathbf{0},\mathbf{g},\mathbf{h},\mathbf{k}\}$	$\frac{[\mathbf{0},\mathbf{f},\mathbf{g}]\cdot[\infty,\mathbf{c},\mathbf{g}]}{[\mathbf{0},\mathbf{c},\mathbf{g}]\cdot[\infty,\mathbf{f},\mathbf{g}]}$	$\{\mathbf{c},\mathbf{f},\mathbf{g}\}$
$\epsilon\cdot\frac{[\infty,\mathbf{k},\mathbf{l}]\cdot[\infty,\mathbf{d},\mathbf{e}]}{[\infty,\mathbf{e},\mathbf{k}]\cdot[\infty,\mathbf{d},\mathbf{l}]}$	$\{\infty,\mathbf{e},\mathbf{g},\mathbf{l}\}$		

This is a list of elements of $\mathcal{HG}_{\mathbf{B},\mathbf{H}}$ together with a certificate for the membership. This certificate are sets of collinear points in the way they are given in Theorem 9.1. The product of all these biquadratic fractions is an element of $\mathcal{HG}_{\mathbf{B},\mathbf{H}}$. After canceling it is

$$\epsilon\cdot\frac{[\mathbf{d},\mathbf{l},\mathbf{m}]\cdot[\infty,\mathbf{d},\mathbf{p}]}{[\mathbf{d},\mathbf{m},\mathbf{p}]\cdot[\infty,\mathbf{d},\mathbf{l}]}.$$

For an instance P and using non-bold letters as point names for the corresponding bold letters in $\mathbf{P}$ and due to Proposition 9.4 we have

$$|d,l,m|\cdot|\infty,d,p|+|d,m,p|\cdot|\infty,d,l|=0.$$

Again, by a 3-termed Grassmann-Plücker relation it holds that

$$|d,l,m|\cdot|d,\infty,p|-|d,l,\infty|\cdot|d,m,p|+|d,l,p|\cdot|d,m,\infty|=0.$$

Adding up both equations yields

$$|d,l,p|\cdot|d,m,\infty|=0.$$

Since we have that $\{\mathbf{d},\mathbf{m},\infty\}\in\mathbf{B}=\mathbf{B}(\mathbf{A})$ and since P is an instance, we know that $|d,m,\infty|\neq 0$ and therefore it must hold that $|d,l,p|=0$ which proofs $\mathcal{A}$.

Generally and roughly speaking, a binomial proof is a collection of biquadratic fractions whose product also has the structure of a biquadratic fraction. This way, an additional collinearity can be concluded. It should be remarked that this proof was obtained by an automatic prover given in [108]. It is based on the theory given in [93] and uses a substitution method for solving the resulting linear equation system. This procedure happens to be good for producing short proofs.

9.3. Interpreting a Biquadratic Proof

Biquadratic proofs are concise and human checkable. However, in the form just presented, there is less geometric intuition attached to them. One could draw a Euclidean picture and interpret bracket as (scaled) oriented triangle areas. However, we think that this is not very intuitive. A concept that is easier to deal with are oriented length ratios. When applicable, the area principle (see Section 3.1) provides an translation from ratios of brackets to oriented length ratios. An analysis of a generic biquadratic fraction

$$\frac{[\mathbf{a},\mathbf{b},\mathbf{x}][\mathbf{a},\mathbf{c},\mathbf{y}]}{[\mathbf{a},\mathbf{b},\mathbf{y}][\mathbf{a},\mathbf{c},\mathbf{x}]}$$

shows that the area principle *can* be applied. Furthermore, it can be applied in two different ways induced by two different *splittings* of the fraction:

$$\frac{[\mathbf{a},\mathbf{b},\mathbf{x}]}{[\mathbf{a},\mathbf{b},\mathbf{y}]}\cdot\frac{[\mathbf{a},\mathbf{c},\mathbf{y}]}{[\mathbf{a},\mathbf{c},\mathbf{x}]} \quad \text{and} \quad \frac{[\mathbf{a},\mathbf{b},\mathbf{x}]}{[\mathbf{a},\mathbf{c},\mathbf{x}]}\cdot\frac{[\mathbf{a},\mathbf{c},\mathbf{y}]}{[\mathbf{a},\mathbf{b},\mathbf{y}]}.$$

We choose to split the biquadratic fractions in the binomial proof of $\mathcal{A}$ in the first way indicated. This results in considering the products of ratios

1. $\frac{[\mathbf{0},\infty,\mathbf{g}]}{[\infty,\mathbf{c},\mathbf{g}]}\cdot\frac{[\infty,\mathbf{c},\mathbf{n}]}{[\mathbf{0},\infty,\mathbf{n}]}$

2. $\frac{[\mathbf{0},\infty,\mathbf{m}]}{[\mathbf{0},\mathbf{c},\ \mathbf{m}]}\cdot\frac{[\mathbf{0},\mathbf{c},\ \mathbf{k}]}{[\mathbf{0},\infty,\mathbf{k}]}$

3. $\frac{[\mathbf{0},\infty,\mathbf{f}]}{[\mathbf{0},\infty,\mathbf{e}]}\cdot\frac{[\mathbf{0},\mathbf{c},\mathbf{e}]}{[\mathbf{0},\mathbf{c},\mathbf{f}]}$

4. $\frac{[\infty,\mathbf{d},\mathbf{k}]}{[\mathbf{d},\mathbf{h},\ \mathbf{k}]}\cdot\frac{[\mathbf{d},\mathbf{h},\ \mathbf{m}]}{[\infty,\mathbf{d},\mathbf{m}]}$

5. $\frac{[\mathbf{0},\infty,\mathbf{n}]}{[\infty,\mathbf{n},\mathbf{p}]}\cdot\frac{[\infty,\mathbf{d},\mathbf{p}]}{[\mathbf{0},\infty,\mathbf{d}]}$

11. $\frac{[\infty,\mathbf{d},\mathbf{g}]}{[\infty,\mathbf{d},\mathbf{e}]}\cdot\frac{[\mathbf{0},\infty,\mathbf{e}]}{[\mathbf{0},\infty,\mathbf{g}]}$

12. $\frac{[\infty,\mathbf{n},\mathbf{p}]}{[\infty,\mathbf{c},\ \mathbf{n}]}\cdot\frac{[\mathbf{0},\mathbf{c},\ \mathbf{n}]}{[\mathbf{0},\mathbf{n},\mathbf{p}]}$

13. $\frac{[\mathbf{0},\mathbf{d},\ \mathbf{k}]}{[\infty,\mathbf{d},\mathbf{k}]}\cdot\frac{[\infty,\mathbf{e},\mathbf{k}]}{[\mathbf{0},\mathbf{e},\ \mathbf{k}]}$

14. $\epsilon\cdot\frac{[\mathbf{0},\infty,\mathbf{k}]}{[\infty,\mathbf{k},\ \mathbf{l}]}\cdot\frac{[\mathbf{k},\mathbf{l},\ \mathbf{m}]}{[\mathbf{0},\mathbf{k},\mathbf{m}]}$

15. $\epsilon\cdot\frac{[\mathbf{0},\mathbf{m},\ \mathbf{p}]}{[\mathbf{0},\infty,\mathbf{m}]}\cdot\frac{[\infty,\mathbf{d},\mathbf{m}]}{[\mathbf{d},\mathbf{m},\ \mathbf{p}]}$

6.	$\frac{[\mathbf{0},\infty,\mathbf{d}]}{[\infty,\mathbf{d},\mathbf{g}]} \cdot \frac{[\infty,\mathbf{f},\mathbf{g}]}{[\mathbf{0},\infty,\mathbf{f}]}$	16.	$\frac{[\mathbf{0},\mathbf{n},\mathbf{p}]}{[\mathbf{0},\mathbf{c},\mathbf{n}]} \cdot \frac{[\mathbf{0},\mathbf{c},\mathbf{m}]}{[\mathbf{0},\mathbf{m},\mathbf{p}]}$
7.	$\epsilon\cdot\frac{[\mathbf{d},\mathbf{h},\mathbf{k}]}{[\mathbf{0},\mathbf{d},\mathbf{k}]} \cdot \frac{[\mathbf{0},\mathbf{k},\mathbf{m}]}{[\mathbf{h},\mathbf{k},\mathbf{m}]}$	17.	$\epsilon\cdot\frac{[\mathbf{d},\mathbf{l},\mathbf{m}]}{[\mathbf{d},\mathbf{h},\mathbf{m}]} \cdot \frac{[\mathbf{h},\mathbf{k},\mathbf{m}]}{[\mathbf{k},\mathbf{l},\mathbf{m}]}$
8.	$\frac{[\mathbf{0},\mathbf{c},\mathbf{h}]}{[\mathbf{0},\mathbf{e},\mathbf{h}]} \cdot \frac{[\mathbf{0},\mathbf{e},\mathbf{k}]}{[\mathbf{0},\mathbf{c},\mathbf{k}]}$	18.	$\frac{[\mathbf{0},\mathbf{e},\mathbf{h}]}{[\mathbf{0},\mathbf{f},\mathbf{h}]} \cdot \frac{[\mathbf{0},\mathbf{c},\mathbf{f}]}{[\mathbf{0},\mathbf{c},\mathbf{e}]}$
9.	$\frac{[\mathbf{0},\mathbf{f},\mathbf{h}]}{[\mathbf{0},\mathbf{c},\mathbf{h}]} \cdot \frac{[\mathbf{0},\mathbf{c},\mathbf{g}]}{[\mathbf{0},\mathbf{f},\mathbf{g}]}$	19.	$\frac{[\mathbf{0},\mathbf{f},\mathbf{g}]}{[\mathbf{0},\mathbf{c},\mathbf{g}]} \cdot \frac{[\infty,\mathbf{c},\mathbf{g}]}{[\infty,\mathbf{f},\mathbf{g}]}$
10.	$\epsilon\cdot\frac{[\infty,\mathbf{k},\mathbf{l}]}{[\infty,\mathbf{e},\mathbf{k}]} \cdot \frac{[\infty,\mathbf{d},\mathbf{e}]}{[\infty,\mathbf{d},\mathbf{l}]}$	20.	$\epsilon\cdot\frac{[\mathbf{d},\mathbf{m},\mathbf{p}]}{[\mathbf{d},\mathbf{l},\mathbf{m}]} \cdot \frac{[\infty,\mathbf{d},\mathbf{l}]}{[\infty,\mathbf{d},\mathbf{p}]}$

where the additional 20th entry is the inverse of the product of the first 19 ones. It has a the same shape, but is not a generating element of $\mathcal{HG}_{\mathbf{B},\mathbf{H}}$. This symmetrizes all following considerations. Every ratio of brackets can be interpreted as an oriented length ratio. From now on, we want to stay in the intuitive world of oriented length ratios. For the moment we forget about the "signs" ϵ, since they can be easily reconstructed. We refer to the above collection, including the 20th entry, as the collection of *split biquadratic fractions* describing the binomial proof of $\mathcal{A}$. It is not at all obvious what canceling of brackets might be in the world of oriented length ratios. So one needs an interpretation of canceling without breaking the interpretation as length ratios. Therefore, the ratios themselves are considered as unbreakable symbols (as it will be done more formally in Definition 10.3). In order to structure the binomial proof, one can find small subpatterns of cancellation. We can even visualize them (see Figure 9.2), which is also the best way to explain them:

Here every edge corresponds to one fraction of a split biquadratic fraction. The number printed at the edges indicates the number of the corresponding biquadratic fraction. The arrows can be considered as being directed from numerators to denominators. Since we neglect sign changes, we can consider the numerator and the denominator to be sets. In Figure 9.2 it is obvious that the arrows arrange themselves in cycles. This is true for every biquadratic proof (see [2]). We want to investigate those cycles and start with 2-cycles. The cycle with vertices $\{\mathbf{0},\mathbf{c},\mathbf{n}\}$ and $\{\mathbf{0},\mathbf{n},\mathbf{p}\}$ can be read as

$$\frac{[\mathbf{0},\mathbf{c},\mathbf{n}]}{[\mathbf{0},\mathbf{n},\mathbf{p}]} \cdot \frac{[\mathbf{0},\mathbf{n},\mathbf{p}]}{[\mathbf{0},\mathbf{c},\mathbf{n}]} = 1,$$

since the brackets in a cycle cancel. Translating this to oriented length ratios states

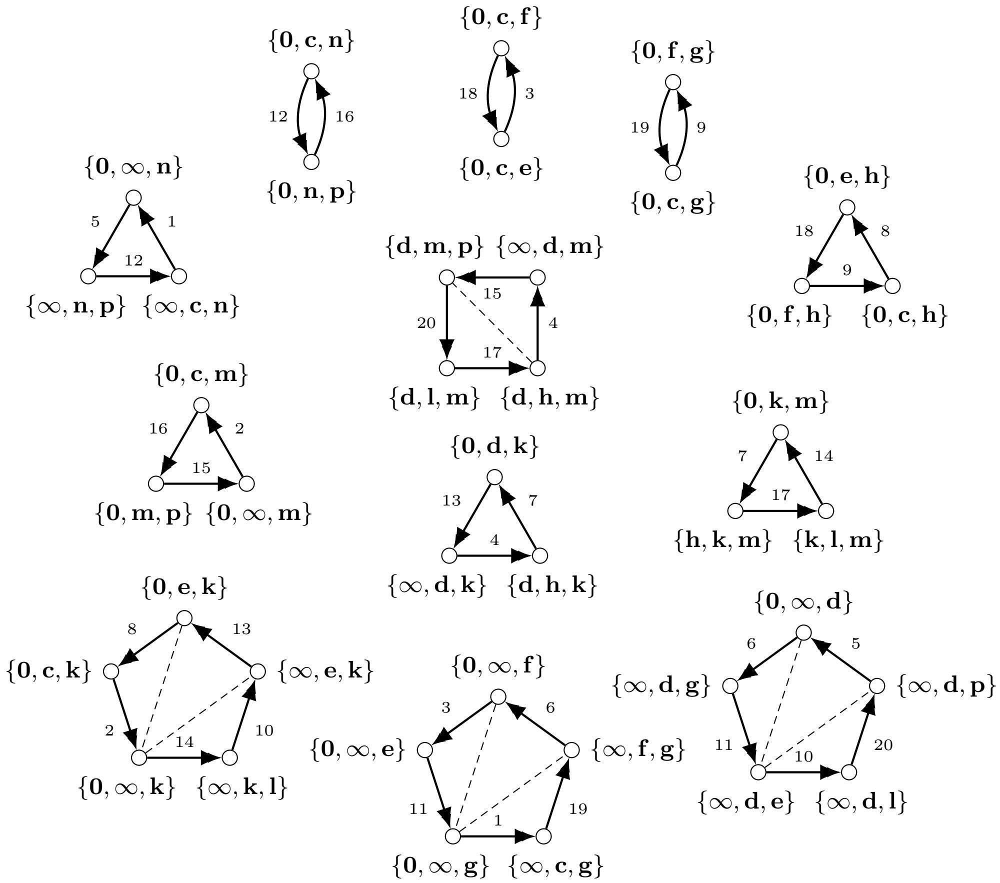


Figure 9.2.: The combinatorics of the edges induced by the split biquadratic fractions describing the binomial proof of $\mathcal{A}$.

that an oriented length ratio multiplied with its inverse equals one. This is intuitive also on the level of oriented length ratios *as well as* on the level of brackets. Therefore, two edges with the same endpoints but pointing in opposite directions can be omitted or added as one likes.

In the following, we will argue that the 3-cycles induce exactly the same structures as either Ceva's or Menelaus's theorem in Section 3.2. Here, in this biquadratic proof of $\mathcal{A}$, there is no need to consider cycles of length greater than three in more detail. This is due to the fact, that additional pairs of arrows pointing in opposite directions can be inserted at the dashed lines in Figure 9.2. These arrows can be introduced since they *can* be interpreted via the area principle. However, this is the point where the interest in Γ-cycles comes form. Before describing Γ-cycles in more detail, we complete the interpretation of the biquadratic proof via length ratios.

For that purpose, observe that we are left with 2-cycles and 3-cycles in Figure 9.2. Each 3-cycle is associated with a Ceva or Menelaus triangle in an instance: let P be an instance which admits Euclidean interpretations of the points. Since incidence relations are projectively invariant, we can furthermore assume that all additional (finitely many) points introduced in the following also admit a Euclidean interpretation. In the instance, we use non-bold letters as point names for the corresponding bold letters in $\mathbf{P}$. Now consider e.g. the two triangles that contain arrows from the biquadratic fraction number six. These triangles are

$$\triangle_1 := (\{\mathbf{f},\mathbf{g},\infty\},\{\mathbf{0},\mathbf{f},\infty\},\{\mathbf{0},\mathbf{g},\infty\}) \text{ and } \triangle_2 := (\{\mathbf{0},\mathbf{d},\infty\},\{\mathbf{d},\mathbf{g},\infty\},\{\mathbf{d},\mathbf{e},\infty\}).$$

The vertices of $\triangle_1$ do have one letter in $\mathbf{P}$ in common and the combinatorics of the area principle (see Section 3.1) applied to the three edges of $\triangle_1$ tell us that in an instance, this is an identity in a Ceva triangle with vertices 0, f and g and centroid ∞ . Similarly, $\triangle_2$ has the combinatorics of a triangle with vertices 0, g and e and a Menelaus line spanned by d and ∞.

Now, the biquadratic fractions tell us, that some length ratios are identical and corresponding triangles can be *glued*: in the case of $\triangle_1$ and $\triangle_2$, both triangles share an edge. The existence of the biquadratic fraction number six

$$\frac{[\mathbf{0},\infty,\mathbf{d}]}{[\infty,\mathbf{d},\mathbf{g}]}\cdot\frac{[\infty,\mathbf{f},\mathbf{g}]}{[\mathbf{0},\infty,\mathbf{f}]}$$

can be interpreted via the area principle and the combinatorics of the construction given in Figure 9.1 or Theorem 9.1:

$$\frac{\overline{0\,h}}{\overline{h\,g}}\cdot\frac{\overline{g\,h}}{\overline{h\,0}}$$

This has to yield 1 by Proposition 9.4. In the present formulation, this is quite obvious. One can easily convince oneself that there can be only two kinds of triangles

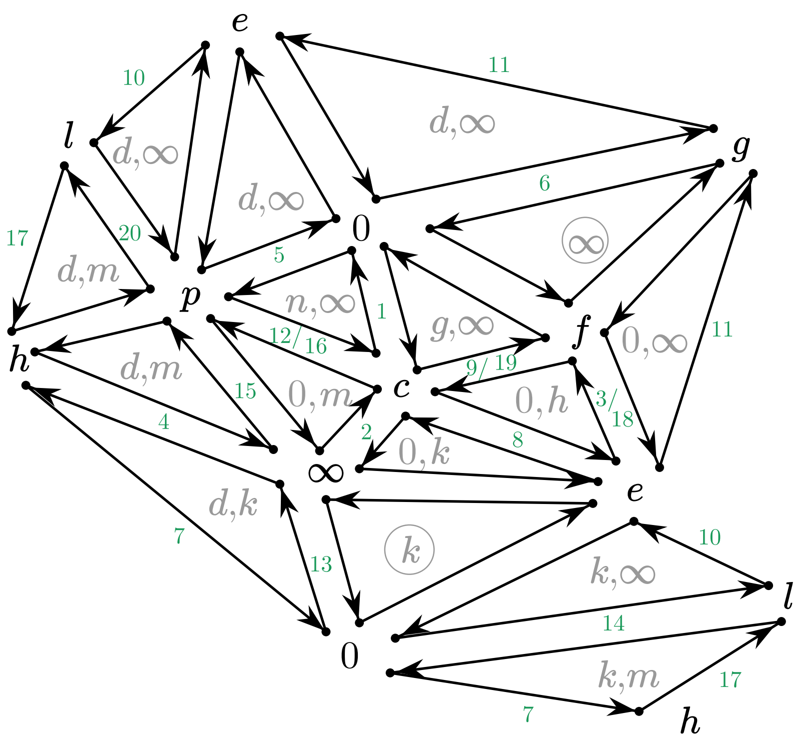


Figure 9.3.: When interpreting the triangles in Figure 9.2 as Ceva and Menelaus triangles (in an instance), the triangles can be considered being glued together via the biquadratic fractions. The numbers indicate which biquadratic fractions induce the glueings of edges.

in Figure 9.2 since these triangle have the property that their edges indicate oriented length ratios. They can be distinguished by the number of point names the vertices have in common.

Translating all triangles in Figure 9.2 to Ceva and Menelaus triangles yields a collection of triangles which are glued together by the interpretation of biquadratic fractions. The combinatorics are shown in Figures 9.3 and 9.4. The arrows are directed in such a way, that in the induced length ratios, the letters at the end of the arrow are located in the numerator and the letters at the arrow tips are located in the denominator of the original biquadratic fraction. Figure 9.3 describes which biquadratic fractions induce the glueings. Two gray letters in a triangle indicate that the triangle is considered to be a Menelaus triangle with vertices given by black labelings. The gray points indicate the points spanning the Menelaus line. A single letter inside a circle indicates the centroid of a Ceva triangle. Edges on the boundary equipped with the same symbols are considered to be glued as well. Therefore, the structure is a topological sphere. The glueings without labeling correspond to dashed lines in Figure 9.2. Edges with two numbers correspond to places where two 2-cycles were glued. E.g. the biquadratic fractions number 12 and 16 are

$$\frac{[\infty,\mathbf{n},\mathbf{p}]}{[\infty,\mathbf{c},\mathbf{n}]}\cdot\frac{[\mathbf{0},\mathbf{c},\mathbf{n}]}{[\mathbf{0},\mathbf{n},\mathbf{p}]} \text{ and } \frac{[\mathbf{0},\mathbf{n},\mathbf{p}]}{[\mathbf{0},\mathbf{c},\mathbf{n}]}\cdot\frac{[\mathbf{0},\mathbf{c},\mathbf{m}]}{[\mathbf{0},\mathbf{m},\mathbf{p}]}$$

and they both express an equality of length ratios with same endpoints and in the product one ratio cancels as a whole. The product implies

$$\frac{|\infty,n,p|}{|\infty,c,n|}\cdot\frac{|0,c,m|}{|0,m,p|}=1 \tag{9.1}$$

in an instance. This identity can easily be checked by means of the area principle: $\frac{[\infty,\mathbf{n},\mathbf{p}]}{[\infty,\mathbf{c},\mathbf{n}]}$ and $\frac{[\mathbf{0},\mathbf{c},\mathbf{m}]}{[\mathbf{0},\mathbf{m},\mathbf{p}]}$ are edges of triangles in Figure 9.2 that express mutually inverse length ratios

$$\frac{\overline{p\,n}}{\overline{n\,c}} \text{ and } \frac{\overline{c\,n}}{\overline{n\,p}}.$$

Since in any biquadratic fraction, both of its ratios always share a common point, an equation corresponding to (9.1) needs to be stated with the help of two biquadratic fractions. For a Euclidean interpretation, the point n of intersection is also important for the combinatorics of Ceva and Menelaus triangles. All points of intersection for all edges are given in Figure 9.3. Therefore seven new points have to be introduced and can be defined as follows:

$$\begin{aligned}
t &= \mathbf{meet}(\mathbf{join}(d,\infty)\mathbf{join}(e,p)), & u &= \mathbf{meet}(\mathbf{join}(c,e),\mathbf{join}(0,k)),\\
v &= \mathbf{meet}(\mathbf{join}(e,0),\mathbf{join}(\infty,k)), & w &= \mathbf{meet}(\mathbf{join}(0,l),\mathbf{join}(\infty,k)),\\
x &= \mathbf{meet}(\mathbf{join}(p,h),\mathbf{join}(d,m)), & y &= \mathbf{meet}(\mathbf{join}(p,0),\mathbf{join}(d,\infty)),\\
a &= \mathbf{meet}(\mathbf{join}(e,k),\mathbf{join}(0,\infty)).
\end{aligned}$$

These intersections exists and do not degenerate since e.g.

$$t = \mathbf{meet}(\mathbf{join}(d,\infty)\mathbf{join}(e,p)) = |d,\ \infty,\ p|\cdot e\ -|d,\ \infty,\ e|\cdot p$$

and $\{\mathbf{d},\infty,\mathbf{p}\}$ and $\{\mathbf{d},\infty,\mathbf{e}\}$ are elements of $\mathbf{B}$.

Based on Figure 9.4, we state either Menelaus's or Ceva's theorem multiply all resulting equations. The arrows pointing in opposite directions along the same edges

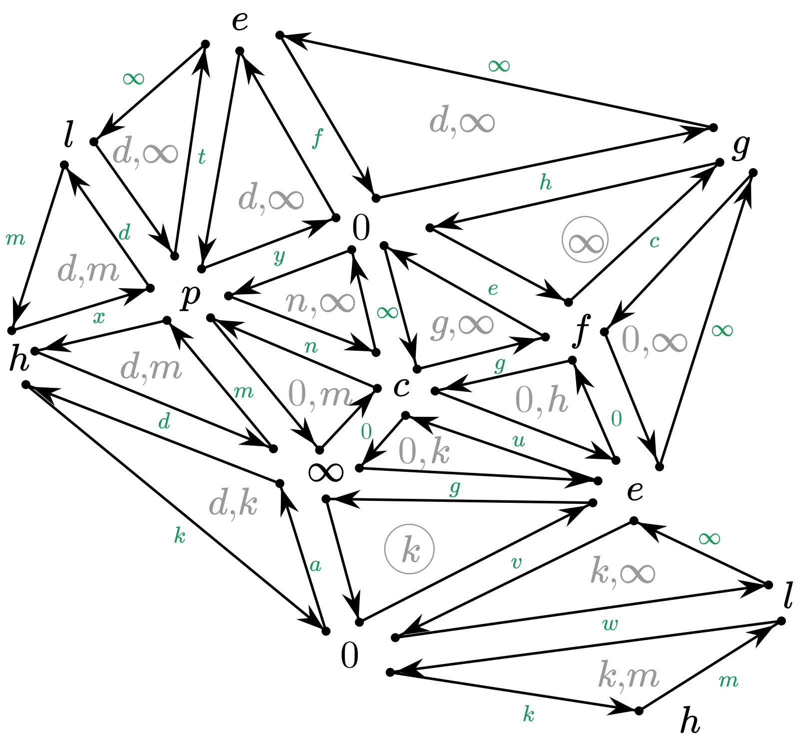


Figure 9.4.: When interpreting the triangles in Figure 9.2 as Ceva and Menelaus triangles (in an instance), the triangles can be considered being glued together via the biquadratic fractions. Interpreting the fractions via the area principle shows that glued edges share the same intermediate point.

imply that in the overall product, length ratios cancel:

$$\frac{\overline{0\,k}}{\overline{k\,h}}\frac{\overline{h\,m}}{\overline{m\,l}}\frac{\overline{l\,w}}{\overline{w\,0}}\cdot\frac{\overline{0\,w}}{\overline{w\,l}}\frac{\overline{l\,\infty}}{\overline{\infty\,e}}\frac{\overline{e\,v}}{\overline{v\,0}}\cdot\frac{\overline{0\,v}}{\overline{v\,e}}\frac{\overline{e\,g}}{\overline{g\,\infty}}\frac{\overline{\infty\,a}}{\overline{a\,0}}\cdot\frac{\overline{0\,a}}{\overline{a\,\infty}}\frac{\overline{\infty\,d}}{\overline{d\,h}}$$
$$\frac{\overline{h\,k}}{\overline{k\,0}}\cdot\frac{\overline{h\,d}}{\overline{d\,\infty}}\frac{\overline{\infty\,m}}{\overline{m\,p}}\frac{\overline{p\,x}}{\overline{x\,h}}\cdot\frac{\overline{h\,x}}{\overline{x\,p}}\frac{\overline{p\,d}}{\overline{d\,l}}\frac{\overline{l\,m}}{\overline{m\,h}}\cdot\frac{\overline{l\,d}}{\overline{d\,p}}\frac{\overline{p\,t}}{\overline{t\,e}}\frac{\overline{e\,\infty}}{\overline{\infty\,l}}\cdot$$
$$\frac{\overline{e\,t}}{\overline{t\,p}}\frac{\overline{p\,y}}{\overline{y\,0}}\frac{\overline{0\,f}}{\overline{f\,e}}\cdot\frac{\overline{0\,y}}{\overline{y\,p}}\frac{\overline{p\,n}}{\overline{n\,c}}\frac{\overline{c\,\infty}}{\overline{\infty\,0}}\cdot\frac{\overline{c\,n}}{\overline{n\,p}}\frac{\overline{p\,m}}{\overline{m\,\infty}}\frac{\overline{\infty\,0}}{\overline{0\,c}}\cdot\frac{\overline{c\,0}}{\overline{0\,\infty}}$$
$$\frac{\overline{\infty\,g}}{\overline{g\,e}}\frac{\overline{e\,u}}{\overline{u\,c}}\cdot\frac{\overline{c\,u}}{\overline{u\,e}}\frac{\overline{e\,0}}{\overline{0\,f}}\frac{\overline{f\,g}}{\overline{g\,c}}\cdot\frac{\overline{c\,g}}{\overline{g\,f}}\frac{\overline{f\,e}}{\overline{e\,0}}\frac{\overline{0\,\infty}}{\overline{\infty\,c}}\cdot\frac{\overline{f\,0}}{\overline{0\,e}}\frac{\overline{e\,\infty}}{\overline{\infty\,g}}$$
$$\frac{\overline{g\,c}}{\overline{c\,f}}\cdot\frac{\overline{f\,c}}{\overline{c\,g}}\frac{\overline{g\,h}}{\overline{h\,0}}\frac{\overline{0\,e}}{\overline{e\,g}}\cdot\frac{\overline{0\,h}}{\overline{h\,g}}\frac{\overline{g\,\infty}}{\overline{\infty\,e}}\frac{\overline{e\,f}}{\overline{f\,0}}=(-1)^{14}=1. \quad (9.2)$$

This also yields a proof for theorem $\mathcal{A}$ called *Ceva-Menelaus proof*: The requirements of $\mathcal{A}$ ensure that (9.2) can be stated everywhere except for the places indicated with highlighted letters d. Define $d' = \textbf{meet}(\textbf{join}(l,p),\textbf{join}(d,m))$ and $d'' = \textbf{meet}(\textbf{join}(l,p),\textbf{join}(d,\infty))$. With those points induced by the triangles derived in Figure 9.4, we can state (9.2) with d' and d'' instead of d. This implies

$$\frac{\overline{p\,d'}}{\overline{d'\,l}}\cdot\frac{\overline{l\,d''}}{\overline{d''\,p}}=1$$

which implies $d' = d''$. By the definition of d' and d'', this also implies $d' = d'' = d$ and hence yields the desired conclusion that d, l and p are collinear.

So the interpretation of a biquadratic proof via the area principle yields another independent proof consisting of Ceva and Menelaus triangles, a glueing pattern and inverse oriented length ratios along the glueings. Observe that the right-hand side of (9.2) have to yield 1, i.e. the overall number of Menelaus configurations has to be an even number. So in order to get a complete cancellation pattern, one has to take sign changes into account.

The Translation Process in the General Case

As discussed in [94, 2], this translation is in principle possible for every biquadratic proof. In order to formulate this statement one needs to specify what is an admissible result of such a translation. The first idea was to require sets of Ceva and Menelaus triangles together with a glueing pattern. In our example, this was possible since we could decompose every cycle in Figure 9.2 into triangles. In an arbitrary translation process, this might not be the case. Starting at this point, there are at least two different options to deal with this problem. In order to formulate these options we introduce some notions in the next section.

9.4. On the Power of Biquadratic Proofs

We remarked earlier that $\mathcal{A}$ is a good example to also show the limitations of the biquadratic proving method. Therefore see Figure 9.5 which is only a slight modification of Figure 9.1. Now the quadrilateral sets induced by d_2, k, m, h and d_2, m, n and p compute $(a + b) + c$ and the quadrilateral sets induced by f, g, h and l and d_1, e, f and l compute $a + (b + c)$. With the help of an computer algebra

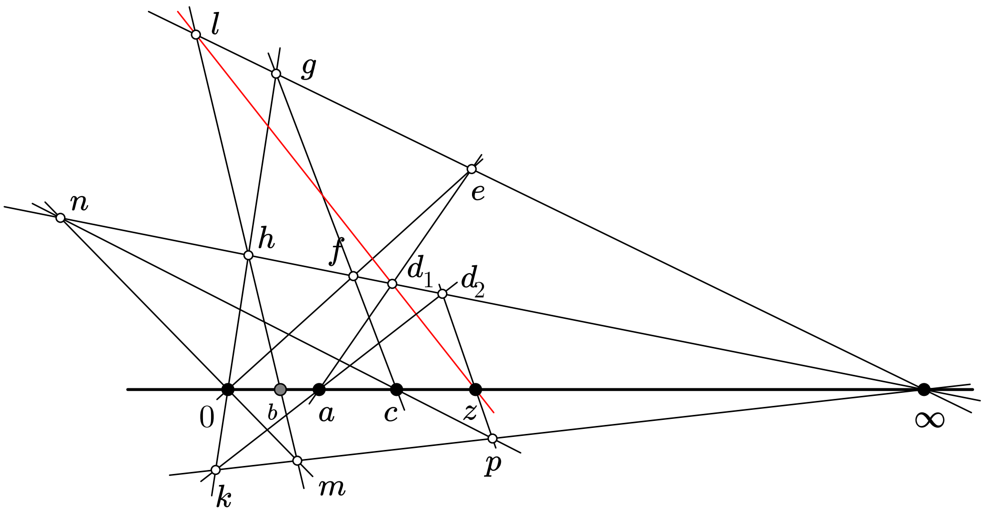


Figure 9.5.: Another version of $(a + b) + c = a + (b + c)$ encoded in geometry via von Staudt constructions.

system one can see that the incidence theorem induced has no biquadratic proof even assuming **B** to be as general as possible. However, there are biquadratic proofs for the fact, that constructive projective addition is well defined and independent of the points responsible for constructing the quadrilateral set. Therefore, using several binomial proofs and inserting additional points still can proof the theorem binomially. To do so, one can insert all gray points from Figure 9.1 and one can show that they coincide with the corresponding ones in Figure 9.5 (which have to be inserted there). Then we can use the binomial proof of $\mathcal{A}$. It was conjectured by Richter-Gebert that this approach might be universal. One does not know how many binomial proof have to be applied and therefore the algorithm has no longer polynomial running time. One cannot expect a better time bound since it was mentioned in Section 7.4 that the realization problem is proven to be NP-hard over the reals.

10. The Base Graph Γ, Irreducible Γ-Cycles and Morphisms

The principle objects of consideration in the present Part II of this thesis are so-called Γ-*cycles*. In their unoriented version they can be considered as cycles in the (undirected) graph Γ defined below:

Definition 10.1. *Let* $\mathbf{P}$ *be a finite set of points. Let* $\mathbf{B}$ *be a set consisting of 3-element subsets of* $\mathbf{P}$*. The* Γ-graph $\Gamma(\mathbf{B})$ of $\mathbf{B}$ *is the (undirected) graph with vertices* $\mathbf{B}$ *and edges*

$$\left\{\{A,B\}\,\middle|\, A,B\in\mathbf{B}, |A\cap B|=2\right\}.$$

A point configuration P *in* $\mathbb{RP}^2$ *is called an* instance *of* $\mathbf{B}$ *if* $[\mathbf{i},\mathbf{j},\mathbf{k}]_P = \Phi^P([\mathbf{i},\mathbf{j},\mathbf{k}]) \neq 0$ *for all* $\{\mathbf{i},\mathbf{j},\mathbf{k}\}\in\mathbf{B}$*. This property is well-defined, since* $\Phi^P([\mathbf{i},\mathbf{j},\mathbf{k}])$ *being zero does not depend on the order of* $\mathbf{i}$*,* $\mathbf{j}$ *and* $\mathbf{k}$*.*

The edges of $\Gamma(\mathbf{B})$ have a combinatorial structure such that in any instance P of $\mathbf{B}$, the area principle can be applied. For each instance P, in matroid theory, the base graph $\Gamma(P)$ is defined. The graph $\Gamma(P)$ has all *bases* as vertices and edges with the same combinatorial definition as above. Bases in this context of a concrete instance are non-collinear triples of points. Obviously, $\Gamma(\mathbf{B})$ is a (induced) subgraph of $\Gamma(P)$. There are some useful results (see [84]) on properties of $\Gamma(P)$. In [38, 40, 41] the concept of matroids is generalized to matroids with coefficients and also infinite matroids. [39] provides a version specialized to the finite case. Different kinds of so-called Tutte groups are defined and morphisms between the groups are constructed. $\Gamma(P)$ can be generalized for the matroids under consideration. The fact that this graph is a a Maurer graph, as introduced in [84], allows for the construction of one of the morphisms described above. This morphism is of particular interest in our considerations since the groups involved are related to the groups considered here. This part of the thesis can be understood as mimicking some of the effects seen there in a setting, where not the full $\Gamma(P)$ is known. E.g. this is the case when considering a projective incidence theorem $\mathcal{T}=(\mathbf{H},\mathbf{B}(\mathbf{A}),C)$ with $\mathbf{B}(\mathbf{A})$ being the generic non-degeneracy conditions depending on $\mathbf{H}$ and defined in Section 7.5. In this case, $\Gamma(\mathbf{B}(\mathbf{A}))$ has only vertices (and edges) that are present *in any instance* of $\mathcal{T}$. The group $\mathcal{BG}_{\mathbf{B}}$ defined in the previous chapter sits between $\mathbb{F}_M^{\mathfrak{B}}$ and $\mathbb{T}_M^{\mathfrak{B}}$ in the language of Dress and Wenzel. The quotient $\mathcal{BG}_{\mathbf{B}}/\mathcal{HG}_{\mathbf{B},\mathbf{H}}$ is essentially $\mathbb{T}_M^{\mathfrak{B}}$.

Similarly, $\mathcal{RG}_{\mathbf{B}}$, which is to be defined, sits between $\mathbb{F}_M$ and $\mathbb{T}_M$. The quotient $\mathcal{RG}_{\mathbf{B}}/\mathcal{E}_{\mathbf{B},\mathbf{H}}$ will be closely related to $\mathbb{T}_M$. We are in the case of finite combinatorial structure with partial information but with at least the intuition that the points correspond to projective configurations.

10.1. Weak Irreducibility

Now consider a translation process as in Chapter 9 for some binomial proof (of a real projective incidence theorem $\mathcal{T} := (\mathbf{H}, \mathbf{B} = \mathbf{B}(\mathbf{A}), C)$). All split biquadratic fractions correspond to (directed) edges in $\Gamma(\mathbf{B}(\mathbf{A}))$. As in Figure 9.2 it also holds in the general case that these edges arrange in cycles which can be considered as (cyclically oriented) cycles in $\Gamma(\mathbf{B}(\mathbf{A}))$. These cycles can in principle have arbitrary length. With the following definition, the underlying cycles are unoriented Γ-cycles.

Definition 10.2. *Let $\Gamma(\mathbf{B})$ be a Γ-graph for some set $\mathbf{B}$. Let $\mathcal{C}$ be a cycle in $\Gamma(\mathbf{B})$. $\mathcal{C}$ is called* unoriented Γ-cycle. *Furthermore, if the subgraph of $\Gamma(\mathbf{B})$ induced by the vertices of $\mathcal{C}$ is again $\mathcal{C}$, we say that $\mathcal{C}$ is* weakly irreducible *and* weakly reducible *otherwise.*

Observe that weakly irreducibility is a very local property and does depend only on the vertices of the cycle itself. It tries not to find a composition in the full graph $\Gamma(\mathbf{B})$. This possibility will be captured in Definition 10.3.

Γ-cycles that are not weakly irreducible have a pair of non-neighboring vertices that differ by exactly one element. We saw an example of an (oriented version of a) weakly reducible Γ-cycles in the 4-cycles and 5-cycles in Figure 9.2. The dashed lines indicate places where edges in $\Gamma(\mathbf{B}(\mathbf{A}))$ are present which imply reducibility. See Figure 10.2 (a) at the end of this chapter for an example of an weakly irreducible Γ-cycle. In the figure, we omit the curly parentheses at the labeling of the vertices in order to get a clean picture. Furthermore, considering the edges of the cycles as ratios allows for applying the area principle (see Section 3.1). When no numerator and denominator are specified, one can only identify points spanning the reference line and the transversal. This is exemplarily shown in Figure 10.2 (b).

In the context of the translation process indicated in Section 9.3, an oriented version of a weakly reducible Γ-cycle can be easily decomposed into smaller cycles along an additional edge. This decomposition is of the same shape as the decomposition along the dotted lines shown in Figure 9.2. There, we were in the lucky position that the resulting Γ-cycles were triangles which have an interpretation as the theorems of Ceva and Menelaus. In general, we are interested in situations where such a simple decomposition is not possible. This is the reason why we are interested in weakly irreducible Γ-cycles. Most of the considerations in the next chapters will refer to them. However, one could resolve the interpretation of biquadratic proofs also in different ways. These ways should be depicted now. Furthermore, a language for oriented Γ-cycle is developed.

10.2. Oriented Γ-Cycles and Strong Irreducibility

To model the considerations about split biquadratic fractions (as they were done in Section 9.3) in terms of *directed* edges in $\Gamma(\mathbf{B})$, we define an algebraic structure $\mathcal{RG}_\mathbf{B}$ which has the directed edges of $\Gamma(\mathbf{B})$ as elements. Furthermore, they vanish when they are traversed in opposite directions (see Definition 10.3). Clearly each set of split biquadratic fractions can be considered to be an element of $\mathcal{RG}_\mathbf{B}$. In the same way, any cycle emerging from the translation process is an element of $\mathcal{RG}_\mathbf{B}$.

Definition 10.3. *Let* $\mathbf{P}$ *be a finite set of points. Let* $\mathbf{B}$ *be a set consisting of 3-element subsets of* $\mathbf{P}$. *The* group of bracket ratios compatible with the area principle $\mathcal{RG}_\mathbf{B}$ *is defined to be the quotient of the multiplicatively written free abelian group generated by the formal (unbreakable) symbols*

$$\left\{ \frac{A}{B} \;\middle|\; \{A,B\} \text{ is an edge in } \Gamma(\mathbf{B}) \right\} \cup \left\{ \;\epsilon\; \right\}$$

and the subgroup generated by

$$\left\{ \frac{A}{B} \cdot \frac{B}{A} \;\middle|\; \{A,B\} \text{ is an edge in } \Gamma(\mathbf{B}) \right\} \cup \left\{ \epsilon^2 \right\}.$$

Furthermore, any element $\overrightarrow{\mathcal{C}} \in \mathcal{RG}_\mathbf{B}$ *that as an representation as*

$$\overrightarrow{\mathcal{C}} = \frac{A_1}{A_2} \cdot \frac{A_2}{A_3} \cdots \quad \frac{A_{k-1}}{A_k} \cdot \frac{A_k}{A_1}$$

is a called a k*-cycle in* $\mathcal{RG}_\mathbf{B}$ *(for some* $k \in \mathbb{N}$*) whenever* $\big(\{A_1,A_2\},\ \{A_2,A_3\},\ \dots,\ \{A_k,A_1\}\big)$ *is a* Γ*-cycle. It is weakly irreducible whenever* $\big(\{A_1,A_2\},\ \{A_2,A_3\},\ \dots,\ \{A_k,A_1\}\big)$ *is weakly irreducible. It is also called (oriented)* Γ*-cycle of length* k*. It is* strongly irreducible *whenever it is not contained in the subgroup of* $\mathcal{RG}_\mathbf{B}$ *that is generated by all* k'*-cycles where* $k' < k$*. A point configuration* P *is called an instance of* $\overrightarrow{\mathcal{C}}$ *whenever it is an instance of* $\mathbf{B}$.

Clearly, whenever an oriented Γ-cycle is strongly irreducible, it is weakly irreducible. Within this setting, one could argue that one should focus on the strongly irreducible k-cycles in $\mathcal{RG}_\mathbf{B}$.

Remark 10.4. Observe that any element of $\mathcal{RG}_{\mathbf{B}}$ can be given a representation which either contains ϵ once or does not contain ϵ. Ignoring ϵ, the rest of the element as the shape $\frac{A_1}{A_2}\cdots\frac{A_{k-1}}{A_k}$ which can be drawn as a directed version of a subgraph of $\Gamma(\mathbf{B})$ where we also allow multiple edges. This yields a graphical representation of many objects in $\mathcal{RG}_{\mathbf{B}}$ which is often quite helpful. The objects $\frac{A_1}{A_2}$ directly correspond to edges and are sometimes called *edges* as well.

10.3. Decomposing Large Γ-cycles within Biquadratic Proofs

Due to [84, 39], when the same definitions are done with underlying graph $\Gamma(P)$ for an *instance* P instead of $\Gamma(\mathbf{B})$, one can cut down the maximal k for which a k-cycle can be strongly irreducible to 3. For a general projective incidence theorem $\mathcal{T} = (\mathbf{H}, \mathbf{B}, C)$ specific informations about the shape of $\Gamma(\mathbf{B})$ are out of reach, especially in the case that $\mathbf{B}$ has very few elements. Those cases can be easily achieved by redefining $\mathbf{B}$ after a (binomial) proof and letting it consist of those elements that are needed for the proof. This may induce big strongly irreducible cycles. The big ambiguity about the shape of $\mathbf{B}$ is another reason, different from the ones given in Chapter 7.5, to restrict ourselves to the cases where $\mathbf{B}$ equals $\mathbf{B}(\mathbf{A})$.

Even in the case that $\mathbf{B} = \mathbf{B}(\mathbf{A})$, one does not have the property of $\Gamma(\mathbf{B}(\mathbf{A}))$ being a Maurer graph. So one cannot apply the same techniques as in the matroidal realizable case in order to control the length of the cycles. It remains an open problem whether for any incidence theorem $\mathcal{T} = (\mathbf{H}, \mathbf{B}(\mathbf{A}), C)$, any biquadratic proof can be transferred to a collection of 3-cycles in $\mathcal{RG}_{\mathbf{B}(\mathbf{A})}$. Observe that therefore, it would be allowed that not every splitting of biquadratic fractions may lead to oriented Γ-cycles that can be written as a product of 3-cycles. There has to be only one splitting which allows for this. It remains also open the stronger problem whether *any* k-cycle in $\mathcal{RG}_{\mathbf{B}(\mathbf{A})}$ can be written as a product of 3-cycles.

We do know cycles where this is possible but this decomposition is not at all easy. More precisely, there is a theorem $(\mathbf{H}, \mathbf{B}(\mathbf{A}), C)$ with a 4-cycle such that the decomposition into 3-cycles that was obtained by solving a linear equation system with Mathematica ([136]) uses 54 triangles. Having this in mind, it could be much more convenient to have a better understanding of the underlying Γ-cycle than to actually carry out the decomposition into 54 triangles.

There is also another way to deal with large cycles as introduced in [2]. Here one adds two new generic points, say $\mathbf{g}$ and $\mathbf{h}$, to the whole theorem. This allows for a decomposition of large cycles but also generates at least $3k-2$ new triangles when decomposing a k-cycle. Within a biquadratic proof, Γ-cycles do look not very complicated. In fact, this is also a benefit of the modeling of an incidence theorem by $(\mathbf{H}, \mathbf{B}, C)$. It has the advantage that we do not have to distinguish between free and dependent points. For dependent points expressed in Grassmann-Cayley algebra, a bracket may look much more complicated and will have more than one evaluation. Therefore, in a setup of Grassmann-Cayley algebra, the canceling of brackets used in biquadratic proofs is less trivial. This cancellation process corresponds to Γ-cycles.

10.4. Some Subgroups Vanishing under Morphisms

In order to describe the relations between $\mathcal{RG}_{\mathbf{B}}$ and $\mathcal{BG}_{\mathbf{B}}$, we consider the group homomorphism $\Phi : \mathcal{RG}_{\mathbf{B}} \to \mathcal{BG}_{\mathbf{B}}$ induced by

$$\epsilon \mapsto \epsilon \qquad \text{and} \qquad \frac{\{\mathbf{b}, \mathbf{a}, \mathbf{c}\}}{\{\mathbf{d}, \mathbf{c}, \mathbf{a}\}} \mapsto \frac{[\mathbf{b}, \mathbf{a}, \mathbf{c}]}{[\mathbf{d}, \mathbf{c}, \mathbf{a}]}$$

where colors only highlight the combinatorics of a given unbreakable symbol in $\mathcal{RG}_{\mathbf{B}}$. Observe that the result indeed only depends on the combinatorics of the argument. The combinatorics are induced by the considerations done in Section 3.1. Furthermore, Φ is well-defined since it vanishes on the defining subgroup. Let

$$\overrightarrow{\mathcal{C}} = \frac{A_1}{A_2} \cdot \frac{A_2}{A_3} \cdots \quad \frac{A_{k-1}}{A_k} \cdot \frac{A_k}{A_1} \in \mathcal{RG}_{\mathbf{B}}$$

be an oriented Γ-cycle. Apparently it holds that

$$\Phi(\overrightarrow{\mathcal{C}}) = \prod_{i=1}^{k} \quad \Phi\left(\frac{A_i}{A_{i+1}}\right) \in \{\epsilon, 1\}.$$

We define $\Phi(\overrightarrow{\mathcal{C}})$ to be the *signature* of $\overrightarrow{\mathcal{C}}$. In order to determine the signature, one has to compare the orientations of canceling brackets in $\Phi(\overrightarrow{\mathcal{C}})$. We will give an easier way to read of the signature of weakly irreducible Γ-cycles later on. We will use the structure introduced when counting them (see Section 12.2). By abuse of notation, we can assume $\Phi(\overrightarrow{\mathcal{C}})$ to lie in $\mathcal{RG}_{\mathbf{B}}$ as well. This allows for defining $\bigcirc_{\mathbf{B}}$ to be the group generated by

$$\left\{ \Phi(\overrightarrow{\mathcal{C}}) \cdot \overrightarrow{\mathcal{C}} \;\middle|\; \overrightarrow{\mathcal{C}} \text{ is oriented } \Gamma\text{-cycle} \right\}$$

which implies

$$\Phi(\,\bigcirc_{\mathbf{B}}\,) = \{1\}.$$

Figure 10.2 (c) highlights the combinatorics which are given in the definition of Φ for each edge in the given example.

10.4.1. A Euclidean Interpretation of Oriented Γ-Cycles

Consider an instance P. We assume that all points have Euclidean interpretations and all intersections defined in the following fit in Euclidean space given by the standard embedding. We write non-bold letters for points in P corresponding to bold indices in $\mathbf{P}$. For $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d} \in \mathbf{P}$ it holds:

$$\Phi^P\left(\frac{\{\mathbf{b},\mathbf{a},\mathbf{c}\}}{\{\mathbf{d},\mathbf{c},\mathbf{a}\}}\right) = \frac{|b,\ a,c|}{|d,\ c,a|}$$

where we use Φ^P also to denote $\Phi^P \circ \Phi$ since no confusion can arise. We apply the area principle and it holds:

$$\frac{|b,\ a,c|}{|d,\ c,a|} = \frac{\overline{b\,p}}{\overline{p\,d}} \tag{10.1}$$

where $p := \mathbf{meet}(\mathbf{join}(b,d),\mathbf{join}(a,c))$.

Definition 10.5. *Let* $\overrightarrow{\mathcal{C}} = \frac{A_1}{A_2}\cdot\frac{A_2}{A_3}\cdots\quad\frac{A_{k-1}}{A_k}\cdot\frac{A_k}{A_1}$ *be an oriented* Γ*-cycle. Let* P *be an instance of* $\overrightarrow{\mathcal{C}}$*. Assume the last entries of the points in* P *equal* 1 *and applying the area principle gives*

$$\Phi^P\left(\frac{A_i}{A_{i+1}}\right) = \frac{\overline{b_i\,p_i}}{\overline{p_i\,d_i}}$$

for finite points p_i *(for* $1 \leq i \leq k$ *and indices considered modulo* k*). Due to* (10.1)*, there is a induced Euclidean theorem stating that*

$$\prod_{i=1}^{k}\frac{\overline{b_i\,p_i}}{\overline{p_i\,d_i}} = \Phi^P(\overrightarrow{\mathcal{C}}) \in \{-1,+1\}.$$

This Euclidean theorem is called a Γ-cycle theorem *induced by* $\overrightarrow{\mathcal{C}}$*.*

Figure 10.2 (d) gives an example of a Γ-cycle theorem. In fact, a Γ-cycle theorem is not trivial in Euclidean space.

Remark 10.6. There are two Γ-cycles of length 3. The Γ-cycle theorems induced by them are exactly the theorems of Ceva and Menelaus.

10.4.2. Including Collinearities

Let $\mathbf{H}$ a set of 3-element subsets of $\mathbf{P}$ such that $\mathbf{B}$ and $\mathbf{H}$ are disjoint. The *group of biquadratic fractions in area form* $\uparrow\!\text{-\,-\,-}\!\downarrow_{\mathbf{B},\mathbf{H}}$ is the subgroup of $\mathcal{RG}_{\mathbf{B}}$ generated by

$$\left\{ \frac{\{\mathbf{a},\mathbf{b},\mathbf{x}\}}{\{\mathbf{a},\mathbf{b},\mathbf{y}\}} \cdot \frac{\{\mathbf{a},\mathbf{c},\mathbf{y}\}}{\{\mathbf{a},\mathbf{c},\mathbf{x}\}} \;\middle|\; \{\mathbf{a},\mathbf{b},\mathbf{c}\} \in \mathbf{H} \text{ or } \{\mathbf{a},\mathbf{x},\mathbf{y}\} \in \mathbf{H} \right\} \tag{10.2}$$

which implies

$$\Phi(\uparrow\!\text{-\,-\,-}\!\downarrow_{\mathbf{B},\mathbf{H}}) \subset \mathcal{HG}_{\mathbf{B},\mathbf{H}}.$$

Let

$$\mathcal{E}_{\mathbf{B},\mathbf{H}} := \langle \bigcirc_{\mathbf{B}} \cup \uparrow\!\text{-\,-\,-}\!\downarrow_{\mathbf{B},\mathbf{H}} \rangle.$$

We say that P is an instance of of $\mathbf{B}$ and $\mathbf{H}$ if $[\mathbf{i},\mathbf{j},\mathbf{k}]_P \neq 0$ for all $\{\mathbf{i},\mathbf{j},\mathbf{k}\} \in \mathbf{B}$ and $[\mathbf{i},\mathbf{j},\mathbf{k}]_P = 0$ for all $\{\mathbf{i},\mathbf{j},\mathbf{k}\} \in \mathbf{H}$. This implies together with Proposition 9.4 that for all instances P it holds that

$$\Phi^P(\mathcal{E}_{\mathbf{B},\mathbf{H}}) = \{1\}.$$

Remark 10.7. Observe that the graphical version of the elements generating $\uparrow\!\text{-\,-\,-}\!\downarrow_{\mathbf{B},\mathbf{H}}$ (see (10.2)) looks like Figure 10.1. The dashed lines indicate places where there is an edge in $\Gamma(\mathbf{B})$. This explains the name of the group $\uparrow\!\text{-\,-\,-}\!\downarrow_{\mathbf{B},\mathbf{H}}$. Observe that multiplying an oriented 4-cycle yields the other splitting of the biquadratic fraction. Observe furthermore that in the left edge, the transversal is given by $\mathbf{a}$ and $\mathbf{b}$ and in

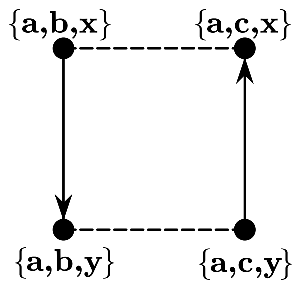


Figure 10.1.: Graphical version of a biquadratic fraction in area form.

the right edge it is given by $\mathbf{a}$ and $\mathbf{c}$. It might be the case, that $\mathbf{H}$ implies that the transversal can be also spanned by some other points. If one wants to express this fact in $\mathcal{E}_{\mathbf{B},\mathbf{H}}$, one can alter the points spanning the transversal step by step giving a graphical picture as $\uparrow\!\text{-\,-\,-}\!\updownarrow\!\text{-\,-\,-}\!\downarrow \cdots \uparrow\!\text{-\,-\,-}\!\downarrow$ (see also Figure 12.3 for an example).

Slight modifications of [39] as well as [2] imply that each biquadratic proof has a preimage under Φ that lies in $\mathcal{E}_{\mathbf{B},\mathbf{H}}$. The formulation in terms of $\mathcal{E}_{\mathbf{B},\mathbf{H}}$ can be considered easier since one has better (geometric) control of the cancellation process. This control is gained via the interpretation as oriented length ratios and the Γ-cycle theorems. In particular, it is easier to construct new theorems within this theory because cancellation can be easily achieved. These theorems can be theorems about oriented length ratios or incidence theorems (along with a biquadratic proof). We will present examples for both cases in Chapter 12 and Chapter 13. Both cases can be regarded as giving a specific representation of an element of $\mathcal{E}_{\mathbf{B},\mathbf{H}}$ by its defining elements. The overall product can again be interpreted geometrically resulting in a derived geometric fact. The approach can be understood as starting with cancellation patterns (or with Γ-cycles) and imposing incidence constraints (requiring $\mathbf{H}$ to contain some elements such that some elements in $\uparrow\dashrightarrow\downarrow_{\mathbf{B},\mathbf{H}}$ are present). The geometric control that one achieves is also a reason for the Cayley factorization to work (see Part III).

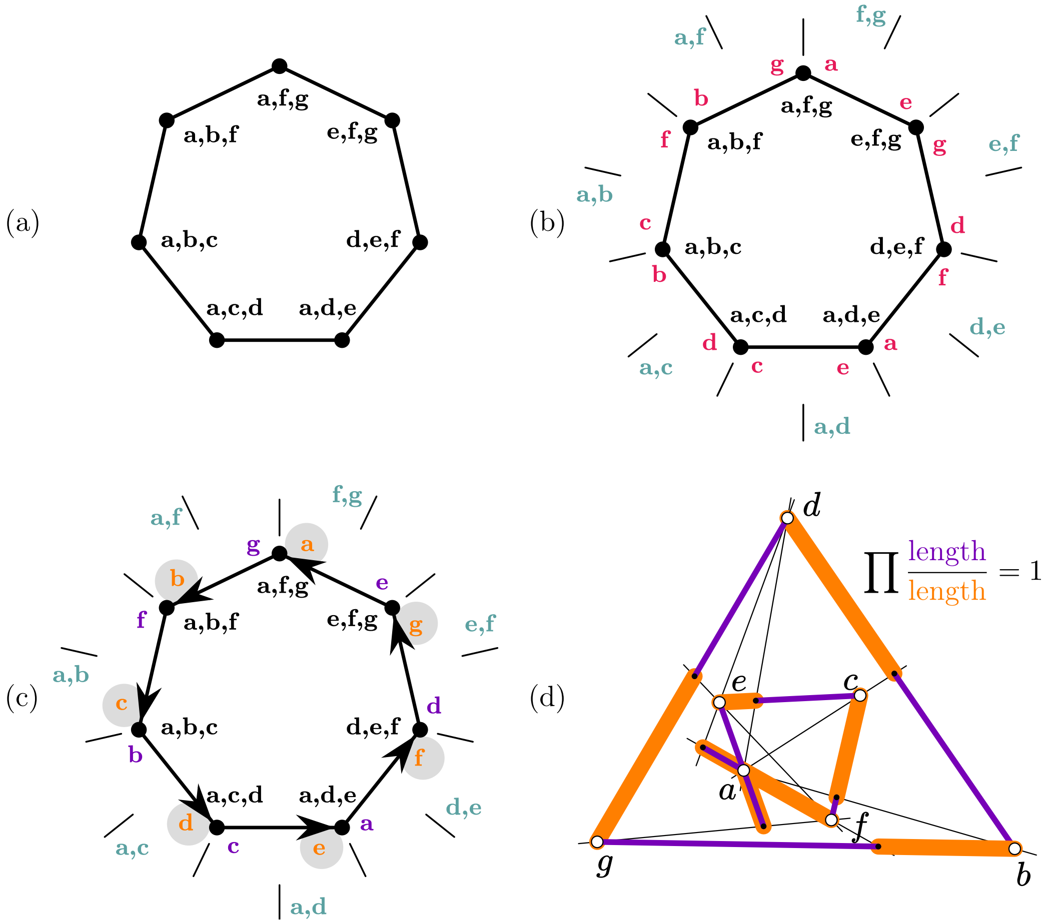


Figure 10.2.: A weakly irreducible Γ-cycle in an unoriented and in an oriented version together with the corresponding Γ-cycle theorem.

11. Counting (Weakly) Irreducible Γ-Cycles

Suppose $\mathcal{C} = (\{A_1, A_2\},\ \{A_2, A_3\},\ \dots\ ,\ \{A_k, A_1\})$ is a weakly irreducible unoriented Γ-cycle. From now on, we will drop the word "weakly", since we will not consider strong irreducibility any longer. The following is joint work with Jürgen Richter-Gebert who invented the table representation of (irreducible) Γ-cycles as shown e.g. in Figure 11.1 (e) and in many other diagrams. In order to structure and enumerate them he made the assumptions given in the following section. By enumerating and investigating all irreducible cycles of fixed length up to 9 he suggested the subscheme depicted in Figure 11.3. The formulation in his enumeration was via the permutation σ which will be used in Section 12.2.

11.1. Assumptions on the Irreducible Γ-Cycles

We will w.l.o.g. require that $\mathbf{P} = \bigcup_{i=1}^{k} A_i$ and that $\mathcal{C}$ has *connected entries*, i.e. that for each $\mathbf{p} \in \mathbf{P}$, the subgraph of $\mathcal{C}$ induced by the vertices $V_{\mathbf{p}}$ containing $\mathbf{p}$

$$V_{\mathbf{p}} = \left\{\, A_i \,\middle|\, \mathbf{p} \in A,\ i \in \{1, \dots, k\} \,\right\}$$

is connected. For a Γ-cycle not meeting this requirement for some $\mathbf{p}$, the entry $\mathbf{p}$ can be given a new name in each connected component. This maintains the property of being a Γ-cycle as well as the irreducibility.

The aim of this Chapter is to give a reasonable counting of irreducible Γ-cycles of fixed length k. Figure 11.5 and 11.6 at the end of the chapter show a resulting listing of irreducible Γ-cycles of length 7 and 8. They use a *table representation* which is to be defined in the following section. In order to avoid case distinctions, from now on, we will assume that $k \geq 4$. Furthermore, some Γ-cycles have to be considered identical, e.g. Γ-cycles that differ only by labeling the entries of the vertices. These assumptions also imply that there is exactly one irreducible Γ-cycle where $V_{\mathbf{a}}$ covers all vertices of $\mathcal{C}$ for some $\mathbf{a} \in \mathbf{P}$:

$$\Big(\{\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2\}, \{\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3\}, \{\mathbf{a}, \mathbf{b}_3, \mathbf{b}_4\}, \dots, \{\mathbf{a}, \mathbf{b}_{k-1}, \mathbf{b}_k\}, \{\mathbf{a}, \mathbf{b}_k, \mathbf{b}_1\}\Big). \qquad (11.1)$$

In order to count the remaining irreducible Γ-cycles we temporally exclude this cycle from the following considerations and each $V_{\mathbf{p}}$ induces a *path* in $\mathcal{C}$.

11.2. Table Representations of Irreducible Γ-Cycles

We chose an orientation $\overrightarrow{\mathcal{C}}$ for $\mathcal{C}$: Let

$$\overrightarrow{\mathcal{C}} = \frac{A_1}{A_2} \cdot \frac{A_2}{A_3} \cdots \quad \frac{A_{k-1}}{A_k} \cdot \frac{A_k}{A_1}.$$

As in the definition of Φ we identify blue-grey, blue-violet and orange entries for each edge (see Figure 11.1 (c) whithin is a continuation and partial restatement of Figure 10.2). Each end of a path induced by $V_{\mathbf{p}}$ corresponds to a blue-violet or to a orange coloring of $\mathbf{p}$. Since this is true for any $\mathbf{p}$ in $\mathbf{P}$, each element of $\mathbf{P}$ is colored in orange exactly once. This implies that $\mathbf{P}$ has exactly k different entries. Furthermore, we can identify the vertices $A_1, A_2, \ldots, A_k$ with its orange labeling (highlighted with grey circles in Figure 11.1(c)). With this identification, the consecutive sequence of vertices vertices $A_1, A_2, \ldots, A_k$ of $\overrightarrow{\mathcal{C}}$ induces a linear order on the elements of $\mathbf{P}$. We will use the notion $\pm l$ to indicate steps of size l ($l \in \mathbb{N}$) in the cyclically closed linear order of $\mathbf{P}$. It is used to produce a *table representation* of the oriented Γ-cycle $\overrightarrow{\mathcal{C}}$. This representation is in essence an incidence matrix (see also Figure 11.1 (e): we label both rows and columns by the elements of $\mathbf{P}$ in the given order. The labels of the columns are considered as vertex labels. We draw a dot at position $(\mathbf{p}, \mathbf{q})$ whenever $\mathbf{p}$ is contained in the defining set of the vertex which is labeled with $\mathbf{q}$. The black lines indicate the paths induced by the $V_{\mathbf{p}}$s.

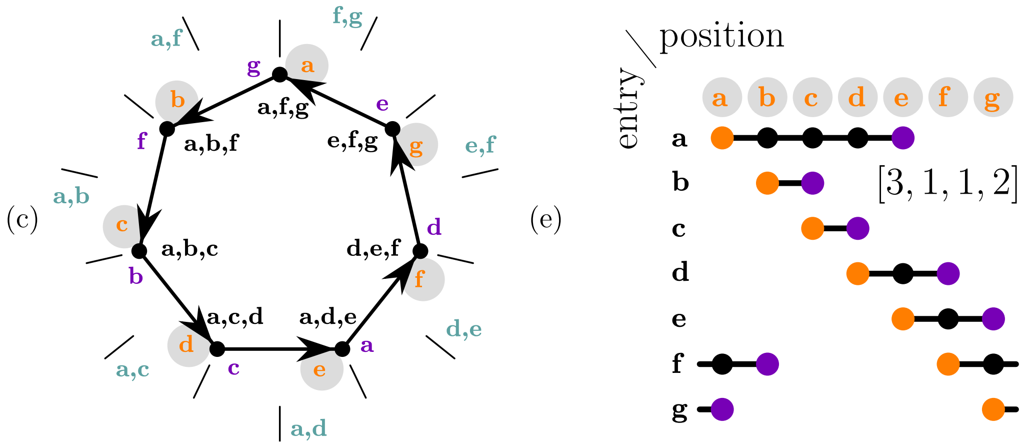


Figure 11.1.: Continuation of Figure 10.2 giving a table representation of the Γ-cycle under consideration.

One can read of the oriented Γ-cycle from the diagram: each column is a vertex and its entries are indicated by dots. Clearly any oriented Γ-cycle can be given a table representation. In Figure 11.1 (e), there is an additional coloring of some dots. The coloring indicates for each column, which entries are colored in blue-violet or orange at the corresponding vertex in the diagram (c). The sequence of numbers given there is detailedly explained below. A rough description is as follows: in the first line, there are $2+3$ dots in the second line there are $2+0$ dots, and so on. This induces a sequence $[3,0,0,1,1,2,0]$. Omitting the 0s gives the sequence depicted in Figure 11.1 (e). The omission is possible since we will see in Figure 11.3 that e.g. the second and the third line are induced by the first one. On the other hand, the restrictions for a table to represent a (not necessarily irreducible) oriented Γ-cycle of length k can be easily expressed (compare Definition 10.2 and 10.1):

- Each column has to contain three dots and therefore the table has to contain $3\,k$ dots.
- For two neighboring columns, there have to be precisely two rows in which both columns contain a dot. Here neighboring also includes a cyclic neighboring, i.e. the first and the last column are considered as neighbors as well.

Taking into account also the specific labeling of the vertices we chose for our restricted class of *irreducible* oriented Γ-cycles, we need to require also:

- In diagonal of the table, dots are drawn, but not in the positions $(\mathbf{p}, \mathbf{p}-1)$ for all $\mathbf{p} \in \mathbf{P}$.

We will argue:

- For all $\mathbf{p} \in \mathbf{P}$ there is a dot at position $(\mathbf{p}, \mathbf{p}+1)$.

In order to see this, suppose the contrary: suppose there is a $\mathbf{p} \in \mathbf{P}$ such that there is no dot in position $(\mathbf{p}, \mathbf{p}-1)$ and $(\mathbf{p}, \mathbf{p}+1)$. Since column $\mathbf{p}$ has to have two dots in common with both vertices $\mathbf{p}-1$ and $\mathbf{p}+1$ and non of these entries can be $\mathbf{p}$ by assumption, vertices $\mathbf{p}-1$ *and* $\mathbf{p}+1$ must have the same entries, say $\mathbf{a}$ and $\mathbf{b}$ in common with vertex $\mathbf{p}$. Therefore, vertices $\mathbf{p}-1$ and $\mathbf{p}+1$ both consist of $\mathbf{a}$ and $\mathbf{b}$ plus another entry and they differ by exactly one element. This is a contradiction to the assumption that $\overrightarrow{\mathcal{C}}$ is irreducible.

All previous assumptions imply that $2\,k$ of the $3\,k$ dots of a Γ-cycle of length k are already fixed. Therefore, each table representation corresponding to an irreducible oriented Γ-cycle contains the dots indicated in Figure 11.2. Since the $V_{\mathbf{p}}$s induce connected components, the entries of row $\mathbf{p}$ have to be given by the column indices

$$\mathbf{p},\ \mathbf{p}+1,\ \mathbf{p}+2,\ \ldots,\ \mathbf{p}+l+1$$

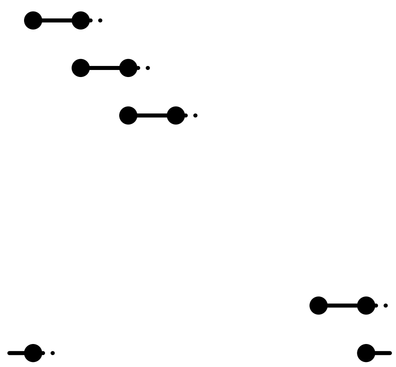


Figure 11.2.: Common subscheme of all table representations of irreducible Γ-cycles.

for some l depending on $\mathbf{p}$ and with $0 \leq l \leq k-3$. Visually speaking, this means that in Figure 11.2 the black lines have to be extended to the right.

Since we have only fixed $2\,k$ positions of dots, there has to be a row $\mathbf{p}$ in which the corresponding l is strictly positive. Since $\overrightarrow{\mathcal{C}}$ is a cycle, we can w.l.o.g. assume that $\mathbf{p}$ is the first element in $\mathbf{P}$ with respect to its induced order. Therefore, the the first row of the table corresponds to the first row in Figure 11.3.

Now consider some $\mathbf{q}$ within the range of $\mathbf{p}+1$ to $\mathbf{p}+l-1$. Due to our previous considerations, there have to be dots at positions $(\mathbf{q}, \mathbf{q})$ and $(\mathbf{q}, \mathbf{q}+1)$. There cannot be more dots in this row since this would imply a dot at position $(\mathbf{q}, \mathbf{q}+2)$. This in not possible since in column $\mathbf{q}+2$ there are already three dots in rows $\mathbf{p}$, $\mathbf{q}+1$ and $\mathbf{q}+2$.

Furthermore, there has to be a dot at position $(\mathbf{p}+l, \mathbf{p}+l+2)$: column $\mathbf{p}+l+1$ as to have two entries in common with column $\mathbf{p}+l+2$. Since there is no dot at position $(\mathbf{p}, \mathbf{p}+l+2)$ and at position $(\mathbf{p}+l+1, \mathbf{p}+l)$ it has to be at position $(\mathbf{p}+l, \mathbf{p}+l+2)$. Now one can restart the game with entry $\mathbf{p}+l$ in the place of $\mathbf{p}$. This implies that the whole table is composed of building blocks as given in Figure 11.3 and there is a sequence $[l_1, l_2, \ldots, l_t]$ of positive numbers that uniquely determines $\overrightarrow{\mathcal{C}}$. Since Figure 11.2 fixes $2\,k$ of the overall $3\,k$ dots, we have

$$\sum_{i=1}^{t} l_i = k.$$

Therefore, $[l_1, \ldots, l_t]$ is a composition of n. Any cyclic shift, e.g. $[l_2, \ldots, l_t, l_1]$, of the ls will also represent $\overrightarrow{\mathcal{C}}$. So in fact, $\overrightarrow{\mathcal{C}}$ corresponds to a cyclic composition of k.

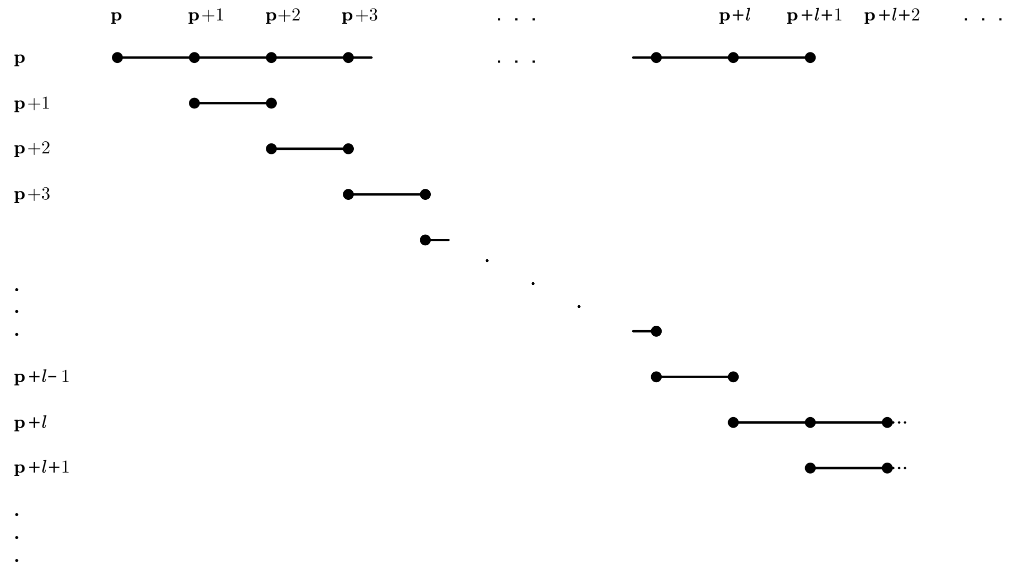


Figure 11.3.: l dots added to Figure 11.2 in row $\mathbf{p}$ determine all other black dots in rows $\mathbf{p}+1$ up to $\mathbf{p}+l+1$ as well as a dot at position $(\mathbf{p}+l, \mathbf{p}+l+2)$.

We summarize:
Each irreducible (oriented) Γ-cycle of length k (with the restrictions done in Section 11.1) corresponds to a cyclic composition of k.

11.3. Counting Irreducible Γ-Cycles

A first step in the direction of counting irreducible Γ-cycles of length k is to give the number of cyclic compositions of k. It is included in the On-Line Encyclopedia of Integer Sequences (OEIS) (as sequence number A008965, see [87]) and a formula is given by

$$\text{number of cyclic compositions of } k \ = \frac{1}{k} \sum_{d|k} \varphi(d) 2^{\frac{k}{d}} \ - 1$$

where φ is Euler's phi function and $d|k$ means that d divides k. In the OEIS there is given another suitable interpretation of this number: it is the number of necklaces where the beads themselves are sets of beads with total number of beads being k. Turning over of the necklace is not allowed. This fits with the action of adding dots to a table as in Figure 11.2 in the allowed way, i.e. locally generating patterns as in Figure 11.3. For future reference we illustrate the connection to number sequence number A000031 (see [88]) in the OEIS: It gives the number of k-bead necklaces with 2 colors when turning over is not allowed. Sequence A008965 is essentially sequence A000031 with an offset of -1. The connection between 2-colored necklaces and cyclic compositions is just an extension of the standard bijection between (ordinary) compositions of the number k and the $k-1$ bit binary numbers to the cyclic case.

In order to count irreducible oriented Γ-cycles we analyze which cyclic compositions encode irreducible oriented Γ-cycles. There are exactly four cyclic compositions that obviously cannot encode irreducible Γ-cycles since their entries are bigger than $k-3$. They are:

$$[k],\ [k-1],\ [k-2,2],\ [k-2,1,1].$$

All remaining cyclic compositions do encode (not necessarily irreducible) Γ-cycles: a table representation of a given cyclic composition can be derived by transferring the first entry of the composition to the first part of the table representation as shown in Figure 11.3. Afterwards, one can do a cyclic shift of the table and of the composition a fill the whole table with dots. Now each column contains three dots and neighboring columns share exactly two dots: due to the translation procedure, it suffices to check this property locally in Figure 11.3. Since it holds true there, each of the remaining cyclic compositions does encode a Γ-cycle. Due to our previous considerations, it remains to identify those cyclic compositions that do not correspond to *irreducible* Γ-cycles in order to find *all* irreducible Γ-cycles (fulfilling the requirements stated in Section 11.1).

Assume that there is a cyclic composition such that the induced Γ-cycle has two vertices V and W that are not neighbored in the cycle but nevertheless have exactly two elements $\mathbf{p}$ and $\mathbf{q}$ in common. Since V and W have at least distance 2 (in the cycle) and since $\mathbf{p}$ and $\mathbf{q}$ are in both columns belonging to V and W and since $\mathbf{p}$ and $\mathbf{q}$ are connected entries, both $\mathbf{p}$ and $\mathbf{q}$ have to belong to at least three vertices. Therefore, $\mathbf{p}$ and $\mathbf{q}$ locally look like the first row in Figure 11.3 and the number of vertices containing $\mathbf{p}$ and $\mathbf{q}$ determined by adding 2 to an entry of the composition. Let l be the entry corresponding to $\mathbf{p}$ in the composition and consider the local situation shown in Figure 11.3. The columns indexed by $\mathbf{p}+2$ up to $\mathbf{p}+l-1$ already contain three dots and none of them can be $\mathbf{q}$ since $\mathbf{q}$ has to be contained in at least three vertices. Therefore, w.l.o.g. V is the vertex belonging to column $\mathbf{p}$ or $\mathbf{p}+1$ and W belongs to column $\mathbf{p}+l$ or $\mathbf{p}+l+1$. In any case, $\mathbf{q}$ has to be in column $\mathbf{p}+l$ or $\mathbf{p}+l+1$ which by our labeling convention of the columns implies that

$$\mathbf{q} = \mathbf{p} + l \quad \text{or} \quad \mathbf{q} = \mathbf{p} + l + 1.$$

In order to have also $\mathbf{q}$ in column $\mathbf{p}$ or $\mathbf{p}+1$ we need to have

$$\mathbf{q} + l_{\mathbf{q}} + 1 = \mathbf{p} \quad \text{or} \quad \mathbf{q} + l_{\mathbf{q}} + 1 = \mathbf{p} + 1$$

where $l_{\mathbf{q}}$ is the entry in the composition that corresponds to $\mathbf{q}$. Combining all possible values for $\mathbf{q}$ and $l_{\mathbf{q}}$ leads to the compositions (see also Figure 11.4)

$$[l, k-l-1, 1],\ [l, 1, k-l-2, 1],\ [l, k-l],\ [l, 1, k-l-1].$$

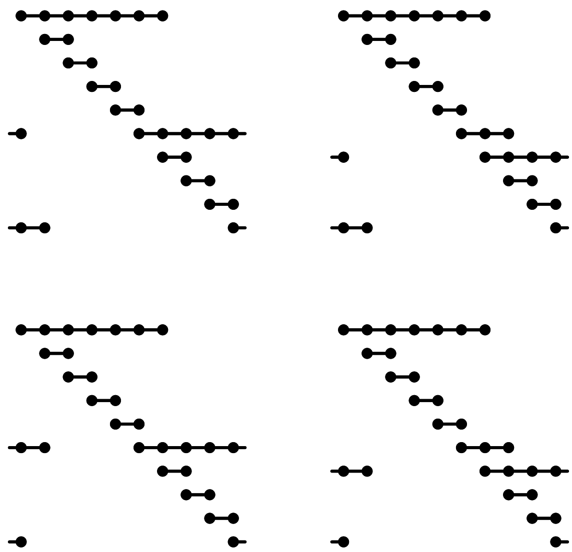


Figure 11.4.: Tables induced by the cyclic compositions $[l, k-l-1, 1]$, $[l, 1, k-l-2, 1]$, $[l, k-l]$ and $[l, 1, k-l-1]$ when $k = 10$ and $l = 5$.

We want to count these compositions which encode *reducible* oriented Γ-cycles. Therefore we have to analyze which parameter range of l is admissible and ensure that we count each reducible oriented Γ-cycle only once. Substituting l for $k-l-1$ in $[l, 1, k-l-1]$ identifies this cyclic composition with the first one (with different parameters). Now all the classes of composition

$$[l, k-l-1, 1],\ [l, 1, k-l-2, 1] \text{ and } [l, k-l]$$

are disjoint. In the following, we count their number of *distinct* elements.

Number of $[\,l,\ k-l-1,\ 1\,]$-Type Cyclic Compositions

We have to ensure that $1 \leq l \leq k-3$ and $1 \leq k-l-1 \leq k-3$. This equivalent to $2 \leq l \leq k-3$. In these case we neither have $l = 1$ nor $k-l-1 = 1$. So each l in the given range gives a different cyclic composition. Their number is

$$k-4.$$

Number of $[\,l,\ k-l\,]$-Type Cyclic Compositions

We have to ensure that $1 \leq l \leq k-3$ and $1 \leq k-l \leq n-3$. This equivalent to $3 \leq l \leq k-3$. We count the lexicographically maximal representatives, which are the ones with $l \geq k-l$. In total there are

$$k-3-\left\lceil \frac{k}{2} \right\rceil + 1 = \left\lfloor \frac{k}{2} \right\rfloor - 2$$

different cyclic compositions of this type.

Number of $[\,l,\ 1,\ k-l-2,\ 1\,]$-Type Cyclic Compositions

We have to ensure that $1 \leq l \leq k-3$ and $1 \leq k-l-2 \leq k-3$. The second equation is redundant. We count the lexicographically maximal representatives, which are the ones with $l \geq k-l-2$. In total there are

$$k-3-\left\lceil \frac{k-2}{2} \right\rceil + 1 = \left\lfloor \frac{k}{2} \right\rfloor - 1$$

different cyclic compositions of this type.

Theorem 11.1. *Up to relabeling, the total number of weakly irreducible oriented Γ-cycles with connected entries of length k (with $k \geq 3$) is*

$$\frac{1}{k}\sum_{d|k} \varphi(d) 2^{\frac{k}{d}} \; - k - 2\left\lfloor \frac{k}{2} \right\rfloor + 3.$$

Proof. The proof for $k = 3$ can be done by hand. For $k \geq 4$ we have:

$$\frac{1}{k}\sum_{d|k} \varphi(d) 2^{\frac{k}{d}} \; - k - 2\left\lfloor \frac{k}{2} \right\rfloor + 3 =$$

$$\underbrace{1}_{\text{cycle as in (11.1)}} + \underbrace{\frac{1}{k}\sum_{d|k} \varphi(d) 2^{\frac{k}{d}} - 1}_{\text{cyclic comp.s}} - \underbrace{4}_{[k],\ [k-1],\ [k-2,2],\ [k-2,1,1]}$$

$$- \underbrace{(k-4)}_{[l,\ k-l-1,\ 1]\text{-type cyclic comp.s}} - \underbrace{(\left\lfloor \frac{k}{2} \right\rfloor - 2)}_{[l,\ k-l]\text{-type cyclic comp.s}} - \underbrace{(\left\lfloor \frac{k}{2} \right\rfloor - 1)}_{[l,\ 1,\ k-l-2,\ 1]\text{-type cyclic comp.s}}$$

□

Corollary 11.2. *Up to relabeling, the total number of unoriented irreducible* Γ*-cycles of length* k *(with* $k \geq 3$*) with connected entries is*

$$\frac{1}{2k} \sum_{d|k} \varphi(d) 2^{\frac{k}{d}} - k - 2 \left\lfloor \frac{k}{2} \right\rfloor + 3 + \begin{cases} 2^{\frac{k-1}{2}} & \textit{if } k \textit{ is odd} \\ 3 \cdot 2^{\frac{k}{2}-2} & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. The case for $k = 3$ can be checked by hand. Now assume $k \geq 4$. Any unoriented Γ-cycle can be oriented arbitrarily and in the case that there is no $\mathbf{p} \in \mathbf{P}$ such that $V_{\mathbf{p}} = \mathbf{P}$, this induces a cyclic composition $[l_1, \ldots, l_t]$. It is not hard to see that the other orientation of the same cycle results in the cyclic composition $[l_t, \ldots, l_1]$. Furthermore, the number of 2-colored necklaces where turning over is allowed, is A000029 in the OEIS. Its formula is given by

$$\frac{1}{2k} \sum_{d|k} \varphi(d) 2^{\frac{k}{d}} + \begin{cases} 2^{\frac{k-1}{2}} & \text{if } k \text{ is odd} \\ 3 \cdot 2^{\frac{k}{2}-2} & \text{if } k \text{ is even.} \end{cases}$$

By another extension of the standard bijection between (ordinary) compositions of the number k and the $k-1$ bit binary numbers we see that, with an offset of -1, sequence A000029 corresponds to the cyclic compositions where reversing the orientation is allowed. One can also check by hand, that all cyclic compositions that had to be excluded in the oriented case give distinct compositions also in the case that turning over is allowed. □

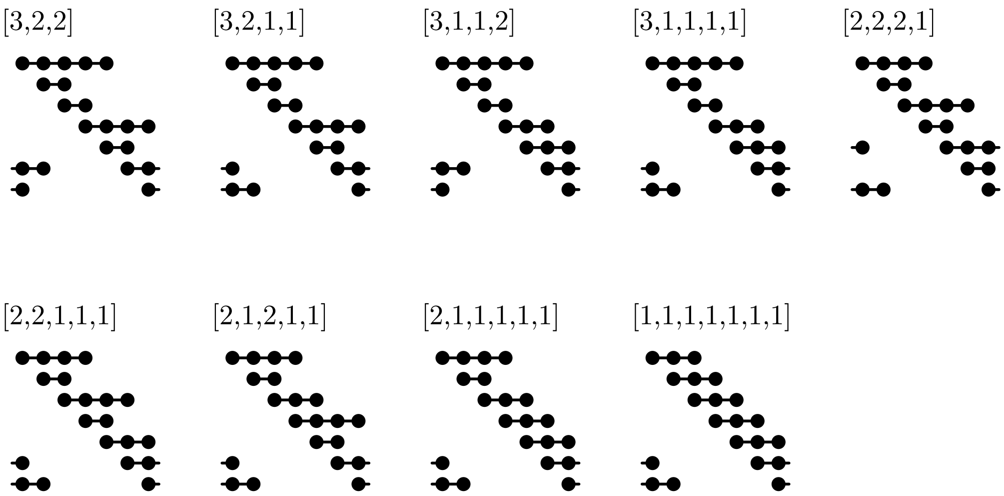


Figure 11.5.: Table representations of the nine irreducible oriented Γ-cycle of length 7 that are not induced by (11.1).

Figure 11.6.: Table representations of the 21 irreducible oriented Γ-cycle of length 8 that are not induced by (11.1).

12. Glueing Γ-Cycles along Their Orbits

12.1. An Example

As already mentioned at the end of Section 10.4.2, Γ-cycles might be more interesting if the entries are considered as dependent points instead of free points. In order to do concrete constructions using a given (oriented) Γ-cycle $\overrightarrow{\mathcal{C}}$ one wants to have information about the length ratios, i.e. about the shape of the Γ-cycle theorem associated with $\overrightarrow{\mathcal{C}}$. In the context of binomial proofs, this can be easily seen: Here edges of Γ-cycles can be glued together only if they can be interpreted as a length ratio along *the same* edge. One can also go the other way round: one can start with a Γ-cycle and impose further incidence relations on the points. This may simplify the statement of the corresponding Γ-cycle theorem in such a way, that an additional incidence relation can be concluded. In this case, one has generated a new incidence theorem. We show the procedure in more detail in an example. Consider the irreducible oriented Γ-cycle associated with the cyclic composition $[3, 1, 1, 2]$ which is illustrated in Figure 10.2 (c) and in Figure 11.1 (c). In the Γ-cycle theorem associated to it and illustrated in Figure 10.2 (d), the length ratios are arranged in two disjoint cycles called *orbits* (see also Section 12.2). We use this terminology since the word "cycle" is already in use. In fact, in Part III, a similar structure will be called "cycle" since no confusion can arise and since there, it is closely related with cycles in a graph theoretic sense.

In an instance as in Figure 10.2 (d), the corresponding Γ-cycle theorem can be read as

$$\frac{\overline{g\,p_1}}{\overline{p_1\,b}} \cdot \frac{\overline{b\,p_2}}{\overline{p_2\,d}} \cdot \frac{\overline{d\,p_3}}{\overline{p_3\,g}} \cdot \frac{\overline{f\,q_1}}{\overline{q_1\,c}} \cdot \frac{\overline{c\,q_2}}{\overline{q_2\,e}} \cdot \frac{\overline{e\,q_3}}{\overline{q_3\,a}} \cdot \frac{\overline{a\,q_4}}{\overline{q_4\,f}} = 1,$$

where the labeling of Figure 12.2 gives definitions for the newly introduced points. Imposing a Ceva configuration on the orbit of length three as shown in Figure 12.2

ensures that

$$\underbrace{\frac{\overline{g\,p_1}}{\overline{p_1\,b}} \cdot \frac{\overline{b\,p_2}}{\overline{p_2\,d}} \cdot \frac{\overline{d\,p_3}}{\overline{p_3\,g}}}_{=1} \cdot \frac{\overline{f\,q_1}}{\overline{q_1\,c}} \cdot \frac{\overline{c\,q_2}}{\overline{q_2\,e}} \cdot \frac{\overline{e\,q_3}}{\overline{q_3\,a}} \cdot \frac{\overline{a\,q_4}}{\overline{q_4\,f}} = 1. \tag{12.1}$$

Therefore, the other orbit on its own yields 1 in the product. A *multi-ratio* is a product

$$\prod_i \frac{\overline{a_i\,x_i}}{\overline{x_i\,a_{i+1}}} \tag{12.2}$$

where the indices are considered cyclically and a_i, x_i can be considered to be Euclidean and such that x_i is on the line spanned by and distinct from a_i and a_{i+1}. A multi-ratio equaling 1 is in non-degenerate situations equivalent to the fact that a triangulation with Ceva and Menelaus triangles (such that the number of Menelaus triangles is even) is possible. For the triangulation we assume that it is compatible with all points of intersection, i.e. when multiplying all theorems corresponding to the Ceva and Menelaus triangles, all ratios except for the ones in (12.2) cancel. For an example, see Figure 12.1. We proceed by demonstrat-

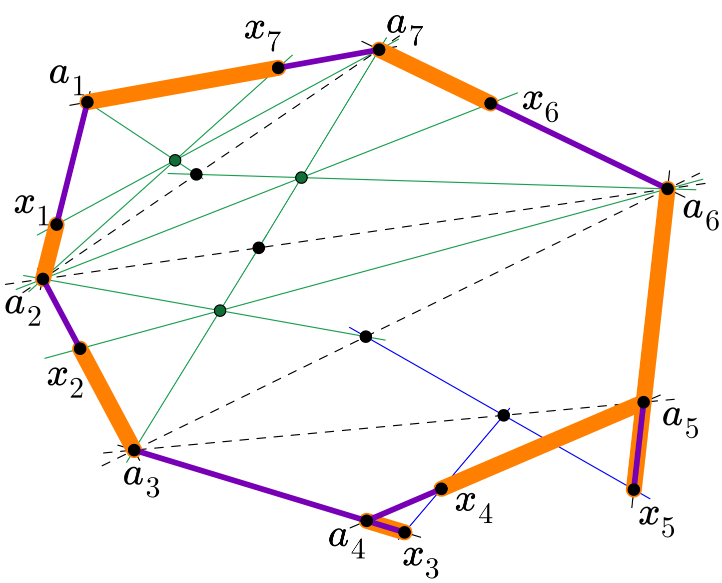


Figure 12.1.: The possibility of triangulating with Ceva and Menelaus configurations is a necessary and sufficient condition for the multi-ratio being 1 (in a non-degenerate instance).

ing this equivalence in the example of the remaining 4-orbit in (12.1). We define $z := \mathbf{meet}(\mathbf{join}(a, c), \mathbf{join}(q_2, q_3))$. This imposes a Menelaus configuration on the triangle with vertices a, e and c. Multiplying the resulting identity

$$\frac{\overline{a\,q_3}}{\overline{q_3\,e}} \cdot \frac{\overline{e\,q_2}}{\overline{q_2\,c}} \cdot \frac{\overline{c\,z}}{\overline{z\,a}} = -1$$

with (12.1) yields

$$\frac{\overline{f\,q_1}}{\overline{q_1\,c}} \cdot \frac{\overline{a\,q_4}}{\overline{q_4\,f}} \cdot \frac{\overline{c\,z}}{\overline{z\,a}} = -1.$$

This equation implies the existence of a Menelaus line and therefore z, q_1 and q_4 have to be collinear. With this construction, we were able to derive an incidence theorem on 16 points very easily (see Figure 12.2).

What we just saw was a Euclidean reasoning for an incidence theorem also holding in projective geometry. Via the area principle, this reasoning can be done via (formal) determinants yielding a more projective reasoning. It can be summarized with Figure 12.3 (where we again omit the curly parentheses at the labeling of the vertices in order to get a clean picture). Black arrows indicate elements of $\uparrow\!\!\dashrightarrow\!\!\downarrow_{\mathbf{B},\mathbf{H}}$ and each set of non-black arrows, which have the same color, corresponds to an element in $\bigcirc_{\mathbf{B}}$. The Γ-cycle we started with is highlighted in blue. The orbits of this cycle can be associated with the length ratio colored in green and in shades of red. However, the graphical version is not enough to see the membership in $\mathcal{E}_{\mathbf{B},\mathbf{H}}$ since one has to also reason about the signature. Multiplying all indicated elements of $\uparrow\!\!\dashrightarrow\!\!\downarrow_{\mathbf{B},\mathbf{H}}$ and all cycles together with their signatures yields an element in $\mathcal{E}_{\mathbf{B},\mathbf{H}}$. In fact, in this example, only the cycles colored in shades of red have signature ϵ. Due to the signatures determined, the overall product

$$\frac{\{\mathbf{a}, \mathbf{q_1}, \mathbf{q_4}\}}{\{\mathbf{c}, \mathbf{q_1}, \mathbf{q_4}\}} \cdot \frac{\{\mathbf{c}, \mathbf{q_2}, \mathbf{q_3}\}}{\{\mathbf{a}, \mathbf{q_2}, \mathbf{q_3}\}}$$

is proven to be an element of $\mathcal{E}_{\mathbf{B},\mathbf{H}}$. Therefore, for any instance P (and non-bold letters for the corresponding points in the instance) we have

$$\Phi^P\left(\frac{\{\mathbf{a}, \mathbf{q_1}, \mathbf{q_4}\}}{\{\mathbf{c}, \mathbf{q_1}, \mathbf{q_4}\}} \cdot \frac{\{\mathbf{c}, \mathbf{q_2}, \mathbf{q_3}\}}{\{\mathbf{a}, \mathbf{q_2}, \mathbf{q_3}\}}\right) = \frac{|a, q_1, q_4|}{|c, q_4, q_1|} \cdot \frac{|c, q_2, q_3|}{|a, q_3, q_2|} = 1$$

which implies the concurrence of the lines $\mathbf{join}(a, c)$, $\mathbf{join}(q_1, q_4)$ and $\mathbf{join}(q_2, q_3)$.

This procedure can be generalized to other oriented Γ-cycles $\overrightarrow{\mathcal{C}}$. First assume that all orbits have length ≥ 3. Here one has to impose incidence relations such that all but one orbit are triangulated with Ceva and Menelaus triangles. Then the

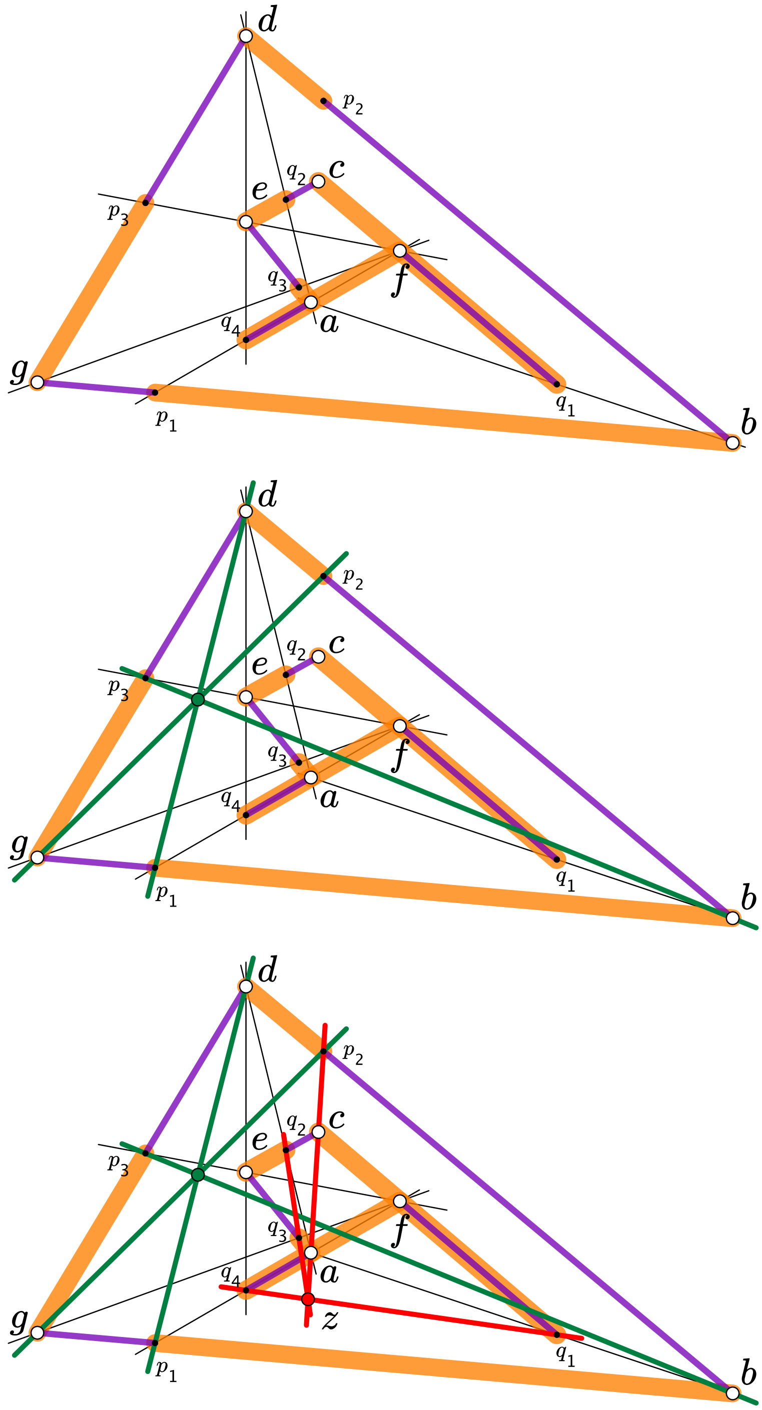


Figure 12.2.: Imposing incidence constraints to the Γ-cycle theorem given in Figure 10.2 results in the incidence theorem stating that the three red lines are concurrent.

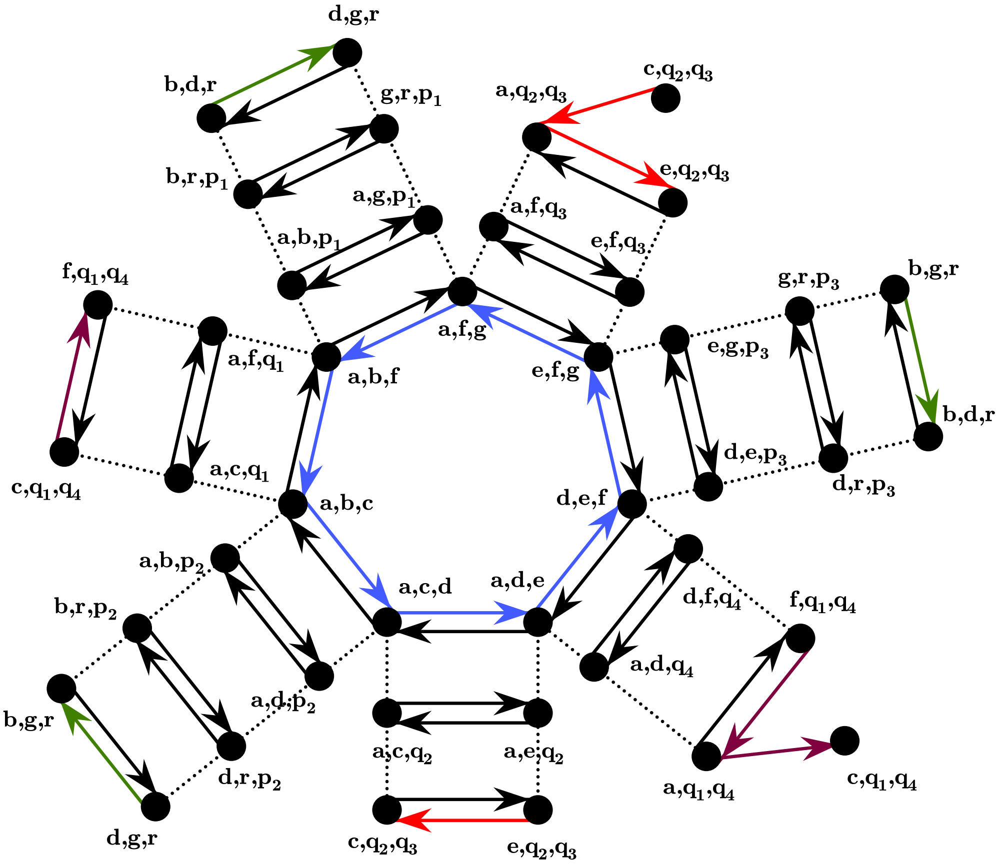


Figure 12.3.: The graphical version of an element of $\mathcal{E}_{\mathbf{B},\mathbf{H}}$. The corresponding element of $\mathcal{E}_{\mathbf{B},\mathbf{H}}$ yields a proof for the incidence theorem given in Figure 12.2.

last orbit can be triangulated with Ceva and Menelaus triangles as well as long as the parity of the total number of Menelaus triangles matches the signature of $\overrightarrow{\mathcal{C}}$. For orbits of length two, the condition, that there exists a triangulation, should be replaced by the fact that the length ratios along both edges of the orbit are identical.

Certainly, this construction may not be possible in any case, since degenerate situations may occur when imposing incidence relations. However, it works in many cases. During this process, there are two steps that are independent of the degenerate situations.

- One needs to know the signature of $\overrightarrow{\mathcal{C}}$.
- One needs to know the number, length and shape of the orbits of $\overrightarrow{\mathcal{C}}$.

We will show how these informations can be easily obtained from encodings of the Γ-cycle in question. Observe that in [2], it is pointed out that glueing the three orbits of length two of the cycle given by the cyclic composition $[1, 1, 1, 1, 1, 1]$, yields Pappos's theorem.

Remark 12.1. Observe that this construction can be generalized in different directions: first, one can close the orbits not only by triangulating them but also by using again more general Γ-cycles. We are not restricted to irreducible Γ-cycle. We can use reducible cycles for closing the orbits and also as a starting point. Of course, we can obtain all reducible Γ-cycles by combining irreducible ones. The larger the orbits of the original cycle, the larger are the straightforward possibilities to close the cycles using irreducible and reducible Γ-cycles. In any cases, informations about signature and orbits are needed.

12.2. Edges of Γ-Cycles and Informations about Orbits

In order to address the problems of signature and orbits, we consider edges in a given irreducible oriented Γ-cycle $\overrightarrow{\mathcal{C}}$ together with its table representation. Assume that for all $\mathbf{p} \in \mathbf{P}$ we have $V_{\mathbf{p}} \neq \mathbf{P}$. For an example see Figure 12.4 which is a continuation of Figure 10.2 and Figure 11.1. To identify the endpoints of the length ratio of an edge $(\mathbf{p}, \mathbf{p}+1)$ (where $\mathbf{p}$ and $\mathbf{p}+1$ are considered to be vertex names), one takes the blue-violet dot from the column labeled $\mathbf{p}$ and the orange dot from the column labeled $\mathbf{p}+1$. Due to the labeling conventions, the orange dot is in position $(\mathbf{p}+1, \mathbf{p}+1)$. For each $\mathbf{p} \in \mathbf{P}$ we define $\sigma(\mathbf{p})$ such that the dot in position $\big(\mathbf{p}, \sigma(\mathbf{p})\big)$ is colored in blue-violet. For considerations similar to the ones done in Section 11.2, σ is a permutation. For each $\mathbf{p} \in \mathbf{P}$, the edge $(\sigma(\mathbf{p}), \sigma(\mathbf{p})+1)$ induces an length ratio whose blue-violet endpoint is $\mathbf{p}$ and whose orange endpoint is $\sigma(\mathbf{p})+1$. Therefore, the endpoints of all length ratios are given by the pairs

$$(\mathbf{p}, \sigma(\mathbf{p})+1).$$

The length ratios arrange in *orbits*. The structure of these orbits can now be defined on a combinatorial level as the cycles of the permutation $\sigma(\mathbf{p}) + 1$. They can be visualized quite well also in the table representation of an irreducible oriented Γ-cycle as it is done in Figure 12.4. For glueing Γ-cycles Theorem 12.2 and Theorem 12.3 might be interesting.

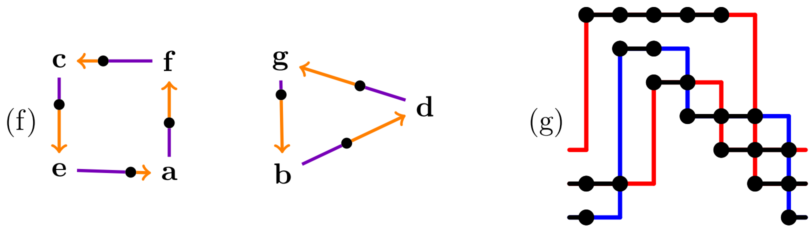


Figure 12.4.: Continuation of Figure 10.2 and Figure 11.1: illustration of the combinatorics of the orbits.

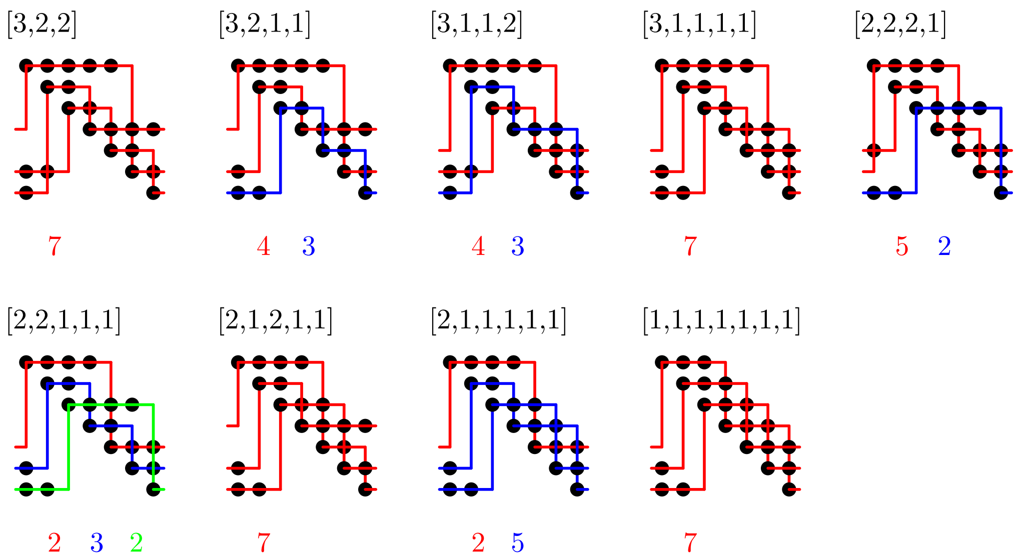


Figure 12.5.: Figure 11.5 with additional information about orbits and its lengths.

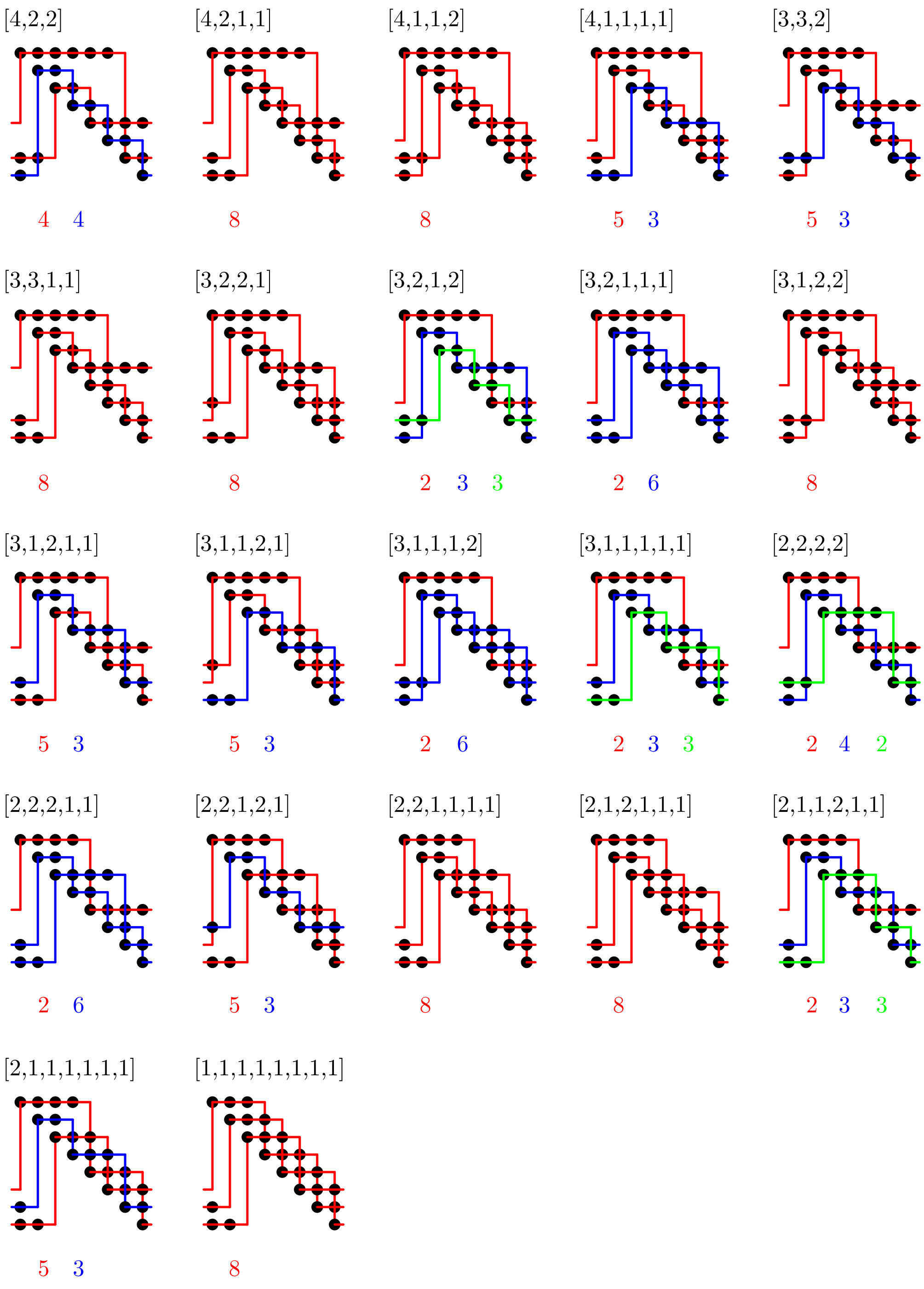


Figure 12.6.: Figure 11.6 with additional information about orbits and its lengths.

Theorem 12.2. *Let $\overrightarrow{\mathcal{C}}$ be an irreducible oriented* Γ*-cycle of length* $k \geq 4$. $\overrightarrow{\mathcal{C}}$ *has at most three orbits.*

Proof. First, assume that for all $\mathbf{p} \in \mathbf{P}$ we have $V_{\mathbf{p}} \neq \mathbf{P}$. We consider the table representation associated with the oriented Γ-cycle $\overrightarrow{\mathcal{C}}$. Due to the possibility of a cyclic shifts, we can w.l.o.g. assume that the first row of the table contains at least three dots and therefore the first rows have a shape as given in Figure 11.3. The labels of the rows and the vertices in this specific table representation induce a *linear* order on the elements of $\mathbf{P}$. Say $\mathbf{p}$ is its first element. Now, the last element is $\mathbf{p} + k - 1$. Using the structure of Figure 11.3 and induction, one sees that in in columns $\mathbf{p} + 2$ up to the last column $\mathbf{p} + (k - 1)$, all dots are located above the diagonal of the table. This means that it holds

$$\sigma^{-1}(\mathbf{x}) \; < \; \mathbf{x} \quad \text{ for } \quad \mathbf{p} + 2 \leq \mathbf{x} \leq \mathbf{p} + (k-1)$$

with respect to the given linear order on $\mathbf{P}$. By letting $\tau(\mathbf{x}) = \sigma(\mathbf{x}) + 1$ this implies

$$\tau^{-1}(\mathbf{x}) = \sigma^{-1}(\mathbf{x} - 1) < \mathbf{x} - 1 \quad \text{ for } \quad \mathbf{p} + 2 \leq \mathbf{x} - 1 \leq \mathbf{p} + (k-1)$$

and therefore

$$\tau^{-1}(\mathbf{x}) < \mathbf{x} \qquad \text{ for } \quad \mathbf{p} + 3 \leq \mathbf{x} \leq \mathbf{p} + (k-1).$$

Thus, there are (at most) three elements $\mathbf{x} \in \mathbf{P}$ such that

$$\tau^{-1}(\mathbf{x}) > \mathbf{x}.$$

We are interested in the cycles of τ. Since the cycles of τ and τ^{-1} are identical and since each cycle of τ^{-1} needs to have an element $\mathbf{x}$ with $\tau^{-1}(\mathbf{x}) > \mathbf{x}$, τ as at most three cycles. Therefore, $\overrightarrow{\mathcal{C}}$ has at most three orbits.

By a similar consideration, also the Γ-cycle with an entry $\mathbf{q}$ contained in every vertex has at most two orbits. To see this, one defines σ on $\mathbf{P} \setminus \{\mathbf{q}\}$ and applies the same considerations as in the previous case. This cycle can be understood a a Γ-cycle in rank 2. □

Theorem 12.3. *Let $\overrightarrow{\mathcal{C}}$ be an irreducible oriented* Γ*-cycle. Assume it corresponds to a cyclic composition* $[l_1, \ldots, l_t]$. *For the signature* $\Phi(\overrightarrow{\mathcal{C}})$ *(see Definition 10.5) holds*

$$\Phi(\overrightarrow{\mathcal{C}}) = \epsilon^t.$$

Proof. Let $\mathbf{p}$ be an element of $\mathbf{P}$ such that in row $\mathbf{p}$ there are $l + 2$ dots for a $l \in \{l_1, \ldots, l_t\}$. Locally, we are in the situation depicted in Figure 11.3. For the edge $(\mathbf{p}+1, \mathbf{p}+2)$ we have

$$\Phi\left(\frac{\mathbf{p}+1}{\mathbf{p}+2}\right) = \Phi\left(\frac{\{\mathbf{x}, \mathbf{p}, \mathbf{p}+1\}}{\{\mathbf{p}+2, \mathbf{p}, \mathbf{p}+1\}}\right) = \epsilon \cdot \frac{[\mathbf{p}, \mathbf{p}{+}1, \quad \mathbf{x}]}{[\mathbf{p}, \mathbf{p}{+}1, \mathbf{p}{+}2]}$$

for some $\mathbf{x} \in \mathbf{P}$. For the edges $(\mathbf{p}+i, \mathbf{p}+i+1)$ $(2 \leq i \leq l)$ we have:

$$\Phi\left(\frac{\mathbf{p}+i}{\mathbf{p}+i+1}\right) = \Phi\left(\frac{\{\mathbf{p}+i-1, \mathbf{p}, \mathbf{p}+i\}}{\{\mathbf{p}+i+1, \mathbf{p}, \mathbf{p}+i\}}\right) = \frac{[\mathbf{p}, \mathbf{p}{+}i-1, \ \ \mathbf{p}{+}i]}{[\mathbf{p}, \mathbf{p}{+}i, \mathbf{p}+i+1]}.$$

Since the edge $(\mathbf{p}+l, \mathbf{p}+l+1)$ is edge $(\mathbf{q}, \mathbf{q}+1)$ for $\mathbf{q} = \mathbf{p}+l$, we can restate the whole game with $\mathbf{q}$ in the role of $\mathbf{p}$. We obtain:

$$\prod_{j=0}^{k-1} \Phi\left(\frac{\mathbf{p}+j}{\mathbf{p}+j+1}\right) = \prod_{r=1}^{t}\left(\epsilon \cdot \frac{[\mathbf{p}_r, \mathbf{p}_r{+}1, \quad \mathbf{x}_r]}{[\mathbf{p}_r, \mathbf{p}_r{+}1, \mathbf{p}_r{+}2]} \cdot \prod_{2 \leq i \leq l_r} \frac{[\mathbf{p}_r, \mathbf{p}_r{+}i-1, \ \ \mathbf{p}_r{+}i]}{[\mathbf{p}_r, \mathbf{p}_r{+}i, \mathbf{p}_r+i+1]}\right)$$

where each $\mathbf{p}_r$ corresponds to l_r $(1 \leq r \leq t)$. Since all determinants are positively oriented with respect to the given cyclic order of $\mathbf{P}$ they cancel and

$$\prod_{j=0}^{k-1} \Phi\left(\frac{\mathbf{p}+j}{\mathbf{p}+j+1}\right) = \epsilon^t$$

proves the statement. □

13. Relations between Γ-Cycles and Results of Grünbaum and Shephard

As previously mentioned in Section 3.1, the usage of the area method as used here is taken from Grünbaum and Shephard [53, 52, 54]. Earlier use of the area and the volume principle was also treated in Section 3.1. In [53] a special kind of theorems about oriented length ratios is investigated: the cyclically ordered vertices $(v_1, \dots, v_k)$ of a k-gon in the affine plane are considered. The authors want to assign (oriented) length ratios to edges or diagonals *in a cyclic way* and in such a fashion that the product of the ratios equals $+1$ or -1. This means cyclically putting Figure 3.2 on the edges or diagonals of the polygon. The transversal is supposed to be spanned by vertices of the k-gon. The formulation of Theorem 3 in [53] summarizes this. It investigates when one can define and assure that

$$\prod_{i=1}^{k} \frac{\overline{v_i\, w_i}}{\overline{w_i\, v_{i+j}}} = (-1)^k \tag{13.1}$$

(indices considered modulo k) for suitable values of j, r and s and with w_i being the intersection of the line spanned by v_i and v_{i+j} and the line spanned by v_{i+r} and v_{i+s}. Hence, all points of Figure 3.2 are shifted cyclically in the k-gon. In order to get a better impression, we give an examples in Figure 13.1). In the proof of the theorem, it is shown, that there is always an underlying $[1, \dots, 1]$-cycle (or a statement involving trivially pairwise canceling length ratios). The different cases in the original formulation of the theorem correspond to different *cyclic* labelings of the points in the same given Γ-cycle. We think that this restriction to perfectly cyclic products of k factors is very strict. At this point, we want to compare this with Γ-cycle theorems in Definition 10.5. They also give theorems about products of k oriented length ratios. In principle, they can be quite asymmetric as the one shown in Figure 10.2. However, there are Γ-cycle theorems that are cyclic but with period bigger than 1. See Figure 13.2 for Γ-cycle theorems corresponding to the cyclic compositions $[3, 3, 3, 3]$ (one orbit) and $[3, 1, 3, 1, 3, 1]$ (three symmetric orbits). The vertices are arranged in a regular manner in order to emphasize the symmetry. Clearly, the theorems are valid for all positions of the vertices except for

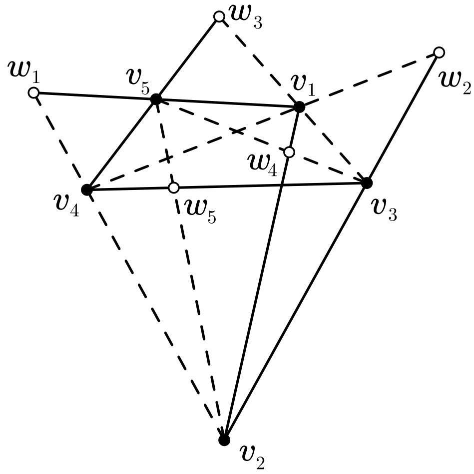


Figure 13.1.: With $k = 5,\ j = 4,\ r = 1$ and $s = 3$, (13.1) holds. By applying the permutation that is given by the cycle $(5, 3, 2, 4)$ to the indices, this is identical to a figure given in [53]. There, the case $k = 5,\ j = 2,\ r = 3$ and $s = 4$ is considered.

degenerate situations. Figure 13.2 should be read as follows: blue-violet and orange line segments supported by the same line form oriented length ratios. The product

$$\prod \frac{\text{length}}{\text{length}} \tag{13.2}$$

equals $+1$ or -1 depending on the signature of the underlying Γ-cycle. Due to Theorem 12.3, the product of length ratios given in (13.2) equals 1 in both cases of Figure 13.2.

Including Reasonable Dependent Points

Based on the structure of $[3, 1, 3, 1, 3, 1]$, one can also construct theorems about oriented length ratios including *depending points* as shown in Figure 13.3. It was achieved by a procedure which can also be described more generality: we replace each edge

$$\frac{\{\mathbf{a}, \mathrm{x}, \mathrm{y}\}}{\{\mathbf{b}, \mathrm{x}, \mathrm{y}\}}$$

of an irreducible oriented Γ-cycle $\overrightarrow{\mathcal{C}}$, which corresponds to a cyclic composition, by the product of two edges

$$\frac{\{\mathbf{a}, \mathrm{x}, \mathrm{y}\}}{\{\mathbf{w}, \mathrm{x}, \mathrm{y}\}} \cdot \frac{\{\mathbf{w}, \mathrm{x}, \mathrm{y}\}}{\{\mathbf{b}, \mathrm{x}, \mathrm{y}\}} \tag{13.3}$$

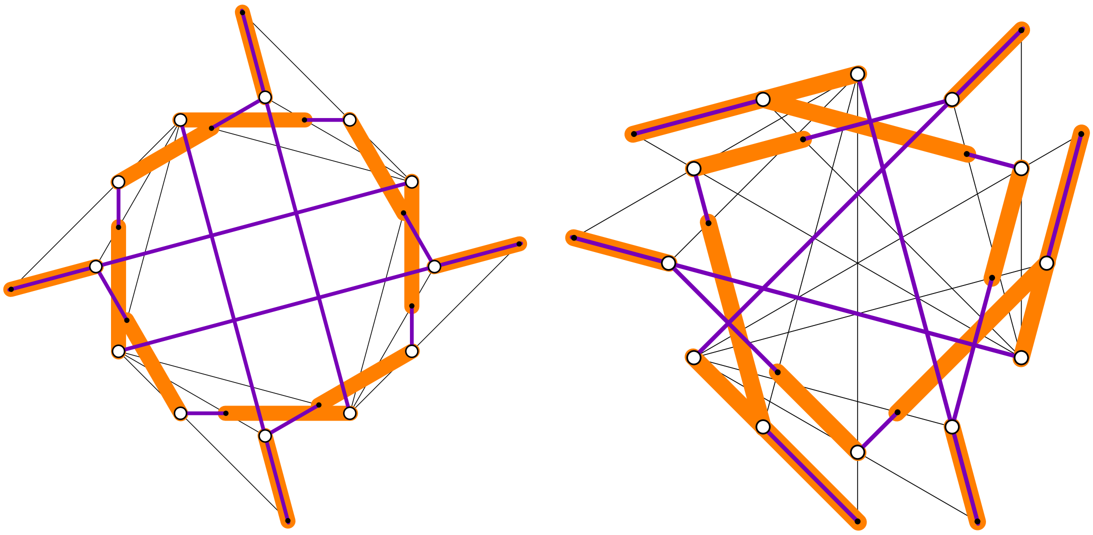


Figure 13.2.: Instances of Γ-cycle theorems corresponding to the cyclic compositions $[3,3,3,3]$ and $[3,1,3,1,3,1]$.

for a newly introduced $\mathbf{w}$. This gives a new (reducible) oriented Γ-cycle. In principle, this induces a Γ-cycle theorem with $2\,k$ length ratios. However, we can also encode additional collinearities in a set $\mathbf{H}$: for each edge, we choose $\{\mathbf{a},\mathbf{w},\mathbf{x}\}$, $\{\mathbf{w},\mathbf{y},\mathbf{b}\} \in \mathbf{H}$ or $\{\mathbf{a},\mathbf{y},\mathbf{w}\}$, $\{\mathbf{w},\mathbf{x},\mathbf{b}\} \in \mathbf{H}$. Assume the latter option is used for some edge. For an instance P, applying Φ^P to (13.3) yields

$$\frac{\overline{a\,y}}{\overline{y\,w}} \cdot \frac{\overline{w\,x}}{\overline{x\,b}}, \tag{13.4}$$

if we are using non-bold letters to identify the corresponding bold letters. Since the underlying cycle $\overrightarrow{\mathcal{C}}$ is irreducible, there is a first vertex following $\{\mathbf{b},\mathbf{x},\mathbf{y}\}$ in the cyclic order of the vertices, that does not contain both elements $\mathbf{b}$ *and* $\mathbf{x}$. Therefore, there are some $\mathbf{u}$ and $\mathbf{v}$ such that either

$$\frac{\{\mathbf{x},\mathrm{b},\mathrm{u}\}}{\{\mathbf{v},\mathrm{b},\mathrm{u}\}} \text{ or } \frac{\{\mathbf{b},\mathrm{x},\mathrm{u}\}}{\{\mathbf{v},\mathrm{x},\mathrm{u}\}}$$

is an edge in the original cycle $\overrightarrow{\mathcal{C}}$. In the general approach given in (13.3), this edge is transferred to

$$\frac{\{\mathbf{x},\mathrm{b},\mathrm{u}\}}{\{\mathbf{w}',\mathrm{b},\mathrm{u}\}}\cdot\frac{\{\mathbf{w}',\mathrm{b},\mathrm{u}\}}{\{\mathbf{v},\mathrm{b},\mathrm{u}\}} \qquad \text{or} \qquad \frac{\{\mathbf{b},\mathrm{x},\mathrm{u}\}}{\{\mathbf{w}',\mathrm{x},\mathrm{u}\}}\cdot\frac{\{\mathbf{w}',\mathrm{x},\mathrm{u}\}}{\{\mathbf{v},\mathrm{x},\mathrm{u}\}} \tag{13.5}$$

for some newly introduced $\mathbf{w}'$. Requiring $\{\mathbf{w}',\mathbf{x},\mathbf{b}\}$, $\{\mathbf{w}',\mathbf{u},\mathbf{v}\}\in\mathbf{H}$ and applying Φ^P to (13.5) yields

$$\frac{\overline{x\,b}}{\overline{b\,w'}}\cdot\frac{\overline{*\,*}}{\overline{*\,*}} \qquad \text{or} \qquad \frac{\overline{b\,x}}{\overline{x\,w'}}\cdot\frac{\overline{*\,*}}{\overline{*\,*}}. \tag{13.6}$$

Hence, in the product of all induced length ratios, two line segments in (13.4) and (13.6) cancel and a theorem on directed length ratios of less than the assumed $2\,k$ ratios is derived. In order to state it correctly, one has to generalize the notion of oriented length ratios along the same supporting line. Therefore we consider numerator and denominator as directed line segments and compare their or orientations. Line segments with the same orientation have positive ratios while line segments with different orientations have negative ratios. In Figure 13.3, the **w**s are indicated by black dots and they are defined in a way that many ratios cancel. The white dots are points associated to the entries of the original irreducible Γ-cycle $\overrightarrow{\mathcal{C}}$. The directions are such that the blue-violet segments point from white points to black points and the orange segments the other way round. For the underlying cycle corresponding to the cyclic composition $[1,\ldots,1]$ it is possible to achieve cancellation everywhere giving theorems involving k length ratios. This results in Hoehn's (first) theorem for k-gons (see [66, 53]). Observe that the Hoehn's second theorem (as given in [53]) can be obtained in essence by a substitution form the first one.

The Selftransversality

The results in [52] can be understood as a generalization of the cyclic products investigated in [53] to higher dimensions. In [54], the general approach is amplified such that the (cyclic) transversal (see Figure 3.1) is spanned by points that are intersections of lines spanned by diagonals of the polygon. Two setups are considered. The *first transversality property* holds whenever

$$\prod_{i=1}^{k}\frac{\overline{v_{i-m}\,w_i}}{\overline{w_i\,v_{i+m}}}=1,$$

where indices considered modulo k. Now, w_i is the intersection of the reference line spanned by v_{i-m} and v_{i+m} and the line spanned by v_i and z_i. Here z_i is the intersection of the line spanned by v_{i-r} and v_{i-s} and the line spanned by v_{i+r} and v_{i+s} for suitable parameters k, m, r and s. The structures for which this property

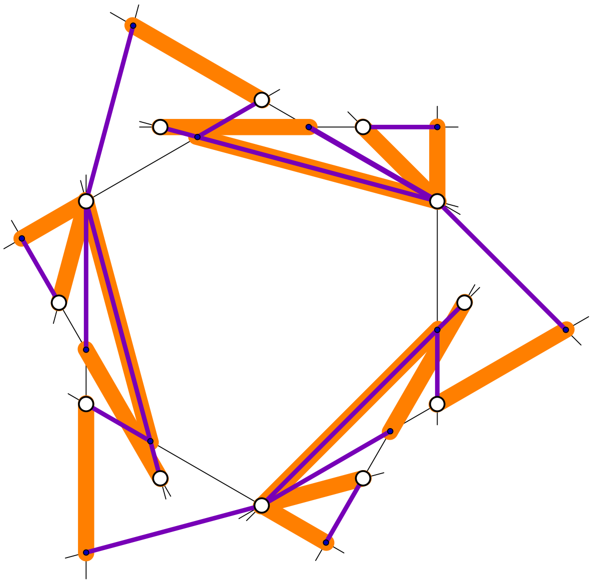


Figure 13.3.: Introducing dependent points in the Γ-cycle associated with $[3, 1, 3, 1, 3, 1]$ yields a statement about 15 oriented length ratios.

holds can be determined and can be stated concisely in [54, Thm. 1]. Figure 13.4 gives examples for the previous and the following transversality property. A further generalization of this property leads to the *second transversality property* stating that

$$\prod_{i=1}^{k} \frac{\overline{v_i\, w_i}}{\overline{w_i\, v_{i+m}}} = 1$$

(indices considered modulo k) where again w_i is the intersection of the reference line spanned by v_i and v_{i+m} and the line spanned by some y_i and z_i. y_i is intersection of $v_{i-t}v_{i-u}$ and $v_{i+m+t}v_{i+m+u}$ and z_i is intersection of $v_{i-r}v_{i-s}$ and $v_{i+m+r}v_{i+m+s}$ for suitable parameters k, m, r, s, t and u. There are several parameter sets for which this property holds. They are listed and proven in [54, Thm. 2–5]. Unfortunately, the systematics of the admissible parameters is not very concise. It is proven to be exhaustive up to $k = 20$. It should be said, that all proofs can be paraphrased in terms of $\mathcal{E}_{\mathbf{B},\mathbf{H}}$ by hand calculation. To do so, one needs in essence three tools which we will describe in the following: first consider a fraction of two determinants given in the proof. As long as they do not differ by exactly one element, the ratio can be written as a product of at most three ratios of determinants differing

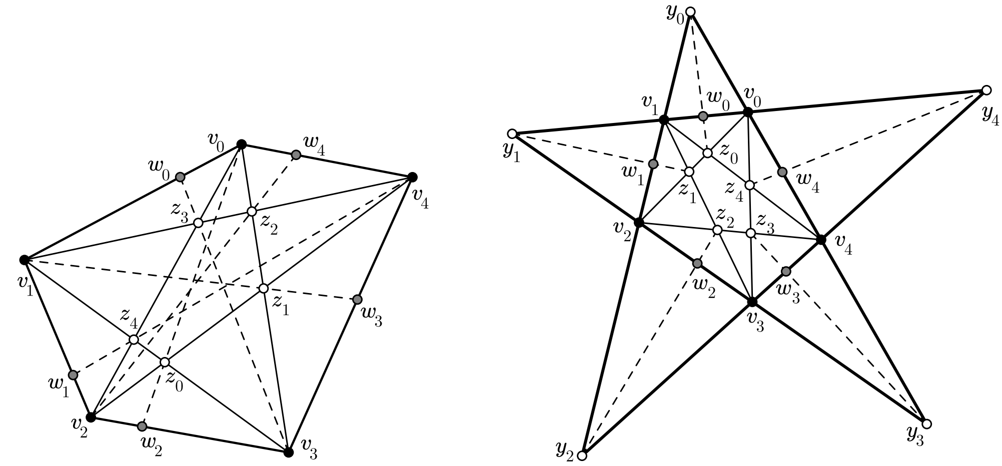


Figure 13.4.: Examples for the 1st and the 2nd transversality property taken form [54] and adjusted to the notation. On the left-hand side, e.g. the values $k = 5$, $m = 2$, $r = 1$ and $s = -1$ correspond to the above description. On the right-hand side $k = 5$, $m = 1$, $r = 1$, $s = -1$, $t = 0$ and $u = 2$ can be taken as parameters.

by exactly one element. E.g. the product $\frac{\{\mathbf{a},\mathbf{b},\mathbf{c}\}}{\{\mathbf{b},\mathbf{c},\mathbf{d}\}} \cdot \frac{\{\mathbf{b},\mathbf{c},\mathbf{d}\}}{\{\mathbf{c},\mathbf{d},\mathbf{e}\}}$ can be used to model a fraction where the determinants differ by two elements (and assuming a non-degenerate situation, i.e. $\{\mathbf{b},\mathbf{c},\mathbf{d}\} \in \mathbf{B}$). This technique is also used in Part III (see Section 16.1) in order to have a geometric interpretation of bracket ratios which cannot be directly interpreted via the area principle. Furthermore, similar to Remark 10.7, the rearrangement of numerators and denominators into other ratios can be done by multiplying a suitable Γ-cycle. The last technique used in the proofs of [54] is the *elimination lemma*. It states for a being the intersection of the line spanned by d and e and the line spanned by f and g, and for two additional points b and c, it holds in non-degenerate situations that

$$\frac{|d,a,b|}{|f,a,c|} = \frac{|d,e,b|}{|f,g,c|} \cdot \frac{|f,g,d|}{|e,d,f|}.$$

This can be modeled as the

$$\frac{\{\mathbf{d},\mathbf{a},\mathbf{b}\}}{\{\mathbf{d},\mathbf{f},\mathbf{a}\}} \cdot \frac{\{\mathbf{d},\mathbf{f},\mathbf{a}\}}{\{\mathbf{f},\mathbf{a},\mathbf{c}\}} \quad \cdot \quad \frac{\{\mathbf{e},\mathbf{d},\mathbf{f}\}}{\{\mathbf{d},\mathbf{e},\mathbf{b}\}} \cdot \frac{\{\mathbf{f},\mathbf{g},\mathbf{c}\}}{\{\mathbf{f},\mathbf{g},\mathbf{d}\}}$$

being in $\mathcal{E}_{\mathbf{B},\mathbf{H}}$. These three tools can be used to transfer any proof given in [54] into a statement in $\mathcal{BG}_{\mathbf{B}}$. The fact that this translation is possible is not surprising and fits with the results [39, 38, 40, 41, 118] of Dress and Wenzel, which treat the relations between different types of Tutte groups. Typically, during the translation process, there will be very symmetric underlying Γ-cycles. Parts of the structure of the proof of the first transversality property can be applied to our standard example (see Figure 10.2, Figure 11.1 and Figure 12.4) giving an identity on oriented length ratios shown in Figure 13.5.

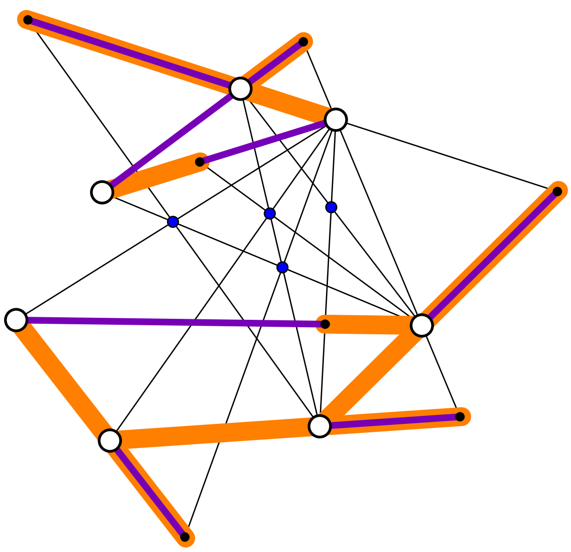


Figure 13.5.: An identity on oriented length ratios inspired by the proof of the first transversality property in [54]. The underlying cycle can be identified with $[3, 1, 1, 2]$ and is also shown in Figure 10.2.

Part III.

Cayley Factorization

14. Statement of the Problem and Progress so Far

This part of the thesis gives a new solution for (a variant of) the Cayley factorization problem: we start by describing the problem. We work in rank 3 as it is done in [104] by Sturmfels and Whiteley. In principle, the ideas can also be generalized to arbitrary rank by in essence adjusting the definitions in Section 16.1. Cayley factorization can be understood as translating algebra to geometry. For another introduction to this more general topic located in the context of Grassmann-Cayley algebra, see again [37, p. 213 ff.]. Remember from Theorem 5.18 that any synthetic construction s (see Definition 5.17) *is* a bracket polynomial B in $\Lambda(\mathbf{P})$. In order to see this, one has to evaluate or expand the synthetic construction (see e.g. Chapter 6). Since s is a construction it has a geometric meaning. Say s can be read as the join of three (dependent) points. In this case, s describes the construction of three points by means of join and meet. They are collinear if and only if B evaluates to zero (for a given instance). Theorem 5.18 also states that this bracket polynomial is multihomogenous and has integer coefficients. In the situation just described, one translates the collinearity of points into the vanishing of a polynomial. One can also go a step further: we can also interpret the equality for any resulting number: s induces a construction of three points by **join** and **meet** based on the coordinates given in P. Computing the determinant of these resulting points will be equivalent to applying Φ^P to B. Interpreting the vanishing of a bracket polynomial is much more projective then the interpretation of resulting numbers, since the vanishing of a multihomogeneous bracket polynomial is a projectively invariant property. Therefore, the vanishing usually gains more attention. In total, we have that s provides a geometric interpretation for the resulting evaluation B. B in turn is multihomogeneous and has integer coefficients. This, and the knowledge of Pascal's theorem (see Section 3.10 and Chapter 6) motivates a first version of the Cayley factorization problem. It is the reversal of the process just described

Problem 14.1 (Cayley factorization). Given a multihomogeneous bracket polynomial B with integer coefficients. Is there a synthetic construction which equals B in the bracket ring? If yes, give one, otherwise output "not Cayley factorable"

There are situations, in which a bracket polynomial is derived in an algebraic manner and a geometric interpretation of it is missing. As mentioned before, one often wants to describe the conditions under which a point configuration P yields

$$\Phi^P(B) = 0. \tag{14.1}$$

Again, due to Section 4.6, this condition describes a projectively invariant property and the first fundamental theorem of invariant theory (Theorem 4.9) is valid. So a geometric interpretation of a multihomogeneous bracket polynomial is in fact a geometric interpretation of a very general concept.

The previously mentioned situations providing B include robotics, statics, rigidity of frameworks and scene analysis (see e.g. [125, 127, 33, 129, 28, 81, 32]). [27, p. 528] has the intention to "make clear why Cayley factorization of invariant polynomial forms is both a crucial and rather difficult problem." In order to underline this intention it provides many examples of possible applications of Cayley factorization within scene analysis and structural mechanics. It is pointed out that the approach of the special case of multilinear Cayley factorization (see below) has its limitations and it is suggested to look "for a alternative approach to the factorization problem, one which will hopefully respect the advice of geometric intuition." We think that our algorithm can go another step in this direction, especially the interpretation of bracket *bi*nomials is very close to geometry. For an example of the quality of interpretation we refer to Section 16.7. Furthermore, using methods given by Whiteley in [134] and also outlined by Sturmfels in [102], it can be embedded in a bigger procedure starting with expressions in coordinates and resulting in synthetic construction. This may help to interpret non-degeneracy conditions in automated geometric theorem proving. Furthermore, most references given elsewhere in the present chapter, as well as [123], also contain introductions and overviews of the development. The problem is also mentioned in the overview [50, Note 10] over Rota's work in invariant theory.

On the concrete problem on Cayley factorization, there exists an algorithm due to White that decides the special case of the multi*linear* problem (see [128, 124]). It is uses the straightening algorithm several times in order to identify possible joins and meets and in order to factor. A concise exposition of this algorithm is also given in [102]. In [80] by Li and Wu one can find algorithms for factoring bracket polynomials up to a specific degree in the brackets. There are also some cases of addressing very special shapes of bracket polynomials, e.g. [106]. The latter appears to be a generalization of phenomena in [27].

We will now focus on the approach considered in [104] by Sturmfels and Whiteley. They give an explicit non-factorable example in order to motivate a variant of the Cayley factorization problem. We choose a different example which also qualifies for this purpose. Consider the bracket polynomial which is inspired by Menelaus's theorem depicted in Figure 3.3:

$$[\mathbf{x_1},\mathbf{x_2},\mathbf{a}][\mathbf{y_1},\mathbf{y_2},\mathbf{b}][\mathbf{z_1},\mathbf{z_2},\mathbf{c}] - [\mathbf{x_1},\mathbf{x_2},\mathbf{b}][\mathbf{y_1},\mathbf{y_2},\mathbf{c}][\mathbf{z_1},\mathbf{z_2},\mathbf{a}] \quad \in \mathcal{B}_{\mathbf{P}}, \tag{14.2}$$

for some set $\mathbf{P}$ containing the distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{y_1}$, $\mathbf{y_2}$, $\mathbf{z_1}$ and $\mathbf{z_2}$. This polynomial is multilinear and an execution of White's algorithm (see [124]) for deciding the multilinear Cayley factorization problem shows that it is not Cayley factorable. Alternatively, one can achieve this insight also by an exhaustive testing of all possible synthetic constructions using each letter of the polynomial exactly once. However, there is an interpretation of the vanishing of (14.2) in a (non-degenerate) instance P:

$$\frac{[\mathbf{x_1},\mathbf{x_2},\mathbf{a}]_P}{[\mathbf{x_1},\mathbf{x_2},\mathbf{b}]_P}\cdot\frac{[\mathbf{y_1},\mathbf{y_2},\mathbf{b}]_P}{[\mathbf{y_1},\mathbf{y_2},\mathbf{c}]_P}\cdot\frac{[\mathbf{z_1},\mathbf{z_2},\mathbf{c}]_P}{[\mathbf{z_1},\mathbf{z_2},\mathbf{a}]_P} = 1.$$

Due to the area principle (see Section 3.1), in the instance (assuming finite positions of the points), this translates into an identity where a product of length ratios along the sides of a triangle equals -1. This is exactly the situation of Menelaus's theorem and the identity can be tested by checking for the existence of a Menelaus line. We state the corresponding construction directly on the symbolic level in the Grassmann-Cayley algebra yielding a synthetic construction:

$$\begin{aligned}
&\big(\mathbf{x_1x_2}\wedge\mathbf{ab}\big)\big(\mathbf{y_1y_2}\wedge\mathbf{bc}\big)\big(\mathbf{z_1z_2}\wedge\mathbf{ca}\big)\\
&=\Big([\mathbf{x_1},\mathbf{x_2},\mathbf{b}]\,\mathbf{a}-[\mathbf{x_1},\mathbf{x_2},\mathbf{a}]\,\mathbf{b}\Big)\Big([\mathbf{y_1},\mathbf{y_2},\mathbf{c}]\,\mathbf{b}-[\mathbf{y_1},\mathbf{y_2},\mathbf{b}]\,\mathbf{c}\Big)\\
&\qquad\qquad\qquad\qquad\Big([\mathbf{z_1},\mathbf{z_2},\mathbf{a}]\,\mathbf{c}-[\mathbf{z_1},\mathbf{z_2},\mathbf{c}]\,\mathbf{a}\Big)\\
&=[\mathbf{a},\mathbf{c},\mathbf{b}]\ \Big([\mathbf{x_1},\mathbf{x_2},\mathbf{a}][\mathbf{y_1},\mathbf{y_2},\mathbf{b}][\mathbf{z_1},\mathbf{z_2},\mathbf{c}]-[\mathbf{x_1},\mathbf{x_2},\mathbf{b}][\mathbf{y_1},\mathbf{y_2},\mathbf{c}][\mathbf{z_1},\mathbf{z_2},\mathbf{a}]\Big).
\end{aligned}$$

This is a bracket monomial multiple of (14.2). In [104] it is shown, that this phenomenon can be generalized to *arbitrary* bracket polynomials qualified for Cayley factorization.

Theorem 14.2 (Sturmfels, Whiteley)**.** *Given a multihomogeneous bracket polynomial B with integer coefficients. Then there exists a bracket monomial M such that the product $M\cdot B$ can be factored yielding a synthetic construction.*

This theorem has consequences for testing (14.1) for an instance P: the equation $\Phi^P(B) = 0$ now is known to be equivalent to a ruler construction, which tests whether three (dependent) points are collinear (or three lines meet in a point), as long as the point-triples corresponding to the brackets of the monomial M of are not collinear. Theorem 14.2 motivates the generalized Cayley factorization problem (see also [102]):

Problem 14.3 (Generalized Cayley Factorization Problem). Given a multihomogenous bracket polynomial B with integer coefficients. Find a synthetic construction such that it equals $M \cdot B$ in the bracket ring with a bracket monomial M and the degree of M in the brackets being *minimal* among all candidates.

We remark that this version of Cayley factorization is called *rational* Cayley factorization in [77] by Li. There, it is considered to be of great importance for actual computations with bracket polynomials. The book also contains a rational Cayley factorization algorithm for *conic geometry*. It treats only configurations where several points lie on a common conic. For the generalized Cayley factorization problem one could argue that it is very restrictive to allow only bracket *monomial* multipliers and not arbitrary non-zero bracket polynomials. However, bracket monomial have some advantages: the first one is that the vanishing of a bracket monomial can be easily interpreted geometrically. Now one could wish to also allow Cayley factorable multipliers. This could be done but this is less convenient for concrete calculations. It is much easier to deal with bracket monomials. Furthermore it is non-trivial to ensure that that a general multiplier monomial does not vanish in $\mathcal{B}_{\mathbf{P}}$. These might be the reasons why the literature states the problem as it does. We follow this approach as well.

We chose a motivating example different from the one given in [104] and [102] since it provides a small example where the multiplier monomial consists of only a single bracket (see also [102, p. 117: Exer. (2)$*$]). The bound for the monomial M given in analysis of the original proof of Theorem 14.2 and for the common special case, that the coefficients of the bracket polynomial are in $\{-1, 1\}$, is (see [104, p. 449])

$$105 \cdot (\# \text{ summands in } B) \cdot (\# \text{ brackets per summand of } B)$$

referring to the specific representation of the input bracket polynomial B. The algorithm given in this thesis will be able to reduce this bound to

$$9 \cdot (\# \text{ summands in } B - 1) \cdot (\# \text{ brackets per summand of } B).$$

The size of the synthetic construction is directly reflected in the degree of the multiplier monomial. Therefore, we obtain a much less complicated construction. The approach is able to explain the introducing example of Pascal's theorem by factoring the corresponding bracket polynomial (see Section 16.7). Furthermore, when locally optimizing the choices in the execution of the algorithm, Pascal's construction can be obtained automatically. In [104], the size of the construction is due to essentially introducing a *global* coordinate system. The bracket polynomial is broken up into elementary arithmetic operations which in turn are mimicked by von Staudt constructions. In order to use this coordinate system, every calculation has to be represented on a single geometric line, along which it is geometrically calculated afterwards. This reduction needs a lot of join and meet operations. In [102, p. 112] it is commented to by one of the authors as being “far too general to be of practical use”. This statement goes with the fact that not a single example of the algorithm is worked out in [104]. Using the letter P for what is called B in the present text, White comments in [126]: “However, M may be of high degree and may be non-multilinear (even if P is multilinear) and the resulting geometric condition equivalent to $M\,P = 0$ may be as uninteresting as the geometric construction corresponding to the complete expansion of P as a determinantal expression in the coordinates of its points. The monomial M may also be thought of as a non-degeneracy condition: M must be non-zero in order for $P = 0$ to imply the desired geometric condition”. And in [104] it is commented by the authors: “Let us remark that our factorization theorem is by no means competitive to the White algorithm because it is too general to be useful for geometrically interpreting concrete algebraic expressions. White’s results, however, suggest that Cayley factorization is not only an elegant mathematical tool but that it might play a substantial role in future "Geometric Algebra" computer systems (cf. Section 3). Our theorem shows that there is no theoretical block to factoring all expressions.”

It is the purpose of this part of the present thesis to reprove Theorem 14.2 giving a better upper bound for the degree of the multiplier monomial. Furthermore, the algorithm given in the proof is concise enough to be implemented in Mathematica using the *Combinatorica* package ([136], see Chapter 18 for samples of the output). The approach can be considered as introducing ad-hoc *local* 1-dimensional coordinate systems whenever the area principle is applied and length ratios are combined. Chapter 15 demonstrates the main ideas with the help of an instructive example. The implementation is able to compute Cayley factorizations of even “medium sized” bracket polynomials. The three following chapters aim to prove the following two statements:

Theorem 14.4 (Binomial (Generalized) Cayley Factorization)**.** *Let $\mathbf{P}$ be a finite set and let $B \in \mathbb{R}[\Delta(\mathbf{P})]$ be a multihomogenous* bi-*nomial of rank* 3 *with coefficients in $\{-1, +1\}$. Assume B's summands do not have a common bracket factor and that they are not in $\langle \mathbf{I}_{repeat} \rangle$. Let $k \geq 2$ be the degree of B in its brackets. There is an algorithm running in time $O(k^3)$ that outputs two simple Grassmann-Cayley expressions*

$$\mathbf{1} = \mathbf{a}\,\mathbf{b} \wedge \mathbf{0}\,\infty \quad \text{with } \mathbf{a}, \mathbf{b}, \mathbf{0}, \infty \in \mathbf{P} \qquad \textit{and} \qquad \mathbf{x} = \mathbf{A} \wedge \infty\,\mathbf{0} \quad \text{with } \mathbf{A} \in \Lambda^2(\mathbf{P}) \text{ representing a line}$$

such that

$$\mathbf{a}\,\mathbf{b} \wedge \mathbf{x} \quad \textit{equals} \quad M \cdot B$$

in $\mathcal{B}_{\mathbf{P}}$ for a bracket monomial $M \in \mathbb{R}[\Delta(\mathbf{P})]$ yielding a Cayley factorization of B (considering M and B as element of $\mathcal{B}_{\mathbf{P}}$). Furthermore, me have

$$\deg(M) \leq 5\,k.$$

Theorem 14.5 ((Generalized) Cayley Factorization)**.** *Let $\mathbf{P}$ be a finite set and let $B \in \mathbb{R}[\Delta(\mathbf{P})]$ be multihomogenous, with integer coefficients and of rank 3. Assume B's summands do not have a common bracket factor. Let $k \geq 2$ be the degree of B in its brackets and $l+1$ its number of summands (which are supposed to be not in $\langle \mathbf{I}_{repeat} \rangle$). There is an algorithm running in time $O(l \cdot k^3)$ that outputs a synthetic construction identical to $M \cdot B$ for a bracket monomial $M \in \mathbb{R}[\Delta(\mathbf{P})]$. Furthermore, the degree of M is bounded by*

$$10\,k\,l + 3\,\alpha$$

in the general case (where α is the sum of the absolute values of the coefficients of B) and by

$$9\,k\,l$$

in the case that the coefficients of B lie in $\{-1, +1\}$.

Restricting k to be bigger than 1 is no big restriction since multihomogenous bracket polynomials of degree 1 in its brackets are rather boring: they are just $\mathbb{Z}$-multiples of a single bracket.

14.1. Conventions about Notions and Evaluations

In Theorem 14.4 and everywhere else in the following, the color in which some letters are printed is *not* part of the name of the point. It is just used to highlight the role, a point plays in the current setup. We will use boldface letters for points in $\Lambda^1(\mathbf{P})$. Whenever the letters $\mathbf{0}$, ∞, $\mathbf{0}'$, ∞', $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}'$ and $\mathbf{b}'$ are used, they are elements of $\mathbf{P}$. Furthermore, we will use schematic diagrams that illustrate constructions in $\Lambda(\mathbf{P})$ (see e.g. Section 16.4). These constructions are labeled with elements of $\Lambda^1(\mathbf{P})$. They are not thought as depicting an instance but to illustrate the combinatorics of the construction. Instead of only giving a formula in $\Lambda(\mathbf{P})$ for constructed points,s we also give the corresponding tensor diagram together with some rules of evaluation explained in Section 6.3. This evaluation yields the result stated there and no intermediate step contains more summands than the result (except for the evaluation in Section 17.1 which can be considered to lie on the borderline of two and three summands, since integers are constructed). In principle one can also perform classical evaluations. The formulas are stated in such a way that evaluating $\mathbf{ab} \wedge \mathbf{cd}$ as $[\mathbf{a}, \mathbf{b}, \mathbf{d}]\mathbf{c} - [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d}$ (where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \Lambda^1(\mathbf{P})$) also yields the result stated. With this information at hand, one can also use software as the GrassmannAlgebra package for Mathematica (see [136]), which is available under `www.grassmannalgebra.info` together with the accompanying book [13], to verify the computations. Observe that in classical evaluations it will happen that the result consists of few (e.g. two) summands but in the steps in between, a lot of summands are produced.

15. Instructive Binomial Example

In this chapter, we introduce the main ideas of the binomial Cayley factorization with the help of an instructive example. The general algorithm will be given in Chapter 16. We follow [1], which was given earlier. Consider the (representation of) the multihomogenous bracket binomial (and let $\mathbf{P} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}\}$)

$$B = [\mathbf{a}, \mathbf{b}, \mathbf{g}][\mathbf{a}, \mathbf{c}, \mathbf{e}][\mathbf{b}, \mathbf{c}, \mathbf{f}][\mathbf{e}, \mathbf{d}, \mathbf{f}][\mathbf{a}, \mathbf{f}, \mathbf{h}] - [\mathbf{a}, \mathbf{e}, \mathbf{g}][\mathbf{a}, \mathbf{b}, \mathbf{c}][\mathbf{f}, \mathbf{b}, \mathbf{h}][\mathbf{f}, \mathbf{c}, \mathbf{e}][\mathbf{a}, \mathbf{d}, \mathbf{f}] \in \mathcal{B}_{\mathbf{P}}.$$

Interpretation as Cycles of Length Ratios In a (non-degenerate) instance P, the vanishing of the coordinatization of B can be written as

$$\frac{[\mathbf{a}, \mathbf{b}, \mathbf{g}]_P}{[\mathbf{a}, \mathbf{e}, \mathbf{g}]_P} \cdot \frac{[\mathbf{a}, \mathbf{c}, \mathbf{e}]_P}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]_P} \cdot \frac{[\mathbf{b}, \mathbf{c}, \mathbf{f}]_P}{[\mathbf{f}, \mathbf{b}, \mathbf{h}]_P} \cdot \frac{[\mathbf{e}, \mathbf{d}, \mathbf{f}]_P}{[\mathbf{f}, \mathbf{c}, \mathbf{e}]_P} \cdot \frac{[\mathbf{a}, \mathbf{f}, \mathbf{h}]_P}{[\mathbf{a}, \mathbf{d}, \mathbf{f}]_P} = 1. \tag{15.1}$$

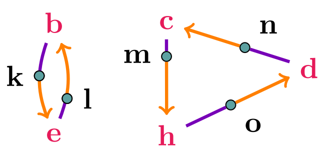
The combinatorics of the given ratios allow us to apply the area principle. These combinatorics are highlighted by rearranging and coloring the letters:

$$\frac{[\mathbf{b}, \mathbf{a}, \mathbf{g}]_P}{[\mathbf{e}, \mathbf{g}, \mathbf{a}]_P} \cdot \frac{[\mathbf{e}, \mathbf{c}, \mathbf{a}]_P}{[\mathbf{b}, \mathbf{a}, \mathbf{c}]_P} \cdot \frac{[\mathbf{c}, \mathbf{b}, \mathbf{f}]_P}{[\mathbf{h}, \mathbf{f}, \mathbf{b}]_P} \cdot \frac{[\mathbf{d}, \mathbf{e}, \mathbf{f}]_P}{[\mathbf{c}, \mathbf{f}, \mathbf{e}]_P} \cdot \frac{[\mathbf{h}, \mathbf{a}, \mathbf{f}]_P}{[\mathbf{d}, \mathbf{f}, \mathbf{a}]_P} = 1. \tag{15.2}$$

In order to apply it in the instance P, one needs to consider the points that are symbolically (i.e. in $\Lambda(\mathbf{P})$) given by

$$\begin{aligned} \mathbf{k} &= \mathbf{ag} \wedge \mathbf{be} \\ \mathbf{l} &= \mathbf{ac} \wedge \mathbf{eb} \\ \mathbf{m} &= \mathbf{bf} \wedge \mathbf{ch} \\ \mathbf{n} &= \mathbf{ef} \wedge \mathbf{dc} \\ \mathbf{o} &= \mathbf{af} \wedge \mathbf{hd}. \end{aligned} \tag{15.3}$$

With this setting, (15.2) reduces to a product of oriented length ratios in the instance P. The combinatorics of the length ratios can be already seen on a symbolic level which is shown below.



(15.4)

One can directly read of the product of oriented length ratios for the instance P (see (15.5)). Without referring to the instance, one can see the *cycles* in which the length ratios arrange. This situation is similar to the case of length ratios arranging themselves in orbits in Chapter 12. Again, we use non-bold letters for points of the instance P and also for points depending on them by a construction. Therefore, (15.1) reads as

$$\frac{\overline{b\,k}}{\overline{k\,e}} \cdot \frac{\overline{e\,l}}{\overline{l\,b}} \quad \cdot \quad \frac{\overline{c\,m}}{\overline{m\,h}} \cdot \frac{\overline{d\,n}}{\overline{n\,c}} \cdot \frac{\overline{h\,o}}{\overline{o\,d}} = 1, \tag{15.5}$$

since the transition from (15.1) to (15.2) was possible without any global sign changes. Furthermore, by grouping the fractions, we highlight the cyclic arrangement of the length ratios.

Shrinking the Length of the Cycles With the help of a Ceva configuration, it is possible to "shrink" the triangle of length ratios. Therefore we define

$$\mathbf{z} := (\mathbf{n}\,\mathbf{h} \wedge \mathbf{o}\,\mathbf{c})\,\mathbf{d} \wedge \mathbf{c}\,\mathbf{h}$$

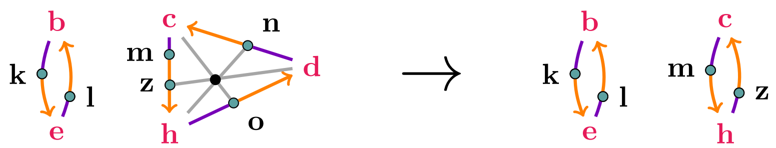
and Ceva's theorem (see Section 3.2) implies

$$\frac{\overline{d\,n}}{\overline{n\,c}} \cdot \frac{\overline{h\,o}}{\overline{o\,d}} = \frac{\overline{h\,z}}{\overline{z\,c}}.$$

This reduces (15.5) to

$$\frac{\overline{b\,k}}{\overline{k\,e}} \cdot \frac{\overline{e\,l}}{\overline{l\,b}} \quad \cdot \quad \frac{\overline{c\,m}}{\overline{m\,h}} \cdot \frac{\overline{h\,z}}{\overline{z\,c}} = 1 \tag{15.6}$$

which also reduces the combinatorics:



We record for the *general case* that plugging in Ceva configurations enables us to combine two length-ratios into one single length-ratio, as long as they have exactly one endpoint in common. Depending on the original distribution of signs induced by applying the area principle, one might want plug in also Menelaus configurations (see again Section 3.2). So by letting $\mathbf{z} = \mathbf{n}\,\mathbf{o} \wedge \mathbf{c}\,\mathbf{h}$ we can obtain $\frac{\overline{d\,n}}{\overline{n\,c}} \cdot \frac{\overline{h\,o}}{\overline{o\,d}} = -\frac{\overline{h\,z}}{\overline{z\,c}}$. Observe that the construction for a Menelaus configuration is less complicated and should be preferred in the general case. However, one has to pay attention to the overall sign and some Ceva configurations may be needed. Since Ceva configurations do not need outer intersections on the edges of a triangles and are therefore easier to draw, we use an example with a Ceva configuration.

Combining 2-Cycles or Cross-Ratios We are left with two 2-cycles of length-ratios. If we look closer at them, we see that in fact, *any* 2-cycle encodes a cross-ratio (see also (3.6)). Therefore, (15.6) can be rewritten as

$$(b, e; k, l) \cdot (c, h; m, z) = 1.$$

In principle, one can test this property by considering both cross-ratios as numbers on projective number lines. In order to calculate with such points, one can use von Staudt's projective multiplication and addition (see Section 3.7). For that purpose one first has to express both numbers as cross-ratios with respect to the *same* number line. This approach will be used in Section 17.2 in order to *add* two cross-ratios.

However, for *multiplying* two cross-ratios, there is a shortcut described as follows: There are well-known ruler-constructions that transfer a cross-ratio to another line. This can be easily seen by the iterative application of the fact that projections of lines do not alter the cross-ratio of points on the line (see Proposition 3.6). Figure 15.1 shows such a construction which concatenates two projections. I.e. there are ruler

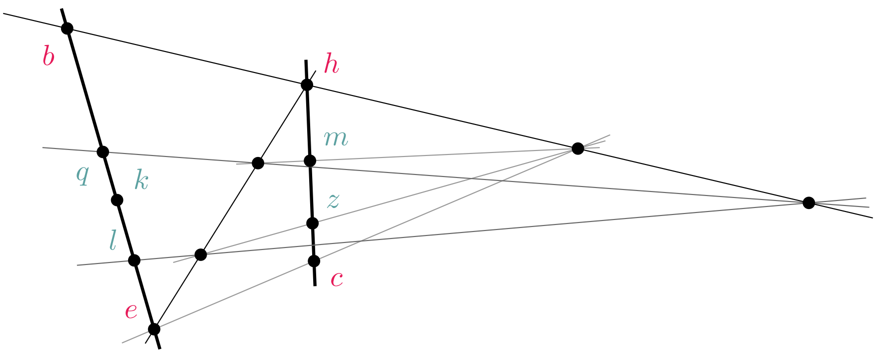


Figure 15.1.: Construction of a point q such that $(c, h; m, z) = (e, b; q, l)$.

constructions for determining a point q such that

$$(c, h; m, z) = (e, b; q, l).$$

This can be read as

$$\frac{\overline{e\,q}}{\overline{q\,b}} \quad = \quad \frac{\overline{e\,l}}{\overline{l\,b}} \cdot \frac{\overline{c\,m}}{\overline{m\,h}} \cdot \frac{\overline{h\,z}}{\overline{z\,c}}. \tag{15.7}$$

Observe that (15.6) contains the right-hand side of this equation. Therefore it can be reduced to

$$1 = \frac{\overline{b\,k}}{\overline{k\,e}} \cdot \frac{\overline{e\,q}}{\overline{q\,b}}. \tag{15.8}$$

This equation holds as soon as $k = q$. Since k is defined to be the intersection of the line spanned by a and g and the line spanned by b and e (see (15.3)), the above equation (15.8) reduces to the collinearity of a, g and q. Altogether, we derived a ruler-construction that is able to test whether (15.1) holds. The complete construction (considered as an element of $\Lambda(\mathbf{P})$) is shown in Figure 15.2.

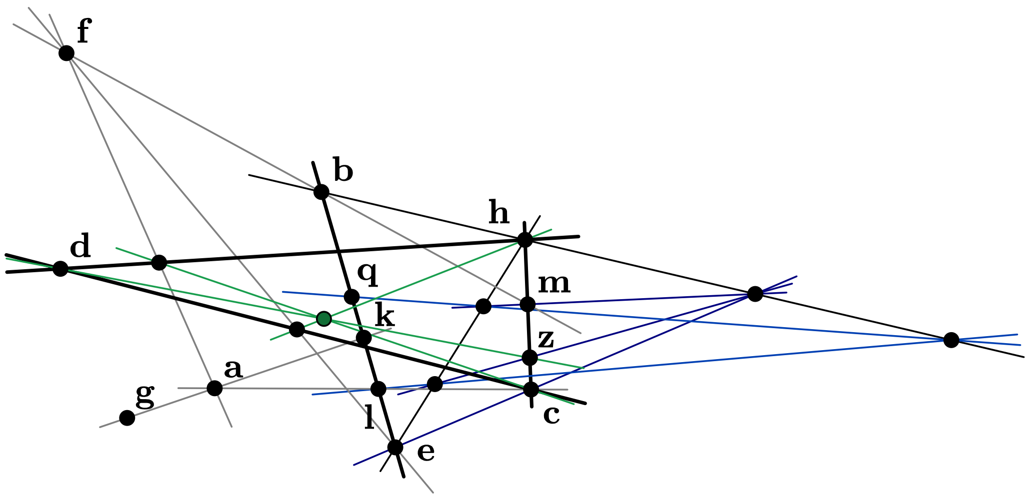


Figure 15.2.: A schematic figure for a generalized Cayley factorization of B.

The fact that this construction has an evaluation as a bracket *monomial* multiple of B is not trivial. All constructions used in this example are (variants of the) building blocks of a more general algorithm stated in Chapter 16. The actual constructions used there are a little bit different since they use already existing points to construct the projections. This keeps the degree of the multiplier monomial small and leads to a case distinction done in Section 16.5. There, each building block comes with an associated evaluation. They are used to express some (intermediate) points as linear combinations of the points spanning the reference line. The coefficients correspond to numerators and denominators indicating length ratios. Furthermore, the constructions in $\Lambda(\mathbf{P})$ meet all requirements for Cayley factorization. I.e. we will prove in the general case that the construction is equal to a multiple of the input polynomial *and* that this multiple is a (non-zero) bracket monomial. In the present example it holds:

$$\begin{aligned} &\mathbf{a\,g} \wedge \Bigg(\Big(((\mathbf{a\,f} \wedge \mathbf{h\,d})\,(\mathbf{b\,h} \wedge \mathbf{e\,d}) \wedge \mathbf{e\,h}\big)\,\big(\mathbf{a\,c} \wedge \mathbf{e\,b}\big) \wedge \mathbf{b\,h}\Big) \\ &\qquad \vee \Big(\big(((\mathbf{b\,f} \wedge \mathbf{c\,h})\,\mathbf{d} \wedge (\mathbf{e\,f} \wedge \mathbf{d\,c})\,\mathbf{h})\,\mathbf{c} \wedge \mathbf{d\,h}\big)\,\big(\mathbf{b\,h} \wedge \mathbf{ed}\big) \wedge \mathbf{e\,h}\Big) \wedge \mathbf{b\,e}\Bigg) \\ &= M \cdot [\mathbf{a},\mathbf{b},\mathbf{g}][\mathbf{a},\mathbf{c},\mathbf{e}][\mathbf{b},\mathbf{c},\mathbf{f}][\mathbf{e},\mathbf{d},\mathbf{f}][\mathbf{a},\mathbf{f},\mathbf{h}] - [\mathbf{a},\mathbf{e},\mathbf{g}][\mathbf{a},\mathbf{b},\mathbf{c}][\mathbf{f},\mathbf{b},\mathbf{h}][\mathbf{f},\mathbf{c},\mathbf{e}][\mathbf{a},\mathbf{d},\mathbf{f}] \end{aligned}$$

for a bracket monomial M of degree 8.

16. Algorithm for Binomials with Coefficients in $\{-1, +1\}$

We generalize the ideas presented in the previous chapter and provide the necessary computations. The previous chapter will serve as a guideline. An abstract graphical picture of all the steps of the binomial algorithm is given in Figure 16.1. Let $\mathbf{P}$ be a finite set and let $B \in \mathbb{R}[\Delta(\mathbf{P})]$ be a multihomogenous binomial with coefficients $+1$ or -1. By switching the positions of two elements in some brackets, we can w.l.o.g. assume that it looks like

$$[*, *, *][*, *, *] \cdots [*, *, *] - [*, *, *][*, *, *] \cdots [*, *, *].$$

Let $k \geq 2$ be the degree of B in its brackets. Assume that in each bracket of B the entries are pairwise distinct elements of $\mathbf{P}$. In Chapter 15 it was crucial that the area principle could be applied. More precisely, in (15.1), each fraction consists of a numerator taken from the first summand and a denominator taken from the second summand. This can be reformulated: each bracket of the first summand of B is matched with a bracket of the second summand such that the area principle can be applied. In the example we were lucky, since such a matching was existent. In the general case, this might not be the case. In this general case, we introduce some slack variables. In order to bound the number of slack variables and the size of the construction afterwards, we show the existence of a special matching in the next section.

16.1. Matching the Brackets of Both Summands

Therefore, let $\mathcal{L}_B$ be a set which, in the most general case, has to be a multiset. It shall contain all brackets of the first summand of B: if there are brackets, which appear more than once in the summand, then add a copy of the bracket for each additional exponent. Analogously, let $\mathcal{M}_B$ be the multiset of brackets of the second summand:

$$B = \underbrace{[*, *, *][*, *, *] \cdots [*, *, *]}_{\text{constitute } \mathcal{L}_B} - \underbrace{[*, *, *][*, *, *] \cdots [*, *, *]}_{\text{constitute } \mathcal{M}_B}. \tag{16.1}$$

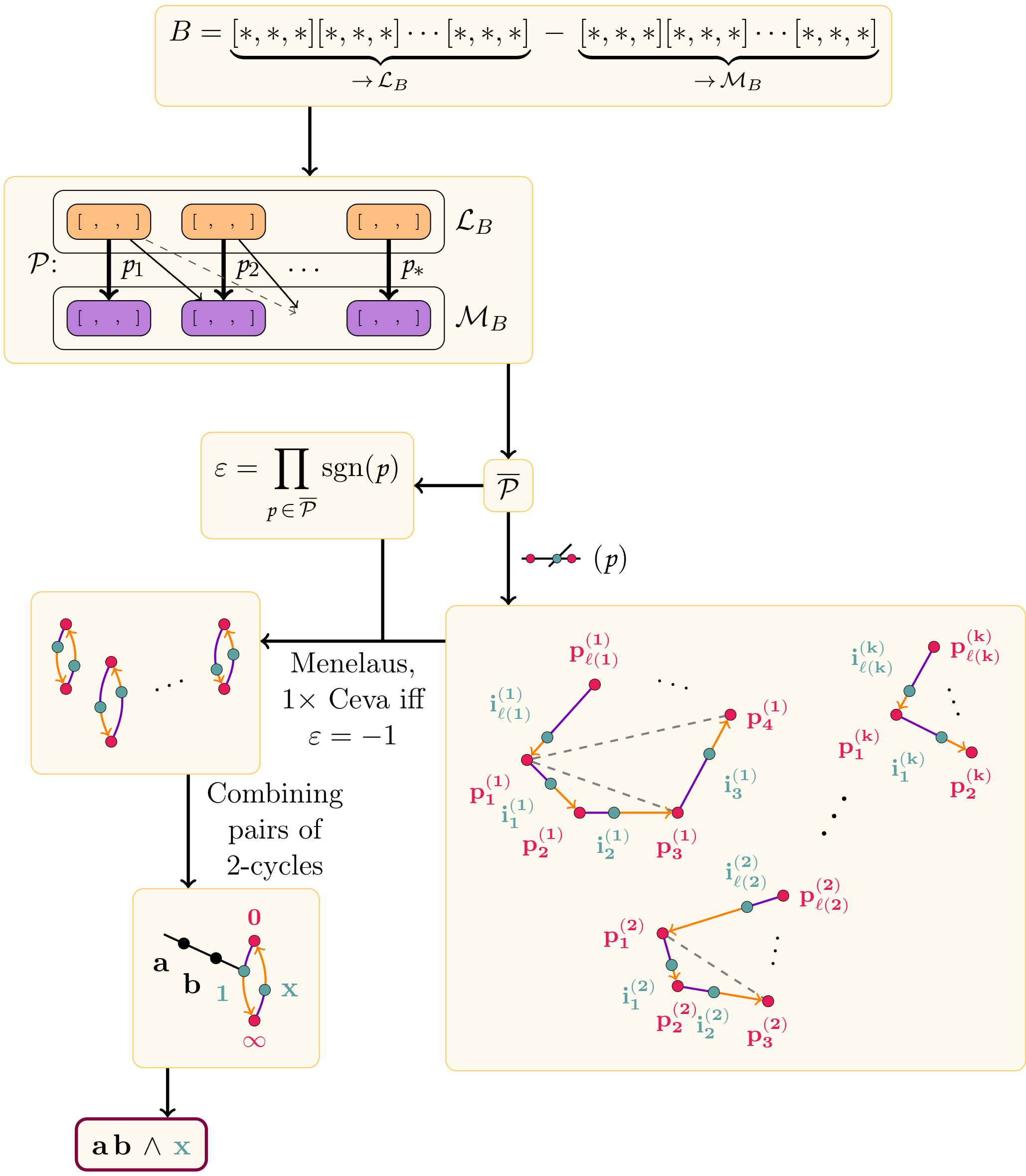


Figure 16.1.: Schematic figure illustrating the general approach.

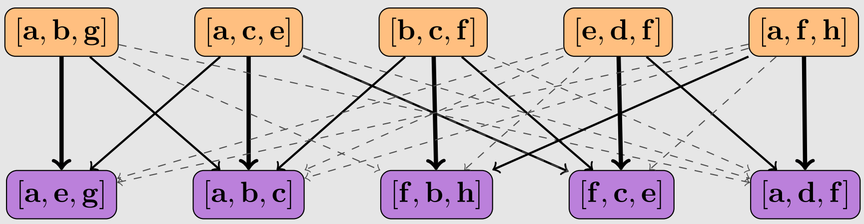
We define a bipartite directed graph $G = (\mathcal{L}_B, \mathcal{M}_B, E)$. The set of edges E contains edges of two types and is the union of

$$E_1 := \Big\{(\ell, m)\Big|\ \ell \in \mathcal{L}_B, m \in \mathcal{M}_B : \text{the brackets } \ell \text{ and } m \text{ have exactly } \textit{two} \text{ entries in common}\Big\}$$

and

$$E_2 := \Big\{(\ell, m)\Big|\ \ell \in \mathcal{L}_B, m \in \mathcal{M}_B : \text{the brackets } \ell \text{ and } m \text{ have exactly } \textit{one} \text{ entry in common}\Big\}.$$

For the example considered in Chapter 15 the graph G is shown below. Straight arrows indicate edges in E_1, dashed ones edges in E_2. The thick edges indicate the selection of fractions in (15.1). They form a perfect matching in G.



In the general case we claim:

Lemma 16.1. *G has a perfect matching, i.e. there is a $\mathcal{P} \subset E$ such that no two edges in $\mathcal{P}$ share a vertex and each vertex is contained in an edge in $\mathcal{P}$.*

In order to prove this lemma we restate a classical result from graph theory originally given in [56]. There are many textbooks including it, e.g. [35]. As a matter of fact, also [31, p. 212] gives a sketch of the proof using vector configurations after translating the problem to linear algebra.

Theorem 16.2 (Hall's Marriage Theorem)**.** *Let $G = (U, V, E)$ be a bipartite graph. For a subset W of U we denote by $N_G(W)$ the* neighborhood *of W in G, i.e. the set of all vertices adjacent to some element of W. G has a perfect matching if and only if*

$$|N_G(W)| \geq |W| \quad \textit{for all } W \subset U.$$

Obviously, the existence of a perfect matching of G is not affected by passing to the underlying undirected graph G' (which is also good for concrete implementations). We use the directed version of the graph because it fits with the interpretation via the area principle where the ratios indicate a *direction* along an edge.

Proof of Lemma 16.1. Let G' be the graph obtained from G by forgetting the orientation of the edges. Let W be a subset of $\mathcal{L}_B$. Contained in $N_{G'}(W)$ are all elements of $\mathcal{M}_B$ that have at least one entry in common with any bracket contained in W. Let v_W be the entries in W:

$$v_W := \bigcup_{[\mathbf{p},\mathbf{q},\mathbf{r}]\,\in W} \{\mathbf{p},\mathbf{q},\mathbf{r}\}.$$

Since B is multihomogenous, the elements of v_W occur in brackets of $\mathcal{M}_B$ as well, namely

$$\sum_{\mathbf{p}\,\in v_W} \deg(B,\mathbf{p}) \;\geq\; 3\cdot|W|$$

times. Since only three of them can share a bracket, at least

$$\frac{1}{3}\sum_{\mathbf{p}\,\in v_W} \deg(B,\mathbf{p}) \;\geq\; |W|$$

elements of $\mathcal{M}_B$ contain elements of v_W. Since all of them are in $N_{G'}(W)$, we have $|N_{G'}(W)| \geq |W|$. $\square$

Let $\mathcal{P}$ denote a perfect matching in G. The edges in $\mathcal{P}\cap E_1$ are compatible with the (combinatorics of the) area principle. Consider an edge $(\ell,m)\in\mathcal{P}\cap E_2$. Here ℓ and m have exactly one entry in common. Therefore, there is a bracket $b\in\mathcal{B}_{\mathbf{P}}$ such that both ordered pairs (ℓ,b) and (b,m) have exactly two elements in common (for an example see Figure 16.2). Let $\overline{\mathcal{P}}$ be the set of ordered pairs $\mathcal{P}\cap E_1$ together with pairs (ℓ,b) and (b,m) for each $(\ell,m)\in\mathcal{P}\cap E_2$.

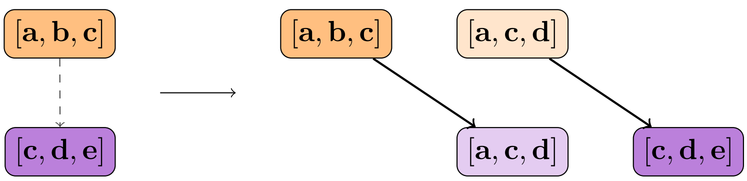


Figure 16.2.: An example on how to resolve an edge in E_2.

Now any pair of brackets in $\overline{\mathcal{P}}$ can be interpreted via the area principle. $\overline{\mathcal{P}}$ is the foundation of all following steps of the algorithm. Obviously we have

$$\left|\overline{\mathcal{P}}\right| \leq 2\,k, \tag{16.2}$$

since k was the degree of B in the brackets and therefore k is also the size of the sets $\mathcal{L}_B$ and $\mathcal{M}_B$. $\overline{\mathcal{P}}$'s connection with B can be summarized as follows:

$$\prod_{(\ell,m)\in\overline{\mathcal{P}}} \ell - \prod_{(\ell,m)\in\overline{\mathcal{P}}} m \quad = \quad N \cdot \left(\prod_{\ell\in\mathcal{L}_B} \ell - \prod_{m\in\mathcal{M}_B} m \right) \quad = \quad N \cdot B \tag{16.3}$$

where N is the bracket monomial consisting of all brackets $\mathscr{b}$ introduced in the resolution of $\mathcal{P} \cap E_2$. Therefore we have

$$\deg(N) \leq k. \tag{16.4}$$

(16.3) holds due to the definition of $\overline{\mathcal{P}}$ and due to (16.1).

Remark 16.3. For the theoretical result, any perfect matching $\mathcal{P}$ in G is good for the algorithm. However, in a concrete implementation one should prefer edges in E_1 over those in E_2, since this reduces the size of the following constructions considerably.

Remark 16.4. If someone, for some reasons, wants to also factor binomials with common bracket factors, say $[\mathbf{a},\mathbf{b},\mathbf{c}]$, he or she can add the two pairs $([\mathbf{a},\mathbf{b},\mathbf{c}],[\mathbf{a},\mathbf{b},\mathbf{d}])$ and $([\mathbf{a},\mathbf{b},\mathbf{d}],[\mathbf{a},\mathbf{b},\mathbf{c}])$ to $\overline{\mathcal{P}}$ (for some $\mathbf{d} \in \mathbf{P}$) and this will do it.

16.2. Geometric Interpretation of the Pairs of Brackets in $\overline{\mathcal{P}}$

In order to state the next definitions and to interpret the bracket pairs $\overline{\mathcal{P}}$, we introduce another notation: For σ being a permutation in the symmetric group $\mathcal{S}_{\{\mathbf{a},\mathbf{b},\mathbf{c}\}}$, we abbreviate $[\sigma(\mathbf{a}),\sigma(\mathbf{b}),\sigma(\mathbf{c})]$ by the notion $[\sigma(\mathbf{a},\mathbf{b},\mathbf{c})]$. With this notion there are σ and τ such that every element p of $\overline{\mathcal{P}}$ can be written as

$$p = \big(\ell, m\big) = \big([\sigma(\mathbf{a},\mathbf{b},\mathbf{c})], [\tau(\mathbf{b},\mathbf{c},\mathbf{d})] \big).$$

Remember that the colors shall only indicate the role of the points and are not part of the names of the points. As in (15.3), we define for each pair $p \in \overline{\mathcal{P}}$ its *edge information*

$$(p) := (\mathbf{a},\ \mathbf{bc} \wedge \mathbf{ad},\ \mathbf{d}).$$

It contains all information about the induced length ratios in an instance (compare (15.5)). Motivated by the expansion $[\mathbf{b},\mathbf{c},\mathbf{d}]\,\mathbf{a} - [\mathbf{b},\mathbf{c},\mathbf{a}]\,\mathbf{d}$ of $\mathbf{bc} \wedge \mathbf{ad}$ we define:

$$\mu_p := [\mathbf{b},\mathbf{c},\mathbf{d}] \quad \text{and} \quad \lambda_p := [\mathbf{b},\mathbf{c},\mathbf{a}]$$

such that

$$(p) = (\mathbf{a},\ \mu_p\,\mathbf{a} - \lambda_p\,\mathbf{d},\ \mathbf{d}).$$

Observe that μ_p differs from m only by $\operatorname{sgn}(\tau)$ and a similar statement holds for λ_p. Note also that there is an ambiguity in the definition of (p), λ_p and μ_p since the order of $\mathbf{b}$ and $\mathbf{c}$ is not determined. In fact, everything what follows in this text works with either one of the orders chosen. The result just differs by a factor of -1. For better readability, we use the notation above which can be thought of choosing an arbitrary order. The notion of *any* (more general and later modified) 3-tuples

$$(\mathbf{a}, \mathbf{x}, \mathbf{b})$$

as also used in the definition of , shall indicate that $\mathbf{x}$ lies on the line spanned by $\mathbf{a}$ and $\mathbf{b}$ where $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{a}, \mathbf{b} \in \mathbf{P}$. In particular, $\mathbf{x}$ has a definition of the shape $* * \wedge \mathbf{a}\,\mathbf{b}$ and an evaluation as linear combination of the two (distinct!) points $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{x}$ evaluates generically to

$$M_{\mathbf{x}}\,(\mu_{\mathbf{x}}\,\mathbf{a} - \lambda_{\mathbf{x}}\,\mathbf{b}), \tag{16.5}$$

for some bracket monomials $M_{\mathbf{x}}$, $\mu_{\mathbf{x}}$ and $\lambda_{\mathbf{x}}$. Observe that the area principle can be applied by considering

$$\frac{[\mathbf{a}, \mathbf{x}, \mathbf{q}]}{[\mathbf{b}, \mathbf{q}, \mathbf{x}]} = \frac{M_{\mathbf{x}}[\mathbf{a}, (\mu_{\mathbf{x}}\,\mathbf{a} - \lambda_{\mathbf{x}}\,\mathbf{b}), \mathbf{q}]}{M_{\mathbf{x}}[\mathbf{b}, \mathbf{q}, (\mu_{\mathbf{x}}\,\mathbf{a} - \lambda_{\mathbf{x}}\,\mathbf{b})]} = -\frac{\lambda_{\mathbf{x}}}{\mu_{\mathbf{x}}} \tag{16.6}$$

for some $\mathbf{q} \in \mathbf{P} \setminus \{\mathbf{a}, \mathbf{b}\}$ evaluated in an instance. Thus, in any representation of the same shape as in (16.5), the ratio of the coefficients of $\mathbf{a}$ and $\mathbf{b}$ in the expansion of $\mathbf{x}$ can be considered as a negated oriented length ratio. This length ratio can be considered as pointing from $\mathbf{a}$ to $\mathbf{b}$. This is the reason why we call 3-tuples in the shape of (16.5) *edges*. The calculations following later, in essence keep track of the exact shape of a specific linear combination describing $\mathbf{x}$. This is done inductively. These conventions on the notation of edges make it easier to keep track of all calculations.

We proceed with the interpretation of a $p \in \overline{\mathcal{P}}$. We learned before that when dealing with the area principle, one has to pay attention to the sign. We define for p given as above

$$\operatorname{sgn}(p) := \operatorname{sgn}(\tau) \cdot \operatorname{sgn}(\sigma).$$

The overall sign is given by

$$\varepsilon := \prod_{p \in \overline{\mathcal{P}}} \operatorname{sgn}(p).$$

This allows us to alter (16.3) in order to obtain

$$\prod_{p \in \overline{\mathcal{P}}} \mu_p \;-\; \varepsilon \cdot \prod_{p \in \overline{\mathcal{P}}} \lambda_p \;=\; \pm 1 \cdot \left(\prod_{(\ell, m) \in \overline{\mathcal{P}}} \ell - \prod_{(\ell, m) \in \overline{\mathcal{P}}} m \right) \;=\; N \cdot B \tag{16.7}$$

with another bracket monomial N whose degree is not bigger than k.

16.3. Detecting Cycles

We saw in (15.4) that the edges corresponding to the length ratios form *cycles*. In what follows we will argue that the same holds in the general case: All information about all length ratios is encoded in

$$\bullet\!\!-\!\!\not\bullet\!-\!\bullet\left(\overline{\mathcal{P}}\right) \quad = \quad \left\{\bullet\!\!-\!\!\not\bullet\!-\!\bullet(p) \mid p \in \overline{\mathcal{P}}\right\}$$

which might be a multiset. We state:

Lemma 16.5. *Forgetting the middle entry in the elements of* $\bullet\!\!-\!\!\not\bullet\!-\!\bullet\left(\overline{\mathcal{P}}\right)$ *indicates the edges of a graph that can be decomposed into edge-disjoint (simple) cycles. The decomposition can be done in linear time with respect to the size of* $\bullet\!\!-\!\!\not\bullet\!-\!\bullet\left(\overline{\mathcal{P}}\right)$.

Proof. In order to see this, choose any $(\ell, m) \in \overline{\mathcal{P}}$. Assume $\bullet\!\!-\!\!\not\bullet\!-\!\bullet\big((\ell, m)\big) = (*, *, \mathbf{a})$. Therefore $\mathbf{a}$ is an entry in m. Since B is multihomogenous and due to the definition of $\overline{\mathcal{P}}$, there must be a bracket

$$\ell' \in \bigcup_{(r,s)\in\overline{\mathcal{P}}} r$$

that contains $\mathbf{a}$ and that is paired with another bracket m' not containing $\mathbf{a}$. I.e. there is a

$$(\ell', m') \in \overline{\mathcal{P}} \quad \text{with} \quad \bullet\!\!-\!\!\not\bullet\!-\!\bullet\big((\ell', m')\big) = (\mathbf{a}, *, *).$$

One can proceed this way until one meets a vertex which was formerly visited. In this case, a cycle is detected and one can delete all corresponding paired brackets from $\overline{\mathcal{P}}$. Also in the new $\overline{\mathcal{P}}$ corresponds to a multihomogenous bracket polynomial and one can restart the same game with a new $(\ell, m) \in \overline{\mathcal{P}}$. $\square$

From now on, we will work on the elements of $\bullet\!\!-\!\!\not\bullet\!-\!\bullet(\overline{\mathcal{P}})$ that *are* organized in cycles $\mathcal{C}_1, \ldots, \mathcal{C}_t$ (see the box on the bottom right in Figure 16.1). By the definition of $\bullet\!\!-\!\!\not\bullet\!-\!\bullet$, they contain incidence information as described in Section 16.2. In particular, any edge is of the shape

$$(\mathbf{a},\, M_{\mathbf{x}}\,(\mu_{\mathbf{x}}\,\mathbf{a} - \lambda_{\mathbf{x}}\,\mathbf{b}),\, \mathbf{b}).$$

16.4. Triangulating Cycles

Assume a cycle contains consecutive edges

$$\big(\mathbf{a},\ M_{\mathrm{x}}\cdot(\mu_{\mathrm{x}}\,\mathbf{a}-\lambda_{\mathrm{x}}\,\mathbf{b}),\ \mathbf{b}\big)\quad\text{and}\quad\big(\mathbf{b},\ M_{\mathrm{y}}\cdot(\mu_{\mathrm{y}}\,\mathbf{b}-\lambda_{\mathrm{y}}\,\mathbf{c}),\ \mathbf{c}\big)$$

with $\mathbf{a}\neq\mathbf{c}$. In this case we combine both edges in a single edge (see Figure 16.3). In contrast to the example in Chapter 15, we prefer using Menelaus configurations

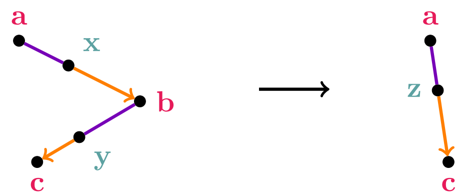
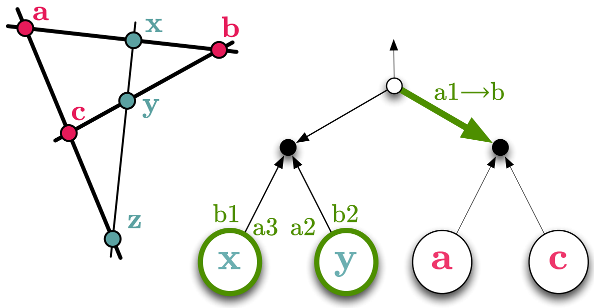


Figure 16.3.: Combinatorics of edges when using Ceva's or Menelaus's construction.

instead of Ceva configurations (see also Figure 3.3). The result is shown in the scheme below. There the resulting formula is a linear combination of the points $\mathbf{a}$ and $\mathbf{c}$ whose coefficients are the (signed) products of the μs and λs. Therefore, $\big(\mathbf{a},\mathrm{z},\mathbf{c}\big)$ meets all requirements to be an edge.

Menelaus

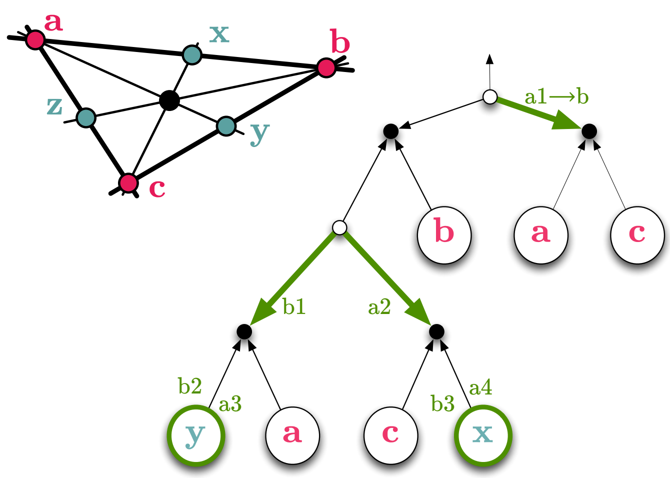
$$\mathrm{z}\ :=\ \mathrm{x}\,\mathrm{y}\wedge\mathbf{a}\,\mathbf{c}\ =\ M_{\mathrm{x}}M_{\mathrm{y}}\cdot[\mathbf{a},\mathbf{b},\mathbf{c}]\big(\mu_{\mathrm{x}}\mu_{\mathrm{y}}\mathbf{a}-\lambda_{\mathrm{x}}\lambda_{\mathrm{y}}\mathbf{c}\big)$$



Using a Ceva configuration instead results in a sign change in the linear combination:

Ceva

$$\mathbf{z} := (\mathbf{y}\,\mathbf{a} \wedge \mathbf{c}\,\mathbf{x})\,\mathbf{b} \wedge \mathbf{a}\,\mathbf{c} \;=\; M_{\mathbf{x}}M_{\mathbf{y}} \cdot [\mathbf{a},\mathbf{b},\mathbf{c}]^2 \big(\mu_{\mathbf{x}}\mu_{\mathbf{y}}\mathbf{a} + \lambda_{\mathbf{x}}\lambda_{\mathbf{y}}\mathbf{c}\big)$$



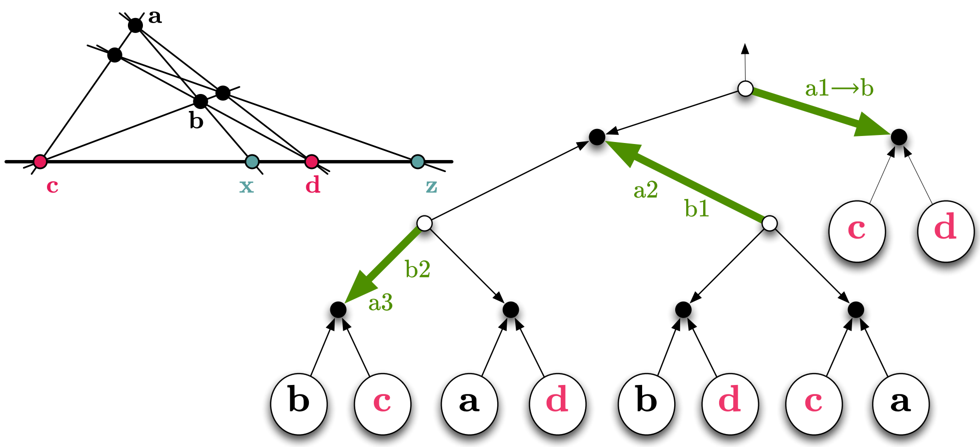
A sign change can also be obtained on a single edge. Suppose the edge is given by

$$(p) = (\mathbf{c}, \quad \mathbf{x} = \mathbf{ab} \wedge \mathbf{cd} = \underbrace{[\mathbf{a},\mathbf{b},\mathbf{d}]}_{\mu_p}\,\mathbf{c} - \underbrace{[\mathbf{a},\mathbf{b},\mathbf{c}]}_{\lambda_p}\,\mathbf{d}, \quad \mathbf{d})$$

for some $p \in \overline{\mathcal{P}}$. The harmonic point construction (see Section 3.8) gives

Harmonic Point

$$\mathbf{z} := (\mathbf{b}\,\mathbf{c} \wedge \mathbf{a}\,\mathbf{d})\,(\mathbf{b}\,\mathbf{d} \wedge \mathbf{c}\,\mathbf{a}) \wedge \mathbf{c}\,\mathbf{d} \;=\; [\mathbf{b},\mathbf{c},\mathbf{d}][\mathbf{a},\mathbf{c},\mathbf{d}]\big(\mu_p\mathbf{c} + \lambda_p\mathbf{d}\big)$$



The processing of the cycles $\mathcal{C}_1, \ldots, \mathcal{C}_t$ detected in Section 16.3 is composed of the building blocks just given and is described as follows:

In the case that $\varepsilon = -1$ we do a case distinction. If there is no cycle of length bigger than 2 then replace any edge by the edge obtained in the harmonic point construction. Else there is a cycle with length bigger than 2. Replace two consecutive edges by the edge obtained in the Ceva construction. If $\varepsilon = 1$ we do nothing.

Now, if there are left any cycles of length bigger than 2 we *triangulate* them by plugging in Menelaus configurations. This means successively replacing consecutive edges by the edge obtained in the Menelaus construction. This results in a collection of 2-cycles. The triangulation can be done in such a way that one edge of the 2-cycle is in its original state, i.e. there is a $p \in \overline{\mathcal{P}}$ such that this edge equals (p).

In any case, we give an evaluation of the points in any resulting 2-cycle: consider e.g. $\mathcal{C}_t \subset (\overline{\mathcal{P}})$. For using (16.7) later on, let $\widetilde{\mathcal{C}}_t$ be the (algorithmic) preimage of $\mathcal{C}_t$ under . After triangulation, $\mathcal{C}_t$ is transferred to a 2-cycle, say

$$\Big((\infty,\ \mathbf{1},\ \mathbf{0})\ ,\ (\mathbf{0},\ \mathbf{x},\ \infty)\Big) \text{ with } (\infty,\ \mathbf{1},\ \mathbf{0}) = (p_{\mathbf{1}}) \text{ for a } p_{\mathbf{1}} \in \overline{\mathcal{P}} \tag{16.8}$$

and with $\infty, \mathbf{0} \in \mathbf{P}$. $\mathbf{1}, \mathbf{x}$ are simple Grassmann-Cayley algebra expressions in $\Lambda^1(\mathbf{P})$. By induction, $\mathbf{x}$ has an evaluation as

$$M_{\mathbf{x}} \cdot \left(\underbrace{\prod_{p \in \widetilde{\mathcal{C}}_t \setminus \{p_{\mathbf{1}}\}} \mu_p}_{=:\mu_{\mathbf{x}}} \ \mathbf{0} \ - \ \underbrace{(\pm 1) \prod_{p \in \widetilde{\mathcal{C}}_t \setminus \{p_{\mathbf{1}}\}} \lambda_p}_{=:\lambda_{\mathbf{x}}} \ \infty \right) \tag{16.9}$$

for a bracket monomial $M_{\mathbf{x}}$. The sign in the above equation equals -1 if the cycle was altered by a construction resulting from $\varepsilon = -1$. Otherwise the sign is $+1$. Observe that $M_{\mathbf{x}}$ is really a bracket *monomial* in $\mathcal{B}_{\mathbf{P}}$, since the monomials in the construction above contain *distinct* elements of $\mathbf{P}$. This is due to the assumption $\mathbf{a} \neq \mathbf{c}$ and the definition of . Furthermore, the cycles detected in Section 16.3 are simple. Observe that we have

$$\deg(M_{\mathbf{x}}) \leq \operatorname{length}(\mathcal{C}_t) - 2 \tag{16.10}$$

if $\mathcal{C}_t$ was not altered by a construction resulting from $\varepsilon = -1$. Otherwise we have

$$\deg(M_{\mathbf{x}}) \leq \big(\operatorname{length}(\mathcal{C}_t) - 2\big) + 2. \tag{16.11}$$

16.5. Combining 2-Cycles or Cross-Ratios

In this section, we give tools to combine two 2-cycles (see Figure 16.4). Inductively, this allows for combining all 2-cycles. Due to the intuition of 2-cycles already provided in Chapter 15 and by abusing the notion, we call the 2-cycles cross-ratios as well. All of the following constructions will result in similar evaluations. Since we want to keep the degree of the multiplier monomial small, we give a lot of different constructions which shall be used depending on the current setup. The setup common to all situations is given as:

$$\Big(\big(\infty,\ \ \mathbf{1}\ = (\mu_{\mathbf{1}}\,\infty\ - \lambda_{\mathbf{1}}\,\mathbf{0}\,),\ \ \mathbf{0}\,\big),\ \big(\mathbf{0},\ \ \mathbf{x}\ = M_{\mathbf{x}}\ (\mu_{\mathbf{x}}\,\mathbf{0}\ - \lambda_{\mathbf{x}}\,\infty\,),\ \ \infty\,\big)\Big)\ \text{and}$$
$$\Big(\big(\infty',\ \ \mathbf{1}' = (\mu_{\mathbf{1}'}\infty' - \lambda_{\mathbf{1}'}\mathbf{0}'),\ \ \mathbf{0}'\big),\ \big(\mathbf{0}',\ \ \mathbf{x}' = M_{\mathbf{x}'}\,(\mu_{\mathbf{x}'}\mathbf{0}' - \lambda_{\mathbf{x}'}\infty'),\ \ \infty'\big)\Big) \tag{16.12}$$

being two 2-cycles. Furthermore, due to (16.8) we can assume that there are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}'$ and $\mathbf{b}'$ in $\mathbf{P}$ such that

$$\mathbf{1} = \mathbf{a}\,\mathbf{b}\,\wedge\,\infty\,\mathbf{0} \qquad \text{and} \qquad \mathbf{1}' = \mathbf{a}'\,\mathbf{b}'\,\wedge\,\infty'\,\mathbf{0}' \tag{16.13}$$

as well as $\mu_{\mathbf{1}'} = [\mathbf{a}', \mathbf{b}', \mathbf{0}']$ and $\lambda_{\mathbf{1}'} = [\mathbf{a}', \mathbf{b}', \infty']$, $\mathbf{0}' \neq \infty'$ and in addition $\{\mathbf{a}', \mathbf{b}', \mathbf{0}'\}$ and $\{\mathbf{a}', \mathbf{b}', \infty'\}$ are pairwise distinct (due to the definition of $\overline{\mathcal{P}}$). Similar properties hold for the points in the first cycle.

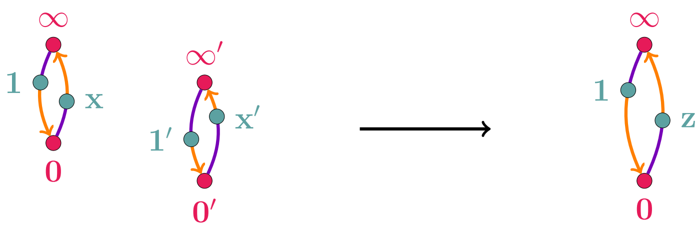
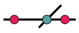


Figure 16.4.: Combinatorics of combining two 2-cycles.

The following tables summarize the constructions for a point $\mathbf{z}$. In this step of the algorithm, both cross-ratios in (16.12) are combined in the single cross-ratio

$$\Big((\infty,\ \mathbf{1},\ \mathbf{0})\ \ ,\ \ (\mathbf{0},\ \mathbf{z},\ \infty)\Big).$$

Observe that also in the case of the resulting 2-cycle, there is a $p_{\mathbf{1}} \in \overline{\mathcal{P}}$ such that $(\infty,\ \mathbf{1},\ \mathbf{0}) =$  $(p_{\mathbf{1}})$. The evaluations of $\mathbf{z}$ given below all are a bracket *monomial* multiple of

$$E^* := M_{\mathbf{x}'} \cdot M_{\mathbf{x}} \left(\mu_{\mathbf{1}'}\mu_{\mathbf{x}}\mu_{\mathbf{x}'}\mathbf{0} - \lambda_{\mathbf{1}'}\lambda_{\mathbf{x}}\lambda_{\mathbf{x}'}\boldsymbol{\infty}\right). \tag{16.14}$$

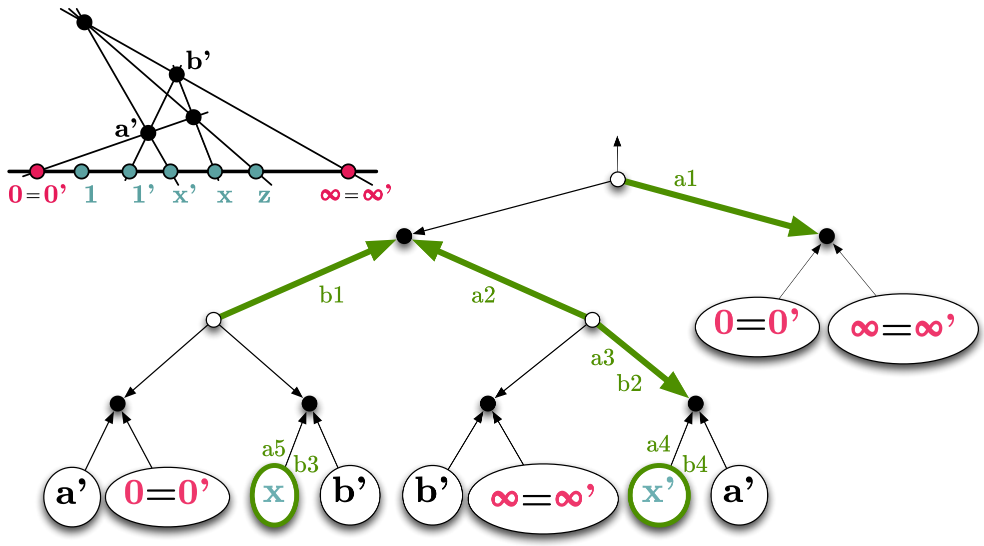
The degree of the multiplier monomial in all cases is

$$\leq 4.$$

By (16.6), the multiplier E^* can be viewed to be a version of (15.7) in Grassmann-Cayley algebra. Observe that the assumptions on the points used in (16.13) imply that the bracket monomial multiplies in the evaluations of the constructions below all do not vanish in $\mathcal{B}_{\mathbf{P}}$. If one would not need a *monomial* multiplier we could save ourselves the case distinction in the case $\left|\{\mathbf{0}, \boldsymbol{\infty}, \mathbf{0}', \boldsymbol{\infty}'\}\right| = 4$. We could construct $\mathbf{z}$ by using the fact that in any instance, the six lines induced by $\mathbf{0}\,\boldsymbol{\infty}$, $\boldsymbol{\infty}'\,\mathbf{0}'$, $\mathbf{0}\,\mathbf{0}'$, $\boldsymbol{\infty}\,\boldsymbol{\infty}'$, $\mathbf{x}\,\mathbf{1}'$ and $\mathbf{x}'\,\mathbf{z}$ are tangent to the same conic. Together with Brianchon's theorem (which is the dual of Pascal's theorem in Section 3.10) this gives a construction for $\mathbf{z}$. However, the resulting multiplier of degree 4 is not a monomial. The four cases for $\left|\{\mathbf{0}, \boldsymbol{\infty}, \mathbf{0}', \boldsymbol{\infty}'\}\right| = 3$ are very similar and therefore only one case is illustrated detailedly. Furthermore, by possibly exchanging the roles of the points $\mathbf{a}'$ and $\mathbf{b}'$ or of the two 2-cycles in question, one of the cases given below is applicable. A remark on the evaluation of the formula given in Case 2 (i): Observe that by $\mathbf{0} = \boldsymbol{\infty}'$ and $\boldsymbol{\infty} = \mathbf{0}'$ it now holds $\mu_{\mathbf{1}'} = [\mathbf{a}', \mathbf{b}', \boldsymbol{\infty}]$ and $\lambda_{\mathbf{1}'} = [\mathbf{a}', \mathbf{b}', \mathbf{0}]$. Similar comments are also appropriate in the other cases.

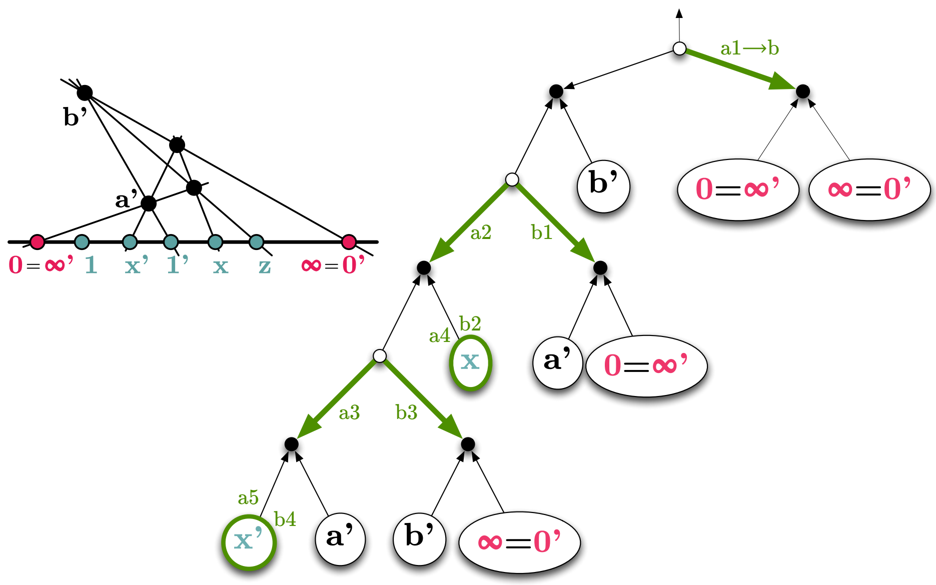
Case 2 (i): $\big|\{\mathbf{0},\infty,\mathbf{0}',\infty'\}\big| = 2$: $\mathbf{0} = \mathbf{0}'$ and $\infty = \infty'$

$$\mathrm{z} := \big(\mathbf{a}'\,\mathbf{0} \wedge \mathrm{x}\,\mathbf{b}'\big)\,\big(\mathbf{b}'\,\infty \wedge \mathrm{x}'\,\mathbf{a}'\big) \;\wedge\; \mathbf{0}\,\infty = [\mathbf{0},\infty,\mathbf{a}'][\mathbf{0},\infty,\mathbf{b}']\,E^*$$



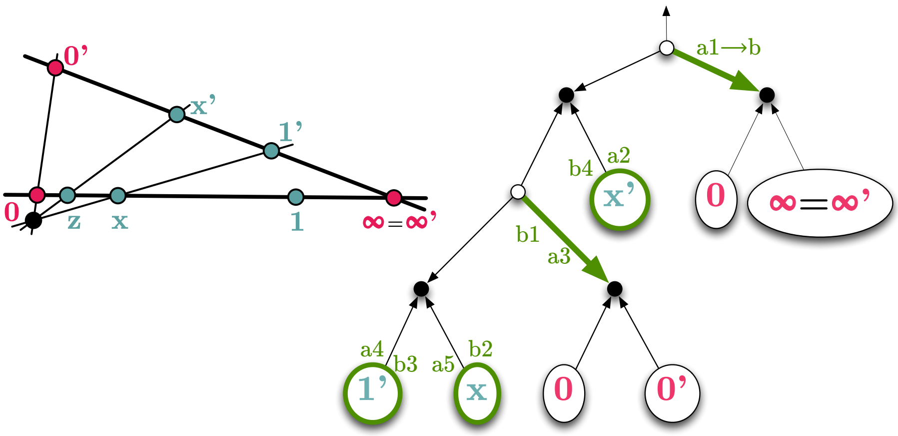
Case 2 (ii): $\big|\{\mathbf{0},\infty,\mathbf{0}',\infty'\}\big| = 2$: $\mathbf{0} = \infty'$ and $\infty = \mathbf{0}'$

$$\mathrm{z} := \Big(\big(\mathrm{x}'\,\mathbf{a}' \wedge \mathbf{b}'\,\infty\big)\,\mathrm{x} \wedge \mathbf{a}'\,\mathbf{0}\Big)\mathbf{b}' \;\wedge\; \mathbf{0}\,\infty = [\mathbf{0},\infty,\mathbf{b}'][\mathbf{0},\infty,\mathbf{a}']\,E^*$$



$$\big|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\big| = 3:$$

Case 3 (i): $\infty = \infty'$: $\mathrm{z} := \big(\mathbf{1}'\,\mathrm{x} \wedge \mathbf{0}\,\mathbf{0}'\big)\,\mathrm{x}' \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \mathbf{0}']^2 E^*$



Case 3 (ii): $\mathbf{0} = \mathbf{0}'$: $\mathrm{z} := \big(\mathbf{1}'\,\mathrm{x} \wedge \infty'\,\infty\big)\,\mathrm{x}' \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \infty']^2 E^*$

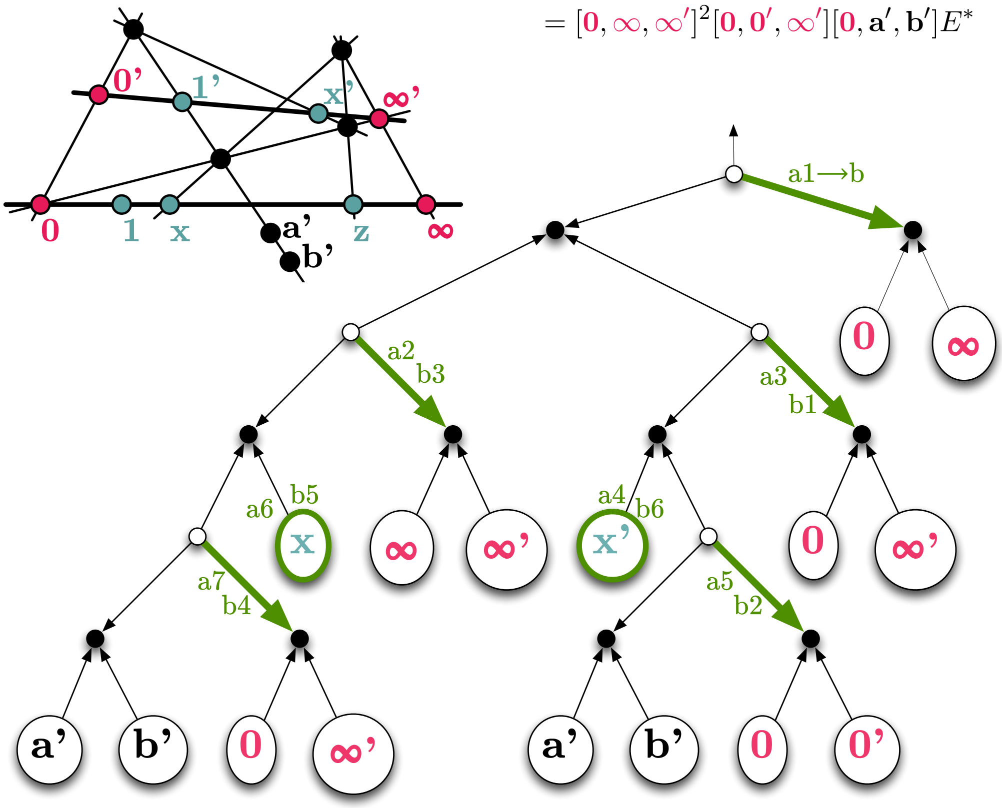
Case 3 (iii): $\infty = \mathbf{0}'$: $\mathrm{z} := \big(\mathrm{x}'\,\mathrm{x} \wedge \mathbf{0}\,\infty'\big)\,\mathbf{1}' \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \infty']^2 E^*$

Case 3 (iv): $\mathbf{0} = \infty'$: $\mathrm{z} := \big(\mathrm{x}'\,\mathrm{x} \wedge \infty\,\mathbf{0}'\big)\,\mathbf{1}' \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \mathbf{0}']^2 E^*$

Case 4 (i): $\left|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\right| = 4$: $|\{\mathbf{a}', \mathbf{b}', \mathbf{0}\}| = 3$

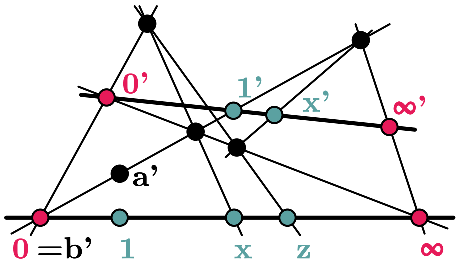
$$\mathrm{z} := \Big((\mathbf{a}'\,\mathbf{b}' \wedge \mathbf{0}\,\infty')\,\mathrm{x} \wedge \infty\,\infty' \Big) \Big(\mathrm{x}'(\mathbf{a}'\,\mathbf{b}' \wedge \mathbf{0}\,\mathbf{0}') \wedge \mathbf{0}\,\infty' \Big) \wedge \mathbf{0}\,\infty$$

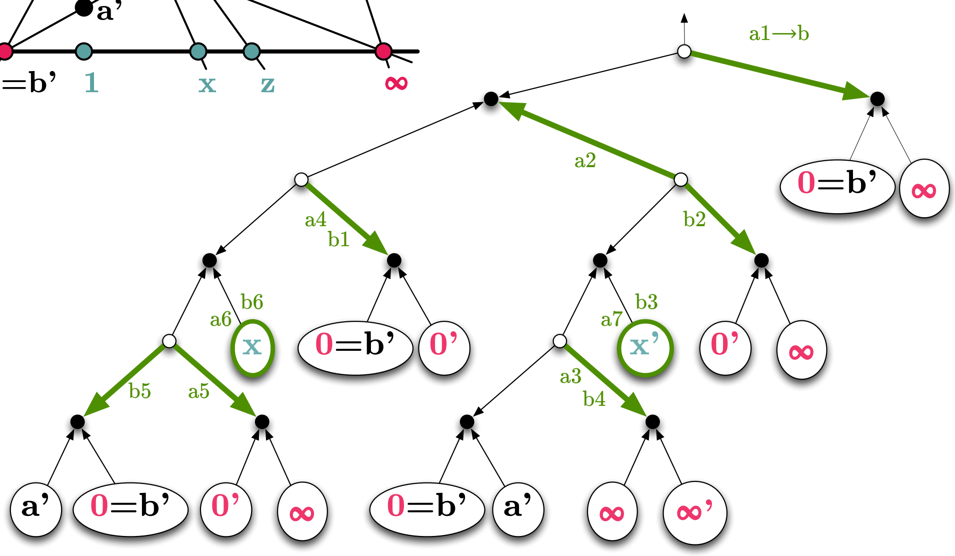
$$= [\mathbf{0}, \infty, \infty']^2 [\mathbf{0}, \mathbf{0}', \infty'][\mathbf{0}, \mathbf{a}', \mathbf{b}']E^*$$



Case 4 (ii): $\left|\{\mathbf{0},\infty,\mathbf{0}',\infty'\}\right| = 4$: $|\{\mathbf{a}',\mathbf{b}',\mathbf{0}\}| = 2$ and $\mathbf{0} = \mathbf{b}'$ and $\infty \neq \mathbf{a}'$

$$\mathrm{z} := \Big(\big(\mathbf{a}'\,\mathbf{0} \wedge \mathbf{0}'\,\infty\big)\,\mathrm{x} \wedge \mathbf{0}\,\mathbf{0}'\Big)\Big(\big(\mathbf{0}\,\mathbf{a}' \wedge \infty\,\infty'\big)\,\mathrm{x}' \wedge \mathbf{0}'\,\infty\Big) \wedge \mathbf{0}\,\infty$$
$$= [\mathbf{0},\infty,\mathbf{0}']^2[\mathbf{0},\infty,\mathbf{a}'][\infty,\mathbf{0}',\infty']E^*$$

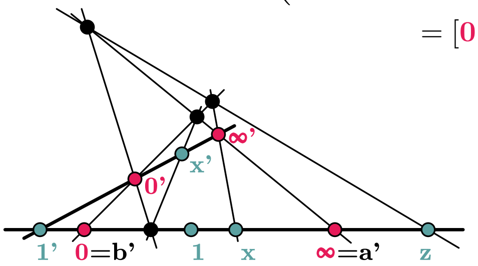


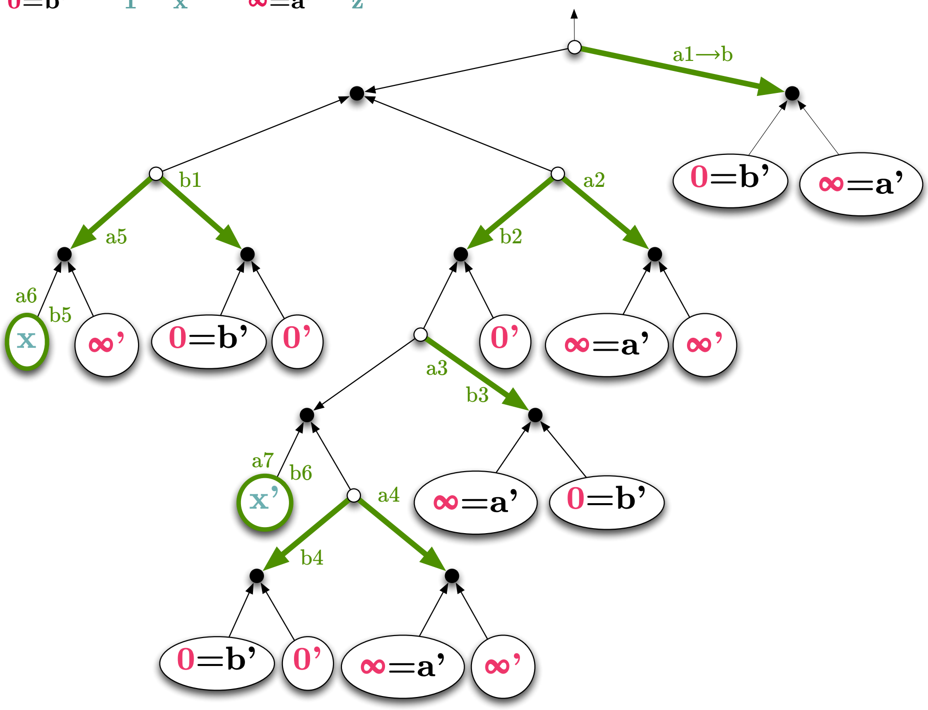


Case 4 (iii): $\left|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\right| = 4$: $\quad |\{\mathbf{a}', \mathbf{b}', \mathbf{0}\}| = 2$ and $\mathbf{0} = \mathbf{b}'$ and $\infty = \mathbf{a}'$

$$\mathbf{z} := \left(\mathbf{x}\,\infty' \wedge \mathbf{0}\,\mathbf{0}'\right)\left(\left(\mathbf{x}'\left(\mathbf{0}\,\mathbf{0}' \wedge \infty\,\infty'\right) \wedge \infty\,\mathbf{0}\right)\mathbf{0}' \wedge \infty\,\infty'\right) \wedge \mathbf{0}\,\infty$$

$$= [\mathbf{0}, \infty, \mathbf{0}'][\mathbf{0}, \infty, \infty'][\mathbf{0}, \mathbf{0}', \infty'][\infty, \mathbf{0}', \infty']E^*$$





In total Obviously, one can combine all cross-ratios resulting from Section 16.4 (by triangulating $\mathcal{C}_1, \ldots, \mathcal{C}_t$) in a single cross-ratio or 2-cycle

$$\Big((\infty,\, \mathbf{1},\, \mathbf{0}) \;\;,\;\; (\mathbf{0},\, \mathrm{x},\, \infty)\Big). \tag{16.15}$$

By induction and by using (16.14) as well as (16.9) for the shape of the cross-ratios resulting from Section 16.4, there is a bracket monomial M_{x} and a $p_{\mathbf{1}} \in \overline{\mathcal{P}}$ with $(\infty,\, \mathbf{1},\, \mathbf{0}) =$ $(p_{\mathbf{1}})$ such that

$$M_{\mathrm{x}} \cdot \left(\prod_{p \in \overline{\mathcal{P}} \setminus \{p_{\mathbf{1}}\}} \mu_p \quad \mathbf{0} \quad - \quad \epsilon \cdot \prod_{p \in \overline{\mathcal{P}} \setminus \{p_{\mathbf{1}}\}} \lambda_p \quad \infty \right). \tag{16.16}$$

For a degree bound on M_{x} we observe that we need to sum up the costs for triangulating the cycles $\mathcal{C}_1, \ldots, \mathcal{C}_t$ (see (16.10) and (16.11)) and for $t-1$ many times combining pairs of 2-cycles. This implies

$$\begin{aligned} \deg(M_{\mathrm{x}}) &\leq \sum_{i=1}^{t} \Big(\mathrm{length}(\mathcal{C}_i) - 2 \Big) \quad + 2 \quad + 4\,(t-1) \\ &= \left|\overline{\mathcal{P}}\right| + 2\,t - 2 \\ &\leq 2\,k + 2\,k - 2 \\ &\leq 4\,k \end{aligned} \tag{16.17}$$

since $\sum_{i=1}^{t} \mathrm{length}(\mathcal{C}_i) = \left|\overline{\mathcal{P}}\right|$ and $t \leq \left\lfloor \frac{|\overline{\mathcal{P}}|}{2} \right\rfloor$ as well as (16.2) which states $\left|\overline{\mathcal{P}}\right| \leq 2\,k$.

16.6. Final Coincidence

Consider the resulting 2-cycle given in (16.15). Due to $(\infty,\, \mathbf{1},\, \mathbf{0}) =$ $(p_{\mathbf{1}})$ we have

$$\mathbf{1} = \mathbf{a}\,\mathbf{b} \wedge \infty\,\mathbf{0}$$

for some $\mathbf{a}$, $\mathbf{b}$ in $\mathbf{P}$ and

$$\mu_{p_{\mathbf{1}}} = [\mathbf{a}, \mathbf{b}, \mathbf{0}] \quad \text{and} \quad \lambda_{p_{\mathbf{1}}} = [\mathbf{a}, \mathbf{b}, \infty].$$

By (16.16) and due to (16.7) it holds

$$
\begin{aligned}
&\mathbf{a}\,\mathbf{b} \wedge \mathbf{x}\\
=&M_{\mathbf{x}} \cdot \left(\prod_{p \in \overline{\mathcal{P}} \setminus \{p_1\}} \mu_p \quad \underbrace{[\mathbf{a}, \mathbf{b}, \mathbf{0}]}_{\mu_{p_1}} \;-\; \epsilon \cdot \prod_{p \in \overline{\mathcal{P}} \setminus \{p_1\}} \lambda_p \quad \underbrace{[\mathbf{a}, \mathbf{b}, \infty]}_{\lambda_{p_1}} \right)\\
=&M_{\mathbf{x}} \cdot N \cdot B
\end{aligned}
$$

and with a bracket monomial N with $\deg(N) \leq k$. Therefore, $\mathbf{a}\,\mathbf{b} \wedge \mathbf{x}$ is a generalized Cayley factorization with multiplier $M_{\mathbf{x}} \cdot N$ with

$$\deg(M_{\mathbf{x}} \cdot N) \leq 5\,k.$$

The algorithm described is linear in k except for the step where it has to find a perfect matching. The construction of G and finding the matching is possible in time $O(k^3)$ (e.g. the classical algorithm using augmented path which is due to [5] runs in the given time). This proves Theorem 14.4.

Remark 16.6. Observe that one can also achieve a linear running time with a weaker upper bound on the degree of the multiplier: Therefore one omits the step in which a perfect matching is obtained. Instead, one pairs a bracket in $\mathcal{L}_B$ with *any* bracket in $\mathcal{M}_B$. In order to apply the area principle, an additional slack variable might has to be introduced for pairs that have no common letter.

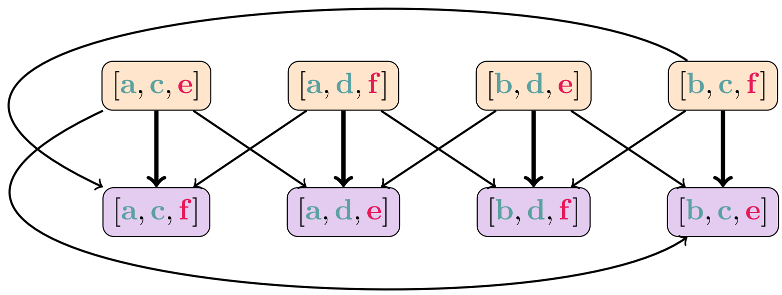
Remark 16.7. It should be remarked that the algorithm is not completely deterministic in the sense that there are various options of choice: in the concrete shape of the pairing $\overline{\mathcal{P}}$, in the concrete triangulation of the cycles and in the order and pattern in which the 2-cycles are combined. This freedom is partially used for local optimizations in the actual implementation presented in Chapter 18.

16.7. Example: A Derivation of Pascal's Theorem

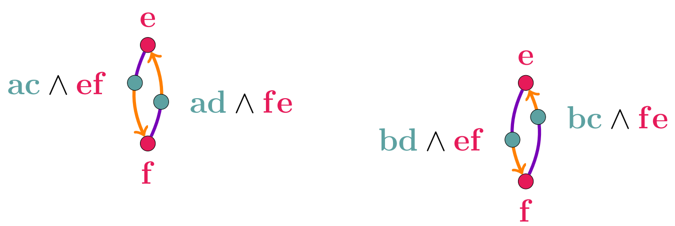
One of the smallest non-trivial example only consisting of two 2-cycles is given by the bracket monomial

$$B := [\mathbf{a},\mathbf{c},\mathbf{e}][\mathbf{a},\mathbf{d},\mathbf{f}][\mathbf{b},\mathbf{c},\mathbf{f}][\mathbf{b},\mathbf{d},\mathbf{e}] - [\mathbf{a},\mathbf{c},\mathbf{f}][\mathbf{a},\mathbf{d},\mathbf{e}][\mathbf{b},\mathbf{c},\mathbf{e}][\mathbf{b},\mathbf{d},\mathbf{f}]. \tag{16.18}$$

Actually, *any* pairing $\overline{\mathcal{P}}$ using only edges in E_1 (i.e. only edges joining brackets with exactly two common entries) of B corresponds to two 2-cycles. B belongs to Pascal's theorem (see Section 3.10 and also Section 6.1) and the existence of a synthetic construction describing the bracket polynomial was one of the motivations for Cayley factorization. This example is also used in [104, p. 449 ff.] in order to motivate the use of Cayley factorization. There it is assumed that that they already possess an efficient subroutine for Cayley factoring non-multilinear bracket polynomials. Afterwards, they use this routine in order to “detect” Pascal’s theorem. We will demonstrate in the following that with suitable choices done, our algorithm can in principle do this. The diagram below indicates all edges in E_1. There are three pairings using only edges in E_1 such that the combination of resulting 2-cycles can be expressed as Case 2 (i) or Case 2 (ii). The other matchings correspond to a case with $\left|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\right| = 4$ which yields a multiplier monomial of bigger degree than the cases indicated by $\left|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\right| = 2$. One of these matchings corresponding to Case 2 (i) or Case 2 (ii) is highlighted by thick edges, the two other ones have a symmetric structure. The highlighted matching shall be called $\overline{\mathcal{P}}$.



$\overline{\mathcal{P}}$ induces $\varepsilon = +1$ and the two 2-cycles



We choose to use Case 2 (i) labeled with $\left|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\right| = 2$: $\mathbf{0} = \mathbf{0}'$ and $\infty = \infty'$. We define

$$\mathbf{z} := \big(\mathbf{b}\,\mathbf{f} \wedge (\mathbf{a}\,\mathbf{d} \wedge \mathbf{f}\,\mathbf{e})\,\mathbf{d}\big)\,\big(\mathbf{d}\,\mathbf{e} \wedge (\mathbf{b}\,\mathbf{c} \wedge \mathbf{f}\,\mathbf{e})\,\mathbf{b}\big) \wedge \mathbf{f}\,\mathbf{e}. \tag{16.19}$$

By the results of the previous sections we have

$$\mathbf{ac} \wedge \mathbf{z} = [\mathbf{f}, \mathbf{e}, \mathbf{b}][\mathbf{f}, \mathbf{e}, \mathbf{d}] \cdot B$$

where the multiplier monomial comes from the combination of 2-cycles. Furthermore, we have

$$(\mathbf{a}\,\mathbf{d} \wedge \mathbf{f}\,\mathbf{e})\,\mathbf{d} = -[\mathbf{f}, \mathbf{e}, \mathbf{d}]\,\mathbf{a}\,\mathbf{d} \quad \text{and} \quad (\mathbf{b}\,\mathbf{c} \wedge \mathbf{f}\,\mathbf{e})\,\mathbf{b} = [\mathbf{f}, \mathbf{e}, \mathbf{b}]\,\mathbf{c}\,\mathbf{b}. \tag{16.20}$$

Therefore, we can substitute (16.20) in (16.19), cancel and conclude

$$-B = \mathbf{ac} \wedge \Big((\mathbf{b}\,\mathbf{f} \wedge \mathbf{a}\,\mathbf{d})\,(\mathbf{d}\,\mathbf{e} \wedge \mathbf{c}\,\mathbf{b}) \wedge \mathbf{f}\,\mathbf{e} \Big) = -(\mathbf{a}\,\mathbf{c} \wedge \mathbf{f}\,\mathbf{e}) \wedge (\mathbf{b}\,\mathbf{f} \wedge \mathbf{a}\,\mathbf{d})\,(\mathbf{d}\,\mathbf{e} \wedge \mathbf{c}\,\mathbf{b}).$$

This is a synthetic construction differing from the one given in Section 3.10 only by a permutation of indices. We have seen similar phenomena also in Section 6.1. We will argue that the given factorization can be considered as an automatic derivation of Pascal's theorem when the algorithm for Cayley factorization is optimized locally. We already did a local optimization when choosing the matching. After choosing Case 2 (i) for the combination of 2-cycles, we had to decide which points should be considered as $\mathbf{a}'$ and $\mathbf{b}'$. Here the given choice should be preferred since it can be locally simplified using (16.20). Now when choosing Case 2 (ii), adjusting the roles of points $\mathbf{a}'$ and $\mathbf{b}'$, allows again for local simplifications of the construction and yields a version of Pascal's construction. We think that this is due to the beauty and symmetry of the conic condition. However, the algorithm for Cayley factorization is close enough to geometry to benefit from this beauty. For further comments on also automatically deriving (16.18) see [104].

17. Generalization for Polynomials with Arbitrary Number of Summands

An intuition for the general algorithm in terms of length ratios (compare Section 15) can be provided as follows: Consider a general bracket polynomial

$$\alpha_1[*,*,*][*,*,*]\cdots[*,*,*]+\cdots+\alpha_l[*,*,*][*,*,*]\cdots[*,*,*] \\ -\alpha_\Omega[*,*,*][*,*,*]\cdots[*,*,*].$$

For an instance P, we have

$$\alpha_1[*,*,*]_P\cdots[*,*,*]_P+\cdots+\alpha_l[*,*,*]_P\cdots[*,*,*]_P-\alpha_\Omega[*,*,*]_P\cdots[*,*,*]_P=0$$

$$\Longleftrightarrow$$

$$\frac{\alpha_1[*,*,*]_P\cdots[*,*,*]_P+\cdots+\alpha_l[*,*,*]_P\cdots[*,*,*]_P}{[*,*,*]_P\cdots[*,*,*]_P}=\alpha_\Omega$$

$$\Longleftrightarrow$$

$$\alpha_1\frac{[*,*,*]_P\cdots[*,*,*]_P}{[*,*,*]_P\cdots[*,*,*]_P}+\cdots+\alpha_l\frac{[*,*,*]_P\cdots[*,*,*]_P}{[*,*,*]_P\cdots[*,*,*]_P}=\alpha_\Omega \tag{17.1}$$

In principle, the algorithm for the binomial case given in Section 16 can interpret each of the summands (without the coefficient) as a product of length ratios. It can output a cross-ratio that contains the same information. They can be combined using the tool of projective addition (see Section 3.7). In order to combine them, we have to consider them as numbers on *the same* projective number line with respect to *the same* basis. This is the very general outline of the present chapter.

To work out the details and to achieve a low total upper bound on the multiplier monomial, we need more than the brief description above. In particular, practical experiments with large bracket polynomials in an implementation of the algorithm in Mathematica ([136]) by the author (see Chapter 18 for more information about the implementation) has shown that it is worth to explicitly factor out some common factors when appropriate (see below). Unfortunately, this makes the exposition a little tedious. Let $\mathbf{P}$ be a finite set and let $B \in \mathbb{R}[\Delta(\mathbf{P})]$ be multihomogenous with coefficients in $\mathbb{Z}$. Let $k \geq 2$ be the degree of B in its brackets and $l+1$ the number of summands, such that no summand has repeating indices in its brackets and such that no difference of two summands lies in $\langle \mathbf{I}_{\text{repeat}} \cup \mathbf{I}_{\text{altern}} \rangle$. (Observe that this does not affect the bounds given in Theorem 14.5.) Furthermore, the summands shall not have a overall bracket monomial factor. By switching the positions of two elements in some brackets we can w.l.o.g. assume that all the coefficients α_j $(1 \leq j \leq l+1)$ are in $\mathbb{N} \setminus \{0\}$ such that B has the following sign pattern:

$$\alpha_1 \underbrace{[*,*,*][*,*,*]\cdots[*,*,*]}_{=:\Pi_1} + \cdots + \alpha_l \underbrace{[*,*,*][*,*,*]\cdots[*,*,*]}_{=:\Pi_l} \\ - \alpha_\Omega \underbrace{[*,*,*][*,*,*]\cdots[*,*,*]}_{=:\Omega}. \qquad (17.2)$$

Consider the l binomials given below. The right-hand sides are versions where the common factors $\mathcal{F}^{(*)}$s are factored out explicitly:

$$\begin{aligned} \Pi_1 - \Omega &= \mathcal{F}^{(1)} \underbrace{(\mathcal{L}^{(1)} - \mathcal{M}^{(1)})}_{=:B_1}, \\ &\vdots \\ \Pi_l - \Omega &= \mathcal{F}^{(l)} \underbrace{(\mathcal{L}^{(l)} - \mathcal{M}^{(l)})}_{=:B_l}. \end{aligned} \qquad (17.3)$$

This implies

$$\mathcal{F}^{(i)} \mathcal{L}^{(i)} = \Pi_i \quad \text{and} \quad \mathcal{F}^{(i)} \mathcal{M}^{(i)} = \Omega. \qquad (17.4)$$

Let k_i denote the degree of B_i $(1 \leq i \leq l)$. Since the summands of B are distinct, all k_is are bigger than 1. Due to Theorem 14.4 and Chapter 16, the B_is can be Cayley factored. In particular, for each $(1 \leq i \leq l)$ there is a 2-cycle

$$\mathcal{C}_i := \Big((\infty_i,\ \mathbf{1}_i = \underbrace{[\mathbf{a}_i, \mathbf{b}_i, \mathbf{0}_i]}_{=:\,\mu_{\mathbf{1}_i}} \infty_i - \underbrace{[\mathbf{a}_i, \mathbf{b}_i, \infty_i]}_{=:\,\lambda_{\mathbf{1}_i}} \mathbf{0}_i \ ,\ \mathbf{0}_i),$$

$$(\mathbf{0}_i,\ \ \mathbf{x}_i = M_{\mathbf{x}_i}(\qquad \mu_{\mathbf{x}_i} \mathbf{0}_i - \qquad \lambda_{\mathbf{x}_i} \infty_i),\ \ \infty_i)\Big).$$

for some $\mathbf{a}_i$ and $\mathbf{b}_i$ in $\mathbf{P}$ with the following properties:

$$\mu_{\mathbf{x}_i} \mu_{\mathbf{1}_i} = N_i \cdot \mathcal{M}^{(i)} \quad \text{and} \quad \lambda_{\mathbf{x}_i} \lambda_{\mathbf{1}_i} = N_i \cdot \mathcal{L}^{(i)}. \qquad (17.5)$$

which implies

$$M_{\mathbf{x}_i} \cdot \left(\mu_{\mathbf{x}_i}\mu_{\mathbf{1}_i} - \lambda_{\mathbf{x}_i}\lambda_{\mathbf{1}_i}\right) = M_{\mathbf{x}_i} \cdot N_i \cdot B_i$$

for suitable bracket monomials N_i such that

$$\deg(M_{\mathbf{x}_i} \cdot N_i) \leq 5\, k_i.$$

Similarly to the situation in Chapter 16, we now are left with l highly preprocessed 2-cycles. In a next step, if necessary, we establish the summands with coefficients $\alpha_i \neq 1$.

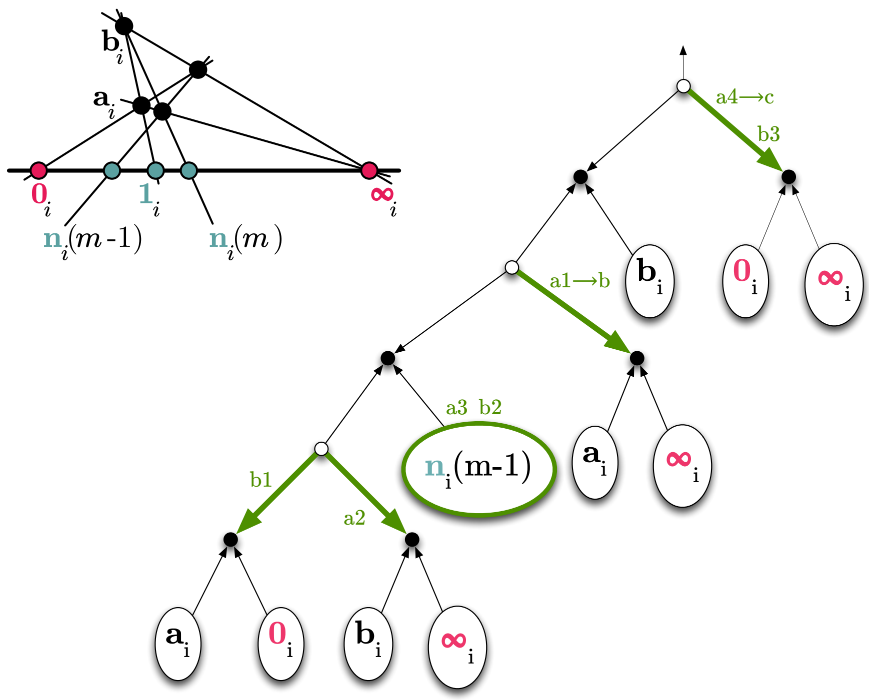
17.1. Constructing the Coefficients

Due to classical literature, the integers can be constructed on projective number lines. This is often referred to as a *harmonic sequence*. We also give an (inductive) evaluation in order to ensure that the multiplier is a bracket *monomial*. Consider a $i \in \{1, \ldots, l\}$ with $\alpha_i \geq 2$. We can recursively construct the natural numbers with respect to the base $\mathbf{0}_i, \infty_i, \mathbf{1}_i$.

Constructing the natural numbers $m \in \mathbb{N} \setminus \{0\}$.

$$\mathbf{n}_i(m) := \begin{cases} \mathbf{1}_i \quad = \mu_{\mathbf{1}_i}\,\infty_i - \lambda_{\mathbf{1}_i}\,\mathbf{0}_i = [\mathbf{a}_i, \mathbf{b}_i, \mathbf{0}_i]\,\infty_i - [\mathbf{a}_i, \mathbf{b}_i, \infty_i]\,\mathbf{0}_i & \text{for } m = 1 \\ \Big(\big(\mathbf{a}_i\ \mathbf{0}_i\ \wedge\ \mathbf{b}_i\ \infty_i\big)\ \mathbf{n}_i(m-1)\ \wedge\ \mathbf{a}_i\ \infty_i\Big)\,\mathbf{b}_i\ \wedge\ \mathbf{0}_i\ \infty_i & \text{for } m \geq 2 \end{cases}$$

$$= (\lambda_{\mathbf{1}_i}[\mathbf{0}_i, \infty_i, \mathbf{a}_i][\mathbf{0}_i, \infty_i, \mathbf{b}_i])^{m-1}\ \left(m\,\mu_{\mathbf{1}_i}\infty_i - \lambda_{\mathbf{1}_i}\,\mathbf{0}_i\right)$$



Now, $\mathbf{n}_i(\alpha_i)$ and $\mathbf{x}_i$ can be considered to represent numbers *on the same* projective number line. A multiplication of both numbers can be achieved by a von Staudt construction (see Section 3.7). We define

$$\mathbf{z}_i := \big(\mathbf{0}_i\,\mathbf{a}_i\ \wedge\ \mathbf{x}_i\,\mathbf{b}_i\big)\ \big(\mathbf{b}_i\,\infty_i\ \wedge\ \mathbf{n}_i(\alpha_i)\,\mathbf{a}_i\big)\ \wedge\ \mathbf{0}_i\,\infty_i.$$

By a pattern matching (substitute $\mathbf{1}_i$ for $\mathbf{1}'$, $\mathbf{x}_i$ for $\mathbf{x}$ and $\mathbf{n}_i$ for $\mathbf{x}'$) with the case described in Case 2 (i) in Section 16.5, we obtain after simplification:

$$\mathbf{z}_i = M_{\mathbf{x}_i}\ (\lambda_{\mathbf{1}_i}[\mathbf{b}_i, \mathbf{0}_i, \infty_i][\mathbf{a}_i, \mathbf{0}_i, \infty_i])^{\alpha_i}\,\mu_{\mathbf{1}_i}\ \Big(\mu_{\mathbf{x}_i}\,\mathbf{0}_i - \alpha_i\,\lambda_{\mathbf{x}_i}\,\infty_i\Big)$$

By replacing each $\mathbf{x}_i$ by $\mathbf{z}_i$ in $\mathcal{C}_i$ whenever the corresponding α_i is bigger than 1, we can w.l.o.g. assume $\mathbf{x}_i = M_{\mathbf{x}_i}\left(\mu_{\mathbf{x}_i}\mathbf{0}_i - \lambda_{\mathbf{x}_i}\infty_i\right)$ such that we have (compare (17.5))

$$\mu_{\mathbf{x}_i}\mu_{\mathbf{1}_i} = N_i \cdot \mathcal{M}^{(i)} \quad \text{and} \quad \lambda_{\mathbf{x}_i}\lambda_{\mathbf{1}_i} = N_i \cdot \alpha_i \cdot \mathcal{L}^{(i)} \tag{17.6}$$

which induces

$$M_{\mathbf{x}_i} \cdot \left(\mu_{\mathbf{x}_i}\mu_{\mathbf{1}_i} - \lambda_{\mathbf{x}_i}\lambda_{\mathbf{1}_i}\right) = M_{\mathbf{x}_i} \cdot N_i \cdot (\alpha_i\,\mathcal{L}^{(i)} - \mathcal{M}^{(i)})$$

with

$$\deg(M_{\mathbf{x}_i} \cdot N_i) \quad \leq \begin{cases} 5\,k_i & \text{if } \alpha_i = 1 \\ 5\,k_i + 3\,\alpha_i + 1 & \text{if } \alpha_i > 1. \end{cases} \tag{17.7}$$

Remark 17.1. There are various possibilities to construct the coefficients α_i. We chose the present one because the degree bound only depends on the α_i and is not too complicated. Of course, there are more efficient methods for constructing the natural numbers than just adding 1s. E.g. we could also use the binary representation together with Horner's method in order to construct $m \in \mathbb{N}$ with a leading monomial of degree not bigger than $8 \cdot \lceil \log_2(m) \rceil$. We will not give any details in order to save space.

17.2. Adding up Two 2-Cycles or Cross-Ratios

We are left with a lot of 2-cycles or cross-ratios $\mathcal{C}_1, \ldots, \mathcal{C}_l$. At this points, literally all comments from the first part of Section 16.5 apply. With the same settings as given there we give constructions for combining two 2-cycles. We do the same case distinction as before. Now the evaluations of $\mathbf{z}$ given below all are a bracket *monomial* multiple of

$$E^{+} = M_{\mathbf{x}'} \cdot M_{\mathbf{x}}\big(\,\lambda_{\mathbf{1}}\mu_{\mathbf{1}'}\mu_{\mathbf{x}'}\mu_{\mathbf{x}}\,\mathbf{0} - (\lambda_{\mathbf{1}'}\lambda_{\mathbf{x}'}\mu_{\mathbf{1}}\mu_{\mathbf{x}} + \lambda_{\mathbf{1}}\lambda_{\mathbf{x}}\mu_{\mathbf{1}'}\mu_{\mathbf{x}'})\,\infty\big). \tag{17.8}$$

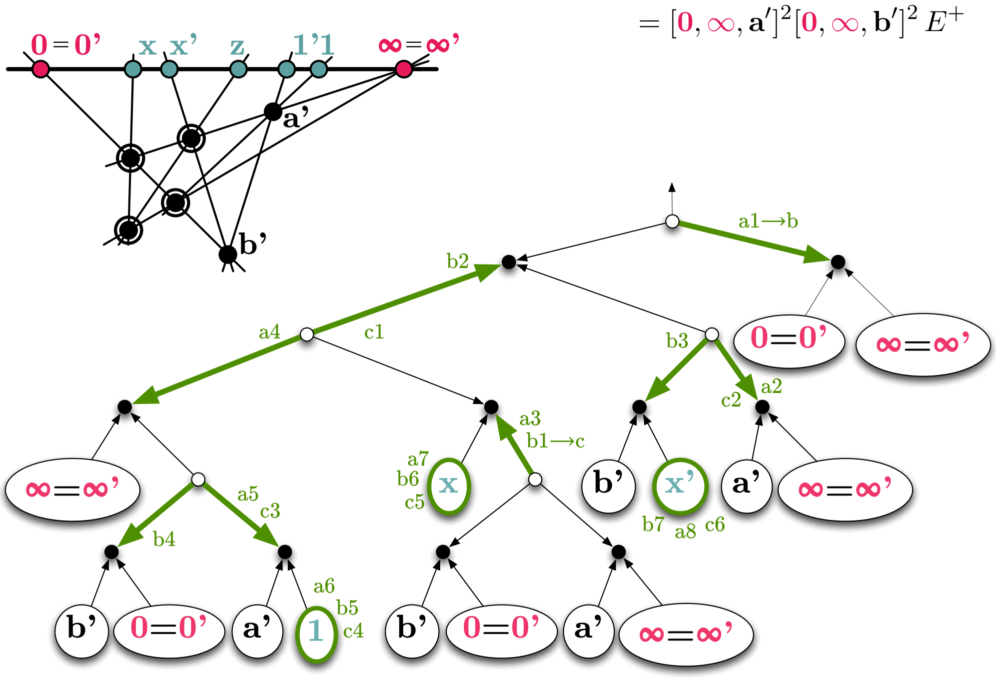
We put on record that the degree of the multiplier monomial is

$$\leq 5. \tag{17.9}$$

The idea behind all of the constructions is just the one given in the beginning of the chapter: We first express both cross-ratios with respect to the same basis. Then we perform a projective addition. Furthermore, we want to keep the construction small (in order to keep the degree of the multiplier small). Therefore, for the addition we reuse points introduced for changing the projective basis. The points marked with a circle in the following diagrams give vertices of a quadrilaterals witnessing the projective addition (compare Section 3.7). In the corresponding quadrilateral a line is missing. Including this line and intersecting it with the line $\mathbf{0}\,\infty$, one can see both parts of the construction: the adjustment of the basis and the projective addition. In order to interpret the construction for Case 2 (ii), one should observe that one can exchange the role of the points $\mathbf{0}'$ and ∞' when simultaneously exchanging points $\mathbf{1}'$ and $\mathbf{x}'$. Therefore the same picture is drawn as in the case where $\mathbf{0} = \mathbf{0}'$ and $\infty = \infty'$. However, one has to take into account that the points $\mathbf{a}'$ and $\mathbf{b}'$ determine $\mathbf{1}'$ (not $\mathbf{x}'$) and therefore both constructions differ. By the same arguments as in Section 16.5, the multipliers do not vanish in $\mathcal{B}_{\mathbf{P}}$. Again, by possibly exchanging the roles of the points $\mathbf{a}'$ and $\mathbf{b}'$ or of the two 2-cycles in question, one of the cases given below is applicable. The possibility of being forced to exchange the roles of the two 2-cycles will make the things more complicated in Section 17.3. Such a situation arises e.g. if $\big|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\big| = 4$, $|\{\mathbf{a}', \mathbf{b}', \mathbf{0}\}| = 2$, $\mathbf{0} = \mathbf{b}'$, $\infty = \mathbf{a}'$, $\{\mathbf{a}, \mathbf{b}\} \neq \{\mathbf{0}', \infty'\}$. Clearly, there are constructions that do not need to exchange the cycles also in this situation. However, those that we know of, all result in a bigger degree of the multiplier monomial of E^+. A remark on the evaluation of the last formula: the result is identical to $-\lambda_{\mathbf{1}}\mu_{\mathbf{1}'}\,E^+$ which might be confusing when identifying λs and μs.

Case 2 (i): $\big|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\big| = 2$: $\mathbf{0} = \mathbf{0}'$ and $\infty = \infty'$

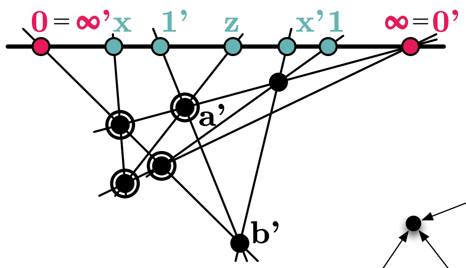
$$\mathbf{z} := \big(\infty\,(\mathbf{b}'\,\mathbf{0} \wedge \mathbf{a}'\,\mathbf{1}) \wedge \mathbf{x}\,(\mathbf{b}'\,\mathbf{0} \wedge \mathbf{a}'\,\infty)\big)\big(\mathbf{b}'\,\mathbf{x}' \wedge \mathbf{a}'\,\infty\big) \wedge \mathbf{0}\,\infty$$
$$= [\mathbf{0}, \infty, \mathbf{a}']^2[\mathbf{0}, \infty, \mathbf{b}']^2\,E^+$$

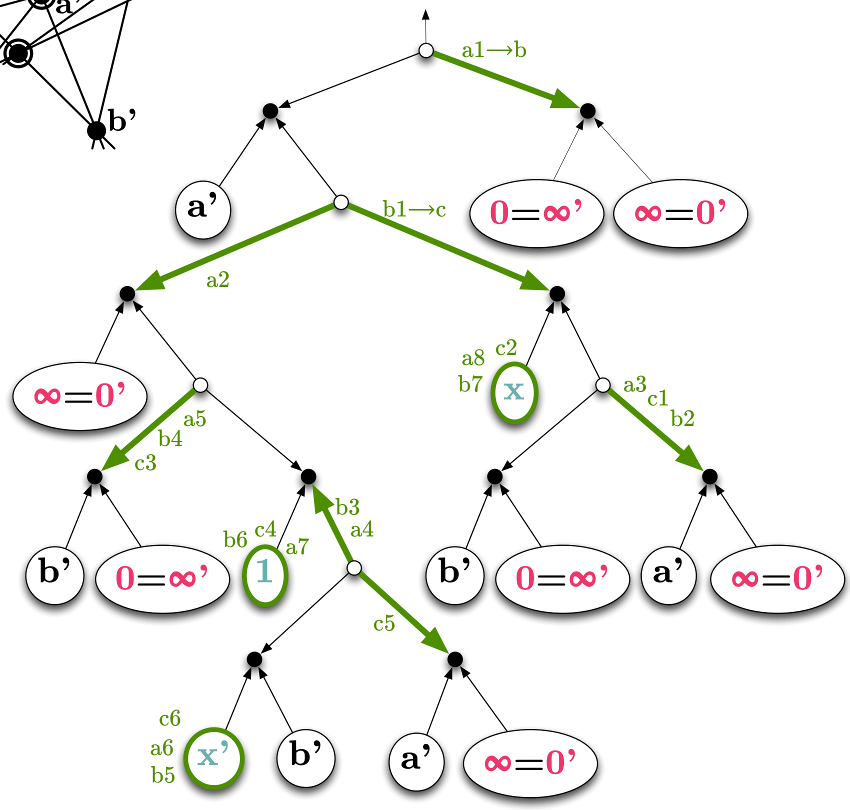


Case 2 (ii): $\left|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\right| = 2$: $\mathbf{0} = \infty'$ and $\infty = \mathbf{0}'$

$$\mathrm{z} := \mathbf{a}' \Big(\infty \left(\mathbf{b}'\, \mathbf{0} \wedge \mathbf{1}\, (\mathrm{x}'\, \mathbf{b}' \wedge \mathbf{a}'\, \infty)\right) \wedge \mathrm{x} \left(\mathbf{b}'\, \mathbf{0} \wedge \mathbf{a}'\, \infty\right) \Big) \wedge \mathbf{0}\, \infty$$

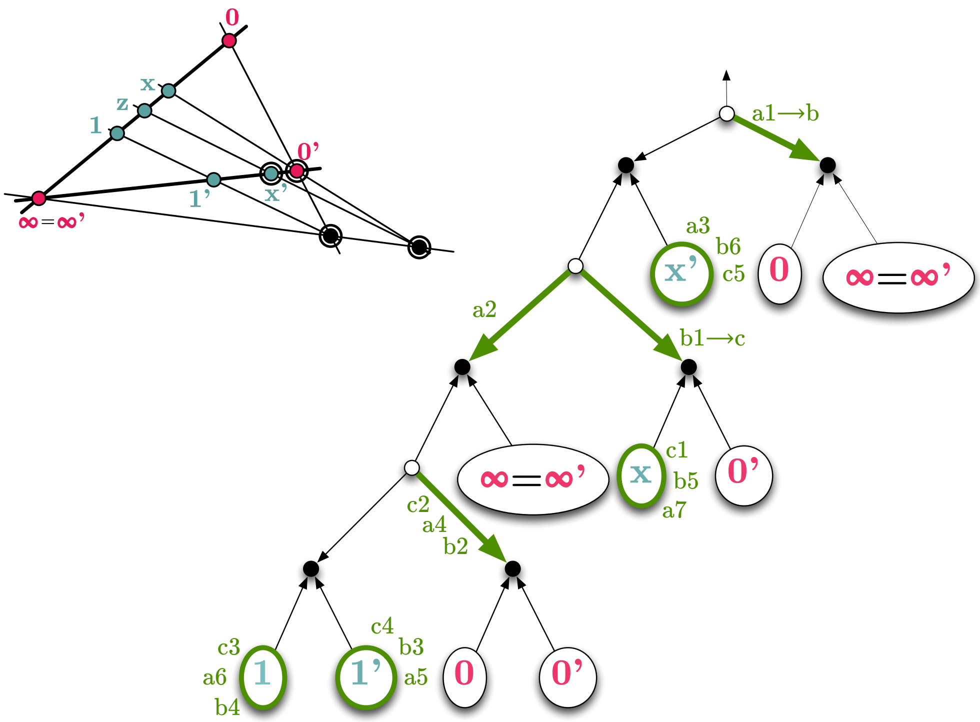
$$= [\mathbf{0}, \infty, \mathbf{a}']^2 [\mathbf{0}, \infty, \mathbf{b}']^2\, E^+$$





$\left|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\right| = 3$:

Case 3(i): $\infty = \infty'$: $\quad \mathbf{z} := \big((\mathbf{1}\,\mathbf{1}' \wedge \mathbf{0}\,\mathbf{0}')\,\infty \wedge \mathbf{x}\,\mathbf{0}'\big)\,\mathbf{x}' \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \mathbf{0}']^3 E^+$



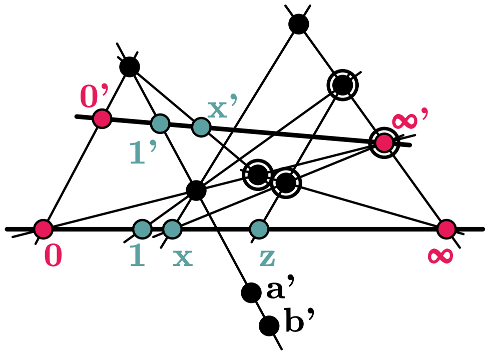
Case 3(ii): $\mathbf{0} = \mathbf{0}'$: $\quad \mathbf{z} := (\mathbf{1}'\,\mathbf{1} \wedge \infty'\,\infty)\,(\mathbf{x}'\,\infty \wedge \infty'\,\mathbf{x}) \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \infty']^3 E^+$

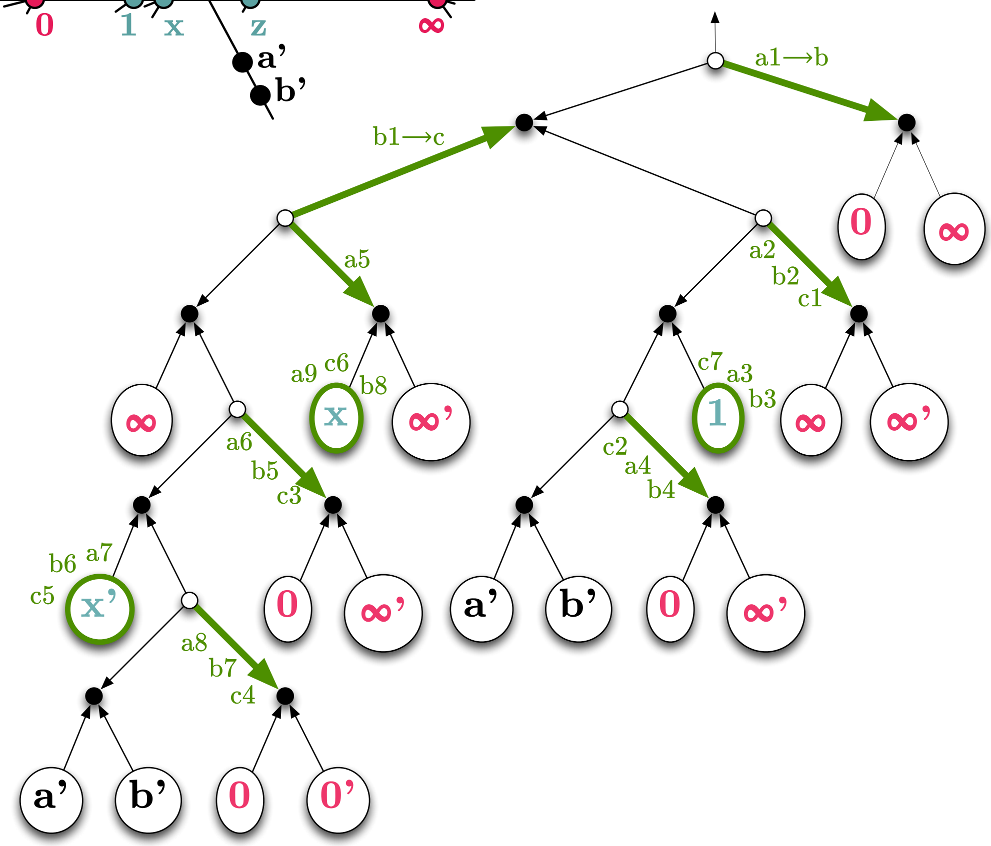
Case 3(iii): $\infty = \mathbf{0}'$: $\quad \mathbf{z} := \big((\mathbf{1}\,\mathbf{x}' \wedge \mathbf{0}\,\infty')\,\infty \wedge \mathbf{x}\,\infty'\big)\,\mathbf{1}' \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \infty']^3 E^+$

Case 3(iv): $\mathbf{0} = \infty'$: $\quad \mathbf{z} := (\mathbf{1}\,\mathbf{x}' \wedge \mathbf{0}'\,\infty)\,(\mathbf{1}'\,\infty \wedge \mathbf{0}'\,\mathbf{x}) \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \mathbf{0}']^3 E^+$

Case 4 (i): $\big|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\big| = 4$: $|\{\mathbf{a}', \mathbf{b}', \mathbf{0}\}| = 3$

$$\mathbf{z} := \Big(\infty\,\Big(\mathbf{x}'\,(\mathbf{a}'\,\mathbf{b}' \wedge \mathbf{0}\,\mathbf{0}') \wedge \mathbf{0}\,\infty'\Big) \wedge \mathbf{x}\,\infty'\Big)\Big((\mathbf{a}'\,\mathbf{b}' \wedge \mathbf{0}\,\infty')\,\mathbf{1} \wedge \infty\,\infty'\Big) \wedge \mathbf{0}\,\infty$$
$$= [\mathbf{0}, \infty, \infty']^3[\mathbf{0}, \mathbf{0}', \infty'][\mathbf{0}, \mathbf{a}', \mathbf{b}']\,E^+$$

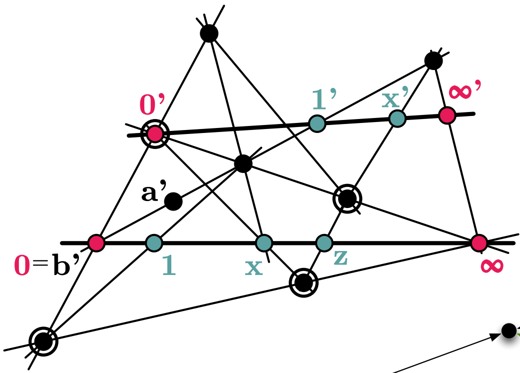


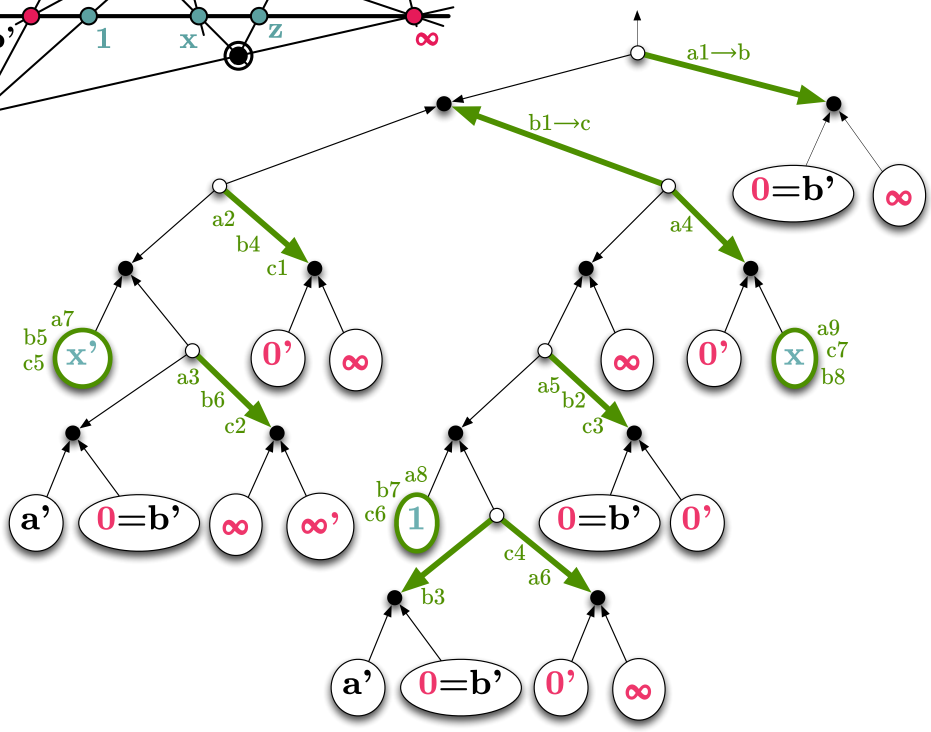


Case 4 (ii): $\big|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\big| = 4$: $|\{\mathbf{a}', \mathbf{b}', \mathbf{0}\}| = 2$ and $\mathbf{0} = \mathbf{b}'$ and $\infty \neq \mathbf{a}'$

$$\mathbf{z} := \Big(\mathbf{x}'\,(\mathbf{a}'\,\mathbf{0} \wedge \infty\,\infty') \wedge \mathbf{0}'\,\infty\Big)\Big(\Big(\mathbf{1}\,(\mathbf{a}'\,\mathbf{0} \wedge \mathbf{0}'\,\infty) \wedge \mathbf{0}\,\mathbf{0}'\Big)\,\infty \wedge \mathbf{0}'\,\mathbf{x}\Big) \wedge \mathbf{0}\,\infty$$

$$= [\mathbf{0}, \infty, \mathbf{0}']^3 [\mathbf{0}, \infty, \mathbf{a}'][\infty, \mathbf{0}', \infty']\, E^+$$

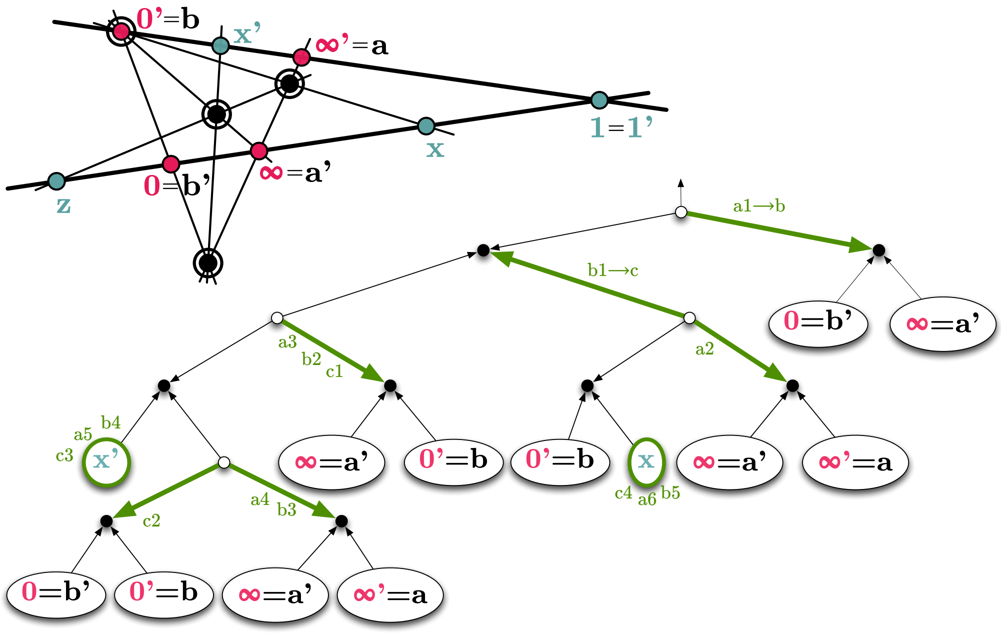




Case 4 (iii): $\left|\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}\right| = 4$:

$|\{\mathbf{a}', \mathbf{b}', \mathbf{0}\}| = 2, \mathbf{0} = \mathbf{b}', \infty = \mathbf{a}', \mathbf{0}' = \mathbf{b}, \infty' = \mathbf{a}$

$$\mathbf{z} := \Big(\mathbf{x}'\,(\mathbf{0}\,\mathbf{0}' \wedge \infty\,\infty') \wedge \infty\,\mathbf{0}'\Big)\Big(\mathbf{0}'\,\mathbf{x} \wedge \infty\,\infty'\Big) \wedge \mathbf{0}\,\infty = [\mathbf{0}, \infty, \mathbf{0}'][\infty, \mathbf{0}', \infty']\,E^{+}$$



17.3. Adding up All 2-Cycles or Cross-Ratios

Unfortunately, in the case of adding up 2-cycles, the result differs much more from the shape of the $\mathbf{x}_i$ in the original $\mathcal{C}_i$s. Therefore, the iterative addition is formulated differently (see Lemma 17.2 below). As mentioned before, the case distinction done in Section 17.2 for combining two 2-cycles implies that in some cases, one has to exchange the roles of both cycles. This implies that one looses the control of which 2-cycles plays which role in the combination. Therefore, the statement and proof of the following lemma have to be formulated without the knowledge of the role the cycles play in the combination.

Lemma 17.2. *Suppose that the cross-ratios $\mathcal{C}_1, \ldots, \mathcal{C}_l$ are combined linearly, i.e. w.l.o.g. we start by adding the 2-cycles $\mathcal{C}_1$ and $\mathcal{C}_2$. Then we add $\mathcal{C}_3, \mathcal{C}_4, \ldots$ by the constructions of Section 17.2. Let A_j be the bracket monomial of degree not bigger than 5 that is produced during the addition of $\mathcal{C}_j$ as multiplier of E^+. The 2-cycle obtained after adding $\mathcal{C}_j$ $(2 \le j \le l)$ can be written as $\big((\boldsymbol{\infty}_{i^*}, \mathbf{1}_{i^*}, \mathbf{0}_{i^*}), \ (\mathbf{0}_{i^*}, \mathbf{x}^{(j)}, \boldsymbol{\infty}_{i^*})\big)$ with some $i^* \in \{1, \ldots, j\}$. For $j \ge 2$ there is an evaluation of $\mathbf{x}^{(j)}$ as:*

$$\mathbf{x}^{(j)} = M_{\mathbf{x}^{(j)}} \cdot \left(\mu_{\mathbf{x}_{i^*}} \lambda_{\mathbf{1}_{i^*}} \prod_{\substack{i=1 \\ i \neq i^*}}^{j} \mathcal{M}^{(i)} \, \mathbf{0}_{i^*} \quad - \quad \left(\sum_{s=1}^{j} \alpha_s \, \mathcal{L}^{(s)} \prod_{\substack{i=1 \\ i \neq s}}^{j} \mathcal{M}^{(i)} \right) N_{i^*} \, \boldsymbol{\infty}_{i^*} \right)$$

such that there is a bracket monomial Q_j of degree $j-2$ with

$$M_{\mathbf{x}^{(j)}} = Q_j \cdot \prod_{\substack{i=1 \\ i \neq i^*}}^{j} N_i \prod_{i=1}^{j} M_{\mathbf{x}_i} \prod_{i=2}^{j} A_i.$$

Proof. We proceed by induction on j. Let $j = 2$. We can w.l.o.g. assume that $i^* = 1$. Due to (17.8), $\mathbf{x}^{(2)}$ is given as

$$\begin{aligned} \mathbf{x}^{(2)} &= A_2 \cdot M_{\mathbf{x}_1} \cdot M_{\mathbf{x}_2} \\ &\quad \cdot \Big(\lambda_{\mathbf{1}_1} \underbrace{\mu_{\mathbf{1}_2} \mu_{\mathbf{x}_2}}_{\mathcal{M}^{(2)} N_2} \mu_{\mathbf{x}_1} \, \mathbf{0}_1 \quad - \big(\underbrace{\lambda_{\mathbf{1}_2} \lambda_{\mathbf{x}_2}}_{\alpha_2 \mathcal{L}^{(2)} N_2} \underbrace{\mu_{\mathbf{1}_1} \mu_{\mathbf{x}_1}}_{\mathcal{M}^{(1)} N_1} + \underbrace{\lambda_{\mathbf{1}_1} \lambda_{\mathbf{x}_1}}_{\alpha_1 \mathcal{L}^{(1)} N_1} \underbrace{\mu_{\mathbf{1}_2} \mu_{\mathbf{x}_2}}_{\mathcal{M}^{(2)} N_2} \big) \boldsymbol{\infty}_1 \Big) \\ &= A_2 \cdot M_{\mathbf{x}_1} \cdot M_{\mathbf{x}_2} \cdot N_2 \big(\mu_{\mathbf{x}_1} \lambda_{\mathbf{1}_1} \mathcal{M}^{(2)} \, \mathbf{0}_1 - (\alpha_2 \, \mathcal{L}^{(2)} \mathcal{M}^{(1)} + \alpha_1 \, \mathcal{L}^{(1)} \mathcal{M}^{(2)}) \, N_1 \, \boldsymbol{\infty}_1 \big). \end{aligned}$$

The underbraces are due to (17.6). This is of the required form. Now let $j \ge 3$. The lemma is considered to be true for $j-1$. Therefore there is a $i^* \in \{1, \ldots, j-1\}$

such that in step j the to 2-cycles

$$\Big((\infty_{i^*},\, \mathbf{1}_{i^*},\, \mathbf{0}_{i^*}),\ (\mathbf{0}_{i^*},\, \mathrm{x}^{(j-1)},\, \infty_{i^*}) \Big) \quad \text{and}$$

$$\mathcal{C}_j = \Big((\infty_j,\ \mathbf{1}_j,\ \mathbf{0}_j\ ),\ (\mathbf{0}_j,\ \mathrm{x}_j,\qquad \infty_j\ ) \Big)$$

are added. With

$$\mu_{\mathrm{x}^{(j-1)}} = \mu_{\mathrm{x}_{i^*}}\, \lambda_{\mathbf{1}_{i^*}} \prod_{\substack{i=1\\ i\neq i^*}}^{j-1} \mathcal{M}^{(i)} \quad \text{and} \quad \lambda_{\mathrm{x}^{(j-1)}} = N_{i^*} \left(\sum_{s=1}^{j-1} \alpha_s\, \mathcal{L}^{(s)} \prod_{\substack{i=1\\ i\neq s}}^{j-1} \mathcal{M}^{(i)} \right)$$

we have

$$\mathrm{x}^{(j-1)} = M_{\mathrm{x}^{(j-1)}} \cdot \big(\mu_{\mathrm{x}^{(j-1)}}\, \mathbf{0}_{i^*} - \lambda_{\mathrm{x}^{(j-1)}}\, \infty_{i^*} \big).$$

We distinguish two cases: *Case 1:* $\mathcal{C}_j$ plays the role of $\Big((\infty',\ \mathbf{1}',\ \mathbf{0}'),\ \ (\mathbf{0}',\ \mathrm{x}',\ \infty') \Big)$ in the addition when using the notions of Section 17.2. Using (17.8) and (17.6) we obtain

$$\begin{aligned}
\mathrm{x}^{(j)} = {} & A_j \cdot M_{\mathrm{x}^{(j-1)}} \cdot M_{\mathrm{x}_j} \cdot \Big(\lambda_{\mathbf{1}_{i^*}} \mu_{\mathbf{1}_j} \mu_{\mathrm{x}_j} \mu_{\mathrm{x}^{(j-1)}}\, \mathbf{0}_{i^*} \\
& \qquad - (\lambda_{\mathbf{1}_j} \lambda_{\mathrm{x}_j} \mu_{\mathbf{1}_{i^*}} \mu_{\mathrm{x}^{(j-1)}} + \lambda_{\mathbf{1}_{i^*}} \lambda_{\mathrm{x}^{(j-1)}} \mu_{\mathbf{1}_j} \mu_{\mathrm{x}_j})\, \infty_{i^*} \Big) \\
= {} & A_j \cdot M_{\mathrm{x}^{(j-1)}} \cdot M_{\mathrm{x}_j} \cdot \\
& \Bigg(\lambda_{\mathbf{1}_{i^*}} \quad \mu_{\mathbf{1}_j} \quad \mu_{\mathrm{x}_j} \quad \mu_{\mathrm{x}_{i^*}} \lambda_{\mathbf{1}_{i^*}} \prod_{\substack{i=1\\ i\neq i^*}}^{j-1} \mathcal{M}^{(i)} \quad \mathbf{0}_{i^*} \\
& \quad -\lambda_{\mathbf{1}_j} \quad \lambda_{\mathrm{x}_j} \quad \mu_{\mathbf{1}_{i^*}} \quad \mu_{\mathrm{x}_{i^*}} \lambda_{\mathbf{1}_{i^*}} \prod_{\substack{i=1\\ i\neq i^*}}^{j-1} \mathcal{M}^{(i)} \quad \infty_{i^*} \\
& \quad -\lambda_{\mathbf{1}_{i^*}}\, N_{i^*} \left(\sum_{s=1}^{j-1} \alpha_s\, \mathcal{L}^{(s)} \prod_{\substack{i=1\\ i\neq s}}^{j-1} \mathcal{M}^{(i)} \right) \mu_{\mathbf{1}_j} \quad \mu_{\mathrm{x}_j} \quad \infty_{i^*} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
= A_j \cdot \quad & Q_{j-1} \cdot \prod_{\substack{i=1\\ i\neq i^*}}^{j-1} N_i \prod_{i=1}^{j-1} M_{\mathbf{x}_i} \prod_{i=2}^{j-1} A_i \quad \cdot M_{\mathbf{x}_j} \cdot \lambda_{\mathbf{1}_{i^*}} \cdot \\
& \Bigg(\quad N_j\,\mathcal{M}^{(j)} \qquad \mu_{\mathbf{x}_{i^*}} \lambda_{\mathbf{1}_{i^*}} \prod_{\substack{i=1\\ i\neq i^*}}^{j-1} \mathcal{M}^{(i)} \quad \mathbf{0}_{i^*} \\
& \quad -N_j\,\alpha_j\,\mathcal{L}^{(j)} \qquad N_{i^*}\,\mathcal{M}^{(i^*)} \prod_{\substack{i=1\\ i\neq i^*}}^{j-1} \mathcal{M}^{(i)} \quad \infty_{i^*} \\
& \quad -N_{i^*} \left(\sum_{s=1}^{j-1} \alpha_s\,\mathcal{L}^{(s)} \prod_{\substack{i=1\\ i\neq s}}^{j-1} \mathcal{M}^{(s)} \right) N_j \quad \mathcal{M}^{(j)} \quad \infty_{i^*} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= Q_j \cdot & \prod_{\substack{i=1\\ i\neq i^*}}^{j} N_i \prod_{i=1}^{j} M_{\mathbf{x}_i} \prod_{i=2}^{j} A_i \quad \cdot \\
& \Bigg(\quad \mu_{\mathbf{x}_{i^*}} \lambda_{\mathbf{1}_{i^*}} \qquad \prod_{\substack{i=1\\ i\neq i^*}}^{j} \mathcal{M}^{(i)} \quad \mathbf{0}_{i^*} \\
& \quad - N_{i^*}\,\alpha_j\,\mathcal{L}^{(j)} \qquad \prod_{i=1}^{j-1} \mathcal{M}^{(i)} \quad \infty_{i^*} \\
& \quad - N_{i^*} \left(\sum_{s=1}^{j-1} \alpha_s\,\mathcal{L}^{(s)} \prod_{\substack{i=1\\ i\neq s}}^{j} \mathcal{M}^{(i)} \right) \infty_{i^*} \Bigg)
\end{aligned}
$$

with $Q_j := Q_{j-1} \cdot \lambda_{\mathbf{1}_{i^*}}$ which by induction has degree $j-3+1 = j-2$. This proves the claim in the first case. In *Case 2*, where $\mathcal{C}_j$ plays the role of $\big((\infty,\, \mathbf{1},\, \mathbf{0}),\ (\mathbf{0},\, \mathbf{x},\, \infty)\big)$, one analogously computes

$$A_j \cdot M_{\mathbf{x}_j} \cdot M_{\mathbf{x}^{(j-1)}} \big(\lambda_{\mathbf{1}_j} \mu_{\mathbf{1}_{i^*}} \mu_{\mathbf{x}^{(j-1)}} \mu_{\mathbf{x}_j}\, \mathbf{0}_j - (\lambda_{\mathbf{1}_{i^*}} \lambda_{\mathbf{x}^{(j-1)}} \mu_{\mathbf{1}_j} \mu_{\mathbf{x}_j} + \lambda_{\mathbf{1}_j} \lambda_{\mathbf{x}_j} \mu_{\mathbf{1}_{i^*}} \mu_{\mathbf{x}^{(j-1)}})\, \infty_j \big).$$

This results in

$$Q_j \cdot \prod_{\substack{i=1\\ i\neq j}}^{j} N_i \prod_{i=1}^{j} M_{\mathbf{x}_i} \prod_{i=2}^{j} A_i \quad \cdot$$
$$\left(\mu_{\mathbf{x}_j} \lambda_{\mathbf{1}_j} \prod_{\substack{i=1\\ i\neq j}}^{j} \mathcal{M}^{(i)}\, \mathbf{0}_j \quad - \quad \left(\sum_{s=1}^{j} \alpha_s\, \mathcal{L}^{(s)} \prod_{\substack{i=1\\ i\neq s}}^{j} \mathcal{M}^{(i)} \right) N_j\, \infty_j \right)$$

with $Q_j := Q_{j-1} \cdot \lambda_{\mathbf{1}_{i^*}}$ which is the desired statement in this case. □

Remark 17.3. It is also possible to use other patterns for combining the 2-cycles (e.g. tree-like structures). One can show that Lemma 17.2 can be adjusted in its formulation and remains still valid. This needs more of bookkeeping of the indices and the multiplier monomial has higher degree. However, in a concrete implementation of the algorithm there might be a potential for optimizing the degree of the multiplier monomial.

17.4. Final Coincidence

Assume that all 2-cycles $\mathcal{C}_1, \ldots, \mathcal{C}_l$ have been combined in a single 2-cycle given by

$$\big((\infty_{i^*}, \mathbf{1}_{i^*}, \mathbf{0}_{i^*}), \quad (\mathbf{0}_{i^*}, \mathbf{x}^{(l)}, \infty_{i^*}) \big).$$

As already declared in (17.1), in an instance testing $B = 0$ can be read as

$$\alpha_1 \frac{\Pi_1}{\Omega} + \cdots + \alpha_l \frac{\Pi_l}{\Omega} = \alpha_\Omega$$

in the corresponding coordination. This equation holds in an instance if and only if the coordinatization of

$$\mathbf{n}_{i^*}(\alpha_\Omega) \quad \text{and} \quad \mathbf{x}^{(l)}$$

are identical points in the projective plane in an instance. We will define $\mathbf{c}$ such that

$$\mathbf{n}_{i^*}(\alpha_\Omega) = \mathbf{c}\, \mathbf{b}_{i^*} \ \wedge\ \mathbf{0}_{i^*}\ \infty_{i^*}$$

by letting

$$\mathbf{c} := \begin{cases} \mathbf{a}_{i^*} & \text{if } \alpha_\Omega = 1 \\ \left(\mathbf{a}_{i^*}\ \mathbf{0}_{i^*} \ \wedge\ \mathbf{b}_{i^*}\ \infty_{i^*}\right)\ \mathbf{n}_{i^*}(\alpha_\Omega - 1)\ \wedge\ \mathbf{a}_{i^*}\ \infty_{i^*} & \text{otherwise.} \end{cases}$$

We have

$$\mathbf{c} = M_{\mathbf{c}} \left((\alpha_\Omega - 1)\mu_{\mathbf{1}_{i^*}} \infty_{i^*} - \lambda_{\mathbf{c}} \, \mathbf{a}_{i^*} \right)$$

with

$$\begin{aligned} \lambda_{\mathbf{c}} &= \begin{cases} -[\mathbf{0}_{i^*}, \infty_{i^*}, \mathbf{b}_{i^*}] & \text{for } \alpha_\Omega \geq 2 \\ 1 & \text{otherwise} \end{cases} \\ M_{\mathbf{c}} &= \begin{cases} (\lambda_{\mathbf{1}_{i^*}} [\mathbf{0}_{i^*}, \infty_{i^*}, \mathbf{a}_{i^*}])^{\alpha_\Omega - 1} [\mathbf{0}_{i^*}, \infty_{i^*}, \mathbf{b}_{i^*}]^{\alpha_\Omega - 2} & \text{for } \alpha_\Omega \geq 2 \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{17.10}$$

For an evaluation of $\mathbf{c}$ observe that the construction is contained in the construction of $\mathbf{n}_{i^*}(\alpha_\Omega)$. Splitting the edges in the tensor diagram representation in the same way as before in the bigger diagram (see Section 17.1) gives the desired representation. With this setting it holds due to Lemma 17.2 and due to the above identities that

$$\begin{aligned} &\mathbf{c}\,\mathbf{b}_{i^*} \wedge \mathbf{x}^{(l)} \\ =& M_{\mathbf{c}} \cdot M_{\mathbf{x}^{(l)}} \cdot \left((\alpha_\Omega - 1)\, \mu_{\mathbf{1}_{i^*}} \, \infty_{i^*} \mathbf{b}_{i^*} - \lambda_{\mathbf{c}} \mathbf{a}_{i^*} \, \mathbf{b}_{i^*} \right) \\ &\qquad \wedge \left(\mu_{\mathbf{x}_{i^*}} \lambda_{\mathbf{1}_{i^*}} \prod_{\substack{i=1 \\ i \neq i^*}}^{l} \mathcal{M}^{(i)} \, \mathbf{0}_{i^*} \; - \; \left(\sum_{s=1}^{l} \alpha_s \, \mathcal{L}^{(s)} \prod_{\substack{i=1 \\ i \neq s}}^{l} \mathcal{M}^{(i)} \right) N_{i^*} \, \infty_{i^*} \right) \\ =& M_{\mathbf{c}} \cdot M_{\mathbf{x}^{(l)}} \cdot \left((\alpha_\Omega - 1) \underbrace{\mu_{\mathbf{1}_{i^*}} \, \mu_{\mathbf{x}_{i^*}}}_{= N_{i^*} \cdot \mathcal{M}^{(i^*)} \text{ by } (17.6)} \lambda_{\mathbf{1}_{i^*}} \prod_{\substack{i=1 \\ i \neq i^*}}^{l} \mathcal{M}^{(i)} \; [\infty_{i^*}, \mathbf{b}_{i^*}, \mathbf{0}_{i^*}] \right. \\ &\qquad - \lambda_{\mathbf{c}} \, \lambda_{\mathbf{1}_{i^*}} \prod_{\substack{i=1 \\ i \neq i^*}}^{l} \mathcal{M}^{(i)} \underbrace{\mu_{\mathbf{x}_{i^*}} [\mathbf{a}_{i^*}, \mathbf{b}_{i^*}, \mathbf{0}_{i^*}]}_{= N_{i^*} \cdot \mathcal{M}^{(i^*)} \text{ by } (17.6)} \\ &\qquad \left. + \lambda_{\mathbf{c}} \left(\sum_{s=1}^{l} \alpha_s \, \mathcal{L}^{(s)} \prod_{\substack{i=1 \\ i \neq s}}^{l} \mathcal{M}^{(i)} \right) N_{i^*} \underbrace{[\mathbf{a}_{i^*}, \mathbf{b}_{i^*}, \infty_{i^*}]}_{= \lambda_{\mathbf{1}_{i^*}}} \right) \end{aligned}$$

$$
\begin{aligned}
=&M_{\mathbf{c}} \cdot M_{\mathbf{x}^{(l)}} \cdot N_{i^*} \cdot \lambda_{\mathbf{1}_{i^*}} \cdot \Bigg((\alpha_\Omega - 1) \prod_{i=1}^{l} \mathcal{M}^{(i)} \quad \underbrace{[\infty_{i^*}, \mathbf{b}_{i^*}, \mathbf{0}_{i^*}]}_{=-\lambda_{\mathbf{c}} \text{ if } \alpha_\Omega \geq 2} \\
&\qquad - \lambda_{\mathbf{c}} \prod_{i=1}^{l} \mathcal{M}^{(i)} \\
&\qquad + \lambda_{\mathbf{c}} \Bigg(\sum_{s=1}^{l} \alpha_s \, \mathcal{L}^{(s)} \prod_{\substack{i=1 \\ i \neq s}}^{l} \mathcal{M}^{(i)} \Bigg) \Bigg) \\
=&M_{\mathbf{c}} \cdot M_{\mathbf{x}^{(l)}} \cdot N_{i^*} \cdot \lambda_{\mathbf{1}_{i^*}} \cdot \lambda_{\mathbf{c}} \cdot \Bigg(-\alpha_\Omega \prod_{i=1}^{l} \mathcal{M}^{(i)} + \sum_{s=1}^{l} \alpha_s \, \mathcal{L}^{(s)} \prod_{\substack{i=1 \\ i \neq s}}^{l} \mathcal{M}^{(i)} \Bigg).
\end{aligned}
$$

This bracket polynomial can be further factored. In order to see this, for $1 \leq s \leq l$ consider

$$
\prod_{\substack{i=1 \\ i \neq s}}^{l} \mathcal{M}^{(i)}.
$$

Observe, that $\mathcal{F}^{(s)}$ (see (17.3)) is a bracket monomial factor of the above. If it was not, there would be a overall bracket monomial factor of B due to the definition of $\mathcal{F}^{(s)}$ as maximal common factor of $\Pi_s - \Omega$. But a bracket monomial factor of B was excluded in the beginning. On the other hand, we have $\mathcal{F}^{(s)} = \frac{\Omega}{\mathcal{M}^{(s)}}$ and therefore

$$
\frac{\prod_{i=1,\, i\neq s}^{l} \mathcal{M}^{(i)}}{\mathcal{F}^{(s)}} = \frac{\prod_{i=1}^{l} \mathcal{M}^{(i)}}{\Omega}
$$

which is independent of the particular s chosen. Therefore we can rewrite

$$
\begin{aligned}
\mathbf{c}\, \mathbf{b}_{i^*} \wedge \mathbf{x}^{(l)} = M_{\mathbf{c}} \cdot M_{\mathbf{x}^{(l)}} \cdot N_{i^*} \cdot \lambda_{\mathbf{1}_{i^*}} \cdot \lambda_{\mathbf{c}} \cdot \frac{\prod_{i=1}^{l} \mathcal{M}^{(i)}}{\Omega} \cdot& \\
\Bigg(\sum_{s=1}^{l} \alpha_s \underbrace{\mathcal{L}^{(s)} \mathcal{F}^{(s)}}_{=\Pi_s} - \alpha_\Omega \, \Omega \Bigg)&
\end{aligned}
\tag{17.11}
$$

and therefore we obtained a generalized Cayley factorization of B.

17.5. The Multiplier of the Factorization

The overall multiplier is due (17.11) and due to Lemma 17.2 given by

$$\begin{aligned} M :=& M_{\mathbf{c}} \cdot M_{\mathbf{x}^{(l)}} \cdot N_{i^*} \cdot \lambda_{\mathbf{1}_{i^*}} \cdot \lambda_{\mathbf{c}} \cdot \frac{\prod_{i=i^*}^{l} \mathcal{M}^{(i)}}{\Omega} \\ =& M_{\mathbf{c}} \cdot Q_l \cdot \prod_{i=1}^{l} (N_i \cdot M_{\mathbf{x}_i}) \prod_{i=2}^{l} A_i . \cdot \lambda_{\mathbf{1}_{i^*}} \cdot \lambda_{\mathbf{c}} \cdot \frac{\prod_{i=1}^{l} \mathcal{M}^{(i)}}{\Omega}. \end{aligned} \tag{17.12}$$

We want to give upper bounds on the degree of M. It remains to plug in the estimates given in (17.7). We distinguish two cases:

Common Special Case
Assume

$$\alpha_1 = \cdots = \alpha_l = \alpha_\Omega = 1.$$

Observe that by switching elements inside of the brackets, this case covers also $\alpha_1, \ldots, \alpha_l, \alpha_\Omega \in \{-1, +1\}$. In this case, M simplifies to

$$M = Q_j \cdot \prod_{i=1}^{l} (N_i \cdot M_{\mathbf{x}_i}) \prod_{i=2}^{l} A_i . \cdot \lambda_{\mathbf{1}_{i^*}} \cdot \frac{\prod_{i=1}^{l} \mathcal{M}^{(i)}}{\Omega}.$$

Due to (17.7) and due to the degree of Q_j given in Lemma 17.2 we have

$$\begin{aligned} \deg(M) \leq& (l-2) \;+\; 5 \sum_{i=1}^{l} k_i \;+\; \sum_{i=2}^{l} \deg(A_i) \;+\; 1 \;+\; \sum_{i=1}^{l} k_i \;-\; k \\ \leq& (l-2) \;+\; 5\, l\, k \;+\; (l-1)\, 5 \;+\; 1 + \; l\, k \;-\; k \\ =& 6\, k\, l + 6\, l - k - 6 \\ \leq& 9\, k\, l \end{aligned}$$

where the estimate on the A_is is due to (17.9) and we use $k \geq 2$.

General Case

In the general case, we can use the representations of $M_{\mathbf{c}}$ and $\lambda_{\mathbf{c}}$ given in (17.10) to bound the overall degree of M as given in (17.12). We get

$$\begin{aligned}
&\deg(M) \leq \\
&\quad \leq 3\,\alpha_\Omega - 4 \;+\; (l-2) \;+\; \sum_{i=1}^{l}(5\,k_i + 3\,\alpha_i + 1) \;+\; 5\,(l-1) \;+\; 2 \;+\; \sum_{i=1}^{l} k_i \;-\; k \\
&\quad \leq 6 \sum_{i=1}^{l} k_i \;+\; 3 \left(\sum_{i=1}^{l} \alpha_i + \alpha_\Omega \right) \;+\; 7\,l \;-\; k \;-\; 9 \\
&\quad \leq 10\,k\,l + 3 \left(\sum_{i=1}^{l} \alpha_i + \alpha_\Omega \right)
\end{aligned}$$

where we again used $k \geq 2$. Also observe that after using the subroutine for the binomial case l times, the algorithm is linear in l. The subroutine runs in $O(k^3)$. Therefore, the overall running time is in $O(l \cdot k^3)$. This proves Theorem 14.5.

18. Examples

In this Chapter we give two examples illustrating the execution of the Cayley factorization algorithm. The output was obtained automatically by an implementation in *Mathematica* ([136]) and using the *Combinatorica* package for finding a bipartite matching and for producing graph representations. The algorithm works exactly as described earlier. Within this description there is some freedom of choice. Within the algorithm, some of this freedom is used in order to construct an optimized matching. Furthermore when factoring binomials, the resulting cycles are triangulated in such a way that the costs of combining the resulting 2-cycles are optimized.

We will give two examples. Both of them have their coefficients in $\{-1, +1\}$ since this is often the case when describing geometric properties. The first example is an extension of the introductive example given in Chapter 15. The second one is given in Section 18.2 and treats the condition for ten points lying on a common cubic. Unfortunately, we do not have enough space to show the complete output.

18.1. Example 1

$$\begin{aligned} B := \ & [\mathbf{a},\mathbf{b},\mathbf{g}][\mathbf{a},\mathbf{c},\mathbf{k}][\mathbf{b},\mathbf{a},\mathbf{c}][\mathbf{e},\mathbf{f},\mathbf{h}][\mathbf{f},\mathbf{d},\mathbf{e}] \\ & +[\mathbf{a},\mathbf{b},\mathbf{g}][\mathbf{a},\mathbf{c},\mathbf{e}][\mathbf{a},\mathbf{f},\mathbf{e}][\mathbf{b},\mathbf{c},\mathbf{k}][\mathbf{h},\mathbf{d},\mathbf{f}] \\ & -[\mathbf{a},\mathbf{b},\mathbf{f}][\mathbf{a},\mathbf{d},\mathbf{f}][\mathbf{a},\mathbf{e},\mathbf{g}][\mathbf{c},\mathbf{b},\mathbf{h}][\mathbf{k},\mathbf{c},\mathbf{e}] \end{aligned}$$

Formatting and pre-processing input:

The bracket polynomial is processed such that it is composed of summands with all but the last coefficient equal to $+1$. The last coefficient is -1. Cayley factorization yields a $\vee$-$\wedge$-sequence to interpret it. Intuition is provided by interpreting the vanishing of the bracket polynomial:

$$B = 0$$

$$\Longleftrightarrow$$

$$\frac{[a,c,k][b,a,c][f,d,e][a,b,g][e,f,h]}{[a,e,g][a,b,f][c,b,h][k,c,e][a,d,f]} + \frac{[a,f,e][h,d,f][b,c,k][a,c,e][a,b,g]}{[a,e,g][a,b,f][c,b,h][k,c,e][a,d,f]} = 1$$

Start of the Main Part of the Algorithm:

Factor all induced fractional equations $\frac{[*,*,*]...[*,*,*]}{[*,*,*]...[*,*,*]} = 1$

——Factorize the fractional equation $\frac{[a,f,e][h,d,f][b,c,k][a,c,e][a,b,g]}{[a,e,g][a,b,f][c,b,h][k,c,e][a,d,f]} = 1$

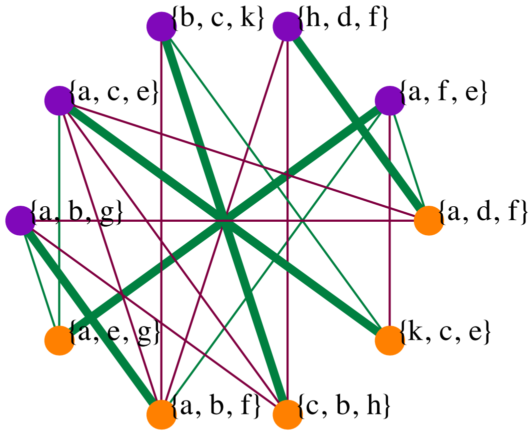
Brackets occurring in the numerator as well as in the denominator can be canceled. The common elements are: $\emptyset$

This reduces the bracket identity in question to:

$$\frac{[a,f,e][h,d,f][b,c,k][a,c,e][a,b,g]}{[a,e,g][a,b,f][c,b,h][k,c,e][a,d,f]} = 1$$

Find a Matching in order to apply the area principle:

Split the big fraction into fractions of the form $\frac{[*,*,*]...[*,*,*]}{[*,*,*]...[*,*,*]}$. This matches each bracket of the numerator with a bracket of the denominator. The aim will be to apply the area principle to each factor $\frac{[*,*,*]...[*,*,*]}{[*,*,*]...[*,*,*]}$. This is possible as soon as both brackets differ by exactly one element. In the graph below, we have one vertex per bracket, but brackets are replaced by corresponding sets of indices. The situations where the area principle can be applied directly are indicated by green edges. The red edges indicate situations where the corresponding fraction can by multiplied by an additional factor $\frac{[*,*,*]...[*,*,*]}{[*,*,*]...[*,*,*]} = 1$ such that the area principle can be applied to the product. E.g. $\frac{[a,b,c]}{[c,d,e]} \cdot \frac{[a,c,d]}{[a,c,d]}$ can be rearranged to $\frac{[a,b,c]}{[a,c,d]} \cdot \frac{[a,c,d]}{[c,d,e]}$. Due to Hall's Marriage Theorem, there exists a perfect matching in the graph below. The green edges are given weights 2 and the red edges weight 1 and a perfect maximum matching with maximal total weight is highlighted by the **thick** edges.



Apply the area principle to fractions corresponding to the matching just found:

Below, reddish letters indicate the end points of the length ratio. The complete formula indicates the intermediate point of the length ratio. E.g. **ab** $\wedge$ **cd**) indicates a length ratio $\frac{\overline{c\,x}}{\overline{x\,d}}$ in the corresponding instance. x is the point corresponding to **ab** $\wedge$ **cd**) in the instance.

$\frac{[\mathbf{a,f,e}]}{[\mathbf{a,e,g}]}$ $\frac{[\mathbf{h,d,f}]}{[\mathbf{a,d,f}]}$ $\frac{[\mathbf{b,c,k}]}{[\mathbf{c,b,h}]}$ $\frac{[\mathbf{a,c,e}]}{[\mathbf{k,c,e}]}$ $\frac{[\mathbf{a,b,g}]}{[\mathbf{a,b,f}]}$

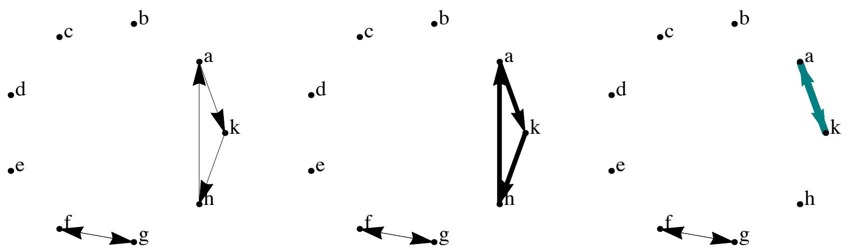
$\rightarrow (\mathbf{ae} \wedge \mathbf{fg})$ $\rightarrow (\mathbf{df} \wedge \mathbf{ha})$ $\rightarrow (\mathbf{bc} \wedge \mathbf{kh})$ $\rightarrow (\mathbf{ce} \wedge \mathbf{ak})$ $\rightarrow (\mathbf{ab} \wedge \mathbf{gf})$

Product of sign-errors occurring when applying the area principle: $\varepsilon = 1$

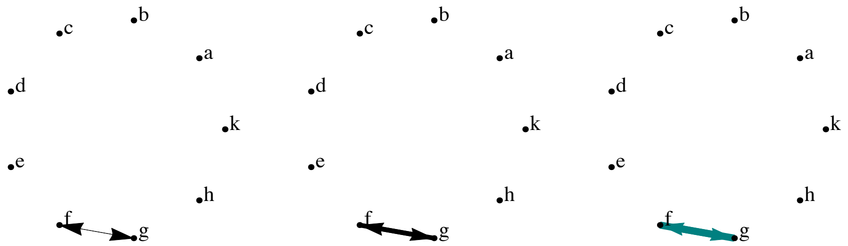
The length ratios induced by applying the area principle to the edges of the matching arrange in cycles. In the following, the corresponding formulas in Grassmann-Cayley algebra are called edges or length ratios (which overloads the notation). The constructions of Ceva and Menelaus are used to combine two edges that share exactly one endpoint into a single edge. Therefore, any cycle of the length ratios can be reduced to a 2-cycle by triangulating the cycle. The combinatorics of the length ratios is depicted in the first diagram appearing below. Also in general the first diagram in the row shows the combinatorics of the length ratios left over. The second diagram shows the triangulation used, and if applicable the edge in which a sign-error-correction ε was included is highlighted in blue. In the last diagram the 2-cycle resulting from the triangulation is highlighted.

Find and triangulate Cycles

Length-ratios left over to be processed: $(\mathbf{ae} \wedge \mathbf{fg})$ $(\mathbf{df} \wedge \mathbf{ha})$ $(\mathbf{bc} \wedge \mathbf{kh})$ $(\mathbf{ce} \wedge \mathbf{ak})$ $(\mathbf{ab} \wedge \mathbf{gf})$
Cycle found:$(\mathbf{ce} \wedge \mathbf{ak})$ $(\mathbf{bc} \wedge \mathbf{kh})$ $(\mathbf{df} \wedge \mathbf{ha})$
Intermediate steps in the triangulation with Menelaus triangles:
$((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka})$



Length-ratios left over to be processed: $(\mathbf{ae} \wedge \mathbf{fg})$ $(\mathbf{ab} \wedge \mathbf{gf})$
Cycle found:$(\mathbf{ae} \wedge \mathbf{fg})$ $(\mathbf{ab} \wedge \mathbf{gf})$

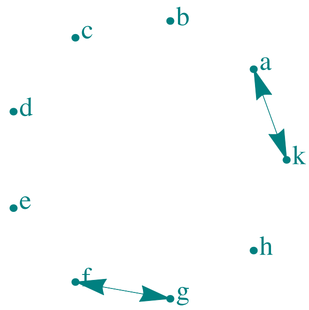


Combination of 2-cycles resulting from triangulation

Two 2-cycles can be reduced to a single 2-cycle whose endpoint are the same as the ones of the starting 2-cycles. In the following, if a 2-cycle is drawn thickly, then another 2-cycle has been reduced to it in the step before. This other 2-cycle has vanished from the drawing in this step before. The case distinction is emphasized by giving the values of $\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}$ (and also the value of $\{\mathbf{a}', \mathbf{b}', \mathbf{0}\}$ if appropriate) in the corresponding situation (with the specified order of elements within the sets). The first complete first formula given the 2-cycles corresponds to the value of $\mathbf{1}$, the second one to $\mathbf{x}$.

Current 2-cycles:

$(\mathbf{ce} \wedge \mathbf{ak})$	$((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka})$
$(\mathbf{ae} \wedge \mathbf{fg})$	$(\mathbf{ab} \wedge \mathbf{gf})$



The following 2-cycles are to be combined:

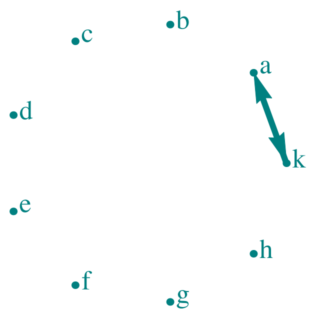
$(\mathbf{ce} \wedge \mathbf{ak}) \qquad ((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka})$
$(\mathbf{ae} \wedge \mathbf{fg}) \qquad (\mathbf{ab} \wedge \mathbf{gf})$

We have $|\{\mathbf{k}, \mathbf{a}, \mathbf{g}, \mathbf{f}\}| = 4$ and $|\{\mathbf{a}, \mathbf{e}, \mathbf{k}\}| = 3$ which indicates Case 4 (i) and results in:

$(\mathbf{ce} \wedge \mathbf{ak}) \qquad (((\mathbf{ae} \wedge \mathbf{kf})((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka}) \wedge \mathbf{af})((\mathbf{ab} \wedge \mathbf{gf})(\mathbf{ae} \wedge \mathbf{kg}) \wedge \mathbf{kf}) \wedge \mathbf{ka})$

Current 2-cycles:

$(\mathbf{ce} \wedge \mathbf{ak}) \qquad (((\mathbf{ae} \wedge \mathbf{kf})((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka}) \wedge \mathbf{af})((\mathbf{ab} \wedge \mathbf{gf})(\mathbf{ae} \wedge \mathbf{kg}) \wedge \mathbf{kf}) \wedge \mathbf{ka})$

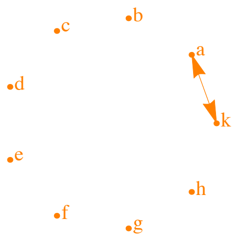


$\frac{[\mathbf{a,f,e}][\mathbf{h,d,f}][\mathbf{b,c,k}][\mathbf{a,c,e}][\mathbf{a,b,g}]}{[\mathbf{a,e,g}][\mathbf{a,b,f}][\mathbf{c,b,h}][\mathbf{k,c,e}][\mathbf{a,d,f}]} = 1$ **whenever the both intermediate points are identical:**

$(\mathbf{ce} \wedge \mathbf{ak})$ $\quad$ $(((\mathbf{ae} \wedge \mathbf{kf})((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka}) \wedge \mathbf{af})((\mathbf{ab} \wedge \mathbf{gf})(\mathbf{ae} \wedge \mathbf{kg}) \wedge \mathbf{kf}) \wedge \mathbf{ka})$

In order to factor the complete bracket polynomial, one has to projectively add

$(\mathbf{ce} \wedge \mathbf{ak})$	$(((\mathbf{ae} \wedge \mathbf{kf})((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka}) \wedge \mathbf{af})((\mathbf{ab} \wedge \mathbf{gf})(\mathbf{ae} \wedge \mathbf{kg}) \wedge \mathbf{kf}) \wedge \mathbf{ka})$



AND the cross-ratios resulting from factorization of the remaining summands rewritten as

$$\frac{[\mathbf{a,c,k}][\mathbf{b,a,c}][\mathbf{f,d,e}][\mathbf{a,b,g}][\mathbf{e,f,h}]}{[\mathbf{a,e,g}][\mathbf{a,b,f}][\mathbf{c,b,h}][\mathbf{k,c,e}][\mathbf{a,d,f}]} = 1$$

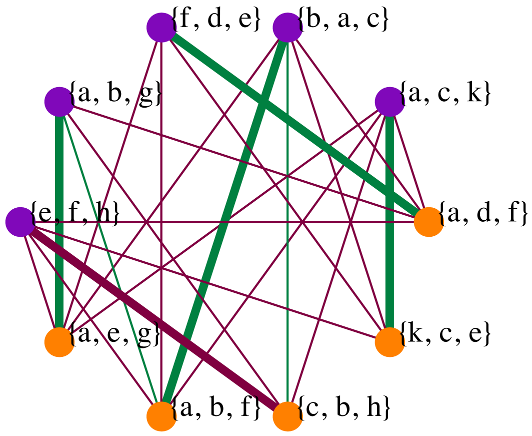
——Factorize the fractional equation $\frac{[\mathbf{a,c,k}][\mathbf{b,a,c}][\mathbf{f,d,e}][\mathbf{a,b,g}][\mathbf{e,f,h}]}{[\mathbf{a,e,g}][\mathbf{a,b,f}][\mathbf{c,b,h}][\mathbf{k,c,e}][\mathbf{a,d,f}]} = 1$

Brackets occurring in the numerator as well as in the denominator can be canceled. The common elements are: $\emptyset$
This reduces the bracket identity in question to:

$$\frac{[\mathbf{a},\mathbf{c},\mathbf{k}][\mathbf{b},\mathbf{a},\mathbf{c}][\mathbf{f},\mathbf{d},\mathbf{e}][\mathbf{a},\mathbf{b},\mathbf{g}][\mathbf{e},\mathbf{f},\mathbf{h}]}{[\mathbf{a},\mathbf{e},\mathbf{g}][\mathbf{a},\mathbf{b},\mathbf{f}][\mathbf{c},\mathbf{b},\mathbf{h}][\mathbf{k},\mathbf{c},\mathbf{e}][\mathbf{a},\mathbf{d},\mathbf{f}]} = 1$$

Find a Matching in order to apply the area principle:

Split the big fraction into fractions of the form $\frac{[*,*,*]\ldots[*,*,*]}{[*,*,*]\ldots[*,*,*]}$. This matches each bracket of the numerator with a bracket of the denominator. The aim will be to apply the area principle to each factor $\frac{[*,*,*]\ldots[*,*,*]}{[*,*,*]\ldots[*,*,*]}$. This is possible as soon as both brackets differ by exactly one element. In the graph below, we have one vertex per bracket, but brackets are replaced by corresponding sets of indices. The situations where the area principle can be applied directly are indicated by green edges. The red edges indicate situations where the corresponding fraction can by multiplied by an additional factor $\frac{[*,*,*]...[*,*,*]}{[*,*,*]...[*,*,*]} = 1$ such that the area principle can be applied to the product. E.g. $\frac{[\mathbf{a,b,c}]}{[\mathbf{c,d,e}]} \cdot \frac{[\mathbf{a,c,d}]}{[\mathbf{a,c,d}]}$ can be rearranged to $\frac{[\mathbf{a,b,c}]}{[\mathbf{a,c,d}]} \cdot \frac{[\mathbf{a,c,d}]}{[\mathbf{c,d,e}]}$. Due to Hall's Marriage Theorem, there exists a perfect matching in the graph below. The green edges are given weights 2 and the red edges weight 1 and a perfect maximum matching with maximal total weight is highlighted by the **thick** edges.



Apply the area principle to fractions corresponding to the matching just found:

Below, reddish letters indicate the end points of the length ratio. The complete formula indicates the intermediate point of the length ratio. E.g. **ab** $\wedge$ **cd**) indicates a length ratio $\frac{\overline{c\,x}}{\overline{x\,d}}$ in the corresponding instance. x is the point corresponding to **ab** $\wedge$ **cd**) in the instance.

$\frac{[\mathbf{a,c,k}]}{[\mathbf{k,c,e}]}$ $\rightarrow$ (**ck** $\wedge$ **ae**)

$\frac{[\mathbf{b,a,c}]}{[\mathbf{a,b,f}]}$ $\rightarrow$ (**ba** $\wedge$ **cf**)

$\frac{[\mathbf{f,d,e}]}{[\mathbf{a,d,f}]}$ $\rightarrow$ (**fd** $\wedge$ **ea**)

$\frac{[\mathbf{a,b,g}]}{[\mathbf{a,e,g}]}$ $\rightarrow$ (**ag** $\wedge$ **be**)

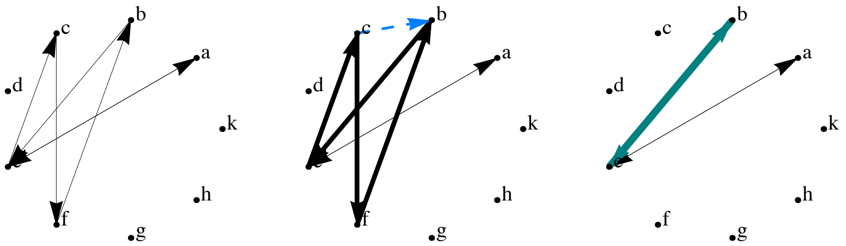
$\frac{[\mathbf{e,f,h}]}{[\mathbf{e,h,b}]} \cdot \frac{[\mathbf{e,h,b}]}{[\mathbf{c,b,h}]}$ $\rightarrow$ (**eh** $\wedge$ **fb**), (**hb** $\wedge$ **ec**)

Product of sign-errors occurring when applying the area principle: $\varepsilon = -1$

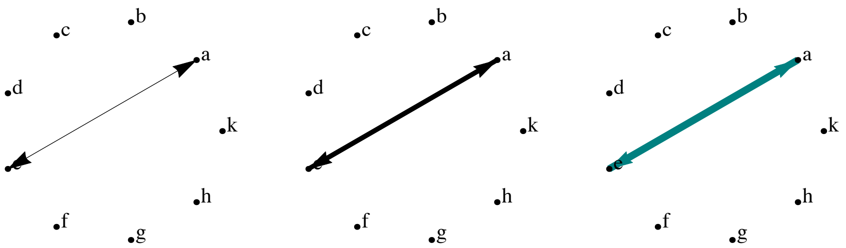
The length ratios induced by applying the area principle to the edges of the matching arrange in cycles. In the following, the corresponding formulas in Grassmann-Cayley algebra are called edges or length ratios (which overloads the notation). The constructions of Ceva and Menelaus are used to combine two edges that share exactly one endpoint into a single edge. Therefore, any cycle of the length ratios can be reduced to a 2-cycle by triangulating the cycle. The combinatorics of the length ratios is depicted in the first diagram appearing below. Also in general the first diagram in the row shows the combinatorics of the length ratios left over. The second diagram shows the triangulation used, and if applicable the edge in which a sign-error-correction ε was included is highlighted in blue. In the last diagram the 2-cycle resulting from the triangulation is highlighted.

Find and triangulate Cycles

Length-ratios left over to be processed: $(\mathbf{ck} \wedge \mathbf{ae})$ $(\mathbf{ba} \wedge \mathbf{cf})$ $(\mathbf{fd} \wedge \mathbf{ea})$ $(\mathbf{ag} \wedge \mathbf{be})$ $(\mathbf{eh} \wedge \mathbf{fb})$ $(\mathbf{hb} \wedge \mathbf{ec})$
Cycle found:$(\mathbf{ag} \wedge \mathbf{be})$ $(\mathbf{hb} \wedge \mathbf{ec})$ $(\mathbf{ba} \wedge \mathbf{cf})$ $(\mathbf{eh} \wedge \mathbf{fb})$
Due to $\varepsilon = -1$ a Ceva configuration is used once in the triangulation. The blue edge reads as $(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb})$
Intermediate steps in the triangulation with Menelaus triangles:
$((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb})$



Length-ratios left over to be processed: $(\mathbf{ck} \wedge \mathbf{ae})$ $(\mathbf{fd} \wedge \mathbf{ea})$
Cycle found:$(\mathbf{ck} \wedge \mathbf{ae})$ $(\mathbf{fd} \wedge \mathbf{ea})$

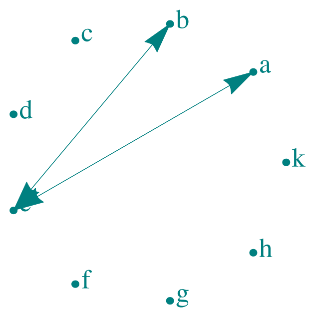


Combination of 2-cycles resulting from triangulation

Two 2-cycles can be reduced to a single 2-cycle whose endpoint are the same as the ones of the starting 2-cycles. In the following, if a 2-cycle is drawn thickly, then another 2-cycle has been reduced to it in the step before. This other 2-cycle has vanished from the drawing in this step before. The case distinction is emphasized by giving the values of $\{\mathbf{0}, \infty, \mathbf{0}', \infty'\}$ (and also the value of $\{\mathbf{a}', \mathbf{b}', \mathbf{0}\}$ if appropriate) in the corresponding situation (with the specified order of elements within the sets). The first complete first formula given the 2-cycles corresponds to the value of $\mathbf{1}$, the second one to $\mathbf{x}$.

Current 2-cycles:

$(\mathbf{ag} \wedge \mathbf{be})$ $((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb})$
$(\mathbf{ck} \wedge \mathbf{ae})$ $(\mathbf{fd} \wedge \mathbf{ea})$



The following 2-cycles are to be combined:

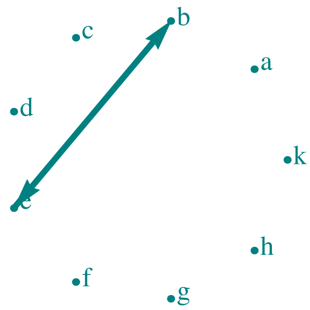
$(\mathbf{ag} \wedge \mathbf{be})$ $((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb})$
$(\mathbf{ck} \wedge \mathbf{ae})$ $(\mathbf{fd} \wedge \mathbf{ea})$

We have $|\{\mathbf{e}, \mathbf{b}, \mathbf{e}, \mathbf{a}\}| = 3$ and $\mathbf{e} = \mathbf{e}$ which indicates Case 3 (ii) and results in:

$(\mathbf{ag} \wedge \mathbf{be})$ $(((\mathbf{ck} \wedge \mathbf{ae})((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb}) \wedge \mathbf{ab})(\mathbf{fd} \wedge \mathbf{ea}) \wedge \mathbf{eb})$

Current 2-cycles:

$(\mathbf{ag} \wedge \mathbf{be})$ $(((\mathbf{ck} \wedge \mathbf{ae})((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb}) \wedge \mathbf{ab})(\mathbf{fd} \wedge \mathbf{ea}) \wedge \mathbf{eb})$

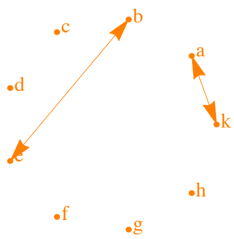


$\frac{[a,c,k][b,a,c][f,d,e][a,b,g][e,f,h]}{[a,e,g][a,b,f][c,b,h][k,c,e][a,d,f]} = 1$ whenever the both intermediate points are identical:

$(\mathbf{ag} \wedge \mathbf{be})$	$(((\mathbf{ck} \wedge \mathbf{ae})((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb}) \wedge \mathbf{ab})(\mathbf{fd} \wedge \mathbf{ea}) \wedge \mathbf{eb})$

In order to factor the complete bracket polynomial, one has to projectively add

$(\mathbf{ce} \wedge \mathbf{ak})$	$(((\mathbf{ae} \wedge \mathbf{kf})((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka}) \wedge \mathbf{af})((\mathbf{ab} \wedge \mathbf{gf})(\mathbf{ae} \wedge \mathbf{kg}) \wedge \mathbf{kf}) \wedge \mathbf{ka})$
$(\mathbf{ag} \wedge \mathbf{be})$	$(((\mathbf{ck} \wedge \mathbf{ae})((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb}) \wedge \mathbf{ab})(\mathbf{fd} \wedge \mathbf{ea}) \wedge \mathbf{eb})$



Adding up all cross-ratios

Similar to the situation before, two 2-cycles can be reduced to a single 2-cycle by projective addition.In the following, if a 2-cycle is drawn thickly, then another 2-cycle has been reduced to it in the step before.This other 2-cycle has vanished from the drawing in this step before.

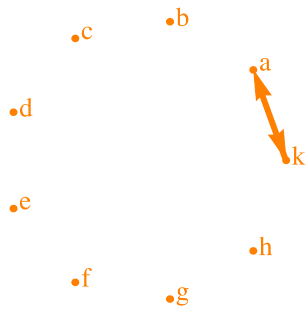
The following 2-cycles are to be combined:

$(\mathbf{ce} \wedge \mathbf{ak})$	$(((\mathbf{ae} \wedge \mathbf{kf})((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka}) \wedge \mathbf{af})((\mathbf{ab} \wedge \mathbf{gf})(\mathbf{ae} \wedge \mathbf{kg}) \wedge \mathbf{kf}) \wedge \mathbf{ka})$
$(\mathbf{ag} \wedge \mathbf{be})$	$(((\mathbf{ck} \wedge \mathbf{ae})((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb}) \wedge \mathbf{ab})(\mathbf{fd} \wedge \mathbf{ea}) \wedge \mathbf{eb})$

We have $|\{\mathbf{k}, \mathbf{a}, \mathbf{e}, \mathbf{b}\}| = 4$ and $|\{\mathbf{a}, \mathbf{g}, \mathbf{k}\}| = 3$ which indicates Case 4 (i) and results in:

Resulting 2-cycles:

$(\mathbf{ce} \wedge \mathbf{ak})$	$((\mathbf{a}((((\mathbf{ck} \wedge \mathbf{ae})((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb}) \wedge \mathbf{ab})(\mathbf{fd} \wedge \mathbf{ea}) \wedge \mathbf{eb})(\mathbf{ag} \wedge \mathbf{ke}) \wedge \mathbf{kb}) \wedge (((\mathbf{ae} \wedge \mathbf{kf})((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka}) \wedge \mathbf{af})((\mathbf{ab} \wedge \mathbf{gf})(\mathbf{ae} \wedge \mathbf{kg}) \wedge \mathbf{kf}) \wedge \mathbf{ka})\mathbf{b})((\mathbf{ag} \wedge \mathbf{kb})(\mathbf{ce} \wedge \mathbf{ak}) \wedge \mathbf{ab}) \wedge \mathbf{ka})$

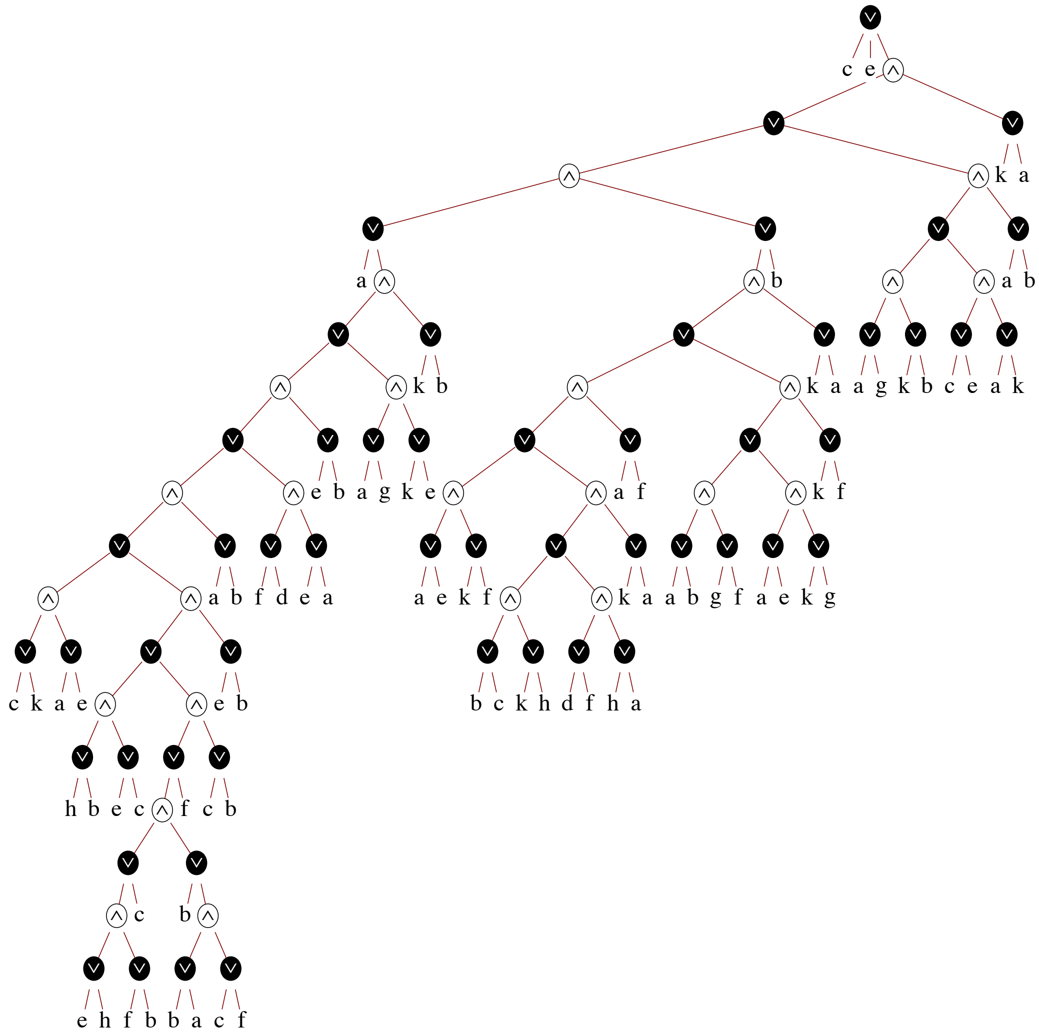


Asking for the final 2-cycle to equal 1 is to ask for the coincidence of both intermediate points of the last remaining length-ratios. Or equivalently:

Result:

$\mathbf{ce}((\mathbf{a}((((\mathbf{ck} \wedge \mathbf{ae})((\mathbf{hb} \wedge \mathbf{ec})(((\mathbf{eh} \wedge \mathbf{fb})\mathbf{c} \wedge \mathbf{b}(\mathbf{ba} \wedge \mathbf{cf}))\mathbf{f} \wedge \mathbf{cb}) \wedge \mathbf{eb}) \wedge \mathbf{ab})(\mathbf{fd} \wedge \mathbf{ea}) \wedge \mathbf{eb})(\mathbf{ag} \wedge \mathbf{ke}) \wedge \mathbf{kb}) \wedge (((\mathbf{ae} \wedge \mathbf{kf})((\mathbf{bc} \wedge \mathbf{kh})(\mathbf{df} \wedge \mathbf{ha}) \wedge \mathbf{ka}) \wedge \mathbf{af})((\mathbf{ab} \wedge \mathbf{gf})(\mathbf{ae} \wedge \mathbf{kg}) \wedge \mathbf{kf}) \wedge \mathbf{ka})\mathbf{b})((\mathbf{ag} \wedge \mathbf{kb})(\mathbf{ce} \wedge \mathbf{ak}) \wedge \mathbf{ab}) \wedge \mathbf{ka})$

Degree of multiplier monomial: 22


c e
k a
a
b
a b
k b
k a a g k b c e a k
e b a g k e
a f
k f
a b f d e a
a e k f
k a a b g f a e k g
c k a e
e b
b c k h d f h a
h b e c
f c b
c
b
e h f b b a c f

18.2. Example 2: Ten Points on a Cubic

Inspired by Pascal's theorem, it is natural to ask for a condition for ten points lying on a common *cubic*. We will factor a bracket polynomial with 20 summands describing this situation. This representation is due to Richter-Gebert and to the knowledge of our research group, it is the shortest known representation of the polynomial. In particular, this bracket polynomial is far from being multilinear and we ask for a ruler construction to interpret it. The conic condition is interesting and of degree two. For cubics, the Cayley factorization might be interesting, since there can arise more coincidences in the execution of the algorithm.

We know of two different synthetic constructions characterizing the situation: the first one is due to [17] and [19]. The construction needs some intermediate steps not carried out in detail in the original references. A straightforward construction, which also fills the gaps and which was obtained by the author, yields a construction where the degree of the multiplier polynomial is 626. However, the computation cannot be carried out in detail and it is unlikely that the multiplier is in fact a *monomial*. The second construction is given in [61] and the degree of the multiplier is 113. The same comments on the shape of the multiplier apply. The following factorization yields a multiplier *monomial* of degree 342. This shows, that in concrete instances, the actual result is much better than the bound given in Theorem 14.5. However, the optimizations done in the concrete implementation turn out to be useful in the present example: a first execution of the algorithm lead to a multiplier monomial of degree 603, a more advanced triangulation of the cycles in the binomial step of the algorithm yielded 544. Here, common factors in the monomial step were not factored out. Instead, they were altered such that they could be treated as to be distinct. The actual factoring out lead to the present degree of 342. The example is given in order to show that polynomials of this can be treated and that the degree of the multiplier is not as big as expected. However, we do not have enough space to give the complete output and large parts are omitted. After handing in this thesis, the complete output will be made available under `http://www-m10.ma.tum.de/CayleyFactorizationExtras`.

$$\begin{aligned}
B := \quad & [0,1,9][0,5,2][6,1,2][6,5,9][1,7,8][2,3,7][4,3,9][4,5,7][4,6,8][0,3,8] \\
& -[6,5,2][0,5,9][0,1,2][9,6,1][1,7,8][2,3,7][4,3,9][4,5,7][4,6,8][0,3,8] \\
& -[0,1,8][3,8,7][0,5,9][1,2,7][1,6,9][3,2,0][4,3,9][4,5,7][4,6,8][5,2,6] \\
& +[0,1,9][0,3,5][0,7,8][1,2,7][1,6,8][2,3,9][4,3,8][4,5,7][4,6,9][5,2,6] \\
& -[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,5][4,3,9][4,5,6][4,7,8][2,6,9] \\
& +[1,2,9][0,3,9][0,5,8][1,0,6][1,7,8][2,3,7][4,3,8][4,5,7][4,6,9][5,2,6] \\
& +[0,1,8][0,3,9][0,5,7][1,2,7][1,6,9][2,3,8][4,3,6][4,5,9][4,7,8][5,2,6] \\
& +[0,1,9][0,3,8][0,5,6][1,2,6][1,7,8][2,3,9][4,3,7][4,6,9][4,5,8][5,2,7] \\
& +[0,1,9][0,3,2][0,5,8][1,2,6][1,7,8][9,3,6][4,3,8][4,5,6][4,7,9][5,2,7] \\
& +[0,3,8][0,5,9][1,0,7][1,6,9][1,2,8][2,3,6][4,3,9][4,5,6][4,7,8][5,2,7] \\
& -[0,1,8][0,3,5][0,9,6][1,2,6][1,7,9][2,3,8][4,3,9][4,5,6][4,7,8][5,2,7] \\
& -[0,1,8][0,3,9][0,5,6][1,2,6][1,7,9][2,3,5][4,3,8][4,5,7][4,6,9][8,2,7] \\
& +[0,1,8][0,3,9][0,5,2][1,2,7][1,6,9][2,3,6][4,3,8][4,5,6][4,7,9][5,8,7] \\
& -[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6] \\
& -[0,1,8][0,3,9][0,5,2][1,2,7][1,6,9][8,3,7][4,3,8][4,5,7][4,6,9][5,2,6] \\
& +[0,1,8][0,3,9][0,5,3][1,2,7][1,6,9][2,7,8][4,3,8][4,5,7][4,6,9][5,2,6] \\
& -[0,3,9][0,5,8][1,0,2][1,6,9][1,7,8][2,3,6][4,3,8][4,5,6][4,7,9][5,2,7] \\
& +[0,1,9][0,3,8][0,5,2][1,2,6][1,7,8][6,3,9][4,3,9][4,5,6][4,7,8][5,2,7] \\
& -[0,1,9][0,3,8][0,5,3][1,2,6][1,7,8][2,9,6][4,3,9][4,5,6][4,7,8][5,2,7] \\
& -[0,1,8][0,3,9][0,5,6][1,2,6][1,7,9][2,3,8][4,3,5][4,6,9][4,7,8][5,2,7]
\end{aligned}$$

Formatting and pre-processing input:

The bracket polynomial is processed such that it is composed of summands with all but the last coefficient equal to $+1$. The last coefficient is -1. Cayley factorization yields a $\vee$-$\wedge$-sequence to interpret it. Intuition is provided by interpreting the vanishing of the bracket polynomial:

$$B = 0$$

$$\Longleftrightarrow$$

$$\frac{[0,1,9][0,5,2][6,1,2][6,5,9][1,7,8][2,3,7][4,3,9][4,5,7][4,6,8][0,3,8]}{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]} +$$

$$\frac{\mathbf{[5,6,2][0,5,9][0,1,2][9,6,1][1,7,8][2,3,7][4,3,9][4,5,7][4,6,8][0,3,8]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[1,0,8][3,8,7][0,5,9][1,2,7][1,6,9][3,2,0][4,3,9][4,5,7][4,6,8][5,2,6]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[0,1,9][0,3,5][0,7,8][1,2,7][1,6,8][2,3,9][4,3,8][4,5,7][4,6,9][5,2,6]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[1,0,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,5][4,3,9][4,5,6][4,7,8][2,6,9]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[1,2,9][0,3,9][0,5,8][1,0,6][1,7,8][2,3,7][4,3,8][4,5,7][4,6,9][5,2,6]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[0,1,8][0,3,9][0,5,7][1,2,7][1,6,9][2,3,8][4,3,6][4,5,9][4,7,8][5,2,6]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[0,1,9][0,3,8][0,5,6][1,2,6][1,7,8][2,3,9][4,3,7][4,6,9][4,5,8][5,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[0,1,9][0,3,2][0,5,8][1,2,6][1,7,8][9,3,6][4,3,8][4,5,6][4,7,9][5,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[0,3,8][0,5,9][1,0,7][1,6,9][1,2,8][2,3,6][4,3,9][4,5,6][4,7,8][5,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[1,0,8][0,3,5][0,9,6][1,2,6][1,7,9][2,3,8][4,3,9][4,5,6][4,7,8][5,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[1,0,8][0,3,9][0,5,6][1,2,6][1,7,9][2,3,5][4,3,8][4,5,7][4,6,9][8,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[0,1,8][0,3,9][0,5,2][1,2,7][1,6,9][2,3,6][4,3,8][4,5,6][4,7,9][5,8,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[1,0,8][0,3,9][0,5,2][1,2,7][1,6,9][8,3,7][4,3,8][4,5,7][4,6,9][5,2,6]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[0,1,8][0,3,9][0,5,3][1,2,7][1,6,9][2,7,8][4,3,8][4,5,7][4,6,9][5,2,6]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[3,0,9][0,5,8][1,0,2][1,6,9][1,7,8][2,3,6][4,3,8][4,5,6][4,7,9][5,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[0,1,9][0,3,8][0,5,2][1,2,6][1,7,8][6,3,9][4,3,9][4,5,6][4,7,8][5,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[1,0,9][0,3,8][0,5,3][1,2,6][1,7,8][2,9,6][4,3,9][4,5,6][4,7,8][5,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}+$$

$$\frac{\mathbf{[1,0,8][0,3,9][0,5,6][1,2,6][1,7,9][2,3,8][4,3,5][4,6,9][4,7,8][5,2,7]}}{\mathbf{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]}}=1$$

Start of the Main Part of the Algorithm:

Factor all induced fractional equations $\frac{[*,*,*]...[*,*,*]}{[*,*,*]...[*,*,*]} = 1$

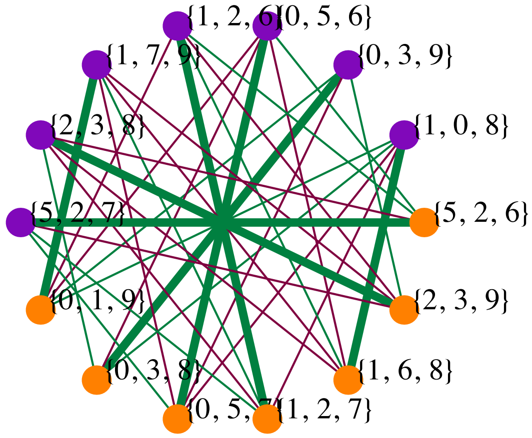
——Factorize the fractional equation

$$\frac{[1,0,8][0,3,9][0,5,6][1,2,6][1,7,9][2,3,8][4,3,5][4,6,9][4,7,8][5,2,7]}{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]} = 1$$

Brackets occurring in the numerator as well as in the denominator can be canceled.
The common elements are: $[\mathbf{3},\mathbf{4},\mathbf{5}], [\mathbf{4},\mathbf{6},\mathbf{9}], [\mathbf{4},\mathbf{7},\mathbf{8}]$
This reduces the bracket identity in question to:

$$\frac{[1,0,8][0,3,9][0,5,6][1,2,6][1,7,9][2,3,8][5,2,7]}{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][5,2,6]} = 1$$

Find a Matching in order to apply the area principle:



Apply the area principle to fractions corresponding to the matching just found:

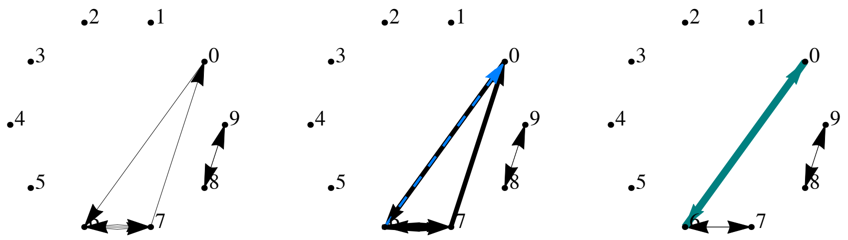
$\frac{[1,0,8]}{[1,6,8]} \to (\mathbf{18} \wedge \mathbf{06})$ $\frac{[0,3,9]}{[0,3,8]} \to (\mathbf{03} \wedge \mathbf{98})$ $\frac{[0,5,6]}{[0,5,7]} \to (\mathbf{05} \wedge \mathbf{67})$ $\frac{[1,2,6]}{[1,2,7]} \to (\mathbf{12} \wedge \mathbf{67})$ $\frac{[1,7,9]}{[0,1,9]} \to (\mathbf{19} \wedge \mathbf{70})$ $\frac{[2,3,8]}{[2,3,9]} \to (\mathbf{23} \wedge \mathbf{89})$
$\frac{[5,2,7]}{[5,2,6]} \to (\mathbf{52} \wedge \mathbf{76})$

Product of sign-errors occurring when applying the area principle: $\varepsilon = -1$

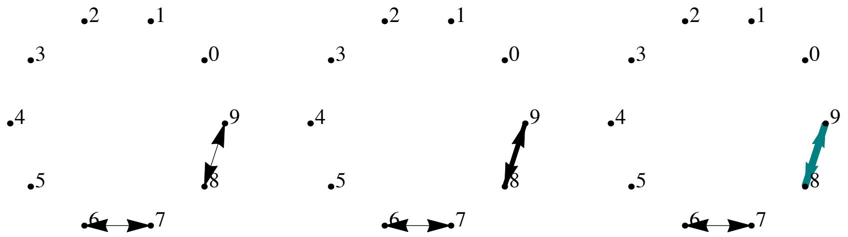
Find and triangulate Cycles

Length-ratios left over to be processed: $(\mathbf{18} \wedge \mathbf{06})$ $(\mathbf{03} \wedge \mathbf{98})$ $(\mathbf{05} \wedge \mathbf{67})$ $(\mathbf{12} \wedge \mathbf{67})$ $(\mathbf{19} \wedge \mathbf{70})$ $(\mathbf{23} \wedge \mathbf{89})$ $(\mathbf{52} \wedge \mathbf{76})$
Cycle found:$(\mathbf{18} \wedge \mathbf{06})$ $(\mathbf{12} \wedge \mathbf{67})$ $(\mathbf{19} \wedge \mathbf{70})$

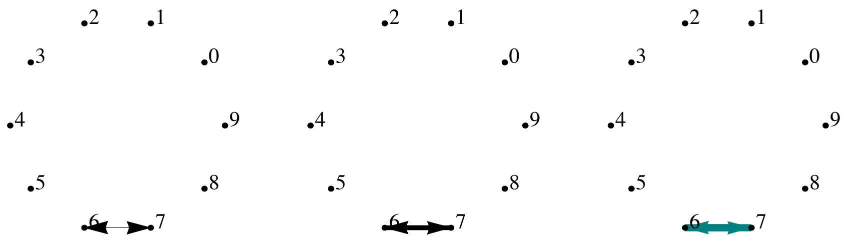
Due to $\varepsilon = -1$ a Ceva configuration is used once in the triangulation. The blue edge reads as $(((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})$



Length-ratios left over to be processed: $(\mathbf{05} \wedge \mathbf{67})$ $(\mathbf{52} \wedge \mathbf{76})$ $(\mathbf{23} \wedge \mathbf{89})$ $(\mathbf{03} \wedge \mathbf{98})$
Cycle found: $(\mathbf{23} \wedge \mathbf{89})$ $(\mathbf{03} \wedge \mathbf{98})$



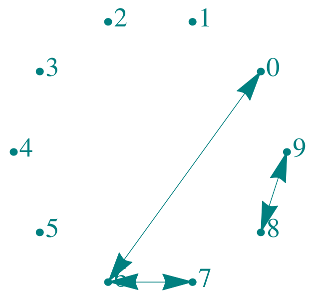
Length-ratios left over to be processed: $(\mathbf{05} \wedge \mathbf{67})$ $(\mathbf{52} \wedge \mathbf{76})$
Cycle found: $(\mathbf{05} \wedge \mathbf{67})$ $(\mathbf{52} \wedge \mathbf{76})$



Combination of 2-cycles resulting from triangulation

Current 2-cycles:

$(\mathbf{18} \wedge \mathbf{06})$	$(((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})$
$(\mathbf{23} \wedge \mathbf{89})$	$(\mathbf{03} \wedge \mathbf{98})$
$(\mathbf{05} \wedge \mathbf{67})$	$(\mathbf{52} \wedge \mathbf{76})$



The following 2-cycles are to be combined:

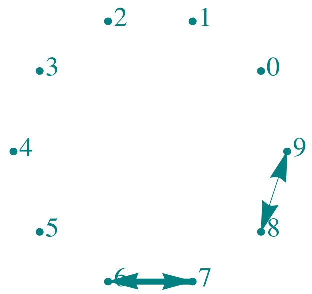
$(\mathbf{05} \wedge \mathbf{67})$ $\quad (\mathbf{52} \wedge \mathbf{76})$
$(\mathbf{18} \wedge \mathbf{06})$ $\quad (((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})$

We have $|\{\mathbf{7}, \mathbf{6}, \mathbf{6}, \mathbf{0}\}| = 3$ and $\mathbf{6} = \mathbf{6}$ which indicates Case 3 (iii) and results in:

$(\mathbf{05} \wedge \mathbf{67})$ $\quad (((((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{70})(\mathbf{18} \wedge \mathbf{06}) \wedge \mathbf{76})$

Current 2-cycles:

$(\mathbf{05} \wedge \mathbf{67})$ $\quad (((((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{70})(\mathbf{18} \wedge \mathbf{06}) \wedge \mathbf{76})$
$(\mathbf{23} \wedge \mathbf{89})$ $\quad (\mathbf{03} \wedge \mathbf{98})$



The following 2-cycles are to be combined:

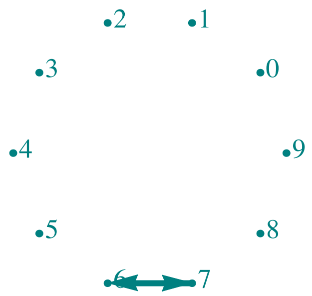
$(\mathbf{05} \wedge \mathbf{67})$ $\quad (((((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{70})(\mathbf{18} \wedge \mathbf{06}) \wedge \mathbf{76})$
$(\mathbf{23} \wedge \mathbf{89})$ $\quad (\mathbf{03} \wedge \mathbf{98})$

We have $|\{7, 6, 9, 8\}| = 4$ and $|\{\mathbf{2}, \mathbf{3}, 7\}| = 3$ which indicates Case 4 (i) and results in:

$$(\mathbf{05} \wedge \mathbf{67}) \qquad (((\mathbf{23} \wedge \mathbf{78})(((((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{70})(\mathbf{18} \wedge \mathbf{06}) \wedge \mathbf{76}) \wedge \mathbf{68})((\mathbf{03} \wedge \mathbf{98})(\mathbf{23} \wedge \mathbf{79}) \wedge \mathbf{78}) \wedge \mathbf{76})$$

Current 2-cycles:

$$(\mathbf{05} \wedge \mathbf{67}) \qquad (((\mathbf{23} \wedge \mathbf{78})(((((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{70})(\mathbf{18} \wedge \mathbf{06}) \wedge \mathbf{76}) \wedge \mathbf{68})((\mathbf{03} \wedge \mathbf{98})(\mathbf{23} \wedge \mathbf{79}) \wedge \mathbf{78}) \wedge \mathbf{76})$$



$\frac{[1,0,8][0,3,9][0,5,6][1,2,6][1,7,9][2,3,8][4,3,5][4,6,9][4,7,8][5,2,7]}{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]} = 1$ **whenever the both intermediate points are identical:**

$$(\mathbf{05} \wedge \mathbf{67}) \qquad (((\mathbf{23} \wedge \mathbf{78})(((((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{70})(\mathbf{18} \wedge \mathbf{06}) \wedge \mathbf{76}) \wedge \mathbf{68})((\mathbf{03} \wedge \mathbf{98})(\mathbf{23} \wedge \mathbf{79}) \wedge \mathbf{78}) \wedge \mathbf{76})$$

——Factorize the fractional equation

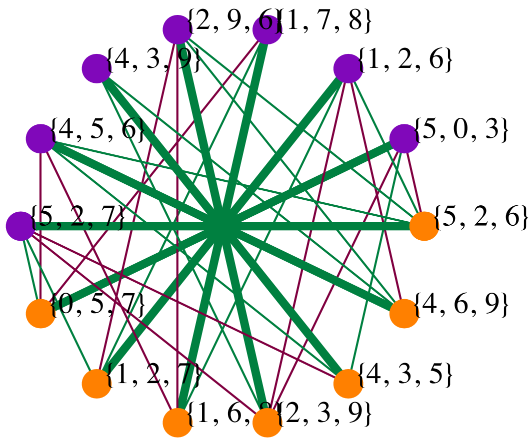
$$\frac{[1,0,9][0,3,8][0,5,3][1,2,6][1,7,8][2,9,6][4,3,9][4,5,6][4,7,8][5,2,7]}{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]} = 1$$

Brackets occurring in the numerator as well as in the denominator can be canceled. The common elements are: $[\mathbf{0}, \mathbf{1}, \mathbf{9}]$, $[\mathbf{0}, \mathbf{3}, \mathbf{8}]$, $[\mathbf{4}, \mathbf{7}, \mathbf{8}]$

This reduces the bracket identity in question to:

$$\frac{[\mathbf{5}, \mathbf{0}, \mathbf{3}][\mathbf{1}, \mathbf{2}, \mathbf{6}][\mathbf{1}, \mathbf{7}, \mathbf{8}][\mathbf{2}, \mathbf{9}, \mathbf{6}][\mathbf{4}, \mathbf{3}, \mathbf{9}][\mathbf{4}, \mathbf{5}, \mathbf{6}][\mathbf{5}, \mathbf{2}, \mathbf{7}]}{[\mathbf{0}, \mathbf{5}, \mathbf{7}][\mathbf{1}, \mathbf{2}, \mathbf{7}][\mathbf{1}, \mathbf{6}, \mathbf{8}][\mathbf{2}, \mathbf{3}, \mathbf{9}][\mathbf{4}, \mathbf{3}, \mathbf{5}][\mathbf{4}, \mathbf{6}, \mathbf{9}][\mathbf{5}, \mathbf{2}, \mathbf{6}]} = 1$$

Find a Matching in order to apply the area principle:



Apply the area principle to fractions corresponding to the matching just found:

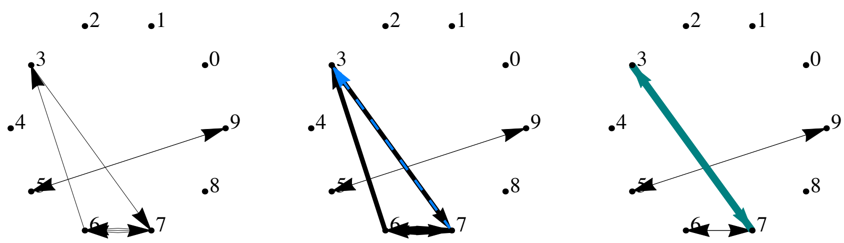
$\frac{[5,0,3]}{[0,5,7]} \to (\mathbf{50} \wedge \mathbf{37})$ $\frac{[1,2,6]}{[1,2,7]} \to (\mathbf{12} \wedge \mathbf{67})$ $\frac{[1,7,8]}{[1,6,8]} \to (\mathbf{18} \wedge \mathbf{76})$ $\frac{[2,9,6]}{[2,3,9]} \to (\mathbf{29} \wedge \mathbf{63})$ $\frac{[4,3,9]}{[4,3,5]} \to (\mathbf{43} \wedge \mathbf{95})$ $\frac{[4,5,6]}{[4,6,9]} \to (\mathbf{46} \wedge \mathbf{59})$

$\frac{[5,2,7]}{[5,2,6]} \to (\mathbf{52} \wedge \mathbf{76})$

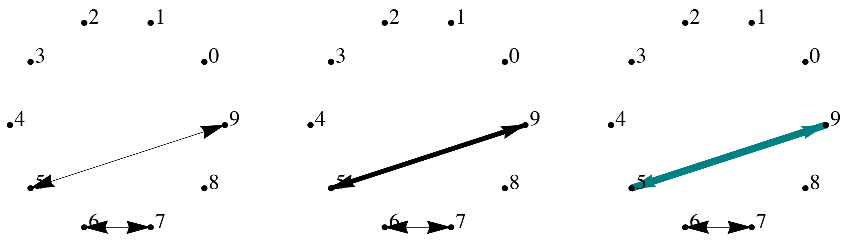
Product of sign-errors occurring when applying the area principle: $\varepsilon = -1$

Find and triangulate Cycles

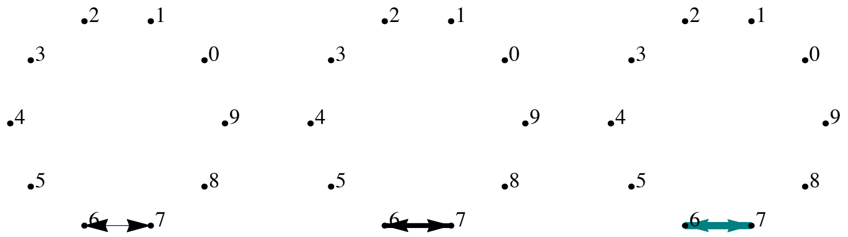
Length-ratios left over to be processed: $(\mathbf{50} \wedge \mathbf{37})$ $(\mathbf{12} \wedge \mathbf{67})$ $(\mathbf{18} \wedge \mathbf{76})$ $(\mathbf{29} \wedge \mathbf{63})$ $(\mathbf{43} \wedge \mathbf{95})$ $(\mathbf{46} \wedge \mathbf{59})$ $(\mathbf{52} \wedge \mathbf{76})$
Cycle found:$(\mathbf{50} \wedge \mathbf{37})$ $(\mathbf{18} \wedge \mathbf{76})$ $(\mathbf{29} \wedge \mathbf{63})$
Due to $\varepsilon = -1$ a Ceva configuration is used once in the triangulation. The blue edge reads as $(((\mathbf{29} \wedge \mathbf{63})\mathbf{7} \wedge \mathbf{3}(\mathbf{18} \wedge \mathbf{76}))\mathbf{6} \wedge \mathbf{73})$



Length-ratios left over to be processed: $(\mathbf{12}\wedge\mathbf{67})$ $(\mathbf{52}\wedge\mathbf{76})$ $(\mathbf{46}\wedge\mathbf{59})$ $(\mathbf{43}\wedge\mathbf{95})$
Cycle found:$(\mathbf{46}\wedge\mathbf{59})$ $(\mathbf{43}\wedge\mathbf{95})$



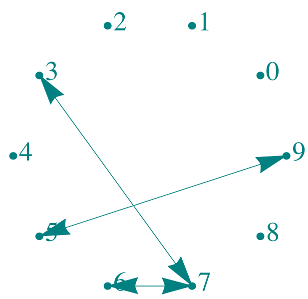
Length-ratios left over to be processed: $(\mathbf{12}\wedge\mathbf{67})$ $(\mathbf{52}\wedge\mathbf{76})$
Cycle found:$(\mathbf{12}\wedge\mathbf{67})$ $(\mathbf{52}\wedge\mathbf{76})$



Combination of 2-cycles resulting from triangulation

Current 2-cycles:

$(\mathbf{50}\wedge\mathbf{37})$	$(((\mathbf{29}\wedge\mathbf{63})\mathbf{7}\wedge\mathbf{3}(\mathbf{18}\wedge\mathbf{76}))\mathbf{6}\wedge\mathbf{73})$
$(\mathbf{46}\wedge\mathbf{59})$	$(\mathbf{43}\wedge\mathbf{95})$
$(\mathbf{12}\wedge\mathbf{67})$	$(\mathbf{52}\wedge\mathbf{76})$



The following 2-cycles are to be combined:

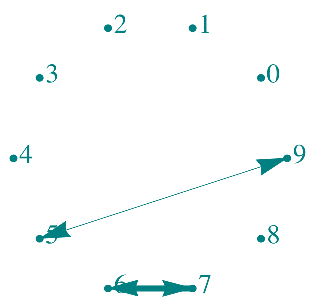
$(\mathbf{12} \wedge \mathbf{67})$ $\quad(\mathbf{52} \wedge \mathbf{76})$
$(\mathbf{50} \wedge \mathbf{37})$ $\quad(((\mathbf{29} \wedge \mathbf{63})\mathbf{7} \wedge \mathbf{3}(\mathbf{18} \wedge \mathbf{76}))\mathbf{6} \wedge \mathbf{73})$

We have $|\{\mathbf{7}, \mathbf{6}, \mathbf{7}, \mathbf{3}\}| = 3$ and $\mathbf{7} = \mathbf{7}$ which indicates Case 3 (ii) and results in:

$(\mathbf{12} \wedge \mathbf{67})$ $\quad(((\mathbf{50}\wedge\mathbf{37})(\mathbf{52}\wedge\mathbf{76})\wedge\mathbf{36})(((\mathbf{29}\wedge\mathbf{63})\mathbf{7}\wedge\mathbf{3}(\mathbf{18}\wedge\mathbf{76}))\mathbf{6}\wedge\mathbf{73})\wedge\mathbf{76})$

Current 2-cycles:

$(\mathbf{46} \wedge \mathbf{59})$ $\quad(\mathbf{43} \wedge \mathbf{95})$
$(\mathbf{12} \wedge \mathbf{67})$ $\quad(((\mathbf{50}\wedge\mathbf{37})(\mathbf{52}\wedge\mathbf{76})\wedge\mathbf{36})(((\mathbf{29}\wedge\mathbf{63})\mathbf{7}\wedge\mathbf{3}(\mathbf{18}\wedge\mathbf{76}))\mathbf{6}\wedge\mathbf{73})\wedge\mathbf{76})$



The following 2-cycles are to be combined:

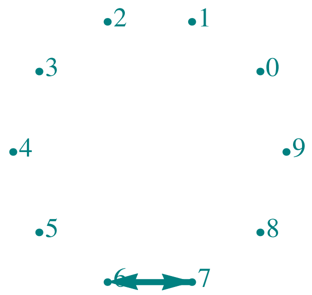
$(\mathbf{12} \wedge \mathbf{67})$ $\quad(((\mathbf{50}\wedge\mathbf{37})(\mathbf{52}\wedge\mathbf{76})\wedge\mathbf{36})(((\mathbf{29}\wedge\mathbf{63})\mathbf{7}\wedge\mathbf{3}(\mathbf{18}\wedge\mathbf{76}))\mathbf{6}\wedge\mathbf{73})\wedge\mathbf{76})$
$(\mathbf{46} \wedge \mathbf{59})$ $\quad(\mathbf{43} \wedge \mathbf{95})$

We have $|\{\mathbf{7}, \mathbf{6}, \mathbf{9}, \mathbf{5}\}| = 4$ and $|\{\mathbf{4}, \mathbf{6}, \mathbf{7}\}| = 3$ which indicates Case 4 (i) and results in:

$(\mathbf{12} \wedge \mathbf{67})$ $\quad(((\mathbf{46} \wedge \mathbf{75})(((\mathbf{50} \wedge \mathbf{37})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{36})(((\mathbf{29} \wedge \mathbf{63})\mathbf{7} \wedge \mathbf{3}(\mathbf{18} \wedge \mathbf{76}))\mathbf{6} \wedge \mathbf{73}) \wedge \mathbf{76}) \wedge \mathbf{65})((\mathbf{43} \wedge \mathbf{95})(\mathbf{46} \wedge \mathbf{79}) \wedge \mathbf{75}) \wedge \mathbf{76})$

Current 2-cycles:

$(\mathbf{12} \wedge \mathbf{67})$ $\quad(((\mathbf{46} \wedge \mathbf{75})(((\mathbf{50} \wedge \mathbf{37})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{36})(((\mathbf{29} \wedge \mathbf{63})\mathbf{7} \wedge \mathbf{3}(\mathbf{18} \wedge \mathbf{76}))\mathbf{6} \wedge \mathbf{73}) \wedge \mathbf{76}) \wedge \mathbf{65})((\mathbf{43} \wedge \mathbf{95})(\mathbf{46} \wedge \mathbf{79}) \wedge \mathbf{75}) \wedge \mathbf{76})$



$\frac{[1,0,9][0,3,8][0,5,3][1,2,6][1,7,8][2,9,6][4,3,9][4,5,6][4,7,8][5,2,7]}{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]} = 1$ **whenever the both intermediate points are identical:**

$(\mathbf{12} \wedge \mathbf{67})$	$(((\mathbf{46} \wedge \mathbf{75})(((\mathbf{50} \wedge \mathbf{37})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{36})(((\mathbf{29} \wedge \mathbf{63})\mathbf{7} \wedge \mathbf{3}(\mathbf{18} \wedge \mathbf{76}))\mathbf{6} \wedge \mathbf{73}) \wedge \mathbf{76}) \wedge \mathbf{65})((\mathbf{43} \wedge \mathbf{95})(\mathbf{46} \wedge \mathbf{79}) \wedge \mathbf{75}) \wedge \mathbf{76})$

——Factorize the fractional equation

$\frac{[0,1,9][0,3,8][0,5,2][1,2,6][1,7,8][6,3,9][4,3,9][4,5,6][4,7,8][5,2,7]}{[0,1,9][0,3,8][0,5,7][1,2,7][1,6,8][2,3,9][4,3,5][4,6,9][4,7,8][5,2,6]} = 1$

Brackets occurring in the numerator as well as in the denominator can be canceled. The common elements are: $[\mathbf{0}, \mathbf{1}, \mathbf{9}], [\mathbf{0}, \mathbf{3}, \mathbf{8}], [\mathbf{4}, \mathbf{7}, \mathbf{8}]$

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[omitted output]
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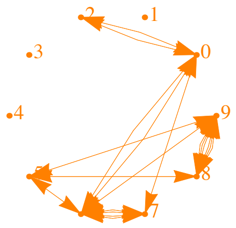
In order to factor the complete bracket polynomial, one has to projectively add

$(\mathbf{05} \wedge \mathbf{67})$	$(((\mathbf{23}\wedge\mathbf{78})(((((\mathbf{19}\wedge\mathbf{70})\mathbf{6}\wedge\mathbf{0}(\mathbf{12}\wedge\mathbf{67}))\mathbf{7}\wedge\mathbf{60})(\mathbf{52}\wedge\mathbf{76})\wedge\mathbf{70})(\mathbf{18}\wedge \mathbf{06}) \wedge \mathbf{76}) \wedge \mathbf{68})((\mathbf{03} \wedge \mathbf{98})(\mathbf{23} \wedge \mathbf{79}) \wedge \mathbf{78}) \wedge \mathbf{76})$
$(\mathbf{12} \wedge \mathbf{67})$	$(((\mathbf{46}\wedge\mathbf{75})(((\mathbf{50}\wedge\mathbf{37})(\mathbf{52}\wedge\mathbf{76})\wedge\mathbf{36})(((\mathbf{29}\wedge\mathbf{63})\mathbf{7}\wedge\mathbf{3}(\mathbf{18}\wedge\mathbf{76}))\mathbf{6}\wedge \mathbf{73}) \wedge \mathbf{76}) \wedge \mathbf{65})((\mathbf{43} \wedge \mathbf{95})(\mathbf{46} \wedge \mathbf{79}) \wedge \mathbf{75}) \wedge \mathbf{76})$

$(12 \wedge 67)$	$(((46\wedge 75)(((05\wedge 27)(52\wedge 76)\wedge 26)(((39\wedge 62)7\wedge 2(18\wedge 76))6\wedge 72)\wedge 76)\wedge 65)((43\wedge 95)(46\wedge 79)\wedge 75)\wedge 76)$
$(12 \wedge 07)$	$((49\wedge 76)((23\wedge 69)((16\wedge 98)((43\wedge 85)((08\wedge 53)((09\wedge 31)((78\wedge 14)((56\wedge 42)(57\wedge 20)\wedge 40)\wedge 10)\wedge 30)\wedge 50)\wedge 80)\wedge 90)\wedge 60)\wedge 70)$
$(39 \wedge 02)$	$(((01\wedge 28)(((57\wedge 40)2\wedge 0(78\wedge 24))4\wedge 20)\wedge 08)((((43\wedge 85)(16\wedge 98)\wedge 95)(03\wedge 58)\wedge 98)(01\wedge 29)\wedge 28)\wedge 20)$
$(39 \wedge 02)$	$(((10\wedge 28)((05\wedge 27)((45\wedge 73)((87\wedge 34)(38\wedge 40)\wedge 30)\wedge 70)\wedge 20)\wedge 08)((16\wedge 98)(10\wedge 29)\wedge 28)\wedge 20)$
$(01 \wedge 89)$	$(((23\wedge 69)(47\wedge 98)\wedge 68)(((46\wedge 59)((03\wedge 98)((57\wedge 80)(52\wedge 06)\wedge 86)\wedge 96)\wedge 56)((16\wedge 98)(43\wedge 85)\wedge 95)\wedge 96)\wedge 98)$
$(18 \wedge 06)$	$(((23\wedge 65)((05\wedge 67)(19\wedge 70)\wedge 60)\wedge 05)(((((47\wedge 58)(27\wedge 81)\wedge 51)((03\wedge 98)(43\wedge 85)\wedge 95)\wedge 91)(26\wedge 15)\wedge 95)(23\wedge 69)\wedge 65)\wedge 60)$
$(18 \wedge 06)$	$(((03\wedge 65)(((96\wedge 04)((12\wedge 67)(19\wedge 70)\wedge 60)\wedge 64)((56\wedge 42)(57\wedge 20)\wedge 40)\wedge 60)\wedge 05)(((23\wedge 89)(43\wedge 95)\wedge 85)(03\wedge 68)\wedge 65)\wedge 60)$
$(23 \wedge 69)$	$(((46\wedge 59)((16\wedge 98)((12\wedge 87)(52\wedge 76)\wedge 86)\wedge 96)\wedge 56)(((09\wedge 51)(43\wedge 95)\wedge 91)(07\wedge 15)\wedge 95)\wedge 96)$
$(47 \wedge 98)$	$(((((12\wedge 67)((18\wedge 76)((93\wedge 62)(03\wedge 28)\wedge 68)\wedge 78)\wedge 68)(52\wedge 76)\wedge 78)((43\wedge 85)(46\wedge 59)\wedge 89)\wedge 97)(05\wedge 87)\wedge 89)$
$(12 \wedge 67)$	$(((48\wedge 57)((07\wedge (18\wedge 76)5)(56\wedge (52\wedge 76)0)\wedge 76)\wedge 56)(43\wedge 75)\wedge 76)$
$(49 \wedge 56)$	$(((23\wedge 68)(43\wedge 65)\wedge 58)(((09\wedge (03\wedge 98)1)(18\wedge (16\wedge 98)0)\wedge 98)(23\wedge 69)\wedge 68)\wedge 65)$
$(03 \wedge 98)$	$(((47\wedge 58)((05\wedge 87)(23\wedge 79)\wedge 89)\wedge 59)(((16\wedge 08)(43\wedge 85)\wedge 05)((17\wedge 82)(19\wedge 20)\wedge 80)\wedge 85)\wedge 89)$
$(46 \wedge 59)$	$((39\wedge (43\wedge 95)2)(25\wedge (26\wedge 95)3)\wedge 95)$
$(47 \wedge 58)$	$((08\wedge (07\wedge 85)3)(35\wedge (43\wedge 85)0)\wedge 85)$
$(46 \wedge 89)$	$(((((43\wedge 95)8\wedge 5(10\wedge 89))9\wedge 85)(((32\wedge 09)(16\wedge 98)\wedge 08)((05\wedge 97)(38\wedge 70)\wedge 90)\wedge 98)\wedge 95)(47\wedge 58)\wedge 98)$
$(46 \wedge 89)$	$(((23\wedge 79)(61\wedge 98)\wedge 78)((43\wedge 95)((09\wedge 51)((78\wedge 14)((57\wedge 40)(12\wedge 07)\wedge 47)\wedge 17)\wedge 57)\wedge 97)\wedge 98)$
$(46 \wedge 89)$	$(((23\wedge 79)(((47\wedge 58)9\wedge 8(43\wedge 95))5\wedge 98)\wedge 78)(((18\wedge 76)((65\wedge 92)(05\wedge 27)\wedge 97)\wedge 96)(12\wedge 67)\wedge 97)\wedge 98)$



Adding up all cross-ratios

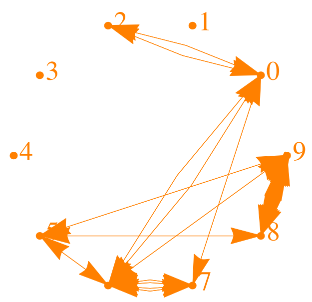
Similar to the situation before, two 2-cycles can be reduced to a single 2-cycle by projective addition. In the following, in each step, the last two 2-cycles in the list are combined.

We have $|\{9, 8, 9, 8\}| = 2$ and $9 = 9$ and $8 = 8$ which indicates Case 2 (i).

Resulting 2-cycles:

$(\mathbf{05} \wedge \mathbf{67})$	$(((\mathbf{23} \wedge \mathbf{78})(((((\mathbf{19} \wedge \mathbf{70})\mathbf{6} \wedge \mathbf{0}(\mathbf{12} \wedge \mathbf{67}))\mathbf{7} \wedge \mathbf{60})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{70})(\mathbf{18} \wedge \mathbf{06}) \wedge \mathbf{76}) \wedge \mathbf{68})((\mathbf{03} \wedge \mathbf{98})(\mathbf{23} \wedge \mathbf{79}) \wedge \mathbf{78}) \wedge \mathbf{76})$
$(\mathbf{12} \wedge \mathbf{67})$	$(((\mathbf{46} \wedge \mathbf{75})(((\mathbf{50} \wedge \mathbf{37})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{36})(((\mathbf{29} \wedge \mathbf{63})\mathbf{7} \wedge \mathbf{3}(\mathbf{18} \wedge \mathbf{76}))\mathbf{6} \wedge \mathbf{73}) \wedge \mathbf{76}) \wedge \mathbf{65})((\mathbf{43} \wedge \mathbf{95})(\mathbf{46} \wedge \mathbf{79}) \wedge \mathbf{75}) \wedge \mathbf{76})$
$(\mathbf{12} \wedge \mathbf{67})$	$(((\mathbf{46} \wedge \mathbf{75})(((\mathbf{05} \wedge \mathbf{27})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{26})(((\mathbf{39} \wedge \mathbf{62})\mathbf{7} \wedge \mathbf{2}(\mathbf{18} \wedge \mathbf{76}))\mathbf{6} \wedge \mathbf{72}) \wedge \mathbf{76}) \wedge \mathbf{65})((\mathbf{43} \wedge \mathbf{95})(\mathbf{46} \wedge \mathbf{79}) \wedge \mathbf{75}) \wedge \mathbf{76})$
$(\mathbf{12} \wedge \mathbf{07})$	$((\mathbf{49} \wedge \mathbf{76})((\mathbf{23} \wedge \mathbf{69})((\mathbf{16} \wedge \mathbf{98})((\mathbf{43} \wedge \mathbf{85})((\mathbf{08} \wedge \mathbf{53})((\mathbf{09} \wedge \mathbf{31})((\mathbf{78} \wedge \mathbf{14})((\mathbf{56} \wedge \mathbf{42})(\mathbf{57} \wedge \mathbf{20}) \wedge \mathbf{40}) \wedge \mathbf{10}) \wedge \mathbf{30}) \wedge \mathbf{50}) \wedge \mathbf{80}) \wedge \mathbf{90}) \wedge \mathbf{60}) \wedge \mathbf{70})$
$(\mathbf{39} \wedge \mathbf{02})$	$(((\mathbf{01} \wedge \mathbf{28})(((\mathbf{57} \wedge \mathbf{40})\mathbf{2} \wedge \mathbf{0}(\mathbf{78} \wedge \mathbf{24}))\mathbf{4} \wedge \mathbf{20}) \wedge \mathbf{08})((((\mathbf{43} \wedge \mathbf{85})(\mathbf{16} \wedge \mathbf{98}) \wedge \mathbf{95})(\mathbf{03} \wedge \mathbf{58}) \wedge \mathbf{98})(\mathbf{01} \wedge \mathbf{29}) \wedge \mathbf{28}) \wedge \mathbf{20})$
$(\mathbf{39} \wedge \mathbf{02})$	$(((\mathbf{10} \wedge \mathbf{28})((\mathbf{05} \wedge \mathbf{27})((\mathbf{45} \wedge \mathbf{73})((\mathbf{87} \wedge \mathbf{34})(\mathbf{38} \wedge \mathbf{40}) \wedge \mathbf{30}) \wedge \mathbf{70}) \wedge \mathbf{20}) \wedge \mathbf{08})((\mathbf{16} \wedge \mathbf{98})(\mathbf{10} \wedge \mathbf{29}) \wedge \mathbf{28}) \wedge \mathbf{20})$
$(\mathbf{01} \wedge \mathbf{89})$	$(((\mathbf{23} \wedge \mathbf{69})(\mathbf{47} \wedge \mathbf{98}) \wedge \mathbf{68})(((\mathbf{46} \wedge \mathbf{59})((\mathbf{03} \wedge \mathbf{98})((\mathbf{57} \wedge \mathbf{80})(\mathbf{52} \wedge \mathbf{06}) \wedge \mathbf{86}) \wedge \mathbf{96}) \wedge \mathbf{56})((\mathbf{16} \wedge \mathbf{98})(\mathbf{43} \wedge \mathbf{85}) \wedge \mathbf{95}) \wedge \mathbf{96}) \wedge \mathbf{98})$
$(\mathbf{18} \wedge \mathbf{06})$	$(((\mathbf{23} \wedge \mathbf{65})((\mathbf{05} \wedge \mathbf{67})(\mathbf{19} \wedge \mathbf{70}) \wedge \mathbf{60}) \wedge \mathbf{05})(((((\mathbf{47} \wedge \mathbf{58})(\mathbf{27} \wedge \mathbf{81}) \wedge \mathbf{51})((\mathbf{03} \wedge \mathbf{98})(\mathbf{43} \wedge \mathbf{85}) \wedge \mathbf{95}) \wedge \mathbf{91})(\mathbf{26} \wedge \mathbf{15}) \wedge \mathbf{95})(\mathbf{23} \wedge \mathbf{69}) \wedge \mathbf{65}) \wedge \mathbf{60})$
$(\mathbf{18} \wedge \mathbf{06})$	$(((\mathbf{03} \wedge \mathbf{65})(((\mathbf{96} \wedge \mathbf{04})((\mathbf{12} \wedge \mathbf{67})(\mathbf{19} \wedge \mathbf{70}) \wedge \mathbf{60}) \wedge \mathbf{64})((\mathbf{56} \wedge \mathbf{42})(\mathbf{57} \wedge \mathbf{20}) \wedge \mathbf{40}) \wedge \mathbf{60}) \wedge \mathbf{05})(((\mathbf{23} \wedge \mathbf{89})(\mathbf{43} \wedge \mathbf{95}) \wedge \mathbf{85})(\mathbf{03} \wedge \mathbf{68}) \wedge \mathbf{65}) \wedge \mathbf{60})$
$(\mathbf{23} \wedge \mathbf{69})$	$(((\mathbf{46} \wedge \mathbf{59})((\mathbf{16} \wedge \mathbf{98})((\mathbf{12} \wedge \mathbf{87})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{86}) \wedge \mathbf{96}) \wedge \mathbf{56})(((\mathbf{09} \wedge \mathbf{51})(\mathbf{43} \wedge \mathbf{95}) \wedge \mathbf{91})(\mathbf{07} \wedge \mathbf{15}) \wedge \mathbf{95}) \wedge \mathbf{96})$
$(\mathbf{47} \wedge \mathbf{98})$	$(((((\mathbf{12} \wedge \mathbf{67})((\mathbf{18} \wedge \mathbf{76})((\mathbf{93} \wedge \mathbf{62})(\mathbf{03} \wedge \mathbf{28}) \wedge \mathbf{68}) \wedge \mathbf{78}) \wedge \mathbf{68})(\mathbf{52} \wedge \mathbf{76}) \wedge \mathbf{78})((\mathbf{43} \wedge \mathbf{85})(\mathbf{46} \wedge \mathbf{59}) \wedge \mathbf{89}) \wedge \mathbf{97})(\mathbf{05} \wedge \mathbf{87}) \wedge \mathbf{89})$

$(\mathbf{12} \wedge \mathbf{67})$	$(((\mathbf{48}\wedge\mathbf{57})((\mathbf{07}\wedge(\mathbf{18}\wedge\mathbf{76})\mathbf{5})(\mathbf{56}\wedge(\mathbf{52}\wedge\mathbf{76})\mathbf{0})\wedge\mathbf{76})\wedge\mathbf{56})(\mathbf{43}\wedge\mathbf{75})\wedge\mathbf{76})$
$(\mathbf{49} \wedge \mathbf{56})$	$(((\mathbf{23}\wedge\mathbf{68})(\mathbf{43}\wedge\mathbf{65})\wedge\mathbf{58})(((\mathbf{09}\wedge(\mathbf{03}\wedge\mathbf{98})\mathbf{1})(\mathbf{18}\wedge(\mathbf{16}\wedge\mathbf{98})\mathbf{0})\wedge\mathbf{98})(\mathbf{23}\wedge\mathbf{69})\wedge\mathbf{68})\wedge\mathbf{65})$
$(\mathbf{03} \wedge \mathbf{98})$	$(((\mathbf{47}\wedge\mathbf{58})((\mathbf{05}\wedge\mathbf{87})(\mathbf{23}\wedge\mathbf{79})\wedge\mathbf{89})\wedge\mathbf{59})(((\mathbf{16}\wedge\mathbf{08})(\mathbf{43}\wedge\mathbf{85})\wedge\mathbf{05})((\mathbf{17}\wedge\mathbf{82})(\mathbf{19}\wedge\mathbf{20})\wedge\mathbf{80})\wedge\mathbf{85})\wedge\mathbf{89})$
$(\mathbf{46} \wedge \mathbf{59})$	$((\mathbf{39}\wedge(\mathbf{43}\wedge\mathbf{95})\mathbf{2})(\mathbf{25}\wedge(\mathbf{26}\wedge\mathbf{95})\mathbf{3})\wedge\mathbf{95})$
$(\mathbf{47} \wedge \mathbf{58})$	$((\mathbf{08}\wedge(\mathbf{07}\wedge\mathbf{85})\mathbf{3})(\mathbf{35}\wedge(\mathbf{43}\wedge\mathbf{85})\mathbf{0})\wedge\mathbf{85})$
$(\mathbf{46} \wedge \mathbf{89})$	$(((((\mathbf{43}\wedge\mathbf{95})\mathbf{8}\wedge\mathbf{5}(\mathbf{10}\wedge\mathbf{89}))\mathbf{9}\wedge\mathbf{85})(((\mathbf{32}\wedge\mathbf{09})(\mathbf{16}\wedge\mathbf{98})\wedge\mathbf{08})((\mathbf{05}\wedge\mathbf{97})(\mathbf{38}\wedge\mathbf{70})\wedge\mathbf{90})\wedge\mathbf{98})\wedge\mathbf{95})(\mathbf{47}\wedge\mathbf{58})\wedge\mathbf{98})$
$(\mathbf{46} \wedge \mathbf{89})$	$((\mathbf{8}(\mathbf{69}\wedge\mathbf{4}(\mathbf{46}\wedge\mathbf{89}))\wedge(((\mathbf{23}\wedge\mathbf{79})(\mathbf{61}\wedge\mathbf{98})\wedge\mathbf{78})((\mathbf{43}\wedge\mathbf{95})((\mathbf{09}\wedge\mathbf{51})((\mathbf{78}\wedge\mathbf{14})((\mathbf{57}\wedge\mathbf{40})(\mathbf{12}\wedge\mathbf{07})\wedge\mathbf{47})\wedge\mathbf{17})\wedge\mathbf{57})\wedge\mathbf{97})\wedge\mathbf{98})(\mathbf{69}\wedge\mathbf{48}))(\mathbf{6}(((\mathbf{23}\wedge\mathbf{79})(((\mathbf{47}\wedge\mathbf{58})\mathbf{9}\wedge\mathbf{8}(\mathbf{43}\wedge\mathbf{95}))\mathbf{5}\wedge\mathbf{98})\wedge\mathbf{78})(((\mathbf{18}\wedge\mathbf{76})((\mathbf{65}\wedge\mathbf{92})(\mathbf{05}\wedge\mathbf{27})\wedge\mathbf{97})\wedge\mathbf{96})(\mathbf{12}\wedge\mathbf{67})\wedge\mathbf{97})\wedge\mathbf{98})\wedge\mathbf{48})\wedge\mathbf{98})$

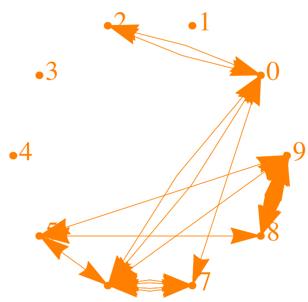


We have $|\{9, 8, 9, 8\}| = 2$ and $9 = 9$ and $8 = 8$ which indicates Case 2 (i).

Resulting 2-cycles:

$(\mathbf{05} \wedge \mathbf{67})$	$(((\mathbf{23}\wedge\mathbf{78})(((((\mathbf{19}\wedge\mathbf{70})\mathbf{6}\wedge\mathbf{0}(\mathbf{12}\wedge\mathbf{67}))\mathbf{7}\wedge\mathbf{60})(\mathbf{52}\wedge\mathbf{76})\wedge\mathbf{70})(\mathbf{18}\wedge\mathbf{06})\wedge\mathbf{76})\wedge\mathbf{68})((\mathbf{03}\wedge\mathbf{98})(\mathbf{23}\wedge\mathbf{79})\wedge\mathbf{78})\wedge\mathbf{76})$
$(\mathbf{12} \wedge \mathbf{67})$	$(((\mathbf{46}\wedge\mathbf{75})(((\mathbf{50}\wedge\mathbf{37})(\mathbf{52}\wedge\mathbf{76})\wedge\mathbf{36})(((\mathbf{29}\wedge\mathbf{63})\mathbf{7}\wedge\mathbf{3}(\mathbf{18}\wedge\mathbf{76}))\mathbf{6}\wedge\mathbf{73})\wedge\mathbf{76})\wedge\mathbf{65})((\mathbf{43}\wedge\mathbf{95})(\mathbf{46}\wedge\mathbf{79})\wedge\mathbf{75})\wedge\mathbf{76})$
$(\mathbf{12} \wedge \mathbf{67})$	$(((\mathbf{46}\wedge\mathbf{75})(((\mathbf{05}\wedge\mathbf{27})(\mathbf{52}\wedge\mathbf{76})\wedge\mathbf{26})(((\mathbf{39}\wedge\mathbf{62})\mathbf{7}\wedge\mathbf{2}(\mathbf{18}\wedge\mathbf{76}))\mathbf{6}\wedge\mathbf{72})\wedge\mathbf{76})\wedge\mathbf{65})((\mathbf{43}\wedge\mathbf{95})(\mathbf{46}\wedge\mathbf{79})\wedge\mathbf{75})\wedge\mathbf{76})$
$(\mathbf{12} \wedge \mathbf{07})$	$((\mathbf{49}\wedge\mathbf{76})((\mathbf{23}\wedge\mathbf{69})((\mathbf{16}\wedge\mathbf{98})((\mathbf{43}\wedge\mathbf{85})((\mathbf{08}\wedge\mathbf{53})((\mathbf{09}\wedge\mathbf{31})((\mathbf{78}\wedge\mathbf{14})((\mathbf{56}\wedge\mathbf{42})(\mathbf{57}\wedge\mathbf{20})\wedge\mathbf{40})\wedge\mathbf{10})\wedge\mathbf{30})\wedge\mathbf{50})\wedge\mathbf{80})\wedge\mathbf{90})\wedge\mathbf{60})\wedge\mathbf{70})$
$(\mathbf{39} \wedge \mathbf{02})$	$(((\mathbf{01}\wedge\mathbf{28})(((\mathbf{57}\wedge\mathbf{40})\mathbf{2}\wedge\mathbf{0}(\mathbf{78}\wedge\mathbf{24}))\mathbf{4}\wedge\mathbf{20})\wedge\mathbf{08})((((\mathbf{43}\wedge\mathbf{85})(\mathbf{16}\wedge\mathbf{98})\wedge\mathbf{95})(\mathbf{03}\wedge\mathbf{58})\wedge\mathbf{98})(\mathbf{01}\wedge\mathbf{29})\wedge\mathbf{28})\wedge\mathbf{20})$

$\mathbf{(39 \wedge 02)}$	$\mathbf{(((10 \wedge 28)((05 \wedge 27)((45 \wedge 73)((87 \wedge 34)(38 \wedge 40) \wedge 30) \wedge 70) \wedge 20) \wedge 08)((16 \wedge 98)(10 \wedge 29) \wedge 28) \wedge 20)}$
$\mathbf{(01 \wedge 89)}$	$\mathbf{(((23 \wedge 69)(47 \wedge 98) \wedge 68)(((46 \wedge 59)((03 \wedge 98)((57 \wedge 80)(52 \wedge 06) \wedge 86) \wedge 96) \wedge 56)((16 \wedge 98)(43 \wedge 85) \wedge 95) \wedge 96) \wedge 98)}$
$\mathbf{(18 \wedge 06)}$	$\mathbf{(((23 \wedge 65)((05 \wedge 67)(19 \wedge 70) \wedge 60) \wedge 05)(((((47 \wedge 58)(27 \wedge 81) \wedge 51)((03 \wedge 98)(43 \wedge 85) \wedge 95) \wedge 91)(26 \wedge 15) \wedge 95)(23 \wedge 69) \wedge 65) \wedge 60)}$
$\mathbf{(18 \wedge 06)}$	$\mathbf{(((03 \wedge 65)(((96 \wedge 04)((12 \wedge 67)(19 \wedge 70) \wedge 60) \wedge 64)((56 \wedge 42)(57 \wedge 20) \wedge 40) \wedge 60) \wedge 05)(((23 \wedge 89)(43 \wedge 95) \wedge 85)(03 \wedge 68) \wedge 65) \wedge 60)}$
$\mathbf{(23 \wedge 69)}$	$\mathbf{(((46 \wedge 59)((16 \wedge 98)((12 \wedge 87)(52 \wedge 76) \wedge 86) \wedge 96) \wedge 56)(((09 \wedge 51)(43 \wedge 95) \wedge 91)(07 \wedge 15) \wedge 95) \wedge 96)}$
$\mathbf{(47 \wedge 98)}$	$\mathbf{(((((12 \wedge 67)((18 \wedge 76)((93 \wedge 62)(03 \wedge 28) \wedge 68) \wedge 78) \wedge 68)(52 \wedge 76) \wedge 78)((43 \wedge 85)(46 \wedge 59) \wedge 89) \wedge 97)(05 \wedge 87) \wedge 89)}$
$\mathbf{(12 \wedge 67)}$	$\mathbf{(((48 \wedge 57)((07 \wedge (18 \wedge 76)5)(56 \wedge (52 \wedge 76)0) \wedge 76) \wedge 56)(43 \wedge 75) \wedge 76)}$
$\mathbf{(49 \wedge 56)}$	$\mathbf{(((23 \wedge 68)(43 \wedge 65) \wedge 58)(((09 \wedge (03 \wedge 98)1)(18 \wedge (16 \wedge 98)0) \wedge 98)(23 \wedge 69) \wedge 68) \wedge 65)}$
$\mathbf{(03 \wedge 98)}$	$\mathbf{(((47 \wedge 58)((05 \wedge 87)(23 \wedge 79) \wedge 89) \wedge 59)(((16 \wedge 08)(43 \wedge 85) \wedge 05)((17 \wedge 82)(19 \wedge 20) \wedge 80) \wedge 85) \wedge 89)}$
$\mathbf{(46 \wedge 59)}$	$\mathbf{((39 \wedge (43 \wedge 95)2)(25 \wedge (26 \wedge 95)3) \wedge 95)}$
$\mathbf{(47 \wedge 58)}$	$\mathbf{((08 \wedge (07 \wedge 85)3)(35 \wedge (43 \wedge 85)0) \wedge 85)}$
$\mathbf{(46 \wedge 89)}$	$\mathbf{((8(69 \wedge 4(46 \wedge 89)) \wedge (((((43 \wedge 95)8 \wedge 5(10 \wedge 89))9 \wedge 85)(((32 \wedge 09)(16 \wedge 98) \wedge 08)((05 \wedge 97)(38 \wedge 70) \wedge 90) \wedge 98) \wedge 95)(47 \wedge 58) \wedge 98)(69 \wedge 48))(6((8(69 \wedge 4(46 \wedge 89)) \wedge (((23 \wedge 79)(61 \wedge 98) \wedge 78)((43 \wedge 95)((09 \wedge 51)((78 \wedge 14)((57 \wedge 40)(12 \wedge 07) \wedge 47) \wedge 17) \wedge 57) \wedge 97) \wedge 98)(69 \wedge 48))(6(((23 \wedge 79)(((47 \wedge 58)9 \wedge 8(43 \wedge 95))5 \wedge 98) \wedge 78)(((18 \wedge 76)((65 \wedge 92)(05 \wedge 27) \wedge 97) \wedge 96)(12 \wedge 67) \wedge 97) \wedge 98) \wedge 48) \wedge 98) \wedge 48) \wedge 98)}$



We have $|\{8, 5, 9, 8\}| = 3$ and $8 = 8$ which indicates Case 3 (iv).

```
[omitted output]
```

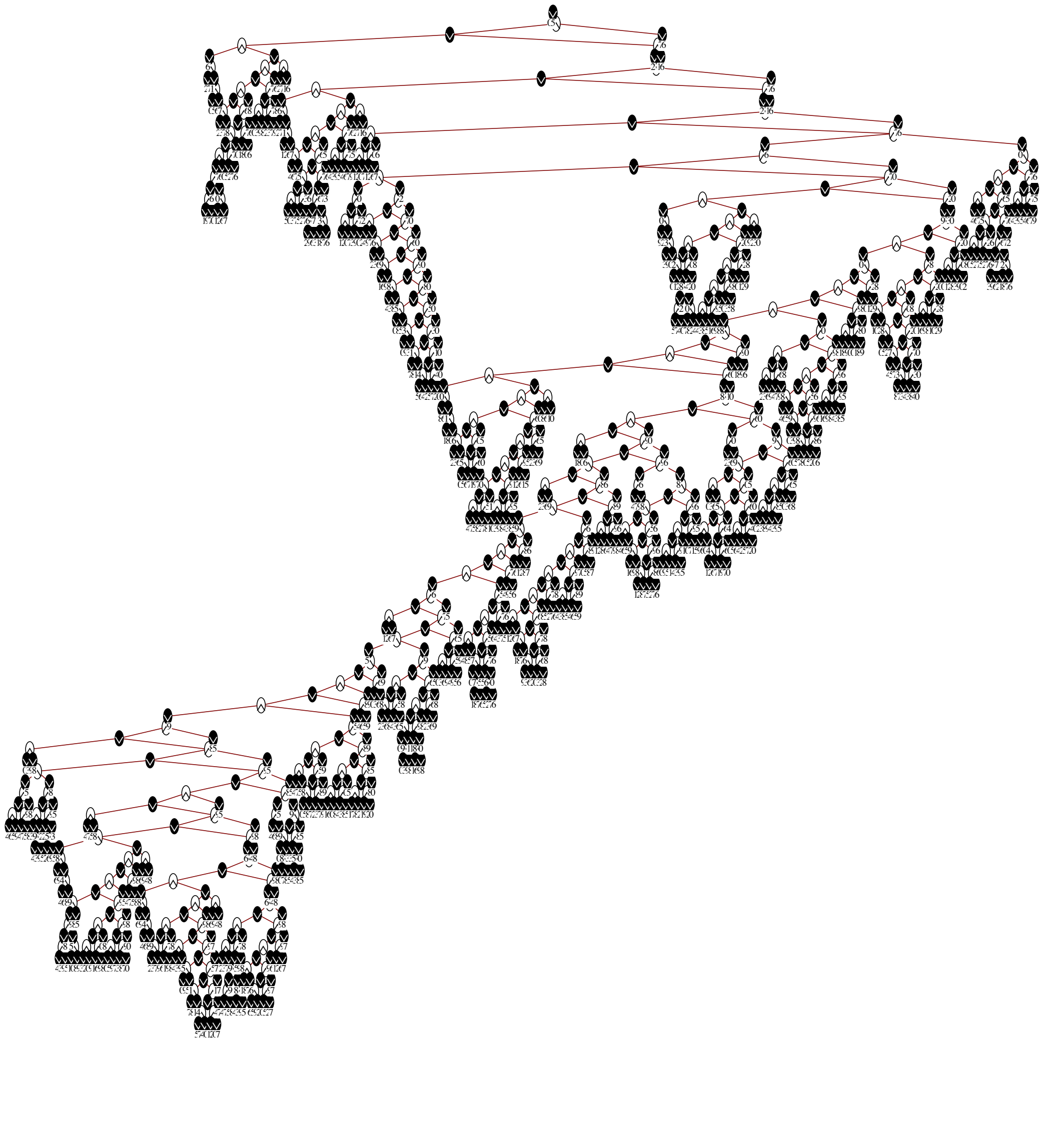
Asking for the final 2-cycle to equal 1 is to ask for the coincidence of both intermediate points of the last remaining length-ratios. Or equivalently:

Result:

$\boldsymbol{05((6(27\wedge 1(05\wedge 67))\wedge(((23\wedge 78)(((((19\wedge 70)6\wedge 0(12\wedge 67))7\wedge 60)(52\wedge 76)\wedge}$
$\boldsymbol{70)(18\wedge 06)\wedge 76)\wedge 68)((03\wedge 98)(23\wedge 79)\wedge 78)\wedge 76)(27\wedge 16))(2((6(27\wedge 1(12\wedge 67))\wedge}$
$\boldsymbol{(((46\wedge 75)(((50\wedge 37)(52\wedge 76)\wedge 36)(((29\wedge 63)7\wedge 3(18\wedge 76))6\wedge 73)\wedge 76)\wedge 65)((43\wedge}$
$\boldsymbol{95)(46\wedge 79)\wedge 75)\wedge 76)(27\wedge 16))(2(((12\wedge 07)(12\wedge 67)\wedge 06)(((((12\wedge 07)(39\wedge 02)\wedge}$
$\boldsymbol{72)0\wedge((49\wedge 76)((23\wedge 69)((16\wedge 98)((43\wedge 85)((08\wedge 53)((09\wedge 31)((78\wedge 14)((56\wedge}$
$\boldsymbol{42)(57\wedge 20)\wedge 40)\wedge 10)\wedge 30)\wedge 50)\wedge 80)\wedge 90)\wedge 60)\wedge 70)2)((0(92\wedge 3(39\wedge 02))\wedge}$
$\boldsymbol{(((01\wedge 28)(((57\wedge 40)2\wedge 0(78\wedge 24))4\wedge 20)\wedge 08)((((43\wedge 85)(16\wedge 98)\wedge 95)(03\wedge 58)\wedge}$
$\boldsymbol{98)(01\wedge 29)\wedge 28)\wedge 20)(92\wedge 30))(9((0(((8(((0(86\wedge 1(18\wedge 06))\wedge(((23\wedge 65)((05\wedge}$
$\boldsymbol{67)(19\wedge 70)\wedge 60)\wedge 05)(((((47\wedge 58)(27\wedge 81)\wedge 51)((03\wedge 98)(43\wedge 85)\wedge 95)\wedge}$
$\boldsymbol{91)(26\wedge 15)\wedge 95)(23\wedge 69)\wedge 65)\wedge 60)(86\wedge 10))(8(((18\wedge 06)(((23\wedge 69)((9(((((12\wedge}$
$\boldsymbol{67)((5(((((03\wedge 98)((((46\wedge 59)(47\wedge 58)\wedge 98)5\wedge((39\wedge(43\wedge 95)2)(25\wedge(26\wedge}$
$\boldsymbol{95)3)\wedge 95)8)(((47\wedge 58)((8(69\wedge 4(46\wedge 89))\wedge(((((43\wedge 95)8\wedge 5(10\wedge 89))9\wedge}$
$\boldsymbol{85)(((32\wedge 09)(16\wedge 98)\wedge 08)((05\wedge 97)(38\wedge 70)\wedge 90)\wedge 98)\wedge 95)(47\wedge 58)\wedge 98)(69\wedge}$
$\boldsymbol{48))(6((8(69\wedge 4(46\wedge 89))\wedge(((23\wedge 79)(61\wedge 98)\wedge 78)((43\wedge 95)((09\wedge 51)((78\wedge}$
$\boldsymbol{14)((57\wedge 40)(12\wedge 07)\wedge 47)\wedge 17)\wedge 57)\wedge 97)\wedge 98)(69\wedge 48))(6(((23\wedge 79)(((47\wedge}$
$\boldsymbol{58)9\wedge 8(43\wedge 95))5\wedge 98)\wedge 78)(((18\wedge 76)((65\wedge 92)(05\wedge 27)\wedge 97)\wedge 96)(12\wedge 67)\wedge}$
$\boldsymbol{97)\wedge 98)\wedge 48)\wedge 98)\wedge 48)\wedge 98)\wedge 95)((46\wedge 89)5\wedge 9((08\wedge(07\wedge 85)3)(35\wedge(43\wedge}$
$\boldsymbol{85)0)\wedge 85))\wedge 85)\wedge 95)\wedge 85)9\wedge(((47\wedge 58)((05\wedge 87)(23\wedge 79)\wedge 89)\wedge 59)(((16\wedge}$
$\boldsymbol{08)(43\wedge 85)\wedge 05)((17\wedge 82)(19\wedge 20)\wedge 80)\wedge 85)\wedge 89)5)(46\wedge 59)\wedge 89)(03\wedge 68)\wedge}$
$\boldsymbol{69)\wedge(((23\wedge 68)(43\wedge 65)\wedge 58)(((09\wedge(03\wedge 98)1)(18\wedge(16\wedge 98)0)\wedge 98)(23\wedge 69)\wedge}$
$\boldsymbol{68)\wedge 65)9)((03\wedge 69)(49\wedge 56)\wedge 59)\wedge 65)\wedge 75)6\wedge(((48\wedge 57)((07\wedge(18\wedge 76)5)(56\wedge}$
$\boldsymbol{(52\wedge 76)0)\wedge 76)\wedge 56)(43\wedge 75)\wedge 76)5)(49\wedge 56)\wedge 76)(12\wedge 87)\wedge 86)\wedge(((((12\wedge}$
$\boldsymbol{67)((18\wedge 76)((93\wedge 62)(03\wedge 28)\wedge 68)\wedge 78)\wedge 68)(52\wedge 76)\wedge 78)((43\wedge 85)(46\wedge 59)\wedge}$
$\boldsymbol{89)\wedge 97)(05\wedge 87)\wedge 89)6)((12\wedge 86)(47\wedge 98)\wedge 96)\wedge 89)\wedge 86)((47\wedge 98)6\wedge 8(((46\wedge}$
$\boldsymbol{59)((16\wedge 98)((12\wedge 87)(52\wedge 76)\wedge 86)\wedge 96)\wedge 56)(((09\wedge 51)(43\wedge 95)\wedge 91)(07\wedge}$
$\boldsymbol{15)\wedge 95)\wedge 96))\wedge 96)\wedge 90)((23\wedge 69)0\wedge 9(((03\wedge 65)(((96\wedge 04)((12\wedge 67)(19\wedge 70)\wedge}$

$\mathbf{60)\wedge 64)((56\wedge 42)(57\wedge 20)\wedge 40)\wedge 60)\wedge 05)(((23\wedge 89)(43\wedge 95)\wedge 85)(03\wedge 68)\wedge 65)\wedge 60))\wedge 60)\wedge 10)\wedge 60)(18\wedge 96)\wedge 90)\wedge(((23\wedge 69)(47\wedge 98)\wedge 68)(((46\wedge 59)((03\wedge 98)((57\wedge 80)(52\wedge 06)\wedge 86)\wedge 96)\wedge 56)((16\wedge 98)(43\wedge 85)\wedge 95)\wedge 96)\wedge 98)0)((18\wedge 90)(01\wedge 89)\wedge 80)\wedge 98)(01\wedge 29)\wedge 28)\wedge(((10\wedge 28)((05\wedge 27)((45\wedge 73)((87\wedge 34)(38\wedge 40)\wedge 30)\wedge 70)\wedge 20)\wedge 08)((16\wedge 98)(10\wedge 29)\wedge 28)\wedge 20)8)((01\wedge 28)(39\wedge 02)\wedge 08)\wedge 20)\wedge 30)\wedge 20)\wedge 70)6\wedge 0(((46\wedge 75)(((05\wedge 27)(52\wedge 76)\wedge 26)(((39\wedge 62)7\wedge 2(18\wedge 76))6\wedge 72)\wedge 76)\wedge 65)((43\wedge 95)(46\wedge 79)\wedge 75)\wedge 76))\wedge 76)\wedge 16)\wedge 76)\wedge 16)\wedge 76)}$

Degree of multiplier monomial: 342



Bibliography

[1] S. Apel. Cayley factorization and the area principle. In *Papers presented at ADG 2012: The 9th International Workshop on Automated Deduction in Geometry, held on September 17–19, 2012 at the University of Edinburgh*, pages 33–41, 2012.

[2] S. Apel and J. Richter-Gebert. Cancellation patterns in automatic geometric theorem proving. In *Automated Deduction in Geometry*, pages 1–33. Springer, 2011.

[3] E. Artin. *Geometric Algebra*. Interscience, 1957.

[4] M. Barnabei, A. Brini, and G.-C. Rota. On the exterior calculus of invariant theory. *Journal of Algebra*, 96(1):120–160, 1985.

[5] C. Berge. Two theorems in graph theory. *Proceedings of the National Academy of Sciences of the United States of America*, 43(9):842—-844, 1957.

[6] J. Blinn. Using tensor diagrams to represent and solve geometric problems. *Siggraph 2002, Course*, 25, 2002.

[7] J. F. Blinn. Jim Blinn's corner-uppers and downers. 2. [graphics]. *Computer Graphics and Applications, IEEE*, 12(3):80–85, 1992.

[8] J. F. Blinn. Polynomial discriminants. 2. Tensor diagrams. *Computer Graphics and Applications, IEEE*, 21(1):86–92, 2001.

[9] J. Bokowski and J. Richter. On the finding of final polynomials. *European Journal of Combinatorics*, 11:21–34, 1990.

[10] J. Bokowski and B. Sturmfels. *Computational Synthetic Geometry*. Springer, 1989.

[11] A. Brini, F. Regonati, and A. G. B. Teolis. Grassmann geometric calculus, invariant theory and superalgebras. In *Algebraic Combinatorics and Computer Science*, pages 151–196. Springer, 2001.

[12] A. Brini and A. G. B. Teolis. Capelli's method of variabili ausiliarie, superalgebras and geometric calculus. In *Invariant Methods in Discrete and Computational Geometry*, pages 59–75. Springer, 1995.

[13] J. Browne. *Grassmann Algebra: Foundations: Exploring Extended Vector Algebra With Mathematica.* CreateSpace Independent Publishing Platform, 2012.

[14] B. Buchberger. Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. *Aequationes mathematicae*, 4(3):374–383, 1970.

[15] B. Buchberger. Grobner bases: An algorithmic method in polynomial ideal theory. *Multidimensional Systems Theory*, pages 184–232, 1985.

[16] J. B. Carrell and J. A. Dieudonné. *Invariant Theory, Old and New.* Academic Press, 1971.

[17] A. Cayley. On the construction of the ninth point of intersection of the cubics which pass through eight given points. In *The Collected Mathematical Papers.* Cambridge University Press, 2009.

[18] W. Chan, G.-C. Rota, and J. A. Stein. The power of positive thinking. In *Invariant Methods in Discrete and Computational Geometry*, pages 1–36. Springer, 1995.

[19] M. Chasles. Construction de la courbe du troisième ordre déterminée par neuf points. *Comptes Rendus des Séances de l'Académie des Sciences*, 36:943–952, 1853.

[20] S.-C. Chou. *Mechanical Geometry Theorem Proving*, volume 41. Springer, 1988.

[21] S.-C. Chou, X.-S. Gao, and J.-Z. Zhang. *Machine Proofs in Geometry: Automated Production of Readable Proofs for Geometry Theorems*, volume 6. World Scientific, 1994.

[22] F. M. Clarke and D. E. Smith. "Essay pour les Coniques" of Blaise Pascal. *Isis*, 10(1):16–20, 1928.

[23] J. Conway and A. Ryba. The Pascal mysticum demystified. *The Mathematical Intelligencer*, pages 1–5, 2012.

[24] H. S. M. Coxeter. *Projective Geometry.* Springer, 2nd edition, 1974.

[25] H. S. M. Coxeter. *The Real Projective Plane.* Springer, 3rd edition, 1993.

[26] H. Crapo. *Computer-Aided Geometric Reasoning: Workshop, 22-26 June 1987, INRIA Sophia-Antipolis (France).* INRIA, 1987.

[27] H. Crapo. Invariant-theoretic methods in scene analysis and structural mechanics. *Journal of Symbolic Computation*, 11(5):523–548, 1991.

[28] H. Crapo. Projective configurations: Their invariants and homology. *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, 95A, Supplement:35–58, 1995.

[29] H. Crapo. On Grassmann's regressive product. In *From Past to Future: Graßmann's Work in Context*, pages 261–274. Springer, 2011.

[30] H. Crapo and J. Richter-Gebert. Automatic proving of geometric theorems. In *Invariant Methods in Discrete and Computational Geometry*, pages 167–196. Springer, 1995.

[31] H. Crapo and G.-C. Rota. *On the Foundations of Combinatorial Theory: Combinatorial Geometries.* MIT Press, Cambridge, 1970.

[32] H. Crapo and G.-C. Rota. The resolving bracket. In *Invariant Methods in Discrete and Computational Geometry*, pages 197–222. Springer, 1995.

[33] H. Crapo and W. Whiteley. Statics of frameworks and motions of panel structures: A projective geometric introduction. *Structural Topology*, 6, 1982.

[34] A. L. Crelle. *Ueber einige Eigenschaften des ebenen geradlinigen Dreiecks rücksichtlich dreier durch die Winkel-Spitzen gezogenen geraden Linien.* 1816.

[35] R. Diestel. *Graph Theory.* Springer, 2nd edition, 2000.

[36] J. Dieudonné. The tragedy of Grassmann. *Linear and Multilinear Algebra*, 8(1):1–14, 1979.

[37] P. Doubilet, G.-C. Rota, and J. Stein. On the foundations of combinatorial theory IX. Combinatorial methods in invariant theory. *Studies in Applied Mathematics*, 53(3):185–216, 1974.

[38] A. W. M. Dress. Duality theory for finite and infinite matroids with coefficients. *Advances in Mathematics*, 59(2):97–123, 1986.

[39] A. W. M. Dress and W. Wenzel. Endliche Matroide mit Koeffizienten. *Bayreuther Mathematische Schriften*, 26:37–98, 1988.

[40] A. W. M. Dress and W. Wenzel. Geometric algebra for combinatorial geometries. *Advances in Mathematics*, 77(1):1–36, 1989.

[41] A. W. M. Dress and W. Wenzel. Grassmann-Plücker relations and matroids with coefficients. *Advances in Mathematics*, 86(1):68–110, 1991.

[42] S. Fèvre and D. Wang. Proving geometric theorems using Clifford algebra and rewrite rules. In *Automated Deduction – CADE-15*, pages 17–32. Springer, 1998.

[43] W. T. Fishback. *Projective and Euclidean Geometry*. John Wiley, 2nd edition, 1969.

[44] H. G. Forder and R. W. Genese. *The Calculus of Extension*. CUP Archive, 1941.

[45] X.-S. Gao, D. Wang, and L. Yang, editors. *Automated Deduction in Geometry: Second International Workshop, ADG'98, Beijing, China, August 1-3, 1998, Proceedings*. Springer, 1999.

[46] H. Gelernter. Realization of a geometry theorem proving machine. In *IFIP Congress*, pages 273–281, 1959.

[47] J. E. Goodman and R. Pollack. On the combinatorial classification of nondegenerate configurations in the plane. *Journal of Combinatorial Theory, Series A*, 29(2):220–235, 1980.

[48] H. Graßmann. *Die Wissenschaft der extensiven Größe oder die Ausdehnungslehre, eine neue mathematische Disciplin: Die lineale Ausdehnungslehre, eine neuer Zweig der Mathematik*. Wigand, 1844.

[49] H. Graßmann. *Die Ausdehnungslehre. Vollständig und in strenger Form bearbeitet*. Verlag von Th. Chr. Fr. Enslin, 1862.

[50] F. D. Grosshans. The work of Gian-Carlo Rota on invariant theory. *Algebra Universalis*, 49(3):213–258, 2003.

[51] F. D. Grosshans, G.-C. Rota, and J. A. Stein. *Invariant Theory and Superalgebras*. American Mathematical Society, 1987.

[52] B. Grünbaum and G. C. Shepard. Ceva, Menelaus, and selftransversality. *Geometriae Dedicata*, 65(2):179–192, 1997.

[53] B. Grünbaum and G. C. Shephard. Ceva, Menelaus, and the area principle. *Mathematics Magazine*, 68(4):254–268, 1995.

[54] B. Grünbaum and G. C. Shephard. Some new transversality properties. *Geometriae Dedicata*, 71(2):179–208, 1998.

[55] G. B. Gurevich. *Foundations of the Theory of Algebraic Invariants*. P. Noordhoff, 1964.

[56] P. Hall. On representatives of subsets. *Journal of the London Mathematical Society*, 10(1):26–30, 1935.

[57] M. J. Hawrylycz. *Geometric Identities in Invariant Theory*. PhD thesis, Massachusetts Institute of Technology, 1995.

[58] O. Hesse. Ueber die Reciprocität der Pascal-Steinerschen und der Kirkman-Cayley-Salmonschen Sätze von dem Hexagrammum mysticum. *Journal für die reine und angewandte Mathematik*, 68:193–207, 1868.

[59] D. Hestenes. *New Foundations for Classical Mechanics*. Springer, 1999.

[60] D. Hestenes. Grassmann's legacy. In *From Past to Future: Graßmann's Work in Context*, pages 243–260. Springer, 2011.

[61] D. Hestenes and R. Ziegler. Projective geometry with Clifford algebra. *Acta Applicandae Mathematica*, 23(1):25–63, 1991.

[62] D. Hilbert. *Grundlagen der Geometrie*. Teubner, 14th edition, 1999.

[63] D. Hilbert and S. Cohn-Vossen. *Geometry and the Imagination*. Chelsea Publishing Company, 1952.

[64] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. *The Annals of Mathematics*, 79(1):109–326, 1964.

[65] W. V. D. Hodge and D. Pedoe. *Methods of Algebraic Geometry*, volume 1. Cambridge University Press, 1994.

[66] L. Hoehn. A Menelaus-type theorem for the pentagram. *Mathematics Magazine*, 66(2):121–123, 1993.

[67] L. Hongbo and C. Minteh. Proving theorems in elementary geometry with Clifford algebraic method. *Chinese Advances in Mathematics*, 26(4):357–371, 1997.

[68] R. Q. Huang, G.-C. Rota, and J. A. Stein. Supersymmetric bracket algebra and invariant theory. In *Topics in Computational Algebra*, pages 193–246. Springer, 1990.

[69] P. Janičić, J. Narboux, and P. Quaresma. The area method. *Journal of Automated Reasoning*, 48(4):489–532, 2012.

[70] D. Kapur. Automated geometric reasoning: Dixon resultants, Gröbner bases, and characteristic sets. In *Automated Deduction in Geometry*, pages 1–36. Springer, 1997.

[71] H. Kraft and C. Procesi. Classical invariant theory, a primer. *Lecture Notes*, 2000.

[72] B. Kutzler and S. Stifter. On the application of Buchberger's algorithm to automated geometry theorem proving. *Journal of Symbolic Computation*, 2(4):389–397, 1986.

[73] H. Li. Automated theorem proving in the homogeneous model with Clifford bracket algebra. In *Applications of Geometric Algebra in Computer Science and Engineering*, pages 69–78. Springer, 2002.

[74] H. Li. Algebraic representation, elimination and expansion in automated geometric theorem proving. In *Automated Deduction in Geometry*, pages 106–123. Springer, 2004.

[75] H. Li. Automated geometric theorem proving, Clifford bracket algebra and Clifford expansions. In *Advances in Analysis and Geometry*, pages 345–363. Springer, 2004.

[76] H. Li. A recipe for symbolic geometric computing: long geometric product, BREEFS and Clifford factorization. In *Proceedings of the 2007 International Symposium on Symbolic and Algebraic Computation*, pages 261–268. ACM, 2007.

[77] H. Li. *Invariant Algebras and Geometric Reasoning.* World Scientific, 2008.

[78] H. Li. Projective geometric theorem proving with Grassmann-Cayley algebra. In *From Past to Future: Graßmann's Work in Context*, pages 275–285. Springer, 2011.

[79] H. Li and Y. Wu. Automated theorem proving in incidence geometry——a bracket algebra based elimination method. In *Automated Deduction in Geometry*, pages 199–227. Springer, 2001.

[80] H. Li and Y. Wu. Automated short proof generation for projective geometric theorems with Cayley and bracket algebras: I. Incidence geometry. *Journal of Symbolic Computation*, 36(5):717–762, 2003.

[81] H. Li, L. Zhao, and Y. Chen. Polyhedral scene analysis combining parametric propagation with calotte analysis. In *Computer Algebra and Geometric Algebra with Applications*, pages 383–402. Springer, 2005.

[82] D. E. Littlewood. Invariant theory, tensors and group characters. *Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences*, pages 305–365, 1944.

[83] S. MacLane. Some interpretations of abstract linear dependence in terms of projective geometry. *American Journal of Mathematics*, 58(1):236–240, 1936.

[84] S. B. Maurer. Matroid basis graphs. I. *Journal of Combinatorial Theory, Series B*, 14(3):216–240, 1973.

[85] T. McMillan and N. L. White. The dotted straightening algorithm. *Journal of Symbolic Computation*, 11(5):471–482, 1991.

[86] A. F. Möbius. *Der barycentrische Calcul*. Johann Ambrosius Barth, 1827.

[87] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. `http://oeis.org/A008965`. Number of necklaces of sets of beads containing a total of n beads.

[88] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. `http://oeis.org/A000031`. Number of n-bead necklaces with 2 colors when turning over is not allowed.

[89] C. B. U. Perwass and D. Hildenbrand. Aspects of geometric algebra in Euclidean, projective and conformal space, 2004.

[90] H.-J. Petsche, A. C. Lewis, J. Liesen, and S. Russ, editors. *From Past to Future: Graßmann's Work in Context: Graßmann Bicentennial Conference, September 2009*. Springer, 2011.

[91] V. L. Popov and E. B. Vinberg. Invariant theory. In *Algebraic Geometry IV*, pages 123–278. Springer, 1994.

[92] J. Richter-Gebert. *On the Realizability Problem of Combinatorial Geometries—Decision Methods*. Dissertation, Technische Hochschule Darmstadt, Fachbereich Mathematik, 1992.

[93] J. Richter-Gebert. Mechanical theorem proving in projective geometry. *Annals of Mathematics and Artificial Intelligence*, 13(1-2):139–172, 1995.

[94] J. Richter-Gebert. Meditations on Ceva's theorem. *The Coxeter Legacy: Reflections and Projections*, 14:227–254, 2006.

[95] J. Richter-Gebert. *Perspectives on Projective Geometry*. Springer, 2011.

[96] J. Richter-Gebert and P. Lebmeir. Diagrams, tensors and geometric reasoning. *Discrete & Computational Geometry*, 42(2):305–334, 2009.

[97] J. Richter-Gebert and D. Wang, editors. *Automated Deduction in Geometry: Third International Workshop, ADG 2000, Zurich, Switzerland, September 25-27, 2000, Revised Papers*. Springer, 2001.

[98] J. F. Ritt. *Differential Algebra.* AMS Bookstore, 1966.

[99] G.-C. Rota and J. A. Stein. Symbolic method in invariant theory. *Proceedings of the National Academy of Sciences*, 83(4):844–847, 1986.

[100] J. Steiner. *Jacob Steiners gesammelte Werke*, volume 1. American Mathematical Society, 1971.

[101] B. Sturmfels. Computational algebraic geometry of projective configurations. *Journal of Symbolic Computation*, 11(5):595–618, 1991.

[102] B. Sturmfels. *Algorithms in Invariant Theory.* Springer, 1993.

[103] B. Sturmfels and N. L. White. Gröbner bases and invariant theory. *Advances in Mathematics*, 76(2):245–259, 1989.

[104] B. Sturmfels and W. Whiteley. On the synthetic factorization of projectively invariant polynomials. *Journal of Symbolic Computation*, 11(5):439–453, 1991.

[105] A. Tarski. A decision method for elementary algebra and geometry. In B. F. Caviness and J. R. Johnson, editors, *Quantifier Elimination and Cylindrical Algebraic Decomposition*, pages 24–84. Springer, 1998.

[106] T.-S. Tay. On the Cayley factorization of calotte conditions. *Discrete & Computational Geometry*, 11(1):97–109, 1994.

[107] H. W. Turnbull. *The Theory of Determinants, Matrices, and Invariants.* Dover Publications, 3rd edition, 1960.

[108] A. Umbach. Automatisches Erzeugen geometrischer Beweise. Diplomarbeit, Eidgenössische Technische Hochschule Zürich, Institut für Theoretische Informatik, 2000.

[109] B. L. Van der Waerden. Über die fundamentalen Identitäten der Invariantentheorie. *Mathematische Annalen*, 95(1):706–735, 1926.

[110] O. Veblen and J. W. Young. *Projective Geometry.* Blaisdell, 1938.

[111] J.-D. Voelke. Le théorème fondamental de la géométrie projective: évolution de sa preuve entre 1847 et 1900. *Archive for History of Exact Sciences*, 62(3):243–296, 2008.

[112] K. G. C. von Staudt. *Beiträge zur Geometrie der Lage.* F. Korn, 1856.

[113] J. Von Zur Gathen and J. Gerhard. *Modern Computer Algebra.* Cambridge University Press, 2013.

[114] D. Wang. Geometry machines: From AI to SMC. In *Artificial Intelligence and Symbolic Mathematical Computation*, pages 213–239. Springer, 1996.

[115] D. Wang, editor. *Automated Deduction in Geometry: International Workshop on Automated Deduction in Geometry, Toulouse, France, September 27-29, 1996: Selected Papers.* Springer, 1997.

[116] D. Wang. Clifford algebraic calculus for geometric reasoning. In *Automated Deduction in Geometry*, pages 115–140. Springer, 1997.

[117] D. Wang. Gröbner bases applied to geometric theorem proving and discovering. In *Gröbner Bases and Applications*, pages 281–304. Cambridge University Press, 1998.

[118] W. Wenzel. *Über die Struktur der Tutte-Gruppe von Matroiden.* Dissertation, Universität Bielefeld, Fakultät für Mathematik, 1986.

[119] H. Weyl. *The Classical Groups: Their Invariants and Representations*, volume 1. Princeton University Press, 1997.

[120] N. L. White. The bracket ring of a combinatorial geometry. I. *Transactions of the American Mathematical Society*, 202:79–95, 1975.

[121] N. L. White. *Combinatorial Geometries.* Cambridge University Press, 1987.

[122] N. L. White. Implementation of the straightening algorithm of classical invariant theory. *Proceedings, Invariant Theory and Tableaux. Minneapolis, MN, Institute for Mathematics and its Applications*, 1988.

[123] N. L. White. Cayley factorization and a straightening algorithm. In *Topics in Computational Algebra*, pages 163–184. Springer, 1990.

[124] N. L. White. Multilinear Cayley factorization. *Journal of Symbolic Computation*, 11(5):421–438, 1991.

[125] N. L. White. Grassmann-Cayley algebra and robotics. *Journal of Intelligent and Robotic Systems*, 11(1-2):91–107, 1994.

[126] N. L. White. A tutorial on Grassmann-Cayley algebra. In *Invariant Methods in Discrete and Computational Geometry*, pages 93–106. Springer, 1995.

[127] N. L. White. Grassmann-Cayley algebra and robotics applications. In *Handbook of Geometric Computing*, pages 629–656. Springer, 2005.

[128] N. L. White and T. McMillan. Cayley factorization. In P. Gianni, editor, *Symbolic and Algebraic Computation*, pages 521–533. Springer, 1989.

[129] N. L. White and W. Whiteley. The algebraic geometry of stresses in frameworks. *SIAM Journal on Algebraic Discrete Methods*, 4(4):481–511, 1983.

[130] W. Whiteley. Logic and invariant theory. I. Invariant theory of projective properties. *Transactions of the American Mathematical Society*, 177:121–139, 1973.

[131] W. Whiteley. Introduction to structural geometry I: Infinitesimal motions and infinitesimal rigidity. *Preprint. Champlain Regional College*, 1977.

[132] W. Whiteley. Logic and invariant theory III: Axiom systems and syzygies. *Journal of the London Mathematical Society (2)*, 15:1–15, 1977.

[133] W. Whiteley. Logic and invariant theory IV. Invariants and syzygies in combinatorial geometry. *Journal of Combinatorial Theory, Series B*, 26(2):251–267, 1979.

[134] W. Whiteley. Invariant computations for analytic projective geometry. *Journal of Symbolic Computation*, 11(5):549–578, 1991.

[135] W. J. Whiteley. *Logic and Invariant Theory.* PhD thesis, Massachusetts Institute of Technology, Department of Mathematics, 1971.

[136] Wolfram Research, Inc. Mathematica Edition: Version 9.0, 2012.

[137] W.-t. Wu. *Mechanical Theorem Proving in Geometrics. Basic Principles.* Springer, 1994.

[138] A. Young. On quantitative substitutional analysis (third paper). *Proceedings of the London Mathematical Society*, s2-28(1):255–292, 1928.

[139] Y. Zou and J. Zhang. Automated generation of readable proofs for constructive geometry statements with the mass point method. In *Automated Deduction in Geometry*, pages 221–258. Springer, 2011.