A Criterion for Invariant Measures of Itô Processes based on the Symbol

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Abstract

An integral criterion for the existence of an invariant measure of an Itô process is developed. This new criterion is based on the probabilistic symbol of the Itô process. In contrast to the standard integral criterion for invariant measures of Markov processes based on the generator, no test functions and hence no information on the domain of the generator is needed.

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1 Introduction

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$ to be a Feller process on $\mathbb{R}^d$ with semigroup $(T_t)_{t\geq 0}$ on $C_0(\mathbb{R}^d)$, i.e.,

$$T_t f(x) = \int_{\mathbb{R}^d} f(y)\mu_t(x, \mathrm{d}y) = \mathbb{E}^x[f(X_t)]$$

where $\mu_t(x, \mathrm{d}y) = \mathbb{P}^x(X_t \in \mathrm{d}y) = \mathbb{P}(X_t \in \mathrm{d}y | X_0 = x)$ are the transition probabilities and $C_0(\mathbb{R}^d)$ are the continuous, real-valued functions on $\mathbb{R}^d$ vanishing at infinity. Then the *infinitesimal generator* $\mathcal{A}$ of X is defined by

$$\mathcal{A}f = \lim_{t\to 0} \frac{T_t f - f}{t}$$

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for all functions f in the domain of $\mathcal{A}$, that is, all f in

$$D(\mathcal{A}) = \left\{ f \in C_0(\mathbb{R}^d),\ \lim_{t\to 0} \frac{T_t f - f}{t} \text{ exists in } \|\cdot\|_\infty \right\}.$$

It is known (see e.g. [17, Thm. 3.37]) that a probability measure μ on $\mathbb{R}^d$ is *invariant* (or *stationary*) for the Feller process X with semigroup $(T_t)_{t\geq 0}$, meaning that

$$\int_{\mathbb{R}^d} T_t f(x) \mathrm{d}\mu(x) = \int_{\mathbb{R}^d} f(x) \mathrm{d}\mu(x), \quad \forall f \in C_0(\mathbb{R}^d), t \geq 0,$$

if and only if

$$\int_{\mathbb{R}^d} \mathcal{A} f(x) \mu(\mathrm{d}x) = 0 \tag{1.1}$$

holds for all f in a core of the generator $\mathcal{A}$.

In the special case that X is a rich Feller process, that is, a Feller process with the property that the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain $D(\mathcal{A})$ of the generator, this generator is a pseudo-differential operator with negative definite symbol $p(x,\xi)$

$$\mathcal{A} f(x) = -\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi) \hat{f}(\xi) \mathrm{d}\xi \tag{1.2}$$

for all $f \in C_c^\infty(\mathbb{R}^d)$ (see e.g. [7, Def. 2.25 and Cor. 2.23]). Hereby the superscript $'$ denotes the transpose of a vector, $\hat{f}(y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix'y} f(x) \mathrm{d}x$ denotes the Fourier transform of f and $C_c^\infty(\mathbb{R}^d)$ is the space of infinitely often continuously differentiable functions on $\mathbb{R}^d$ with compact support. Thus in this case Equations (1.1) and (1.2) together yield that if μ is an invariant law for X, then for all f in $C_c^\infty(\mathbb{R}^d)$ one has

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi) \hat{f}(\xi) \mathrm{d}\xi\, \mu(\mathrm{d}x) = 0. \tag{1.3}$$

Conversely, if (1.3) holds for all f in a core $D \subset C_c^\infty(\mathbb{R}^d) \cap D(\mathcal{A})$ of $\mathcal{A}$, then μ is an invariant law for X.
Further, formally applying Fubini's theorem on Equation (1.3) leads to

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix\xi} p(x,\xi) \mu(\mathrm{d}x) = 0, \quad \text{for } \lambda\text{-all } \ \xi \in \mathbb{R}^d, \tag{1.4}$$

where λ denotes the Lebesgue measure. Observe that (1.4) is an equation directly relating the symbol to the invariant law and does not involve any test functions. This is a big advantage for application of (1.4) compared to (1.1) where we started from.

As a trivial example of application, consider L to be a Lévy process, i.e. we have $p(x,\xi) = \psi_L(\xi)$ where ψ_L denotes the Lévy-Khintchine exponent of L. Then (1.4) is equivalent to $\psi_L(\xi)\phi_\mu(\xi) = 0$ where ϕ_μ denotes the characteristic function of μ. Since this characteristic function is continuous and takes the value 1 at 0 this yields that $\psi_L(\xi) = 0$ for ξ in some neighborhood of 0. This implies that $\psi_L(\xi) = 0$ for all ξ (cf. [22, Lemma 13.9]). Hence $L_t = 0$. Indeed this is the only Lévy process which admits an invariant law (cf. [22, Exercise 19.6]). On the contrary, if $L_t = 0$, (1.4) follows immediately.

Instead of making the above computations rigorous in the case of rich Feller processes, in this paper we will consider a wider class of processes. Therefore recall that in the case of general Markov processes, necessity of Equation (1.1) for μ to be invariant is still given. E.g. [11, Proposition 9.2] states that if a distribution μ is invariant for a Markov process X then (1.1) holds for all f in the domain of the generator of X. Remark that one part of the literature on Markov processes (and so [11]) defines the generator on functions with bounded support, i.e. in $C_b(\mathbb{R}^d) \supset C_0(\mathbb{R}^d)$ which does not fit into the setting we described above and which we will use throughout this paper.
For the converse direction in the Markovian setting, that is, to show sufficiency of (1.1) for μ to be invariant, further assumptions on the generator are needed as discussed e.g. in [10] and [4]. This is the reason why sometimes (e.g. in [1]) a probability measure μ is called *infinitesimal invariant* for a given generator and domain, if and only if (1.1) is fulfilled for all f in the domain of the generator.

General Markov processes do not necessarily have an associated symbol. Hence in this paper we restrict ourselves to Itô processes as they are defined below (Def. 2.2). This class includes the rich Feller processes, but is much more general. For these Itô processes we derive the relation between the symbol of an Itô process as defined below and its invariant law. In particular, our aim is to show in as much generality as possible, that for an Itô process X with symbol $p(x, \xi)$ Equation (1.4) holds if (and only if) μ is an (infinitesimal) invariant law for X.

The paper is outlined as follows. In Section 2 we recall the necessary definitions of Itô processes and symbols as they will be used throughout this paper. Section 3 then shows necessity of (1.4) for μ to be an invariant law for a wide class of Itô processes. Several examples are given and some special cases are studied. Sufficiency of (1.4) for μ to be infinitesimal invariant is then treated in Section 4 and again it is illustrated by special cases. Some rather technical proofs for results in Section 3 have been postponed to the closing Section 5.

2 Preliminaries

In 1998 Jacob came up with the idea to use a probabilistic approach in order to calculate the so-called symbol of a stochastic process [14]. This probabilistic formula was generalized in the same year to rich Feller processes by Schilling [23]. Let us recall the definition.

Definition 2.1. Let $(X_t)_{t\geq 0}$ be a Markov process in $\mathbb{R}^d$. Define for every $x, \xi \in \mathbb{R}^d$ and $t \geq 0$ the quantity

$$\lambda_\xi(x,t) := \frac{\mathbb{E}^x \mathrm{e}^{i(X_t - x)'\xi} - 1}{t}.$$

We call $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by

$$p(x,\xi) := \lim_{t \downarrow 0} \frac{\mathbb{E}^x \mathrm{e}^{i(X_t - x)'\xi} - 1}{t} = \lim_{t \downarrow 0} \lambda_\xi(x,t) \tag{2.1}$$

the *probabilistic symbol* of X if the limit exists for every x, ξ.

For rich Feller processes satisfying the growth condition

$$\sup_{x\in\mathbb{R}^d} |p(x,\xi)| \leq c(1+\|\xi\|^2), \qquad \xi\in\mathbb{R}^d \tag{2.2}$$

for $\|\cdot\|$ denoting an arbitrary submultiplicative norm, the probabilistic symbol and the symbol in Equation (1.2) coincide. This is why we use the letter p and call the new object again 'symbol'. Remark that (2.2) is a standard condition in this context. We refer to [7] for a comprehensive overview on Feller processes and their symbol.
The class of rich Feller processes includes Lévy processes as special case. For these the symbol only depends on ξ and coincides with the *Lévy exponent*, i.e.

$$p(x,\xi) = \psi_L(\xi) := -\log \mathbb{E}[e^{iL_1'\xi}] = -i\ell'\xi + \frac{1}{2}\xi' Q\xi - \int_{\mathbb{R}^d} (\mathrm{e}^{i\xi' y} - 1 - i\xi' y 1_{\{\|y\|<1\}}(y) N(\mathrm{d}y),$$

where $(L_t)_{t\geq 0}$ is a Lévy process with *characteristic triplet* (ℓ, Q, N). For details on Lévy processes in particular we refer to [22].

On the other hand, every rich Feller process is an Itô process (cf. [26, Thm. 3.9]) in the following sense. It is this class we are dealing with in the present paper.

Definition 2.2. An *Itô process* is a strong Markov process, which is a semimartingale w.r.t. every $\mathbb{P}^x$ having semimartingale characteristics of the form

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\ \mathrm{d}s, \qquad && j=1,...,d \\ C_t^{jk}(\omega) &= \int_0^t Q^{jk}(X_s(\omega))\ \mathrm{d}s, \qquad && j,k=1,...,d \\ \nu(\omega;\mathrm{d}s,\mathrm{d}y) &= N(X_s(\omega),\mathrm{d}y)\ \mathrm{d}s \end{aligned} \tag{2.3}$$

for every $x\in\mathbb{R}^d$ with respect to a fixed cut-off function χ. Here $\ell(x) = (\ell^{(1)}(x),...,\ell^{(d)}(x))'$ is a vector in $\mathbb{R}^d$, $Q(x)$ is a positive semi-definite matrix and N is a Borel transition kernel such that $N(x,\{0\})=0$. We call ℓ, Q and $n := \int_{y\neq 0}(1\wedge\|y\|^2)\ N(\cdot,\mathrm{d}y)$ the *differential characteristics* of the process.

Usually we will have to impose the following condition on the differential characteristics of Itô processes.

Definition 2.3. Let X be a Markov process and $f:\mathbb{R}^d\to\mathbb{R}$ be a Borel-measurable function. Then f is called X*-finely continuous* (or *finely continuous*, for short) if the function

$$t\mapsto f(X_t) = f\circ X_t \tag{2.4}$$

is right continuous at zero $\mathbb{P}^x$-a.s. for every $x\in\mathbb{R}^d$.

Remark 2.4. Fine continuity is usually introduced in a different way (see [6, Section II.4] and [12]). However, by [6, Thm. 4.8] this is equivalent to (2.4). It is this kind of right continuity which we will use in our proofs. Let us mention that this assumption is very weak, even weaker than ordinary continuity.

In [8] the class of Itô processes in the sense of Definition 2.2 has been characterized as the set of solutions of very general SDEs. In particular, as mentioned already, the class of Itô processes contains the class of rich Feller processes. The following example (cf. [25, Example 5.2]) shows that this inclusion is strict: The process

$$X_t^x = \begin{cases} x+t & \text{under } \mathbb{P}^x \text{ for } x<0, \\ 0 & \text{under } \mathbb{P}^x \text{ for } x=0, \\ x-t & \text{under } \mathbb{P}^x \text{ for } x>0, \end{cases} \quad t \geq 0,$$

is an Itô process with bounded and finely continuous differential characteristics which is not Feller.

For Itô processes we can compute the probabilistic symbol. In particular we even have the following connection of probabilistic symbol and generator.

Lemma 2.5. *If the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain $D(\mathcal{A})$ of the generator $\mathcal{A}$ of an Itô process X, the representation* (1.2) *holds for every $f \in C_c^\infty(\mathbb{R}^d)$, where $p(x,\xi)$ is the probabilistic symbol.*

Proof. An operator $\overline{\mathcal{A}}$ with domain $D(\overline{\mathcal{A}})$ is called the extended generator of X, if $D(\overline{\mathcal{A}})$ consists of those Borel measurable functions f for which there exists a $(\mathcal{B}^d)^*$-measurable function $\overline{\mathcal{A}}f$ such that the process

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \overline{\mathcal{A}}f(X_s)\ \mathrm{d}s$$

is a local martingale (cf. [9, Def. 7.1]). Here, $(\mathcal{B}^d)^*$ denotes as usual the universally measurable sets (see e.g. [6, Section 0.1]). We have $C_c^\infty(\mathbb{R}^d) \subset D(\mathcal{A}) \subset D(\overline{\mathcal{A}})$ and $\overline{\mathcal{A}}|(D(\mathcal{A})) = \mathcal{A}$ by Dynkin's formula. By [9, Thm. 7.16] we obtain that

$$\begin{aligned} -\overline{\mathcal{A}}f(x) &= -\sum_{j\leq d} \ell^{(j)}(x) D_j f(x) - \frac{1}{2}\sum_{j,k\leq d} Q^{j,k} D_{j,k} f(x) \\ &\qquad - \int_{y\neq 0} f(x+y) - f(x) - \chi(y)\sum_{j\leq d} y^{(j)} D_j f(x) N(x,\mathrm{d}y) \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\widehat{f}(\xi)\ \mathrm{d}\xi \end{aligned}$$

for every $f \in C_c^\infty(\mathbb{R}^d)$ which is (1.2). □

To end this section, let us also mention that there exists a formula to calculate the symbol even for the wider class of homogeneous diffusions with jumps in the sense of [15] (cf. [27, Thm. 3.6]). However, this formula uses stopping times and can not be used in our considerations. In the proof of Theorem 3.3 below we have to use the classical version of the probabilistic symbol presented above.
For the even wider class of Hunt semimartingales the limit (2.1) is not defined and hence the symbol does not exist any more (cf. [19]).

3 Necessity

We start by showing the necessity of (1.4) for μ to be an invariant law.

Theorem 3.1. *Let $(X_t)_{t\geq 0}$ be an Itô process with generator $\mathcal{A}$ whose domain $D(\mathcal{A})$ contains the test functions $C_c^\infty(\mathbb{R}^d)$ and with symbol $p(x,\xi)$. Assume μ is an invariant measure for X such that $\int_{\mathbb{R}^d} |p(x,\xi)|\mu(\mathrm{d}x) < \infty$. Then*

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = 0 \quad \forall \xi \in \mathbb{R}^d.$$

Proof. It is well known, that for an invariant measure μ it holds

$$\int_{\mathbb{R}^d} \mathcal{A}f(x)\mu(\mathrm{d}x) = 0 \quad \text{for all} \quad f \in D(\mathcal{A}).$$

By Lemma 2.5 the generator $\mathcal{A}$ admits the representation (1.2) for all $f \in C_c^\infty(\mathbb{R}^d)$. Hence using Fubini's theorem we obtain for all $f \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} 0 = \int_{\mathbb{R}^d} \mathcal{A}f(x)\mu(\mathrm{d}x) &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\hat{f}(\xi)\mathrm{d}\xi\mu(\mathrm{d}x) \\ &= -\int_{\mathbb{R}^d} \hat{f}(\xi)\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x)\mathrm{d}\xi. \end{aligned}$$

This yields that for λ-a.a. ξ it holds $\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = 0$. Since by hypothesis $p(x,\xi)$ is absolutely integrable with respect to μ, its Fourier transform with respect to μ is continuous. This gives the claim. □

Example 3.2. Let $(X_t)_{t\geq 0}$ be a generalized Ornstein-Uhlenbeck process, defined as the unique solution of

$$\mathrm{d}X_t = X_{t-}\mathrm{d}U_t + \mathrm{d}L_t, \quad t \geq 0,$$

for two independent Lévy processes $(U_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$. It has been shown in [3, Thm. 3.1] that X is a Feller process, that the domain of its generator contains $C_c^\infty(\mathbb{R})$ and that $C_c^\infty(\mathbb{R})$ is a core for the generator.
Further it follows from the results in [24] that the symbol of X is given by

$$p(x,\xi) = \psi_U(x\xi) + \psi_L(\xi), \quad x,\xi \in \mathbb{R}.$$

Assume μ is a probability measure on $\mathbb{R}$ such that $\int x^2\mu(\mathrm{d}x) < \infty$. Then due to the specific form of the symbol $\int_{\mathbb{R}^d} |p(x,\xi)|\mu(\mathrm{d}x) < \infty$ is automatically fulfilled. Hence we see from Theorem 3.1 that

$$\int_{\mathbb{R}} \mathrm{e}^{ix\xi}\psi_U(x\xi)\mu(\mathrm{d}x) = -\psi_L(\xi)\phi_\mu(\xi) \tag{3.1}$$

is a necessary condition for μ to be invariant for X. Remark that Equation (3.1) has also been obtained in [3, Thm. 4.1].

In general we have only little or no information on the domain of the generator of an Itô process which makes the above theorem inapplicable. We will see in the following, that it is possible to obtain similar results without any information on the domain of the generator by using the probabilistic definition of the symbol directly. The first case we consider is the Itô process with bounded and finely continuous differential characteristics. Examples include Feller processes satisfying the growth condition (2.2). Although the boundedness assumption seems to be rather restrictive, this class of processes already contains various interesting examples, which are used in stochastic modeling and mathematical statistics (see e.g. Example 3.5 below).

Theorem 3.3. *Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Itô process with bounded, finely continuous differential characteristics which admits an invariant law μ and whose symbol is given by $p(x,\xi)$, $x,\xi\in\mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = 0 \quad \textit{for all} \ \ \xi \in \mathbb{R}^d.$$

For the proof of Theorem 3.3, we need the following Lemma, which also shows the form of the symbol in the given setting.

Lemma 3.4. *Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Itô process with bounded, finely continuous differential characteristics. For every $\xi\in\mathbb{R}^d$ the limit*

$$p(x,\xi) := \lim_{t\downarrow 0} \frac{\mathbb{E}^x \mathrm{e}^{i(X_t-x)'\xi} - 1}{t} = \lim_{t\downarrow 0} \lambda_\xi(x,t)$$

exists and the functions λ_ξ are globally bounded in x (and t) for every $\xi\in\mathbb{R}^d$. As the limit we obtain

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(\mathrm{e}^{iy'\xi} - 1 - iy'\xi\cdot\chi(y)\right) N(x,\mathrm{d}y). \tag{3.2}$$

The proof of Lemma 3.4 is postponed to Section 5.

Proof of Theorem 3.3. Let X_∞ be a random variable such that $\mu = \mathcal{L}(X_\infty)$ where "$\mathcal{L}$" stands for "law of". Then using Lemma 3.4 we obtain by Lebesgue's dominated convergence theorem

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) &= \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} \lim_{t\to 0} \mathbb{E}^x \left[\frac{\mathrm{e}^{i(X_t-x)'\xi}-1}{t}\right]\mu(\mathrm{d}x) \\ &= \lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi}\,\mathbb{E}^x[\mathrm{e}^{i(X_t-x)'\xi}-1]\mu(\mathrm{d}x),\end{aligned}$$

with

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi}\mathbb{E}^x\left[\mathrm{e}^{i(X_t-x)'\xi}-1\right]\mu(\mathrm{d}x) &= \int_{\mathbb{R}^d} \mathbb{E}^x\left[\mathrm{e}^{iX_t'\xi}\right]\mu(\mathrm{d}x) - \int_{\mathbb{R}^d}\mathrm{e}^{ix'\xi}\mu(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \mathbb{E}\left[\mathrm{e}^{iX_t'\xi}|X_0=x\right]\mu(\mathrm{d}x) - \mathbb{E}\left[\mathrm{e}^{iX_\infty'\xi}\right] \\ &= \mathbb{E}\left[\mathrm{e}^{iX_\infty'\xi}\right] - \mathbb{E}\left[\mathrm{e}^{iX_\infty'\xi}\right] \\ &= 0.\end{aligned}$$

□

The following example is taken from [16, Section 5.7]. It is derived by a transformation from a classical example due to Barndorff-Nielsen [5].

Example 3.5. Let $(X_t)_{t\geq 0}$ be the unique solution of the SDE

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \sigma(X_t)\ \mathrm{d}W_t, \quad t \geq 0,$$

with $X_0 = x_0$, a standard Brownian motion $(W_t)_{t\geq 0}$,

$$b(x) = -\left(\vartheta + \frac{c^2}{2\cosh(x)}\right)\frac{\sinh(x)}{\cosh^2(x)} \quad \text{and} \quad \sigma(x) = \frac{c}{\cosh(x)}$$

where $\vartheta, c > 0$. For $x_0 \in \mathbb{R}$ the *scale density* and the *speed density* of X are then given by

$$s(x) := \exp\left(-2\int_{x_0}^{x} \frac{b(u)}{\sigma^2(x)}\ \mathrm{d}u\right) \quad \text{and} \quad m(x) := \frac{1}{\sigma^2(x)s(x)}.$$

Since $\int s(x)\ \mathrm{d}x = \infty$ while $M := \int m(x)\ \mathrm{d}x < \infty$ we are in the setting of [16, Section 5.2]. There, the authors restate the well-known fact that the unique stationary distribution of the process X in this case admits the density

$$\pi(x) = \frac{m(x)}{M}.$$

By our above result, this means

$$\int \mathrm{e}^{ix\xi}(|\sigma(x)|^2 |\xi|^2 - ib(x)\xi)\pi(x)\ \mathrm{d}x = 0$$

since $p(x,\xi) = |\sigma(x)|^2 |\xi|^2 - ib(x)\xi$ is the symbol of X by [24, Thm. 3.1].

3.1 Lévy driven SDEs

In general, we can not drop the boundedness assumption on the differential characteristics which we have used in Theorem 3.3. This assumption corresponds to bounded coefficients of the SDEs whose solutions are the considered Itô processes. However, in some cases we are able to generalize our result as shown in the following proposition where a linearly growing cofficient is allowed. Another possible extension of Theorem 3.3 will be stated in Proposition 3.12 below.

Proposition 3.6. *Let $(X_t)_{t\geq 0}$ be the unique solution of the SDE*

$$\mathrm{d}X_t = -aX_t\mathrm{d}t + \Phi(X_{t-})\mathrm{d}L_t, \quad t \geq 0,$$

where $a \in \mathbb{R}$, $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is bounded and locally Lipschitz continuous and $(L_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^n$ satisfying $\mathbb{E}\|L_1\| < \infty$. Then X is an Itô process and for every $\xi \in \mathbb{R}^d$ the limit $p(x,\xi) = \lim_{t\downarrow 0} \lambda_\xi(x,t)$ exists and the functions λ_ξ are globally bounded in x (and t) for every $\xi \in \mathbb{R}^d$. Furthermore if μ is an invariant law of X, then

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi}(\psi_L(\Phi(x)'\xi) + iax'\xi)\mu(\mathrm{d}x) = 0 \quad \textit{for all } \xi \in \mathbb{R}^d.$$

The proof of Proposition 3.6 is postponed to Section 5.

Remarks 3.7. (i) The structure of the symbol in Proposition 3.6 is not surprising. It is exactly what one would get for the generalized symbol which uses stopping times (see [27, Thm. 3.6]). However, it is important to see that for the classic probabilistic symbol, as we have to use it in our context, the convergence of the $\lambda_\xi(x,t)$ is uniform.

(ii) The class of Itô processes studied in the above Proposition is a subset of the class of solutions of delay equations for which stationarity was treated in [21]. Remark that even in their general paper, the authors have to impose the boundedness condition on Φ [21, Assumption 4.1(c)].

(iii) Although the assumption of L having a finite first moment seems very restrictive, it can not be released.
As an example consider the Lévy-driven Ornstein-Uhlenbeck (OU) process

$$X_t = \mathrm{e}^{-\lambda t}\left(X_0 + \int_{(0,t]} \mathrm{e}^{\lambda s}\mathrm{d}L_s\right), \quad t \geq 0,$$

for $\lambda > 0$ and a symmetric α-stable, real-valued Lévy process $(L_t)_{t\geq 0}$, $\alpha \in [1,2]$. Then $(X_t)_{t\geq 0}$ solves the SDE $\mathrm{d}X_t = -\lambda X_{t-}\mathrm{d}t + \mathrm{d}L_t$ and by results in [24] its symbol is given by

$$p(x,\xi) = i\lambda x\xi + \psi_L(\xi) = i\lambda x\xi + |\xi|^\alpha.$$

Since L is α-stable, we have $E|L_1|^r < \infty$ if and only if $r < \alpha$. Hence in particular $E\log^+|L_1| < \infty$, such that by [18, Thm. 2.1] the process $(X_t)_{t\geq 0}$ admits a stationary solution with distribution $\mathcal{L}(X_\infty)$. By [2, Thm. 3.1] it further follows that $E|X_\infty|^r < \infty$ for all $r < \alpha$ and - as it was already remarked in [2] - this result is sharp. Hence $\mathcal{L}(X_\infty)$ does not necessarily have a finite α-th moment and the integral (1.4) does not necessarily exist.

In [1] the authors studied the absolutely continuous invariant measures of solutions of SDEs of the form

$$\mathrm{d}X_t = \sqrt{2a_1}\mathrm{d}W_t + \beta(X_t)\mathrm{d}t + a_2\mathrm{d}Z_t, \quad t \geq 0, \tag{3.3}$$

for suitable coefficients, a standard Brownian motion $(W_t)_{t\geq 0}$ and a pure-jump process $(Z_t)_{t\geq 0}$ with stable (type) Lévy measure.

For such SDEs with additive Lévy noise, we obtain the following proposition whose proof can be found in Section 5. Remark that this proposition would follow directly from Proposition 3.6, if we imposed $\mathbb{E}\|L\| < \infty$ and $\mathbb{E}\|Z\| < \infty$.

Proposition 3.8. *Let $(X_t)_{t\geq 0}$ be the unique solution of the SDE*

$$\mathrm{d}X_t = b\mathrm{d}Z_t + \Phi(X_{t-})\mathrm{d}L_t, \quad t \geq 0,$$

where $(L_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are independent Lévy processes which are n- respectively d-dimensional, $b \in \mathbb{R}$ and $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is bounded and locally Lipschitz. Then X is an Itô process, for every $\xi \in \mathbb{R}^d$ the limit $p(x,\xi) = \lim_{t\downarrow 0}\lambda_\xi(x,t)$ exists and the functions λ_ξ

are globally bounded in x (and t) for every $\xi \in \mathbb{R}^d$. Furthermore if μ is an invariant law of X, then

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} \left(\psi_L(\Phi(x)'\xi) + \psi_Z(b\xi)\right)\mu(\mathrm{d}x) = 0 \quad \textit{for all} \;\; \xi \in \mathbb{R}^d.$$

Further, restricting to the class of processes whose absolutely continuous invariant measures have been studied in [1], we obtain the following corollary.

Corollary 3.9. *Let $(X_t)_{t\geq 0}$ be the unique solution of the SDE (3.3) where $a_1, a_2 \geq 0$, $a_1 + a_2 > 0$, $\beta : \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable, locally Lipschitz, bounded and its Fourier transform exists, $(W_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued standard Brownian motion, and $(Z_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued pure-jump process with Lévy measure $\nu_\alpha(\mathrm{d}y) := |y|^{-(d+\alpha)}\mathrm{d}y$, for $\alpha \in (0,2)$. Assume $\mu(\mathrm{d}x) = \rho(x)\mathrm{d}x$ is invariant for X. Then*

$$(a_1|\xi|^2 - a_2 c_\alpha |\xi|^\alpha)\hat{\rho}(\xi) + i\xi' \cdot \widehat{\beta\rho}(\xi) = 0, \quad \xi \in \mathbb{R}^d, \tag{3.4}$$

where $c_\alpha = \int_{\mathbb{R}^d\setminus\{0\}} (\cos(u'y) - 1)\nu_\alpha(\mathrm{d}y)$ for u some unit vector in $\mathbb{R}^d$ and $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$.

Proof. Let $\mu(\mathrm{d}x) = \rho(x)\mathrm{d}x$ be invariant for X. Since β is bounded and locally Lipschitz, and Z and W only act additively, we can apply Proposition 3.8 in the given setting and obtain that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\rho(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi}(-i\beta(x)'\xi + a_1|\xi|^2 - a_2 c_\alpha|\xi|^\alpha)\rho(x)\mathrm{d}x \\ &= -i\xi' \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi}\beta(x)\rho(x)\mathrm{d}x + (a_1|\xi|^2 - a_2 c_\alpha|\xi|^\alpha) \int_{\mathbb{R}^d} \mathrm{e}^{ix\xi}\rho(x)\mathrm{d}x. \end{aligned}$$

Substituting ξ by $-\xi$ we further observe that this is equivalent to

$$0 = i\xi'\widehat{\beta\rho}(\xi) + (a_1|\xi|^2 - a_2 c_\alpha|\xi|^\alpha)\hat{\rho}(\xi)$$

which is (3.4). □

Remark 3.10. In [1, Prop. 3.1] an invariance condition for the type of process considered in Corollary 3.9 is given. Unfortunately, in their computations, the authors missed to use complex conjugates when applying Parseval's identity, resulting in a wrong sign [1, between Eqs. (3.4) and (3.5)]. The (corrected) condition stated there is then automatically fulfilled if (3.4) holds.

Another interesting class of processes in our setting are processes with factorizing symbol as they appear in the following corollary.

Corollary 3.11. *Let $(X_t)_{t\geq 0}$ be the unique solution of the SDE*

$$\mathrm{d}X_t = \Phi(X_{t-})\mathrm{d}L_t, \quad t \geq 0,$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is bounded and locally Lipschitz continuous and $(L_t)_{t\geq 0}$ is a symmetric α-stable, real-valued Lévy process, $\alpha \in (0,1)$.
Assume X has an absolutely continuous invariant law $\mu(\mathrm{d}x) = \rho(x)\mathrm{d}x$. Then $\Phi(x)\rho(x) = 0$ for λ-a.a. x.

Proof. We know from Proposition 3.6 that the symbol of X is given by $p(x,\xi) = |\Phi(x)|^\alpha|\xi|^\alpha$. Suppose now that X had an invariant law $\mu(\mathrm{d}x) = \rho(x)\mathrm{d}x$. By our above results this yields

$$0 = \int \mathrm{e}^{ix\xi} p(x,\xi)\mu(\mathrm{d}x) = |\xi|^\alpha \int \mathrm{e}^{ix\xi}|\Phi(x)|^\alpha \rho(x)\mathrm{d}x \quad \forall \xi \in \mathbb{R}.$$

Thus $f(\xi) := \int \mathrm{e}^{ix\xi}|\Phi(x)|^\alpha\rho(x)\mathrm{d}x = 0$ for ξ non-zero. Since Φ was assumed to be bounded, the product $|\Phi(x)|^\alpha\rho(x) \leq C \cdot \rho(x)$ is integrable. Hence its Fourier transform is in C_0. Thus $f(0) = 0$ which gives the claim. □

3.2 Diffusions

In case of a Brownian motion as driving process instead of a general Lévy process, we can even drop the boundedness condition on the coefficient Φ.

Proposition 3.12. *Let $(X_t)_{t\geq 0}$ be the unique solution of the SDE*

$$\mathrm{d}X_t = -aX_t\mathrm{d}t + \Phi(X_t)\mathrm{d}W_t, \quad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is an n-dimensional standard Brownian motion, $a \in \mathbb{R}$ and $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is continuously differentiable with bounded derivative. Then X is an Itô process, for every $\xi \in \mathbb{R}^d$ the limit $p(x,\xi) = \lim_{t\downarrow 0} \lambda_\xi(x,t)$ exists and the functions λ_ξ are globally bounded in x (and t) for every $\xi \in \mathbb{R}^d$. Furthermore if μ is an invariant law for X, then

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi}\left(|\Phi(x)'\xi|^2 + iax'\xi\right)\mu(\mathrm{d}x) = 0 \quad \textit{for all } \ \xi \in \mathbb{R}^d.$$

We postpone the proof of this Proposition to Section 5.

Example 3.13. Consider the Ornstein-Uhlenbeck (OU) process driven by a one-dimensional, standard Brownian motion $(W_t)_{t\geq 0}$ with parameters $\lambda > 0$, $\sigma > 0$ and starting random variable X_0, independent of $(W_t)_{t\geq 0}$, which is given by

$$X_t = \mathrm{e}^{-\lambda t}\left(X_0 + \int_{(0,t]} \mathrm{e}^{\lambda s}\sigma\mathrm{d}W_s\right), \quad t \geq 0.$$

This process is a special case of the generalized OU process introduced in Example 3.2. In particular X solves the SDE $\mathrm{d}X_t = -\lambda X_{t-}\mathrm{d}t + \sigma\mathrm{d}W_t$ such that we can now obtain directly from Proposition 3.12 that the symbol of the OU process is

$$p(x,\xi) = i\lambda x\xi + |\sigma\xi|^2.$$

It is well-known that X admits a stationary distribution $\mu = \mathcal{L}(X_\infty)$ which is normal with mean 0 and variance $\frac{\sigma^2}{\lambda}$. Using the symbol and this stationary distribution yields

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{e}^{ix\xi} p(x,\xi)\mu(\mathrm{d}x) &= \lambda\xi E[iX_\infty \mathrm{e}^{iX_\infty\xi}] + \sigma^2\xi^2 E[\mathrm{e}^{iX_\infty\xi}] \\ &= \lambda\xi\phi'_{X_\infty}(\xi) + \sigma^2\xi^2\phi_{X_\infty}(\xi) \\ &= -\xi^2\sigma^2 \exp\left(-\frac{\sigma^2}{2\lambda}\xi^2\right) + \sigma^2\xi^2 \exp\left(-\frac{\sigma^2}{2\lambda}\xi^2\right) \\ &= 0\end{aligned}$$

such that Equation (1.4) is fulfilled.

Example 3.14. Let $(X_t)_{t\geq 0}$ be the stochastic exponential of a Brownian motion $(W_t)_{t\geq 0}$ with variance σ^2, i.e. $X_t = 1 + \int_{(0,t]} X_{t-}\mathrm{d}W_t$, then we have [20, Theorem II.37]

$$X_t = \exp\left(W_t - \frac{1}{2}\sigma^2 t\right), \quad t \geq 0.$$

From Proposition 3.12 we obtain the corresponding symbol of the stochastic exponential as

$$p(x,\xi) = x^2\xi^2.$$

Now, if X had a stationary distribution $\mu = \mathcal{L}(X_\infty)$ with finite second moment, this would fulfill (1.4), i.e.

$$0 = \int_{\mathbb{R}} \mathrm{e}^{ix\xi} x^2\xi^2 \mu(\mathrm{d}x) = \xi^2 \phi''_{X_\infty}(\xi)$$

and hence we had $\phi''_{X_\infty}(\xi) = 0$ for all ξ which is only possible if $\phi_{X_\infty} = 1$ for all ξ. Thus μ had to be the Dirac measure at 0. But obviously $X_t > 0$ for all $t \geq 0$ which leads to a contradiction.

4 Sufficiency

As mentioned in the Introduction, for Markov processes which are not rich Feller, Equation (1.1) is in general not sufficient to prove invariance of the law μ. Therefore we restrict ourselves in this section to infinitesimal invariant laws, i.e. to laws which fulfill (1.1).

Theorem 4.1. *Let $(X_t)_{t\geq 0}$ be an Itô process with generator $\mathcal{A}$ whose domain $D(\mathcal{A})$ contains the test functions $C_c^\infty(\mathbb{R}^d)$ and with symbol $p(x,\xi)$. Assume there exists a probability measure μ such that $\int_{\mathbb{R}^d} |\mathrm{e}^{ix'\xi} p(x,\xi)|\mu(\mathrm{d}x) < \infty$ and $\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = 0$. Then*

$$\int_{\mathbb{R}^d} \mathcal{A}f(x)\mu(\mathrm{d}x) = 0 \quad \textit{for all} \quad f \in C_c^\infty(\mathbb{R}^d).$$

Proof. By Lemma 2.5 the generator $\mathcal{A}$ admits the representation (1.2) for all $f \in C_c^\infty(\mathbb{R}^d)$. Hence using Fubini's theorem we obtain for all $f \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} 0 &= -\int_{\mathbb{R}^d} \hat{f}(\xi) \int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x)\mathrm{d}\xi \\ &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\hat{f}(\xi)\mathrm{d}\xi\mu(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \mathcal{A}f(x)\mu(\mathrm{d}x). \end{aligned}$$

□

The above theorem can easily be adapted to specific classes of symbols. We illustrate this with the following corollary.

Corollary 4.2. *Let $(X_t)_{t\geq 0}$ be the unique solution of the SDE*

$$\mathrm{d}X_t = -aX_t\mathrm{d}t + \Phi(X_t)\mathrm{d}W_t, \quad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, $a \in \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded derivative and such that $|\Phi(x)| \leq K|x|^{\kappa/2}$ for some constant K and some $\kappa \in [1,2]$. Further suppose that the domain $D(\mathcal{A})$ of the generator of X contains the test functions $C_c^\infty(\mathbb{R}^d)$. Assume there exists a probability distribution μ such that $\int_{\mathbb{R}^d} \mathrm{e}^{ix'\xi} p(x,\xi)\mu(\mathrm{d}x) = 0$ and $\int \|x\|^\kappa \mu(\mathrm{d}x) < \infty$. Then

$$\int_{\mathbb{R}^d} \mathcal{A}f(x)\mu(\mathrm{d}x) = 0 \quad \textit{for all} \quad f \in C_c^\infty(\mathbb{R}^d).$$

Proof. We know from Proposition 3.12 that the symbol of X is given by $p(x,\xi) = |\Phi(x)'\xi|^2 + iax'\xi$. Hence we have $\int_{\mathbb{R}^d} |\mathrm{e}^{ix'\xi} p(x,\xi)|\mu(\mathrm{d}x) \leq \int_{\mathbb{R}^d} |p(x,\xi)|\mu(\mathrm{d}x) < \infty$. By Theorem 4.1 this gives the claim. □

Example 4.3. Let $(X_t)_{t\geq 0}$ be a generalized Ornstein-Uhlenbeck process, as defined in Example 3.2. Then by the same arguments as in Example 3.2 we see from Theorem 4.1 together with [17, Thm. 3.37] that

$$\int_{\mathbb{R}} \mathrm{e}^{ix\xi} \psi_U(x\xi)\mu(\mathrm{d}x) = -\psi_L(\xi)\phi_\mu(\xi), \quad \xi \in \mathbb{R}, \tag{4.5}$$

is also sufficient for μ to be an invariant law for X with finite second moment.
In the special case of the Ornstein-Uhlenbeck process as introduced in Example 3.13 it is sufficient to suppose μ to be integrable and Equation (3.1) reduces to

$$-\lambda\xi\phi_\mu'(\xi) = \sigma^2\xi^2\phi_\mu(\xi), \quad \xi \in \mathbb{R}.$$

This differential equation can be uniquely solved by $\phi_\mu(\xi) = \exp(-\frac{\sigma^2}{2\lambda}\xi^2)$ (compare Example 3.13).

5 Proofs

Proof of Lemma 3.4. We give the one dimensional proof, since the multidimensional version works alike; only the notation becomes more involved. Let $x, \xi \in \mathbb{R}$. First we use Itô's formula under the expectation and obtain

$$\frac{1}{t}\mathbb{E}^x\left(\mathrm{e}^{i(X_t-x)\xi}-1\right) = \frac{1}{t}\mathbb{E}^x\left(\int_{0+}^t i\xi\mathrm{e}^{i(X_{s-}-x)\xi}\,\mathrm{d}X_s\right) \tag{I}$$

$$+\frac{1}{t}\mathbb{E}^x\left(\frac{1}{2}\int_{0+}^t -\xi^2\mathrm{e}^{i(X_{s-}-x)\xi}\,\mathrm{d}[X,X]_s^c\right) \tag{II}$$

$$+\frac{1}{t}\mathbb{E}^x\left(\mathrm{e}^{-ix\xi}\sum_{0<s\le t}\left(\mathrm{e}^{i\xi X_s}-\mathrm{e}^{i\xi X_{s-}}-i\xi\mathrm{e}^{i\xi X_{s-}}\Delta X_s\right)\right). \tag{III}$$

In what follows we will deal with the terms one-by-one. To calculate term (I) we use the canonical decomposition of a semimartingale (see [15, Thm. II.2.34]) which we write as follows

$$X_t = X_0 + X_t^c + \int_0^t \chi(y)y\left(\mu^X(\cdot;\mathrm{d}s,\mathrm{d}y)-\nu(\cdot;\mathrm{d}s,\mathrm{d}y)\right) + \check{X}_t(\chi) + B_t(\chi), \tag{5.6}$$

where $\check{X}_t = \sum_{s\le t}(\Delta X_s(1-\chi(\Delta X_s)))$. Therefore, term (I) can be rewritten as

$$\frac{1}{t}\mathbb{E}^x\Big(\int_{0+}^t i\xi\mathrm{e}^{i(X_{s-}-x)\xi}\,\mathrm{d}\Big(\underbrace{X_t^c}_{\text{(IV)}}+\underbrace{\int_0^t \chi(y)y\left(\mu^X(\cdot;\mathrm{d}s,\mathrm{d}y)-\nu(\cdot;\mathrm{d}s,\mathrm{d}y)\right)}_{\text{(V)}}$$

$$+\underbrace{\check{X}_t(\chi)}_{\text{(VI)}}+\underbrace{B_t(\chi)}_{\text{(VII)}}\Big)\Big)$$

We use the linearity of the stochastic integral mapping. First we prove for term (IV)

$$\mathbb{E}^x\int_{0+}^t i\xi\mathrm{e}^{i(X_{s-}-x)\xi}\,\mathrm{d}X_s^c = 0.$$

The integral $\mathrm{e}^{i(X_{t-}-x)\xi}\bullet X_t^c := \int_{0+}^t \mathrm{e}^{i(X_{s-}-x)\xi}\,\mathrm{d}X_s^c$ is a local martingale, since X_t^c is a local martingale. To see that it is indeed a martingale, we calculate

$$\left[\mathrm{e}^{i(X-x)\xi}\bullet X^c, \mathrm{e}^{i(X-x)\xi}\bullet X^c\right]_t = \int_0^t (\mathrm{e}^{i(X_s-x)\xi})^2\,\mathrm{d}[X^c,X^c]_s = \int_0^t \left((\mathrm{e}^{i(X_s-x)\xi})^2 Q(X_s)\right)\mathrm{d}s.$$

The last term is uniformly bounded in ω and therefore, finite for every $t \ge 0$. Hence, $\mathrm{e}^{i(X_t-x)\xi}\bullet X_t^c$ is an L^2-martingale which is zero at zero and therefore, its expected value is constantly zero.
The same is true for the integrand (V): We show that the function $H_{x,\xi}(\omega,s,y) := \mathrm{e}^{i(X_{s-}-x)\xi}\cdot y\chi(y)$ is in the class F_p^2 of Ikeda and Watanabe (see [13, Section II.]), that is,

$$\mathbb{E}^x\int_0^t\int_{y\neq 0}\left|\mathrm{e}^{i(X_{s-}-x)\xi}\cdot y\chi(y)\right|^2\nu(\cdot;\mathrm{d}s,\mathrm{d}y) < \infty.$$

In order to prove this, we observe

$$\mathbb{E}^x \int_0^t \int_{y\neq 0} \left|\mathrm{e}^{i(X_{s-}-x)\xi}\right|^2 \cdot |y\chi(y)|^2 \, \nu(\cdot;\mathrm{d}s,\mathrm{d}y) = \mathbb{E}^x \int_0^t \int_{y\neq 0} |y\chi(y)|^2 \ N(X_s,\mathrm{d}y)\ \mathrm{d}s.$$

Since we have by hypothesis $\left\|\int_{y\neq 0}(1\wedge y^2)\ N(\cdot,\mathrm{d}y)\right\|_\infty < \infty$ this expected value is finite. Therefore, the function $H_{x,\xi}$ is in F_p^2 and we conclude that

$$\begin{aligned}\int_0^t \mathrm{e}^{i(X_{s-}-x)\xi}\ \mathrm{d}\left(\int_0^s \int_{y\neq 0} \chi(y)y\ (\mu^X(\cdot;dr,\mathrm{d}y) - \nu(\cdot;dr,\mathrm{d}y))\right)\\ = \int_0^t \int_{y\neq 0} \left(\mathrm{e}^{i(X_{s-}-x)\xi}\chi(y)y\right)(\mu^X(\cdot;\mathrm{d}s,\mathrm{d}y) - \nu(\cdot;\mathrm{d}s,\mathrm{d}y))\end{aligned}$$

is a martingale. The last equality follows from [15, Thm. I.1.30].

Now we deal with term (II). Here we have

$$[X,X]_t^c = [X^c,X^c]_t = C_t = (Q(X_t)\bullet t)$$

and therefore,

$$\frac{1}{2}\int_{0+}^t -\xi^2\mathrm{e}^{i(X_{s-}-x)\xi}\ \mathrm{d}[X,X]_s^c = -\frac{1}{2}\xi^2\int_0^t \mathrm{e}^{i(X_{s-}-x)\xi}Q(X_s)\ \mathrm{d}s. \tag{5.7}$$

Since Q is finely continuous and bounded we obtain by dominated convergence

$$-\lim_{t\downarrow 0}\frac{1}{2}\xi^2\frac{1}{t}\mathbb{E}^x\int_0^t \mathrm{e}^{i(X_s-x)\xi}Q(X_s)\ \mathrm{d}s = -\frac{1}{2}\xi^2 Q(x).$$

For the finite variation part of the first term, i.e, (VII), we obtain analogously

$$\lim_{t\downarrow 0} i\xi\frac{1}{t}\mathbb{E}^x\int_0^t \mathrm{e}^{i(X_s-x)\xi}\ell(X_s)\ \mathrm{d}s = i\xi\ell(x). \tag{5.8}$$

Finally we have to deal with the various jump parts. At first we write the sum in (III) as an integral with respect to the jump measure μ^X of the process:

$$\begin{aligned}
&\mathrm{e}^{-ix\xi}\sum_{0<s\le t}\left(\mathrm{e}^{iX_s\xi} - \mathrm{e}^{iX_{s-}\xi} - i\xi\mathrm{e}^{i\xi X_{s-}}\Delta X_s\right)\\
&\quad= \mathrm{e}^{-ix\xi}\sum_{0<s\le t}\left(\mathrm{e}^{iX_{s-}\xi}(\mathrm{e}^{i\xi\Delta X_s} - 1 - i\xi\Delta X_s)\right)\\
&\quad= \int_{]0,t]\times\mathbb{R}^d}\left(\mathrm{e}^{i(X_{s-}-x)\xi}(\mathrm{e}^{i\xi y} - 1 - i\xi y)1_{\{y\neq 0\}}\right)\ \mu^X(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\quad= \int_{]0,t]\times\{y\neq 0\}}\left(\mathrm{e}^{i(X_{s-}-x)\xi}(\mathrm{e}^{i\xi y} - 1 - i\xi y\chi(y) - i\xi y\cdot(1-\chi(y)))\right)\ \mu^X(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\quad= \int_{]0,t]\times\{y\neq 0\}}\left(\mathrm{e}^{i(X_{s-}-x)\xi}(\mathrm{e}^{i\xi y} - 1 - i\xi y\chi(y))\right)\ \mu^X(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\qquad+ \int_{]0,t]\times\{y\neq 0\}}\left(\mathrm{e}^{i(X_{s-}-x)\xi}(-i\xi y\cdot(1-\chi(y)))\right)\ \mu^X(\cdot;\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

The last term cancels with the one we left behind from (I), given by (VI). For the remainder-term we get:

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x \int_{]0,t]\times\{y\neq 0\}} \left(\mathrm{e}^{i(X_{s-}-x)\xi}(\mathrm{e}^{i\xi y}-1-i\xi y\chi(y))\right)\,\mu^X(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\quad=\frac{1}{t}\mathbb{E}^x \int_{]0,t]\times\{y\neq 0\}} \left(\mathrm{e}^{i(X_{s-}-x)\xi}(\mathrm{e}^{i\xi y}-1-i\xi y\chi(y))\right)\,\nu(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\quad=\frac{1}{t}\mathbb{E}^x \int_{]0,t]\times\{y\neq 0\}} \underbrace{\left(\mathrm{e}^{i(X_{s-}-x)\xi}(\mathrm{e}^{i\xi y}-1-i\xi y\chi(y))\right)}_{:=g(s-,\cdot)}\,N(X_s,\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}$$

Here we have used the fact that it is possible to integrate with respect to the compensator of a random measure instead of the measure itself, if the integrand is in F_p^1 (see [13, Section II.3]). The function $g(s,\omega)$ is measurable and bounded by our assumption, since $\left|\mathrm{e}^{i\xi y}-1-i\xi y\chi(y)\right|\leq C_\xi\cdot(1\wedge|y|^2)$, for a constant $C_\xi>0$. Hence $g\in F_p^1$.
Again by bounded convergence we obtain

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^x\int_0^t \mathrm{e}^{i(X_s-x)\xi}\int_{y\neq 0}\left(\mathrm{e}^{iy\xi}-1-iy\xi\chi(y)\right)\,N(X_s,\mathrm{d}y)\,\mathrm{d}s\\
&\qquad=\int_{y\neq 0}\left(\mathrm{e}^{iy\xi}-1-iy\xi\chi(y)\right)\,N(x,\mathrm{d}y).
\end{aligned}\tag{5.9}$$

This is the last part of the symbol. Here we have used the continuity assumption on $N(x,\mathrm{d}y)$.
Considering the above calculations, in particular (5.7), (5.8) and (5.9) we obtain

$$\begin{aligned}
\left|\frac{\mathbb{E}^x\mathrm{e}^{i(X_t-x)'\xi}-1}{t}\right| &= \left|i\xi\frac{1}{t}\mathbb{E}^x\int_0^t \mathrm{e}^{i(X_{s-}-x)\xi}\ell(X_s)\,\mathrm{d}s-\frac{1}{2}\xi^2\frac{1}{t}\mathbb{E}^x\int_0^t \mathrm{e}^{i(X_{s-}-x)\xi}Q(X_s)\,\mathrm{d}s\right.\\
&\quad\left.+\frac{1}{t}\mathbb{E}^x\int_0^t \mathrm{e}^{i(X_{s-}-x)\xi}\int_{y\neq 0}\left(\mathrm{e}^{iy\xi}-1-iy\xi\chi(y)\right)\,N(X_s,\mathrm{d}y)\,\mathrm{d}s\right|\\
&\leq|\xi|\,\frac{t}{t}\,\|\ell\|_\infty+\xi^2\frac{t}{2t}\,\|Q\|_\infty+C_\xi\frac{t}{t}\left\|\int_{y\neq 0}(1\wedge|y|^2)\,N(\cdot,\mathrm{d}y)\right\|_\infty,
\end{aligned}$$

a bound which is uniform in t and x. □

For the proof of Proposition 3.6 we need the following lemma. Observe that for $\kappa\geq 2$ and in the one-dimensional case, this lemma follows directly from [20, Thm. V.67].

Lemma 5.1. *Let $\kappa\geq 1$ and suppose $(L_t)_{t\geq 0}$ is a Lévy process such that $\mathbb{E}[\|L_1\|^\kappa]<\infty$. Assume X_0 is a random variable, independent of L, such that $\mathbb{E}[\|X_0\|^\kappa]<\infty$. Then the process $(X_t)_{t\geq 0}$ defined by*

$$X_t=X_0-a\int_{(0,t]}X_{s-}\mathrm{d}s+\int_{(0,t]}\Phi(X_{s-})\mathrm{d}L_s$$

where $\Phi:\mathbb{R}^d\to\mathbb{R}^{d\times n}$ is bounded, locally Lipschitz and $a\in\mathbb{R}$, fulfills

$$\mathbb{E}[\sup_{0\leq t\leq 1}\|X_t\|^\kappa]<\infty.$$

Proof. Observe that

$$\|X_t\|^\kappa \quad \leq \quad 4^\kappa \|X_0\|^\kappa + 4^\kappa |a|^\kappa \left\| \int_{(0,t]} X_{s-} \mathrm{d}s \right\|^\kappa + 2^\kappa \left\| \int_{(0,t]} \Phi(X_{s-}) \mathrm{d}L_s \right\|^\kappa$$

and hence for any $0 \leq s \leq 1$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq s} \|X_t\|^\kappa\right] \\
&\leq 4^\kappa \mathbb{E}[\|X_0\|^\kappa] + 4^\kappa |a|^\kappa \mathbb{E}\left[\sup_{0\leq t\leq s} \left\| \int_{(0,t]} X_u \mathrm{d}u \right\|^\kappa\right] + 2^\kappa \mathbb{E}\left[\sup_{0\leq t\leq s} \left\| \int_{(0,t]} \Phi(X_{u-}) \mathrm{d}L_u \right\|^\kappa\right].
\end{aligned}$$

By [20, Lemma on bottom of page 345] we have that

$$\mathbb{E}\left[\sup_{0\leq t\leq s} \| \int_{(0,t]} X_u \mathrm{d}u \|^\kappa\right] \leq \int_{(0,s]} E\left[\|X_u\|^\kappa\right] \mathrm{d}u.$$

On the other hand it follows from an easy multivariate extension of [2, Lemma 6.1] that $\mathbb{E}[\sup_{0\leq t\leq 1} \| \int_{(0,t]} \Phi(X_{u-}) \mathrm{d}L_u \|^\kappa]$ is finite, say $\leq K$, under the given conditions. Thus

$$\mathbb{E}\left[\sup_{0\leq t\leq s} \|X_t\|^\kappa\right] \quad \leq \quad 4^\kappa \mathbb{E}\left[\|X_0\|^\kappa\right] + 2^\kappa K + 4^\kappa |a|^\kappa \int_{(0,s]} \mathbb{E}\left[\sup_{0\leq v\leq u} \|X_v\|^\kappa\right] \mathrm{d}u.$$

Now it follows from Gronwall's inequality (cf. [20, Thm. V.68]) that

$$\mathbb{E}\left[\sup_{0\leq t\leq 1} \|X_t\|^\kappa\right] \leq (4^\kappa \mathbb{E}\left[\|X_0\|^\kappa\right] + 2^\kappa K) \mathrm{e}^{4^\kappa |a|^\kappa} < \infty$$

as we had to show. □

Proof of Proposition 3.6. It is well known that the given SDE has a unique solution under the given conditions (cf. e.g. [15, IX.6.7.]). To keep notation simple, we give only the proof for $d = n = 1$. Fix $x, \xi \in \mathbb{R}$ and apply Itô's formula to the function $\exp(i(\cdot - x)\xi)$:

$$\begin{aligned}
\frac{1}{t}\mathbb{E}^x\left(\mathrm{e}^{i(X_t - x)\xi} - 1\right) = \frac{1}{t}\mathbb{E}^x \Bigg(&\int_{0+}^t i\xi\, \mathrm{e}^{i(X_{s-}-x)\xi}\, \mathrm{d}X_s - \frac{1}{2}\int_{0+}^t \xi^2\, \mathrm{e}^{i(X_{s-}-x)\xi}\, \mathrm{d}[X,X]_s^c \\
&+ \mathrm{e}^{-ix\xi} \sum_{0<s\leq t} \left(\mathrm{e}^{iX_s\xi} - \mathrm{e}^{iX_{s-}\xi} - i\xi \mathrm{e}^{iX_{s-}\xi}\Delta X_s\right) \Bigg).
\end{aligned} \tag{5.10}$$

For the first term we get

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x \int_{0+}^t \left(i\xi\, \mathrm{e}^{i(X_{s-}-x)\xi}\right) \mathrm{d}X_s \\
&\quad = \frac{1}{t}\mathbb{E}^x \int_{0+}^t \left(i\xi\, \mathrm{e}^{i(X_{s-}-x)\xi}\right) \mathrm{d}\left(\int_0^s \Phi(X_{r-})\, \mathrm{d}L_r\right) - \frac{1}{t}\mathbb{E}^x \int_{0+}^t \left(i\xi\, \mathrm{e}^{i(X_{s-}-x)\xi} a X_{s-}\right) \mathrm{d}s \\
&\quad = \frac{1}{t}\mathbb{E}^x \int_{0+}^t \left(i\xi\, \mathrm{e}^{i(X_{s-}-x)\xi}\Phi(X_{s-})\right) \mathrm{d}(\ell s) && (5.11) \\
&\qquad + \frac{1}{t}\mathbb{E}^x \int_{0+}^t \left(i\xi\, \mathrm{e}^{i(X_{s-}-x)\xi}\Phi(X_{s-})\right) \mathrm{d}\left(\sum_{0<r\leq s} \Delta L_r 1_{\{|\Delta Z_r|\geq 1\}}\right) && (5.12) \\
&\qquad - \frac{1}{t}\mathbb{E}^x \int_{0+}^t \left(i\xi\, \mathrm{e}^{i(X_{s-}-x)\xi} a X_{s-}\right) \mathrm{d}s, && (5.13)
\end{aligned}$$

where we have used the Lévy-Itô decomposition of the Lévy process. Since the integrand is bounded, the martingale parts of the Lévy process yield martingales whose expected value is zero.

Now we deal with (5.12). Adding this integral to the third expression on the right-hand side of (5.10) we obtain

$$\begin{aligned}\frac{1}{t}\,\mathbb{E}^x \sum_{0<s\le t} &\left(\mathrm{e}^{i(X_{s-}-x)\xi}\left(\mathrm{e}^{i\Phi(X_{s-})\Delta L_s\xi}-1-i\xi\Phi(X_{s-})\Delta L_s 1_{\{|\Delta X_s|<1\}}\right)\right)\\ &\xrightarrow{t\downarrow 0}\int_{\mathbb{R}\setminus\{0\}}\left(\mathrm{e}^{i\Phi(x)y\xi}-1-i\xi\Phi(x)y1_{\{|y|<1\}}\right)N(\mathrm{d}y).\end{aligned}$$

The calculation above uses the same well known results about integration with respect to integer valued random measures as the proof of Lemma 3.4. In the case of a Lévy process the compensator is of the form $\nu(\cdot\,;\mathrm{d}s,\mathrm{d}y)=N(\mathrm{d}y)\,\mathrm{d}s$, see [13, Example II.4.2].

For the first drift part (5.11) we obtain

$$\frac{1}{t}\,\mathbb{E}^x\int_{0+}^t\left(i\xi\cdot\mathrm{e}^{i(X_{s-}-x)\xi}\Phi(X_{s-})\ell\right)\,\mathrm{d}s=i\xi\ell\cdot\mathbb{E}^x\frac{1}{t}\int_0^t\left(\mathrm{e}^{i(X_s-x)\xi}\Phi(X_s)\right)\,\mathrm{d}s\xrightarrow{t\downarrow 0}i\xi\ell\Phi(x).$$

To deal with the second expression on the right-hand side of (5.10), we first have to calculated the square bracket of the process

$$[X,X]_t^c=\left(\left[\int_0^\cdot\Phi(X_{r-})\mathrm{d}L_r,\ \int_0^\cdot\Phi(X_{r-})\mathrm{d}L_r\right]_t^c\right)=\int_0^t\Phi(X_{s-})^2\,\mathrm{d}(Qs).$$

Let us remark that $\int aX_s\,\mathrm{d}s$ is negligible in calculating the square bracket $[X,X]_t$ since it is quadratic pure jump by [20, Thm. II.26]. Now we can calculate the limit for the second term of (5.10)

$$\begin{aligned}\frac{1}{2t}\,\mathbb{E}^x\int_{0+}^t\left(-\xi^2\,\mathrm{e}^{i(X_{s-}-x)\xi}\right)\,\mathrm{d}[X,X]_s^c&=\frac{1}{2t}\,\mathbb{E}^x\int_{0+}^t\left(-\xi^2\,\mathrm{e}^{i(X_{s-}-x)\xi}\right)\,\mathrm{d}\left(\int_0^s(\Phi(X_{r-}))^2Q\,\mathrm{d}r\right)\\ &=-\frac{1}{2}\,\xi^2Q\,\mathbb{E}^x\left(\frac{1}{t}\int_0^t\left(\mathrm{e}^{i(X_s-x)\xi}\Phi(X_s)^2\,\mathrm{d}s\right)\right)\qquad(5.14)\\ &\xrightarrow{t\downarrow 0}-\frac{1}{2}\,\xi^2Q\,\Phi(x)^2.\end{aligned}$$

While in these three parts, due to the boundedness of Φ, the uniform boundedness of the approximants is trivially seen, we have to be a bit more careful in dealing with the term (5.13): we use the Lemma 5.1 and the fact $\sup_{0\le t\le 1}\mathbb{E}|X_t|\le\mathbb{E}[\sup_{0\le t\le 1}|X_t|]$. In order to show that

$$ai\xi\mathbb{E}^x\int_0^1\mathrm{e}^{i(X_{ts}-x)\xi}X_{(ts)-}\,\mathrm{d}s\underset{t\downarrow 0}{\longrightarrow}ai\xi x$$

in a uniformly bounded way, we consider

$$\begin{aligned}\mathbb{E}^x&\int_0^1\left|\mathrm{e}^{i(X_{st}-x)\xi}X_{st}-\mathrm{e}^{i(X_{st}-x)\xi}x+\mathrm{e}^{i(X_{st}-x)\xi}x-x\right|\,\mathrm{d}s\\ &=\mathbb{E}^x\int_0^1\left|\mathrm{e}^{i(X_{st}-x)\xi}(X_{st}-x)+(\mathrm{e}^{i(X_{st}-x)\xi}-1)x\right|\,\mathrm{d}s.\end{aligned}$$

By $\mathbb{E}^x |X_{st} - x| \leq c < \infty$ we can interchange the order of integration. In the end we obtain

$$\begin{aligned}
p(x,\xi) &= -i\ell(\Phi(x)\xi) + iax\xi + \frac{1}{2}(\Phi(x)\xi)Q(\Phi(x)\xi) \\
&\qquad - \int_{y\neq 0} \left(\mathrm{e}^{i(\Phi(x)\xi)y} - 1 - i(\Phi(x)\xi)y \cdot 1_{\{|y|<1\}}(y)\right) N(\mathrm{d}y) \\
&= \psi_L(\Phi(x)\xi) + iax\xi.
\end{aligned}$$

Let us remark that in the multi-dimensional case the matrix $\Phi(x)$ has to be transposed, that is, the symbol of the solution is $\psi_L(\Phi(x)'\xi) + iax'\xi$.

The result now follows as in the proof of Theorem 3.3. □

Proof of Proposition 3.8. In order to prove this result we can mimic the previous proof. In this case, $a = 0$, the driving Lévy process is $(Z', L')' \in \mathbb{R}^{d+n}$ and the bounded coefficient is $(b \cdot I_d, \Phi(x)) \in \mathbb{R}^{d\times(d+n)}$ where I_d denotes the d-dimensional identity matrix. Since a is zero the respective part of the proof - the one where the moment assumption is needed - can be omitted. □

Proof of Proposition 3.12. The proof works perfectly analogue to the one of Proposition 3.6 with the following exception: from [20, Thm. V.67] we obtain that $\sup_{0\leq t\leq 1} E(X_t)^2$ is finite. This is needed in order to obtain the convergence in (5.14) in a uniformly bounded way. In the present setting, Q is the identity matrix. □

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