

Universal quantum state merging

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Dedicated to the memory of Rudolf Ahlswede

Abstract

We determine the optimal entanglement rate of quantum state merging when assuming that the state is unknown except for its membership in a certain set of states. We find that merging is possible at the lowest rate allowed by the individual states. Additionally, we establish a lower bound for the classical cost of state merging under state uncertainty. To this end we give an elementary proof for the cost in case of a perfectly known state which makes no use of the “resource framework”. As applications of our main result, we determine the capacity for one-way entanglement distillation if the source is not perfectly known. Moreover, we give another achievability proof for the entanglement generation capacity over compound quantum channels.

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1 Introduction

Quantum state merging was introduced by Horodecki, Oppenheim, and Winter [19, 20] in order to quantify the amount of partial quantum information contained in bipartite quantum states. I.e. for a bipartite i.i.d. quantum source with generic state ρ_{AB} shared by communication parties A (“sender”) and B (“receiver”), we want to know how much quantum communication is needed per copy when transferring A’s share to B so that source output is completely available to B.

A convenient way of measuring quantum communication within this scenario is quantifying the entanglement cost (cf. Ref. [20]): The parties A and B are free to use local operations together with certain exchange of classical messages (LOCC) and moreover they may use preexistent pure entanglement. The protocol performs state merging and produces/returns pure entanglement. The optimal rate for this task was determined in Ref. [20] as the conditional von Neumann entropy $S(A|B)$. In this way, the conditional von Neumann entropy obtains an operational interpretation as the net amount of entanglement resources needed to merge the states. Moreover, the puzzling fact that for some states $S(A|B) < 0$ can occur can be interpreted naturally within the state merging paradigm: Merging protocols achieving negative rates produce rather than consume entanglement during the process.

Additionally, the optimal (i.e. the lowest possible) classical communication rate for a merging procedure achieving quantum rate $S(A|B)$ was determined in Ref. [20] as well. It turned out that $I(A; E)$, the quantum mutual information between the A -part and an environment purifying ρ_{AB} is optimal in this case.

Another important aspect is that many other protocols can be derived (mostly by reduction) from quantum state merging. Here we just mention some of the examples from [20] like distributed compression, quantum source coding with side information at the decoder, and entanglement generation over quantum multiple access channels.

However, these results rely on the assumption of idealized conditions. The authors of Ref. [20] assumed the source to be memoryless and perfectly known. Both of these conditions will hardly be fulfilled in real-life communication settings.

In this paper, we drop the second condition and determine the optimal average cost of entanglement per copy under partial ignorance of the state to be merged. We consider a scenario, where statistical properties of the ensemble emitted by the source are not perfectly known to the merging partners. Rather it is assumed, that they only know that the state belongs to a certain set of states. Thus they have to use a protocol which works well for every member of this set. This model can be seen as a source analogue to the notion of compound quantum channels which were considered in Refs.[7] and [8].

Our main technical result is a generalization of the original one-shot bound given in Ref. [20], which respects state uncertainty. The question of the optimal classical communication cost in this case is addressed as well.

The results of this paper gather their relevance from the fact, that other related communication protocols can be obtained by modifying state merging protocols. Our generalization to sets of states can be used to generate protocols which are successful in the corresponding “compound” scenarios. These in turn are stepping stones to tackle the much more involved “arbitrarily varying” models. If one considers, for example, the problem of determining capacities of arbitrarily varying channels, it is well known that good codes for particular compound channels can be transformed in good random codes for arbitrarily varying

channels via Ahlswede’s robustification technique [3]. The robustification technique can be applied in the quantum case as well. It is exactly this idea that was employed in Ref. [4] in order to determine the random code capacity for entanglement transmission over arbitrarily varying quantum channels. This in turn can be used to show that either the deterministic classical capacity of the arbitrarily varying quantum channel is zero or the deterministic and random code capacities for entanglement transmission of these channels are equal, a quantum version of Ahlswede’s famous dichotomy [2].

We mention this here, because this is up to date the only method allowing us to prove such results. The ingenious and very direct method to prove the coding theorem for classical arbitrarily varying channels developed by Csiszár and Narayan [11] does not carry over to the quantum case.

1.1 Related work

The present result relies, as it was in the single state case, on a variant of the so-called decoupling approach, an idea which originally appeared in Ref. [25] and was successfully applied to several scenarios. The idea is, in short, to consider not only the bipartite states to merge, but purifications of them, where the purifying systems are not allowed to be affected by A or B . In this way, the question of success of the procedure is broken down to successful decoupling of the subsystems under control of A from the purifying environment. Techniques which were developed earlier [7, 8] for proving coding theorems for compound quantum channels based on the decoupling approach, can be used here as well.

The quantum state merging protocol can be further generalized, by replacing the classical communication channels involved by quantum channels. This leads to the so-called fully quantum Slepian Wolf or “mother” protocol [1], which together with a corresponding “father” protocol forms the head of a whole hierarchy of quantum protocols.

1.2 Outline

In Section 2, we provide precise definitions for the model considered in this work. At the end of the section, our main result is stated. Section 3 contains the technical groundwork for the proof of our main result. There, we generalize the original one-shot result for single states from Ref. [20] to the case, where the set of possible states to merge is finite. With these results at hand, we prove our main result in Section 4, where we first establish the direct part in case that the set of possible states to merge is finite. Then we extend this result to arbitrary sets of states using finite approximations in the set of quantum states. The converse statement directly carries over from the known result for single states. Section 5 is devoted to the classical communication cost of quantum state merging. There we review the single state case and add an elementary proof to the corresponding result from Ref. [20]. Unfortunately, the protocol class used to establish the achievability proof for the quantum cost turns out to be too narrow. We point out, that contrary to the single state case, it is suboptimal regarding the classical communication requirements.

We conclude our work by demonstrating some applications of our main result in Sect. 6, where we determine the entanglement distillation capacity in case, that the source from which is distilled is not perfectly known. Finally, we give another proof for the direct part of the entanglement generation coding theorem for compound quantum channels. There we use the correspondence between distillation of entanglement from quantum states and entanglement generation over quantum channels.

1.3 Notations and Conventions

All the Hilbert spaces which appear in this work are assumed to be finite dimensional and over the field of complex numbers. For any two Hilbert spaces \mathcal{H} and \mathcal{K} , $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denotes the set of linear operators mapping \mathcal{H} to \mathcal{K} and $\mathcal{B}(\mathcal{H})$ denotes the set of linear operators on \mathcal{H} . The set of states (i.e. positive semidefinite operators of trace one) on \mathcal{H} is denoted by $\mathcal{S}(\mathcal{H})$. With a Hilbert space \mathcal{K} , the set of channels (i.e. completely positive (cp) and trace preserving maps) from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ is denoted by $\mathcal{C}(\mathcal{H}, \mathcal{K})$, the set of trace non-increasing cp maps by $\mathcal{C}^\downarrow(\mathcal{H}, \mathcal{K})$. With a little abuse of notation, we write $id_{\mathcal{H}}$ for the

identical channel on $\mathcal{B}(\mathcal{H})$. Because we mainly deal with systems containing several relevant subsystems, we freely make use of the following convention: An Hilbert space \mathcal{H}_{XYZ} is always thought to be the space of a composite system consisting of systems with Hilbert spaces \mathcal{H}_X , \mathcal{H}_Y and \mathcal{H}_Z . We use a similar notation for states of composite systems. A state denoted ρ_{XY} for instance is a bipartite state with marginals ρ_X and ρ_Y and so on. Pure states on \mathcal{H} are identified with state vectors, e.g. the symbol ψ sometimes denotes the state $|\psi\rangle\langle\psi|$ and sometimes a state vector $\psi \in \mathcal{H}$ corresponding to $|\psi\rangle\langle\psi|$. The fidelity is defined by

$$F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$$

for quantum states ρ and σ on a Hilbert space \mathcal{H} . We frequently use the fact that if one of the input states is pure, the fidelity takes the form of an inner product

$$F(\rho, |\psi\rangle\langle\psi|) = \langle\psi, \rho\psi\rangle. \quad (1)$$

For other properties of the fidelity see Ref. [21]. The von Neumann entropy of a state ρ is defined

$$S(\rho) := -\text{tr}(\rho \log \rho)$$

where $\log(\cdot)$ denotes the base two logarithm throughout this work (accordingly $\exp(\cdot)$ is defined to base two as well). For certain other information quantities we choose a notation which indicates the states on which they are evaluated. For a state ρ_{XY} on \mathcal{H}_{XY} we denote the quantum mutual information by

$$I(X; Y, \rho_{XY}) := S(\rho_X) + S(\rho_Y) - S(\rho_{XY}),$$

and the conditional von Neumann entropy by

$$S(X|Y, \rho_{XY}) := S(\rho_{XY}) - S(\rho_Y).$$

For a channel $\mathcal{N} \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ and a state $\rho \in \mathcal{S}(\mathcal{H})$, the coherent information is denoted by

$$I_c(\rho, \mathcal{N}) := S(\mathcal{N}(\rho)) - S((\text{id}_{\mathcal{H}} \otimes \mathcal{N})(|\varphi\rangle\langle\varphi|)),$$

where φ is an arbitrary purification of ρ on $\mathcal{H} \otimes \mathcal{H}$. We further denote the hermitian conjugate of an operator a by a^* and the complex conjugate of a complex number z by \bar{z} . We use $[N]$ as the shortcut for the set $\{1, \dots, N\}$ for $N \in \mathbb{N}$.

Concluding this section, we specify the notion of one-way LOCC channels. As references, we recommend Ref. [22] (were the following definitions can be found stated in the Heisenberg picture), and the more recent treatment Ref. [9]. Readers not familiar with LOCC channels may also consult the appendix on the same topic included in this paper, where the following definitions are stated more extensively.

A *quantum instrument* (or just *instrument*) on a Hilbert space \mathcal{H} can be defined as a family $\{\mathcal{T}_k\}_{k=1}^D \subset \mathcal{C}^\downarrow(\mathcal{H}, \mathcal{K})$ of trace non-increasing cp maps with an output space \mathcal{K} such that their sum is a channel, i.e. $\sum_{k=1}^D \mathcal{T}_k(\cdot)$ is trace preserving. We will only consider finite families (i.e. D finite) in this paper. For bipartite Hilbert spaces \mathcal{H}_{AB} and \mathcal{K}_{AB} , a channel $\mathcal{N} \in \mathcal{C}(\mathcal{H}_{AB}, \mathcal{K}_{AB})$ is called an *LOCC channel with one-way classical communication from A to B* (or *A \rightarrow B one-way LOCC* for short), if it is a combination of an instrument $\{\mathcal{A}_k\}_{k=1}^D \subset \mathcal{C}^\downarrow(\mathcal{H}_A, \mathcal{K}_A)$ on A's systems and a family of quantum channels $\{\mathcal{B}_k\}_{k=1}^D$ on B's systems in the following manner. To each member \mathcal{A}_k of the instrument there is assigned a channel \mathcal{B}_k resulting in the form

$$\mathcal{N}(\rho) = \sum_{k=1}^D \mathcal{A}_k \otimes \mathcal{B}_k(\rho) \quad (\rho \in \mathcal{S}(\mathcal{H}_{AB})). \quad (2)$$

The interpretation of (2) is, that B chooses a channel for his system which depends on which of the D operations has been realized on A's system.

The amount of $A \rightarrow B$ classical communication required for application of \mathcal{M} is therefore determined by the possible measurement outcomes assigned to the operations $\mathcal{A}_1, \dots, \mathcal{A}_D$, i.e. a message of length $\lceil \log D \rceil$ bits has to be communicated.

2 Definitions and main result

Let $\mathcal{X} \subseteq \mathcal{S}(\mathcal{H}_{AB})$ be a set of bipartite states with subsystems distributed to (possibly) distant communication partners A and B. An (l, k_l) -merging for \mathcal{X} is an one-way LOCC channel

$$\mathcal{M}_l : \mathcal{B}(\mathcal{K}_{AB}^0) \otimes \mathcal{B}(\mathcal{H}_{AB}^{\otimes l}) \rightarrow \mathcal{B}(\mathcal{K}_{AB}^1) \otimes \mathcal{B}(\mathcal{H}_{B'B}^{\otimes l}),$$

with local operations on the A - and the B -subscripted spaces and classical $A \rightarrow B$ communication, where $\mathcal{K}_A^i \simeq \mathcal{K}_B^i$ for $i = 0, 1$ and $k_l := \dim \mathcal{K}_A^0 / \dim \mathcal{K}_A^1$. A real number R is called an achievable *entanglement rate* for \mathcal{X} , if there exists a sequence of (l, k_l) -mergings with

1. $\limsup_{l \rightarrow \infty} \frac{1}{l} \log(k_l) \leq R$
2. $\inf_{\mathcal{X}_p} F(\mathcal{M}_l \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_0^l \otimes \psi_{ABE}^{\otimes l}), \phi_1^l \otimes \psi_{B'BE}^{\otimes l}) \rightarrow 1$ for $l \rightarrow \infty$.

where $\phi_0^l \in \mathcal{S}(\mathcal{K}_{AB}^{0,l})$ and $\phi_1^l \in \mathcal{S}(\mathcal{K}_{AB}^{1,l})$ are maximally entangled states on their spaces. We demand that the Schmidt ranks of these states do not grow more than exponentially fast for $l \rightarrow \infty$, i.e. $\dim \mathcal{K}_A^{0,l}, \dim \mathcal{K}_A^{1,l} \leq 2^{lC}$ for all $l \in \mathbb{N}$ and some constant $C > 0$. Note that the fraction $\dim \mathcal{K}_A^{0,l} / \dim \mathcal{K}_A^{1,l}$ equals, by definition, the fraction of the Schmidt ranks of the input and output entanglement resources ϕ_0^l and ϕ_1^l . Therefore, the expression $\frac{1}{l} \log(k_l)$ corresponds to the number of maximally entangled qubits (ebits) per input copy consumed (or gathered) by the action of \mathcal{M}_l .

The infimum in the second condition is evaluated over a set \mathcal{X}_p which contains a purification ψ_{ABE} on a space \mathcal{H}_{ABE} for each ρ_{AB} in \mathcal{X} . $\psi_{B'BE}$ is the state ψ_{ABE} where the A-part is located on a Hilbert space $\mathcal{H}_{B'}$ under B's control. The fidelity measure in 2.) is independent of the choice of the purifications (which will be shown in the next section). We frequently use the abbreviation

$$F_m(\rho_{AB}, \mathcal{M}) := F(\mathcal{M} \otimes id_{\mathcal{H}_E}(\phi_0 \otimes \psi_{ABE}), \phi_1 \otimes \psi_{B'BE})$$

for a state ρ_{AB} and a merging channel \mathcal{M} for ρ_{AB} and frequently not specify the space \mathcal{H}_E explicitly. The maximally entangled input and output states ϕ_0 and ϕ_1 are considered to be determined by \mathcal{M} . The optimal entanglement rate $C_m(\mathcal{X})$, i.e.

$$C_m(\mathcal{X}) := \inf\{R : R \text{ is an achievable entanglement rate for } \mathcal{X}\}$$

is called the *merging cost* of \mathcal{X} .

The main result of this paper is the following theorem, which quantifies the merging cost of any set \mathcal{X} of bipartite states.

Theorem 1. *Let $\mathcal{X} \subset \mathcal{S}(\mathcal{H}_{AB})$ be a set of states on \mathcal{H}_{AB} . Then*

$$C_m(\mathcal{X}) = \sup_{\rho \in \mathcal{X}} S(A|B; \rho) \tag{3}$$

holds.

To prove the achievability part of the above Theorem 1 we show that we find universal protocols for state merging within the class of LOCC operations which was used by the authors of Ref. [20]. We give a brief outline of our proof of Theorem 1. In Sect. 3.1 we state and prove some important facts about the fidelity measure under consideration. We follow this path and recall the decoupling lemma given in Ref. [20] in Sect. 3.2. On this basis we establish a one-shot bound for finite sets of states in Section 3.3. To this end we utilize techniques developed in Refs. [7] and [8] for proving coding theorems for compound quantum channels. In Sect. 4.2 we provide the direct part of our merging theorem for finite sets of states and extend these results to arbitrary sets in Sect. 4.3. The converse theorem easily carries over from the one given in Ref. [20], and we just provide the missing link in Sect. 4.4.

3 One-shot result

3.1 Properties of the fidelity measure

In this section we aim to prove some important properties of the merging fidelity.

Lemma 1. *Let $\mathcal{M} : \mathcal{B}(\mathcal{K}_{AB}^0 \otimes \mathcal{H}_{AB}) \rightarrow \mathcal{B}(\mathcal{K}_{AB}^1 \otimes \mathcal{H}_{B'B})$ be a channel, $\phi_0 \in \mathcal{S}(\mathcal{K}_{AB}^0)$, and $\phi_1 \in \mathcal{S}(\mathcal{K}_{AB}^1)$ maximally entangled states. Then the following assertions hold*

1. For any state $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$ on \mathcal{H}_{AB} with purification $\psi_{ABE} \in \mathcal{S}(\mathcal{H}_{ABE})$,

$$F(\mathcal{M} \otimes id_{\mathcal{H}_E}(\phi_0 \otimes \psi_{ABE}), \phi_1 \otimes \psi_{B'BE}) = \sum_{z=1}^Z |\text{tr}(p_z \rho_{AB})|^2$$

holds, where p_1, \dots, p_Z are elements of $\mathcal{B}(\mathcal{H}_{AB})$ which depend on \mathcal{M}, ϕ_0 and ϕ_1 .

2. Merging fidelity is a convex function of the input state. For any two states ρ_1 and ρ_2 on \mathcal{H}_{AB} and $\lambda \in [0, 1]$

$$F_m(\lambda \rho_1 + (1 - \lambda) \rho_2, \mathcal{M}) \leq \lambda F_m(\rho_1, \mathcal{M}) + (1 - \lambda) F_m(\rho_2, \mathcal{M})$$

holds.

Proof. Let

$$\mathcal{M}(\cdot) := \sum_{z=1}^Z m_z(\cdot) m_z^*$$

be a Kraus decomposition of \mathcal{M} with operators $m_z \in \mathcal{B}(\mathcal{K}_{AB}^0 \otimes \mathcal{H}_{AB}, \mathcal{K}_{AB}^1 \otimes \mathcal{H}_{B'B})$ for every $z \in \{1, \dots, Z\}$. We define channels \mathcal{V} and \mathcal{W} which incorporate the input and output states ϕ_0 and ϕ_1 . Let $\mathcal{V} \in \mathcal{C}(\mathcal{H}_{AB}, \mathcal{K}_{AB}^1 \otimes \mathcal{H}_{B'B})$ be the channel constituted by Kraus operators $v_z \in \mathcal{B}(\mathcal{H}_{AB}, \mathcal{K}_{AB}^1 \otimes \mathcal{H}_{B'B})$ defined by

$$v_z x := m_z(\phi_0 \otimes x)$$

for every $1 \leq z \leq Z$, $x \in \mathcal{H}_{AB}$ and $\mathcal{W}(\cdot) := w(\cdot) w^*$ with

$$w x := \phi_1 \otimes (U \otimes \mathbb{1}_{\mathcal{H}_B}) x$$

for every $x \in \mathcal{H}_{AB}$. Here, $U \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_{B'})$ is the isometry which identifies \mathcal{H}_A and $\mathcal{H}_{B'}$. With these definitions at hand we have

$$\begin{aligned} & F(\mathcal{M} \otimes id_{\mathcal{H}_E}(\phi_0 \otimes \psi_{ABE}), \phi_1 \otimes \psi_{B'BE}) \\ &= F(\mathcal{V} \otimes id_{\mathcal{H}_E}(\psi_{ABE}), \mathcal{W} \otimes id_{\mathcal{H}_E}(\psi_{ABE})) \\ &= \sum_{z=1}^Z \langle (w \otimes \mathbb{1}_{\mathcal{H}_E}) \psi, (v_z \otimes \mathbb{1}_{\mathcal{H}_E}) |\psi\rangle \langle \psi| (v_z \otimes \mathbb{1}_{\mathcal{H}_E})^* (w \otimes \mathbb{1}_{\mathcal{H}_E}) \psi \rangle. \end{aligned} \quad (4)$$

The r.h.s. of (4) is due to the fact that the fidelity admits a representation in terms of an inner product if one of the inputs is pure, see eq. (1). Each of the summands on the r.h.s. of eq. (4) can be written as

$$\begin{aligned} \langle \psi, (w^* v_z \otimes \mathbb{1}_{\mathcal{H}_E}) |\psi\rangle \langle \psi| (v_z^* w \otimes \mathbb{1}_{\mathcal{H}_E}) \psi \rangle &= \langle \psi, (w^* v_z \otimes \mathbb{1}_{\mathcal{H}_E}) \psi \rangle \overline{\langle \psi, (w^* v_z \otimes \mathbb{1}_{\mathcal{H}_E}) \psi \rangle} \\ &= |\text{tr}((w^* v_z \otimes \mathbb{1}_{\mathcal{H}_E}) |\psi\rangle \langle \psi|)|^2 \\ &= |\text{tr}(w^* v_z \rho_{AB})|^2. \end{aligned} \quad (5)$$

Inserting the r.h.s. of eq. (5) into (4) yields

$$F(\mathcal{M} \otimes id_{\mathcal{H}_E}(\phi_0 \otimes \psi_{ABE}), \phi_1 \otimes \psi_{B'BE}) = \sum_{z=1}^Z |\text{tr}(w^* v_z \rho_{AB})|^2,$$

which is the desired result, if we set $p_z = w^* v_z$ for every z . The second assertion of the lemma is a direct consequence of the first one together with the fact that the fidelity takes only values in $[0, 1]$. \square

3.2 Protocol and decoupling for single states

In this section we briefly recall a result given in Ref. [20] which marks the starting point for our investigations. Fortunately, the protocol constructed there, which is of relatively simple structure, can be modified for our purposes. Let d_A be the dimension of the Hilbert space \mathcal{H}_A . For an integer $0 < L \leq d_A$ we use the term *L-merging* if we speak of a channel

$$\mathcal{M} : \mathcal{B}(\mathcal{H}_{AB}) \rightarrow \mathcal{B}(\mathcal{K}_{AB}) \otimes \mathcal{B}(\mathcal{H}_{B'B})$$

which is of the form

$$\mathcal{M}(\rho) = \sum_{k=0}^D a_k \otimes u_k(\rho) a_k^* \otimes u_k^*, \quad (6)$$

for every $\rho \in \mathcal{S}(\mathcal{H}_{AB})$. Here D is defined $D := \lfloor \frac{d_A}{L} \rfloor$ and \mathcal{K}_A and \mathcal{K}_B are Hilbert spaces with $\dim \mathcal{K}_A = \dim \mathcal{K}_B = L$ and $\mathcal{K}_A \subseteq \mathcal{H}_A$ is a subspace of \mathcal{H}_A , where

- $\{a_k\}_{k=0}^D \subset \mathcal{B}(\mathcal{H}_A, \mathcal{K}_A)$ is a set of rank L partial isometries (except a_0 which has rank $d_A - L \cdot D < L$) with pairwise orthogonal initial subspaces (in the following, we call such channels *L-instrument* for short).
- $\{u_k\}_{k=0}^D \subset \mathcal{B}(\mathcal{H}_B, \mathcal{K}_B \otimes \mathcal{H}_{B'B})$ is a family of isometries.

We abbreviate the corresponding operation with $\mathcal{A}_k := a_k(\cdot) a_k^*$ for every k . Let ψ_{ABE} be a purification of ρ_{AB} on a Hilbert space \mathcal{H}_{ABE} . For notational simplicity we define abbreviations

$$p_k := \text{tr}(a_k \rho_A a_k^*) \quad \text{and} \quad \rho_{AE}^k := \text{tr}_{\mathcal{H}_B}(\mathcal{A}_k \otimes id_{\mathcal{H}_E}(\psi_{ABE})).$$

for every $k \in \{0, \dots, D\}$. The following lemma is taken from Ref. [20], we repeat it here including a sketch of the proof which we give for the convenience of the reader.

Lemma 2 (cf. Ref. [20], Prop. 3). *Let ρ_{AB} be a bipartite state on \mathcal{H}_{AB} and $\{a_k\}_{k=0}^D \subset \mathcal{B}(\mathcal{H}_A, \mathcal{K}_A)$ an *L-instrument*. There exists a family $\{u_k\}_{k=0}^D$ of isometries completing $\{a_k\}_{k=0}^D$ to an *L-merging* \mathcal{M} which satisfies*

$$F(\mathcal{M} \otimes id_{\mathcal{H}_E}(\psi_{ABE}), \phi_L \otimes \psi_{B'BE}) \geq 1 - \tilde{Q},$$

where \tilde{Q} is defined by

$$\tilde{Q} := 2 \left(p_0 + \sum_{k=1}^D \left\| \rho_{AE}^k - \frac{L}{d_A} \pi_L \otimes \rho_E \right\|_1 \right). \quad (7)$$

Here, the state ϕ_L is maximally entangled on \mathcal{K}_{AB} and π_L denotes the maximally mixed state on \mathcal{K}_A (i.e. $\pi_L := \frac{1_{\mathcal{K}_A}}{L}$).

In the following proof, the well known relations (see Ref. [17])

$$F(\rho, \sigma) \geq 1 - \|\rho - \sigma\|_1 \quad \text{and} \quad (8)$$

$$\|\rho - \sigma\|_1 \leq 2\sqrt{1 - F(\rho, \sigma)} \quad (9)$$

between trace distance and fidelity of any two states ρ and σ on a Hilbert space \mathcal{H} are used.

Proof. For every k , $0 \leq k \leq D$, the (sub-normalized) state $\mathcal{A}_k \otimes id_{\mathcal{H}_E}(\psi_{ABE})$ is a purification of ρ_{AE}^k and $\phi_L \otimes \psi_{B'BE}$ is a purification of $\pi_L \otimes \rho_E$. These facts and Uhlmann's theorem [27] (see Ref. [21] for the finite dimensional version) guarantee that for every $k \in \{0, \dots, D\}$ there exists an isometry $u_k : \mathcal{H}_B \rightarrow \mathcal{K}_B \otimes \mathcal{H}_{B'B}$ satisfying

$$F(\mathcal{A}_k \otimes \mathcal{U}_k \otimes id_{\mathcal{H}_E}(\psi_{ABE}), \phi_L \otimes \psi_{B'BE}) = F(\rho_{AE}^k, \pi_L \otimes \rho_E), \quad (10)$$

where $\mathcal{U}_k(\cdot) := u_k(\cdot)u_k^*$. The rest is mostly done by lower bounding the fidelity in terms of the trace distance. Given the case that $p_k > 0$ for k , using (8) we have

$$\begin{aligned} F(\rho_{AE}^k, \pi_L \otimes \rho_E) &= p_k F\left(\frac{1}{p_k} \rho_{AE}^k, \pi_L \otimes \rho_E\right) \\ &\geq p_k - \|\rho_{AE}^k - p_k \pi_L \otimes \rho_E\|_1. \end{aligned} \quad (11)$$

In case that $p_k = 0$ for k , $F(\rho_{AE}^k, \pi_L \otimes \rho_E) = 0$. Taking the sum over all k we arrive at

$$F\left(\sum_{k=0}^D \mathcal{A}_k \otimes \mathcal{U}_k \otimes id_{\mathcal{H}_E}(\psi_{ABE}), \phi_L \otimes \psi_{B'BE}\right) = \sum_{k=0}^D F(\rho_{AE}^k, \pi_L \otimes \rho_E) \quad (12)$$

$$\geq 1 - \sum_{k=0}^D \|\rho_{AE}^k - p_k \pi_L \otimes \rho_E\|_1 \quad (13)$$

$$\geq 1 - 2p_0 - \sum_{k=1}^D \|\rho_{AE}^k - p_k \pi_L \otimes \rho_E\|_1. \quad (14)$$

Eq. (12) follows from the linearity of the fidelity in one of the inputs given the other one is pure and (10). For (13) we used (11) along with the fact that $\sum_{k=0}^D \mathcal{A}_k$ is a channel implying $\sum_k p_k = 1$. The r.h.s. of (14) holds because the trace distance of any two states is upper bounded by 2 which ensures

$$\|\rho_{AE}^0 - p_0 \pi_L \otimes \rho_E\|_1 \leq 2p_0.$$

It remains to show that $\|\rho_{AE}^k - p_k \pi_L \otimes \rho_E\|_1 \leq 2 \cdot \|\rho_{AE}^k - \frac{L}{d_A} \pi_L \otimes \rho_E\|_1$, which can be seen as follows. It holds that

$$\begin{aligned} \|\rho_{AE}^k - p_k \pi_L \otimes \rho_E\|_1 &\leq \|\rho_{AE}^k - \frac{L}{d_A} \pi_L \otimes \rho_E\|_1 + |p_k - \frac{L}{d_A}| \\ &\leq 2 \cdot \|\rho_{AE}^k - \frac{L}{d_A} \pi_L \otimes \rho_E\|_1, \end{aligned}$$

where the first inequality is obtained by adding a zero and applying the triangle inequality together with the fact that every quantum state has trace norm one. The second line is by monotonicity of the trace norm under the action of channels. \square

3.3 One shot bound for finite sets of states

In this section we consider a finite set $\mathcal{X} := \{\rho_{AB,i}\}_{i=1}^N$ of states on \mathcal{H}_{AB} and derive a bound for the minimal merging fidelity of the states in \mathcal{X} which is based on Lemma 2. The main ingredient for the proof is the observation, that a good merging scheme for the averaged state

$$\bar{\rho}_{AB} := \frac{1}{N} \sum_{i=1}^N \rho_{AB,i} \quad (15)$$

will be good for every single member of \mathcal{X} . This is due to convexity of the merging fidelity (see Lemma 1). Now let $\psi_{ABE,i}$ be any purification of $\rho_{AB,i}$ on \mathcal{H}_{ABE} for every $i \in [N]$. The state

$$|\bar{\psi}_{ABR}\rangle \langle \bar{\psi}_{ABR}| := \frac{1}{N} \sum_{i,j=1}^N |\psi_{ABE,i}\rangle \langle \psi_{ABE,j}| \otimes |e_i\rangle \langle e_j| \quad (16)$$

with $\{e_i\}_{i=1}^N$ being an orthonormal basis in \mathbb{C}^N is a purification of $\bar{\rho}_{AB}$ on \mathcal{H}_{ABR} with $\mathcal{H}_R := \mathcal{H}_E \otimes \mathbb{C}^N$. The following lemma provides a lower bound for the fidelity of an L -merging of $\bar{\rho}_{AB}$ in terms of quantities determined by the states in \mathcal{X} .

Lemma 3. *Let $\{\rho_{AB,i}\}_{i=1}^N$ be a set of states on \mathcal{H}_{AB} . Then for the corresponding averaged state $\bar{\rho}_{AB}$ and purifications $\psi_{ABE,1}, \dots, \psi_{ABE,N}$, Lemma 2 also holds with \tilde{Q} replaced by*

$$Q := 2 \left(p_0 + \frac{1}{N} \sum_{k=1}^D \sum_{i,j=1}^N \sqrt{L_{ij} \cdot T_{ij}^{(k)}} \right)$$

where $L_{ij} := L \cdot \min_{m \in \{i,j\}} \{\text{rank}(\rho_{E,m})\}$ and

$$T_{ij}^{(k)} := \left\| \rho_{AE,ij}^k - \frac{L}{d_A} \pi_L \otimes \rho_{E,ij} \right\|_2^2.$$

Here we used the definitions

$$\begin{aligned} \psi_{ABE,ij} &:= |\psi_{ABE,i}\rangle \langle \psi_{ABE,j}|, \quad \rho_{E,ij} := \text{tr}_{\mathcal{H}_{AB}}(\psi_{ABE,ij}), \quad \text{and} \\ \rho_{AE,ij}^k &:= \text{tr}_{\mathcal{H}_B}((a_k \otimes \mathbb{1}_{\mathcal{H}_{BE}})\psi_{ABE,ij}(a_k^* \otimes \mathbb{1}_{\mathcal{H}_{BE}})) \end{aligned}$$

for $i, j \in [N], k \in [D]$.

Proof. Define

$$\bar{\rho}_R := \text{tr}_{\mathcal{H}_{AB}}(\bar{\psi}_{ABR}), \quad \text{and} \quad \bar{\rho}_{AR}^k := \text{tr}_{\mathcal{H}_B}((a_k \otimes \mathbb{1}_{\mathcal{H}_{BR}})\bar{\psi}_{ABR}(a_k^* \otimes \mathbb{1}_{\mathcal{H}_{BR}}))$$

for every $k \in [D]$. We bound the trace distance terms on the r.h.s. of (7) for ρ_{AB} with its purification

introduced in eq. (16). Explicitly, for every $k \in [D]$, we have

$$\begin{aligned}
& \left\| \bar{\rho}_{AR}^k - \frac{L}{d_A} \pi_L \otimes \bar{\rho}_R \right\|_1 \\
& \stackrel{(a)}{=} \left\| \frac{1}{N} \sum_{i,j=1}^N \left\{ \rho_{AE,ij}^k - \frac{L}{d_A} \pi_L \otimes \rho_{E,ij} \right\} \otimes |e_i\rangle \langle e_j| \right\|_1 \\
& \stackrel{(b)}{\leq} \frac{1}{N} \sum_{i,j=1}^N \left\| \left\{ \rho_{AE,ij}^k - \frac{L}{d_A} \pi_L \otimes \rho_{E,ij} \right\} \otimes |e_i\rangle \langle e_j| \right\|_1 \\
& \stackrel{(c)}{\leq} \frac{1}{N} \sum_{i,j=1}^N \left\| \rho_{AE,ij}^k - \frac{L}{d_A} \pi_L \otimes \rho_{E,ij} \right\|_1 \\
& \stackrel{(d)}{\leq} \frac{1}{N} \sum_{i,j=1}^N \sqrt{L_{ij}} \left\| \rho_{AE,ij}^k - \frac{L}{d_A} \pi_L \otimes \rho_{E,ij} \right\|_2.
\end{aligned}$$

where $L_{ij} := L \cdot \min\{\text{rank}(\rho_{E,i}), \text{rank}(\rho_{E,j})\}$ for every $1 \leq i, j \leq N$. The above (in)equalities are justified by the following arguments. (a) by definition of $\bar{\rho}_R$ and $\bar{\rho}_{AR}^k$, (b) by use of the triangle inequality and (c) because the trace norm is multiplicative with respect to tensor products and the equality $\| |e_i\rangle \langle e_j| \|_1 = 1$ for all $1 \leq i, j \leq N$. The well known relation $\|x\|_1 \leq \sqrt{r} \|x\|_2$ between the trace- and Hilbert-Schmidt norms with r being the rank of x justifies (d), if the rank of the matrix

$$\rho_{AE,ij}^k - \frac{L}{d_A} \pi_L \otimes \rho_{E,ij}$$

is smaller or equal than L_{ij} for all $i, j \in [N]$. This is fulfilled, which can be seen as follows. Let with an orthonormal basis $\{f_k\}_{k=1}^{\dim \mathcal{H}_E}$ of \mathcal{H}_E ,

$$\psi_{ABE,i} := \sum_{k=1}^{r_i} \psi_{AB,k}^{(i)} \otimes f_k \quad (17)$$

be a Schmidt decomposition of $\psi_{ABE,i}$ for every $1 \leq i \leq N$, with the Schmidt coefficients incorporated in the first tensor factors. This is always possible since we are free in the choice of the purifications. Using (17), one can verify, that

$$\begin{aligned}
& \rho_{AE,ij}^k - \frac{L}{d_A} \pi_L \otimes \rho_{E,ij} \\
& = \sum_{k=1}^{r_i} \sum_{l=1}^{r_j} \left(a_k \text{tr}_{\mathcal{H}_B} (|\psi_{AB,k}^{(i)}\rangle \langle \psi_{AB,l}^{(j)}|) a_k^* - \langle \psi_{AB,l}^{(j)}, \psi_{AB,k}^{(i)} \rangle \frac{L}{d_A} \pi_L \right) \otimes |f_k\rangle \langle f_l|.
\end{aligned}$$

holds for every $i, j \in [N]$. This expression can be interpreted as an $r_i \times r_j$ block matrix with each block an $L \times L$ matrix. It has therefore rank smaller or equal $L \cdot \min\{r_i, r_j\}$. \square

Let $L \in \{1, \dots, d_A\}$ be fixed and an arbitrary but fixed L -instrument $\mathcal{A} := \{\mathcal{A}_k\}_{k=1}^D \subset \mathcal{C}^\downarrow(\mathcal{H}_A, \mathcal{K}_A)$ be given. Every unitary $v \in \mathfrak{U}(\mathcal{H}_A)$ defines a channel $\mathcal{V} \in \mathcal{C}(\mathcal{H}_A)$ via $\mathcal{V}(\cdot) := v(\cdot)v^*$. With these definitions, for every v , we get an L -instrument $\mathcal{A}(v)$ with

$$\mathcal{A}(v) := \{\mathcal{A}_k \circ \mathcal{V}\}_{k=0}^D.$$

Every collection of isometric channels $\{\mathcal{U}_k\}_{k=0}^D \subset \mathcal{C}(\mathcal{H}_B, \mathcal{K}_B \otimes \mathcal{H}_{B'B})$ completes $\mathcal{A}(v)$ to an L -merging

$$\sum_{k=0}^D \mathcal{A}_k \circ \mathcal{V} \otimes \mathcal{U}_k(\cdot) \quad (18)$$

We define the function

$$F_m(\rho, \mathcal{A}(v)) := \max_{\{\mathcal{U}_k\}_{k=0}^D} F_m(\rho_{AB}, \sum_{k=1}^D \mathcal{A}_k \circ \mathcal{V} \otimes \mathcal{U}_k(\cdot)) \quad (19)$$

for every $v \in \mathfrak{U}(\mathcal{H}_A)$, $\rho \in \mathcal{S}(\mathcal{H}_{AB})$. The maximization in (19) is over all collections $\{\mathcal{U}_k\}_{k=0}^D \subset \mathcal{C}(\mathcal{H}_B, \mathcal{K}_B \otimes \mathcal{H}_{B'B})$ of isometric channels.

The expected merging fidelity under random selection of such L -mergings according to the normalized Haar measure on $\mathfrak{U}(\mathcal{H}_A)$ is bounded in the following lemma, which is the key technical result for the proof of the merging theorem.

Lemma 4. *For $L \in \{1, \dots, d_A\}$, a set $\{\rho_{AB,i}\}_{i=1}^N$ of states on \mathcal{H}_{AB} and $\psi_{ABE,i}$ a purification of $\rho_{AB,i}$ on \mathcal{H}_{ABE} for each i , we have*

$$\int_{\mathfrak{U}(\mathcal{H}_A)} F_m(\bar{\rho}_{AB}, \mathcal{A}(v)) dv \geq 1 - 2 \left(\frac{L}{d_A} + 2 \cdot \sum_{i=1}^N \sqrt{L \cdot \text{rank}(\rho_{E,i}) \|\rho_{B,i}\|_2^2} \right) \quad (20)$$

where the integration is with respect to the normalized Haar measure on $\mathfrak{U}(\mathcal{H}_A)$.

To prove the claim of Lemma 4 the following two lemmas are needed.

Lemma 5 (Ref. [7], Lemma 3.2). *Let L and D be $N \times N$ -matrices with nonnegative entries such that*

$$L_{jl} \leq L_{jj}, \quad L_{jl} \leq L_{ll} \quad \text{and} \quad D_{ij} \leq \max\{D_{ii}, D_{jj}\}$$

for all $i, j \in \{1, \dots, N\}$. Then

$$\sum_{i,j=1}^N \frac{1}{N} \sqrt{L_{ij} D_{ij}} \leq 2 \sum_{i=1}^N \sqrt{L_{ii} D_{ii}}$$

Lemma 6. *Let τ and ξ be elements of a bipartite Hilbert space $\mathcal{H} \otimes \mathcal{H}'$. Then*

$$\|\text{tr}_{\mathcal{H}'}(|\tau\rangle\langle\xi|)\|_2^2 \leq \max_{\chi \in \{\tau, \xi\}} \|\text{tr}_{\mathcal{H}'}(|\chi\rangle\langle\chi|)\|_2^2$$

Proof of Lemma 6. Choose an orthonormal basis $\{e_m\}_{m=1}^d$ in \mathcal{H}' where $d := \dim(\mathcal{H}')$. The elements φ and ψ can be decomposed in the form

$$\begin{aligned} \varphi &= \sum_{m=1}^d \varphi_m \otimes e_m & \text{and} \\ \psi &= \sum_{m=1}^d \psi_m \otimes e_m \end{aligned}$$

with suitable elements $\varphi_1, \dots, \varphi_d$ and ψ_1, \dots, ψ_d in \mathcal{H} . With these decompositions

$$\text{tr}_{\mathcal{H}'}(|\varphi\rangle\langle\psi|) = \sum_{m,n=1}^d |\varphi_m\rangle\langle\psi_n| \cdot \text{tr}(|e_m\rangle\langle e_n|).$$

Therefore

$$\|\mathrm{tr}_{\mathcal{H}'}(|\varphi\rangle\langle\psi|)\|_2^2 = \left\| \sum_{m=1}^d |\varphi_m\rangle\langle\psi_m| \right\|_2^2 \quad (21)$$

$$= \left| \mathrm{tr} \left(\sum_{m,n=1}^d (|\varphi_m\rangle\langle\psi_m|)^* (|\varphi_n\rangle\langle\psi_n|) \right) \right| \quad (22)$$

$$= \left| \sum_{m,n=1}^d \langle\varphi_m, \varphi_n\rangle \langle\psi_n, \psi_m\rangle \right|. \quad (23)$$

To show the assertion of the lemma consider $2d \times d$ matrices X and Y with entries $X_{mn} := \langle\varphi_m, \varphi_n\rangle$ resp. $Y_{mn} := \langle\psi_m, \psi_n\rangle$ for $0 < m, n \leq d$. Then the r.h.s. of (23) can be read as $\mathrm{tr}(XY)$, and we have

$$\begin{aligned} \left| \sum_{m,n=1}^d \langle\varphi_m \varphi_n\rangle \langle\psi_n, \psi_m\rangle \right| &= |\mathrm{tr}(XY)| \\ &\leq \|X\|_2 \|Y\|_2 \\ &\leq \max_{Z \in \{X, Y\}} \|Z\|_2^2, \end{aligned} \quad (24)$$

where the r.h.s. of (24) is an application of the Cauchy-Schwarz inequality. It is easy to see that $\|X\|_2^2 = \|\mathrm{tr}_{\mathcal{H}'}(|\varphi\rangle\langle\varphi|)\|_2^2$ and $\|Y\|_2^2 = \|\mathrm{tr}_{\mathcal{H}'}(|\psi\rangle\langle\psi|)\|_2^2$, so we are done. \square

Proof of Lemma 4. First we have to convince ourselves, that $F_m(\bar{\rho}_{AB}, \mathcal{A}(\cdot))$ depends measurably on $v \in \mathfrak{U}(\mathcal{H}_A)$. For each fixed set $\{\mathcal{U}_k\}_{k=0}^D$, the function $F_m(\bar{\rho}_{AB}, \sum_{k=1}^D \mathcal{A}_k \circ \mathcal{V} \otimes \mathcal{U}_k)$ clearly is continuous in v , therefore, $F_m(\bar{\rho}_{AB}, \mathcal{A}(v))$ as a maximum over such functions is lower semicontinuous, which implies its measurability.

Using Lemma 3 we get

$$F_m(\bar{\rho}_{AB}, \mathcal{A}(v)) \geq 1 - Q_v$$

with error

$$Q_v := 2 \left(p_0^v + \frac{1}{N} \sum_{k=1}^D \sum_{i,j=1}^N \sqrt{L_{ij} \cdot T_{ij,v}^{(k)}} \right).$$

Here $p_0^v := \mathrm{tr}((\mathcal{A}_0 \circ \mathcal{V})(\rho_A))$,

$$T_{ij,v}^{(k)} := \left\| \rho_{AE,ij,v}^k - \frac{L}{d_A} \pi_L \otimes \rho_{E,ij} \right\|_2^2$$

and

$$\rho_{AE,ij,v}^k := \mathcal{A}_k \circ \mathcal{V}(\mathrm{tr}_{\mathcal{H}_B}(\psi_{ABE,ij})).$$

By virtue of Jensen's inequality

$$\int_{\mathfrak{U}(\mathcal{H}_A)} Q_v \, dv \leq 2 \left(\int_{\mathfrak{U}(\mathcal{H}_A)} p_0^v \, dv + \frac{1}{N} \sum_{k=1}^D \sum_{i,j=1}^N \left(L_{ij} \cdot \int_{\mathfrak{U}(\mathcal{H}_A)} T_{ij,v}^{(k)} \, dv \right)^{\frac{1}{2}} \right)$$

holds. It remains to bound the expectations in the right hand side of the above inequality. This was already done in Lemma 6 of Ref. [20]. We have

$$\int_{\mathfrak{M}(\mathcal{H}_A)} T_{ij,v}^k dv \leq \frac{L^2}{d_A^2} \|\text{tr}_{\mathcal{H}_B}(|\psi_{ABE,i}\rangle\langle\psi_{ABE,j}|)\|_2^2, \text{ and } \int_{\mathfrak{M}(\mathcal{H}_A)} p_0^v dv \leq \frac{L}{d_A}. \quad (25)$$

Abbreviating $D_{ij} := \|\text{tr}_{\mathcal{H}_B}(|\psi_{ABE,i}\rangle\langle\psi_{ABE,j}|)\|_2^2$ for every $i, j \in [N]$, (25) implies

$$\int_{\mathfrak{M}(\mathcal{H}_A)} Q_v dv \leq 2 \left(\frac{L}{d_A} + \frac{1}{N} \sum_{k=1}^D \sum_{i,j=1}^N \sqrt{L_{ij} \cdot \frac{L^2}{d_A^2} D_{ij}} \right) \quad (26)$$

$$\leq 2 \left(\frac{L}{d_A} + \frac{1}{N} \sum_{i,j=1}^N \sqrt{L_{ij} D_{ij}} \right). \quad (27)$$

The second inequality follows from the fact that the summands on the r.h.s. of (26) are independent of k and $D \frac{L}{d_A} \leq 1$ by definition of D . By definition of L_{ij} , clearly $L_{ij} = \min\{L_{ii}, L_{jj}\}$ for all i, j and so the first assumption of Lemma 5 is fulfilled. The second assumption (i.e. $D_{ij} \leq \max\{D_{ii}, D_{jj}\}$) holds by Lemma 6. Using Lemma 5, we obtain

$$\int_{\mathfrak{M}(\mathcal{H}_A)} Q_v dv \leq 2 \left(\frac{L}{d_A} + 2 \sum_{i=1}^N \sqrt{L \cdot \text{rank}(\rho_{E,i}) \|\rho_{B,i}\|_2^2} \right).$$

Note that we replaced $\|\rho_{AE,i}\|_2$ by $\|\rho_{B,i}\|_2$ for every i , which is admissible, because they are complementary marginals of a pure state [5]. \square

Corollary 2. *Lemma 4 provides the desired bound on the worst-case merging fidelity for finite sets. If we choose \mathcal{M} to be composed of the L -instrument $\mathcal{A}(\tilde{v})$ for some \tilde{v} which fulfills the bound on the right hand side of (20), and $\{\mathcal{U}_k\}_{k=1}^D$ which is a maximizer realizing $F_m(\bar{\rho}_{AB}, \mathcal{A}(\tilde{v}))$ for \tilde{v} (see eq. (19)), we have*

$$F_m(\bar{\rho}_{AB}, \mathcal{M}) \geq 1 - 2 \left(\frac{L}{d_A} + 2 \sum_{i=1}^N \sqrt{L \cdot \text{rank}(\rho_{E,i}) \|\rho_{B,i}\|_2^2} \right)$$

which implies, together with the convexity property of F_m (see Lemma 1),

$$\min_{i \in [N]} F_m(\rho_{AB,i}, \mathcal{M}) \geq 1 - 2N \left(\frac{L}{d_A} + 2 \sum_{i=1}^N \sqrt{L \cdot \text{rank}(\rho_{E,i}) \|\rho_{B,i}\|_2^2} \right).$$

4 Proof of the merging theorem

4.1 Typical subspaces

Here we state some properties of frequency typical projections which will be needed in the achievability proof. The concept of typicality is standard in classical and quantum information theory. Therefore we provide just the needed properties which can be found (along with basic definitions) in Ref. [7] (see Ref.[10] for the properties of types and typical sequences).

Lemma 7. *There exists a real number $c > 0$ such that for every Hilbert space \mathcal{H} of dimension d the following holds: For each state ρ on \mathcal{H} , $\delta \in (0, \frac{1}{2})$ and $l \in \mathbb{N}$ there is a projection $q_{\delta,l} \in \mathcal{B}(\mathcal{H}^{\otimes l})$ (its so-called frequency typical projection) with*

1. $\text{tr}(q_{\delta,l}\rho^{\otimes l}) \geq 1 - 2^{-l(c\delta^2 - h(l))}$
2. $q_{\delta,l}\rho^{\otimes l}q_{\delta,l} \leq 2^{-l(S(\rho) - \varphi(\delta))}q_{\delta,l}$
3. $2^{l(S(\rho) - \varphi(\delta) - h(l))} \leq \text{rank}(q_{\delta,l}) \leq 2^{l(S(\rho) + \varphi(\delta))}$

where the functions $\varphi(\delta) \rightarrow 0$ for $\delta \rightarrow 0$ and $h(l) \rightarrow 0$ for $l \rightarrow \infty$. Explicitly they are given by

$$h(l) = \frac{d}{l} \log(d+1) \quad \text{and} \quad \varphi(\delta) = -\delta \log \frac{\delta}{d}$$

for all $l \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$.

4.2 Proof of the direct part in case of finite sets of states

In this section we prove the optimal merging rate theorem using our one-shot result from Lemma 4. We first consider a finite set $\mathcal{X} := \{\rho_{AB,i}\}_{i=1}^N \subset \mathcal{S}(\mathcal{H}_{AB})$ with purifications $\psi_{ABE,1}, \dots, \psi_{ABE,N} \in \mathcal{H}_{ABE}$. For these states we introduce some sort of ‘‘typical reductions’’. We define

$$\tilde{\psi}_{ABE,i,\delta}^l := \frac{1}{\sqrt{w_{i,\delta,l}}} \tilde{q}_{i,\delta}^l \psi_{ABE,i}^{\otimes l},$$

where $w_{i,\delta,l} := \text{tr}(\tilde{q}_{i,\delta}^l \psi_{ABE,i}^{\otimes l})$,

$$\tilde{\rho}_{B,i,\delta}^l := \text{tr}_{\mathcal{H}_{AE}^{\otimes l}}(\tilde{\psi}_{ABE,i,\delta}^l), \quad \text{and} \quad \tilde{\rho}_{E,i,\delta}^l := \text{tr}_{\mathcal{H}_{AB}^{\otimes l}}(\tilde{\psi}_{ABE,i,\delta}^l).$$

for all $i \in \{1, \dots, N\}$, $l \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$. Here $\tilde{q}_{i,\delta}^l$ is given by the typical projectors $q_{A,i}$, $q_{B,i}$ and $q_{E,i}$ of the corresponding marginals of $\psi_{ABE,i}$

$$\tilde{q}_i := q_{A,i} \otimes q_{B,i} \otimes q_{E,i}$$

(here and in the following, the indices δ, l, i are sometimes omitted for the sake of brevity). The following lemma provides some bounds needed later

Lemma 8. *With the definitions given above, we have*

1. $w_{i,\delta,l} \geq 1 - 4 \cdot 2^{-l(c\delta^2 - h(l))}$
2. $\|\tilde{\rho}_{B,i,\delta}^l\|_2 \leq w_{i,\delta,l}^{-1} 2^{-\frac{l}{2}(S(\rho_{B,i}) - 3\varphi(\delta) - h(l))}$
3. $\text{rank}(\tilde{\rho}_{E,i,\delta}^l) \leq 2^{l(S(\rho_{AB,i}) + \varphi(\delta))}$

for all $i \in \{1, \dots, N\}$, $\delta \in (0, \frac{1}{2})$ and $l \in \mathbb{N}$.

Note, that the functions φ and h in Lemma 7 depend on the dimensions of the individual Hilbert space, however the above lemma clearly holds if we take the functions φ and h in Lemma 7 with $d = \dim(\mathcal{H}_{ABE})$.

Proof. 1.) Some simple algebra shows that

$$\begin{aligned} \tilde{q} &= \mathbb{1}_{ABE} - q_A \otimes q_B^\perp \otimes \mathbb{1}_E - q_A^\perp \otimes \mathbb{1}_B \otimes q_E \\ &\quad - \mathbb{1}_A \otimes q_B \otimes q_E^\perp - q_A^\perp \otimes q_B^\perp \otimes q_E^\perp \\ &\geq \mathbb{1}_{ABE} - q_A^\perp \otimes \mathbb{1}_{BE} - \mathbb{1}_A \otimes q_B^\perp \otimes \mathbb{1}_E \\ &\quad - 2(\mathbb{1}_{AB} \otimes q_E^\perp) \end{aligned}$$

holds. Therefore

$$w_{i,\delta,l} = \text{tr}(\tilde{q}_i \psi_{ABE,i}^{\otimes l}) \quad (28)$$

$$\geq 1 - \text{tr}(q_{A,i}^\perp \rho_{A,i}^{\otimes l}) - \text{tr}(q_{B,i}^\perp \rho_{A,i}^{\otimes l}) - 2\text{tr}(q_{E,i}^\perp \rho_{A,i}^{\otimes l}) \quad (29)$$

$$\geq 1 - 4 \cdot 2^{-(c\delta^2 - h(l))}. \quad (30)$$

2.) We first show, that

$$\text{tr} \left(\text{tr}_{\mathcal{H}_{AE}^{\otimes l}} (\tilde{q}^l \psi_{ABE}^{\otimes l} \tilde{q}^l)^2 \right) \leq \text{tr} \left((q_B^l \rho_B^{\otimes l} q_B^l)^2 \right) \quad (31)$$

holds. Note, that

$$\text{tr}_{\mathcal{H}_{AE}^{\otimes l}} \left((q_A^l \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^l) \psi_{ABE}^{\otimes l} (q_A^l \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^l) \right) = \text{tr}_{\mathcal{H}_{AE}^{\otimes l}} \left((q_A^l \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^l) \psi_{ABE}^{\otimes l} \right). \quad (32)$$

Additionally, we have $\text{tr}_{\mathcal{H}_{AE}^{\otimes l}} \left((q_A^l \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^l) \psi_{ABE}^{\otimes l} \right) \leq \rho_B^{\otimes l}$, because

$$\begin{aligned} \rho_B^{\otimes l} - \text{tr}_{\mathcal{H}_{AE}^{\otimes l}} \left((q_A^l \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^l) \psi_{ABE}^{\otimes l} \right) &= \text{tr}_{\mathcal{H}_{AE}^{\otimes l}} \left(q_A^{\perp l} \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^l \psi_{ABE}^{\otimes l} \right) \\ &\quad + \text{tr}_{\mathcal{H}_{AE}^{\otimes l}} \left(q_A^l \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^{\perp l} \psi_{ABE}^{\otimes l} \right) \\ &\quad + \text{tr}_{\mathcal{H}_{AE}^{\otimes l}} \left(q_A^{\perp l} \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^{\perp l} \psi_{ABE}^{\otimes l} \right), \end{aligned}$$

where all of the summands on the r.h.s. are nonnegative operators. Therefore

$$\begin{aligned} \text{tr} \left(\text{tr}_{\mathcal{H}_{AE}^{\otimes l}} (\tilde{q}_i^l \psi_{ABE,i}^{\otimes l} \tilde{q}_i^l)^2 \right) &= \text{tr} \left(\left(q_B^l \text{tr}_{\mathcal{H}_{AE}^{\otimes l}} (q_A^l \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^l \psi_{ABE,i}^{\otimes l}) q_B^l \right)^2 \right) \\ &\leq \text{tr} \left((q_B^l \rho_B^{\otimes l} q_B^l) (q_B^l \text{tr}_{\mathcal{H}_{AE}^{\otimes l}} (q_A^l \otimes \mathbb{1}_{\mathcal{H}_B} \otimes q_E^l \psi_{ABE,i}^{\otimes l}) q_B^l) \right) \\ &\leq \text{tr} \left((q_B^l \rho_B^{\otimes l} q_B^l)^2 \right), \end{aligned}$$

which proves eq. (31). The above inequalities rely on the fact, that $\text{tr}(A(\cdot))$ and $q_B^l(\cdot)q_B^l$ are positive maps, if A is a nonnegative operator. Finally we arrive at

$$\begin{aligned} \|\tilde{\rho}_{B,i,\delta}^l\|_2^2 &= w_{i,\delta,l}^{-2} \text{tr} \left(\text{tr}_{\mathcal{H}_{AE}^{\otimes l}} (\tilde{q}_{i,\delta}^l \psi_{ABE,i,\delta}^{\otimes l} \tilde{q}_{i,\delta}^l)^2 \right) \\ &\leq w_{i,\delta,l}^{-2} \text{tr} \left((q_{B,i,\delta}^l \rho_{B,i,\delta}^{\otimes l} q_{B,i,\delta}^l)^2 \right) \quad (33) \end{aligned}$$

$$\leq w_{i,\delta,l}^{-2} \text{tr}(q_{B,i,\delta}^l) \cdot 2^{-2l(S(\rho_{B,i}) - \varphi(\delta))} \quad (34)$$

$$\leq w_{i,\delta,l}^{-2} 2^{-l(S(\rho_{B,i}) - 3\varphi(\delta))}$$

where the r.h.s. of eq. (33) follows from (31), and (34) results from Lemma 7.2 applied twice. The last of the above inequalities follows from Lemma 7.3 .

3.) follows from the third claim in Lemma 7 and the fact that $S(\rho_{AB,i}) = S(\rho_{E,i})$ holds. \square

Theorem 3. For a finite collection $\mathcal{X} := \{\rho_{AB,i}\}_{i=1}^N$ of states on \mathcal{H}_{AB} , it holds

$$C_m(\mathcal{X}) \leq \max_{1 \leq i \leq N} S(A|B; \rho_{AB,i}).$$

Proof. The proof is similar to the corresponding one given in Ref. [20], but uses the one-shot bound given in Lemma 4. We show, that the for every $\epsilon > 0$, the number $\max_{i \in [N]} S(A|B; \rho_{AB,i}) + \epsilon$ is an achievable

rate for a merging of \mathcal{X} . First assume, that $\max_{i \in [N]} S(A|B; \rho_{AB,i}) < 0$. Let $\delta \in (0, \frac{1}{2})$ such that $\frac{\epsilon}{5} < \varphi(\delta)$. It suffices to consider ϵ with $0 < \epsilon < |\max_{1 \leq i \leq N} S(A|B, \rho_{AB,i})|$. Define

$$L_l := \left\lceil \exp \left(-l \left(\max_{i \in [N]} S(A|B; \rho_{AB,i}) + \epsilon \right) \right) \right\rceil.$$

According to Lemma 4 along with Corollary 2, there is an L_l -merging \mathcal{M}_l which fulfills

$$\min_{i \in [N]} F(\mathcal{M}_l \otimes id_{\mathcal{H}_E^{\otimes l}}(\tilde{\psi}_{ABE,i,\delta}^l), \phi_{L_l} \otimes \tilde{\psi}_{B'BE,i,\delta}^l) \geq 1 - NQ$$

with

$$Q := 2 \left(\frac{L_l}{\dim(\mathcal{H}_A^{\otimes l})} + 2 \sum_{i=1}^N \sqrt{L_l \cdot \text{rank}(\tilde{\rho}_{E,i,\delta}^l) \|\tilde{\rho}_{B,i,\delta}^l\|_2^2} \right). \quad (35)$$

With help of Lemma 8 it is easy to bound the summands on the r.h.s. of eq. (35). Explicitly it holds

$$\begin{aligned} \frac{L_l}{\dim(\mathcal{H}_A^{\otimes l})} &\leq \frac{L_l}{\text{tr}(q_{A,i})} \leq 2^{-6l\varphi(\delta)}, \\ \sqrt{L_l \cdot \text{rank}(\tilde{\rho}_{E,i,\delta}^l) \|\tilde{\rho}_{B,i,\delta}^l\|_2^2} &\leq \frac{2^{-\frac{1}{2}\varphi(\delta)}}{1 - 4 \cdot 2^{-l(c\delta^2 - h(l))}}. \end{aligned}$$

Therefore

$$\min_{i \in [N]} F(\mathcal{M}_l \otimes id_{\mathcal{H}_E^{\otimes l}}(\tilde{\psi}_{ABE,i,\delta}^l), \phi_{L_l} \otimes \tilde{\psi}_{B'BE,i,\delta}^l) \geq 1 - \tilde{f}(l, N, \delta)$$

holds, where

$$\tilde{f}(l, N, \delta) := 2N \left(2^{-6l\varphi(\delta)} + 2N \frac{2^{-\frac{1}{2}\varphi(\delta)}}{1 - 4 \cdot 2^{-l(c\delta^2 - h(l))}} \right) \quad (36)$$

for $l, N \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$. The desired bound for the merging fidelity of the original set \mathcal{X} of states follows from Winter's gentle measurement Lemma (cf. Ref. [28], Lemma 9). Explicitly, it holds

$$\min_{i \in [N]} F(\rho_{AB,i}^{\otimes l}, \mathcal{M}_l) \geq 1 - f(l, N, \delta). \quad (37)$$

where $f(l, N, \delta) := 2\sqrt{\tilde{f}(l, N, \delta)} - 2\sqrt{32 \cdot 2^{-l(c\delta^2 - h(l))}}$. It remains to consider the case $\max_{i \in \{1, \dots, N\}} S(A|B; \rho_{AB,i}) \geq 0$. The above argument can be used with additional assistance of a sufficient amount of entanglement shared by the merging partners. Let ϕ_K be a maximally entangled state shared by A and B of Schmidt rank $K := 2^{\lceil \max_{i \in [N]} S(A|B, \rho_{AB,i}) \rceil + 1}$ then for every i the state

$$\phi_K \otimes \rho_{AB,i}$$

has negative conditional von Neumann entropy. Therefore the above argument holds for these states giving an L_l -merging $\tilde{\mathcal{M}}_l$ with

$$L_l = \exp \left(-l \left(\max_{1 \leq i \leq N} S(A|B, \rho_{AB,i}) - \left[\max_{1 \leq i \leq N} S(A|B, \rho_{AB,i}) \right] - 1 + \epsilon \right) \right) \quad (38)$$

and $\min_{i \in [N]} F_m((\phi_K \otimes \rho_{AB,i})^{\otimes l}, \tilde{\mathcal{M}}_l)$ is lower bounded by a function as on the r.h.s. of eq. (37). Some unitaries which rearrange the tensor factors do the rest. Because

$$\frac{1}{l} \log \left(\frac{K^l}{L_l} \right) = \max_{i \in [N]} S(A|B, \rho_{AB,i}) + \epsilon + o(l^0) \quad (39)$$

we are done □

4.3 Proof of the direct part for arbitrary sets of states

In this section we aim to show that the achievability part of Theorem 1 does hold for any arbitrary set \mathcal{X} of states as well. This can be achieved by approximating \mathcal{X} by a sequence of (finite) nets and using the result obtained in the previous sections. The argument parallels the one given in case of compound quantum channels in Ref. [8].

A τ -net in $\mathcal{S}(\mathcal{H})$ is a finite set $\{\rho_i\}_{i=1}^N$ such that for each state ρ on \mathcal{H} there is at least one $i \in \{1, \dots, N\}$ with $\|\rho - \rho_i\|_1 < \tau$. We find such a finite set for every $\tau > 0$ due to compactness of $\mathcal{S}(\mathcal{H})$. For our proof we have to ensure, that we find τ -nets with cardinality upper bounded in an appropriate sense. This is the claim of the next lemma, which is a special case of Lemma 2.6 in Ref. [23].

Lemma 9. *For any $\tau \in (0, 1]$ there is a τ -net $\{\rho_i\}_{i=1}^N$ in $\mathcal{S}(\mathcal{H})$ with cardinality*

$$N \leq \left(\frac{3}{\tau}\right)^{2d^2}$$

Proof. The proof is exactly the same as the one given in Ref. [7] with the sets and norms replaced by the ones which are treated here. \square

Let $\mathcal{X} \subseteq \mathcal{S}(\mathcal{H}_{AB})$ be an arbitrary set of states on \mathcal{H}_{AB} . For a $\frac{\tau}{2}$ -net $\tilde{\mathcal{X}}_\tau$, which fulfills the bound given in Lemma 9, i.e.

$$|\tilde{\mathcal{X}}_\tau| \leq \left(\frac{6}{\tau}\right)^{2d_{AB}^2}$$

where $d_{AB} := \dim(\mathcal{H}_{AB})$, we define the set

$$\mathcal{X}_\tau := \{\rho_i \in \tilde{\mathcal{X}}_\tau : \exists \rho \in \mathcal{X} \text{ with } \|\rho_i - \rho\|_1 < \frac{\tau}{2}\}. \quad (40)$$

The following lemma provides some statements concerning τ -nets needed later.

Lemma 10. *Let $\mathcal{X} \subseteq \mathcal{S}(\mathcal{H}_{AB})$ be a set of bipartite states on \mathcal{H}_{AB} and \mathcal{X}_τ , for $\tau \in (0, \frac{1}{e}]$, the set defined in (40). It holds*

1. $|\mathcal{X}_\tau| \leq \left(\frac{6}{\tau}\right)^{2d_{AB}^2}$,
2. For every $\rho \in \mathcal{S}(\mathcal{H}_{AB})$ there is a state ρ_i in \mathcal{X}_τ satisfying

$$\|\rho^{\otimes l} - \rho_i^{\otimes l}\|_1 < l \cdot \tau,$$

3. $|\sup_{\rho \in \mathcal{X}} S(A|B, \rho) - \max_{\rho_i \in \mathcal{X}_\tau} S(A|B, \rho_i)| \leq \tau + 2 \cdot \tau \log\left(\frac{d_{AB}}{\tau}\right)$, and

4. Let \mathcal{M} be any merging operation for states on \mathcal{H}_{AB} . Then

$$\min_{\rho_i \in \mathcal{X}_\tau} F_m(\rho_i^{\otimes l}, \mathcal{M}) \geq 1 - \epsilon \Rightarrow \inf_{\rho \in \mathcal{X}} F_m(\rho^{\otimes l}, \mathcal{M}) \geq 1 - 2\sqrt{\epsilon} - 4\sqrt{l \cdot \tau} \quad (41)$$

Proof. The first assertion is obvious from the definition of \mathcal{X}_τ together with Lemma 9. The argument which proves the second one is exactly the same as done in Ref. [7] for channels. The third claim is a consequence of Fannes' inequality. Namely, to every positive real number τ we find states ρ' in \mathcal{X} and ρ_i in \mathcal{X}_τ such that

$$\|\rho' - \rho_i\|_1 < \tau \quad (42)$$

and

$$S(A|B, \rho') \geq \sup_{\rho \in \mathcal{X}} S(A|B, \rho) - \tau. \quad (43)$$

Eq. (42) implies

$$S(A|B, \rho') - S(A|B, \rho_i) \leq 2\tau \log\left(\frac{d_{AB}}{\tau}\right)$$

via twofold application of Fannes inequality [15]. Therefore

$$\sup_{\rho \in \mathcal{X}} S(A|B, \rho) - \tau \leq S(A|B, \rho') \quad (44)$$

$$\leq S(A|B, \rho_i) + 2\tau \log\left(\frac{d_{AB}}{\tau}\right). \quad (45)$$

which proves the assertion. To verify the last claim of the lemma we first fix a purification corresponding to every member of \mathcal{X}_τ (remember that we are free in our choice of the purifications). Let $\psi_{ABE,i}$ be a purification of $\rho_{AB,i}$ on \mathcal{H}_{ABE} for $1 \leq i \leq N$. Let ρ_{AB} an arbitrary element of \mathcal{X} , then we find at least one element of \mathcal{X}_τ satisfying

$$\|\rho_{AB,i} - \rho_{AB}\|_1 < \tau. \quad (46)$$

As a consequence of Uhlmann's theorem, there exists a purification ψ_{ABE} of ρ_{AB} on \mathcal{H}_{ABE} such that

$$F(\rho_{AB}^{\otimes l}, \rho_{AB,i}^{\otimes l}) = F(\psi_{ABE}^{\otimes l}, \psi_{ABE,i}^{\otimes l}). \quad (47)$$

Now let ϕ_0 and ϕ_1 the maximally entangled input and output states associated with \mathcal{M} , then

$$F_m(\rho_{AB}^{\otimes l}, \mathcal{M}) \quad (48)$$

$$= F(\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_0 \otimes \psi_{ABE}^{\otimes l}), \phi_1 \otimes \psi_{B'BE}^{\otimes l}) \quad (49)$$

$$\geq 1 - \|\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_0 \otimes \psi_{ABE}^{\otimes l}) - \phi_1 \otimes \psi_{B'BE}^{\otimes l}\|_1 \quad (50)$$

where the last inequality follows from the bound given in eq. (8). By an application of the triangle inequality, the trace distance on the r.h.s. of eq. (50) is upper bounded by

$$\begin{aligned} \|\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_0 \otimes \psi_{ABE}^{\otimes l}) - \phi_1 \otimes \psi_{B'BE}^{\otimes l}\|_1 &\leq \|\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_0 \otimes \psi_{ABE,i}^{\otimes l}) - \phi_1 \otimes \psi_{B'BE,i}^{\otimes l}\|_1 \\ &\quad + \|\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_0 \otimes (\psi_{ABE}^{\otimes l} - \psi_{ABE,i}^{\otimes l}))\|_1 \\ &\quad + \|\phi_1 \otimes (\psi_{B'BE}^{\otimes l} - \psi_{B'BE,i}^{\otimes l})\|_1. \end{aligned} \quad (51)$$

By monotonicity of the trace distance under the use of channels and eq. (46), each of the two last summands can be upper bounded by $\|\psi_{ABE,i}^{\otimes l} - \psi_{ABE}^{\otimes l}\|_1$, and

$$\|\psi_{ABE,i}^{\otimes l} - \psi_{ABE}^{\otimes l}\|_1 \leq 2\sqrt{1 - F(\rho_{AB,i}^{\otimes l}, \rho_{AB}^{\otimes l})} \quad (52)$$

$$\begin{aligned} &\leq 2\sqrt{\|\rho_{AB,i}^{\otimes l} - \rho_{AB}^{\otimes l}\|_1} \\ &\leq 2\sqrt{l\tau} \end{aligned} \quad (53)$$

holds. Eq. (52) is justified by (47) along with the relation given in eq. (9), and (53) is by the second claim of the present lemma. The first summand is upper bounded by

$$\|\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_0 \otimes \psi_{ABE,i}^{\otimes l}) - \phi_1 \otimes \psi_{B'BE,i}^{\otimes l}\|_1 \leq 2\sqrt{\epsilon} \quad (54)$$

again with eq. (9) and the assumptions. Eqns. (51), (53) and (54) justify

$$\inf_{\rho_{AB} \in \mathcal{X}} F_m(\rho_{AB}^{\otimes l}, \mathcal{M}) \geq 1 - 2\sqrt{\epsilon} - 4\sqrt{l\tau} \quad (55)$$

□

Theorem 4. *Let $\mathcal{X} \subset \mathcal{S}(\mathcal{H}_{AB})$ be a set of states on \mathcal{H}_{AB} . For the merging cost of \mathcal{X} it holds*

$$C_m(\mathcal{X}) \leq \sup_{\rho \in \mathcal{X}} S(A|B, \rho). \quad (56)$$

Proof. We show that

$$\sup_{\rho \in \mathcal{X}} S(A|B, \rho) + \epsilon$$

is an achievable rate for every ϵ satisfying $0 < \epsilon < |\sup_{\rho \in \mathcal{X}} S(A|B, \rho)|$. Fix $\tau \in (0, \frac{1}{e})$ for the moment and consider the corresponding set \mathcal{X}_τ given in (40) which approximates \mathcal{X} . According to the proof of Theorem 3 we find, for l large enough, an (l, k_l) -merging with

$$\begin{aligned} k_l &\leq \exp \left(l \left(\max_{1 \leq i \leq N_\tau} S(A|B, \rho_i) + \frac{\epsilon}{2} \right) \right) \\ &\leq \exp \left(l \left(\sup_{\rho \in \mathcal{X}} S(A|B, \rho) + \frac{\epsilon}{2} + \tau + 2\tau \log \frac{\dim(\mathcal{H}_{AB})}{\tau} \right) \right), \end{aligned} \quad (57)$$

where the second inequality is from Lemma 10. Another consequence of Lemma 10 is the inequality

$$\inf_{\rho \in \mathcal{X}} F_m(\rho^{\otimes l}, \mathcal{M}_l) \geq 1 - 2\sqrt{f(l, N_\tau, \delta)} - 4\sqrt{l \cdot \tau}. \quad (58)$$

If we now choose a sequence $\{\tau_l\}_{l \in \mathbb{N}}$ such that $\lim_{l \rightarrow \infty} \tau_l = 0$ and $\lim_{l \rightarrow \infty} \sqrt{l \cdot \tau_l} = 0$ hold, and additionally N_{τ_l} is growing polynomially (which is possible because Lemma 9 holds), then (57) and (58) show that $\sup_{\rho \in \mathcal{X}} S(A|B, \rho) + \epsilon$ is achievable. □

4.4 Proof of the converse part

Because we have shown that any rate above the least upper bound of the entanglement costs of the members of \mathcal{X} achievable, our converse follows immediately from the original converse for single states from Ref. [20]. The argument given there is based on the fact that entanglement measures must be monotone under LOCC operations along with an application of Fannes' inequality. As the proof is carried out in detail there, we just extend the argument to our present case.

Let $\delta > 0$ and χ_{AB} a member of \mathcal{X} which satisfies

$$S(A|B, \chi_{AB}) \geq \sup_{\rho \in \mathcal{X}} S(A|B, \rho) - \delta. \quad (59)$$

Following the argument of the single state converse, we arrive at

$$\begin{aligned} \frac{1}{l} \log(k_l) &\geq S(A|B, \chi_{AB}) - g(l)2\sqrt{\epsilon}(1 - \log(2\sqrt{\epsilon})) \\ &= \sup_{\rho \in \mathcal{X}} S(A|B, \rho) - \delta - g(l)2\sqrt{\epsilon}(1 - \log(2\sqrt{\epsilon})) \end{aligned} \quad (60)$$

with a function g which is $O(1)$ for $l \rightarrow \infty$. Therefore the entanglement cost of \mathcal{X} is least $\sup_{\rho \in \mathcal{X}} S(A|B, \rho) - \delta$ for every $\delta > 0$.

5 Classical communication cost of state merging

Having determined the optimal entanglement cost of a state merging process, we consider the classical cost of state merging in this section. By classical cost, we mean the rate of classical communication from A to B , which is at least required for an asymptotically perfect merging process. More precisely, if $\{\mathcal{M}_l\}_{l=1}^\infty$ is a sequence of $A \rightarrow B$ one-way LOCCs for a set \mathcal{X} , where A distinguishes a number of D_l measurement outcomes (see Section 2, eq. (2)) within the application of \mathcal{M}_l , the classical cost is given by

$$R_c = \limsup_{l \rightarrow \infty} \frac{1}{l} \log D_l.$$

In case of a single state ρ_{AB} , the minimum rate of classical communication for merging protocols achieving entanglement rate $R_q = S(A|B, \rho_{AB})$ was determined in Ref. [20] as $R_c = I(A; E, \rho_{AE})$, where ρ_{AE} is the marginal on the subsystems belonging to A and E of an arbitrary purification ψ_{ABE} of ρ_{AB} . In this section we deal with the case of a set of states to be merged and for the sake of simplicity, we restrict ourselves to finite sets of states. Clearly, the classical communication cost of a merging procedure for a set \mathcal{X} of states is lower bounded by the maximum of the communication costs for the individual states in \mathcal{X} . This is a direct consequence of the known result for single states, which was given in Ref. [20]. The original proof given there is based on properties of the closely related ‘‘mother protocol’’ [1] and general assertions within the resource framework from Ref. [14]. Here, we give a more elementary proof for the reader not familiar with the results of Refs. [14] and [1]. Moreover, this result and a converse statement for the case that A and B are restricted to L -mergings show, that the protocol class we considered to show achievability of the merging cost, is suboptimal regarding the classical cost.

Proposition 5 (cf. Ref. [20], Theorem 8). *Let $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$ be a bipartite state with purification ψ_{ABE} on a space \mathcal{H}_{ABE} and $\epsilon \in (0, 1)$. If $\mathcal{M}(\cdot) := \sum_{k=1}^D \mathcal{A}_k \otimes \mathcal{B}_k(\cdot)$ is an $A \rightarrow B$ one-way LOCC such that*

$$F(\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_K \otimes \psi_{ABE}^{\otimes l}), \phi_L \otimes \psi_{B'BE}^{\otimes l}) \geq 1 - \epsilon \quad (61)$$

holds with maximally entangled states ϕ_K, ϕ_L of Schmidt rank K resp. L , then

$$\frac{1}{l} \log(D) \geq I(A; E, \rho_{AE}) - 6\sqrt{\epsilon} \left(\frac{1}{l} \log(KL) + \log \dim \mathcal{H}_{AB} \right) - 3\eta(2\sqrt{\epsilon}) \quad (62)$$

holds, where the function η is defined on $[0, 1]$ by

$$\eta(x) := \begin{cases} -x \log x & 0 < x \leq \frac{1}{e} \\ \frac{\log e}{e} & \frac{1}{e} < x \leq 1 \end{cases} \quad (63)$$

and $\eta(0) := 0$.

Proof. The proof is inspired by ideas from Ref. [18]. Fix $\epsilon \in (0, 1)$ and $l \in \mathbb{N}$. Let $\phi_K \in \mathcal{K}_{AB}^0$ and $\phi_L \in \mathcal{K}_{AB}^1$ maximally entangled input resp. output states of the protocol such that with notations

$$\psi_0 := \phi_K \otimes \psi_{ABE}^{\otimes l}, \quad \text{and} \quad \psi_1 := \phi_L \otimes \psi_{B'BE}^{\otimes l}$$

eq. (61) reads

$$F(\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_0), \psi_1) \geq 1 - \epsilon. \quad (64)$$

We use the abbreviations $\mathcal{H}_{BE}^0 := \mathcal{K}_B^0 \otimes \mathcal{H}_{BE}^{\otimes l}$, $p_k := \text{tr}(\mathcal{A}_k \otimes id_{\mathcal{H}_{BE}^0}(\psi_0))$ for $k \in [D]$, and $T = \{k \in [D] : p_k \neq 0\}$. It is well known, that the von Neumann entropy is an almost convex function, i.e. for a state $\bar{\rho}$ defined as a mixture $\bar{\rho} := \sum_{i=1}^N p_i \rho_i$ of quantum states,

$$S(\bar{\rho}) \leq H(p_1, \dots, p_N) + \sum_{i=1}^N p_i S(\rho_i)$$

holds, where $H(p_1, \dots, p_N)$ is the Shannon entropy of the probability distribution on $[N]$ given by p_1, \dots, p_N . Using this fact, we obtain the lower bound

$$\begin{aligned} \log D &\geq H(p_1, \dots, p_D) \\ &\geq S\left(\sum_{k \in T} \mathcal{A}_k \otimes id_{\mathcal{H}_{BE}^0}(\psi_0)\right) - \sum_{k \in T} p_k S\left(\frac{1}{p_k} \mathcal{A}_k \otimes id_{\mathcal{H}_{BE}^0}(\psi_0)\right) \end{aligned} \quad (65)$$

on $\log D$. We separately bound the terms on the r.h.s. of eq. (65). With definitions $\pi_{K,A} := \text{tr}_{\mathcal{K}_B^0}(\phi_K)$, $\pi_{K,B} := \text{tr}_{\mathcal{K}_A^0}(\phi_K)$ and $\pi_{L,A} := \text{tr}_{\mathcal{K}_B^1}(\phi_L)$ (these are maximally mixed states of rank K resp. L) and $\mathcal{A}(\cdot) := \sum_{k \in T} \mathcal{A}_k(\cdot)$, we obtain

$$S\left(\sum_{k \in T} \mathcal{A}_k \otimes id_{\mathcal{H}_{BE}^0}(\psi_0)\right) \geq S(\pi_{K,B} \otimes \rho_{BE}^{\otimes l}) - S(\mathcal{A}(\pi_{K,A} \otimes \rho_A^{\otimes l})) \quad (66)$$

$$\geq \log K + lS(\rho_{BE}) - \log L - \Delta_1(\epsilon) \quad (67)$$

$$= \log \frac{K}{L} - lS(\rho_A) - \Delta_1(\epsilon) \quad (68)$$

where $\Delta_1(\cdot) := 2\sqrt{\cdot} \log(L) + \eta(2\sqrt{\cdot})$. Here eq. (66) is by the Araki-Lieb inequality [5], and eq. (68) is due to the fact that $S(\rho_A) = S(\rho_{BE})$ holds. Eq. (67) is justified as follows. Using the relation between fidelity and trace distance from (9) along with the fact, that the latter is monotone under taking partial traces, (64) implies

$$\|\mathcal{A}(\pi_{K,A} \otimes \rho_A^{\otimes l}) - \pi_{L,A}\|_1 \leq 2\sqrt{\epsilon}. \quad (69)$$

This, via application of Fannes' inequality leads to

$$S(\mathcal{A}(\pi_{K,A} \otimes \rho_A^{\otimes l})) \leq S(\pi_{L,A}) - 2\sqrt{\epsilon} \log L - \eta(2\sqrt{\epsilon}), \quad (70)$$

where η is the function defined in (63). To bound the second term on the r.h.s. of (65), we use Stinespring extensions of the individual trace decreasing channels which constitute \mathcal{M} . Let for each $k \in [D]$,

$$v_k : \mathcal{K}_A^0 \otimes \mathcal{H}_A^{\otimes l} \rightarrow \mathcal{K}_A^1 \otimes \mathcal{H}_{C'}$$

be a Stinespring extension of \mathcal{A}_k and

$$u_k : \mathcal{K}_B^0 \otimes \mathcal{H}_B^{\otimes l} \rightarrow \mathcal{K}_B^1 \otimes \mathcal{H}_{B'B}^{\otimes l} \otimes \mathcal{H}_{C''} \quad (71)$$

be a Stinespring extension of \mathcal{B}_k . Here $\mathcal{H}_{C'}$ is a Hilbert space associated to A and $\mathcal{H}_{C''}$ belongs to B . We fix notations $\mathcal{V}_k(\cdot) := v_k(\cdot)v_k^*$ and $\mathcal{U}_k := u_k(\cdot)u_k^*$ and denote the normalized outputs of these extensions by

$$\gamma_k := \frac{1}{p_k} \mathcal{V}_k \otimes \mathcal{U}_k \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_0) \quad (72)$$

for every $k \in T$. Note that $\mathcal{V}_1, \dots, \mathcal{V}_D$ are trace decreasing, while $\mathcal{U}_1, \dots, \mathcal{U}_D$ are channels. For every $k \in T$, we have

$$\begin{aligned} S\left(\frac{1}{p_k} \mathcal{A}_k \otimes id_{\mathcal{H}_{BE}^0}(\psi_0)\right) &= S\left(\frac{1}{p_k} \text{tr}_{\mathcal{H}_{C'}} \mathcal{V}_k \otimes id_{\mathcal{H}_{BE}^0}(\psi_0)\right) \\ &= S\left(\frac{1}{p_k} \text{tr}_{\mathcal{H}_{C'}} \mathcal{V}_k \otimes \mathcal{U}_k \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_0)\right) \\ &= S(\text{tr}_{\mathcal{H}_{C'}} \gamma_k), \end{aligned} \quad (73)$$

where the second equality is by the fact that u_k is an isometry and consequently the action of \mathcal{U}_k does not change the entropy. Note, that (64) implies, because fidelity is linear in the first input here, existence of a positive number c_k for every $k \in T$, such that

$$F\left(\frac{1}{p_k}\mathcal{A}_k \otimes \mathcal{B}_k \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_0), \psi_1\right) = 1 - c_k \quad (74)$$

and $\sum_{k \in T} p_k c_k \leq \epsilon$ hold. Because γ_k is a purification of $\frac{1}{p_k}\mathcal{A}_k \otimes \mathcal{B}_k \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_0)$ and ψ_1 is already pure, Uhlmann's Theorem ensures existence of a pure state φ_k on $\mathcal{H}_{C'} \otimes \mathcal{H}_{C''}$ with

$$\begin{aligned} F(\gamma_k, \psi_1 \otimes \varphi_k) &= \max\{|\langle \gamma_k, \sigma \rangle|^2 : \sigma \text{ purification of } \psi_0 \text{ on } \mathcal{K}_{AB}^1 \otimes \mathcal{H}_{B'B}^{\otimes l} \otimes \mathcal{H}_{C'} \otimes \mathcal{H}_{C''}\} \\ &= F\left(\frac{1}{p_k}\mathcal{A}_k \otimes \mathcal{B}_k \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_0), \psi_1\right) \end{aligned} \quad (75)$$

for every $k \in T$. From eqns. (74) and (75) we conclude, again via the well known relation between fidelity and trace distance from (9),

$$\|\gamma_k - \psi_1 \otimes \varphi_k\|_1 \leq 2\sqrt{c_k}, \quad (76)$$

which implies, again via Fannes' inequality and monotonicity of the trace distance under partial tracing

$$\begin{aligned} S(\text{tr}_{\mathcal{H}_{C'}} \gamma_k) &\leq S(\psi_1 \otimes \text{tr}_{\mathcal{H}_{C'}} \varphi_k) + \Delta_2(c_k) \\ &\leq S(\text{tr}_{\mathcal{H}_{C'}} \varphi_k) + \Delta_2(c_k). \end{aligned} \quad (77)$$

where $\Delta_2(\cdot) = 2\sqrt{\cdot} \log(\dim \mathcal{H}_{AB}^2 \dim \mathcal{H}_{C''}) + \eta(2\sqrt{\cdot})$. Consequently, we have

$$\begin{aligned} \sum_{k \in T} p_k S\left(\frac{1}{p_k}\mathcal{A}_k \otimes id_{\mathcal{H}_{BE}^0}(\psi_0)\right) &= \sum_{k \in T} p_k S(\text{tr}_{\mathcal{H}_{C'}} \gamma_k) \\ &\leq \sum_{k \in T} p_k S(\text{tr}_{\mathcal{H}_{C'}} \varphi_k) + \Delta_2(\epsilon). \end{aligned} \quad (78)$$

The above equality is by (73), the inequality follows by (77) and the fact, that that Δ_2 is monotone and concave (see the definition of η in 63)). It remains to bound $\sum_{k \in T} p_k S(\text{tr}_{\mathcal{H}_{C'}} \varphi_k)$. Abbreviating $\mathcal{H}_{AE}^1 := \mathcal{K}_A^1 \otimes \mathcal{H}_E^{\otimes l} \otimes \mathcal{H}_{C'}$, an argument very similar to the one above gives (again via (76) and an application of Fannes' inequality) the bound

$$\begin{aligned} S(\text{tr}_{\mathcal{H}_{AE}^1}(\gamma_k)) &\geq S(\text{tr}_{\mathcal{H}_{AE}^1}(\psi_1 \otimes \varphi_k)) - \Delta_3(c_k) \\ &= S(\pi_{L,B} \otimes \rho_{B'B}^{\otimes l} \otimes \text{tr}_{\mathcal{H}_{C'}} \varphi_k) - \Delta_3(c_k) \end{aligned} \quad (79)$$

with the function $\Delta_3(\cdot) := 2\sqrt{\cdot}(\log(K) + l \log(\dim \mathcal{H}_{AB} \cdot \dim \mathcal{H}_{C''})) + 2\eta(\sqrt{\cdot})$. And, using monotonicity and concavity of Δ_3 together with (79), we obtain

$$\sum_{k \in T} p_k S(\text{tr}_{\mathcal{H}_{AE}^1}(\gamma_k)) \geq \log(L) + lS(\rho_{AB}) + \sum_{k \in T} p_k S(\text{tr}_{\mathcal{H}_{C'}} \varphi_k) - \Delta_3(\epsilon) \quad (80)$$

where we used, that $S(\rho_{B'B}) = S(\rho_{AB})$ holds. If we now look at $\sum_{k=1}^D \mathcal{V}_k \otimes \mathcal{U}_k \otimes id_{\mathcal{H}_E^{\otimes l}}(\cdot)$ as an one-way LOCC-channel with local operations on systems belonging to A and E on one side and B on the other

side which 3 the pure input state ψ_0 to the state described by the pure state mixture $\sum_{k \in T} p_k \gamma_k$, we have

$$\begin{aligned}
S(\pi_K \otimes \rho_B^{\otimes l}) &= S(\text{tr}_{\mathcal{K}_A^0 \otimes \mathcal{H}_{AE}^{\otimes l}} \psi_0) \\
&= S\left(\text{tr}_{\mathcal{H}_{AE}^l} \left(\sum_{k=1}^D \mathcal{V}_k \otimes \text{id}_{\mathcal{H}_{BE}^0}(\psi_0)\right)\right) \\
&\geq \sum_{k \in T} p_k S\left(\frac{1}{p_k} \text{tr}_{\mathcal{H}_{AE}^l} \mathcal{V}_k \otimes \text{id}_{\mathcal{H}_{BE}^0}(\psi_0)\right) \\
&= \sum_{k \in T} p_k S\left(\frac{1}{p_k} \text{tr}_{\mathcal{H}_{AE}^l} \mathcal{V}_k \otimes \mathcal{U}_k \otimes \text{id}_{\mathcal{H}_E^{\otimes l}}(\psi_0)\right) \tag{81} \\
&= \sum_{k \in T} p_k S\left(\text{tr}_{\mathcal{H}_{AE}^l} \gamma_k\right). \tag{82}
\end{aligned}$$

The second of the above equalities is due to the fact, that $\sum_{k=1}^D \mathcal{V}_k(\cdot)$ is trace preserving, the inequality is by concavity of the von Neumann entropy. Eq. (81) is because the von Neumann entropy is not changed by application of unitary channels in the input. The last equality is by the definitions introduced in (72). With (80), (82) and the equality $S(\rho_{AB}) = S(\rho_E)$, we obtain

$$S(\pi_K \otimes \rho_B^{\otimes l}) \geq \log(L) + lS(\rho_E) + \sum_{k \in T} p_k S(\text{tr}_{\mathcal{H}_{C'}} \varphi_k) - \Delta_3(\epsilon). \tag{83}$$

Rearranging the terms in inequality (83) and using (78) leads to the bound

$$\sum_{k \in T} p_k S\left(\frac{1}{p_k} \mathcal{A}_k \otimes \text{id}_{\mathcal{H}_{BE}^0}(\psi_0)\right) \leq \log \frac{K}{L} + l(S(\rho_{AE}) - S(\rho_E)) + \Delta_2(\epsilon) + \Delta_3(\epsilon). \tag{84}$$

Here, we additionally used the fact, that $S(\rho_B) = S(\rho_{AE})$ holds. Combining the bounds from (67) and (84) with (65), we arrive at

$$\frac{1}{l} \log D \geq I(A; E, \rho_{AE}) - \frac{1}{l}(\Delta_1(\epsilon) + \Delta_2(\epsilon) + \Delta_3(\epsilon)). \tag{85}$$

In fact, we find Stinespring extensions on spaces $\mathcal{H}_{C'}$ and $\mathcal{H}_{C''}$ with

$$\dim \mathcal{H}_{C'} = K \cdot L \cdot \dim \mathcal{H}_A^l \tag{86}$$

$$\dim \mathcal{H}_{C''} = K \cdot L \cdot \dim \mathcal{H}_B^{2l} \dim \mathcal{H}_A^l. \tag{87}$$

Using the definition of Δ_1, Δ_2 and Δ_3 with the above dimensions, we conclude

$$\frac{1}{l} \log D \geq I(A; E, \rho_{AE}) - 6\sqrt{\epsilon} \left(\frac{\log KL}{l} + \log \dim \mathcal{H}_{AB} \right) - 3\eta(2\sqrt{\epsilon}), \tag{88}$$

which we aimed to prove. \square

Remark 1. *It is worth noting here, that the lower bound for the classical cost established in the proof of Proposition 5 does not explicitly rely on the entanglement rate of the protocol. Consequently, there is no chance to significantly reduce the required classical communication by admitting a higher entanglement rate, as long as one demands the protocol to be asymptotically perfect.*

In contrast to the above result, the following lemma indicates the limitations of the class of protocols used for establishing the achievability of the merging cost.

Lemma 11. Let $\{\rho_{AB,i}\}_{i=1}^N$ be a set of states on \mathcal{H}_{AB} . For every $\epsilon \in (0, 1)$ and $\delta > 0$, there exists a number $l_0(\epsilon, \delta)$, such that if $l > l_0$ and $\mathcal{M}(\cdot) := \sum_{k=1}^D \mathcal{A}_k \otimes \mathcal{B}_k(\cdot)$ is an L -merging for states on $\mathcal{H}_{AB}^{\otimes l}$ for some $L \in \{1, \dots, \dim(\mathcal{H}_A^{\otimes l})\}$ with

$$\min_{1 \leq i \leq N} F_m(\rho_{AB,i}^{\otimes l}, \mathcal{M}) \geq 1 - \epsilon, \quad (89)$$

then

$$\frac{1}{l} \log(D) \geq \max_{1 \leq i \leq N} S(\rho_{A,i}) + \frac{1}{l} \log \frac{K}{L} - \delta \quad (90)$$

holds.

Proof. First we consider for an arbitrary but fixed number $l \in \mathbb{N}$ and an arbitrary single state ρ_{AB} . Let $M \subset [D]$ be a set of indices which fulfills

$$F \left(\sum_{k \in M} \mathcal{A}_k \otimes \mathcal{B}_k \otimes id_{\mathcal{H}_E^{\otimes l}}(\phi_K \otimes \psi_{ABE}^{\otimes l}), \phi_L \otimes \psi_{B'BE}^{\otimes l} \right) \geq 1 - \epsilon$$

We use abbreviations

$$\psi_0 := \phi_K \otimes \psi_{ABE}^{\otimes l} \quad \text{and} \quad \rho_0 := \text{tr}_{\mathcal{K}_B^0 \otimes \mathcal{H}_{BE}^{\otimes l}}(\psi_0) = \pi_K \otimes \rho_A^{\otimes l}$$

Without any loss we assume that M contains no index k with $\text{tr}(\mathcal{A}_k(\rho_0)) = 0$. Because we are concerned with an L -merging for ψ_0 here, we have

$$\mathcal{A}_k(\cdot) = u_k p_k(\cdot) p_k^* u_k^*$$

for every k in M where $\{p_k\}_{k \in M}$ is a set of mutually orthogonal projections of rank L . We have

$$\text{tr}(\mathcal{A}_k(\rho_0)) = \text{tr}(p_k \rho_0),$$

and

$$\text{tr}(q \rho_0) = \sum_{k \in M} \text{tr}(\mathcal{A}_k(\rho_0)), \quad (91)$$

where we used the definition $q := \sum_{k \in M} p_k$. It holds

$$\begin{aligned} 1 - \epsilon &\leq \sum_{k \in M} F(\mathcal{A}_k \otimes \mathcal{B}_k \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_0), \phi_L \otimes \psi_{B'BE}^{\otimes l}) \\ &\leq \sum_{k \in M} F(\mathcal{A}_k(\rho_0), \pi_L) \end{aligned} \quad (92)$$

$$\leq \sum_{k \in M} \text{tr}(\mathcal{A}_k(\rho_0)) \quad (93)$$

$$= \text{tr}(q \rho_0). \quad (94)$$

Here, (92) follows from the monotonicity of the fidelity under partial traces, (93) by the fact that it is homogeneous in its inputs. The last equality is by (91). We may w.l.o.g. assume, that ρ_0 is of the form $\phi_0^{\otimes l} \otimes \psi_{ABE}^{\otimes l}$ with some maximally entangled state ϕ_0 , otherwise one could add a maximally entangled system to achieve this. In this case, Eq. (94) would hold with the projector $\mathbb{1} \otimes q$ instead of q , and this can be done without changing in the asymptotic rates. The well known fact, that subspaces of large

probability, asymptotically, cannot have dimension substantially smaller than the typical subspace (see Ref. [10], Lemma 2.14) guarantees

$$\frac{1}{l} \log \text{tr}(q) \geq S(\pi_0) + S(\rho_A) - \delta \quad (95)$$

if l is sufficiently large. If we take into account, that q is a sum of $|M|$ mutually orthogonal projections of rank L (i.e. $\text{tr}(q) = L \cdot |M|$), we have

$$\frac{1}{l} \log |M| \geq S(\rho_A) - \frac{1}{l} \log \frac{K}{L} - \delta. \quad (96)$$

If we now consider a set $\mathcal{X} := \{\rho_{AB,i}\}_{i=1}^N$ and repeat the above argument with sets M_1, \dots, M_N for this case we arrive at

$$\frac{1}{l} \log D \geq \frac{1}{l} \log \max_{1 \leq i \leq N} |M_i| \geq \max_{1 \leq i \leq N} S(\rho_{A,i}) - \frac{1}{l} \log \frac{K}{L} - \delta \quad (97)$$

which concludes our proof. \square

Theorem 6 (classical cost of L-merging). *Let $\mathcal{X} := \{\rho_{AB,i}\}_{i=1}^N$ be a set of bipartite states on \mathcal{H}_{AB} and $\delta > 0$. For a merging procedure, where A and B are restricted to L -mergings (together with adding some further input pure entanglement) and entanglement rate*

$$R_q = \max_{1 \leq i \leq N} S(A|B, \rho_{AB,i}) + \delta \quad (98)$$

is achieved, the optimal rate of classical communication is

$$R_c = \max_{1 \leq i \leq N} S(\rho_{A,i}) + \max_{1 \leq i \leq N} S(A|B, \rho_{AB,i}) + \delta.$$

Proof. The converse statement follows directly from Lemma 11. If $\{\mathcal{M}_l\}_{l=1}^\infty$ is a merging which fulfills the assumptions of the Theorem, then

$$F(\mathcal{M}_l \otimes id_{\mathcal{H}_E}^{\otimes l}(\phi_{K_l} \otimes \psi_{ABE,i}^{\otimes l}), \phi_{L_l} \otimes \psi_{B'BE}^{\otimes l}) \geq 1 - o(l^0)$$

with maximally entangled states ϕ_{K_l} resp. ϕ_{L_l} of Schmidt ranks for K_l and L_l for every $i \in [N]$, $l \in \mathbb{N}$, and

$$\limsup_{l \rightarrow \infty} \frac{1}{l} \log \left(\frac{K_l}{L_l} \right) = \max_{1 \leq i \leq N} S(A|B, \rho_{AB,i}) + \delta \quad (99)$$

hold. With (99) and Lemma 11 it follows

$$\limsup_{l \rightarrow \infty} \frac{1}{l} \log(D_l) \geq \max_{1 \leq i \leq N} S(\rho_{A,i}) + \max_{1 \leq i \leq N} S(A|B, \rho_{AB,i}) + \delta.$$

To prove achievability, we step back to Section 3.2. Because A and B are using an L_l -merging for every l , the distinct number of measurement results A has to communicate to B is given by

$$D_l = \frac{\dim \mathcal{H}_A^{\otimes l}}{L_l}.$$

The argument in Section 3.2 shows, that the desired quantum rate can be achieved by choosing L -mergings for the mixtures

$$\bar{\rho}_{AB}^l := \frac{1}{N} \sum_{i=1}^N \tilde{\rho}_{AB,i}^l,$$

where $\tilde{\rho}_{AB,i}^l$ is the $\frac{\delta}{2}$ -typically reduced state for $\phi_K \otimes \rho_{AB,i}$ for every $l \in \mathbb{N}$ some $\delta \in (0, \frac{1}{2})$. We can therefore assume $\mathcal{H}_A^{\otimes l}$ to be restricted to the support of $\tilde{\rho}_A^l$. Clearly, it holds

$$\begin{aligned} \text{rank } \tilde{\rho}_A^l &\leq \sum_{i=1}^N \text{rank } \tilde{\rho}_{A,i}^l \\ &\leq N \cdot \max_{1 \leq i \leq N} \text{rank } \tilde{\rho}_{A,i}^l \\ &\leq N \cdot \exp \left(l \left(\max_{1 \leq i \leq N} S(\pi_K \otimes \rho_{A,i}) + \frac{\delta}{2} \right) \right). \end{aligned}$$

Therefore

$$D_l \leq \frac{N}{L_l} \cdot \exp \left(l \left(S(\pi_K) + \max_{1 \leq i \leq N} S(\rho_{A,i}) + \frac{\delta}{2} \right) \right) \quad (100)$$

and

$$\begin{aligned} \frac{1}{l} \log(D_l) &\leq \max_{1 \leq i \leq N} S(\rho_{A,i}) + \log \frac{1}{l} \left(\frac{K_l}{L_l} \right) + \frac{N}{l} + \frac{\delta}{2} \\ &\leq \max_{1 \leq i \leq N} S(\rho_{A,i}) + \max_{1 \leq i \leq N} S(A|B, \rho_{AB,i}) + \delta \end{aligned} \quad (101)$$

if l is large enough. \square

The converse statement in the preceding Theorem is more strict than the one given in Prop. 5. The following example shows, that there are sets \mathcal{X} , where the optimal classical cost is surely not achieved by using L -mergings. However, here we achieve the desired classical rate just by simple modifications of the protocol.

Example 7. Consider the set $\{\rho_{AB,1}, \rho_{AB,2}\} \subset \mathcal{S}(\mathcal{H}_{AB})$ consisting of two members $\rho_{AB,1} = \phi_L$ and $\rho_{AB,2} = \pi_M \otimes \pi_M$, where ϕ_L is a maximally entangled state of Schmidt rank L on a subspace of \mathcal{H}_{AB} and π_M is the maximally mixed state. We assume, that $L > M$ and

$$\text{supp}(\rho_{A,1}) \perp \text{supp}(\rho_{A,2}) \quad (102)$$

holds. In this case, we have

$$\max_{i=1,2} I(A; E, \rho_{AE,i}) = S(\rho_{A,2}) + S(A|B, \rho_{AB,2}) \quad (103)$$

$$< S(\rho_{A,1}) + S(A|B, \rho_{AB,2}) \quad (104)$$

$$= \max_{i=1,2} S(\rho_{A,i}) + \max_{i=1,2} S(A|B, \rho_{AB,i}). \quad (105)$$

Since the supports of the A -marginals are orthogonal, A can perfectly distinguish his parts of the states (using one copy) and therefore get state knowledge. The rest is done by tracing out remaining entanglement to make both mergings have the same entanglement cost.

6 Applications

In this section we give some indications how the result obtained so far has impact on other problems in quantum Shannon theory. As an example we provide another achievability proof for the entanglement generating capacity of a compound quantum channel with uninformed users. The original proof[8] was based on an one-shot result for entanglement transmission, a closely related concept (actually their capacities were shown to be equal). Here we follow another line of reasoning, namely we use the close

correspondence between the task of distilling entanglement from bipartite sources and generating entanglement over quantum channels. To this end we prove a compound version of the so-called hashing bound which is known as a prominent lower bound on distillable entanglement for perfectly known states [13]. For convenience we restrict ourselves to the case of finite sets of states and finite compound channels. The results are easily generalized to arbitrary sets using approximation techniques as it was done in Sect. 4.3.

6.1 Entanglement distillation under state uncertainty

Following Ref. [13], we define a (l, k_l) -protocol for one-way distillation of states on \mathcal{H}_{AB} as a combination of an instrument $\{\mathcal{A}_k\}_{k=1}^D \subset \mathcal{C}^\downarrow(\mathcal{H}_A^{\otimes l}, \mathcal{K}^l)$ and a set of quantum channels $\{\mathcal{B}_k\}_{k=1}^D \subset \mathcal{C}(\mathcal{H}_B^{\otimes l}, \mathcal{K}^l)$ of the form

$$\mathcal{T} = \sum_{k=1}^D \mathcal{A}_k \otimes \mathcal{B}_k,$$

such that $\dim(\mathcal{K}^l) = k_l$. For a set $\mathcal{X} \subset \mathcal{S}(\mathcal{H}_{AB})$ of states on \mathcal{H}_{AB} a nonnegative number R is an achievable (one-way) entanglement distillation rate, if there is a sequence $\{\mathcal{T}_l\}_{l=1}^\infty$ of (l, k_l) -entanglement distillation protocols such that

1. $\liminf_{l \rightarrow \infty} \frac{1}{l} \log(k_l) \geq R$
2. $\lim_{l \rightarrow \infty} \inf_{\rho \in \mathcal{X}} F(\mathcal{T}_l(\rho^{\otimes l}), \phi_l) = 1$

where ϕ_l is a maximally entangled state on $\mathcal{K}^l \otimes \mathcal{K}^l$. The number

$$D_{\rightarrow}(\mathcal{X}) := \sup\{R : R \text{ is an achievable rate for one-way entanglement distillation}\}.$$

is called the (one way) *entanglement capacity* of \mathcal{X} . The following lemma is a compound analog to Theorem 3.1 in Ref [13].

Lemma 12. *Let $\mathcal{X} := \{\rho_i\}_{i=1}^N \subset \mathcal{S}(\mathcal{H}_{AB})$ be a (finite) set of bipartite states on \mathcal{H}_{AB} . Then*

$$D_{\rightarrow}(\mathcal{X}) \geq - \max_{1 \leq i \leq N} S(A|B, \rho_i) \quad (106)$$

Proof. It suffices to consider the case of a set with $\max_{1 \leq i \leq N} S(A|B, \rho_i) < 0$, since rate 0 can always be achieved by using a trivial protocol which distills no entanglement at all. Let $\mathcal{M} := \sum_{k=1}^D \mathcal{A}_k \otimes \mathcal{U}_k$ be an L -merging for \mathcal{X} satisfying

$$\min_{1 \leq i \leq N} F(\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_{ABE,i}^{\otimes l}), \phi_l \otimes \psi_{B'BE,i}^{\otimes l}) \geq 1 - \epsilon. \quad (107)$$

Then $\mathcal{T}(\cdot) := \sum_{k=1}^D \mathcal{A}_k \otimes \mathcal{R}_k(\cdot)$ with $\mathcal{R}_k := \text{tr}_{\mathcal{H}_{B'BE}} \circ (\mathcal{U}_k \otimes id_{\mathcal{H}_E^{\otimes l}})$ for every k is a one-way entanglement distillation protocol for \mathcal{X} satisfying

$$\begin{aligned} & F(\mathcal{T}(\rho_i^{\otimes l}), \phi_l) \\ & \geq F(\mathcal{M} \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_{ABE,i}^{\otimes l}), \phi_l \otimes \psi_{B'BE,i}^{\otimes l}) \\ & \geq 1 - \epsilon. \end{aligned} \quad (108)$$

for every $1 \leq i \leq N$. Eq. (108) is justified by the fact that taking partial traces cannot decrease fidelity. Following the proof of Theorem 4, we find for $\epsilon > 0$ and $l \in \mathbb{N}$ large enough an L_l -merging \mathcal{M}_l for \mathcal{X} such that

$$L_l \geq \left\lceil \exp \left(-l \left(\max_{1 \leq i \leq N} S(A|B, \rho_i) + \epsilon + o(l^0) \right) \right) \right\rceil \quad (109)$$

and

$$\min_{1 \leq i \leq N} F(\mathcal{M}_l \otimes id_{\mathcal{H}_E^{\otimes l}}(\psi_{ABE,i}^{\otimes l}), \phi_l \otimes \psi_{B'BE,i}^{\otimes l}) \geq 1 - o(l^0). \quad (110)$$

holds. Eqns. (108) and (111) give

$$\min_{1 \leq i \leq N} F(\mathcal{T}_l(\rho_i^{\otimes l}), \phi_l) \geq 1 - o(l^0). \quad (111)$$

The achievability of $-\max_{1 \leq i \leq N} S(A|B, \rho_i)$ follows from (109) and (111). \square

The above lemma provides the main building block for determining the one-way entanglement capacity for sets of states, which is done in the following theorem.

Theorem 8. *Let $\mathcal{X} := \{\rho_i\}_{i=1}^N \subset \mathcal{S}(\mathcal{H}_{AB})$. Then*

$$D_{\rightarrow}(\mathcal{X}) = \lim_{l \rightarrow \infty} \frac{1}{l} D^{(1)}(\mathcal{X}^{\otimes l}) \quad (112)$$

with

$$D^{(1)}(\mathcal{X}) := -\min_{\mathcal{T}} \max_{1 \leq i \leq N} \sum_{j: \lambda_j^{(i)} \neq 0} \lambda_j^{(i)} S(A|B, \rho_j^{(i)}) \quad (113)$$

where the minimization is over quantum instruments \mathcal{T} of the form $\mathcal{T} := \{\mathcal{T}_j\}_{j=1}^J$ on \mathcal{H}_A with definitions

$$\lambda_j^{(i)} := \text{tr}(\mathcal{T}_j(\text{tr}_{\mathcal{H}_B} \rho_i)) \quad \text{and} \quad \rho_j^{(i)} := \frac{1}{\lambda_j^{(i)}} \mathcal{T}_j \otimes id_{\mathcal{H}_B}(\rho_i) \quad (114)$$

for $1 \leq j \leq J$ and $1 \leq i \leq N$ with $\lambda_j \neq 0$. In fact, we can restrict ourselves to $J \leq \dim(\mathcal{H}_A)^2$ (see [13]).

Remark 2. *One easily verifies, that the limit in (112) exists. Clearly,*

$$D^{(1)}(\mathcal{X}^{\otimes k}) + D^{(1)}(\mathcal{X}^{\otimes l}) \leq D^{(1)}(\mathcal{X}^{\otimes(k+l)}) \quad (115)$$

holds for any $k, l \in \mathbb{N}$, because if $\mathcal{T}^{(k)}$ and $\mathcal{T}^{(l)}$ are instruments on $\mathcal{H}_A^{\otimes k}$ resp. $\mathcal{H}_A^{\otimes l}$, then $\mathcal{T}^{(k)} \otimes \mathcal{T}^{(l)}$ is an instrument on $\mathcal{H}_A^{\otimes(k+l)}$. The rest is by Fekete's Lemma [16].

Proof of Theorem 8. We begin with the direct part of the Theorem. Our proof parallels the one given in Ref. [13] for the single state case. However, for the direct part, we use Lemma 12 instead of the single state hashing bound. To prove achievability, let $\mathcal{T} := \{\mathcal{T}_j\}_{j=1}^J$ be any instrument on \mathcal{H}_A , $\mathcal{P} := \{\mathcal{P}_j\}_{j=1}^J$ a set of channels of the form

$$\mathcal{P}_j(\chi) := \chi \otimes |e_j\rangle \langle e_j| \quad (116)$$

for every $\chi \in \mathcal{S}(\mathcal{H}_B)$ and $1 \leq j \leq J$, where e_1, \dots, e_J are members of an orthonormal basis of a Hilbert space $\mathcal{H}_{B'}$ located at B 's site. Define states

$$\tilde{\rho}_i := \sum_{j=1}^J \mathcal{T}_j \otimes \mathcal{P}_j(\rho_i) = \sum_{j: \lambda_j^{(i)} \neq 0} \lambda_j^{(i)} \rho_j^{(i)} \otimes |e_j\rangle \langle e_j|$$

for $1 \leq i \leq N$. These preprocessed states have conditional von Neumann entropy

$$S(A|BB', \tilde{\rho}_i) = \sum_{j: \lambda_j^{(i)} \neq 0} \lambda_j^{(i)} S(A|B, \rho_j^{(i)}).$$

Direct application of Lemma 12 gives achievability. The converse statement can be proven just by the same arguments as given in Ref. [13], we give the proof for convenience. We consider an arbitrary (l, k_l) one-way distillation protocol with rate R , given by a LOCC channel with $A \rightarrow B$ classical communication

$$\mathcal{T}(\cdot) := \sum_{j=1}^J \mathcal{T}_j \otimes \mathcal{R}_j(\cdot)$$

with $\mathcal{T}_j \in \mathcal{C}^\downarrow(\mathcal{H}_A^{\otimes l}, \mathcal{K})$ and $\mathcal{R}_j \in \mathcal{C}(\mathcal{H}_B^{\otimes l}, \mathcal{K})$, $1 \leq j \leq J$, such that for a given $\tau \in (0, \frac{1}{2})$

$$F(\mathcal{T}(\rho_i^{\otimes l}), \phi) \geq 1 - \tau \quad (i \in \{1, \dots, N\}) \quad (117)$$

holds, where ϕ is a maximally entangled state on $\mathcal{K} \otimes \mathcal{K}$ and $\dim \mathcal{K} = \lfloor 2^{lR} \rfloor$. We fix notations

$$\begin{aligned} \lambda_j^{(i)} &:= \text{tr}(\mathcal{T}_j \otimes \mathcal{R}_j(\rho_i^{\otimes l})), \quad \text{and} \quad \omega_j^{(i)} := \frac{1}{\lambda_j^{(i)}} \mathcal{T}_j \otimes \mathcal{R}_j(\rho_i^{\otimes l}), \\ \rho_j^{(i)} &:= \frac{1}{\lambda_j^{(i)}} \mathcal{T}_j \otimes \text{id}_{\mathcal{H}_B^{\otimes l}}(\rho_i^{\otimes l}) \end{aligned}$$

for $i \in [N]$, $j \in [J]$ with $\lambda_j^{(i)} \neq 0$. Application of \mathcal{T} on ρ_i results in the state

$$\Omega^{(i)} := \sum_{j: \lambda_j^{(i)} \neq 0} \lambda_j^{(i)} \omega_j^{(i)}.$$

Using the relation from (9), (117) implies, that

$$\|\Omega^{(i)} - \phi\|_1 \leq 2\sqrt{\tau}$$

holds for all $i \in [N]$, which leads us to

$$|S(A|B, \Omega^{(i)}) - S(A|B, \phi)| \leq \epsilon \quad (118)$$

with $\epsilon := 2(2\sqrt{\tau} \log(\dim \mathcal{K}^2) + \eta(2\sqrt{\tau}))$ via twofold application of Fannes' inequality. Eq. (118) along with $S(A|B, \phi) = -l \cdot R$ implies

$$lR \leq -S(A|B, \Omega^{(i)}) + 4\sqrt{\tau} \cdot lR + 2\eta(2\sqrt{\tau}). \quad (119)$$

Moreover, we have

$$\begin{aligned} S(A|B, \Omega^{(i)}) &\geq \sum_{j: \lambda_j^{(i)} \neq 0} \lambda_j^{(i)} S(A|B, \omega_j^{(i)}) \\ &\geq \sum_{j: \lambda_j^{(i)} \neq 0} \lambda_j^{(i)} S(A|B, \rho_j^{(i)}), \end{aligned} \quad (120)$$

where the first inequality is by concavity of the map $\rho \mapsto S(A|B, \rho)$ for quantum states, the second is by application of the quantum data processing inequality. Combining (119) and (120), we obtain

$$\begin{aligned} lR &\leq -\max_{i \in [N]} \sum_{j: \lambda_j^{(i)} \neq 0} \lambda_j^{(i)} S(A|B, \rho_j^{(i)}) + 4\sqrt{\tau}l \cdot R + 2\eta(2\sqrt{\tau}) \\ &\leq -\min_{\mathcal{T}} \max_{i \in [N]} \sum_{j: \lambda_j^{(i)} \neq 0} \lambda_j^{(i)} S(A|B, \rho_j^{(i)}) + 4\sqrt{\tau}l \cdot R + 2\eta(2\sqrt{\tau}) \\ &\leq D^{(1)}(\mathcal{X}^{\otimes l}) + 4\sqrt{\tau}l \cdot R + 2\eta(2\sqrt{\tau}) \end{aligned}$$

□

Remark 3. *Theorem 8 shows, that one may have to pay an additional price for imperfect knowledge of the state. Namely, the capacity for a set \mathcal{X} is, in general, strictly smaller than the minimum over the single-state capacities of the individual states in \mathcal{X} , as can be seen from eq. (113).*

6.2 Entanglement generation over compound quantum channels

Finally, we give another proof for the direct part of the coding theorem for entanglement generation over compound channels, which was originally given in Ref. [8], Theorem 13. We first recall some definitions from Ref. [8]. Let \mathfrak{J} be a compound quantum channel generated by a set $\mathfrak{J} \subseteq \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$ of channels. We consider the uninformed user scenario, where precise knowledge about the identity of the channel is available neither to encoder nor decoder. An *entanglement generating (l, k_l) -code* for \mathfrak{J} is a pair $(\mathcal{R}^l, \varphi^l)$ where $\mathcal{R}^l \in \mathcal{C}(\mathcal{H}_B^{\otimes l}, \mathcal{K}^l)$ is a channel with $k_l = \dim \mathcal{K}^l$ and φ^l a pure state on $\mathcal{K}^l \otimes \mathcal{H}_A^{\otimes l}$. A positive number R is an *achievable rate for entanglement generation* over \mathfrak{J} if there is a sequence of (l, k_l) -entanglement generating codes satisfying

1. $\liminf_{l \rightarrow \infty} \frac{1}{l} \log k_l \geq R$, and
2. $\liminf_{l \rightarrow \infty, N \in \mathfrak{J}} F(\phi_l, (id_{\mathcal{K}^l} \otimes \mathcal{R}^l \circ \mathcal{N}^{\otimes l})(\varphi_l)) = 1$, where ϕ_l denotes a maximally entangled state on $\mathcal{K}^l \otimes \mathcal{K}^l$.

The number

$$E(\mathfrak{J}) := \sup\{R : R \text{ is an achievable rate for entanglement generation over } \mathfrak{J}\}.$$

is called the *entanglement generating capacity* of \mathfrak{J} .

Theorem 9 (cf. Ref. [8], Th. 13). *Let $\mathfrak{J} := \{\mathcal{N}_i\}_{i=1}^N$ be a finite compound quantum channel, $\mathfrak{J} \subset \mathcal{C}(\mathcal{H}_A, \mathcal{H}_B)$. Then*

$$E(\mathfrak{J}) \geq \lim_{l \rightarrow \infty} \frac{1}{l} \max_{\rho \in \mathcal{S}(\mathcal{H}_A^{\otimes l})} \min_{1 \leq i \leq N} I_c(\rho, \mathcal{N}_i^{\otimes l}) \quad (121)$$

holds

Proof. First note that the limit on the r.h.s of (121) exists by standard arguments (see Ref. [8], Remark 2). We just have to prove that the number

$$\min_{1 \leq i \leq N} I_c(\rho, \mathcal{N}_i) - \epsilon$$

is an achievable rate for every state ρ on \mathcal{H}_A and every $\epsilon > 0$, the rest is by standard blocking arguments. There is nothing to prove for sets with $\min_{1 \leq i \leq N} I_c(\rho, \mathcal{N}_i) \leq 0$. Therefore let ρ be a state on \mathcal{H}_A with $\min_{1 \leq i \leq N} I_c(\rho, \mathcal{N}_i) > 0$. Consider the set $\mathcal{X} := \{\rho_i\}_{i=1}^N$ of bipartite states in \mathcal{H}_{AB} , where ρ_i is defined

$$\rho_i := (id_{\mathcal{H}_A} \otimes \mathcal{N}_i)(\chi) \quad (122)$$

for $1 \leq i \leq N$. Here χ is the pure state on $\mathcal{H}_A \otimes \mathcal{H}_A$ such that the partial trace over any of the two subsystems results in the state ρ . We show that a good entanglement distillation protocol for the set \mathcal{X} of bipartite states generated by \mathfrak{J} implies the existence of a good entanglement generation code for \mathfrak{J} . Following the proof of Lemma 12, there exists an (l, k_l) -distillation protocol $\mathcal{T} = \sum_{k=0}^D \mathcal{A}_k \otimes \mathcal{R}_k$ for \mathcal{X} with $\mathcal{A}_k \in \mathcal{C}(\mathcal{H}_A^{\otimes l}, \mathcal{K}^l)$ and $\mathcal{R}_k \in \mathcal{C}(\mathcal{H}_B^{\otimes l}, \mathcal{K}^l)$ for $k \in \{1, \dots, D\}$ with D determined by $\dim \mathcal{H}_A$ and $\dim \mathcal{K}^l$ such that

$$\dim \mathcal{K}^l \geq \left\lfloor \exp \left(l \left(\min_{1 \leq i \leq N} I_c(\rho, \mathcal{N}_i) - \epsilon \right) \right) \right\rfloor \quad (123)$$

and

$$\min_{1 \leq i \leq N} F(\mathcal{T}(\rho_i), \phi_l) \geq 1 - o(l^0) \quad (124)$$

with ϕ_l being the maximally entangled state on \mathcal{K}^l . Notice, that in eq. (123), we used the identity

$$I_c(\rho, \mathcal{N}_i) = -S(A|B, \rho_i)$$

for every $i \in \{1, \dots, N\}$. The definitions given in eq. (122) imply

$$\mathcal{A}_k \otimes \mathcal{R}_k(\rho) = (id_{\mathcal{K}^l} \otimes \mathcal{R}_k \circ \mathcal{N}_i)(\mathcal{A}_k \otimes id_{\mathcal{H}_A^{\otimes l}}(\chi))$$

for every $0 \leq k \leq D$ and $1 \leq i \leq N$. Therefore,

$$\begin{aligned} F(\mathcal{T}(\rho_i), \phi_l) &= \sum_{k=0}^D F(id_{\mathcal{K}^l} \otimes \mathcal{R}_k \circ \mathcal{N}_i^{\otimes l}(\mathcal{A}_k \otimes id_{\mathcal{H}_A^{\otimes l}}(\chi)), \phi_l) \\ &= \sum_{k: p_k \neq 0} p_k F(id_{\mathcal{K}^l} \otimes \mathcal{R}_k \circ \mathcal{N}_i^{\otimes l}(\varphi_k), \phi_l) \end{aligned} \quad (125)$$

holds for every i , where we used the definitions

$$p_k := \text{tr}(\mathcal{A}_k(\rho)), \text{ and } \varphi_k := \frac{1}{p_k}(\mathcal{A}_k \otimes id_{\mathcal{H}_A^{\otimes l}})(\chi)$$

for $p_k \neq 0$, $0 \leq k \leq D$. Notice, that $\varphi_0, \dots, \varphi_D$ are pure states, because the operations \mathcal{A}_k are pure since they arise from an L -merging (see the proof of Lemma 12). Again because the fidelities are affine functions of the first input, (124) and (125) imply

$$\sum_{k: p_k \neq 0} p_k F\left(id_{\mathcal{K}^l} \otimes \mathcal{R}_k \circ \frac{1}{N} \sum_{i=1}^N \mathcal{N}_i^{\otimes l}(\varphi_k), \phi_l\right) \geq 1 - o(l^0). \quad (126)$$

The r.h.s. of equation (125) is, in fact, an average of fidelities of entanglement generating codes $(\mathcal{R}_1, \varphi_1), \dots, (\mathcal{R}_D, \varphi_D)$ with probabilities p_1, \dots, p_D . This implies the existence of a number $k' \in \{1, \dots, D\}$ such that with $\varphi := \varphi_{k'}$ and $\mathcal{R} := \mathcal{R}_{k'}$

$$\min_{1 \leq i \leq N} F(id_{\mathcal{K}^l} \otimes \mathcal{R} \circ \mathcal{N}_i^{\otimes l}(\varphi), \phi_l) \geq 1 - o(l^0) \quad (127)$$

holds. Eqns. (127) and (123) show that

$$\min_{1 \leq i \leq N} I_c(\rho, \mathcal{N}_i) - \epsilon$$

is an achievable rate. \square

To conclude this section we compare the proof of Theorem 9 given above with the one given in Ref. [8]. The original achievability proof relies on the fact that good entanglement generation codes can be deduced from entanglement transmission codes working good on maximally mixed states on certain subspaces of the input space of the channels. The passage to arbitrary states is done by a compound version of the so-called BSST-Lemma [6]. Indeed, one of the results from Ref. [8] is that the entanglement transmission capacity $\mathcal{Q}(\mathfrak{J})$ equals $E(\mathfrak{J})$ for every compound channel \mathfrak{J} .

The proof given above follows a more direct route by taking advantage of a direct correspondence between entanglement distillation from bipartite states and entanglement generation over quantum channels, which is very close even in the compound setting. In this way, we have demonstrated, that quantum state merging provides a genuine approach to problems of entanglement generation over quantum channels even in the compound setting.

7 Conclusion

In this work, we have extended the concept of quantum state merging to the case, where the users are partially ignorant of the parameters which describe the state they keep. We have determined the optimal entanglement cost of state merging in this setting, and found out that, in principle, a merging process is possible with the worst case merging cost in the set representing this uncertainty. We also derived a lower bound on the classical cost for merging with state uncertainty, based on an elementary proof of the corresponding result for single states. Whether or not this bound is achievable in general, is left as an open question. In particular, we have shown, that the class of protocols (called “ L -mergings” in this work), which contains protocols optimal for the quantum as well the classical part of the state merging problem in case of perfectly known states is suboptimal in its classical costs for situations with state uncertainty. However, in some special cases, protocols which are minor variations of the L -merging concept achieve this bound.

Despite this, the protocol preserved its good reputation as a communication primitive regarding the quantum performance. We were able, to apply our results to prove corresponding assertions in other communication settings as entanglement distillation under state uncertainty as well as entanglement generation under channel uncertainty. To apply these results in more complicated situations as multiuser settings (e.g. entanglement generation over quantum multiple access channels) is an interesting topic for further research activities.

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8 Appendix: LOCC Channels

In this section, we give a short account to the class of one-way LOCC channels which we use in our considerations. For further information, we recommend the survey article by Keyl [22] (and references therein). A more recent general treatment can be found in Ref. [9].

Crucial for the definition of LOCC channels is the concept of an instrument. Instruments (or operation valued measures [12]) were introduced to model the situation, where a measurement is made, and not only the measurement results but also the state transformations according to the measurement values are taken into account. To each measurement result i , there is assigned a positive trace non-increasing cp map \mathcal{I}_i which transforms the input state. In this paper, we restrict ourselves to finite sets of possible measurement results.

Definition 1. A (finite) instrument \mathcal{A} is a map

$$\begin{aligned} \mathcal{A} : I &\rightarrow \mathcal{C}^\downarrow(\mathcal{H}, \mathcal{K}) \\ i &\mapsto \mathcal{A}_i \end{aligned} \quad (i \in I)$$

with a finite index set I and Hilbert spaces \mathcal{H}, \mathcal{K} , such that $\sum_{i \in I} \mathcal{A}_i$ is trace preserving. The instrument \mathcal{A} is completely determined by the family $\{\mathcal{A}_i\}_{i \in I}$. We will sometimes write $\mathcal{A} = \{\mathcal{A}_i\}_{i \in I}$ to denote the instrument \mathcal{A} .

For bipartite systems, an instrument at, say, A 's (the sender's) site can be combined with a parameter-dependent channel use, which is defined by a function

$$\begin{aligned} \mathcal{B} : I &\rightarrow \mathcal{C}(\mathcal{H}_B, \mathcal{K}_B) \\ i &\mapsto \mathcal{B}_i \end{aligned} \quad (i \in I),$$

i.e. each \mathcal{B}_i is a completely positive and trace preserving map. A one-way LOCC channel is then defined as a combination of an instrument and a parameter-dependent channel. This leads to the following definition.

Definition 2. A channel $\mathcal{N} \in \mathcal{C}(\mathcal{H}_{AB}, \mathcal{K}_{AB})$ is called $A \rightarrow B$ one-way LOCC channel, if it takes the form

$$\mathcal{N}(\rho) = \sum_{i \in I} \mathcal{A}_i \otimes \mathcal{B}_i(\rho) \quad (\rho \in \mathcal{S}(\mathcal{H}_{AB})), \quad (128)$$

where $\mathcal{A} = \{\mathcal{A}_i\}_{i \in I}$, $\mathcal{A}_i \in \mathcal{C}^\downarrow(\mathcal{H}_A, \mathcal{K}_A)$, is an instrument and $\{\mathcal{B}_i\}_{i \in I}$ is a parameter-dependent channel.

A one-way LOCC can also again be considered as a ‘‘one-way local’’ instrument[9] with members $\{\mathcal{A}_i \otimes \mathcal{B}_i\}_{i \in I}$. There is a convenient way of handling one-way LOCCs. One can equivalently write the instrument \mathcal{A} used on A 's site in channel form

$$\mathcal{A}(\rho) = \sum_{i \in I} \mathcal{A}_i(\rho) \otimes |e_i\rangle \langle e_i| \quad (\rho \in \mathcal{S}(\mathcal{H}_A))$$

with an orthonormal basis $\{|e_i\rangle\}_{i \in I} \subset \mathbb{C}^{|I|}$. If the basis is assigned to a system on B 's site (which models a classical communication and coherent storage of the measurement results at the receiver's system), the parameter-dependent channel can be written in the form

$$\mathcal{B}(\rho) := \sum_{i \in I} |e_i\rangle \langle e_i| \otimes \mathcal{B}_i(\rho) \quad (\rho \in \mathcal{S}(\mathcal{H}_B))$$

(this map may not be trace-preserving). Then we have for $\rho \in \mathcal{S}(\mathcal{H}_{AB})$

$$\begin{aligned} \mathcal{N}(\rho) &= (id_{\mathcal{K}_A} \otimes \mathcal{B}) \circ (\mathcal{A} \otimes id_{\mathcal{H}_B})(\rho) \\ &= \sum_{j, i \in I} \mathcal{A}_i \otimes \mathcal{B}_j(\rho) \otimes |e_i\rangle \langle e_i| |e_j\rangle \langle e_j| \\ &= \sum_{i \in I} \mathcal{A}_i \otimes \mathcal{B}_i(\rho) \otimes |e_i\rangle \langle e_i|, \end{aligned}$$

where the second line includes a permutation of the tensor factors. Tracing out the classical information exchanged within the application of the map (i.e. the system with space $\mathbb{C}^{|I|}$) leads back to the form given in Eq. (128). The more general class of two-way LOCC channels exhibits a more intricate definition for which we refer to Refs. [22], [9].

Moreover, Def. 2 should not be confused with the definition of the class of separable channels. A channel $\mathcal{M} \in \mathcal{C}(\mathcal{H}_{AB}, \mathcal{K}_{AB})$ is called *separable*, if it takes the form

$$\mathcal{M}(\rho) = \sum_{i \in I} \mathcal{A}_i \otimes \mathcal{B}_i(\rho) \quad (\rho \in \mathcal{S}(\mathcal{H}_{AB})) \quad (129)$$

where $\mathcal{A}_i \in \mathcal{C}^\downarrow(\mathcal{H}_A, \mathcal{K}_A)$ and $\mathcal{B}_i \in \mathcal{C}^\downarrow(\mathcal{H}_B, \mathcal{K}_B)$ for all $i \in I$. From eqns. (128) and (129), the difference between the one-way LOCC and separable channels can be observed. While separable channels allow general trace decreasing cp maps for both parties, the receiver party is restricted to usage of trace preserving cp maps (i.e. channels) in the one-way LOCC class of channels.

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