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The geometry of Newton strata in the reduction modulo p of Shimura varieties of PEL type

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THE GEOMETRY OF NEWTON STRATA IN THE REDUCTION MODULO p OF SHIMURA VARIETIES OF PEL TYPE

BY PAUL HAMACHER

Abstract. In this thesis we study the Newton stratification on the reduction of Shimura varieties of PEL type with hyperspecial level structure. Our main result is a formula for the dimension of Newton strata and the description of their closure, where the dimension formula was conjectured by Chai. As a key ingredient of its proof we calculate the dimension of some Rapoport-Zink spaces. Our result yields a dimension formula, which is similar but not equivalent to the formula conjectured by Rapoport.

As an interesting application to deformation theory, we determine the dimension and closure of Newton strata on the algebraisation of the deformation space of a Barsotti-Tate group with (P)EL structure. Our result on the closure of a Newton stratum generalises conjectures of Grothendieck and Koblitz.

Contents

1. Introduction

1.1. **Background.** The theory of Shimura varieties goes back to the research on modular functions and forms in the 19$^{\text{th}}$ century. (Meromorphic) modular forms can be realised as meromorphic differential forms on the compactification of the classical modular curve $Y_0(N) = \{(j(\tau), j(N\tau) \mid \tau \in \mathbb{C}\}$, which is the earliest (and easiest) example of a Shimura variety. In order to extend this theory to functions of several variables, higher dimensional analogues of modular curves were introduced, ultimately culminating in the definition of a Shimura variety by Deligne.

Interestingly, many Shimura varieties can be described as a moduli space of abelian varieties with additional structure, making them more accessible. We consider the case of Shimura varieties of PEL type, which represent a moduli problem of abelian varieties equipped with **p**olarisation, **e**ndomorphisms and **l**evel structure. Under a certain unramifiedness condition this moduli problem can be defined (and represented) over $O_E \otimes \mathbb{Z}_{(p)}$ where O_E is the ring of integers of a certain number field E, which depends on the Shimura variety (see [Kot85] ch. 5).

In this thesis we consider the special fibre $\mathscr{A}_0$ of this moduli space. While for a complex abelian variety A the Barsotti-Tate group $A[p^\infty]$ of p-power torsion points does only depend on the dimension of A, there are many possibilities for the isomorphism (or even isogeny) class of the Barsotti-Tate group of an abelian variety over a field of characteristic p. The stratification corresponding to the isogeny class of Barsotti-Tate groups (with endomorphisms and polarisation) $\underline{A}[p^\infty] = (A[p^\infty], \lambda_{|A[p^\infty]}, \iota_{|A[p^\infty]})$ of geometric points $\underline{A} = (A, \lambda, \iota, \eta)$ of $\mathscr{A}_0$ is called the Newton stratification. This stratification is a main tool when one studies the arithmetic properties of PEL-Shimura varieties. For example, the Newton stratification is used to derive Kottwitz's formula for the local zeta function of PEL-Shimura varieties with good reduction ([Kot92]), in the proof of the local Langlands correspondence by Harris and Taylor ([HT01]) and in the proof of Fargues that the supercuspidal part of the cohomology of some unramified Rapoport-Zink spaces realises local Langlands correspondences ([Far04]).

Let us illustrate the above notions through the example of the modular curve $Y_0(N)$. The curve $Y_0(N)$ parametrises the elliptic curves over $\mathbb{C}$ with level-N-structure (i.e. a trivialisation $(\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$ of the N-torsion points of E). This moduli problem can be defined (and represented) over $\mathbb{Z}[N^{-1}]$, which defines a canonical integral model of $Y_0(N)$. The Newton stratification of its fibre over $\mathbb{F}_p$ consists of two strata. One stratum is open and dense and parametrises the elliptic curves with level-N-structure over $\mathbb{F}_p$ which are ordinary (i.e. $E[p](\overline{\mathbb{F}}_p) \cong \mathbb{Z}/p\mathbb{Z}$) and the other stratum is zero-dimensional and parametrises the elliptic curves with level-N-structure which are supersingular (i.e. $E[p](\overline{\mathbb{F}}_p) = 0$).

The aim of this thesis is to determine the closure of the Newton strata in $\mathscr{A}_0$ and their dimension. Partial results were obtained by Oort ([Oor00]), who determined the closure and dimension of Newton strata in the Siegel moduli space, by Bültel and Wedhorn ([BW06]) in the case of unitary Shimura varieties of type $(n-1, 1)$ and Wedhorn showed in [Wed99] that the "ordinary" Newton stratum is open and dense. The last result was recently generalised by Wortmann ([Wor]) to Shimura varieties of Hodge type.

1.2. **The main results.** We fix a prime p and denote by σ the Frobenius automorphism over $\mathbb{Q}_p$. We denote by $\breve{\mathbb{Q}}_p := \widehat{\mathbb{Q}}_p^{\mathrm{nr}}$ the completion of the maximal unramified extension of $\mathbb{Q}_p$. Let $\mathscr{D}$ be a PEL-Shimura datum unramified at p as in [Kot85] ch. 5 such that the associated linear algebraic group G is connected. We denote by $\mathscr{A}_0$ the reduction modulo p of the associated moduli space defined by Kottwitz in [Kot85].

We call Barsotti-Tate groups with PEL-structure induced by $\mathscr{D}$ "Barsotti-Tate groups with $\mathscr{D}$-structure" (for a more precise definition see section 6.2). By Dieudonné theory their isogeny classes correspond to a certain finite subset $B(\mathsf{G}_{\mathbb{Q}_p}, \mu)$ of the set $B(\mathsf{G}_{\mathbb{Q}_p})$ of σ-conjugacy classes in $\mathsf{G}(\breve{\mathbb{Q}}_p)$. For $b \in B(\mathsf{G}_{\mathbb{Q}_p}, \mu)$ denote by $\mathscr{A}_0^b$ the associated Newton stratum of $\mathscr{A}_0$. It is known by Viehmann and Wedhorn that $\mathscr{A}_0^b$ is non-empty ([VW13] Thm. 11.1).

The set $B(\mathsf{G}_{\mathbb{Q}_p})$ is equipped with a partial order, which is given in group theoretic terms. In the "classical" case of Barsotti-Tate groups without additional structure, i.e. $\mathsf{G}_{\mathbb{Q}_p} = \mathrm{GL}_n$, we have the following description of this order. By a result of Dieudonné, the set $B(\mathsf{G}_{\mathbb{Q}_p})$ equals the set of (concave) Newton polygons over $[0, n]$. Then $b' \leq b$ iff the polygons have the same endpoint and b lies above b' (for more details see section 2.2). It is known that the closure of $\mathscr{A}_0^b$ in $\mathscr{A}_0$ is contained in $\mathscr{A}_0^{\leq b} := \bigcup_{b' \leq b} \mathscr{A}_0^{b'}$ by a theorem of Rapoport and Richartz ([RR96] Thm. 3.6). Their result generalises Grothendieck's specialisation theorem which states that (concave) Newton polygons only "go down" under specialisation.

The primary goal of this paper is the following theorem.

Theorem 1.1. *(1) $\mathscr{A}_0^{\leq b}$ is equidimensional of dimension*

$$\langle \rho, \mu + \nu(b) \rangle - \frac{1}{2}\operatorname{def}(b) \tag{1.1}$$

where ρ denotes the half-sum of (absolute) positive roots of G, μ is the cocharacter induced by $\mathscr{D}$ and $\nu(b)$ and $\operatorname{def}(b)$ denote the Newton point resp. the defect of b (cf. section 2.2).

(2) $\mathscr{A}_0^{\leq b}$ is the closure of $\mathscr{A}_0^b$ in $\mathscr{A}_0$.

In the case of the Siegel moduli variety this theorem was proven by Oort (cf. [Oor00] Cor. 3.5).

In the general PEL-case the dimension formula (1.1) proves a conjecture of Chai. In [Cha00] Question 7.6 he conjectured a formula for the codimension of $\mathscr{A}_0^b$ using his notion of chains of Newton points. We prove the equivalence of his conjecture and our dimension formula in section 2.4.

The most important ingredient of the proof of the above theorem is the following result, which is also interesting on its own right.

Theorem 1.2. *Let $\mathscr{M}_G(b, \mu)$ be the underlying reduced subscheme of the Rapoport-Zink space associated to an unramified Rapoport-Zink datum (cf. Def. 7.8).*

(1) The dimension of $\mathscr{M}_G(b, \mu)$ equals

$$\langle \rho, \mu - \nu_G(b) \rangle - \frac{1}{2}\operatorname{def}_G(b) \tag{1.2}$$

(2) If b is superbasic (cf. Def. 2.6) then the connected components of $\mathscr{M}_G(b, \mu)$ are projective.

This theorem is already known in the case of moduli spaces of Barsotti-Tate groups without endomorphism structure by results of Viehmann ([Vie08a], [Vie08b]).

We also show that the same dimension formula holds for affine Deligne-Lusztig varieties, which are a function field analogue of Rapoport-Zink spaces. This result will not be needed for the proof of the above theorems, but its proof gives a short and comprehensible blueprint for the proof of Theorem 1.2.

Theorem 1.3. *Let G be a reductive group over a finite field and $X_\mu(b)$ be an affine Deligne-Lusztig variety in the affine Grassmannian of G (cf. section 3.1). Then*

$$\dim X_\mu(b) = \langle \rho, \mu - \nu_G(b) \rangle - \operatorname{def}_G(b).$$

This proposition generalises the work of Görtz, Haines, Kottwitz and Reuman [GHKR06] and Viehmann [Vie06] which proved that the above proposition holds in the split case.

The dimension formula for Rapoport-Zink spaces and affine Deligne-Lusztig varieties is not identical with the formula conjectured by Rapoport in [Rap05] p. 296. We give a comparison of these two formulas in Remark 2.13.

1.3. **Application to deformation theory.** Let $\underline{X}$ be a Barsotti-Tate group with $\mathscr{D}$-structure over a perfect field k_0 of characteristic p. We denote by $\mathrm{Def}(\underline{X})$ the deformation functor of $\underline{X}$. It is known that $\mathrm{Def}(\underline{X})$ is representable, we denote by $\mathscr{S}_{\underline{X}}$ its algebraisation. By a result of Drinfeld there exists a (unique) algebraisation of the universal deformation of $\underline{X}$ to a Barsotti-Tate group with $\mathscr{D}$-structure over $\mathscr{S}_{\underline{X}}$ (for more details see section 6.3). This induces a Newton stratification on $\mathscr{S}_{\underline{X}}$ for which we use the analogous notation as above. We derive the following theorem from Theorem 1.1 by using a Serre-Tate argument (see section 8.1).

Theorem 1.4. *Denote by b_0 the isogeny class of $\underline{X}$ and let $b \in B(\mathsf{G}_{\mathbb{Q}_p}, \mu)$ with $b \geq b_0$.*

(1) $\mathscr{S}_{\underline{X}}^{\leq b}$ is equidimensional of dimension

$$\langle \rho, \mu + \nu_G(b) \rangle - \frac{1}{2} \operatorname{def}_G(b).$$

(2) $\mathscr{S}_{\underline{X}}^{\leq b}$ is the closure of $\mathscr{S}_{\underline{X}}^{b}$ in $\mathscr{S}_{\underline{X}}$.

In the case of Barsotti-Tate groups without additional structure and for polarised Barsotti-Tate groups (without endomorphism structure) this was also proven by Oort ([Oor00] Thm. 3.2 and Thm. 3.3).

We give an intrinsic characterisation of the Barsotti-Tate groups with additional structure which is induced by some PEL-Shimura datum $\mathscr{D}'$ (and thus Theorem 1.4 applies) in section 7.4. We show that one basically has to exclude case D in Kottwitz's notation and hence call these groups of type (AC).

1.4. **Conjectures of Grothendieck and Koblitz.** Let $\underline{X}_0$ and $\underline{X}_\eta$ be two Barsotti-Tate groups with EL structure or PEL structure of type (AC). We say that $\underline{X}_0$ is the specialisation of $\underline{X}_\eta$ if there exists an integral local scheme S of characteristic p and a Barsotti-Tate group with (P)EL structure $\underline{X}$ over S which has generic fibre $\underline{X}_\eta$ and special fibre $\underline{X}_0$.

Now assume that $\underline{X}_0$ is a specialisation of $\underline{X}_\eta$ and denote by b_0 and b their respective isogeny classes. Then [RR96] Thm. 3.6 states that $b_0 \leq b$. In the case of Barsotti-Tate groups without additional structure this is a result of Grothendieck known as Grothendieck's specialisation theorem. Grothendieck conjectured in a letter to Barsotti (see e.g. the appendix of [Gro74]) that the converse of his specialisation theorem also holds true. He writes "The necessary conditions (1) (2) that G' is a specialisation of G are also sufficient. In other words, taking the formal modular deformation in char. p (over a modular formal variety S [...]) and the BT group G over S thus obtained, we want to know if for every sequence of rational numbers $(\lambda_i)_i$ which satisfies (1) and (2), these numbers occur as the sequence of slopes of a fibre of G at some point S." Here he considers the isogeny class b of a Barsotti-Tate group via the family of the slopes of their Newton polygon and the conditions (1) and (2) reformulate to $b_0 \leq b$ where b_0 denotes the isogeny class of G'.

The following generalisation follows from Theorem 1.4, as it is a reformulation of the non-emptiness of Newton strata. In particular, it was already proven by Oort for Barsotti-Tate groups without additional structure and for polarised Barsotti-Tate groups ([Oor00], Thm. 6.2, Thm. 6.3).

Proposition 1.5. *Let $\underline{X}$ be a Barsotti-Tate group with EL structure or PEL structure of type (AC) and let b_0 denote its isogeny class. For any isogeny class b with $b \geq b_0$ there exists a deformation of $\underline{X}$ which has generically isogeny class b.*

Motivated by Grothendieck's conjecture, Koblitz conjectured in [Kob75] p. 211 that "all totally ordered sequences of Newton polygons can be realized by successive specialisations of principally polarised abelian varieties". In other words, if $\mathscr{A}_0$ is the Siegel moduli space and $b_1 > \ldots > b_h$

is a chain of isogeny classes (or equivalently a chain of symmetric Newton polygons with slopes between 0 and 1) then

$$\overline{\mathscr{A}_0^{b_1} \cap \overline{\mathscr{A}_0^{b_2} \cap \ldots}} \cap \mathscr{A}_0^{b_h} \neq \emptyset$$

The second assertion of Theorem 1.1 implies that the analogue holds for arbitrary Shimura varieties $\mathscr{A}_0$ of PEL-type, as the left hand side equals $\mathscr{A}_0^{b_h}$.

1.5. **Overview.** We first show Theorem 1.3. Its proof is a generalisation of the proof in the split case. First we reduce the theorem to the case where $G = \mathrm{Res}_{k'/k}\, GL_h$ and b is superbasic. The reduction step is almost literally the same as in [GHKR06], we give an outline of the proof and explain how one has to modify the proof of [GHKR06]. Then we focus on proving the theorem in this special case. For this we generalise the proof of Viehmann in [Vie06]. We decompose the affine Deligne-Lusztig variety using combinatorial invariants called extended EL-charts, which generalise the notion of extended semi-modules considered in [Vie06] for $G = GL_h$, and calculate the dimension of each part by generalising the computations in the GL_h-case.

The proof of Theorem 1.1 follows an idea of Viehmann. By an analogous argument as in [Vie13] we prove in section 8 that the dimension formula as well as the closure relations follow if we show that $\dim \mathscr{A}_0^{\leq b}$ is less or equal than the term (1.1). By the work of Mantovan ([Man05]) each Newton stratum is in a finite to finite correspondence with the product of a (truncated) Rapoport-Zink space and a so-called central leaf inside the Newton stratum. In particular the dimension of a Newton stratum is the sum of the dimension of a central leaf and a Rapoport-Zink space. Here a central leaf is defined as the locally closed subset of $\mathscr{A}_0(\overline{\mathbb{F}}_p)$ where $\underline{A}[p^\infty] \cong \underline{X}$ for a fixed Barsotti-Tate group with $\mathscr{D}$-structure $\underline{X}$. In sections 9 and 10 we calculate the dimension of the central leaves, thus reducing Theorem 1.1 to the claim that $\mathscr{M}_G(b,\mu)$ has dimension less or equal than the term (1.2).

We construct a correspondence between $\mathscr{M}_G(b,\mu)$ and a disjoint union of Rapoport-Zink spaces associated to data with superbasic σ-conjugacy classes in section 12 by using similar moduli spaces as in [Man08]. In section 13 we translate the dimension of the fibres of the correspondence into group theoretical terms and calculate it in section 14. This allows us to reduce to the case of a superbasic Rapoport-Zink datum of EL type in section 15. In section 16 we prove Theorem 1.2 in this special case thus finishing the proof of Theorem 1.1.

The reduction step which reduces the task of estimating the dimension of Rapoport-Zink spaces to the case of a superbasic datum of EL type is an analogy of the reduction step in the case of affine Deligne-Lusztig varieties. After introducing the notion of "numerical dimension", we easily translate the proof for affine Deligne-Lusztig varieties to Rapoport-Zink spaces. The proof of Theorem 1.2 in the superbasic EL case in section 16 follows the proof in [Vie08a] replacing semi-modules by EL-charts.

The article is subdivided as follows. In section 2 we recall the group theoretical notions and results which will be used throughout this article. We prove Theorem 1.3 in the sections 3 and 4 using the calculations done in section 5. The rest of the paper then focuses on proving Theorem 1.1. The sections 6 - 7.3 are mostly recapitulations of already known facts. In sections 8 to 10 we consider the Newton stratification on the special fibre of Shimura varieties. We give an overview of the relationship between the Theorems 1.1, 1.2 and 1.4 in section 11. In the subsequent sections we exclusively deal with the geometry of Rapoport-Zink spaces.

Notation 1.6. As we will work over p-adic fields as well as over Laurent series fields, there does not exist a sustained notation in this paper. However, the following notation will be used most of the time and we will indicate when it has a different meaning.

For any non-archimedean local field F we denote by O_F its ring of integers, by k_F its residue field and $q_F := \#k_F$. We denote $\Gamma_F := \mathrm{Gal}(\bar{k}_F, k_F) = \mathrm{Aut}_{F,\mathrm{cont}}(\breve{F})$ and by $\sigma_F \in \Gamma_F$ the Frobenius automorphism. When working over a ground field F_0 we will often abbreviate $\Gamma := \Gamma_{F_0}$ and $\sigma := \sigma_{F_0}$; unless said otherwise we assume that our ground field is $\mathbb{Q}_p$. For any ring R we denote by $W(R)$ its Witt-vectors. We denote by k an arbitrary algebraically closed field of characteristic p and $L = W(k)_{\mathbb{Q}}$, $O_L = W(k)$ or $L = k(\!(t)\!), O_L = k[\![t]\!]$. Starting in section 12 we will assume $k = \overline{\mathbb{F}}_p$.

In most cases we will denote objects defined over global fields by letters in sans serif while we use the usual italic letters for objects defined over local fields.

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2. Group theoretic preliminaries

2.1. **Reductive group schemes over the ring of integers of a local field.** We recall the following definitions from [SGA3].

Definition 2.1. Let S be an arbitrary scheme. A group scheme $G \to S$ is called reductive if the structure morphism is smooth and affine and for every geometric point $\bar{s}$ of S the linear algebraic group $G_{\bar{s}}$ is reductive.

Definition 2.2. Let $G \to S$ be a reductive group scheme.

(1) A maximal torus of G is a subtorus $T \subset G$ such that $T_{\bar{s}}$ is a maximal torus of $G_{\bar{s}}$ for all geometric points $\bar{s}$ of S.
(2) A Borel subgroup of G is a subgroup $B \subset G$ such that $B_{\bar{s}}$ is a Borel subgroup of $G_{\bar{s}}$ for all geometric points $\bar{s}$ of S.

In the case where $S = \operatorname{Spec} R$ is the spectrum of a local ring we make the following definitions. A reductive group scheme G over S is called *split* if it contains a maximal split torus and it is called *quasi-split* if it contains a Borel subgroup. The notions of split resp. quasi-split reductive group schemes also exist over arbitrary bases, but are a bit more complicated.

We fix a non-archimedean local field F_0 and a uniformizer ε and we denote $\Gamma := \Gamma_{F_0}$. Let G be a reductive group scheme over O_{F_0}. Then it is automatically quasi-split and splits over a finite unramified extension O_F of O_{F_0} (cf. [VW13] A.4). We fix $T \subset B \subset G$, where T is a maximal torus and B a Borel subgroup of G. Denote

$$\begin{aligned} X^*(T) &= \underline{\mathrm{Hom}}(T, \mathbb{G}_m) \\ X_*(T) &= \underline{\mathrm{Hom}}(\mathbb{G}_m, T) \end{aligned}$$

These sheaves become constant after base change to O_F, thus we regard them as finite groups with $\mathrm{Aut}_{O_{F_0}}(O_F)$ action. We obtain canonical isomorphisms of Galois-modules

$$X_*(T) \cong X_*(T_{F_0}) \cong X_*(T_{k_{F_0}})$$

(and analogously for $X^*(T)$) where we also identify $\mathrm{Gal}(F/F_0) = \mathrm{Aut}_{O_{F_0}}(O_F) = \mathrm{Gal}(k_F/k_{F_0})$. We denote by R (resp. $R^\vee$) the set of absolute roots (resp. coroots) of G with respect to T, that is the lifts of the absolute roots (resp. coroots) of G_{k_F} to O_F. This definition coincides with the definition of roots (resp. coroots) of G_{O_F} given in [SGA3], Exp. XXII, ch. 1. In particular, the absolute roots (resp. coroots) of G also coincide with the absolute roots (resp. coroots) of G_{F_0} w.r.t. the identifications above. We denote by R^+, Δ^+ and $R^{\vee,+}, \Delta^{\vee,+}$ the system of positive/simple roots resp. positive/simple coroots determined by B. Let ρ be the half-sum of all positive roots. We denote by $\pi_1(G)$ the fundamental group of G, i.e. the quotient of $X_*(T)$ by the coroot lattice.

The Weyl group of G is defined as the quotient $W := (\mathrm{Norm}_G\, T)/T$. It is represented by a finite étale scheme which becomes constant after base change to O_F. Thus we may identify $W = (\mathrm{Norm}_G\, T)(O_F)/T(O_F)$ with the canonical Galois action. In particular W is canonically isomorphic to the absolute Weyl groups of $G_{k_{F_0}}$ and G_{F_0} equipped with Galois action.

We denote by $\tilde{W} := \mathrm{Norm}_G(T)(\breve{F}_0)/T(O_{\breve{F}_0}) \cong \mathrm{Norm}_G\, T(F)/T(O_F)$ the (absolute) extended affine Weyl group of G, equipped with the canonical Galois action. We will often consider an element $x \in \tilde{W}$ as an element of $G(F)$ by which we mean an arbitrary lift of x. We have $\tilde{W} \cong W \rtimes X_*(T)$ canonically; denote by ε^μ the image of a cocharacter μ in $\tilde{W}$. The canonical inclusion of the affine Weyl group W_a into $\tilde{W}$ yields a short exact sequence

$$0 \longrightarrow W_a \longrightarrow \tilde{W} \longrightarrow \pi_1(G) \longrightarrow 0.$$

The isomorphism $\tilde{W} \cong W \rtimes X_*(T)$ defines an action of $\tilde{W}$ on the apartment $\mathfrak{a} := X_*(T)_{\mathbb{R}}$ by affine linear maps. As W_a acts simply transitively on the set of alcoves in $\mathfrak{a}$, the stabilizer $\Omega \subset \tilde{W}$ of a fixed "base" alcove defines a right-splitting of the exact sequence above. We choose the alcove in the anti-dominant chamber whose closure contains the origin as the base alcove. This alcove corresponds to the Iwahori subgroup $\mathscr{I}$ of $G(\breve{F}_0)$ which is defined as the pre-image of $B(\overline{\mathbb{F}}_p)$ w.r.t. the canonical projection $G(O_{\breve{F}_0}) \twoheadrightarrow G(\overline{\mathbb{F}}_p)$. We define the length function on $\tilde{W}$ by

$$\ell(w\tau) = \ell(w).$$

for $w \in W_a, \tau \in \Omega$. In particular, the elements of length 0 are precisely those which are contained in Ω.

Remark 2.3. As one deduces from the root datum of G, its generic fibre is an unramified reductive linear algebraic group (i.e. quasi-split and splits over an unramified extension field). On the other any unramified reductive linear algebraic group over F_0 has an integral reductive model given by the Bruhat-Tits group scheme associated to a hyperspecial point. Furthermore this model is unique up to isomorphism as a reductive group scheme over O_{F_0} is uniquely determined by its root datum (cf. [SGA3] Exp. XXIII Cor. 5.1).

2.2. **σ-conjugacy classes.** We briefly recall Kottwitz's classification of σ-conjugacy classes in the case of unramified groups. The main reference for this subsection is the article of Rapoport and Richartz [RR96].

We keep the notation of the previous subsection. Let k be an algebraically closed extension field of k_{F_0} and let $L = W(k)_{\mathbb{Q}}$ resp. $L = k(\!(t)\!)$. Recall that two elements $b, b' \in G(L)$ are called σ-conjugated if there exists an element $g \in G(L)$ such that $b' = gb\sigma(g)^{-1}$. The equivalence classes with respect to this relation are called σ-conjugacy classes; we denote the σ-conjugacy class of an element $b \in G(L)$ by $[b]$. Let $B(G)$ denote the set of all σ-conjugacy classes in $G(L)$. By [RR96] Lemma 1.3 the sets of σ-conjugacy classes does not depend on the choice of k (up to canonical bijection), so this notation is without ambiguity.

Kottwitz assigns in [Kot85] to each σ-conjugacy class $[b]$ two functorial invariants

$$\nu_G(b) \in X_*(T)^{\Gamma}_{\mathbb{Q},\mathrm{dom}}$$

$$\kappa_G(b) \in \pi_1(G)_\Gamma,$$

which are called the Newton point resp. the Kottwitz point of $[b]$. Those two invariants determine $[b]$ uniquely.

Recall that we have the Cartan decomposition

$$G(L) = \bigsqcup_{\mu \in X_*(T)_{\mathrm{dom}}} G(O_L)\varepsilon^{\mu} G(O_L).$$

An estimate for ν_G and κ_G on a $G(O_L)$-double coset is given by the generalised Mazur inequality. Before we can state it, we need to introduce some more notation. We equip $X_*(T)_{\mathbb{Q}}$ with a partial order $\preceq$ where we say that $\mu' \preceq \mu$ if $\mu - \mu'$ is a linear combination of positive coroots with positive (rational) coefficients. For any cocharacter $\mu \in X_*(T)$ we denote by $\bar{\mu}$ the the average of its Γ-orbit.

Proposition 2.4 ([RR96], Thm. 4.2)**.** *Let $b \in G(O_L)\varepsilon^{\mu}G(O_L)$ for $\mu \in X_*(T)_{\mathrm{dom}}$. Then the following assertions hold.*

(1) We have $\nu_G(b) \preceq \mu$.
(2) The Kottwitz point $\kappa_G(b)$ equals the image of μ in $\pi_1(G)_\Gamma$.

Definition 2.5. (1) We define the partial order $\leq$ on $B(G)$ by

$$[b'] \leq [b] :\Leftrightarrow \nu_G(b') \preceq \nu_G(b) \text{ and } \kappa_G(b') = \kappa_G(b).$$

(2) We denote

$$\begin{aligned} B(G,\mu) &= \{[b] \in B(G) \mid \nu_G(b) \preceq \mu \text{ and } \kappa_G(B) \text{ is the image of } \mu \text{ in } \pi_1(G)_\Gamma\} \\ &= \{[b] \in B(G) \mid [b] \preceq [\mu(\varepsilon)]\}. \end{aligned}$$

By the generalised Mazur inequality $B(G,\mu)$ contains all σ-conjugacy classes which intersect $G(O_L)\varepsilon^{\mu}G(O_L)$ non-emptily. It is known that the converse is also true. Many authors have worked on this conjecture, the result in the generality we need was proven by Kottwitz and Gashi ([Kot03] ch. 4.3, [Gas10] Thm. 5.2).

To every σ-conjugacy class $[b]$ one associates linear algebraic groups M_b and J_b which are defined over $\mathbb{Q}_p$. The group M_b is defined as the centraliser of $\nu_G(b)$ in $G_{\mathbb{Q}_p}$. So in particular M_b is a standard Levi subgroup of $G_{\mathbb{Q}_p}$. Kottwitz showed that the intersection of $M_b(L)$ and $[b]$ is non-empty ([Kot85] ch. 6). The group J_b represents the functor

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} L) \mid gb = b\sigma(g)\}.$$

This group is an inner form of M_b which (up to canonical isomorphism) does not depend on the choice of the representative of $[b]$ ([Kot85] § 5.2).

Definition 2.6. Let $[b] \in B(G)$.

(1) $[b]$ is called basic if $\nu_G(b)$ is central.
(2) $[b]$ is called superbasic if every intersection with a proper Levi subgroup of $G_{\mathbb{Q}_p}$ is empty.

We note that M_b is a proper subgroup of $G_{\mathbb{Q}_p}$ if and only if $[b]$ is not basic. As $M_b(L)$ intersects $[b]$ non-trivially, this observation shows that every superbasic σ-conjugacy class is also basic. We have a bijection between the basic σ-conjugacy classes of G and $\pi(G)_\Gamma$ induced by the Kottwitz point ([Kot85] Prop. 5.6).

Finally we can define the last group theoretic invariant which appears in the dimension formula.

Definition 2.7. Let $[b] \in B(G)$. We define the defect of $[b]$ by

$$\mathrm{def}_G(b) := \mathrm{rk}\, G_{\mathbb{Q}_p} - \mathrm{rk}\, J_b$$

We finish this subsection with two important examples of σ-conjugacy classes, which we will use later on.

2.2.1. *Application to F_0-spaces.* Recall that an F_0-space over k is a pair (V,Φ) of a finite dimensional vector space V over L together with a bijective σ-semilinear map $V\to V$. In the equal characteristic case F_0-spaces are often called "local shtukas" and in the unequal characteristic case one uses the term "σ-isocrystal". The dimension of V is called the height of (V,Φ).

Let $G=\mathrm{GL}_n$. We have a one-to-one correspondence

$$\begin{array}{rcl} B(G) & \leftrightarrow & \{F_0-\text{spaces over } k \text{ of height } n\}/\cong \\ [b] & \mapsto & (L^n, b\sigma). \end{array}$$

The above bijection is easy to see, as a base change of $(L^n,b\sigma)$ by the matrix g replaces b with $gb\sigma(g)^{-1}$.

Now we choose T to be the diagonal torus and B to be the Borel subgroup of upper triangular matrices. Then we have canonical isomorphisms $X_*(T)_{\mathbb{Q}}^{\Gamma}=X_*(T)_{\mathbb{Q}}=\mathbb{Q}^n$ and $\pi_1(G)_\Gamma=\pi_1(G)=\mathbb{Z}$. The first isomorphism identifies

$$X_*(T)_{\mathbb{Q},\mathrm{dom}}=\{\nu=(\nu_1,\ldots,\nu_n)\in\mathbb{Q}^n\mid \nu_1\geq\ldots\geq\nu_n\}.$$

Then $\nu_G(b)=(\nu_1,\ldots,\nu_n)$ is the vector of Newton slopes of the F-space $(L^n,b\sigma)$ given in descending order. Of course, this already determines $[b]$ uniquely. The Kottwitz point is given by

$$\kappa_G(b)=\mathrm{val}\det b=\nu_1+\ldots+\nu_n.$$

2.2.2. *Application to F_0-spaces with F-action.* If we consider F_0-spaces with additional structure, the isomorphism classes will still naturally correspond to σ-conjugacy classes of a reductive group. The most important example for us is the following.

Let F/F_0 be a finite unramified field extension of degree d and let $G=\mathrm{Res}_{O_F/O_{F_0}}\mathrm{GL}_n$. Similar as above one sees that $[b]\mapsto((F\otimes L)^n, b(1\otimes\sigma))$ defines a bijection between $B(G)$ and the isomorphism classes of F_0-spaces (V,Φ) over k of height n together with an F_0-linear F-action $\iota: F\hookrightarrow \mathrm{Aut}(V,\Phi)$. We have a canonical isomorphism $F\otimes L\cong\prod_{\tau:F\hookrightarrow L}L$ and likewise

$$(L\otimes_{F_0}F)^n\cong\prod_{\tau:F\hookrightarrow L}L^n=:\prod_{\tau:F\hookrightarrow L}N_\tau$$

where the product runs over all F_0-*linear* embeddings of F into L. Then σ defines a bijection of N_τ onto $N_{\sigma\tau}$ and any element $b\in G(L)$ stabilizes the N_τ. Fixing an embedding $\tau:F\hookrightarrow L$, we thus obtain an equivalence of categories

$$\begin{array}{rcl} \{F_0-\text{spaces over } k \text{ of height } n \text{ with } F-\text{action}\} & \to & \{F-\text{spaces over } k \text{ of height } \frac{n}{d}\} \\ (V,\Phi,\iota) & \mapsto & (V_\tau,\Phi^d). \end{array}$$

Using that in $\mathrm{GL}_n(L)$ any element g is σ^d-conjugate to $\sigma(g)$ (this holds as every σ^d-conjugacy class contains a σ-stable element, e.g. a suitable lift of an element in $\tilde{W}$) one sees that the isomorphism class of the object on the right hand side does not depend on the choice of τ. Hence if we denote the Newton slopes of (N_τ,Φ^d) by $(\lambda_1,\ldots,\lambda_n)$ the slopes of the F_0-space (V,Φ) (forgetting the F-action) equals

$$(\underbrace{\frac{\lambda_1}{d},\ldots,\frac{\lambda_1}{d}}_{d\text{ times}},\ldots,\underbrace{\frac{\lambda_n}{d},\ldots,\frac{\lambda_n}{d}}_{d\text{ times}}).$$

We choose T to be the diagonal torus and B to be the Borel subgroup of upper triangular matrices. Then $X_*(T)\cong\prod_{\tau:F\hookrightarrow L}\mathbb{Z}^n$ canonically identifying

$$X_*(T)_{\mathbb{Q},\mathrm{dom}}^{\Gamma}=\{\nu=((\nu_1,\ldots,\nu_n))_\tau\in\prod_{\tau:F\hookrightarrow L}\mathbb{Q}^n\mid\nu_1\geq\ldots\geq\nu_n\}$$

Then by functoriality $\nu_G(b)=((\frac{\lambda_1}{d},\ldots,\frac{\lambda_n}{d}))_\tau$.

2.3. **A formula for the defect.** We keep the notation above. Furthermore, let $\underline{\omega}_1, \ldots, \underline{\omega}_l$ be the sums over all elements in a Galois orbit of absolute fundamental weights of G. Recall that we have an embedding $\pi_1(G) \cong \Omega \hookrightarrow \tilde{W}$. For $\varpi \in \pi_1(G)$ let $\dot{\varpi}$ be its image in $\tilde{W}$. Then by construction $\dot{\varpi}$ is basic and $\kappa_G(\dot{\varpi})$ is the image of ϖ in $\pi_1(G)_\Gamma$.

Proposition 2.8. *Let $b \in G(L)$. Then*

$$\mathrm{def}_G(b) = 2 \cdot \sum_{i=1}^{n} \{\langle \underline{\omega}_i, \nu_G(b) \rangle\}$$

where $\{\cdot\}$ denotes the fractional part of a rational number.

The proposition is a generalisation of [Kot06] Cor. 1.10.2. The proof of the proposition given here is a generalisation of Kottwitz's proof. The calculations will use the following combinatorial result.

Lemma 2.9. *Let $\Psi = (V, R)$ be a reduced root system with basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. We fix the following notations, which will be used until the end of the proof of this lemma.*

$P^\vee :=$ coweight lattice of Ψ.
$Q^\vee :=$ coroot lattice of Ψ.
$\pi_1 := P^\vee / Q^\vee$ denotes the fundamental group.
$I := \mathrm{Aut}\,\Psi$.
$\omega_1, \ldots, \omega_l :=$ fundamental weights of Ψ.
$\varpi_1, \ldots, \varpi_l :=$ images of $\omega_1^\vee, \ldots, \omega_l^\vee$ in π_1.
$\chi_1, \ldots, \chi_l$ are the characters of π_1 defined by $\chi_j(\varpi) := \exp(2\pi i \cdot \langle \underline{\omega}_j, \varpi \rangle)$.
$\Xi := \chi_1 \oplus \cdots \oplus \chi_l$ seen as $I \ltimes \pi_1$-representation. Here the action of I is given by the permutation of the χ_i according to its action on the fundamental weights.

Moreover, we consider $V_{\mathbb{C}}^\vee$ as $I \ltimes \pi_1$-representation, where π_1 acts on $V_{\mathbb{C}}^\vee$ via

$$\pi_1 \cong \Omega \subset W \ltimes P^\vee \twoheadrightarrow W$$

where Ω is the subset of all elements which fix the simple affine roots of Ψ and the last arrow is the canonical projection. Then

$$V_{\mathbb{C}}^\vee \cong \Xi$$

Remark 2.10. If Ψ comes from a root datum of a reductive group G then in general $\pi_1(G)$ is only a subgroup of π_1. We have equality if and only if $P^\vee = X^\vee$, i.e. G is adjoint.

Proof. The assertion that Ξ and $V_{\mathbb{C}}^\vee$ are isomorphic as representations of π_1 was proven in [Kot06] Lemma 4.1.1. In particular, it proves our assertion in the case where I is trivial.

Now the decomposition of Ψ according the the isomorphism class of the connected components of its Dynkin diagram also induces compatible direct sum decompositions of I, π_1, $V_{\mathbb{C}}^\vee$ and Ξ. Thus we may assume that all connected components of its Dynkin diagram are isomorphic. Now let $\Psi = \Psi_0^n$ with Ψ_0 irreducible. Then $I \ltimes \pi_1 = (S_n \ltimes I(\Psi_0)^n) \ltimes \pi_1(\Psi_0)^n = S_n \ltimes (I(\Psi_0) \ltimes \pi_1(\Psi_0))^n$ with the canonical action on $V^\vee = V_{\mathbb{C}}^\vee(\Psi_0)^n$ and $\Xi = \Xi(\Psi_0)^n$. Hence we may assume that $\Psi = \Psi_0$ is irreducible. Because of Kottwitz's result we have to check the assertion only in the cases where I is non-trivial, i.e. when the Dynkin diagram is of type A_l, D_l or E_6.

Consider the space V' of affine linear functions on $V_{\mathbb{C}}^\vee$. Since $V_{\mathbb{C}}^\vee$ is self-contragredient, we have $V' \cong \mathbf{1} \oplus V_{\mathbb{C}}^\vee$ as $I \ltimes \pi_1$ -representations. Let $\Xi' := \mathbf{1} \oplus \Xi$. We show that V' and Ξ' are isomorphic by calculating their characters. It is obvious how to calculate the character of Ξ'. For the character of V' we use that the action of $I \ltimes \pi_1$ permutes the simple affine roots. We obtain for $\tau \cdot \varpi \in I \ltimes \pi_1$

$$\begin{aligned} \mathrm{tr}(\tau \cdot \varpi \mid V') &= \#\text{ simple affine roots fixed by } \tau \cdot \varpi, \\ \mathrm{tr}(\tau \cdot \varpi \mid \Xi') &= 1 + \sum_{i; \tau(\alpha_i) = \alpha_i} \chi_i(\varpi). \end{aligned}$$

Now all data we need to calculate the right hand sides are given in [Bou68] and thus the claim is reduced to some straight forward calculations.

For his we use the notation of [Bou68] ch. VI, planche I,IV and V. That is $\alpha_0, \alpha_1, \ldots, \alpha_l$ denote the simple affine roots, where α_0 is the unique root which is not finite and ω_i (in Bourbaki's notation $\bar{\omega}_i$) denotes the fundamental weight associated to α_i. Note that we already know from Kottwitz's result that

$$\mathrm{tr}(1 \cdot \varpi \mid V') = \mathrm{tr}(1 \cdot \varpi \mid \Xi')$$

for all $\varpi \in \pi_1$. Moreover,

$$\mathrm{tr}(\tau \cdot 1 \mid V') = \mathrm{tr}(\tau \cdot 1 \mid \Xi')$$

(for all $\tau \in I$) is an easy consequence of the above description of the traces.

Case 1: Ψ is of type A_l. We have $\pi_1 = \{1, \varpi_1, \ldots, \varpi_l\} \cong \mathbb{Z}/(l+1)\mathbb{Z}$ with $\varpi_k = \varpi_1^k$. The group π_1 acts on V via

$$\varpi_k(\alpha_j) = \alpha_{j+k \bmod l+1}$$

and $I =: \{1, \tau\}$ acts via

$$\tau(\alpha_j) = \alpha_{-j \bmod l+1}.$$

We obtain

$$\tau\varpi_k(\alpha_j) = \alpha_{-j-k \bmod l+1}$$

Thus

$$\mathrm{tr}(\tau \cdot \varpi^k(\alpha_j)|V') = \begin{cases} 1 & \text{if } l \text{ is even,} \\ 0 & \text{if } l \text{ is odd}, k \text{ is odd,} \\ 2 & \text{if } l \text{ is odd}, k \text{ is even.} \end{cases}$$

Now τ acts fixed point free on the set of simple affine roots if l is even and has the unique fixed point $\alpha_{\frac{l+1}{2}}$ if l is odd. Thus

$$\mathrm{tr}(\tau \cdot \varpi_k \mid \Xi') = 1 = \mathrm{tr}(\tau \cdot \varpi_k(\alpha_j)|V')$$

if l is even and

$$\begin{aligned} \mathrm{tr}(\tau \cdot \varpi_k \mid \Xi') &= 1 + \chi_{\frac{l+1}{2}}(\varpi_k) \\ &= 1 + \chi_{\frac{l+1}{2}}(\varpi_1)^k \\ &= 1 + \exp(2\pi i k \cdot \langle \omega_{\frac{l+1}{2}}, \omega_1^\vee \rangle) \\ &= 1 + \exp(2\pi i k \cdot \langle \omega_{\frac{l+1}{2}}, \frac{1}{l+1}(l\alpha_1^\vee + (l-1)\alpha_2^\vee + \ldots + \frac{l+1}{2}\alpha_{\frac{l+1}{2}}^\vee + \ldots + \alpha_l^\vee)\rangle \\ &= 1 + (-1)^k \\ &= \mathrm{tr}(\tau \cdot \varpi^k(\alpha_j)|V') \end{aligned}$$

if l is odd.

Case 2: Ψ is of type D_4. We have $\pi_1 = \{1, \varpi_1, \varpi_3, \varpi_4\} \cong \mathbb{Z}/2\mathbb{Z}$ and I is isomorphic to the symmetric group $S_{\{1,3,4\}}$ acting on π_1 and the set of simple affine roots by the canonical permutation of indices. In particular every conjugacy class of $I \ltimes \pi_1$ has a representative of the form $\tau \cdot \varpi_4$ or $\tau \cdot 1$ (with $\tau \in I$). Thus it suffices to check that

$$\mathrm{tr}(\tau \cdot \varpi_4 \mid V') = \mathrm{tr}(\tau \cdot \varpi_4 \mid \Xi').$$

Now [Bou68] tells us that

$$\tau \cdot \varpi_4(\alpha_j) = \tau(\alpha_{4-j}) = \alpha_{\tau(4-j)}$$

and

$$\chi_j(\varpi_4) = \exp(2\pi i \cdot \langle \omega_j, \frac{1}{2}(\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + 2\alpha_4^\vee)\rangle) = (-1)^j.$$

Thus,

$$\begin{aligned}
\operatorname{tr}(\tau\cdot\varpi_4 \mid V') &= 1+\delta_{\tau(1),3}+\delta_{\tau(3),1}\\
&= 1+(1-\delta_{\tau(1),1}-\delta_{\tau(1),4})+(1-\delta_{\tau(3),3}-\delta_{\tau(3),4})\\
&= 2-\delta_{\tau(1),1}-\delta_{\tau(3),3}+(1-\delta_{\tau(1),4}-\delta_{\tau(3),4})\\
&= 2-\delta_{\tau(1),1}-\delta_{\tau(3),3}+\delta_{\tau(4),4}\\
&= \operatorname{tr}(\tau\cdot\varpi_4 \mid \Xi').
\end{aligned}$$

Case 3: Ψ is of type D_l, $l>4$ is even. Similar as above, we have $\pi_1=\{1,\varpi_1,\varpi_{l-1},\varpi_l\}\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $I\cong S_{\{l-1,l\}}=:\{1,\tau\}$ with the canonical action an π_1 and the simple affine roots. Thus it suffices to consider the traces of $\tau\cdot\varpi_1$ and $\tau\cdot\varpi_l$. Now

$$\tau\cdot\varpi_l(\alpha_j)=\tau(\alpha_{l-j})=\begin{cases}\alpha_{l-1} & \text{if } j=0,\\ \alpha_l & \text{if } j=1,\\ \alpha_{l-j} & \text{otherwise}\end{cases}$$

and

$$\tau\cdot\varpi_1(\alpha_j)=\begin{cases}\alpha_1 & \text{if } j=0,\\ \alpha_0 & \text{if } j=1,\\ \alpha_l & \text{otherwise.}\end{cases}$$

Thus, $\operatorname{tr}(\tau\cdot\varpi_l\mid V')=1$ and $\operatorname{tr}(\tau\cdot\varpi_1\mid V')=l-1$. Now

$$\begin{aligned}
\chi_j(\varpi_1) &= \exp(2\pi i\cdot\langle\omega_j,\alpha_1^\vee+\dots\alpha_{l-2}^\vee+\frac{1}{2}(\alpha_{l-1}^\vee+\alpha_l^\vee)\rangle)\\
&= \begin{cases}1 & \text{if } j\le l-2\\ -1 & \text{if } j\ge l-1\end{cases}\\
\chi_j(\varpi_l) &= \exp(2\pi i\cdot\langle\omega_j,\frac{1}{2}(\alpha_1^\vee+2\alpha_2^\vee+\dots+(l-2)\alpha_{l-2}^\vee+\frac{l-2}{2}\alpha_{l-1}^\vee+\frac{l}{2}\alpha_l^\vee)\rangle)\\
&= \begin{cases}(-1)^j & \text{if } j\le l-2\\ (-1)^{j-\frac{l}{2}} & \text{if } j\ge l-1.\end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{tr}(\tau\cdot\varpi_1\mid\Xi') &= 1+\sum_{j=1}^{l-2}\chi_j(\varpi_1)=l-1=\operatorname{tr}(\tau\cdot\varpi_1\mid V')\\
\operatorname{tr}(\tau\cdot\varpi_l\mid\Xi') &= 1+\sum_{j=1}^{l-2}\chi_j(\varpi_l)=1=\operatorname{tr}(\tau\cdot\varpi_1\mid V').
\end{aligned}$$

Case 4: Ψ is of type D_l, $l>4$ is odd. We have $\pi_1=\{1,\varpi_1,\varpi_{l-1},\varpi_l\}\cong\mathbb{Z}/4\mathbb{Z}$ with $\varpi_1=\varpi_l^2$ and $I\cong S_{\{l-1,l\}}=:\{1,\tau\}$ with the canonical action on π_1 and V'. As above, we only have to consider $\tau\cdot\varpi_1$ and $\tau\cdot\varpi_l$. As

$$\varpi_l(\alpha_j)=\begin{cases}\alpha_{l-j} & \text{if } j\le l-2,\\ \alpha_0 & \text{if } j=l-1,\\ \alpha_1 & \text{if } j=l,\end{cases}$$

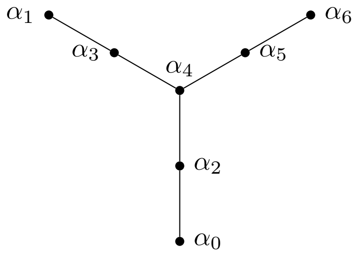
and $\varpi_1=\varpi_j^2$, we obtain

$$\begin{aligned}
\tau\cdot\varpi_1(\alpha_j) &= \begin{cases}\alpha_1 & \text{if } j=0,\\ \alpha_0 & \text{if } j=1,\\ \alpha_j & \text{otherwise,}\end{cases}\\
\tau\cdot\varpi_l(\alpha_j) &= \begin{cases}\alpha_{j+l-1} & \text{if } j\le 1,\\ \alpha_{l-j} & \text{if } 2\le j\le l-2\\ \alpha_{j-(l-1)} & \text{if } l-1\le j.\end{cases}
\end{aligned}$$

Thus, $\mathrm{tr}(\tau \cdot \varpi_1 \mid V') = l-1$ and $\mathrm{tr}(\tau \cdot \varpi_l \mid V') = 0$. Now $\chi_j(\varpi_1)$ and $\chi_j(\varpi_l)$ are as in case 3 (with the convention $(-1)^{1/2} = i$), thus

$$\begin{aligned} \mathrm{tr}(\tau \cdot \varpi_1 \mid \Xi') &= 1 + \sum_{j=1}^{l-2} \chi_j(\varpi_1) = l - 1 = \mathrm{tr}(\tau \cdot \varpi_1 \mid V') \\ \mathrm{tr}(\tau \cdot \varpi_l \mid \Xi') &= 1 + \sum_{j=1}^{l-2} \chi_j(\varpi_l) = 0 = \mathrm{tr}(\tau \cdot \varpi_l \mid V'). \end{aligned}$$

Case 5: Ψ is of type E_6. We have $\pi_1 = \{1, \varpi_1, \varpi_6\} \cong \mathbb{Z}/3\mathbb{Z}$ and $I \cong \mathbb{Z}/2\mathbb{Z} =: \{1, \tau\}$. The group $I \ltimes \pi_1$ acts on V' as follows. Consider the completed Dynkin diagram



Now π_1 acts by rotation and I by reflection at the vertical middle axis. Thus for any $\varpi \in \pi_1$ the action of $\tau \cdot \varpi$ fixes precisely one "branch" of this diagram and hence fixes three simple affine roots. We have

$$\begin{aligned} \chi_j(\varpi_1) &= \exp(2\pi i \cdot \langle \omega_j, \frac{1}{3}(4\alpha_1^\vee + 3\alpha_2^\vee + 5\alpha_3^\vee + 6\alpha_4^\vee + 2\alpha_6^\vee)\rangle \\ &= \begin{cases} 1 & \text{if } j = 2 \text{ or } j = 4, \\ \exp(\frac{2}{3}\pi i) & \text{if } j = 3 \text{ or } j = 6, \\ \exp(\frac{4}{3}\pi i) & \text{if } j = 1 \text{ or } j = 5, \end{cases} \end{aligned}$$

thus $\chi_2 = \chi_4 = 1$. Now

$$\mathrm{tr}(\tau \cdot \varpi \mid \Xi') = 1 + \chi_2(\varpi) + \chi_4(\varpi) = 3 = \mathrm{tr}(\tau \cdot \varpi \mid V').$$

$\square$

The following result from Bruhat-Tits theory is needed to relate the rank of J_b with the twisted Galois action on $X_*(T)_{\mathbb{Q}}$, which is characterized by the previous lemma.

Lemma 2.11. *Let H be a reductive group over a quasi-local field F_0 and A a maximal split torus of $H_{\breve{F}_0}$ defined over F_0. If its apartment $\mathfrak{a}$ contains a $\mathrm{Gal}(F_0^{\mathrm{nr}}/F_0)$-stable alcove C, then A contains a maximal split torus of H.*

Proof. Let $\mathscr{B}(\cdot)$ denote the Bruhat-Tits building of a given group. We identify $\mathscr{B}(H)$ with the $\mathrm{Gal}(F_0^{\mathrm{nr}}/F_0)$-fixed points of $\mathscr{B}(H_{\breve{F}_0})$. Then $\mathscr{B}(H) \cap C$ is a non-empty open subset of $\mathscr{B}(H)$. In particular we have

$$\dim \mathfrak{a}^{\mathrm{Gal}(F^{\mathrm{nr}}/F)} \geq \dim C^{\mathrm{Gal}(F^{\mathrm{nr}}/F)} = \dim \mathscr{B}(H) = \mathrm{rk}\, H.$$

Thus A contains a maximal split torus of H. $\square$

Proof of Proposition 2.8. First we note that the equation does not change if we replace G by M_b^{ad} (cf. [Kot06] Lemma 1.9.1). Thus we may assume that b is basic and G of adjoint type. Hence $[b]$ is uniquely determined by its Kottwitz point, so we may even assume that b is a representative of an element $\tilde{w} \in \Omega$ in the normaliser $(\mathrm{Norm}_G\, T)(L)$.

Now conjugation with b fixes T and the standard Iwahori subgroup $\mathscr{I}$. By Lemma 2.11 the twist of T by b is a maximal torus of J_b which contains a maximal split torus. Now the automorphism of $\mathfrak{a}$ induced by conjugation with $\tilde{w}$ equals the finite Weyl group part of $\tilde{w}$, which we denote by w. Thus $\operatorname{rk} J_b = \dim \mathfrak{a}^{w\sigma}$ and we obtain

$$\operatorname{def}_G(b) = \dim \mathfrak{a}^{\sigma} - \dim \mathfrak{a}^{w\sigma}.$$

As the action of Γ factorises through the automorphism group of the root datum of G, we have an isomorphism $\mathfrak{a} \cong \chi_1 \oplus \cdots \oplus \chi_l$ of $\pi_1(G) \rtimes \Gamma$-representations by Lemma 2.9. Let $\chi_1, \ldots, \chi_m$ be a Γ-orbit with σ mapping χ_j to χ_{j+1}. Let $v = (v_1, \ldots, v_m) \in \chi_1 \oplus \cdots \oplus \chi_m$, then

$$(w\sigma)(v) = (\chi_1(\tilde{w}) \cdot v_n, \chi_2(\tilde{w}) \cdot v_1, \ldots, \chi_m(\tilde{w}) \cdot v_{n-1}).$$

We see that v is fixed by $w\sigma$ if and only if $v_1 = \chi_1(\tilde{w}) \cdot \ldots \cdot \chi_m(\tilde{w}) \cdot v_1$ and $v_j = \chi_j(v_{j-1})$ for $j > 1$. Thus the subspace of $\chi_1 \oplus \cdots \oplus \chi_m$ of vectors fixed by $w\sigma$ has dimension 1 if $\chi_1(\tilde{w}) \cdot \ldots \cdot \chi_m(\tilde{w}) = 1$ and dimension 0 otherwise. We obtain

$$\operatorname{rk} J_b = \dim \mathfrak{a}^{w\sigma} = \#\{i \mid \langle \nu_G(b), \underline{\omega}_i \rangle \in \mathbb{Z}\}.$$

As the $\Omega \rtimes \Gamma$-representation $\mathfrak{a}$ (and thus $\chi_1 \oplus \cdots \oplus \chi_n$) is self-contragredient, we have

$$\begin{aligned} 2\sum_{i=1}^{l} \{\langle \underline{\omega}_i, \nu_G(b) \rangle\} &= \sum_{i=1}^{l} \{\langle \underline{\omega}_i, \nu_G(b) \rangle\} + \sum_{i=1}^{l} \{\langle -\underline{\omega}_i, \nu_G(b) \rangle\} \\ &= l - \#\{i \mid \langle \nu_G(b), \underline{\omega}_i \rangle \in \mathbb{Z}\} \\ &= \operatorname{rk} G - \operatorname{rk} J_b \\ &= \operatorname{def}_G(b). \end{aligned}$$

□

Using the previous proposition and the self-contragredience of Ξ, we obtain the following reformulation of the dimension formula for affine Deligne-Lusztig varieties and Rapoport-Zink spaces

Corollary 2.12. *Let $[b] \in B(G, \mu)$ and ν its Newton point. Then*

$$\langle \rho, \mu - \nu \rangle - \frac{1}{2} \operatorname{def}_G(b) = \sum_{j=1}^{n} \lfloor \langle \underline{\omega}_j, \mu - \nu \rangle \rfloor.$$

Proof.

$$\begin{aligned} \langle \rho, \mu - \nu \rangle - \frac{1}{2} \operatorname{def}_G(b) &= \langle \rho, \mu - \nu \rangle - \sum_{i=1}^{n} \{\langle \underline{\omega}_i, \nu \rangle\} \\ &= \langle \rho, \mu - \nu \rangle - \sum_{i=1}^{n} \{-\langle \underline{\omega}_i, \nu \rangle\} \\ &= \sum_{i=1}^{n} \langle \underline{\omega}_i, \mu - \nu \rangle - \sum_{i=1}^{n} \{\langle \underline{\omega}_i, -\nu \rangle\} \\ &= \sum_{i=1}^{n} \lfloor \langle \underline{\omega}_i, \mu - \nu \rangle \rfloor. \end{aligned}$$

□

Remark 2.13. We note that this formula is not equivalent to the dimension formula conjectured by Rapoport in [Rap05], p. 296

$$\langle 2\rho, \overline{\mu} - \nu \rangle + \sum_{j=1}^{n} \lfloor -\langle \omega_{F_0,j}, \bar{\mu} - \nu \rangle \rfloor,$$

where $\omega_{F_0,1}, \dots, \omega_{F_0,n}$ are the relative fundamental weights. Note that Rapoport's formula becomes correct if one replaces $\omega_{F_0,j}$ by $\underline{\omega}_j$.

2.4. **Chains of Newton points.** Our calculations in the previous subsection allows us to reformulate the dimension formula (1.1) in terms of Chai's chains of Newton points. We first recall some of his notions and results from [Cha00].

Denote by $\mathscr{N}(G)$ the image of ν_G. For $\nu \in \mathscr{N}(G)$ and $[b] \in B(G)$ with $\nu_G(b) = \nu$ the image $\mathscr{N}(G)_{\leq \nu}$ of the set $\{[b'] \in B(G) \mid \nu_G(b') \leq \nu, \kappa_G(b') = \kappa_G(b)\}$ in $\mathscr{N}(G)$ only depends on ν ([Cha00] Prop. 4.4). For elements $\nu'' \leq \nu'$ in $\mathscr{N}(G)_{\leq \nu}$ define

$$[\nu'', \nu'] = \{\xi \in \mathscr{N}(G)_{\leq \nu} \mid \nu'' \leq \xi \leq \nu'\}.$$

We note that $[\nu'', \nu']$ does not change if we replace ν by ν', thus it is independent of the choice of ν. We denote by $\mathrm{length}([\nu', \nu])$ the maximum of all integers n such that there exists a chain $\nu_0 \leq \ldots \leq \nu_n$ in $[\nu', \nu]$.

Chai gave a formula for $\mathrm{length}([\nu', \nu])$ but made a small mistake in his calculations. In the formula at the bottom of page 982 one has to replace the relative fundamental weights $\omega_{F_0,j}$ by the sum of elements of a Galois orbit of absolute fundamental weights $\underline{\omega}_j$. As the $\omega_{F_0 \cdot j}$ and $\underline{\omega}_j$ are scalar multiples of each other, the other assertions in [Cha00] section 7 and the proofs remain valid.

Proposition 2.14 (cf. [Cha00] Thm. 7.4 (iv))**.** *Let $\nu \in \mathscr{N}(G)$ and $\nu' \in \mathscr{N}(G)_{\leq \nu}$. Then*

$$\mathrm{length}([\nu', \nu]) = \sum_{j=1}^{n} \lceil \langle \underline{\omega}_j, \nu \rangle - \langle \underline{\omega}_j, \nu' \rangle \rceil.$$

Now we can reformulate the dimension formula of Theorem 1.1 in a more elegant terms.

Corollary 2.15. *Let $[b] \in B(G, \mu)$ and ν be its Newton point. Then*

$$\langle \rho, \mu + \nu \rangle - \frac{1}{2} \mathrm{def}_G(b) = 2\langle \rho, \mu \rangle - \mathrm{length}([\nu, \mu])$$

Proof.

$$\begin{aligned} \langle \rho, \mu + \nu \rangle - \frac{1}{2} \mathrm{def}_G(b) &= 2\langle \rho, \mu \rangle - (\langle \rho, \mu - \nu \rangle + \frac{1}{2} \mathrm{def}_G(b)) \\ &\overset{\text{Prop. 2.8}}{=} 2\langle \rho, \mu \rangle - \sum_{j=1}^{n} \lceil \langle \underline{\omega}_j, \mu - \nu \rangle \rceil \\ &\overset{\text{Prop. 2.14}}{=} 2\langle \rho, \mu \rangle - \mathrm{length}([\nu, \mu]). \end{aligned}$$

□

3. Dimension of ADLVs: Reduction to the superbasic case

3.1. **Notation.** Let l_0 be a finite field of characteristic p and let $\bar{l}_0$ be an algebraic closure of l_0. We consider a connected reductive group G over l_0. By a theorem of Steinberg, G is quasi-split. Let l be a finite subfield of $\bar{l}_0$ such that G_l is split. We fix $T \subset B \subset G$, where T a maximal torus which splits over k' and B a Borel subgroup of G. We denote $K = G(\bar{l}_0[\![t]\!])$ and by $\mathcal{G}r$ the affine Grassmannian of G. Let $F_0 = l_0(\!(t)\!)$, $F = l(\!(t)\!)$ be the Laurent series fields.

For $\mu \in X_*(T)_{\mathrm{dom}}$ and $b \in G(\bar{l}_0(\!(t)\!))$ the affine Deligne-Lusztig variety is the locally closed subset

$$X_\mu(b)(\bar{l}_0) = \{g \cdot K \in \mathcal{G}r(\bar{l}_0); g^{-1} b \sigma(g) \in K \mu(t) K\}.$$

We equip $X_\mu(b)$ with reduced structure, making it a scheme which is locally of finite type over $\bar{l}_0$. We note that the isomorphism class of $X_\mu(b)$ does not change if we replace b by a σ-conjugate. Indeed, if $b' = hb\sigma(h)^{-1}$ then we have an isomorphism $X_\mu(b) \xrightarrow{\sim} X_\mu(b'), gK \mapsto h \cdot g$.

3.2. **The correspondence to the superbasic case.** The aim of this section is to prove the following proposition.

Proposition 3.1. *Assume Theorem 1.3 is true for each affine Deligne-Lusztig variety $X_\mu(b)$ with $G \cong \mathrm{Res}_{l/l_0} \mathrm{GL}_h$ and b superbasic. Then it is true in general.*

As mentioned in the introduction, we follow the proof given in [GHKR06] for split groups. First we have to fix some more notation. Let

k	be an algebraically closed field extension of $\bar{l}_0$. As mentioned in the introduction, we denote $L = k(\!(t)\!)$ and $O_L = k[\![t]\!]$.
K	denote $G(O_L)$ (considered as subgroup of $G(L)$.
$P = MN$	be a parabolic subgroup of G containing B. We denote by M the corresponding Levi subgroup containing T and by N the unipotent radical of P.
$\mathcal{G}r, \mathcal{G}r_P, \mathcal{G}r_M$	denote the affine Grassmannians of G, P and M respectively.
$\mathcal{G}r^\omega, \mathcal{G}r_M^\omega$	denote the geometric connected component of $\mathcal{G}r$ resp. $\mathcal{G}r_M$ corresponding to $\omega \in \pi_1(G)$ resp. $\omega \in \pi_1(M)$. (cf. [PR08] Thm. 0.1)
x_λ	denote the image of $\lambda(t)$ in $\mathcal{G}r(\bar{k})$ for $\lambda \in X_*(T)$. We use x_0 as "base point" of $\mathcal{G}r(\bar{k})$. For $g \in G(L)$ we write gx_0 for the translate of x_0 w.r.t. the obvious $G(L)$-action on $\mathcal{G}r(k)$.
$X_*(T)_{\mathrm{dom}}$	be the subset of $X_*(T)$ of cocharacters which are dominant w.r.t. $T \subset B \subset G$.
$X_*(T)_{M-\mathrm{dom}}$	be the subset of $X_*(T)$ of cocharacters which are dominant w.r.t. $T \subset B \cap M \subset M$.
R_N	denote the set of roots of T in $\mathrm{Lie}\, N$.
ρ	denote the half-sum of all positive roots in G.
ρ_N	denote the half-sum of all elements of R_N.
$\rho_M = \rho - \rho_N$	denote the half-sum of all positive roots in M.

Furthermore the canonical morphisms $P \hookrightarrow G$ and $P \to M$ induce morphisms of ind-schemes

$$\begin{tikzcd} & \mathcal{G}r_P \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & \\ \mathcal{G}r_M & & \mathcal{G}r_G. \end{tikzcd}$$

The idea of the proof for Proposition 3.1 is to consider the image of an affine Deligne-Lusztig variety $X_\mu(b)$ in $\mathcal{G}r_M$ under the above correspondence, assuming that $b \in M(L)$. We want to show that the image is a union of affine Deligne-Lusztig varieties, which we will later assume to be superbasic and relate the dimension of $X_\mu(b)$ to the dimension of its image.

Let us study the diagram more thoroughly. Certainly π is surjective and ι is bijective on geometric points by the Iwasawa decomposition of G. Thus ι is a decomposition of $\mathcal{G}r$ into locally closed subsets by Lemma 3.2 below. In particular we see that $X_\mu^{P\subset G}(b) := \iota^{-1}(X_\mu(b))$ is also locally of finite type and has the same dimension as $X_\mu(b)$.

Lemma 3.2. *Let $i : I \hookrightarrow H$ be a closed embedding of connected algebraic groups. Then the induced map on the identity components of the affine Grassmannians $i_{\mathcal{G}} : \mathcal{G}r_I^0 \to \mathcal{G}r_H^0$ is an immersion.*

Proof. First recall the following result in the proof of Thm. 4.5.1 of [BD] (see also [Gör10], Lemma 2.12): In the case where H/I is quasi-affine (resp. affine), the induced morphism $\mathcal{G}r_I \to \mathcal{G}r_H$ is an immersion (resp. closed immersion). So we want to replace I by a suitable closed subgroup I' which is small enough such that H/I' is quasi-affine, yet big enough such that the immersion $\mathcal{G}r_{I'}^0 \hookrightarrow \mathcal{G}r_I^0$ is surjective.

Now let

$$0 \longrightarrow R(I)_u \longrightarrow I \longrightarrow I_1 \longrightarrow 0$$

be the decomposition of I into a unipotent and a reductive group. We denote by I_1^{der} the derived group of I_1 and by $R(I_1)$ its radical. As I_1/I_1^{der} is affine, the canonical morphism $\mathcal{G}r^0_{I_1^{\mathrm{der}}} \to \mathcal{G}r^0_{I_1}$ is a closed immersion. Using that $I_1 = R(I_1)\cdot I_1^{\mathrm{der}}$, we see that it is also surjective.

We denote $I' := I \times_{I_1} I_1^{\mathrm{der}}$. As $\pi_1(I) = \pi_1(I_1)$, the canonical morphism $\mathcal{G}r^0_{I'} \to \mathcal{G}r^0_I$ is the pullback of $\mathcal{G}r^0_{I_1^{\mathrm{der}}} \hookrightarrow \mathcal{G}r^0_I$ and hence also a surjective immersion. Furthermore I' has no non-trivial homomorphisms to $\mathbb{G}_m$, hence the quotient H/I' is quasi-affine and $\mathcal{G}r^0_{I'} \to \mathcal{G}r^0_H$ is an immersion. Altogether we have

$$\begin{tikzcd} \mathcal{G}r^0_{I'} \arrow[r, hook, "\text{surj. immersion}"'] \arrow[rr, bend left, "\text{immersion}"] & \mathcal{G}r^0_I \arrow[r, "i_{\mathcal{G}}\ \text{monomorphism}"'] & \mathcal{G}r^0_H, \end{tikzcd}$$

which proves that $i_{\mathcal{G}}$ is an immersion. □

3.3. **The dimension of fibers of π.** In order to the determine the dimension of $X_\mu^{P\subset G}(b)$, we want to calculate the dimension of its fibres under π and its image. For this we need a few auxiliary results.

We fix a dominant, regular, σ-stable coweight $\lambda_0 \in X_*(T)$. We denote for $m \in \mathbb{Z}$

$$N(m) := \lambda_0(t)^m N(O_L)\lambda_0(t)^{-m}$$

Then we have a chain of inclusions $\ldots \supset N(-1) \supset N(0) \supset N(1) \supset \ldots$ and moreover $N(L) = \bigcup_{i\in\mathbb{Z}} N(i)$. Furthermore, we note that $N(-m)/N(n)$ has a canonical structure of a variety for $m, n > 0$.

Definition 3.3. (1) A subset Y of $N(L)$ is called *admissible* if there exist $m, n > 0$ such that $Y \subset N(-m)$ and it is the pre-image of a locally closed subset of $N(-m)/N(n)$ under the canonical projection $N(-m) \twoheadrightarrow N(-m)/N(n)$. For admissible $Y \subset N(L)$ we define the dimension of Y by

$$\dim Y = \dim Y/N(n) - \dim N(0)/N(n).$$

(2) A subset Y of $N(L)$ is called *ind-admissible* if $Y \cap N(-m)$ is admissible for every $m > 0$. For any ind-admissible $Y \subset N(L)$ we define

$$\dim Y = \sup \dim(Y \cap N(-m)).$$

Lemma 3.4. *Let $m \in M(L)$ and $\nu \in X_*(T)^\Gamma_{M-\mathrm{dom},\mathbb{Q}}$ be its Newton point. We denote $f_m : N(L) \to N(L), n \mapsto n^{-1}m^{-1}\sigma(n)m$. Then for any admissible subset Y of $N(L)$ the pre-image $f_m^{-1}Y$ is ind-admissible and*

$$\dim f_m^{-1}Y - \dim Y = \langle \rho, \nu - \nu_{\mathrm{dom}}\rangle.$$

Moreover, f_m is surjective.

Proof. This assertion is the analogue of Prop. 5.3.1 in [GHKR06]. Note that R_N is σ-stable and thus the sets $N[i]$ defined in the proof of Prop. 5.3.2 in [GHKR06] are σ-stable. □

We denote by $p_M : X_*(T) \twoheadrightarrow \pi_1(M)$ the canonical projection.

Definition 3.5. (1) For $\mu \in X_*(T)_{\mathrm{dom}}$ let

$$S_M(\mu) := \{\mu_M \in X_*(T)_{M-\mathrm{dom}};\ N(L)x_{\mu_M} \cap Kx_\mu \neq \emptyset\}.$$

(2) For $\mu \in X_*(T)_{\mathrm{dom}}$, $\kappa \in \pi_1(M)_I$ let

$$S_M(\mu,\kappa) := \{\mu_M \in S_M(\mu);\ \text{the image of } p_M(\mu_M) \text{ in } \pi_1(M) \text{ is } \kappa\}$$

(3) For $\mu \in X_*(T)_{\mathrm{dom}}$ let

$$\begin{aligned} \Sigma(\mu) &:= \{\mu' \in X_*(T);\ \mu'_{\mathrm{dom}} = \mu\} \\ \Sigma(\mu)_{M-\mathrm{dom}} &:= \Sigma(\mu) \cap X_*(T)_{M-\mathrm{dom}} \end{aligned}$$

We denote by $\Sigma(\mu)_{M-\max}$ the set of maximal elements in $\Sigma(\mu)_{M-\mathrm{dom}}$ w.r.t. the Bruhat order corresponding to M.

Lemma 3.6. *For any $\mu \in X_*(T)_{\mathrm{dom}}$ we have inclusions*

$$\Sigma(\mu)_{M-\max} \subset S_M(\mu) \subset \Sigma(\mu)_{M-\mathrm{dom}}.$$

Moreover these sets have the same image in $\pi_1(M)_I$. In particular, $S_M(\mu,\kappa)$ is non-empty if and only if κ lies in the image of $\Sigma(\mu)_{M-\mathrm{dom}}$.

Proof. This is (a slightly weaker version of) Lemma 5.4.1 of [GHKR06] applied to $G_{k'}$. □

Definition 3.7. Let $\mu \in X_*(T)_{\mathrm{dom}}$ and $\mu_M \in \Sigma(\mu)$. We write

$$d(\mu,\mu_M) := \dim(N(L)x_{\mu_M} \cap Kx_\mu).$$

We can extend the definition above to arbitrary elements of $\mathcal{G}r_M(\overline{k})$. Multiplication by an element $k_M \in K_M$ induces an isomorphism $N(L)x_{\mu_M} \cap Kx_\mu \xrightarrow{\sim} N(L)k_M x_{\mu_M} \cap Kx_\mu$, thus we have for each $m \in K_M \mu_M(t) K_M$

$$\dim(N(L)mx_0 \cap Kx_\mu) = d(\mu,\mu_M).$$

Lemma 3.8. *Let $\mu \in X_*(T)_{\mathrm{dom}}$. Then for all $\mu_M \in S_M(\mu)$ we have*

$$d(\mu,\mu_M) \leq \langle \rho, \mu + \mu_M \rangle - 2\langle \rho_M, \mu_M \rangle$$

If $\mu_M \in \Sigma(\mu)_{M-\max}$ this is an equality.

Proof. This is Cor. 5.4.4 of [GHKR06] applied to $G_{k'}$. □

For $b \in M(L)$, $\mu_M \in X_*(T)_{M-\mathrm{dom}}$ we denote by $X^M_{\mu_M}(b)$ the corresponding affine Deligne-Lusztig variety in the affine Grassmannian of M. On the contrary $X_{\mu_M}(b)$ still denotes the affine Deligne-Lusztig variety in $\mathcal{G}r$, assuming that $\mu_M \in X_*(T)_{\mathrm{dom}}$.

Proposition 3.9. *Let $b \in M(L)$ be basic. We denote by $\kappa \in \pi_1(M)_\Gamma$ its Kottwitz point and by $\nu \in X_*(T)^\Gamma_{\mathbb{Q},M-\mathrm{dom}}$ its Newton point.*

(1) The image of $X^{P\subset G}_\mu(b)$ under π is contained in

$$\bigcup_{\mu_M \in S_M(\mu,\kappa)} X^M_{\mu_M}(b).$$

(2) Denote by $\beta : X^{P\subset G}_\mu(b) \to \bigcup_{\mu_M \in S_M(\mu,\kappa)} X^M_{\mu_M}(b)$ the restriction of π. For $\mu_M \in S_M(\mu,\kappa)$ and $x \in X^M_{\mu_M}(b)$ the fibre $\beta^{-1}(x)$ is non-empty and of dimension

$$d(\mu,\mu_M) + \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle - \langle 2\rho_N, \nu \rangle.$$

(3) For all $\mu_M \in S_M(\mu,\kappa)$ the set $\beta^{-1}(X^M_{\mu_M}(b))$ is locally closed in $X^{P\subset G}_\mu(b)$ and

$$\dim \beta^{-1}(X^M_{\mu_M}(b)) = \dim X^M_{\mu_M}(b) + d(\mu,\mu_M) + \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle - \langle 2\rho_N, \nu \rangle.$$

(4) If $X_\mu(b)$ is non-empty it has dimension

$$\sup\{\dim X^M_{\mu_M}(b) + d(\mu,\mu_M);\ \mu_M \in S_M(\mu,\kappa)\} + \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle - \langle 2\rho_N, \nu \rangle.$$

Proof. This is the analogue of [GHKR06], Prop. 5.6.1. The proof of (1)-(3) is the same as in [GHKR06]; as this is the centrepiece of this section, we give a sketch of the proof for the readers convenience. Let $x = gx_0 \in X_\mu(b)(k)$. We write $g = mn$ with $m \in M(L)$, $n \in N(L)$. Then

$$n^{-1}m^{-1}b\sigma(m)\sigma(n) = g^{-1}b\sigma(g) \in K\mu(t)K \tag{3.1}$$

As $N(L) \subset P(L)$ is a normal subgroup, this implies

$$N(L) \cdot (m^{-1}b\sigma(m)) \cap K\mu(t)K \neq \emptyset.$$

Thus $m^{-1}b\sigma(m) \in K\mu_M(t)K$ for a unique $\mu_M \in S_M(\mu)$, i.e. $\alpha(x) \in X^M_{\mu_M}(b)(k)$ proving (1).

Now let $x = mx_0 \in X^M_{\mu_M}(b)(k)$ and $b' = m^{-1}b\sigma(m)$. Then $(\beta^{-1}(x))(k)$ is the set of all mnx_0 satisfying (3.1), which is equivalent to

$$(n^{-1}b'\sigma(n)b'^{-1})b' \in K\mu(t)K.$$

Thus

$$(\beta^{-1}(x))(k) \cong f_b^{-1}(K\mu(t)Kb'^{-1} \cap N(L))/N(0).$$

Hence we get

$$\begin{array}{rcl}\dim \beta^{-1}(x) & \overset{\text{Lem. 3.4}}{=} & \dim(K\mu(t)Kb'^{-1} \cap N(L)) - \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle \\ & = & (N(L)b'x_0 \cap Kx_\mu) + \dim(b'N(0)b'^{-1}) - \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle \\ & = & d(\mu, \mu_M) - \langle 2\rho_N, \nu \rangle + \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle,\end{array}$$

where the second equality is true because $N(L)b'x_0 \cap Kx_\mu \cong (K\mu(t)Kb'^{-1} \cap N(L))/b'N(0)b'^{-1}$. As k can be chosen arbitrarily, this gives (2). Now (3) follows from (2) because source and target of β are locally of finite type over $\bar{l}_0$.

Finally we prove (4). Since

$$X_\mu^{P \subset G}(b) = \bigcup_{\mu_M \in S_M(\mu, \kappa)} \beta^{-1}(X^M_{\mu_M}(b))$$

is a decomposition into locally closed subsets, we have

$$\dim X_\mu(b) = \dim X_\mu^{P \subset G}(b) = \sup\{\dim X^M_{\mu_M}(b);\ \mu_M \in S_M(\mu, \kappa)\}.$$

Applying (3) to this formula finishes the proof. ☐

3.4. **The reduction step.** The previous proposition already implies the main part of Proposition 3.1 as follows:

Proposition 3.10. *Let $b \in M(L)$ be basic. Assume that Theorem 1.3 holds true for $X^M_{\mu_M}(b)$ for every $\mu_M \in S_M(\mu, \kappa)$. Then it is also true for $X_\mu(b)$.*

Proof. This is a consequence of Lemma 3.8 and Proposition 3.9. Its proof is literally the same as the proof of its analogue Prop. 5.8.1 in [GHKR06]. ☐

In particular, replacing b by a σ-conjugate if necessary, we may choose M such that b is superbasic in M. Thus we have reduced Theorem 1.3 to the case where b is superbasic. Now it is only left to show that we may assume $G = \mathrm{Res}_{k'/k} \mathrm{GL}_h$.

For this we show that it suffices to prove Theorem 1.3 for the adjoint group G^{ad}. We denote by subscript "ad" the image of elements of $G(L)$ resp. $X_*(T)$ resp. $\pi_1(G)$ in $G^{\mathrm{ad}}(L)$ resp. $X_*(T^{\mathrm{ad}})$ resp. $\pi_1(G^{\mathrm{ad}})$. For $\omega \in \pi_1(G)$ we write $X_\mu(b)^\omega := X_\mu(b) \cap \mathcal{G}r^\omega$. Then it is easy to see that if $X_\mu(b)^\omega$ is non-empty,

$$X_\mu(b)^\omega \cong X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^{\omega_{\mathrm{ad}}}. \tag{3.2}$$

Now in Lemma 3.1.1 of [CKV] it is proven that if G is of adjoint type and contains a superbasic element $b \in G(L)$, then

$$G \cong \prod_{i=1}^{r} \operatorname{Res}_{l_i/l_0} \mathrm{PGL}_{h_i},$$

where the l_i are finite field extensions of l_0. As

$$X_{(\mu_i)_{i=1}^r}((b_i)_{i=1}^r) \cong \prod_{i=1}^{r} X_{\mu_i}(b_i)$$

it suffices to prove Theorem 1.3 for b superbasic and $G \cong \operatorname{Res}_{l/l_0} \mathrm{PGL}_h$. Using the isomorphism (3.2) again, we may also assume $G \cong \operatorname{Res}_{l/l_0} \mathrm{GL}_h$, which finishes the proof of Proposition 3.1.

4. Dimension of ADLVs: The superbasic case

4.1. **Notation and Conventions.** We keep the notation of the previous section with $G = \operatorname{Res}_{l/l_0} \mathrm{GL}_h$. We denote $d := [l : l_0] = [F : F_0]$ and identify $I := \mathrm{Gal}(F/F_0) = \mathbb{Z}/d\mathbb{Z}$ such that σ is mapped to 1. Let T be the diagonal torus and B the Borel subgroup of lower triangular matrices in G.

We fix a superbasic element $b \in G(L)$ with Newton point $\nu \in X_*(T)_{\mathbb{Q}}^{\Gamma}$ and a cocharacter $\mu \in X_*(T)$. Using the formula of Corollary 2.12, we have to show that if $X_\mu(b)$ is non-empty, then

$$\dim X_\mu(b) = \sum_{i=1}^{h-1} \lfloor \langle \underline{\omega}_i, \mu - \nu \rangle \rfloor.$$

For every l-algebra R the R-valued points of G are given by $G(R) \cong \mathrm{End}_{l \otimes_{l_0} R}(l \otimes_{l_0} R^h)$. We denote $N = k \otimes_{l_0} L^h$, which is canonically isomorphic to the direct sum $\bigoplus_{\tau \in I} N_\tau$ of isomorphic copies of L^h. The Frobenius element σ acts via the canonical action of $\mathrm{Aut}_{F_0}(L)$ on N, for all $\tau \in I$ we have $\sigma : N_\tau \xrightarrow{\sim} N_{\tau+1}$. We fix a basis $(e_{\tau,i})_{i=1}^h$ of the N_τ such that $\varsigma(e_{\tau,i}) = e_{\varsigma\tau,i}$ for all $\varsigma \in I$. For $\tau \in I, l \in \mathbb{Z}, i = 1, \ldots, h$ denote $e_{\tau,i+l\cdot h} := t^l \cdot e_{\tau,i}$. Then each $v \in N_\tau$ can be written uniquely as infinite sum

$$v = \sum_{n \gg -\infty} a_n \cdot e_{\tau,n}$$

with $a_n \in k$.

Now we denote by M^0 the O_L-submodule of N generated by the $e_{\tau,i}$ for $i \geq 0$. With respect to our choice of basis, K is the stabilizer of M^0 in $G(L)$ and $g \mapsto gM^0$ defines a bijection

$$\mathcal{G}r(k) \cong \{M = \prod_{\tau \in I} M_\tau;\ M_\tau \text{ is a lattice in } N_\tau\}.$$

Suppose we are given two lattices $M, M' \subset L^h$. By the elementary divisor theorem we find a basis $v_1, \ldots, v_n$ of M and a unique tuple of integers $a_1 \leq \ldots \leq a_n$ such that $t^{a_1}v_1, \ldots, t^{a_n}v_n$ from a basis of M'. We define the cocharacter $\mathrm{inv}(M, M') : \mathbb{G}_m \to \mathrm{GL}_h, x \mapsto \mathrm{diag}(x^{a_1}, \ldots, x^{a_n})$. If we write $M' = gM$ with $g \in GL_h(L)$ we may equivalently define $\mathrm{inv}(M, M')$ to be the unique cocharacter of the diagonal torus which is dominant w.r.t. the Borel subgroup of lower triangular matrices and satisfies $g \in \mathrm{GL}_h(O_L)\,\mathrm{inv}(M, M')(t)\,\mathrm{GL}_h(O_L)$.

In terms of the notation introduced above we have

$$X_\mu(b)(k) \cong \{(M_\tau \subset N_\tau \text{ lattice})_{\tau \in I};\ \mathrm{inv}(M_\tau, b\sigma(M_{\tau-1})) = \mu_\tau\}.$$

Definition 4.1. (1) We call a tuple of lattices $(M_\tau \subset N_\tau)_{\tau \in I}$ a *G-lattice*.

(2) We define the *volume* of a G-lattice $M = gM^0$ to be the tuple
$$\mathrm{vol}(M) = (\mathrm{val}\det g_\tau)_{\tau\in I}.$$
Similarly, we define the volume of M_τ to be $\mathrm{val}\det g_\tau$. We call M *special* if $\mathrm{vol}(M) = (0)_{\tau\in I}$.

The assertion that b is superbasic is equivalent to ν being of the form $(\frac{m}{d\cdot h}, \frac{m}{d\cdot h}, \ldots, \frac{m}{d\cdot h})$ with $(m,h) = 1$. Then by [Kot03], Lemma 4.4 $X_\mu(b)$ is non-empty if and only if ν and μ have the same image in $\pi_1(G)_I$, which is equivalent to $\sum_{\tau\in I, i=1,\ldots h} \mu_{\tau,i} = m$. We assume that this equality holds from now on.

Since the affine Deligne-Lusztig varieties of two σ-conjugated elements are isomorphic, we can assume that b is the form $b(e_{\tau,i}) = e_{\tau,i+m_\tau}$ where $m_\tau = \sum_{i=1}^h \mu_{\tau,i}$ (cf. [CKV] Lemma 3.2.1). We could have chosen any tuple of integers (m_τ) such that $\sum_{\tau\in I} m_\tau = m$ but this particular choice has the advantage that the components of any G-lattice in $X_\mu(b)$ have the same volume. In general,
$$\begin{aligned}\mathrm{vol}\, M_\tau - \mathrm{vol}\, M_{\tau-1} &= (\mathrm{vol}\, M_\tau - \mathrm{vol}\, b\sigma(M_{\tau-1})) + (\mathrm{vol}\, b\sigma(M_{\tau-1}) - \mathrm{vol}\, M_{\tau-1}) \\ &= (\sum_{i=1}^h \mu_{\tau,i}) - m_\tau.\end{aligned}$$

Furthermore we have for each central cocharacter $\nu' \in X_*(S)$ the obvious isomorphism
$$X_\mu(b) \xrightarrow{\sim} X_{\mu+\nu'}(\nu'(t)\cdot b).$$
So we may (and will) assume that all entries of μ are non-negative (w.r.t. the standard identification $T \cong (\mathbb{Z}^h)^I$), which amounts to saying that we have $b\sigma(M) \subset M$ for G-lattices $M \in X_\mu(b)(\overline{k})$.

Recall that the geometric connected components of $\mathcal{G}r$ are in bijection with $\pi_1(G) = \mathbb{Z}^I$. This bijection is given by mapping a G-lattice to its volume. Thus the subsets of G-lattices in $\mathcal{G}r$ resp. $X_\mu(b)$ obtained by restricting the value of the volume of the components is open and closed. Denote by $X_\mu(b)^i \subset X_\mu(b)$ the subset of all G-lattices M such that M_0 (or equivalently every M_τ) has volume i. Let $\pi \in J_b(F_0)$ be the element with $\pi(e_{\tau,i}) = e_{\tau,i+1}$ for all $\tau \in I, i \in \mathbb{Z}$. Then $g\cdot K \mapsto \pi g \cdot K$ defines an isomorphism $X_\mu(b)^i \xrightarrow{\sim} X_\mu(b)^{i+1}$. Thus $\dim X_\mu(b) = \dim X_\mu^0(b)$ so that it is enough to consider the subset of special lattices.

4.2. **Decomposition of $X_\mu(b)^0$.** In order to calculate the dimension of the affine Deligne-Lusztig variety, we decompose $X_\mu(b)^0$ as follows. Denote
$$\begin{aligned}\mathscr{I}_\tau : N_\tau \setminus \{0\} &\to \mathbb{Z}\\ \sum_{n\gg-\infty} a_n\cdot e_{\tau,n} &\mapsto \min\{n\in\mathbb{Z}; a_n \neq 0\}.\end{aligned}$$
Note that $\mathscr{I}_\tau$ satisfies the strong triangle inequality for every τ. We denote $N_{hom} := \coprod_{\tau\in I}(N_\tau \setminus \{0\})$, analogously M_{hom}, and define the index map
$$\mathscr{I} := \sqcup \mathscr{I}_\tau : N_{hom} \to \coprod_{\tau\in I}\mathbb{Z}$$
For $M \in X_\mu(b)^0(k)$, we define
$$A(M) := \mathscr{I}(M_{hom})$$
and a map $\varphi(M) : \coprod_{\tau\in I}\mathbb{Z} \to \mathbb{Z}$ by
$$\varphi(M)(a) = \max\{n \in \mathbb{N}_0; \exists v \in M_{hom} \text{ with } \mathscr{I}_\tau(v) = a, t^{-n}b\sigma(v) \in M_{hom}\}.$$
Now we decompose $X_\mu^0(b)$ such that $(A(M),\varphi(M))$ is constant on each component. We postpone the description of the invariants (A,φ) and their properties into the next chapter, as we will also use them to calculate the dimension of Rapoport-Zink spaces.

We fix a lattice $M \in X_\mu(b)^0(k)$. First we show that $(A(M), \varphi(M))$ is an extended EL-chart (cf. Definition 5.10). We define $B(M) := A(M) \setminus (A(M) + h)$.

Lemma 4.2. *Let $a_{(\tau)} \in A(M)$ such that $c \in A(M)$ for every $c \geq a_{(\tau)}$. Then $\{v \in N_\tau; \mathscr{I}_\tau(v) \geq a\} \subset M_\tau$.*

Proof. We denote $M' := \{v \in N_\tau;\ \mathscr{I}_\tau(v) \geq a\}$ and $M'' := M' \cap M_\tau$. For $b = a, \ldots, a+h-1$ choose $v_b \in M''$ with $\mathscr{I}_\tau(v_b) = b$. Obviously we can write any element $x \in M'$ in the form

$$x = \sum_{b=a}^{a+h-1} \alpha_b \cdot v_b + x'$$

with $\alpha_b \in k$ and $x' \in t \cdot M' = \{v \in N_\tau;\ \mathscr{I}(v) \geq a+h\}$. Thus $M' = M'' + t \cdot M'$ and the claim follows by Nakayama's lemma. □

Lemma 4.3. *Let $M \in X_\mu(b)^0(k)$. Then $(A, \varphi) := (A(M), \varphi(M))$ is an extended EL-chart for μ.*

Proof. Let us first check that A is a normalized EL-chart. It is stable under f and the addition of h since

$$\mathscr{I}(t \cdot v) = \mathscr{I}(v) + h \tag{4.1}$$

$$\mathscr{I}(b\sigma(v)) = f(\mathscr{I}(v)) \tag{4.2}$$

and $t \cdot M \subset M$ and $b\sigma(M) \subset M$. The fact that A is bounded from below is obvious. As $\{g(e_{0,i}) \mid 1 \leq i \leq h\}$ is a basis of M_0 we have

$$B_{(0)} = \{\mathscr{I}(g(e_{0,i} \mid 1 \leq i \leq h\},$$

hence

$$0 = \operatorname{val}\det g_0 = \sum_{i=1}^{h} \left\lfloor \frac{\mathscr{I}(g(e_{0,i}))}{h} \right\rfloor = \sum_{b_{(0)} \in B_{(0)}} \left\lfloor \frac{b}{h} \right\rfloor = h \cdot (\sum_{b_{(0)} \in B_{(0)}} b - \frac{h(h-1)}{2})$$

and thus A is indeed a normalized EL-chart.

Now φ satisfies property (a) of Definition 5.10 by definition. To see that it satisfies (b) and (c), fix $a \in A$ and let $v \in M_{hom}$ such that $\mathscr{I}(v) = a$ and $t^{-\varphi(a)} \cdot b\sigma(v) \in M$. Then $t^{-\varphi(a)+1} b\sigma(t \cdot v) = t^{-\varphi(a)} \cdot b\sigma(v) \in M$, proving (b) and $f(a) - \varphi(a) \cdot h = \mathscr{I}(t^{-\varphi(a)} \cdot b\sigma(v)) \in A$ which implies the inequality part of (c). Let $a \in A$ such that $c \in A$ for every $c \geq a$. We choose an element $v' \in M_{hom}$ with $\mathscr{I}(v') = f(a) - \operatorname{ht} f(a)$ and denote $v = t^{\operatorname{ht} f(a)} \cdot (b\sigma)^{-1}(v')$. Then $\mathscr{I}(v) = a$, thus $v \in M$ by Lemma 4.2 and $t^{-\operatorname{ht} f(a)} \cdot b\sigma(v) = v' \in M$. Thus $\varphi(a) = \operatorname{ht} f(a)$. To verify that φ has property (d), we fix $\tau \in I$ and define for $a \in \mathbb{Z}_{(\tau)} \cup \{-\infty\}, n \in \mathbb{N}$ the k-vector space

$$V'_{a,n} := \{v \in M_\tau \mid v = 0 \text{ or } \mathscr{I}(v) \geq a, t^{-n} \cdot b\sigma(v) \in M\}$$

and $V_{a,n} := V'_{a,n}/V'_{a,n+1}$. Now associate to every $c \in \{a' \in A_{(\tau)};\ a' \geq a\} \cap \varphi^{-1}(\{n\})$ an element $v_c \in M_\tau$ with $\mathscr{I}(v_c) = c$ and $t^{-n} \cdot v_c \in M$. Using the strong triangle inequality for $\mathscr{I}_\tau$, we see that the images v_c in $V_{a,n}$ are linearly independent. Thus

$$\dim V_{a,n} \geq \#\{a' \in A_{(\tau)} \mid a' \geq a, \varphi(a') = n\}.$$

By counting dimensions in a suitable finite dimensional quotient of $V'_{a,0}$, we see that this is in fact an equality. Now the images of the $t \cdot v_c$ in $V_{a+h,n+1}$ are also linearly independent, thus

$$\#\{a' \in A_{(\tau)} \mid a' \geq a, \varphi(a') = n\} \leq \#\{a' \in A_{(\tau)} \mid a' \geq a+h, \varphi(a') = n+1\}.$$

Now it remains to show that (A, φ) has Hodge point μ. But

$$\#\{i \mid \mu_{\tau,i} = n\} = \dim V_{-\infty,n} - \dim V_{-\infty,n-1} = \#(A_{(\tau)} \cap \varphi^{-1}(\{n\})) - \#(A_{(\tau)} \cap \varphi^{-1}(\{n-1\})).$$

□

This proof also shows that $A(M)$ is an EL-chart for every G-lattice $M \subset N$ and $A(M)$ is normalized if and only if M is special. For any extended EL-chart (A, φ) for μ we denote

$$\mathscr{S}_{A,\varphi} = \{M \in \mathcal{G}r;\ (A(M), \varphi(M)) = (A, \varphi)\}.$$

Since the Hodge point of M and $(A(M), \varphi(M))$ coincide by the lemma above, we have indeed $\mathscr{S}_{A,\varphi} \subset X_\mu(b)^0$.

Lemma 4.4. *The $\mathscr{S}_{A,\varphi}$ define a decomposition of $X_\mu(b)^0$ into finitely many locally closed subsets. In particular,* $\dim X_\mu(b)^0 = \max_{(A,\varphi)} \dim \mathscr{S}_{A,\varphi}$.

Proof. By Lemma 4.3, $X_\mu(b)^0$ is the (disjoint) union of the $\mathscr{S}_{A,\varphi}$ and as there are only finitely many extended EL-charts for μ by Corollary 5.18, this union is finite. It remains to show that $\mathscr{S}_{A,\varphi}$ is locally closed. One shows that the condition $(A(M)_{(\tau)}, \varphi(M)_{|A(M)_{(\tau)}}) = (A_{(\tau)}, \varphi_{|A_{(\tau)}})$ is locally closed analogously to the proof of Lemma 4.2 in [Vie06]. Then it follows that $\mathscr{S}_{A,\varphi}$ is locally closed as it is the intersection of finitely many locally closed subsets. □

Proposition 4.5. *Let (A, φ) be an extended EL-chart for μ and $\mathscr{V}(A, \phi)$ be as in Definition 5.19. There exists an open subscheme $U_{A,\varphi} \subseteq \mathbb{A}^{\mathscr{V}(A,\varphi)}$ and a morphism $U_{A,\varphi} \to \mathscr{S}_{A,\varphi}$ which is bijective on geometric points. In particular,* $\dim \mathscr{S}_{A,\varphi} = \#\mathscr{V}(A, \varphi)$.

Proof. The proof is almost the same as of Thm. 4.3 in [Vie06]. We give an outline of the proof and explain how to adapt the proof of Viehmann to our more general notion.

For any l_0-algebra R and $x \in R^{\mathscr{V}(A,\varphi)} = \mathbb{A}^{\mathscr{V}(A,\varphi)}(R)$ we denote the coordinates of x by $x_{a,c}$. We associate to every x a set of elements $\{v(a) \in N_{hom}; a \in A\}$ which satisfies the following equations.

If $a = y := \max\{b \in B_{(0)}\}$ then

$$v(a) = e_a + \sum_{(a,c) \in \mathscr{V}(A,\varphi)} x_{a,c} \cdot v(c).$$

For any other element $a \in B$ we want

$$v(a) = v' + \sum_{(a,c) \in \mathscr{V}(A,\varphi)} x_{a,c} \cdot v(c)$$

where $v' = t^{-\varphi(a')} \cdot b\sigma(v(a'))$ for a' being minimal satisfying $f(a') - \varphi(a') \cdot h = a$. At last, if $a \notin B$, we impose

$$v(a) = t \cdot v(a - h) + \sum_{(a,c) \in \mathscr{V}(A,\varphi)} x_{a,c} \cdot v(c).$$

Claim 1. The set $\{v(a) \mid a \in A\}$ is uniquely determined by the equations above.

Hence the rule $x \mapsto M(x) := \langle v(a);\, a \in A\rangle_{O_L}$ is well-defined and as it is obviously functorial, induces a morphism $\mathbb{A}^{\mathscr{V}(A,\varphi)} \to \mathcal{G}r$. But the image of this morphism is in general not contained in $\mathscr{S}_{A,\varphi}$, we only have the following assertions:

Claim 2. For every $x \in \mathbb{A}^{\mathscr{V}(A,\varphi)}(k)$ we have $A(M(x)) = A$ and $\varphi(M(x))(a) \geq \varphi(a)$ for every $a \in A$.

Claim 3. The pre-image $U(A, \varphi)$ of $\mathscr{S}_{A,\varphi}$ is non-empty and open in $\mathbb{A}^{\mathscr{V}(A,\varphi)}$.

Now the fact that the restriction $U(A, \varphi) \to \mathscr{S}_{A,\varphi}$ of above morphism defines a bijection of k-valued points follows from the following assertion.

Claim 4. Let $M \subset N$ be a special G-lattice such that $(A(M), \varphi(M)) = (A, \varphi)$. Then there exists a unique set of elements $\{v(a) \mid a \in A\} \subset M$ satisfying the equations above.

It remains to prove the four claims. But their proofs are literally the same as in [Vie06] if one replaces "$a+m$" and "$a+im$" by "$f(a)$" respectively "$f^i(a)$". □

Now we can finish the proof of Theorem 1.3.

Corollary 4.6. *We have*

$$\dim X_\mu(b)^0 = \sum_{j=1}^{n} \lfloor \langle \underline{\omega}_j, \mu - \nu \rangle \rfloor.$$

Proof. By Proposition 4.5 we have

$$\dim X_\mu(b)^0 = \max\{\#\mathscr{V}(A,\varphi) \mid (A,\varphi) \text{ is an extended EL-chart for } \mu\}.$$

By Proposition 5.20 and Theorem 5.21 we have indeed

$$\max\{\#\mathscr{V}(A,\varphi) \mid (A,\varphi) \text{ is an extended EL-chart for } \mu\} = \sum_{j=1}^{n} \lfloor \langle \underline{\omega}_j, \mu - \nu \rangle \rfloor$$

□

5. Extended EL-charts

In this section we consider the combinatorial invariants, which are used to calculate the dimension of affine Deligne-Lusztig varieties and Rapoport-Zink spaces in the superbasic case. We fix an extension l/l_0 of finite fields and denote by I its Galois group. Let $d := [l : l_0]$, then $I \cong \mathbb{Z}/d\mathbb{Z}$.

5.1. **Combinatorics for $G = \mathrm{Res}_{l/l_0} \mathrm{GL}_n$.** Let $T \subset B \subset G$ denote the diagonal torus and the Borel subgroup of lower triangular matrices. As G splits over F, the action of $\mathrm{Gal}(\bar{l}_0, l_0)$ on $X_*(T)$ factorizes over I. We identify $X_*(T) = \prod_{\tau\in I} \mathbb{Z}^h$ with I acting by cyclically permuting the factors. This yields an identification of $X_*(T)^I$ with $\mathbb{Z}^h$ such that

$$X_*(T)^I \hookrightarrow X_*(T), \nu \mapsto (\nu)_{\tau\in I}.$$

Furthermore, we denote for an element $\mu \in X_*(T)$ by $\underline{\mu} \in X_*(T)^I$ the sum of all I-translates of μ. We impose the same notation as above for $X_*(T)_\mathbb{Q} = \prod_{\tau\in I} \mathbb{Q}^h$ and $X_*(S)_\mathbb{Q} = \mathbb{Q}^h$.

We note that an element $\nu \in \mathbb{Q}^h$ is dominant if $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_h$ and $\mu \in \prod_{\tau\in I} \mathbb{Q}^h$ is dominant if μ_τ is dominant for every $\tau \in I$.

The Bruhat order is defined on $X_*(T)_{\mathrm{dom}}$ such that an element μ' dominates μ if and only if $\mu' - \mu$ is a non-negative linear combination of relative resp. absolute positive coroots. We write $\mu \preceq \mu'$ in this case. This motivates the following definition. For $\nu, \nu' \in \mathbb{Q}^h$ we write $\nu \preceq \nu'$ if

$$\begin{aligned} \sum_{i=1}^{j} \nu_i &\geq \sum_{i=1}^{j} \nu'_i \quad \text{for all } j < n \\ \sum_{i=1}^{n} \nu_i &= \sum_{i=1}^{n} \nu'_i. \end{aligned}$$

For $\mu, \mu' \in \prod_{\tau\in I} \mathbb{Q}^h$ we write $\mu \preceq \mu'$ if $\mu_\tau \preceq \mu'_\tau$ for every $\tau \in I$. If ν and ν' resp. μ and μ' are both dominant, this order coincides with the Bruhat order.

Let

$$\begin{aligned} (\mathbb{Q}^h)^0 &:= \{\nu \in \mathbb{Q}^h \mid \nu_1 + \ldots + \nu_n = 0\} \cong X_*(T^{\mathrm{der}})^I_\mathbb{Q} \\ (\prod_{\tau\in I} \mathbb{Q}^h)^0 &:= \{\mu \in \prod_{\tau\in i} \mathbb{Q}^h \mid \mu_\tau \in (\mathbb{Q}^h)^0 \text{ for all } \tau\} \cong X_*(T^{\mathrm{der}})_\mathbb{Q} \end{aligned}$$

where T^{der} is the pre-image of T in the derived group G^{der}. We denote by $\omega_{F_0,i}$ resp. $\underline{\omega}_i$ the relative fundamental weights of G^{der} resp. the sum over all elements of the Galois orbit of absolute fundamental weights lying over $\omega_{F_0,i}$. After renumbering the fundamental weights if necessary, we have for $\nu \in (\mathbb{Q}^h)^0$ and $\mu \in (\prod_{\tau\in I}\mathbb{Q}^h)^0$

$$\begin{aligned} \langle \omega_{F_0,i}, \nu\rangle &= -\sum_{j=1}^{i} \nu_j \\ \langle \underline{\omega}_i, \mu\rangle &= -\sum_{j=1}^{i} \underline{\mu}_j \end{aligned}$$

Definition 5.1. (1) We define for $\nu,\nu' \in \mathbb{Q}^h$ with $\nu'-\nu \in (\mathbb{Q}^h)^0$

$$\delta(\nu,\nu') := \sum_{i=1}^{h-1} \lfloor \langle \omega_{F_0,i}, \nu'-\nu\rangle \rfloor.$$

(2) For $\mu,\mu' \in \prod_{\tau\in I}\mathbb{Q}^h$ with $\mu'-\mu \in (\prod_{\tau\in I}\mathbb{Q}^h)^0$ let

$$\delta_G(\mu,\mu') := \sum_{i=1}^{h-1} \lfloor \langle \underline{\omega}_i, \mu'-\mu\rangle \rfloor = \delta(\underline{\mu},\underline{\mu}').$$

We now give a geometric interpretation of $\delta(\nu,\nu')$ in terms of polygons in a special case that covers all applications in this paper.

Definition 5.2. To an element $\nu \in \mathbb{Q}^h$ we associate a polygon $P(\nu)$ which is defined over $[0,h]$ with starting point $(0,0)$ and slope ν_i over $(i-1,i)$. We also denote by $P(\nu)$ the corresponding piecewise linear function on $[0,h]$.

Let $\nu,\nu' \in \mathbb{Q}^h$ with $\nu \preceq \nu'$ and $\nu' \in \mathbb{Z}^h$. Then

$$\begin{aligned} \delta(\nu,\nu') &= \sum_{i=1}^{h-1} \lfloor \langle \omega_{F_0,i}, \nu'-\nu\rangle \rfloor \\ &= \sum_{i=1}^{h-1} \lfloor P(\nu)(i) - P(\nu')(i) \rfloor \\ &= \sum_{i=1}^{h-1} \lfloor P(\nu)(i) \rfloor - P(\nu')(i). \end{aligned}$$

Now $\nu \preceq \nu'$ amounts to saying that $P(\nu)$ lies above $P(\nu')$ and that these two polynomials have the same endpoint. It follows from the first assertion of the lemma that $\delta(\nu,\nu')$ is equal to the number of lattice points which are on or below $P(\nu)$ and above $P(\nu')$.

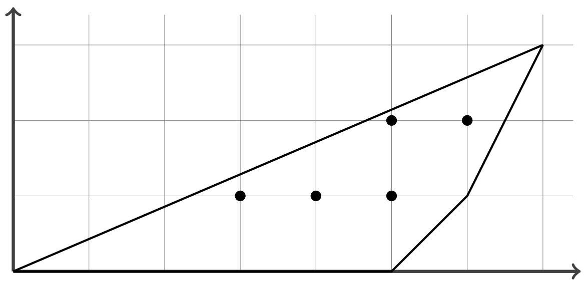


FIGURE 1. Geometric interpretation of $\delta(\nu,\nu') = 5$ for $\nu = (\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7}), \nu' = (0,0,0,0,0,1,2)$.

Finally we prove two lemmas, which we will use in section 5.3. The reader may skip the rest of this subsection for the moment.

Lemma 5.3. *Let $\nu \in \mathbb{Z}^h$. Then*

$$\delta(\nu, \nu_{\mathrm{dom}}) = \sum_{1 \leq i < j \leq h} \max\{\nu_i - \nu_j, 0\}.$$

Proof. The assertion follows from the following observation. If $\nu' \in \mathbb{Z}^h$ with $\nu'_i > \nu'_{i+1}$ and we swap these coordinates, then $P(\nu')(i)$ is reduced by the difference of these two values while the value of $P(\nu')$ at all other integers remains the same. Now ν_{dom} is obtained from ν by carrying out the above transposition repeatedly until the coordinates are in increasing order. Since we have swapped the coordinates ν_i and ν_j during this construction if and only if $i < j$ and $\nu_i > \nu_j$ we get the above formula. □

Lemma 5.4. *Let $\nu \in \mathbb{Z}^h$ be dominant, $1 \leq i \leq j \leq h, \beta \in \mathbb{Z}_{\geq 0}$ and*

$$\nu' := (\nu_1, \ldots, \nu_{i-1}, \nu_i - \beta, \nu_{i+1}, \ldots, \nu_{j-1}, \nu_j + \beta, \nu_{j+1}, \ldots, \nu_h).$$

Then

$$\delta(\nu, \nu'_{\mathrm{dom}}) = \left(\sum_{k=1}^{\beta} \sum_{l=\nu_i-\beta}^{\nu_j-1} \#\{n;\, \nu_n = k + l\} \right) - \beta.$$

Proof. Obviously we have

$$\delta(\nu, \nu') = (j - i) \cdot \beta = \left(\sum_{i \leq n \leq j} \beta \right) - \beta$$

and by the previous lemma

$$\delta(\nu', \nu'_{\mathrm{dom}}) = \sum_{n<i:\, \nu_i-\beta<\nu_n} (\nu_n - (\nu_i - \beta)) + \sum_{n>j:\, \nu_n<\nu_j+\beta} (\nu_j + \beta - \nu_n).$$

Using $\delta(\nu, \nu'_{\mathrm{dom}}) = \delta(\nu, \nu') + \delta(\nu', \nu'_{\mathrm{dom}})$ one easily deduces the above assertion. □

Corollary 5.5. *Let $\nu, \nu' \in \mathbb{Z}^h$ be dominant with $\nu \preceq \nu'$ such that the multiset of their coordinates differs by only two elements. Say n_2, n_3 in the multiset of coordinates of ν are replaced by n_1, n_4 in the multiset of coordinates of ν' with $n_1 \leq n_2 \leq n_3 \leq n_4$. Then*

$$\delta(\nu, \nu') = \sum_{k=0}^{n_4-n_3-1} \sum_{l=0}^{n_4-n_2-1} |\{n;\, \nu'_n = n_4 - k - l - 1\}|.$$

Proof. The assertion is just a reformulation of the previous lemma. □

5.2. **Definition and basic properties of extended EL-charts.** We keep the notation from the previous subsection. Furthermore we fix a tuple of integers $(m_\tau)_{\tau\in I}$ such that $m := \sum_{\tau \in I} m_\tau$ is positive and coprime to h. We denote $\nu := (\frac{m}{d \cdot h}, \ldots, \frac{m}{d \cdot h}) \in \mathbb{Z}^h$.

Definition 5.6. Let $\mathbb{Z}^{(d)} := \coprod_{\tau \in I} \mathbb{Z}_{(\tau)}$ be the disjoint union of d isomorphic copies of $\mathbb{Z}$. For $a \in \mathbb{Z}$ we denote by $a_{(\tau)}$ the corresponding element of $\mathbb{Z}_{(\tau)}$ and write $|a_{(\tau)}| := a$. We equip $\mathbb{Z}^{(d)}$ with a partial order "$\leq$" defined by

$$a_{(\tau)} \leq c_{(\varsigma)} :\Leftrightarrow a \leq c \text{ and } \tau = \varsigma$$

and a $\mathbb{Z}$-action given by

$$a_{(\tau)} + n = (a + n)_{(\tau)}.$$

Furthermore we define a function $f : \mathbb{Z}^{(d)} \to \mathbb{Z}^{(d)}, a_{(\tau)} \mapsto (a + m_{\tau+1})_{(\tau+1)}$.

We impose the notation that for any subset $A \subset \mathbb{Z}^{(d)}$ we write $A_{(\tau)} := A \cap \mathbb{Z}_{(\tau)}$.

Definition 5.7. (1) An *EL-chart* in $(\mathbb{Z}^{(d)}, f, h)$ is a non-empty subset $A \subset \mathbb{Z}^{(d)}$ which is bounded from below, stable under f and satisfies $A + h \subset A$.
(2) Let A be an EL-chart and $B = A \setminus (A + h)$. We say that A is *normalized* if $\sum_{b_{(0)} \in B_{(0)}} b = \frac{h \cdot (h-1)}{2}$.

Our next aim is to give a characterization of EL-charts. For this let A be an EL-chart and $B := A \setminus (A + h)$. Obviously $\# B = d \cdot h$. We define a sequence $b_0, \ldots b_{d \cdot h - 1}$ of distinct elements of B as follows. Denote by b_0 the minimal element of $B_{(0)}$. If b_i is already defined, we denote by b_{i+1} the unique element which can be written as

$$b_{i+1} = f(b_i) - \mu'_{i+1} \cdot h$$

for some $\mu'_{i+1} \in \mathbb{Z}$. These elements are indeed distinct: If $b_i = b_j$ then obviously $i \equiv j \mod d$ and then $b_{i+k \cdot d} \equiv b_i + k \cdot m \mod h$ implies that $i = j$ as m and h are coprime. This reasoning also shows that if we define $b_{d \cdot h}$ according to the recursion formula above, we get $b_{d \cdot h} = b_0$. Therefore we will consider the index set of the b_i and μ'_i as $\mathbb{Z}/dh\mathbb{Z}$. We define

$$\operatorname{succ}(b_i) := b_{i+1}$$

and call $\mu' = (\mu'_i)_{i=1,\ldots,d \cdot h}$ the type of A.

At some point, it may be helpful to distinguish the b_i's and μ'_i's of different components. For this we may change the index set to $I \times \{1, \ldots, h\}$ via

$$\begin{array}{rcl} b_{\tau,i} & := & b_{\tau+(i-1)d} \\ \mu'_{\tau,i} & := & \left\{ \begin{array}{ll} \mu'_{\tau+(i-1)d} & \text{if } \tau \neq 0 \\ \mu'_{id} & \text{if } \tau = 0 \end{array} \right. . \end{array}$$

Here we choose that standard set of representatives $\{0, \ldots, d-1\} \subset \mathbb{Z}$ for I.

With the change of notation we have that $b_{\tau,i} \in B_{(\tau)}$ for all i, τ and that $b_{0,1}$ is the minimal element of $B_{(0)}$ and we have the recursion formula

$$\begin{array}{rcll} b_{\tau+1,i} & = & f(b_{\tau,i}) - \mu'_{\tau+1,i} h & \text{if } \tau \neq d-1 \\ b_{0,i} & = & f(b_{d-1,i-1}) - \mu'_{0,i-1} h. & \end{array}$$

Lemma 5.8. *(1) For every EL-chart A there exists a unique integer n such that A + n is normalized.*
(2) Mapping an EL-chart to its type induces a bijection between normalized EL-charts and the set $\{\mu' \in \prod_{\tau \in I} \mathbb{Z}^h; \underline{\mu}' \succeq \underline{\nu}\}$.

Proof. (1) In order to obtain a normalized EL-chart, we have to choose

$$n = \frac{1}{h} \cdot \left(\frac{h \cdot (h-1)}{2} - \sum_{b_{(0)} \in B_{(0)}} b \right).$$

Since by definition every residue modulo h occurs exactly once in $B_{(0)}$, this is indeed an integer.
(2) Since an EL-chart A is uniquely determined by $A \setminus (A+h)$ which is, up to $\mathbb{Z}$-action, uniquely determined by the type of A, we know the type induces an injection on the set of normalized EL-charts into $\prod_{\tau \in I} \mathbb{Z}^h$. For $1 \leq k \leq h-1$ we have

$$\begin{array}{rc} & b_0 \leq b_{kd} \\ \Leftrightarrow & b_0 \leq b_0 + k \cdot d \cdot m - \sum_{i=1}^{k} (\underline{\mu'}_i) \cdot h \\ \Leftrightarrow & \sum_{i=1}^{k} \underline{\mu}'_i \leq k \cdot \frac{m}{h} \\ \Leftrightarrow & \sum_{i=1}^{k} \underline{\mu}'_i \leq \sum_{i=1}^{k} \underline{\nu}_i \end{array}$$

Similarly one shows the equivalence of $b_0 = b_{hd}$ and $\sum_{i=1}^h \underline{\mu}'_i = \sum_{i=1}^h \underline{\nu}_i$. Thus if μ' is the type of an EL-chart, we have $\underline{\mu}' \succeq \underline{\nu}$. On the other hand this also shows that any such μ' is the type of some EL-chart and thus by (1) also the type of a normalized EL-chart. □

Definition 5.9. For $a \in A$ we call $\mathrm{ht}(a) := \max\{n \in \mathbb{N}_0; a - n \cdot h \in A\}$ the *height of* a.

Definition 5.10. (1) An *extended EL-chart* is a pair (A, φ) where A is a normalized EL-chart and $\varphi : \mathbb{Z}^{(d)} \to \mathbb{N}_0 \cup \{-\infty\}$ such that the following conditions hold for every $a \in \mathbb{Z}^{(d)}$.
   (a) $\varphi(a) = -\infty$ if and only if $a \notin A$.
   (b) $\varphi(a + h) \geq \varphi(a) + 1$.
   (c) $\varphi(a) \leq \mathrm{ht}\, f(a)$ if $a \in A$ with equality if $\{c \in \mathbb{Z}^{(d)} \mid c \geq a\} \subset A$.
   (d) $\#\{c \in \mathbb{Z}^{(d)} \mid c \geq a, \varphi(c) = n\} \leq \#\{c \in \mathbb{Z}^{(d)} \mid c \geq a + h, \varphi(c) = n + 1\}$ for all $n \in \mathbb{N}_0$.

(2) An extended EL-chart is called *cyclic* if equality holds in (c) for every $a \in A$.
(3) The *Hodge-point* of an extended EL-chart (A, φ) is the dominant cocharacter $\mu \in \prod_{\tau \in I} \mathbb{Z}^h$ for which the coordinate n occurs with multiplicity $\#\left(A_{(\tau)} \cap \varphi^{-1}(\{n\})\right) - \#\left(A_{(\tau)} \cap \varphi^{-1}(\{n-1\})\right)$ in μ_τ. We also say that (A, φ) is an extended EL-chart for μ.

Remark 5.11. Because of condition (c) we have $\#\left(A_{(\tau)} \cap \varphi^{-1}(\{n\})\right) = h$ for every $\tau \in I$ and sufficiently large n. Thus the Hodge point is indeed an element of $\prod_{\tau \in I} \mathbb{Z}^h$.

Except for condition (d) the definition of an EL-chart is obviously a generalisation of Definition 3.4 in [Vie06]. As we will frequently refer to Viehmann's paper we give an equivalent condition for (d) which is easily seen to be a generalisation of condition (4) in her definition. However, we will not use this assertion in the sequel.

Lemma 5.12. *For every (A, φ) satisfying conditions (a)-(c) of Definition 5.10, (d) may equivalently be replaced with the following condition. For every τ, we can write $A_{(\tau)} = \bigcup_{l=1}^h \{a_j^{\tau,l}\}_{j=0}^\infty$ with*

(a) *$\varphi(a_{j+1}^{\tau,l}) = \varphi(a_j^{\tau,l}) + 1$*
(b) *If $\varphi(a_j^{\tau,l} + h) = \varphi(a_j^{\tau,l}) + 1$, then $a_{j+1}^{\tau,l} = a_j^{\tau,l} + h$, otherwise $a_{j+1}^{\tau,l} > a_j^{\tau,l} + h$.*

Then the Hodge point is the dominant cocharacter associated to $(a_0^{\tau,l})_{\substack{l=1,\ldots,h \\ \tau \in I}}$.

Proof. If we have a decomposition of A as above, it is obvious that (d) is true. Now let (A, φ) be an extended EL-chart. We construct the sequences $(a_j^{\tau,l})_{j \in \mathbb{N}}$ separately for each τ. So fix $\tau \in I$. We construct the sequences by induction on the value of φ. Take every element of A for which φ has minimal value as initial element for some sequence. Now if we have sorted all elements $a \in A$ with $\varphi(a) \leq n$ in sequences $(a_j^{\tau,l})_{j \in \mathbb{N}}$ we proceed as follows. Condition (d) of Definition 5.10 guarantees that we can continue all our sequences such that they satisfy (a) and (b). If there are still some $a \in A$ with $\varphi(a) = n + 1$ which are not already an element of a sequence, we take them as initial objects for some sequences. Since $\#\left(\varphi^{-1}(\{n\}) \cap A_{(\tau)}\right) = h$ for $n \gg 0$, we get indeed h sequences. □

Lemma 5.13. *Let A be an EL-chart of type μ'. There exists a unique φ_0 such that (A, φ_0) is a cyclic extended EL-chart. The Hodge point of (A, φ_0) is μ'_{dom}.*

Proof. The function $\varphi_0 : \mathbb{Z}^{(d)} \to \mathbb{N} \cup \{-\infty\}$ is uniquely determined by equality in (c) and condition (a). For any $a \in A$ we get $\varphi_0(a + h) = \varphi_0(a) + 1$, which proves (b) and (d). The second assertion follows from $\mu'_{i+1} = \varphi_0(b_i)$. □

The following construction will help us to deduce assertions for general extended EL-charts from the assertion in the cyclic case.

Notation 5.14. For $\mathbf{i}, \mathbf{n} \in \mathbb{Z}^d$ we write $\mathbf{i} \leq \mathbf{n}$ if this inequality holds true coordinate-by-coordinate.

Definition 5.15. Let (A,φ) be an extended EL-chart and (A,φ_0) the cyclic extended EL-chart associated to A. For any $\tau\in I$ we denote

$$\{x_{\tau,1},\dots,x_{\tau,n_\tau}\}=\{a\in A_{(\tau)}\mid \varphi(a+h)>\varphi(a)+1\}$$

where the $x_{\tau,i}$ are arranged in decreasing order. We write $\mathbf{n}:=(n_\tau)_{\tau\in I}$. For $0\leq\mathbf{i}\leq\mathbf{n}$ let

$$\varphi_{\mathbf{i}}=\begin{cases}-\infty & \text{if } a\notin A,\\ \varphi_0(a) & \text{if } a\in A_{(\tau)} \text{ and } i_\tau=0\\ \varphi(a) & \text{if } a\in A_{(\tau)}, i_\tau>0 \text{ and } a\geq x_{\tau,i_\tau},\\ \varphi(a+h)-1 & \text{otherwise.}\end{cases}$$

We call the family $(A,\varphi_{\mathbf{i}})_{0\leq\mathbf{i}\leq\mathbf{n}}$ the *canonical deformation* of (A,φ).

One easily checks that the $(A,\varphi_{\mathbf{i}})$ are indeed extended EL-charts (the properties (a)-(d) of Definition 5.10 follow from the analogous properties of (A,φ)) and that $\varphi_{\mathbf{i}}=\varphi_0$ for $\mathbf{i}=(0)_{\tau\in I}$ and $\varphi_{\mathbf{i}}=\varphi$ for $\mathbf{i}=\mathbf{n}$. Denote the Hodge-point of $(A,\varphi_{\mathbf{i}})$ by $\mu^{\mathbf{i}}$.

We note that one can define the $\varphi_{\mathbf{i}}$ recursively. Let $\varsigma\in I$ and $0\leq\mathbf{i}\leq\mathbf{i}'\leq\mathbf{n}$ with $i'_\varsigma=i_\varsigma+1$ and $i'_\tau=i_\tau$ for $\tau\neq\varsigma$. We denote $\alpha:=\varphi(x_{\varsigma,i_\varsigma}+h)-(\varphi(x_{\varsigma,i_\varsigma})+1)$. Then

$$\varphi_{\mathbf{i}'}(a)=\begin{cases}\varphi_{\mathbf{i}}(a)-\alpha & \text{if } a=x_{\varsigma,i_\varsigma},x_{\varsigma,i_\varsigma}-h,\dots,x_{\varsigma,i_\varsigma}-\mathrm{ht}(x_{\varsigma,i_\varsigma})\cdot h\\ \varphi_{\mathbf{i}}(a) & \text{otherwise}\end{cases}$$

Lemma 5.16. *Let (A,φ) be an extended EL-chart of type μ' with Hodge point μ. Then $\mu'_{\mathrm{dom}}\preceq\mu$. Furthermore, we have $\mu'_{\mathrm{dom}}=\mu$ if and only if (A,φ) is cyclic.*

Proof. We have already shown that $\mu=\mu'_{\mathrm{dom}}$ if (A,φ) is cyclic in Lemma 5.13. It suffices to show $\mu^{\mathbf{i}}\prec\mu^{\mathbf{i}'}$ for all pairs $\mathbf{i},\mathbf{i}'$ such that $i'_\varsigma=i_\varsigma+1$ for some $\varsigma\in I$ and $i'_\tau=i_\tau$ for $\tau\neq\varsigma$. From the description of $\varphi_{\mathbf{i}'}$ above we see that we get $\mu^{\mathbf{i}'}$ from $\mu^{\mathbf{i}}$ by replacing two coordinates in $\mu^{\mathbf{i}}_\varsigma$ and permuting its coordinates if necessary to get a dominant cocharacter. Using the same notation as above, we replace

$$\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\mathrm{ht}(x_{\varsigma,i_\varsigma}),\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha+1\}$$

with

$$\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha-\mathrm{ht}(x_{\varsigma,i_\varsigma}),\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})+1\}.$$

Since

$$\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\mathrm{ht}(x_{\varsigma,i_\varsigma}),\ \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha+1\in(\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha-\mathrm{ht}(x_{\varsigma,i_\varsigma}),\varphi_{\mathbf{i}'}(x_{\varsigma,i_\varsigma})+1),$$

we get $\mu^{\mathbf{i}}\prec\mu^{\mathbf{i}'}$. ☐

Deducing the following corollaries from Lemma 5.16 is literally the same as the proofs of Cor. 3.7 and Lemma 3.8 in [Vie06]. We give the proofs for the reader's convenience.

Corollary 5.17. *If μ is minuscule, then all extended EL-charts for μ are cyclic.*

Proof. Let (A,φ) be an extended EL-chart for μ and let μ' be the type of (A,φ). Since μ is minuscule, $\mu'_{\mathrm{dom}}\preceq\mu$ implies $\mu'_{\mathrm{dom}}=\mu$. Hence the assertion follows from Lemma 5.16. ☐

Corollary 5.18. *There are only finitely many extended EL-charts for μ.*

Proof. As a consequence of Lemma 5.16 there are only finitely many possible types of extended EL-charts with Hodge point μ. If we fix such a type, the EL-chart A is uniquely determined. The value of the function φ is uniquely determined by A for all but finitely many elements and for each such element, φ can only take finitely many values by the inequality of Definition 5.10 (c). ☐

5.3. **Combinatorics of extended EL-charts.** The stratum of an affine Deligne-Lusztig variety resp. Rapoport-Zink space corresponding to an extended EL-chart (A,φ) is isomorphic to an open subset of an affine space resp. isomorphic to an affine space whose coordinates corresponds to a certain subset $\mathscr{V}(A,\varphi) \subset A \times A$. The aim of this section is to determine the maximal cardinality of $\mathscr{V}(A,\varphi)$ for an extended EL-chart for μ and thus determine the dimension of the top-dimensional stratum.

Definition 5.19. Let (A,φ) be an extended EL-chart for μ. We define

$$\mathscr{V}(A,\varphi) = \{(a,c) \in A \times A \mid a < c, \varphi(a) > \varphi(c) > \varphi(a-h)\}$$

Proposition 5.20. *Let (A,φ) the cyclic extended EL-chart of type μ. Then*

$$\#\mathscr{V}(A,\varphi) \geq \delta_G(\nu,\mu)$$

.

Proof. First we show that the right hand side of the inequality counts the number of positive integers n such that $b_0 + n \notin A_{(0)}$. Indeed, as $A_{(0)} + h \subset A_{(0)}$, we have

$$\begin{aligned}
\#\{n \in \mathbb{N} \mid b_0 + n \notin A_{(0)}\} &= \frac{1}{h} \cdot \sum_{j=1}^{h-1} (b_{j\cdot d} - b_0 - j) \\
&= \frac{1}{h} \cdot \sum_{j=1}^{h-1} \left(f^{j\cdot d}(b_0) - \left(\sum_{i=1}^{j\cdot d} \mu_i \cdot h \right) - b_0 - j \right) \\
&= \frac{1}{h} \cdot \sum_{j=1}^{h-1} \left(b_0 + j \cdot m - \left(\sum_{i=1}^{j} \underline{\mu}_i \cdot h \right) - b_0 - j \right) \\
&= \frac{1}{h} \cdot \sum_{j=1}^{h-1} \big(j \cdot (m-1) - h \cdot P(\underline{\mu})(i) \big) \\
&= \frac{(m-1)(h-1)}{2} - \sum_{i=1}^{h-1} P(\underline{\mu})(i) \\
&= \sum_{i=1}^{h-1} (\lfloor P(\underline{\nu})(i) \rfloor - P(\underline{\mu})(i)) \\
&= \delta_G(\nu,\mu)
\end{aligned}$$

Now we construct an injective map from $\{n \in \mathbb{N} \mid b_0 + n \notin A\}$ into $\mathscr{V}(A,\varphi)$. For this we remark that $(b_i, b_i + n) \in \mathscr{V}(A,\varphi)$ if and only if $b_i + n \in A$ and $b_{i+1} + n \notin A$. Thus $n \mapsto (b_i, b_i + n)$ where $i = \max\{i = 1, \ldots, h-1 \mid b_i + n \in A\}$ gives us the injection we sought. Note this map is well-defined since for any $n \in \mathbb{N}$ and maximal element b of B we have $b + n \in A$. □

Theorem 5.21. *Let (A,φ) be an extended EL-chart for μ. Then $\#\mathscr{V}(A,\varphi) \leq \delta_G(\nu,\mu)$.*

Proof. We assume first that (A,φ) is cyclic with type μ'. Then

$$\begin{aligned}
\#\mathscr{V}(A,\varphi) &= \#\{(b_i, a) \in B \times A \mid b_i < a, \varphi(a) < \varphi(b_i)\} \\
&= \sum_{(b_i,b_j):\, b_i < b_j, \mu'_{i+1} > \mu'_{j+1}} \mu'_{i+1} - \mu'_{j+1} \\
&\quad + \#\{(b_i, b_j + \alpha h) \mid, \alpha \in \mathbb{N}, b_j < b_i < b_j + \alpha h, \mu'_{i+1} > \mu'_{j+1} + \alpha\}
\end{aligned}$$

We refer to these two summands by S_1 and S_2.

For each $\tau \in I$ denote by $(\tilde{b}_{\tau,1}, \tilde{\mu}_{\tau+1,1}), \ldots, (\tilde{b}_{\tau,h}, \tilde{\mu}_{\tau+1,h})$ the permutation of $(b_{\tau,1}, \mu_{\tau+1,1}), \ldots,$ $(b_{\tau,h}, \mu_{\tau+1,h})$ where the $(\tilde{b}_{\tau,i})_i$ are arranged in increasing order. From the ordering we obtain for any $1 \leq j \leq h, \tau \in I$

$$\sum_{i=1}^{j} \tilde{b}_{\tau,i} \leq \sum_{i=1}^{j} \operatorname{succ}(\tilde{b}_{\tau-1,i})$$

and thus

$$\sum_{i=1}^{j} |\tilde{b}_{\tau,i}| - |\tilde{b}_{\tau-1,i}| \leq j \cdot m_\tau - \sum_{i=1}^{j} \tilde{\mu}_{\tau,i} \cdot h.$$

Adding these inequalities for all $\tau \in I$ and rearranging the terms we obtain $\sum_{i=1}^{j} \underline{\tilde{\mu}}_j \leq j \cdot \frac{m}{h}$. Thus $\nu \preceq \overline{\tilde{\mu}} \preceq \overline{\mu}$.

Using this notation we can simplify

$$\begin{aligned} S_1 &= \sum_{\substack{i<j \\ \tau\in I}} \max\{\tilde{\mu}_i - \tilde{\mu}_j, 0\} \\ &= \sum_{\tau \in I} \delta_G(\tilde{\mu}_\tau, \tilde{\mu}_{\tau\,\mathrm{dom}}) \\ &= \delta_G(\tilde{\mu}, \mu), \end{aligned}$$

where the second line holds because of Lemma 5.3.

We have now reduced the claim to $S_2 \leq \delta_G(\nu,\mu) - \delta_G(\tilde{\mu},\mu)$, which is equivalent to $S_2 \leq \delta_G(\nu,\tilde{\mu})$. Now

$$\begin{aligned} S_2 &= \sum_{i=2}^{h}\sum_{j=1}^{i-1}\sum_{\tau\in I} \#\{\alpha \in \mathbb{Z} \mid \tilde{b}_{\tau,i} < \tilde{b}_{\tau,j} + \alpha h, \tilde{\mu}_{\tau+1,i} > \tilde{\mu}_{\tau+1,j} + \alpha\} \\ &= \sum_{i=2}^{h}\sum_{j=1}^{i-1}\sum_{\tau\in I} \#\{\alpha \in \mathbb{Z} \mid \tilde{b}_{\tau,i} - \tilde{b}_{\tau,j} < \alpha h < \tilde{\mu}_{\tau+1,i}h - \tilde{\mu}_{\tau+1,j}h\} \\ &\leq \sum_{i=2}^{h}\sum_{j=1}^{i-1}\sum_{\tau\in I} \#\{\alpha \in \mathbb{Z} \mid 0 < \alpha h < (\tilde{b}_{\tau,j} - \tilde{\mu}_{\tau+1,j}h) - (\tilde{b}_{\tau,i} - \tilde{\mu}_{\tau+1,i}h)\} \\ &= \sum_{i=2}^{h}\sum_{j=1}^{i-1}\sum_{\tau\in I} \max\{\lfloor \tfrac{\operatorname{succ}(\tilde{b}_{\tau,j}) - \operatorname{succ}(\tilde{b}_{\tau,i})}{h} \rfloor, 0\} \end{aligned}$$

Recall that $\delta_G(\nu,\tilde{\mu})$ counts the lattice points between the polynomials associated to $\underline{\nu}$ and $\underline{\tilde{\mu}}$. So it is enough to construct a decreasing sequence (with respect to $\preceq$) of $\widehat{\psi}^i \in \mathbb{Q}^h$ for $i = 1, \ldots, h$ with $\widehat{\psi}^1 = \underline{\nu}$ and $\widehat{\psi}^h = \underline{\tilde{\mu}}$ such that there are at least

$$\sum_{j=1}^{i-1}\sum_{\tau\in I} \max\{\lfloor \tfrac{\operatorname{succ}(\tilde{b}_{j,\tau}) - \operatorname{succ}(\tilde{b}_{i,\tau})}{h} \rfloor, 0\}$$

lattice points which are on or below $P(\widehat{\psi}^{i-1})$ and above $P(\widehat{\psi}^i)$.

We define a bijection $\operatorname{succ}_i : B \to B$ as follows: For $j > i, \tau \in I$ let $\operatorname{succ}_i(\tilde{b}_{j,\tau}) = \operatorname{succ}(\tilde{b}_{j,\tau})$. For $j \leq i$ define $\operatorname{succ}_i(\tilde{b}_{\tau,j})$ such that for every $\tau \in I$ the tuple $(\operatorname{succ}_i(\tilde{b}_{\tau,j}))_{j=1}^i$ is the permutation of $(\operatorname{succ}(\tilde{b}_{\tau,j}))_{j=1}^i$ which is arranged in increasing order. Let $\psi^i \in \prod_{\tau\in I} \mathbb{Q}^h$ be defined by $\operatorname{succ}_i(\tilde{b}_{\tau,j}) = f(\tilde{b}_{\tau,j}) - \psi^i_{\tau+1,j} \cdot h$ and $\widehat{\psi}^i = \underline{\psi^i}$. By definition we have

$$\psi^i_{\tau+1,j} = \frac{m_{\tau+1}}{h} - \frac{|\operatorname{succ}_i(\tilde{b}_j)| - |\tilde{b}_j|}{h}$$

and thus $\widehat{\psi}^1 = \underline{\tilde{\mu}}$ and $\widehat{\psi}^h = \underline{\nu}$.

We have the following recursive construction of succ_i. Let $i_0 \leq i$ be minimal such that $\mathrm{succ}_{i-1}(\tilde{b}_{\tau,i_0}) \leq \mathrm{succ}(\tilde{b}_{\tau,i})$. Then

$$\mathrm{succ}_i(\tilde{b}_{\tau,j}) = \begin{cases} \mathrm{succ}_{i-1}(\tilde{b}_{\tau,j}) & \text{if } j < i_0 \\ \mathrm{succ}(\tilde{b}_{\tau,i}) & \text{if } j = i_0 \\ \mathrm{succ}_{i-1}(\tilde{b}_{\tau,j-1}) & \text{if } i_0 < j \leq i \\ \mathrm{succ}_{i-1}(\tilde{b}_{\tau,j}) & \text{if } j > i \end{cases}.$$

Now

$$\begin{aligned} P(\widehat{\psi}^i)(j) - P(\widehat{\psi}^{i-1})(j) &= \sum_{\tau\in I}\sum_{k=1}^{j}(\widehat{\psi}^i_{\tau,k} - \widehat{\psi}^{i-1}_{\tau,k}) \\ &= \sum_{\tau\in I}\sum_{k=1}^{j}\frac{1}{h}(\mathrm{succ}_{i-1}(\tilde{b}_{\tau,k}) - \mathrm{succ}_i(\tilde{b}_{\tau,k})) \\ &= \sum_{\tau\in I}\frac{1}{h}\left(\sum_{k=1}^{j}\mathrm{succ}_{i-1}(\tilde{b}_{\tau,k}) - \sum_{k=1}^{j}\mathrm{succ}_i(\tilde{b}_{\tau,k})\right). \end{aligned}$$

By the recursive formula above the right hand side equals zero if $j \geq i$ and

$$\sum_{\tau\in I}\max\{0, \tfrac{\mathrm{succ}_{i-1}(\tilde{b}_{\tau.j})-\mathrm{succ}(\tilde{b}_{\tau,i})}{h}\}$$

if $j < i$. Thus there are at least

$$\sum_{\tau\in I}\sum_{j<i}\max\{0, \lfloor\tfrac{\mathrm{succ}_{i-1}(\tilde{b}_{\tau.j})-\mathrm{succ}(\tilde{b}_{\tau,i})}{h}\rfloor\} = \sum_{\tau\in I}\sum_{j<i}\max\{0, \lfloor\tfrac{\mathrm{succ}(\tilde{b}_{\tau.j})-\mathrm{succ}(\tilde{b}_{\tau,i})}{h}\rfloor\}$$

lattice points which are above $P(\widehat{\psi}^i)$ and on or below $P(\widehat{\psi}^{i-1})$, which finishes the proof for a cyclic EL-chart.

For a non-cyclic EL-chart (A,φ) consider the canonical deformation $((A,\varphi_{\mathbf{i}}))_{\mathbf{i}}$ (see Definition 5.15). The theorem is proven by induction on $\mathbf{i}$. For $\mathbf{i} = (0)_{\tau\in I}$ the extended EL-chart is cyclic and the claim is proven above. Now the induction step requires that we show that the claim remains true if increase a single coordinate of $\mathbf{i}$ by one. Let $\mathbf{i}' \leq \mathbf{n}$ with $i'_\varsigma = i_\varsigma + 1$ for some $\varsigma \in I$ and $i'_\tau = i_\tau$ for $\tau \neq \varsigma$. For convenience, we introduce the notations

$$\begin{aligned} \alpha &:= \varphi(x_{\varsigma,i_\varsigma} + h) - (\varphi(x_{\varsigma,i_\varsigma}) + 1) \\ n &:= \mathrm{ht}(x_{\varsigma,i_\varsigma}) \\ \mu^i &:= \mu^{\mathbf{i}}_\varsigma \\ \mu^{i'} &:= \mu^{\mathbf{i}'}_\varsigma. \end{aligned}$$

Then the right hand sides of the formula (16.1) for $\mu^{\mathbf{i}'}$ and $\mu^{\mathbf{i}}$ differ by

$$\langle\rho,\mu^{\mathbf{i}'}\rangle - \langle\rho,\mu^{\mathbf{i}}\rangle = \delta_G(\mu^{\mathbf{i}'},\mu^{\mathbf{i}}).$$

Recall the explicit description of the difference between $\varphi_{\mathbf{i}}$ and $\varphi_{\mathbf{i}'}$ resp. $\mu^{\mathbf{i}}$ and $\mu^{\mathbf{i}'}$ which we gave right before resp. in the proof of Lemma 5.16. Then Corollary 5.5 implies

$$\delta_G(\mu^{\mathbf{i}'},\mu^{\mathbf{i}}) = \delta(\mu^{i'},\mu^i) \quad = \quad (\sum_{k=0}^{\alpha-1}\sum_{l=0}^{n}\#\{j \mid \mu^i_j = \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - k - l\}) - \min\{\alpha, n+1\}.$$

We denote this term by Δ. We have to show that $\Delta \geq |\mathscr{V}(A,\varphi_{\mathbf{i}'})| - |\mathscr{V}(A,\varphi_{\mathbf{i}})|$. Now

$$\begin{aligned} \mathscr{V}(A,\varphi_{\mathbf{i}'}) \setminus \mathscr{V}(A,\varphi_{\mathbf{i}}) &= D_1 \cup D_3 \\ \mathscr{V}(A,\varphi_{\mathbf{i}}) \setminus \mathscr{V}(A,\varphi_{\mathbf{i}'}) &= D_2 \end{aligned}$$

where

$$\begin{aligned}
D_1 &= \{(x_{\varsigma,i_\varsigma}+h,c)\in A\times A \mid c > x_{\varsigma,i_\varsigma}+h, \varphi_{\mathbf{i}'}(x_{\varsigma,i_\varsigma}+h) > \varphi_{\mathbf{i}'}(c) > \varphi_{\mathbf{i}'}(x_{\varsigma,i_\varsigma})\}\\
D_2 &= \left\{(x_{\varsigma,i_\varsigma}-nh,c)\in A\times A \mid \begin{array}{l} c > x_{\varsigma,i_\varsigma}-nh, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}-nh) > \varphi_{\mathbf{i}}(c),\\ \varphi_{\mathbf{i}'}(x_{\varsigma,i_\varsigma}-nh) \leq \varphi_{\mathbf{i}'}(c)\end{array}\right\}\\
D_3 &= \left\{(b,x_{\varsigma,i_\varsigma}-\delta h)\in B\times A \mid \begin{array}{l} b\neq x_{\varsigma,i_\varsigma}-nh, b < x_{\varsigma,i_\varsigma}-\delta h, \varphi_{\mathbf{i}'}(b) > \varphi_{\mathbf{i}'}(x_{\varsigma,i_\varsigma}-\delta h),\\ \varphi_{\mathbf{i}}(b)\leq \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}-\delta h)\end{array}\right\}.
\end{aligned}$$

Thus we get $\#\mathscr{V}(A,\varphi_{\mathbf{i}'}) - \#\mathscr{V}(A,\varphi_{\mathbf{i}}) = S_1 - S_2 + S_3$ with

$$\begin{aligned}
S_1 &= \#\{a\in A \mid a > x_{\varsigma,i_\varsigma}+h, \varphi_{\mathbf{i}}(a)\in[\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha+1,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})]\}\\
S_2 &= \#\{a\in A \mid a > x_{\varsigma,i_\varsigma}-nh, \varphi_{\mathbf{i}}(a)\in[\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha-n,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-n-1]\}\\
S_3 &= \#\{(b,\delta)\in B\times\{0,\dots,n\} \mid b\neq x_{\varsigma,i_\varsigma}-nh, b < x_{\varsigma,i_\varsigma}-\delta h,\\
&\qquad \varphi_{\mathbf{i}}(b)\in[\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\delta-\alpha+1,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\delta]\}.
\end{aligned}$$

Now let

$$\begin{aligned}
C_1 &= \{a\in A \mid a \leq x_{\varsigma,i_\varsigma}+h, \varphi_{\mathbf{i}}(a)\in[\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha+1,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})]\}\\
C_2 &= \{a\in A \mid a \leq x_{\varsigma,i_\varsigma}-nh, \varphi_{\mathbf{i}}(a)\in[\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha-n,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-n-1]\}.
\end{aligned}$$

As $\varphi_{\mathbf{i}}(x+h)=\varphi_{\mathbf{i}}(x)+1$ for all $x\in A$ with $x\leq x_{\varsigma,i_\varsigma}$, we have $C_2+(n+1)h\subset C_1$. We denote $C_3 := C_1\setminus(C_2+(n+1)h)$. Then

$$\begin{aligned}
C_3 &= \left\{b+\delta h \mid \begin{array}{l} b\in B, \delta\in\{0,\dots,n\}, b\leq x_{\varsigma,i_\varsigma}+h-\delta h,\\ \varphi_{\mathbf{i}}(b)\in[\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\delta-\alpha+1,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\delta]\end{array}\right\}\\
&= \left\{b+\delta h \mid \begin{array}{l} b\in B\setminus\{x_{\varsigma,i_\varsigma}-nh\}, \delta\in\{0,\dots,n\}, b\leq x_{\varsigma,i_\varsigma}+h-\delta h,\\ \varphi_{\mathbf{i}}(b)\in[\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\delta-\alpha+1,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\delta]\end{array}\right\}\\
&\quad \cup\{x_{\varsigma,i_\varsigma}-nh+\delta h \mid \delta=1,\dots,\min\{\alpha,n+1\}\}.
\end{aligned}$$

In particular, we have $\#C_3\geq S_3+\min\{\alpha,n+1\}$.

Altogether, we get

$$\begin{aligned}
\Delta &= (\sum_{k=0}^{\alpha-1}\sum_{l=0}^{n}\#\{j;\mu_j^i=\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-k-l\}) - \min\{\alpha,n+1\}\\
&= \sum_{k=0}^{\alpha-1}\sum_{l=0}^{n}\left(\#\left(A_{(\varsigma)}\cap\varphi_{\mathbf{i}}^{-1}(\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-k-l\})\right) - \#\left(A_{(\varsigma)}\cap\varphi_{\mathbf{i}}^{-1}(\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-k-l-1\})\right)\right)\\
&\quad -\min\{\alpha,n+1\}\\
&= \sum_{k=0}^{\alpha-1}\left(\#\left(A_{(\varsigma)}\cap\varphi_{\mathbf{i}}^{-1}(\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-k\})\right) - \#\left(A_{(\varsigma)}\cap\varphi_{\mathbf{i}}^{-1}(\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-k-n-1\})\right)\right)\\
&\quad -\min\{\alpha,n+1\}\\
&= \#\left(A_{(\varsigma)}\cap\varphi_{\mathbf{i}}^{-1}([\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha+1,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})])\right)\\
&\quad -\#\left(A_{(\varsigma)}\cap\varphi_{\mathbf{i}}^{-1}([\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha-n,\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-n-1])\right) - \min\{\alpha,n+1\}\\
&= (S_1+\#C_1)-(S_2+\#C_2)-\min\{\alpha,n+1\}\\
&= S_1-S_2+\#C_3-\min\{\alpha,n+1\}\\
&\geq S_1-S_2+S_3.
\end{aligned}$$

□

6. Shimura varieties of PEL type with good reduction

6.1. **Moduli spaces of abelian varieties.** We recall Kottwitz's definition of an integral PEL-Shimura datum with parahoric level structure at p (which we will abbreviate to "unramified

PEL-Shimura datum") and the associated moduli spaces. They were defined in [Kot85], which is also the main reference for this section.

The moduli problem is given by a datum $\mathscr{D} = (\mathsf{B}, {}^*, \mathsf{V}, \langle\ ,\ \rangle, \mathsf{O}_\mathsf{B}, \Lambda_0, h)$. To this datum one associates an linear algebraic group G over $\mathbb{Q}$, a conjugacy class $[\mu_h]$ of cocharacters of $\mathsf{G}_\mathbb{C}$ and a number field E. This data have the following meaning.

- B is a finite dimensional semi-simple $\mathbb{Q}$-algebra such that $\mathsf{B}_{\mathbb{Q}_p}$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.
- * is a positive involution of B over $\mathbb{Q}$.
- V is a non-zero finitely generated left-B-module.
- $\langle\ ,\ \rangle$ is a symplectic form of the underlying $\mathbb{Q}$-vector space of B which is B-adapted, i.e. for all $v, w \in \mathsf{V}$ and $b \in \mathsf{B}$
$$\langle bv, w\rangle = \langle v, b^* w\rangle.$$
- O_B is a $\mathbb{Z}_{(p)}$-order of B, whose p-adic completion $\mathsf{O}_{\mathsf{B},p}$ is a maximal order of $\mathsf{B}_{\mathbb{Q}_p}$.
- Λ_0 is a lattice in $\mathsf{V}_{\mathbb{Q}_p}$ which is self dual for $\langle\ ,\ \rangle$ and preserved under the action of O_B.
- The group G represents the functor
$$\mathsf{G}(R) = \{g \in \mathrm{GL}_\mathsf{B}(\mathsf{V} \otimes R) \mid \exists c(g) \in R^\times : \langle g(v), g(w)\rangle = c(g) \cdot \langle v, w\rangle\}.$$
We assume henceforth that G is connected.
- $h : \mathbb{C} \to \mathrm{End}_B(\mathsf{V})_\mathbb{R}$ is a homomorphism of algebras with involution (where on the left hand side the involution is the complex conjugation and on the right hand side the involution maps a homomorphism to its adjoint with respect to $\langle\ ,\ \rangle$) such that the form $\langle v, h(i) \cdot w\rangle$ on $\mathrm{End}_\mathsf{B}(\mathsf{V})_\mathbb{R}$ is positive definite.
- Let $\mathsf{V}_\mathbb{C} = \mathsf{V}^0 \oplus \mathsf{V}^1$, where V^0 resp. V^1 is the subspace where $h(z)$ acts by $\bar{z}$ resp. z. We define μ_h to be the cocharacter of $\mathsf{G}_\mathbb{C}$ which acts with weight 0 on V^0 and with weight 1 on V^1. Then $[\mu_h]$ is defined as the $\mathsf{G}(\mathbb{C})$-conjugacy class of μ_h.
- E is the field of definition of $[\mu_h]$.

Definition 6.1. We call a datum $\mathscr{D}$ as above an unramified PEL-Shimura datum. The field E is called reflex field of $\mathscr{D}$.

Let $K^p \subset \mathsf{G}(\mathbb{A}^p)$ be an open compact subgroup. We consider functor $\mathscr{A}_{\mathscr{D},K^p}$ which associates to an $\mathsf{O}_\mathsf{E} \otimes \mathbb{Z}_{(p)}$-algebra R the set of isomorphism classes of tuples $(A, \lambda, \iota, \eta)$ where

- A is a projective abelian scheme over $\operatorname{Spec} R$.
- λ is a polarisation of A of degree prime to p.
- $\iota : \mathsf{O}_\mathsf{B} \to \mathrm{Aut}(A)$ is a homomorphism satisfying the following two conditions. For every $a \in \mathsf{O}_\mathsf{B}$ we have the compatibility of λ and ι
$$\iota(a) = \lambda^{-1} \circ \iota(a^*)^\vee \circ \lambda \tag{6.1}$$
and ι satisfies the Kottwitz determinant condition. That is, we have an equality of characteristic polynomials
$$\mathrm{char}(\iota(a)|\operatorname{Lie} A) = \mathrm{char}(a|\mathsf{V}^1). \tag{6.2}$$
The polynomial on the right hand side has actually coefficients in $\mathbb{Z}_{(p)}$, but we consider it as element of $R[X]$ via the structural morphism.
- η is a level structure of type K^p in the sense of [Kot85] §5.

Two such tuples $(A, \lambda, \iota, \eta), (A', \iota', \lambda', \eta')$ are isomorphic if there exists an isogeny $A \to A'$ of degree prime to p which commutes with ι, carrying λ into a $\mathbb{Z}_{(p)}^\times$-scalar multiple of λ' and carrying η to η'.

If K^p is small enough, this functor is representable by a smooth quasi-projective $\mathsf{O}_{\mathsf{E}} \otimes \mathbb{Z}_{(p)}$-scheme. Henceforth we will always assume that this is the case.

We fix $\mathscr{D}, K^p$ as above and choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ of the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ into an algebraic closure of $\mathbb{Q}_p$. We denote by v the place of E over p which is given by this embedding and by κ its residue field. We write

$$\mathscr{A}_0 := \mathscr{A}_{\mathscr{D},K^p,0} = \mathscr{A}_{\mathscr{D},K^p} \times \bar{\kappa}$$

for the geometric special fibre of $\mathscr{A}_{\mathscr{D},K^p}$ at v. We fix an isomorphism $\bar{\kappa} \cong \bar{\mathbb{F}}_p$ and denote by $\underline{A}^{\mathrm{univ}}$ the universal object over $\mathscr{A}_0$.

6.2. **Barsotti-Tate groups with $\mathscr{D}$-structure.** Let R be a $\overline{\mathbb{F}}_p$-algebra and $(A, \iota, \lambda, \eta) \in \mathscr{A}_0(R)$. Then by functoriality we obtain additional structure on the Barsotti-Tate group $X = A[p^\infty]$ of A. That is, we get an action of $\mathsf{O}_{\mathsf{B},p}$ on $A[p^\infty]$ and a polarisation up to $\mathbb{Z}_p^\times$-scalar multiple, satisfying the same compatibility conditions as ι and λ.

Notation 6.2. We call a polarisation up to $\mathbb{Z}_p^\times$-scalar multiple as above a $\mathbb{Z}_p^\times$-homogeneous polarisation. We will use the analogous notion for bilinear forms which are defined up $\mathbb{Z}_p^\times$- or $\mathbb{Q}_p^\times$-scalar multiple.

Definition 6.3. Let X be a Barsotti-Tate group over a κ-algebra R, together with a homomorphism $\iota : \mathsf{O}_{\mathsf{B},p} \to \operatorname{End} X$ and a $\mathbb{Z}_p^\times$-homogeneous polarisation $\lambda : X \to X^\vee$. The tuple $\underline{X} = (X, \iota, \lambda)$ is called a Barsotti-Tate group with $\mathscr{D}$-structure if the following conditions are satisfied.

(a) Let $\mathsf{B}_{\mathbb{Q}_p} = \prod B_i$ be a decomposition into simple factors and $\mathsf{V}_{\mathbb{Q}_p} = \prod V_i$, $\mathsf{O}_{\mathsf{B},p} = \prod O_{B_i}$,the induced decompositions. Denote by ϵ_i the multiplicative unit in B_i and let $X_i := \operatorname{im} \epsilon_i$. Then X_i is a Barsotti-Tate group of height $\dim_{\mathbb{Q}_p} V_i$.
(b) $\iota(a) = \lambda^{-1} \circ \iota(a^*)^\vee \circ \lambda$.
(c) $\operatorname{char}(\iota(a)| \operatorname{Lie} X) = \operatorname{char}(a|\mathsf{V}_v^1)$ where V_v^1 denotes the v-adic completion of V^1.

Remark 6.4. The condition that X_i is a Barsotti-Tate group is automatic, cf. section 7.1.

Some of the data above are superfluous for the general study of Barsotti-Tate groups with $\mathscr{D}$-structure. Therefore we introduce the following (simpler) objects.

Definition 6.5. (1) Let B be a finite product of matrix algebras over finite unramified field extensions of $\mathbb{Q}_p$ and $O_B \subset B$ be a maximal order. We call a Barsotti-Tate group with O_B-action a Barsotti-Tate group with EL structure.
(2) Let O_B, B be as above and * be a $\mathbb{Q}_p$-linear involution of B which stabilizes O_B. We call a Barsotti-Tate group with polarisation λ and O_B-action ι satisfying

$$\iota(a) = \lambda^{-1} \circ \iota(a^*)^\vee \circ \lambda$$

a Barsotti-Tate group with PEL structure.
(3) We call a tuple (B, O_B) resp. $(B, O_B, *)$ as above an EL-datum resp. a PEL-datum.

Notation 6.6. When we want to consider the EL case and the PEL case simultaneously, we will write the additional data in brackets, e.g. "Let $\underline{X} = (X, \iota, (\lambda))$ be a Barsotti-Tate group with (P)EL structure."

Definition 6.7. Let $\underline{X} = (X, \iota, (\lambda))$ and $\underline{X}' = (X', \iota', (\lambda'))$ be two Barsotti-Tate groups with (P)EL structure.

(1) A morphism $\underline{X} \to \underline{X}'$ is a homomorphism of Barsotti-Tate groups $X \to X'$ which commutes with the O_B-action and the polarisation in the PEL case.
(2) An isogeny $\underline{X} \to \underline{X}'$ is an isogeny $X \to X'$ which commutes with the O_B-action and in the PEL case also commutes with the polarisation up to $\mathbb{Q}_p^\times$-scalar. The scalar given in the PEL case is called the similitude factor of the isogeny.

We note that an isogeny of Barsotti-Tate groups with PEL structure is not necessarily a homomorphism.

6.3. **Deformation theory.** Let $\underline{X} = (X, \iota, (\lambda))$ be a Barsotti-Tate group with (P)EL structure over a perfect field k_0 of characteristic p. We briefly recall the construction of its universal deformation $\underline{X}^{\mathrm{univ}}$. By [Ill85], Cor. 4.8 the deformation space $\mathrm{Def}(X)$ is pro-representable by $\mathrm{Spf}\, k[\![X_1, \ldots, X_{d\cdot(n-d)}]\!]$ where n denotes the height of X and d its dimension. In order to describe $\mathrm{Def}(\underline{X})$ we use the following result of Drinfeld. For any Artinian local k_0-algebra A and Barsotti-Tate groups X', X'' over A the canonical map

$$\mathrm{Hom}(X', X'') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Hom}(X'_{k_0}, X''_{k_0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism. Now the condition that an element of $\mathrm{Hom}(X', X'') \otimes \mathbb{Q}_p$ is a homomorphism is closed on $\mathrm{Spec}\, A$ by [RZ96], Prop. 2.9. Thus $\mathrm{Def}(\underline{X})$ is a closed subfunctor of $\mathrm{Def}(X)$ and in particular pro-representable by $\mathrm{Spf}\, \mathscr{R}_{\underline{X}}$ for some adic ring $\mathscr{R}_{\underline{X}}$. Denote by $\underline{X}^{\mathrm{def}}$ the universal object over $\mathrm{Spf}\, \mathscr{R}_X$. By a result of Messing ([Mes72] Lemma II.4.16) for any I-adic ring R the functor

$$\begin{array}{rcl} \{\text{Barsotti-Tate groups over } \mathrm{Spec}\, R\} & \to & \{\text{Barsotti-Tate groups over} \, \mathrm{Spf}\, R\} \\ X' & \mapsto & (X' \mod I^n)_{n\in\mathbb{N}} \end{array}$$

is an equivalence of categories. Thus $\underline{X}^{\mathrm{def}}$ induces a Barsotti-Tate group with (P)EL structure $\underline{X}^{\mathrm{univ}}$ over $\mathrm{Spec}\, \mathscr{R}_{\underline{X}}$. We note that the same construction also works for Barsotti-Tate groups with PEL structure with $\mathbb{Z}_p^\times$-homogeneous polarisation and in particular yields a canonically isomorphic deformation functor.

Definition 6.8. We call $\underline{X}^{\mathrm{univ}}$ the universal deformation of $\underline{X}$. We denote $\mathscr{S}_{\underline{X}} = \mathrm{Spec}\, \mathscr{R}_{\underline{X}}$

The Serre-Tate theorem states that the canonical homomorphism $\mathrm{Def}(A) \to \mathrm{Def}(A[p^\infty])$ is an isomorphism for every abelian variety A over an algebraically closed field k of characteristic p. We obtain the following corollary.

Proposition 6.9. *Let $x = \underline{A} \in \mathscr{A}_0(\overline{\mathbb{F}}_p)$ and $\underline{X} := \underline{A}[p^\infty]$. Then morphism $\mathscr{R}_{\underline{X}} \to \widehat{\mathscr{O}}_{\mathscr{A}_0,x}$ induced by the deformation $(A^{\mathrm{univ}}[p^\infty])_{\mathrm{Spec}\, \widehat{\mathscr{O}}_{\mathscr{A}_0,x}}$ is an isomorphism and the pull-back of $\underline{X}^{\mathrm{univ}}$ equals $(A^{\mathrm{univ}}[p^\infty])_{\mathrm{Spec}\, \widehat{\mathscr{O}}_{\mathscr{A}_0,x}}$.*

7. Barsotti-Tate groups with additional structure

7.1. **Decomposition of Barsotti-Tate groups with (P)EL structure.** In this subsection we recall a mechanism that will often allow us to reduce to the case where $B = F$ is a finite unramified field extension of $\mathbb{Q}_p$. Even though this mechanism is well known (see for example [Far04]), I could not find a reference for its proof.

Lemma 7.1. *Let X be a Barsotti-Tate group over a scheme S.*

(1) Assume we have a subalgebra $O_1 \times \ldots \times O_r \subset \mathrm{End}\, X$. Denote by ϵ_i the multiplicative unit in O_i and let $X_i := \mathrm{im}\, \epsilon_i$. Then $X = X_1 \times \ldots \times X_r$ and the X_i are Barsotti-Tate groups over S.

(2) Assume there is a subalgebra $\mathrm{M}_d(O) \subset \mathrm{End}\, X$. Let

$$\epsilon = \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix} \in \mathrm{M}_d(O)$$

and $X' = \mathrm{im}\, \epsilon$. Then X' is a Barsotti-Tate group with O-action and $X \cong (X')^d$ compatible with $\mathrm{M}_d(O)$-action.

Proof. The first assertion of (1) is obvious. The sheaves X_i are p-divisible and p-torsion because X is. It remains to show that $X_i[p]$ is representable by a finite locally free group scheme over S. Let $\epsilon_i' := \sum_{j\neq i} \epsilon_j$. Then $X_i = \ker \epsilon_i'$, in particular $X_i[p]$ is represented by a closed subgroup scheme of $X[p]$. Let $\mathscr{E}, \mathscr{E}_i$ be coherent $\mathscr{O}_S$-algebras such that $X[p] = \underline{\mathrm{Spec}}\mathscr{E}$ and $X_i[p] = \underline{\mathrm{Spec}}\mathscr{E}_i$. Then the closed embedding $X_i[p] \hookrightarrow X[p]$ induces a surjection $\mathscr{E} \twoheadrightarrow \mathscr{E}_i$ and $\epsilon_{i|X[p]} : X[p] \twoheadrightarrow X_i[p]$ induces a splitting of this surjection. Thus $\mathscr{E}_i$ is a direct summand of $\mathscr{E}$ as $\mathscr{O}_S$-module, in particular it is again locally free.

Now the isomorphism $X \cong (X')^d$ is the standard Morita argument and the fact that X' is a Barsotti-Tate group can be shown by the same argument as above. □

Corollary 7.2. *(1) Let (B, O_B) be an unramified EL-datum and let $B = \prod B_i$ be a decomposition into simple factors and $O_{B_i} = O_B \cap B_i$. Let $\underline{X}$ be a Barsotti-Tate group with (B, O_B)-EL structure. Then $\underline{X}$ decomposes as $\underline{X} = \prod \underline{X}_i$ where $\underline{X}_i$ is a Barsotti-Tate group with (B_i, O_{B_i})-EL structure. This defines an equivalence of categories*

$$\left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (B, O_B)\text{-EL structure} \end{array} \right\} \cong \prod_i \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (B_i, O_{B_i})\text{-EL structure} \end{array} \right\}$$

that preserves isogenies.

(2) Let (B, O_B) be an unramified EL-datum with $B \cong \mathrm{M}_d(F)$ simple. Let $\underline{X}$ be a Barsotti-Tate group with (B, O_B)-EL structure. Then $O_B \cong \mathrm{M}_d(O_F)$ and $\underline{X} \cong (\underline{X}')^d$ where $\underline{X}'$ is a Barsotti-Tate group with (F, O_F)-EL structure. This defines an equivalence of categories

$$\left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (B, O_B)\text{-EL structure} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (F, O_F)\text{-EL structure} \end{array} \right\}$$

which preserves isogenies.

Definition 7.3. Let X be a Barsotti-Tate group with (B, O_B)-EL structure. We define the relative height of X as the tuple

$$\operatorname{relht} X = \left(\frac{\operatorname{ht} X_1}{[B_1 : \mathbb{Q}_p]^{\frac{1}{2}}}, \ldots, \frac{\operatorname{ht} X_r}{[B_r : \mathbb{Q}_p]^{\frac{1}{2}}} \right)$$

with X_i, B_i as above.

Let $(B, O_B, ^*)$ be a PEL-datum. Following [RZ96] ch. A, we decompose $(B, ^*) = \prod B_i$ where B_i is isomorphic to one of the following.

(I) $\mathrm{M}_d(F) \times \mathrm{M}_d(F)^{\mathrm{opp}}$ where F is an unramified p-adic field and $(a, b)^* = (b, a)$.

($\mathrm{II_C}$) $\mathrm{M}_d(F)$ with F as above, $a^* = a^t$.

($\mathrm{II_D}$) $\mathrm{M}_d(F)$ with F as above, $a^* = J^{-1}a^*J$ with $J^tJ = -1$.

(III) $\mathrm{M}_d(F)$ with $\mathbb{Q}_p \subset F' \subset F$ finite unramified field extensions, $[F : F'] = 2$ and $a = \bar{a}^t$ where $\bar{\cdot}$ denotes the non-trivial F'-automorphism of F, acting on $M_d(F)$ component-by-component.

We call the algebras with involution which are isomorphic to one of the above indecomposable. Note that we also have the analogous decomposition for O_B.

Definition 7.4. A PEL-datum $(B, O_B, ^*)$ is of type (AC) if no factors of type ($\mathrm{II_D}$) appear in the decomposition of $(B, ^*)$.

Now Lemma 7.1 implies the following result.

Corollary 7.5. *(1) Let $(B, O_B, ^*)$ be an unramified PEL-datum of type (AC) and let $(B, ^*) = \prod B_i$ be a decomposition into indecomposable factors. Let $\underline{X}$ be a Barsotti-Tate group with $(B, O_B, ^*)$-PEL structure. Then $\underline{X}$ decomposes as $\underline{X} = \prod \underline{X}_i$ where $\underline{X}_i$ is a Barsotti-Tate* *group with $(B_i, O_{B_i},^*)$-PEL structure. This defines an equivalence of categories*

$$\left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (B, O_B, {}^*)\text{-PEL structure} \end{array} \right\} \cong \prod_i \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (B_i, O_{B_i}, {}^*)\text{-PEL structure} \end{array} \right\}$$

and also a bijection of isogenies on the left hand side with tuples of isogenies on the right hand side which have the same similitude factor.

(2) Let $(B, O_B,^*)$ be an unramified PEL-datum of type (AC) with $(B,^*)$ indecomposable and let $\underline{X}$ be a Barsotti-Tate group with $(B, O_B,^*)$-PEL structure. Using the notation above, we may describe $\underline{X}$ as follows.

(I) $\underline{X} \cong (\underline{X}')^d \times (\underline{X}'^\vee)^d$ where X' is a Barsotti-Tate group with (F, O_F)-EL structure and the polarisation is given by $\lambda(a,b) = (b, -a)$.

(II$_C$) $\underline{X} \cong (\underline{X}')^d$ where X' is a Barsotti-Tate group with (F, O_F, id)-PEL structure.

(III) $\underline{X} \cong (\underline{X}')^d$ where X' is a Barsotti-Tate group with $(F, O_F, \bar{\cdot})$-PEL structure.

We obtain equivalences of categories

$$\begin{array}{cccc} (I) & \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (B, O_B, {}^*)\text{-PEL structure} \end{array} \right\} & \cong & \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (F, O_F)\text{-EL structure} \end{array} \right\} \\ (II_C) & \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (B, O_B, {}^*)\text{-PEL structure} \end{array} \right\} & \cong & \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (F, O_F, \mathrm{id})\text{-PEL structure} \end{array} \right\} \\ (III) & \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (B, O_B, {}^*)\text{-PEL structure} \end{array} \right\} & \cong & \left\{ \begin{array}{l} \text{Barsotti-Tate groups with} \\ (F, O_F, \bar{\cdot})\text{-PEL structure} \end{array} \right\} \end{array}$$

and in the cases (II$_C$) and (III) also a bijection of the sets of isogenies. This is also true in case (I) if one fixes the similitude factor.

Proof. Write $X_i = \operatorname{im} \epsilon_i$ as in Lemma 7.1. Let $\underline{X} = (X, \iota, \lambda)$. Then

$$\operatorname{im} \lambda_{|X_i} = \operatorname{im}(\lambda \circ \epsilon_i) = \operatorname{im}(\epsilon_i^* \circ \lambda) = \operatorname{im}(\epsilon_i \circ \lambda) = X_i,$$

which proves (1).

For the second assertion assume first that $(B, {}^*)$ is of type (II$_\mathrm{C}$) or (III). Let $\epsilon_{i,j}$ be the matrix with 1 as (i,j)-th entry and 0 otherwise and denote $X_i := \operatorname{im}(\epsilon_{i,i})$. Then we have by Lemma 7.1 that $X = \prod X_i$ with O_F-equivariant isomorphisms $\epsilon_{i,j} : X_j \xrightarrow{\sim} X_i$. Now the same argument as in (1) shows $\operatorname{im} \lambda_{|X_i} = X_i$ (consider $\epsilon_{i,i}$) and $X_i \cong X_j$ as Barsotti-Tate groups with PEL structure (consider $\epsilon_{i,j}$). If $(B, {}^*)$ is of type (I) let ϵ_0 and ϵ_1 be the units of the $\mathrm{M}_d(O_F)$-factors. Decompose $X = X_0 \oplus X_1$ as in Corollary 7.2. Then

$$\operatorname{im} \lambda_{|X_i} = \operatorname{im}(\lambda \circ \epsilon_i) = \operatorname{im}(\epsilon_i^* \circ \lambda) = \operatorname{im}(\epsilon_{1-i} \circ \lambda) = X_{1-i}$$

Now the claim follows from Corollary 7.2. □

7.2. **Dieudonné theory for Barsotti-Tate groups with additional structure.** We fix a perfect field k_0 of characteristic p and denote $L_0 = W(k_0)_\mathbb{Q}$. Recall that one has an equivalence of categories

$$\begin{array}{rcl} \{\text{Barsotti-Tate groups over } k_0\} & \to & \{\text{Dieudonné modules, finite and free as } W(k_0)\text{-module}\} \\ X & \mapsto & M(X) \end{array}$$

which preserves height and dimension. Let X be a Barsotti-Tate group over k_0 with (P)EL structure. The action of O_B corresponds by functoriality to an O_B-action on $M(X)$. Write $M(X) = (\Lambda, \Phi)$. In the PEL case λ corresponds to a symplectic form $\langle\ ,\ \rangle$ on V which satisfies

$$\langle \Phi(v), \Phi(w) \rangle_\lambda = p \cdot \sigma(\langle v, w \rangle_\lambda) \quad \text{for all } v, w \in \Lambda. \tag{7.1}$$

Recall that the above Dieudonné functor induces an equivalence of categories

$$\begin{array}{rcl} \{\text{Barsotti-Tate groups over } k_0 \text{ up to isogeny}\} & \to & \{k_0-\text{isocrystals } (V, \Phi) \text{ with slopes in } [0,1]\} \\ X & \to & N(X). \end{array}$$

By the same argument as above, the O_B-action on X corresponds to a B-action on $N(X)$. In the PEL-case the $\mathbb{Q}_p^\times$-homogeneous polarisation corresponds to a $\mathbb{Q}_p^\times$-homogeneous polarisation $\langle\ ,\ \rangle$ on $N(X)$. This motivates the following definition.

Definition 7.6. (1) Let $\mathscr{B} = (B, O_B, (^*))$ be a (P)EL-datum. A $\mathscr{B}$-isocrystal is a pair (N, Φ) such that

- N is a finite-dimensional L_0-vector space with B-action.
- In the PEL case N is equipped with a perfect anti-symmetric B-compatible $\mathbb{Q}_p$-linear pairing $\langle\ ,\ \rangle$.
- $\Phi : N \to N$ is a σ-semilinear bijection which commutes with B-action and in the PEL-case satisfies

$$\langle \Phi(v), \Phi(w)\rangle = p^a\langle v, w\rangle \tag{7.2}$$

for some fixed integer a.

(2) Let $N = \prod N_i$ be the decomposition of N induced by the decomposition $B = \prod B_i$ into simple algebras. Then the tuple $\mathrm{relht}(N, \Phi) := (\frac{\dim_L N_i}{[B_i:\mathbb{Q}_p]^{\frac{1}{2}}})_i$ is called relative height of (N, Φ).

(3) A $\mathscr{B}$-isocrystal (N, Φ) is called relevant, if there exists a O_B-stable lattice $\Lambda \subset N$ which in the PEL-case is self-dual w.r.t. some representative of $\langle\ ,\ \rangle$.

Lemma 7.7. *(1) Assume $\mathscr{B}$ is an EL-datum and let (N, Φ) be a $\mathscr{B}$-isocrystal. Then there exists a unique B-module V such that $N \cong V \otimes L$. In particular (N, Φ) is relevant, Λ may be chosen to be defined over $\mathbb{Z}_p$ and is unique up to O_B-linear isomorphism.*

(2) Let $\mathscr{B}$ be a PEL-datum of type (AC) and let (N, Φ) be a relevant $\mathscr{B}$-isocrystal. Then there exists a unique B-module V with $\mathbb{Q}_p^\times$-homogeneous polarisation, depending only on the relative height of (N, Φ), such that $N \cong V \otimes L$. Furthermore, Λ can be chosen to be defined over $\mathbb{Z}_p$ and is unique up to O_B-linear isometry.

Proof. For the proof of part (1) we may assume that $B = F$ is a finite unramified field extension of $\mathbb{Q}_p$ by Morita equivalence. Then $F \otimes L \cong \prod_{\tau:F\hookrightarrow L} L$ inducing a decomposition $N \cong \prod_{\tau:F\hookrightarrow L} N_\tau$. As Φ induces a σ-linear bijection $N_\tau \to N_{\sigma\circ\tau}$, we have that all N_τ have dimension $\mathrm{relht}(N, \Phi)$ over L, thus $N \cong L \otimes F^{\mathrm{relht}(N,\Phi)}$. Obviously $O_L \otimes O_F^{\mathrm{relht}(N,\Phi)}$ is an O_F-stable lattice in N and any O_F-stable lattice of N is isomorphic to it. Now part (2) follows from part (1) and [Kot85] Lemma 7.2 and Remark 7.5. □

Let $\mathscr{B} = (B, O_B, (^*))$ be a (P)EL-datum, let V be a (polarised) left-B-module and let $\Lambda \subset V$ be a (self-dual) O_B-stable lattice in V. Let $\mathbf{n} = (\frac{\dim_{\mathbb{Q}_p} V_i}{[B_i:\mathbb{Q}_p]^{\frac{1}{2}}})_i$. Denote by G the group scheme of O_B-linear automorphisms (similitudes) of Λ. If (N, Φ) is a $\mathscr{B}$-isocrystal of relative height $\mathbf{n}$ we may choose an isomorphism $N \cong V \otimes L$ and write $\Phi = b\sigma$ with $b \in G(L)$. Choosing another isomorphism $N \cong V \otimes L$ replaces b by a σ-conjugate, thus we obtain a bijection

$$\begin{aligned} B(G) &\to \{\text{relevant } \mathscr{B}-\text{isocrystals of relative height } \mathbf{n}\}/\cong \\ [b] &\mapsto (V_L, b\sigma). \end{aligned}$$

We warn the reader that in the PEL-case the above bijection requires twisting $\langle\ ,\ \rangle$ by a scalar, which is unique up to $\mathbb{Q}_p^\times$, so that (7.2) is satisfied (see [RZ96] 1.38). This scalar would not necessarily exist if k was not algebraically closed. Combining this bijection with the Dieudonné functor, we obtain an injective map

$$\left\{\begin{array}{c}\text{Barsotti-Tate groups over } k \text{ with } \mathscr{B}-\text{(P)EL structure} \\ \text{of relative height } \mathbf{n} \text{ up to isogeny}\end{array}\right\} \hookrightarrow B(G).$$

In the case $k = \overline{\mathbb{F}}_p$ denote by $C(G)$ the $G(O_L)$-σ-conjugacy classes in $G(L)$. By a similar argument one gets an injection

$$\{\text{Barsotti-Tate groups over } \overline{\mathbb{F}}_p \text{ with } \mathscr{B}-\text{(P)EL structure of relative height } \mathbf{n}\} \hookrightarrow C(G)$$

where in the PEL case the polarisation is meant to be defined up to $\mathbb{Z}_p^\times$-scalar. For $b \in G(L)$ we denote by $[\![b]\!]$ its $G(O_L)$-σ-conjugacy class.

7.3. **Rapoport-Zink spaces.** We restrict the definition of a Rapoport-Zink space in [RZ96] to our case, that is with unramified (P)EL-datum and hyperspecial level structure.

Definition 7.8. (1) An unramified Rapoport-Zink datum of type EL is a tuple $(B, O_B, V, \mu_{\overline{\mathbb{Q}}_p}, \Lambda_0, [b])$ where
- $\mathscr{B} = (B, O_B)$ is an EL-datum.
- V is a finite left-B-module,
- Λ_0 is an O_B-stable lattice in V,
- $\mu_{\overline{\mathbb{Q}}_p}$ is a conjugacy class of cocharacters $\mu_{\overline{\mathbb{Q}}_p} : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, where $G = \underline{\mathrm{Aut}}_{O_B}(\Lambda_0)$.
- $[b] \in B(G, \mu_{\overline{\mathbb{Q}}_p})$

which satisfy the conditions given below.

(2) An unramified Rapoport-Zink datum of type PEL is a tuple $\hat{\mathscr{D}} = (B, O_B, *, V, \langle\ ,\ \rangle, \mu_{\overline{\mathbb{Q}}_p}, \Lambda_0, [b])$ where
- $\mathscr{B} = (B, O_B, ^*)$ is a PEL-datum,
- V is a finite left-B-module with a B-adapted symplectic form $\langle\ ,\ \rangle$ on the underlying $\mathbb{Q}_p$-vector space,
- Λ_0 is an O_B-stable self-dual lattice in V,
- $\mu_{\overline{\mathbb{Q}}_p}$ is a conjugacy class of cocharacters $\mu_{\overline{\mathbb{Q}}_p} : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, where G is the linear algebraic group with R-valued points (for every $\mathbb{Z}_p$-algebra R)
$$G(R) \quad = \quad \{g \in \underline{\mathrm{Aut}}_{O_B}(\Lambda_0)(R) \mid \langle g(v_1), g(v_2)\rangle = c(g)\langle v_1, v_2\rangle \text{ for some } c(g) \in R^\times\}$$
- $[b] \in B(G, \mu_{\overline{\mathbb{Q}}_p})$.

which satisfy the conditions below.

(3) The field of definition E of $\mu_{\overline{\mathbb{Q}}_p}$ is called the local Shimura field.

(4) A Rapoport-Zink datum as above is called superbasic if $[b]$ is superbasic.

Let $(V_L, \Phi) := (L \otimes_{\mathbb{Q}_p} V, b(\sigma \otimes \mathrm{id}))$ denote the $\mathscr{B}$-isocrystal associated to $[b]$. The additional conditions we impose on the data are the following.

(a) G is connected.

(b) The Newton slopes of (V_L, Φ) are in $[0, 1]$.

(c) The weight decomposition of V_L w.r.t. $\mu_{\overline{\mathbb{Q}}_p}$ contains only the weights 0 and 1, denote V^0 and V^1 the corresponding weight spaces.

(d) In the PEL-case, we have $c(\mu(p)) = p$.

The first condition implies that G is a reductive group scheme, in particular E is a finite unramified field extension of $\mathbb{Q}_p$. We fix a Killing pair $T \subset B$ of G. Denote by μ the dominant element in $\mu_{\overline{\mathbb{Q}}_p}$ and let $V_E = V^0 \oplus V^1$ be the decomposition into weight spaces w.r.t. μ.

Now the $\mathscr{B}$-isocrystal (V, Φ) gives rise to an isogeny class of Barsotti-Tate groups with (P)EL structure. We fix an element $\underline{\mathbb{X}}$ of this isogeny class.

Definition 7.9. Let $Nilp_{O_L}$ denote the full subcategory of schemes over O_L on which p is locally nilpotent. Then the Rapoport-Zink space associated to a datum $\hat{\mathscr{D}}$ as above is the functor $\mathscr{M}_G(b, \mu) : Nilp_{O_L} \to Sets, S \mapsto \mathscr{M}_G(b, \mu)(S)$ given by the following data up to isomorphism.

- A Barsotti-Tate group with (P)EL structure $\underline{X}$ over S with associated (P)EL-datum $\mathscr{B}$. Moreover the Kottwitz determinant condition holds, i.e.
$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = \mathrm{char}(a \mid V^1).$$

- An isogeny $\mathbb{X}_{\overline{S}} \to \underline{X}_{\overline{S}}$, where $\overline{S}$ denotes the closed subscheme of S defined by $p\mathscr{O}_S$.

Example 7.10. Let $\mathscr{D} = (\mathsf{B},^*, \mathsf{V}, \langle\ ,\ \rangle, \mathsf{O}_\mathsf{B}, \Lambda_0, h)$ be a PEL-Shimura datum and $[b]$ be an isogeny class of Barsotti-Tate groups with $\mathscr{D}$-structure. Then $(\mathsf{B}_{\mathbb{Q}_p}, {}^*, \mathsf{V}_{\mathbb{Q}_p}, \langle\ ,\ \rangle, \mathsf{O}_{\mathsf{B},p}, \Lambda_0, [\mu_h], [b])$ is an unramified Rapoport-Zink datum and $\mathscr{M}_G(b,\mu)$ is the moduli functor parametrizing Barsotti-Tate groups with $\mathscr{D}$-structure inside the isogeny class $[b]$.

Theorem 7.11 ([RZ96], Thm. 3.25)**.** *The functor $\mathscr{M}_G(b,\mu)$ is representable by a formal scheme locally of finite type.*

Notation 7.12. From now on we denote by $\mathscr{M}_G(b,\mu)$ the *underlying reduced subscheme* of the moduli space above.

By [RZ96] § 3.23(b) the Kottwitz determinant condition can be reformulated as follows. Let $B = \prod B_i = \prod_i \mathrm{M}_{d_i}(F_i)$ be a decomposition of B into simple factors inducing $X = \prod X_i$ and $V^1 = \prod V_i^1$. Furthermore we decompose $\operatorname{Lie} X$ and V_i^1 according to the O_{F_i}- resp. F_i-action.

$$\begin{aligned} \operatorname{Lie} X_i &= \prod_{\tau: F_i \hookrightarrow L} (\operatorname{Lie} X_i)_\tau \\ V_i^1 &= \prod_{\tau: F_i \hookrightarrow L} V_{i,\tau}^1 \end{aligned}$$

Now X satisfies the determinant condition if and only if $(\operatorname{Lie} X_i)_\tau$ has rank $\dim_{\mathbb{Q}_p} V_{i,\tau}^1$.

We obtain that by Dieudonné theory the k-valued points of $\mathscr{M}_G(b,\mu)$ correspond to O_B-stable lattices in V_L which are self-dual up to $L^\times$-scalar and satisfy $p\Lambda \subset b\sigma(\Lambda) \subset \Lambda, \dim_k(\Lambda/b\sigma(\Lambda))_\tau = \dim V_{i,1,\tau}$. By [Kot92] Cor. 7.3 and Rem. 7.5 the group $G(L)$ acts transitively on the set of O_B-stable lattices in V_L which are self-dual up to a constant, thus

$$\begin{aligned} \mathscr{M}_G(b,\mu)(k) &= \{g\Lambda_0 \mid g \in G(L), gb\sigma(g)^{-1} \in G(O_L)\mu(p)G(O_L)\} \\ &= \{g \in G(L)/G(O_L) \mid gb\sigma(g)^{-1} \in G(O_L)\mu(p)G(O_L)\}. \end{aligned}$$

7.4. **Comparison of PEL structure and $\mathscr{D}$-structure.** The aim of this subsection is to show the following proposition.

Proposition 7.13. *Any Barsotti-Tate group with $\mathscr{D}$-structure is a Barsotti-Tate group with PEL structure of type (AC) with $\mathbb{Z}_p^\times$-homogeneous polarisation. On the other hand, for every Barsotti-Tate group with PEL structure of type (AC) with $\mathbb{Z}_p^\times$-homogeneous polarisation over a connected $\overline{\mathbb{F}}_p$-scheme there exists an unramified PEL-Shimura datum $\mathscr{D}'$ such that the PEL structure defines a $\mathscr{D}'$-structure.*

The construction of the PEL-Shimura datum associated to the Barsotti-Tate group with PEL structure consists of two steps. First we construct a local datum similar to a Rapoport-Zink datum and then we show that it can be obtained by localizing a PEL-Shimura datum. The first step also works for Barsotti-Tate groups with EL structure and will be needed later. Thus we include this case in our construction.

Let $\underline{X}$ be a Barsotti-Tate group with EL structure or with PEL structure of type (AC). As in section 7.2 we associate (V,Λ) and G to $\underline{X}$. It remains to construct a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters such that $\underline{X}$ satisfies the Kottwitz determinant condition. For this we give an explicit description of the group G.

Assume first that we are in the EL-case. Let $B = \prod B_i$ be a decomposition into simple factors, inducing a decomposition $\Lambda = \prod \Lambda_i$ into O_{B_i}-modules. Then

$$G = \prod_i \mathrm{GL}_{O_{B_i}}(\Lambda_i),$$

thus it suffices to give a description of G in the case where B is simple. By Morita equivalence we may furthermore assume that $B = F$ is a finite unramified field extension of $\mathbb{Q}_p$. By choosing an O_F-basis $e_1, \ldots, e_n$ of Λ, we obtain an isomorphism

$$G \cong \mathrm{Res}_{O_F/\mathbb{Z}_p} \mathrm{GL}_n .$$

We denote $\mathrm{GL}_{O_F,n} = \mathrm{Res}_{O_F/\mathbb{Z}_p} \mathrm{GL}_n$.

If $\underline{X}$ is a Barsotti-Tate group with PEL structure of type (AC) a similar construction works. Let $(B, {}^*) = \prod B_i$ be the decomposition into indecomposable factors and $\Lambda = \prod_i \Lambda_i$ accordingly. We denote by G_i the group schemes of similitudes of Λ_i. Then

$$G = (\prod_i G_i)^1 := \{(g_i)_i \in \prod_i G_i \mid c(g_i) = c(g_j) \text{ for all } i, j\}.$$

Hence it suffices to give a description of G in the case where $(B, {}^*)$ is indecomposable. Using the notation of section 7.1 we may replace $\langle\ ,\ \rangle$ by a $\mathrm{M}_d(F)$-hermitian form on V without changing G ([Knu91] Thm. 7.4.1). Furthermore we assume $d = 1$ by hermitian Morita equivalence ([Knu91] Thm. 9.3.5), i.e. $B = F \times F$ in case (I) and $B = F$ in cases ($\mathrm{II_C}$) and (III).

We first consider case (I). We decompose $\Lambda = \Lambda' \oplus \Lambda''$ with the first resp. second factor of $O_F \times O_F$ acting on Λ' resp. Λ''. Then $\langle\ ,\ \rangle$ induces an isomorphism $\Lambda' \cong (\Lambda'')^\vee$ of O_F-modules. Now

$$\begin{aligned} G &= \{(g', g'') \in \underline{\mathrm{Aut}}(\Lambda') \times \underline{\mathrm{Aut}}(\Lambda''); g'' = c(g) \cdot g'^{-t}\} \\ &\cong \underline{\mathrm{Aut}}(\Lambda') \times \mathbb{G}_m \\ &\cong \mathrm{GL}_{O_F,n} \times \mathbb{G}_m. \end{aligned}$$

Here $\mathbb{G}_m$ parametrizes the similitude factor. By choosing dual bases of Λ' and Λ'', we may also consider G as a closed subgroup of $\mathrm{GL}_{O_F,n} \times \mathrm{GL}_{O_F,n}$.

Now assume that $(B, {}^*)$ is of the form ($\mathrm{II_C}$). By the uniqueness assertion of Lemma 7.7 there exists an O_F-basis $x_1, \ldots, x_n$ of Λ such that

$$\langle x_i, x_j \rangle = \delta_{i,n+1-j} \text{ for } i \leq j.$$

This identifies G with the closed subgroup $\mathrm{GSp}_{O_F,n}$ of $\mathrm{GL}_{O_F,n}$, which is given by

$$\mathrm{GSp}_{O_F,n}(R) = \{g \in \mathrm{GL}_n(F \otimes R) \mid gJ_a g^t = c(g) J_a \text{ for some } c(g) \in R^\times\},$$

where J_a denotes the matrix

$$\begin{array}{c} \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix} \\ \underbrace{\hphantom{-1\quad\iddots\quad-1}}_{\frac{n}{2}}\quad\underbrace{\hphantom{1\quad\iddots\quad1}}_{\frac{n}{2}} \end{array}$$

If $(B, {}^*)$ is of type (III), we choose a basis $x_1, \ldots, x_n$ of Λ such that

$$\langle x_i, x_j \rangle = u \cdot \delta_{i,n+1-j}$$

where $u \in O_F^\times$ with $\sigma_{F'}(u) = -u$. This defines a closed embedding of G into $\mathrm{GL}_{O_F,n}$ with image

$$\mathrm{GU}_{O_F,n}(R) = \{g \in \mathrm{GL}(F \otimes R) \mid gJ\sigma_{F'}(g)^t = c(g) J \text{ for some } c(g) \in R^\times\}.$$

Here J denotes the matrix with ones on the anti-diagonal and zeros anywhere else.

We are interested in the $G(\overline{\mathbb{Q}}_p)$-conjugacy classes of $\mathrm{Hom}(\mathbb{G}_m, G_{\overline{\mathbb{Q}}_p})$. As G is reductive, the conjugacy classes are in canonical one-to-one correspondence with the subset of dominant elements $X_*(T)_{\mathrm{dom}}$ of $X_*(T)$ for some choice $T \subset B \subset G$ of a maximal torus T and a Borel subgroup B. For $G = \mathrm{GL}_{O_F,n}, G = \mathrm{GSp}_{O_F,n}, G = \mathrm{GU}_{O_F,n}$ we choose T to be the diagonal torus and B to be the subgroup of upper triangular matrices. In case (I) take T and B to be induced by our choice for $\mathrm{GL}_{O_F,n}$ and the isomorphism $G \cong \mathrm{GL}_{O_F,n} \times \mathbb{G}_m$. With respect to the canonical embedding $G \hookrightarrow \mathrm{GL}_{O_F,n} \times \mathrm{GL}_{O_F,n}$ this means that B denotes the Borel subgroup of pairs (g', g'') where the first factor is an upper triangular matrix (or equivalently the second factor is a lower triangular matrix).

Using the canonical identification $X_*(T) \cong \prod \mathbb{Z}^n$ in the $\mathrm{GL}_{O_F,n}$ case we obtain in the EL-case

$$X_*(T)_{\mathrm{dom}} = \{\mu \in \prod_{\tau: F \hookrightarrow \mathbb{Q}_p} \mathbb{Z}^n \mid \mu_{\tau,1} \geq \ldots \geq \mu_{\tau.n}\}$$

and in the PEL case

$$\begin{array}{llcl}
\text{(I)} & X_*(T)_{\mathrm{dom}} & = & \{\mu = (\mu', \mu'') \in (\prod_\tau \mathbb{Z}^n)^2 \mid \mu'_{\tau,1} \geq \ldots \geq \mu'_{\tau,n}, \mu'_{\tau,i} + \mu''_{\tau,i} = c(\mu), c(\mu) \in \mathbb{Z}\}. \\
(\mathrm{II_C}) & X_*(T)_{\mathrm{dom}} & = & \{\mu \in \prod_\tau \mathbb{Z}^n \mid \mu_{\tau,1} \geq \ldots \geq \mu_{\tau,n}, \mu_{\tau,i} + \mu_{\tau,n+1-i} = c(\mu) \text{ with } c(\mu) \in \mathbb{Z}\}. \\
\text{(III)} & X_*(T)_{\mathrm{dom}} & = & \{\mu \in \prod_\tau \mathbb{Z}^n \mid \mu_{\tau,1} \geq \ldots \geq \mu_{\tau,n}, \mu_{\tau,i} + \mu_{\tau+\sigma_{F'},n+1-i} = c(\mu) \text{ with } c(\mu) \in \mathbb{Z}\}.
\end{array}$$

For details, see the appendix.

Lemma 7.14. *Let $\underline{X}$ be a Barsotti-Tate group with EL structure or PEL structure of type (AC) and V, Λ, G be as above. There exists a (unique) $[\mu]$ with weights 0 and 1 on $V_{\overline{\mathbb{Q}}_p}$ such that $\underline{X}$ satisfies the determinant condition, i.e.*

$$\mathrm{char}(\iota(a) | \operatorname{Lie} X) = \mathrm{char}(a | V^1)$$

Proof. We start with the EL-case. By the virtue of section 7.1 we may assume without loss of generality that $B = F$ is an unramified field extension of $\mathbb{Q}_p$. Recall that the determinant condition is equivalent to

$$\dim(\operatorname{Lie} X)_\tau = \dim_{\mathbb{Q}_p} V^1_\tau = \#\{i \mid \mu_{i,\tau} = 1\}.$$

Thus $\mu = ((\underbrace{1, \ldots, 1,}_{\dim(\operatorname{Lie} X)_\tau}\ 0, \ldots, 0))_{\tau: F \hookrightarrow \overline{\mathbb{Q}}_p}$.

In the PEL-case assume without loss of generality that $(B, {}^*)$ is indecomposable and that $B = F$ resp. $B = F \times F$ in case (I). We exclude case (I) for the moment. Denote by n be the relative height of $\underline{X}$. As in the EL-case, we need μ to satisfy

$$\dim(\operatorname{Lie} X)_\tau = \#\{i \mid \mu_{i,\tau} = 1\}.$$

Now use the argument given in [RZ96] § 3.23(c). The above condition is open and closed on S and local for the étale topology. Hence we assume that $S = \operatorname{Spec} k$ is the spectrum of an algebraically closed field of characteristic p. Let $E(X)$ be the universal extension of X. By functoriality we obtain an O_B-action on $E(X)$ and an isomorphism $E(X) \stackrel{\sim}{\to} E(X^\vee)$. Now by Lemma 7.7 the crystal induced by X over k is a free $W(k) \otimes O_F$-module, thus the value of this crystal at $\operatorname{Spec} k$ is a free $k \otimes O_F$-module. The value at $\operatorname{Spec} k$ is the Lie-algebra of the universal extension $E(X)$ of X. Thus

$$\dim(\operatorname{Lie} E(X))_\tau = n.$$

Now we have a short exact sequence of $O_F \otimes k$-modules

$$0 \to (\operatorname{Lie} X^\vee)^\vee \to \operatorname{Lie} E(X) \to \operatorname{Lie} X \to 0. \tag{7.3}$$

Thus

$$\dim(\operatorname{Lie} X)_\tau = \dim((\operatorname{Lie} X^\vee)^\vee)_\tau = \begin{cases} n - \dim(\operatorname{Lie} X)_\tau & \text{in case } (\mathrm{II_C}) \\ n - \dim(\operatorname{Lie} X)_{\tau+\sigma_{F'}} & \text{in case (III)} \end{cases},$$

which is equivalent to the constraint $c(\mu) = 1$.

In case (I) let $V = V' \oplus V''$ and $\mathrm{Lie}\, X = (\mathrm{Lie}\, X)' \oplus (\mathrm{Lie}\, X)$ be induced by the decomposition of B resp. O_B into two factors. The relative height of $\underline{X}$ is of the form (n, n). The determinant condition is equivalent to

$$\begin{array}{rcl} \dim(\mathrm{Lie}\, X)'_\tau & = & V'^1_\tau \\ \dim(\mathrm{Lie}\, X)''_\tau & = & V''^1_\tau. \end{array}$$

Now the side condition is $\dim V'^1_\tau = \dim V''^0_\tau$, or equivalently $c(\mu) = 1$. Indeed by (7.3) we have

$$\dim(\mathrm{Lie}\, X)''_\tau + \dim(\mathrm{Lie}\, X)'_\tau = n.$$

□

Proof of Proposition 7.13. We have associated a datum $(B, {}^*, V, \langle\ ,\ \rangle, O_B, \Lambda, [\mu])$ to $\underline{X}$ so far. It remains to show that there exists a PEL-Shimura datum $(\mathsf{B}, {}^*, \mathsf{V}, \langle\ ,\ \rangle', \mathsf{O}_\mathsf{B}, \Lambda_0, h)$ such that

- $(B, {}^*) \cong (\mathsf{B}_{\mathbb{Q}_p}, {}^*)$, inducing $O_B = \mathsf{O}_{\mathsf{B},p}$.
- $(V, \langle\ ,\ \rangle) \cong (\mathsf{V}_{\mathbb{Q}_p}, \langle\ ,\ \rangle')$ inducing $\Lambda_0 \cong \Lambda$, $[\mu] = [\mu_h]$.

This is proven in [VW13] Lemma 10.4. On the contrary, the assertion that the PEL structure induced by $\mathscr{D}$ is of type (AC) is well-known to be equivalent to the condition that G is connected.

□

8. The Newton stratification

8.1. **Comparison between the Newton stratifications on $\mathscr{A}_0$ and on $\mathscr{S}_{\underline{X}}$.** Let $\underline{X}$ be a Barsotti-Tate group with EL structure or PEL structure of type (AC) of fixed relative height over an $\mathbb{F}_p$-scheme S. As in the previous section this yields a reductive group scheme G over $\mathbb{Z}_p$ and we choose $T \subset B \subset G$ as above. Now for every geometric point $\bar{s}$ of S the isogeny class of $\underline{X}_{\bar{s}}$ defines a σ-conjugacy class of $G_{\mathbb{Q}_p}$ and thus $\underline{X}$ induces functions

$$\begin{array}{rcl} b_{\underline{X}} : \{\text{ geometric points of } S\} & \to & B(G) \\ \nu_{\underline{X}} : \{\text{ geometric points of } S\} & \to & X_*(T)^\Gamma_{\mathbb{Q},\mathrm{dom}} \\ \kappa_{\underline{X}} : \{\text{ geometric points of } S\} & \to & \pi_1(G)_\Gamma. \end{array}$$

Lemma 8.1 ([RR96], Thm. 3.6)**.** *The map $b_{\underline{X}}$ is lower semi continuous.*

Definition 8.2. For $\underline{X}$ as above and $b \in B(G)$ we consider

$$\begin{array}{rcl} S^b & := & \{s \in S; b_{\underline{X}}(\bar{s}) = b\} \\ S^{\leq b} & := & \{s \in S; b_{\underline{X}}(\bar{s}) \leq b\} \end{array}$$

as locally closed subschemes with reduced structure. The S^b are called Newton strata and the $S^{\leq b}$ are called closed Newton strata.

Now the isogeny classes of Barsotti-Tate groups with $\mathscr{D}$-structure correspond to $B(G, \mu)$ (see for example [VW13] ch. 8). Hence κ is locally constant on $\mathscr{A}_0$, thus it suffices to consider the Newton points to distinguish the Newton strata. We will also write $\mathscr{A}_0^{\nu_G(b)}$ instead of $\mathscr{A}_0^b$ whenever it is convenient.

Proposition 8.3. *Assume Theorem 1.1 holds true. Let $\underline{X}$ be a Barsotti-Tate group with EL structure or PEL structure of type (AC) over a perfect field k_0; denote by G the associated reductive group over $\mathbb{Z}_p$ denote by $b_0 \in B(G)$ the isogeny class of $\underline{X}_{\bar{k}_0}$. For any $b \in B(G)$ which corresponds to an isogeny class of Barsotti-Tate groups with (P)EL structure the following assertions hold.*

(1) $\mathscr{S}_{\underline{X}}^b$ is non-empty if and only if $b \geq b_0$.

(2) If $\mathscr{S}_{\underline{X}}^{\leq b}$ is non-empty then it is and equidimensional of dimension

$$\langle \rho_G, \mu + \nu_G(b) \rangle - \frac{1}{2} \operatorname{def}_G(b).$$

(3) $\mathscr{S}_{\underline{X}}^{\leq b}$ is the closure of $\mathscr{S}_{\underline{X}}^{b}$ in $\mathscr{S}_{\underline{X}}$.

Proof. As mentioned in section 6.3, we may assume in the PEL case that the polarisation is $\mathbb{Z}_p^\times$-homogeneous. In the EL case we may replace $\underline{X}$ by $\underline{X} \times \underline{X}^\vee$ with the obvious $\mathbb{Z}_p^\times$-homogeneous PEL structure by Corollary 7.5. We note that the similitude factor is constant modulo $\mathbb{Z}_p^\times$, as it is determined by the Kottwitz point, which is constant on $\mathscr{S}_{\underline{X}}$ by [RR96] Thm. 3.8. Also, we may replace $\underline{X}$ by $\underline{X}_{\bar{k}_0}$ and thus assume that k_0 is algebraically closed.

Now by Proposition 7.13 $\underline{X}$ is a Barsotti-Tate group with $\mathscr{D}'$-structure for a suitable unramified PEL-Shimura datum $\mathscr{D}'$. Thus by [VW13] Thm. 10.2 there exists $x \in \mathscr{A}_{\mathscr{D}',0}(k_0)$ such that the associated Barsotti-Tate group with $\mathscr{D}$-structure is isomorphic to $\underline{X}$. Now the assertions follows from Theorem 1.1 by applying Proposition 6.9 to x. □

8.2. **The Newton polygon stratification.** Our medium-term goal is to generalise the following improvement of de Jong-Oort's purity theorem by Yang to Barsotti-Tate groups with (P)EL structure. It will be our main tool to compare the two assertions of Theorem 1.1.

Proposition 8.4 (cf. [Yan11] Thm. 1.1)**.** *Let S be a locally noetherian connected $\mathbb{F}_p$-scheme and X be a Barsotti-Tate group over S. If there exists a neighbourhood U of a point $s \in S$ such that the Newton polygons $\mathrm{NP}(X)(x)$ of X over all points $x \in U \setminus \overline{\{s\}}$ have a common break point, then either $\operatorname{codim}_U(\overline{\{s\}}) \leq 1$ or $\mathrm{NP}(X)(s)$ has the same break point.*

Remark 8.5. The original formulation of this theorem gives the assertion for arbitrary F-crystals. For simplicity, we will work with Barsotti-Tate groups instead. However, we remark that our generalisation also works in the setting of F-crystals.

We will generalise the above proposition by comparing the Newton stratification on S to a stratification given by a family of Newton polygons. We consider the following stratification.

Definition 8.6. Let $\underline{X}$ be a Barsotti-Tate group with (P)EL structure over a connected $\mathbb{F}_p$-scheme S and $(B, O_B, (^*))$ the associated (P)EL-datum. Let $B = \prod B_i$ be the decomposition into simple algebras and $X = \prod \underline{X}_i$ be as in Lemma 7.1. Denote by $\mathrm{NP}(\underline{X}) = (\mathrm{NP}(\underline{X}_i))_i$ the family of (classical) Newton polygons associated to the $\underline{X}_i$. We call the decomposition of S according to the invariant $\mathrm{NP}(\underline{X})$ the Newton polygon stratification.

Remark 8.7. One might also think of considering the stratification given by the Newton polygon of X, i.e. the stratification given by of the isogeny class of $X_{\bar{s}}$ (forgetting the (P)EL-structure) for geometric points $\bar{s}$ of S. We warn the reader that this stratification is in general coarser than the Newton stratification.

Now the aim of this subsection is to show that the Newton polygon stratification and the Newton stratification on S coincide. It follows from the definition that the Newton polygon stratification is at worst coarser than the Newton stratification. We reformulate the invariant NP in group theoretic terms to show that they are in fact equal.

Let $\underline{X}, S$ be as above and assume in the PEL-case that $(B, O_B, ^*)$ is of type (AC). Let V, Λ and G be associated to $\underline{X}$ as in subsection 7.4. Define $H := \prod \mathrm{GL}_{\mathbb{Z}_p} \Lambda_i$ with canonical embedding $i : G \hookrightarrow H$. Then the Newton polygon stratification is given by the invariant

$$\mathrm{NP}(\bar{s}) := \nu_H(i(b_{\underline{X}}(\bar{s}))).$$

Lemma 8.8. *The Newton stratification and the Newton polygon stratification on S coincide.*

Proof. We start with the EL-case. Obviously it suffices to show the claim when B is simple, by Morita equivalence we may further assume that $B = F$ is a finite unramified field extension of $\mathbb{Q}_p$. This case the assertion was already proven in section 2.2.2.

In the PEL-case, i factorizes as $G \overset{i_1}{\hookrightarrow} \mathrm{GL}_{O_B}(\Lambda) \overset{i_2}{\hookrightarrow} H$. We have already seen that i_2 separates Newton points, thus it suffices to prove this for i_1. But we have already noted in section 7.4 that i_1 induces an injective map on the dominant cocharacters thus in particular separates Newton points. □

8.3. **Break points of Newton polygons and Newton points.** In order to generalise Proposition 8.4 to the PEL-case, we need the group theoretic description of break points.

Definition 8.9. Let G be a reductive group over $\mathbb{Z}_p$, $T \subset B \subset G$ be a maximal torus and an Borel subgroup of G and let $\Delta^+_{\mathbb{Q}_p}(G)$ denote the set of simple relative roots of $G_{\mathbb{Q}_p}$. For any $\nu \in \mathscr{N}(G)$ and $\beta \in \Delta^+_{\mathbb{Q}_p}(G)$ we make the following definitions.

(1) We say that ν has a break point at β if $\langle \nu, \beta\rangle > 0$. We denote by $J(\nu)$ the set of break points of ν.
(2) Let $\omega_\beta^\vee$, be the relative fundamental coweight of G^{ad} corresponding to β and let $pr_\beta : X_*(T)^\Gamma \to X_*(T^{\mathrm{ad}})^\Gamma \to \mathbb{Q}\cdot\omega_\beta$ be the orthogonal projection (cf. [Cha00] chapter 6). If $j \in J(\nu)$, we call $(\beta, pr_\beta(\nu))$ the break point of ν at β.

Example 8.10. Consider the case $G = GL_h$ with diagonal torus T and Borel of upper triangular matrices B. Then the simple roots are

$$\beta_i : \mathrm{diag}(t_1, \dots, t_n) \mapsto t_{i+1} - t_i.$$

Hence $J(\nu)$ is the set of all j such that $\nu_j > \nu_{j+1}$. In terms of the Newton polygon P associated to ν, this is the set of all places where the slope on the left and the slope on the right do not coincide.

We use the standard identification $X_*(T^{\mathrm{ad}})_{\mathbb{Q}} \cong \mathbb{Q}^n/\mathbb{Q}$ where $\mathbb{Q} \hookrightarrow \mathbb{Q}^n$ is the diagonal embedding. We write $[\nu_1, \dots, \nu_n]$ for $(\nu_1, \dots, \nu_n) \mod \mathbb{Q}$. Now $\omega^\vee_{\beta_j} = [1, \dots, 1, 0, \dots, 0]$ and a short calculation shows

$$\begin{aligned} pr_\beta(\nu) &= [\underbrace{\frac{\nu_1 + \dots + \nu_j}{j}, \dots, \frac{\nu_1 + \cdot + \nu_j}{j}}_{j \text{ times}}, \underbrace{\frac{\nu_{j+1} + \dots + \nu_n}{n-j}, \dots, \frac{\nu_{j+1} + \dots + \nu_n}{n-j}}_{n-j \text{ times}}] \\ &= (\frac{\nu_1 + \ldots + \nu_j}{j} - \frac{\nu_{j+1} + \ldots + \nu_n}{n-j}) \cdot \omega^\vee_{\beta_j} \\ &= (\frac{n}{j(n-j)} P(j) - \frac{1}{n-j} P(h)) \cdot \omega^\vee_{\beta_j}. \end{aligned}$$

In particular if $\nu, \nu' \in \mathscr{N}(G)$ with corresponding Newton polygons P, P' satisfying $P(n) = P'(n)$ we have $pr_{\beta_j}(\nu) = pr_{\beta_j}(\nu')$ if and only if $P(j) = P'(j)$. Thus, under the premise that $P(n) = P'(n)$, the notion of a common break point for $\nu, \nu' \in \mathscr{N}(\mathrm{GL}_h)$ coincides with the classical definition for Newton polygons.

Lemma 8.11. *In the situation of the previous subsection, let β a relative root of $G_{\mathbb{Q}_p}$. There exists a relative root β' of $H_{\mathbb{Q}_p}$ such that for any subset $\{\nu_i\} \subset \mathscr{N}(G)$ the ν_i have a common break point at β if and only if the $i(\nu_i)$ have a common break point at β'.*

Proof. We proceed similarly as in Lemma 8.8. In the EL-case we reduce to the case $B = F$ which follows from the explicit description given in section 2.2.2. In the PEL-case it thus suffices to consider the embedding $i_1 : G \hookrightarrow \mathrm{GL}_{O_F}\Lambda$. We note that the simple roots of G are precisely the restriction of the simple roots of $\mathrm{GL}_{O_F}\Lambda$. Denote by $R'(\beta)$ the set of simple roots of $\mathrm{GL}_{O_F}\Lambda$ above β. Obviously we have for any $\beta' \in R'(\beta)$

$$\langle \nu, \beta\rangle = \langle i_1(\nu), \beta'\rangle.$$

Thus ν has a break point at β if and only if $i_1(\nu)$ has a breakpoint at some (or equivalently every) element of $R'(\beta)$. Now it remains to show that for $\nu_1, \nu_2 \in \mathscr{N}(\beta)$ we have $pr_\beta(\nu_1) = pr_\beta(\nu_2) \Leftrightarrow pr_{\beta'}(i_1(\nu_1)) = pr_{\beta'}(i_1(\nu_2))$. As all functions appearing in the term are linear, it suffices to show

$$pr_\beta(\nu) = 0 \Leftrightarrow pr_{\beta'}(i_1(\nu)) = 0.$$

By the definition of the projections $pr_\beta, pr_{\beta'}$ this is equivalent to

$$\nu \in \sum_{\alpha \in \Delta^+_{\mathbb{Q}_p}(G)\setminus\beta} \mathbb{R}\alpha^\vee \Leftrightarrow i_1(\nu) \in \sum_{\alpha' \in \Delta^+_{\mathbb{Q}_p}(\mathrm{GL}_{O_F}\Lambda)\setminus\beta'} \mathbb{R}\alpha'^\vee.$$

This follows from the fact that $\alpha^\vee$ is a scalar multiple of the sum of the coroots corresponding to elements in $R'(\alpha)$, which is easily checked using the data given in the appendix. □

8.4. **The relationship between the dimension of Newton strata and their position relative to each other.** Now we can finally formulate and prove the generalisation of Proposition 8.4.

Proposition 8.12. *Let S be a connected locally noetherian $\mathbb{F}_p$-scheme and $\underline{X}$ be a Barsotti-Tate group over S with EL structure or PEL structure of type (AC). If there exists a neighbourhood U of a point $s \in S$ such that the Newton polygons $\nu_{\underline{X}}(x)$ of $\underline{X}$ over all points $x \in U \setminus \overline{\{s\}}$ have a common break point, then either* $\mathrm{codim}_U(\overline{\{s\}}) \leq 1$ *or $\nu_{\underline{X}}(s)$ has the same break point.*

Proof. By Lemma 8.8 and 8.11 it suffices to prove the assertion for a family (X_i) of Barsotti-Tate groups without additional structure. But this obviously follows from Proposition 8.4. □

We now proceed to one of the central pieces of the proof of the main theorem.

Proposition 8.13. *We fix an unramified PEL-Shimura datum $\mathscr{D}$ and assume that we have that for any $b \in B(G,\mu)$*

$$\dim \mathscr{A}_0^b \leq \langle \rho, \mu + \nu_G(b)\rangle - \frac{1}{2}\operatorname{def}_G(b). \tag{8.1}$$

Then Theorem 1.1 holds true.

Proof. This proof is the same as the proof of the analogous assertion in the equal characteristic case considered in [Vie13]. As for the importance of this assertion, we give the full proof anyway.

Note that by Corollary 2.15 the inequality on the dimension is equivalent to $\operatorname{codim} \mathscr{A}_0^\nu \geq \operatorname{length}([\nu,\mu])$. We prove the claim by induction on $\operatorname{length}([\nu,\mu])$. For $\operatorname{length}([\nu,\mu]) = 0$, i.e. $\nu = \mu$ the dimension formula of Theorem 1.1 certainly holds true and the density of $\mathscr{A}_0^\mu$ is known by Wedhorn [Wed99]. Now fix an integer i and assume the statement holds true for all $\nu \leq \mu$ and with $\operatorname{length}([\nu,\mu]) < i$. Let ν' with $\operatorname{length}([\nu',\mu]) = i$. We fix an element $\nu \in [\nu',\mu]$ such that $\operatorname{length}([\nu',\nu]) = 1$. Then $\operatorname{length}([\nu,\mu]) = i-1$ and by [Vie13] Lemma 5 (or more precisely the remark after this lemma), we have

$$\mathscr{N}_{\leq\nu'} = \{\xi \in \mathscr{N}_{\leq\nu} \mid pr_\beta(\xi) < \lambda\}$$

for some break point (β,λ) of ν. In particular, $pr_\beta(\nu') < \lambda$.

Let η be a maximal point of $\mathscr{A}_0^{\leq\nu'}$ (i.e. the generic point of an irreducible component). By applying Proposition 8.12 to $S = \operatorname{Spec} \mathscr{O}_{\mathscr{A}_0^{\leq\nu},\eta}$ and $s = \eta$ we see that every irreducible component of $\mathscr{A}_0^{\leq\nu'}$ has at most codimension 1 in $\mathscr{A}_0^{\leq\nu}$. By (8.1) and induction hypothesis we have $\dim \mathscr{A}_0^{\leq\nu'} < \dim \mathscr{A}_0^\nu$, thus $\mathscr{A}_0^{\leq\nu'}$ is of pure codimension $\operatorname{length}([\nu',\mu])$ in $\mathscr{A}_0$. By (8.1) we know that for any $\xi < \nu'$ that $\operatorname{codim} \mathscr{A}_0^\xi \geq \operatorname{length}([\xi,\mu]) > \operatorname{length}([\nu',\mu])$, thus $\mathscr{A}_0^{\nu'}$ is dense in $\mathscr{A}_0^{\leq\nu'}$ and Theorem 1.1 follows. □

9. Central leaves of $\mathscr{A}_0$

In the previous proposition we reduced Theorem 1.1 to an estimate of the dimension of Newton strata in $\mathscr{A}_0$. In the special case of the Siegel moduli variety, Oort has calculated the dimension of Newton strata by writing irreducible components "almost" as a product of so-called central leaves and isogeny leaves and calculating the dimension of these. We will use a similar approach to prove the estimate, applying Mantovan's theorem which implies that a Newton stratum of a PEL-Shimura variety is in finite-to-finite correspondence with the product of a central leaf with a truncated Rapoport-Zink space (cf. Proposition 9.4). First, let us recall the definition and the basic properties of central leaves.

9.1. **The geometry of central leaves.**

Definition 9.1. Let $\underline{X}$ be a completely slope divisible Barsotti-Tate group with $\mathscr{D}$-structure over k with isogeny class b (Recall that by [OZ02] Lemma 1.4 a Barsotti-Tate group X over k is slope divisible if and only if it is isomorphic to a direct sum of isoclinic Barsotti-Tate groups which are defined over a finite field). The corresponding central leaf is defined as

$$C_{\underline{X}} := \{x \in \mathscr{A}_0 \mid \underline{A}^{\mathrm{univ}}_{\overline{x}}[p^\infty] \cong \underline{X}_{\overline{k(x)}}\}$$

The central leaf is a smooth closed subscheme of $\mathscr{A}_0^b$ by [Man05] Prop. 1. The fact that $C_{\underline{X}}$ is closed in the Newton stratum is proven by writing $C_{\underline{X}}$ as a union of irreducible components of

$$C_X := \{x \in \mathscr{A}_0 \mid A^{\mathrm{univ}}_{\overline{x}}[p^\infty] \cong X_{\overline{k(x)}}\},$$

which is closed in $\mathscr{A}_0^b$ by a result of Oort. By [VW13] Thm. 10.2 the central leaves are non-empty. Furthermore, the dimension of $C_{\underline{X}}$ only depends on b.

Proposition 9.2. *Let $\mathscr{D}' = (B, *, V, \langle\,,\rangle, O_B, \Lambda_0, h)$ be a second unramified Shimura datum which agrees with $\mathscr{D}$ except for h and let $K^p \subset G(\mathbb{A}^p)$ be a sufficiently small open compact subgroup. Denote by $\mathscr{A}_0'$ the special fibre of the associated moduli space. We consider two Barsotti-Tate groups $\underline{X} = (X, \iota, \lambda)$ and $\underline{X}' = (X', \iota', \lambda')$ over k equipped with $\mathscr{D}$- resp. $\mathscr{D}'$-structure and assume that we have an isogeny $\rho : \underline{X} \to \underline{X}'$. Denote by C_X and $C_{X'}$ the associated central leaves in $\mathscr{A}_0$ resp. $\mathscr{A}_0'$. Then*

$$\dim C_{\underline{X}} = \dim C_{\underline{X}'}.$$

Remark 9.3. (1) Oort proved the analogous assertion for moduli spaces of (not necessarily principally) polarised abelian varieties ([Oor04] Thm. 3.13). Our proof is a generalisation of his proof.
(2) In the case $\mathscr{D} = \mathscr{D}'$ the assertion was already proved by Mantovan. In the proof of [Man04] Prop. 4.7 she showed the proposition only for some special cases of PEL-Shimura data, but her proof can be generalised to arbitrary unramified PEL-Shimura data using [Man05] ch.4 and 5.

Proof. First we fix some notation. Let $n' = \dim_{\mathbb{Q}} V$. Oort showed in [Oor04] Cor. 1.7 that there exists an integer $N(n')$ such that any two Barsotti-Tate groups X, X' of height n' over an algebraically closed field with $X[p^{N(n')}] \cong X'[p^{N(n')}]$ are isomorphic. Denote $H := \ker \rho$, furthermore $i := \deg \rho$ and $n = N(n') + i$. Choose $x = (A, \iota, \lambda, \eta) \in C_{\underline{X}}$ and let $x' = (A/G, \iota', \lambda', \eta)$. We denote by C_x and $C_{x'}$ the connected components of the central leaves containing x resp. x'. We denote the corresponding universal abelian varieties with additional structure by

$$\begin{aligned} \underline{M} &\to C_x \\ \underline{M}' &\to C_{x'} \end{aligned}$$

and

$$\begin{aligned} \underline{Y} &= \underline{M}[p^\infty] \\ \underline{Y}' &= \underline{M}'[p^\infty]. \end{aligned}$$

Using a slight generalisation of [Oor04] Lemma 1.4 (the same proof still works) there is exists a scheme $T \to C_x$ finite and surjective such that $\underline{M}[p^n]_T$ is constant. We assume that T is irreducible. The abelian variety with additional structure $\underline{M}_T/H_T$ defines a morphism $f : T \to \mathscr{A}'_0$. Using [Oor04] Cor. 1.7, we see that f factorises over C_X; as T is irreducible it thus factors over $C_{x'}$.

We now show that f is quasi-finite. We denote by $\varphi : \underline{Y}_T \to (\underline{M}_T/H_T)[p^\infty]$ the isogeny constructed above and choose ψ such that $\varphi \circ \psi = \psi \circ \varphi = p^n$. Let $u \in C_x$ and $S \subset T_u$ be a reduced and irreducible subscheme. Then

$$\psi_S : (\underline{Y}'_u)_S = (\underline{M}_T/H_T)[p^\infty]_S \to X_S.$$

By arguing as in [Oor04] § 1.11 one checks that the kernel of ψ_S is constant. Thus

$$\underline{M}_S \cong \underline{M}'_S/\ker\psi_S$$

is also constant, i.e. the image of S in C_x is a single point. As T is finite over C_x, this implies that S is a single point.

So we get $\dim C_x = \dim T \leq \dim C_{x'}$. As the assertion of the proposition is symmetric in X and X', the claim follows. □

Proposition 9.4. *Let $b \in B(G,\mu)$ and let $\underline{\mathbb{X}}$ be a slope divisible Barsotti-Tate group over $\overline{\mathbb{F}}_p$ with $\mathscr{D}$-structure in the isogeny class b. Denote by $\mathscr{M}_G(b,\mu)$ the Rapoport-Zink space associated to $\underline{\mathbb{X}}$. Then*

$$\dim \mathscr{A}_0^b = \dim C_{\underline{\mathbb{X}}} + \dim \mathscr{M}_G(b,\mu)$$

Proof. By [Man05] ch. 5 there exists a finite surjective map $\pi_N : J_{m,b} \times \mathscr{M}_b^{n,d} \to \mathscr{A}_0^b$ $(m,n,d \gg 0)$. Here $J_{m,b}$ is an Igusa variety over $C_{\underline{\mathbb{X}}}$ (cf. [Man05] ch. 4), which in particular means finite étale over $C_{\underline{\mathbb{X}}}$, and $\mathscr{M}_b^{n,d}$ are truncated Rapoport-Zink spaces (cf. [Man05] ch. 5), which are quasi-compact and of the same dimension of $\mathscr{M}_G(b,\mu)$ for $n,d \gg 0$. Hence

$$\dim \mathscr{A}_0^b = \dim J_{m,b} + \dim \mathscr{M}_b^{n,d} = \dim C_{\underline{\mathbb{X}}} + \dim \mathscr{M}(b,\mu).$$

□

9.2. **Results on Ekedahl-Oort-strata.** Proposition 9.4 reduces the dimension formula of Theorem 1.1 to computing the dimension of the central leaves and of Rapoport-Zink spaces. Because of Proposition 9.2 it suffices to calculate the dimension of the central leaf for one representative of each isogeny class. In order to find a suitable central leaf we consider of the Ekedahl-Oort stratification, which is studied in great detail in the paper [VW13] of Viehmann and Wedhorn. We recall some of their notions and results.

For a Barsotti-Tate group with $\mathscr{D}$-structure $\underline{X}$ over k we denote by $\mathrm{tc}(\underline{X})$ the isomorphism class of $\underline{X}[p]$. The map tc has the following group theoretic interpretation (cf. [VW13] ch. 8). By Dieudonné theory the isomorphism classes of truncated Barsotti-Tate groups with $\mathscr{D}$-structure correspond to $G(O_L)$-σ-conjugacy classes in $G_1\backslash G(L)/G_1$ where $G_1 := \ker(G(O_L) \twoheadrightarrow G(k))$. Now the truncation map tc is given by the canonical projection.

$$C(G) \twoheadrightarrow \{\sigma - \text{conjugacy classes in } G_1\backslash G(L)/G_1\}$$

The Ekedahl-Oort stratification is the decomposition of $\mathscr{A}_0(\overline{\mathbb{F}}_p)$ given by the invariant tc. The Ekedahl-Oort strata are locally closed subsets of $\mathscr{A}_0(\overline{\mathbb{F}}_p)$ and are indexed by elements $w \in W$ which are the shortest element in the coset $W_{\sigma^{-1}(\mu)}w$. Here $W_{\sigma^{-1}(\mu)}$ denotes the Weyl group of the centralizer of $\sigma^{-1}(\mu)$. We denote the Ekedahl-Oort stratum corresponding to w by $\mathscr{A}_{0,w}$.

Proposition 9.5 ([VW13] Thm. 11.1 and Prop. 11.3)**.** *$\mathscr{A}_{0,w}$ is non-empty and of pure dimension $\ell(w)$.*

As we want to calculate the dimension of central leaves, we are interested in Ekedahl-Oort strata corresponding to p-torsion subgroups (with additional structure) which determine their Barsotti-Tate group with $\mathscr{D}$-structure uniquely.

Definition 9.6. (1) A Barsotti-Tate group with $\mathscr{D}$-structure $\underline{X}$ over k is called minimal if every Barsotti-Tate group with $\mathscr{D}$-structure $\underline{Y}$ with $\underline{X}[p] \cong \underline{Y}[p]$ is isomorphic to $\underline{X}$.
(2) An Ekedahl-Oort stratum is called minimal if the fibre of $\underline{A}^{\mathrm{univ}}[p^\infty]$ over some (or equivalently every) point of it is minimal.

Surely an Ekedahl-Oort stratum is a central leaf if and only if it is minimal and the corresponding Barsotti-Tate group over $\overline{\mathbb{F}}_p$ is slope-divisible. Viehmann and Wedhorn show in their paper that every isogeny class contains a minimal Barsotti-Tate group with $\mathscr{D}$-structure, more precisely they show that this is true for the stronger notion of fundamental elements. We recall their definition.

Definition 9.7. (1) Let P be a semi-standard parabolic subgroup of G_{O_F}, denote by N its unipotent radical and let M be the Levi factor containing T_{O_F}. Furthermore denote by $\bar{N}$ the unipotent radical of the opposite parabolic. We denote

$$\mathscr{I}_M = \mathscr{I} \cap M(O_L), \quad \mathscr{I}_N = \mathscr{I} \cap N(O_L), \quad \mathscr{I}_{\bar{N}} = \mathscr{I} \cap \bar{N}(O_L).$$

Then an element $x \in \tilde{W}$ is called P-fundamental if

$$\begin{aligned} \sigma(x\mathscr{I}_M x^{-1}) &= \mathscr{I}_M, \\ \sigma(x\mathscr{I}_N x^{-1}) &\subseteq \mathscr{I}_N, \\ \sigma(x\mathscr{I}_{\bar{N}} x^{-1}) &\supseteq \mathscr{I}_{\bar{N}}. \end{aligned}$$

(2) We call an element $x \in \tilde{W}$ fundamental if it is P-fundamental for some semi-standard parabolic subgroup $P \subset G_{O_F}$.
(3) We call $[\![c]\!] \in C(G)$ fundamental if it contains a fundamental element of $\tilde{W}$. A Barsotti-Tate group with $\mathscr{D}$-structure over $\overline{\mathbb{F}}_p$ is called fundamental if the corresponding element of $C(G)$ is fundamental.
(4) An Ekedahl-Oort stratum is called fundamental if the fibre of $\underline{A}^{\mathrm{univ}}[p^\infty]$ over some point of it is fundamental.

By [VW13] Rem. 9.8 every fundamental Barsotti-Tate group with $\mathscr{D}$-structure is also minimal. As mentioned above they also show the following assertion.

Proposition 9.8 ([VW13] Thm. 9.18)**.** *Let G and μ be the reductive group scheme and the cocharacter associated to an unramified PEL-Shimura datum. Then for every $b \in B(G,\mu)$ there exists a fundamental element $x \in \tilde{W}$ such that $x \in b$ and $x \in W\mu'$ for some minuscule cocharacter μ'. Furthermore the Barsotti-Tate group with $\mathscr{D}$-structure associated with $[\![x]\!]$ is completely slope divisible.*

The slope divisibility is not mentioned in [VW13], but following their construction of x one easily checks that the induced Barsotti-Tate group with $\mathscr{D}$-structure is completely slope divisible.

10. The dimension of certain Ekedahl-Oort-strata

In order to apply the formula $\dim \mathscr{A}_{0,w} = \ell(w)$ to fundamental Ekedahl-Oort strata, we have to explain how to compute w, or rather $\ell(w)$, if one is given a fundamental element $x \in \tilde{W}$. This can be done by the algorithm provided by Viehmann in the proof of [Vie] Thm. 1.1. Before we apply this algorithm, we recall some combinatorics of the Weyl group which it uses. As the considerations in this section are purely group theoretic, we allow G to be an arbitrary reductive group scheme over the ring of integers O_F of a local field. We use the notation introduced in section 2.1.

10.1. **Shortest elements in cosets of the extended affine Weyl group.** We denote by S resp. S_a the set of simple reflections in W resp. W_a. For $J \subset S_a$ let $W_J \subset \tilde{W}$ be the subgroup generated by J. Then every right (resp. left) W_J-coset of $\tilde{W}$ contains a unique shortest element. We denote by $\tilde{W}^J$ (resp. ${}^J\tilde{W}$) the set of those elements. Then every $x \in \tilde{W}$ can be written uniquely as $x = x^J \cdot w_J = {}_Jw \cdot {}^Jx$ with $x^J \in \tilde{W}^J, w_J \in W_J, {}_Jw \in {}_JW, {}^Jx \in {}^J\tilde{W}$. We have $\ell(x^J) + \ell(w_J) = \ell(x) = \ell({}_Jw) + \ell({}^Jx)$. (cf. [DDPW08] Prop. 4.16)

Moreover there exists a unique shortest element for every double coset $W_J\backslash\tilde{W}/W_K$. we denote the set of all shortest elements in their respective double coset by ${}^J\tilde{W}^K$. Let $u \in {}^J\tilde{W}^K$ and $K' := K \cap u^{-1}Ju$. Then $W_{K'} = W_K \cap u^{-1}W_J u$ and every element $x \in W_J u W_K$ can be written uniquely as $x = w_J u w_{K'}$ with $w_J \in W_J$ and $w_{K'} \in W_{K'}$. Moreover, we have $\ell(x) = \ell(w_J) + \ell(u) + \ell(w_{K'})$. (cf. [DDPW08] Lemma 4.17 and Thm. 4.18)

Of course the above statements also hold for $J, K \subset S$ and W instead of $\tilde{W}$. We denote the respective sets of shortest elements by $W^J, {}^JW$ resp. ${}^JW^K$.

For $\mu \in X_*(T)_{\mathrm{dom}}$ denote by τ_μ the shortest element of $W\varepsilon^\mu W$. Then τ_μ is of the form $x_\mu \cdot \varepsilon^\mu$ with $x_\mu \in W$. Let M_μ be the centralizer of μ and $W_\mu = \{w \in W \mid w(\mu) = \mu\}$ the Weyl group of M_μ. We have the following useful lemmas.

Lemma 10.1. $W_\mu = W_{S\cap\tau_\mu^{-1}S\tau_\mu} = W \cap \tau_\mu^{-1}W\tau_\mu$.

Proof. It suffices to show that for $s \in S$ we have $s(\mu) = \mu$ if and only if $s \in \tau_\mu^{-1}S\tau_\mu$. If $s(\mu) = \mu$ then $\tau_\mu s \tau_\mu^{-1} \in W$. Thus

$$\ell(\tau_\mu s\tau_\mu^{-1}) = \ell(\tau_\mu s\tau_\mu^{-1}\tau_\mu) - \ell(\tau_\mu) = \ell(\tau_\mu s) - \ell(\tau_\mu) = \ell(\tau_\mu) + \ell(s) - \ell(\tau_\mu) = \ell(s) = 1$$

and hence $\tau_\mu s\tau_\mu^{-1} \in S$. On the other hand we have

$$\tau_\mu s\tau_\mu^{-1} = x_\mu p^\mu s p^{-\mu} x_\mu^{-1} = x_\mu s x_\mu^{-1} p^{x_\mu(s(\mu)-\mu)}.$$

Thus $\tau_\mu s\tau_\mu^{-1} \in S$ implies $s(\mu) - \mu = 0$. □

We denote $S_\mu := S \cap \tau_\mu^{-1}S\tau_\mu$.

Lemma 10.2. *Let $J, K \subset S_a, u \in {}^J\tilde{W}^K$. Denote by $K' := K \cap u^{-1}Ju$ and let $w \in W_{K'}$. Then*

$$\ell(uwu^{-1}) = \ell(w).$$

Proof. We have

$$\begin{aligned} l(w) &= \ell(uw) - \ell(u) \\ &= \ell((uwu^{-1})u) - \ell(u) \\ &= \ell(uwu^{-1}) + \ell(u) - \ell(u) \\ &= \ell(uwu^{-1}) \end{aligned}$$

□

Corollary 10.3. *For $x \in \tilde{W}_\mu$ we have*

$$\ell(x_\mu x x_\mu^{-1}) = \ell(x)$$

Proof. This is a consequence of $x_\mu x x_\mu^{-1} = \tau_\mu x \tau_\mu$ and the preceding two lemmas. □

10.2. **$G(O_L)$-σ-conjugacy classes in $\widetilde{W}$.** We have the following result on $G(O_L)$-σ-conjugacy classes of $G_1\backslash G(L)/G_1$.

Proposition 10.4 ([Vie] Thm. 1.1)**.** *Let $\mathscr{T} = \{(w,\mu) \in W \times X_*(T)_{\mathrm{dom}} \mid w \in {}^{\sigma^{-1}(S_\mu)}W\}$. Then the map assigning to (w,μ) the $G(O_L)$-σ-conjugacy class of $G_1 w\tau_\mu G_1$ is a bijection between $\mathscr{T}$ and the $G(O_L)$-σ-conjugacy classes in $G_1\backslash G(L)/G_1$.*

In the case where G is given by an unramified PEL-Shimura datum, the stratum $\mathscr{A}_{0,w}$ corresponds to the $G(O_L)$-σ-conjugacy class of $G_1 w\tau_{\mu_h} G_1$. The proof of the above theorem provides an algorithm which determines the pair (w,μ) associated to the $G(O_L)$-σ-conjugacy class of $G_1 b G_1$ for any $b \in G(L)$. For the special case that $b \in \tilde{W}$ the algorithm simplifies as follows.

(1) Denote by μ' the image of b under the canonical projection $\tilde{W} \twoheadrightarrow X_*(T), w\cdot\varepsilon^\lambda \mapsto \varepsilon^\lambda$ and let $\mu = \mu'_{\mathrm{dom}}$. Then $b \in W\tau_\mu W$, thus we may write
$$b = w\tau_\mu w'$$
with $w, w' \in W$, $\ell(b) = \ell(w) + \ell(\tau_\mu) + \ell(w')$. Now replace b by its σ-conjugate
$$\sigma^{-1}(w')bw^{-1} = \sigma^{-1}(w')w\tau_\mu =: b_0\tau_\mu.$$
(2) Define sequences of subsets $J_i, J'_i \subset S$ and sequences of elements $u_i \in W, b_i \in W_{J'_i}$ as follows;
  (a) $J_0 = J'_0 = S$,
  $J_1 = \sigma^{-1}(S_\mu)$ and $J'_1 = x_\mu S_\mu x_\mu^{-1}$,
  $J_i = J'_{i-1} \cap u_i J_1 u_{i-1}^{-1}$ and $J'_i = x_\mu \sigma(u_{i-1} J_i u_{i-1}^{-1}) x_\mu^{-1}$.
  (b) $u_0 = 1$ and b_0 as above. Let δ_i be the shortest length representative of $W_{J_i}\delta_{i-1}W_{J_i}$ in $W_{J'_{i-1}}$. Then $u_i = u_{i-1}\delta_i$ and b_i is defined as follows. Decompose
  $$b_{i-1} = w_i\delta_i w'_i$$
  with $w_i \in J_i$, $w'_i \in J'_i$ such that $\ell(b_{i-1}) = \ell(w_i) + \ell(\delta_i) + \ell(w'_i)$. Then
  $$b_i = w'_i \cdot x_\mu \sigma(u_i w_i u_i^{-1}) x_\mu^{-1}$$
  The above sequences satisfy the following properties. The sequences (J_i) and (J'_i) are descending and $u_i \in {}^{J_1}W^{J'_i}$.

(3) Let n be sufficiently large such that $J_n = J_{n+1}, J'_n = J'_{n+1}$. Then (w,μ) is given by $w = u_n$ and μ as in step 1.

We note that b and $w\tau_\mu$ are $G(O_L)$-σ-conjugate.

Proposition 10.5. *Let $b, (w,\mu)$ as above. Then*
$$\ell(b) \geq \ell(w\tau_\mu).$$

Proof. We follow the algorithm above. We have
$$\ell(b_0\tau_\mu) = \ell(\sigma^{-1}(w')w\tau_\mu) \leq \ell(w') + \ell(w) + \ell(\tau_\mu) = \ell(b).$$
Next, we show that $\ell(u_i b_i) \leq \ell(u_{i-1}b_{i-1})$ for all i. Note that
$$\ell(u_{i-1}b_{i-1}) = \ell(u_{i-1}) + \ell(b_{i-1}) = \ell(u_{i-1}) + \ell(w_i) + \ell(\delta_i) + \ell(w'_i) = \ell(u_i) + \ell(w_i) + \ell(w'_i).$$
Thus we have to show that $\ell(b_i) \leq \ell(w_i) + \ell(w'_i)$. Now
$$\begin{aligned} \ell(b_i) &= \ell(w'_i x_\mu \sigma(u_i w_i u_i^{-1}) x_\mu^{-1}) \\ &\leq \ell(w'_i) + \ell(x_\mu \sigma(u_i w_i u_i^{-1}) x_\mu^{-1}) \\ &\overset{\text{Cor. 10.3}}{=} \ell(w'_i) + \ell(\sigma(u_i w_i u_i^{-1})) \\ &= \ell(w'_i) + \ell(u_i w_i u_i^{-1}) \\ &\overset{\text{Lemma 10.2}}{=} \ell(w'_i) + \ell(w_i). \end{aligned}$$

Altogether, we have

$$\ell(w\tau_\mu) = \ell(w) + \ell(\tau_\mu) = \ell(u_n) + \ell(\tau_\mu) = \ell(u_n b_n) - \ell(b_n) + \ell(\tau_\mu) \leq \ell(b_0) + \ell(\tau_\mu) \leq \ell(b).$$

□

10.3. **Fundamental Ekedahl-Oort strata.** It is well-known that one has for any $x \in \tilde{W}$ the inequality $\ell(x) \geq \langle 2\rho, \nu_G(x)\rangle$. One calls x a σ-straight element if equality holds, or equivalently if

$$\ell(x \cdot \sigma(x) \cdot \ldots \cdot \sigma^n(x)) = (n+1) \cdot \ell(x)$$

for every non-negative integer n (see for example [He] Lemma 8.1).

Lemma 10.6. *Let G be a reductive group scheme with extended affine Weyl group $\tilde{W}$. Then any fundamental element $x \in \tilde{W}$ is σ-straight.*

Proof. In the split case this was proven in [GHKR10] Prop.13.1.3. Our proof uses the same idea but is a bit more technical than the proof in the split case. We show

$$\ell(x \cdot \sigma(x) \cdot \ldots \cdot \sigma^n(x)) = (n+1) \cdot \ell(x)$$

via induction on n. For $n = 0$ the assertion is tautological. For $n \geq 1$ let $x_{(n)} = \sigma(x) \cdot \ldots \sigma^n(x)$. As x is fundamental we deduce

$$\sigma(x\mathscr{I}_M x^{-1}) = \mathscr{I}_M \Rightarrow \mathscr{I}_{\sigma^k(M)} = \sigma^k(x^{-1})\mathscr{I}_{\sigma^{k-1}(M)}\sigma^k(x) \text{ for all } k \Rightarrow x_{(n)}^{-1}\mathscr{I}_M x_{(n)} = \mathscr{I}_{\sigma^n(M)} \subset \mathscr{I},$$

$$\sigma(x\mathscr{I}_{\bar{N}} x^{-1}) \supseteq \mathscr{I}_{\bar{N}} \Rightarrow \mathscr{I}_{\sigma^k(\bar{N})} \supseteq \sigma^k(x^{-1})\mathscr{I}_{\sigma^{k-1}(\bar{N})}\sigma^k(x) \text{ for all } k \Rightarrow x_{(n)}^{-1}\mathscr{I}_{\bar{N}} x_{(n)} \subseteq \mathscr{I}_{\sigma^n(\bar{N})} \subset \mathscr{I}.$$

Hence by Iwahori factorisation

$$x\mathscr{I}x_{(n)} = x\mathscr{I}_N\mathscr{I}_M\mathscr{I}_{\bar{N}}x_{(n)} = (x\mathscr{I}_N x^{-1})xx_{(n))}(x_{(n)}^{-1}\mathscr{I}_M x_{(n)})(x_{(n)}^{-1}\mathscr{I}_{\bar{N}}x_{(n)}) \subseteq \mathscr{I}xx_{(n)}\mathscr{I}$$

which implies

$$\ell(x \cdot x_{(n)}) = \ell(x) + \ell(x_{(n)}) = \ell(x) + n \cdot \ell(x) = (n+1) \cdot \ell(x).$$

□

Corollary 10.7. *We fix $b \in B(G, \mu)$.*

(1) Let $x \in \tilde{W}$ and $\mu' \in X_*(T)$ be as in Proposition 9.8. Denote by $\mathscr{A}_0'$ the special fibre of the moduli space associated to the datum $(\mathsf{B}, \mathsf{O}_\mathsf{B}, \mathsf{V}, \langle\ ,\ \rangle, \Lambda_0, \mu'_{\mathrm{dom}})$ and by $\mathscr{A}_{0,w}'$ the Ekedahl-Oort stratum corresponding to $\mathrm{tc}(x)$. Then $\dim \mathscr{A}_{0,w}' = \langle 2\rho, \nu_G(b)\rangle$.
(2) Any central leaf in $\mathscr{A}_0^b$ has dimension $\langle 2\rho, \nu_G(b)\rangle$.
(3) We have $\dim \mathscr{A}_0^b = \dim \mathscr{M}_G(b, \mu) + \langle 2\rho, \nu_G(b)\rangle$.

Proof. We have $\dim \mathscr{A}_{0,w} = \ell(w) = \ell(w\tau_\mu)$. Now $\nu_G(w\tau_\mu) = \nu_G(b)$, thus $\ell(w\tau_\mu) \geq \langle 2\rho, \nu_G(b)\rangle$. On the other hand we have $\ell(w\tau_\mu) \leq \ell(x) = \langle 2\rho, \nu_G(b)\rangle$ by Proposition 10.5 and the previous lemma, thus proving part (1). The second part is a direct consequence of part (1) and Proposition 9.2 and the last assertion follows by Prop. 9.4. □

11. Epilogue

Now we have reduced the task of proving the main theorems to proving the following proposition.

Proposition 11.1. *(1) We have*

$$\dim \mathscr{M}_G(b, \mu) \leq \langle \rho, \mu - \nu_G(b)\rangle - \frac{1}{2}\operatorname{def}_G(b) \tag{11.1}$$

(2) Assume that b is superbasic. Then the connected components of $\mathscr{M}_G(b, \mu)$ are projective.

We recall the reduction steps we made so far for the readers convenience. First we reduced Theorem 1.1 to the inequality (8.1) in Proposition 8.13. Assuming that the above proposition holds true, we obtain the inequality (8.1) by applying the above estimate for $\dim \mathscr{M}_G(b,\mu)$ to Corollary 10.7 (3). Now Theorem 1.4 follows from Theorem 1.1 by Proposition 8.3.

In return, applying the dimension formula for $\mathscr{A}_0^b$ of Theorem 1.1 to Corollary 10.7 (3) yields the first assertion of Theorem 1.2. The other assertion is identical to the second assertion of the proposition above.

12. The correspondence between the general and superbasic Rapoport-Zink spaces

12.1. **Simple RZ-data and Rapoport-Zink spaces over perfect fields.** We call an unramified Rapoport-Zink datum $(B, O_B, (*), V, (\langle\ ,\ \rangle), \Lambda_0)$ *simple* if $B = F$ is an unramified field extension. In [Far04] § 2.3.7 Fargues gives an open and closed embedding

$$\mathscr{M}_G(b,\mu) \hookrightarrow \prod_i \mathscr{M}_{G_i}(b_i,\mu_i)$$

of an arbitrary Rapoport-Zink space into a product of Rapoport-Zink spaces associated to simple data using assertions as in section 7.1. In particular,

$$\dim \mathscr{M}_G(b,\mu) = \sum_i \dim \mathscr{M}_{G_i}(b_i,\mu_i)$$

and one obtains an analogous formula for the right hand side of the dimension estimate of Proposition 11.1. Thus it suffices to prove the proposition for simple Rapoport-Zink data.

We note that as a consequence of [RR96] Lemma 1.3 the reduced subscheme of a Rapoport-Zink space can be defined over $\overline{\mathbb{F}}_p$, thus we can make the following assumption.

Notation 12.1. In the following we will assume that $k = \overline{\mathbb{F}}_p$.

In order to estimate the dimension of the Rapoport-Zink spaces we will use some methods which only work for schemes defined over a finite field. So we consider the more general setup as in section 7.3 where the RZ-spaces are defined over perfect fields. We fix a perfect field k_0 and denote $L_0 = W(k_0)_{\mathbb{Q}}$.

Definition 12.2. A simple Rapoport-Zink datum relative to L_0 is a datum $\hat{\mathscr{D}} = (F, O_F, (^*), V, (\langle\ ,\ \rangle), \Lambda_0, [b])$ as in Definition 7.8 but $[b]$ denotes a σ-conjugacy class in $G(L_0)$ which is assumed to be decent, i.e. it contains an element b satisfying

$$(b\sigma)^s = \nu_G(b)(p^s)\sigma \tag{12.1}$$

for some natural number s.

As in the case of an algebraically closed field, one can associate an isomorphism class of $\mathscr{B}$-isocrystals to $[b]$ ([RZ96] Lemma 3.37). Now let $(\mathbb{X}, \iota_{\mathbb{X}}, (\lambda_{\mathbb{X}}))$ be a Barsotti-Tate group with (P)EL structure with $\mathscr{B}$-isocrystal $(L_0 \otimes V, b(\sigma \otimes \mathrm{id}))$.

Proposition 12.3 ([RZ96], Cor. 3.40)**.** *Assume that L_0 contains the local Shimura field and $\mathbb{Q}_{p^s}$ (where s is chosen as in (12.1)). Then the associated functor $\mathscr{M}_G(b,\mu)$ defined as in Definition 7.9 is representable by a formal scheme formally locally of finite type over* $\operatorname{Spf} W(k_0)$.

We note that every class $[b] \in B(G)$ contains an element b_0 which satisfies the decency equation (12.1) above. By [RZ96] Cor. 1.9 we have $b_0 \in \mathbb{Q}_{p^s}$ thus every Rapoport-Zink space can be defined over a finite field k_0.

12.2. **Construction of the correspondence.** We fix a simple Rapoport-Zink datum $\hat{\mathscr{D}} = (B, O_B, (^*), V, (\langle\ ,\ \rangle), \Lambda_0, \mu)$. Let $P \subset G$ be a standard parabolic subgroup defined over $\mathbb{Z}_p$ and $M \subset P$ be its Levi subgroup which contains T. We will later assume that $[b]$ induces a superbasic σ-conjugacy class in $M(L)$. However, our construction works in greater generality.

By [SGA3] Exp. XXVI ch. 1 the pairs $M \subset P$ as above are given by the root datum of G in analogy with reductive group over fields. Using the explicit description of the root data in the appendix, one sees that there exists a decomposition $\Lambda_0 = \bar{\Lambda}_1 \oplus \ldots \oplus \bar{\Lambda}_r$ such that

$$\begin{aligned} M &= \{g \in G; g(\bar{\Lambda}_i) = \bar{\Lambda}_i\} \\ P &= \{g \in G; g(\Lambda_i) = \Lambda_i\}, \end{aligned}$$

where $\Lambda_i = \bar{\Lambda}_1 \oplus \ldots \oplus \bar{\Lambda}_i$. We get in the EL-case

$$M \cong \prod_{i=1}^{r'} \mathrm{GL}_{O_F, n_i},$$

where $n_i = \dim_F \overline{V_i}$ and $r' = r$.

In the PEL-case we can assume that for any i

$$\bar{\Lambda}_i^{\perp} = \bar{\Lambda}_1 \oplus \ldots \oplus \bar{\Lambda}_{r-i-1} \oplus \bar{\Lambda}_{r-i+1} \oplus \ldots \oplus \bar{\Lambda}_r.$$

We denote $r' = \lfloor \frac{r}{2} \rfloor + 1$. Then we have

$$M \cong \begin{cases} \prod_{i=1}^{r'-1} \mathrm{GL}_{O_F, n_i} \times \mathbb{G}_m & \text{if } r \text{ is even} \\ \prod_{i=1}^{r'-1} \mathrm{GL}_{O_F, n_i} \times \mathrm{GU}_{O_F, n_{r'}} & \text{if } r \text{ is odd, } ^* = \mathrm{id} \\ \prod_{i=1}^{r'-1} \mathrm{GL}_{O_F, n_i} \times \mathrm{GSp}_{O_F, n_{r'}} & \text{if } r \text{ is odd, } ^* = \bar{\cdot}. \end{cases}$$

We also write this decomposition as $M = \prod M_i$ with $M_{r'}$ denoting the right factor in the PEL-case.

After replacing b by a $G(L)$-σ-conjugate, we may assume that $b \in M(L)$. Denote by $[b]_M$ the σ-conjugacy class in $M(L)$. After σ-conjugating by an element of $M(L)$ we furthermore assume that b satisfies a decency equation (12.1). Denote by b_i the images of b in M_i. Also note that the b_i satisfy the same decency equation in M_i as b.

Let k_0 be finite field containing k_F and $\mathbb{F}_{p^s}$ where the s is given by the decency equation of b. We choose Barsotti-Tate groups with (P)EL structure $\overline{\mathbb{X}}_i$ over k_0 with $\mathscr{B}$-isocrystals isomorphic to $(V_{i,K_0}, b\sigma)$ having the property that in the PEL-case the pairing $\langle\ ,\ \rangle$ induces an isomorphism $\overline{\mathbb{X}}_i \cong \overline{\mathbb{X}}_{r-i}^{\vee}$. In particular the isogeny class of $\mathbb{X} = \overline{\mathbb{X}}_1 \oplus \ldots \oplus \overline{\mathbb{X}}_r$ corresponds to the isomorphism class of $(V, b\sigma)$ (with additional structure). We denote $\mathbb{X}_i = \overline{\mathbb{X}}_1 \oplus \ldots \oplus \overline{\mathbb{X}}_i$.

The following objects will allow us to relate $\mathscr{M}_G(b, \mu)$ to the (underlying reduced subschemes of) Rapoport-Zink spaces corresponding to the σ-conjugacy classes $[b_i]_{M_i}$.

Definition 12.4. Using the notation introduced above we make the following definitions.

(1) Let $\mathscr{M}_P(b, \mu)$ be the functor associating to each scheme S over k_0 the following data up to isomorphism.
   - $(X, \rho) \in \mathscr{M}_G(b, \mu)(S)$ and
   - a filtration $X_\bullet = (X_1 \subset \ldots \subset X_r = X)$ of X such that the restriction $\rho_{|\mathbb{X}_i}$ defines a quasi-isogeny onto X_i.

(2) Similarly, let $\mathscr{M}_M(b, \mu)$ be the functor associating the following data (up to isomorphism) to a scheme S over k_0.

- $(X,\rho) \in \mathscr{M}_G(b,\mu)(S)$ and
- a direct sum decomposition $X = \bar{X}_1 \oplus \ldots \oplus \bar{X}_r$ such that the restriction $\rho_{|\overline{\mathbb{X}_i}}$ defines a quasi-isogeny onto $\bar{X}_i$.

We note that if $(X, X_\bullet, \rho) \in \mathscr{M}_P(b,\mu)(S)$ then $X_i = \rho(\mathbb{X}_i)$. Thus the filtration $X_\bullet$ is uniquely determined by (X,ρ) if it exists, i.e. if $\rho(\mathbb{X}_i)$ is a Barsotti-Tate group. Following the proof of [Man08], Prop. 5.1, we conclude that $i_P : \mathscr{M}_P(b,\mu) \to \mathscr{M}_G, (X, X_\bullet, \rho) \mapsto (X,\rho)$ is the decomposition of $\mathscr{M}_G(b,\mu)$ into locally closed subsets given by the invariant $(\mathrm{ht}(\rho_{|\mathbb{X}_i}))_i$. In particular, $\mathscr{M}_P(b,\mu)$ is locally of finite type and $\dim \mathscr{M}_P(b,\mu) = \dim \mathscr{M}_G(b,\mu)$

Now the $\mathbb{X}_i$ are O_F-stable and in the PEL-case $\lambda_{\mathbb{X}}$ induces isomorphisms $\mathbb{X}_i \stackrel{\sim}{\to} (\mathbb{X}/\mathbb{X}_{r-i})^\vee$. As ρ is compatible with the PEL structure, the analogous compatibility assertion also holds for the filtration $X_\bullet$ with respect to λ and ι. Hence we have a canonical O_F-action and polarisation on $\bigoplus_{i=1}^r X_i/X_{i-1}$ (where $X_0 := 0$). Thus we get a morphism

$$p_M : \mathscr{M}_P(b,\mu) \to \mathscr{M}_M(b,\mu), (X, X_\bullet, \rho) \mapsto (\bigoplus_{i=1}^r X_i/X_{i-1}, \oplus\bar{\rho}_i)$$

where the $\bar{\rho}_i$ denote the induced isogenies on the subquotients. So we have a correspondence

$$\begin{tikzcd} & \mathscr{M}_P(b,\mu) \arrow[dl, "p_M"'] \arrow[dr, "i_P"] & \\ \mathscr{M}_M(b,\mu) & & \mathscr{M}_G(b,\mu) \end{tikzcd} \tag{12.2}$$

Now we want to describe $\mathscr{M}_M(b,\mu)$ in terms of Rapoport-Zink spaces. Analogously to our consideration above we get that for $(\bigoplus_i \overline{X_i}, \rho) \in \mathscr{M}_M(b,\mu)(S)$ the $\bar{X}_i$ are O_F-stable and that λ induces isomorphisms $\overline{X_i} \stackrel{\sim}{\to} \bar{X}_{r-i}^\vee$. As in [RV] ch. 5.2 we get an isomorphism

$$\begin{array}{ccc} \mathscr{M}_M(b,\mu) & \stackrel{\sim}{\longrightarrow} & \coprod_{\mu_M \in I_{\mu,b,M}} \mathscr{M}_{M,b,\mu_M} \\ (\bigoplus_i \bar{X}_i, \rho) & \mapsto & (\bar{X}_i, \rho_{|\overline{\mathbb{X}}_i})_i \end{array}$$

where $I_{\mu,b,M}$ denotes the $M(\overline{\mathbb{Q}}_p)$-conjugacy classes of cocharacters $\mu_M : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to M_{\overline{\mathbb{Q}}_p}$ which are contained in the conjugacy class of μ and satisfy $b \in B(M,\mu_M)$ and $\mathscr{M}_{M,b,\mu_M}$ is defined by

$$\mathscr{M}_{M,b,\mu_M} = \prod_{i=1}^{r'} \mathscr{M}_{M_i}(b_i, \mu_{M,i})$$

Now consider again the diagram (12.2). As we have $\dim \mathscr{M}_P(b,\mu) = \dim \mathscr{M}_G(b,\mu)$, it remains to calculate the dimension of the fibres of p_M to relate the dimension of $\mathscr{M}_G(b,\mu)$ to the dimension of $\mathscr{M}_M(b,\mu)$.

For this we consider the k-valued points of the diagram (12.2) with the natural action of $\Gamma_0 = \mathrm{Gal}(k/k_0)$ on the set of points. By section 7.3 we have

$$\begin{array}{rcl} \mathscr{M}_G(b,\mu)(k) & = & \{g \in G(L)/G(O_L) \mid gb\sigma(g)^{-1} \in G(O_L)p^\mu G(O_L)\} \\ \mathscr{M}_P(b,\mu)(k) & = & \{g \in P(L)/P(O_L) \mid gb\sigma(g)^{-1} \in G(O_L)p^\mu G(O_L)\} \\ \mathscr{M}_M(b,\mu_M)(k) & = & \{m \in M(L)/M(O_L) \mid mb\sigma(m)^{-1} \in G(O_L)p^\mu G(O_L)\} \end{array}$$

and

$$\mathscr{M}_{M,b,\mu_M}(k) = \{m \in M(L)/M(O_L) \mid mb\sigma(m)^{-1} \in M(O_L)p^{\mu_M} M(O_L)\}.$$

with the canonical Γ_0-action.

Now the diagram (12.2) induces

$$
\begin{tikzcd}
& \mathscr{M}_P(b,\mu)(k) \arrow[dl, "p_M : mnP(O_L)\mapsto mM(O_L)"'] \arrow[dr, "i_P : gP(O_L)\mapsto gG(O_L)"] & \\
\mathscr{M}_M(b,\mu)(k) & & \mathscr{M}_G(b,\mu)(k)
\end{tikzcd}
\tag{12.3}
$$

Here p_M uses the decomposition $P \cong M \times N$ where N denotes the unipotent radical of P.

Proposition 12.5. *Let $x \in \mathscr{M}_{M,b,\mu_M}$. Then*

$$\dim p_M^{-1}(x) = \langle \rho, \mu - \nu_G(b) \rangle - \langle \rho_M, \mu_M \rangle.$$

where ρ resp. ρ_M denotes the half-sum of all positive (absolute) roots in G resp. M.

We denote $\mathscr{F} := p_M^{-1}(x)$. Choose an element $m \in M(L)$ such that $x = mM(O_L)$. If we replace Λ_0 by $m \cdot \Lambda_0$, we may assume that $m = \mathrm{id}$. Note that this replaces b by a σ-conjugate. However, the formula above shows that the dimension of $\mathscr{F}$ does not change if we replace b by a σ-conjugate.

13. Numerical dimension of subsets of $N(L)$

Let $\tilde{\mathscr{F}}$ denote the pre-image of $\mathscr{F}(k)$ in $N(L)$ with respect to the canonical projection $N(L) \twoheadrightarrow N(L)/N(O_L)$. In this section we define the notion of "numerical dimension" for certain subsets of $N(L)$ and show that with respect to this notion the dimension of $\tilde{\mathscr{F}}$ coincides with the dimension of $\mathscr{F}$. We will calculate the numerical dimension of $\tilde{\mathscr{F}}$ and thus prove Proposition 12.5 in the next section.

13.1. **The concept of numerical dimension.** Let us first sketch the idea behind the numerical dimension. The starting point is the following result of Lang and Weil. If X is variety over $\mathbb{F}_q$ of dimension d then the cardinality of $X(\mathbb{F}_{q^s})$ grows with the same speed as $q^{s\cdot d}$. In particular the dimension of X is uniquely determined by the $\mathrm{Gal}(k/\mathbb{F}_q)$-set $X(k)$. Now if we tried to calculate the dimension of $\mathscr{F}$ by applying the theorem and counting points, we would run into two problems.

First of all $\mathscr{F}$ is only locally of finite type. We will solve this problem by defining an exhausting filtration of $N(L)$ by so-called bounded subsets which correspond to closed quasi-compact (and thus finite type) subschemes of $\mathscr{F}$.

The second problem is that Galois-sets without a geometric background are in general too badly behaved to use them. Here the general idea is that by applying the theorem of Lang and Weil two times, one sees that any finite type scheme Y with $Y(k) \cong X(k)$ as $\mathrm{Gal}(k/\mathbb{F}_q)$-sets has the same dimension as X. This applies to our situation as follows.

Let $(\mathscr{F}_m)_{m\in\mathbb{N}}$ be the above mentioned filtration by finite type subschemes and $(\tilde{\mathscr{F}}_m)_{m\in\mathbb{N}}$ be the corresponding bounded subsets in $N(L)$. Unfortunately, it is not known (and probably not true) whether there is any good structure of an ind-scheme of ind-finite type on $N(L)/N(O_L)$ which induces a useful scheme structure on $\tilde{\mathscr{F}}_m/N(O_L)$. Therefore we will use the following work-around. We replace $\mathscr{F}_m$ its image $c(\mathscr{F}_m)$ under the conjugation by a suitable element of $T(L)$ such that $c(\mathscr{F}_m)$ is contained in $N(O_L)$. Here we have a canonical structure of affine spaces on the quotients $N(O_L/p^j)$ given by the truncated p-adic loop groups. We will prove that $c(\tilde{\mathscr{F}}_m)$ is the full pre-image of a locally closed subvariety $\bar{Y}$ of $N(O_L/p^j)$ if j is big enough. Thus we can define the numerical dimension of $\tilde{\mathscr{F}}_m$ via the dimension of $\bar{Y}$.

We note that $\tilde{\mathscr{F}}_m/N(O_L)$ will in general not be isomorphic to $\bar{Y}(k)$ (yet it will be the analogue of an affine fibration over $\bar{Y}(k)$). This is because we have shrunken $\tilde{\mathscr{F}}_m$ by applying c and have only divided out a subgroup of $N(O_L)$ instead of $N(O_L)$ itself. We can (and will) compensate this deviations by adding a constant to $\dim \bar{Y}$ depending on c and subtracting the dimension of $N(O_L/p^j)$ as variety.

This definition of numerical dimension will give us the same value as we would get if we were counting points (cf. Proposition 13.18), but has the advantage that it also allows us to use some tools from geometry.

13.2. **Notation and conventions.** Let G be a reductive group scheme over $\mathbb{Z}_p$. We impose the same notions as in section 2.1. Moreover, we denote $I := \mathrm{Gal}(O_F/\mathbb{Z}_p)$. In particular the action of Γ on $X^*(T)$ and $X_*(T)$ factorizes through I. For any $\alpha \in R$ denote by $\mathfrak{g}^\alpha$ the corresponding weight space in the Lie algebra of G_{O_F} and by U_α the corresponding root subgroup. By the uniqueness of U_α ([SGA3] Exp. XXII Thm. 1.1) the action of an element $\tau \in I$ on G_{O_F} maps U_α isomorphically onto $U_{\tau(\alpha)}$.

We fix a standard parabolic subgroup $P = MN$ of G. We write $K = G(O_L), K_M = M(O_L)$ and $K_N = N(O_L)$. Now

$$\mathrm{Lie}\, P_{O_F} = \bigoplus_{\alpha \in R'} \mathfrak{g}^\alpha$$

for a closed I-stable subset $R' \subset R$. We denote $R_N := \{\alpha \in R^+ \mid -\alpha \notin R'\}$. Then multiplication in G defines an isomorphism of schemes

$$N_{O_F} \cong \prod_{\alpha \in R_N} U_\alpha \tag{13.1}$$

where the product is taken with respect to an arbitrary (but fixed) total order on R_N (cf. [SGA3] Exp. XXVI ch. 1).

Let δ_N be the sum of all fundamental coweights corresponding to simple roots in R_N. For $i \geq 1$ we define the group schemes

$$\begin{aligned} N[i] &:= \prod_{\substack{\alpha \in R_N \\ \langle \alpha, \delta_N \rangle \geq i}} U_\alpha \subseteq N \\ N\langle i \rangle &:= N[i] \big/ N[i+1] \end{aligned}$$

We note that the sets $\{\alpha \in R_N \mid \langle \alpha, \delta_N \rangle \geq i\}$ are I-stable, thus the I-action permutes the U_α in the above product. Now the commutator $[u_\alpha, u_\beta]$ of two elements $u_\alpha \in U_\alpha, u_\beta \in U_\beta$ is contained in $\prod U_{i\alpha + j\beta}$, which is contained in $N[i]$ if U_α and U_β are. We conclude that the I-action on N stabilizes $N[i]$. Thus $N[i]$ descents to $\mathbb{Z}_p$ and a posteriori also $N\langle i \rangle$. Also note that the canonical isomorphism of schemes

$$N\langle i \rangle \cong \prod_{\substack{\alpha \in R_N \\ \langle \alpha, \delta_N \rangle = i}} U_\alpha$$

is in fact an isomorphism of group schemes.

Let λ be a regular dominant coweight of T defined over $\mathbb{Z}_p$. For $i \in \mathbb{Z}$ we define $N(i) := \lambda(p^i) N(O_L) \lambda(p^{-i})$. This defines an exhausting filtration

$$\ldots N(-2) \supset N(-1) \supset N(0) \supset N(1) \supset N(2) \supset \ldots$$

of $N(L)$. For every subset $Y \subset N(L)$ we denote $Y(i) := \lambda(p^i) Y \lambda(p^{-i})$.

For any algebraic group H over $\mathbb{Z}_p$ and any $i \in \mathbb{N}$ we denote $H_i = \ker(H(O_L) \twoheadrightarrow H(O_L/p^i O_L))$. In particular, we get a second filtration

$$N(0) = N_0 \supset N_1 \supset N_2 \supset \ldots$$

of $N(0)$.

Remark 13.1. The isomorphism (13.1) induces an homeomorphism

$$N(L) \cong \prod_{\alpha \in R_N} L$$

which yields identifications

$$\begin{aligned} N(j) &= \prod_{\alpha \in R_N} p^{j\langle \alpha, \delta_N \rangle} O_L \\ N_j &= \prod_{\alpha \in R_N} p^j O_L \end{aligned}$$

Definition 13.2. A subset $Y \subset N(L)$ is called bounded if it is contained in $N(-j)$ for an $j \in \mathbb{Z}$.

Lemma 13.3. *A subset is $Y \subset N(L)$ is bounded if and only if it is bounded as a subset of $G(L)$ in the sense of Bruhat and Tits.*

Proof. To avoid confusion we temporarily call Y "BT-bounded" if it is bounded in the sense of Bruhat and Tits. By definition a subset $Y \subset N(L)$ is BT-bounded if and only if $\mathrm{val}_p(f(Y))$ is bounded from below for every $f \in \Gamma(G, \mathscr{O}_G)$, or equivalently for every $f \in \Gamma(N, \mathscr{O}_N)$. Using the isomorphism $N_L \cong \mathbb{A}_L^{R_N}$, we may reformulate the condition as follows. The set $Y \subset L^{R_N}$ is BT-bounded if and only if $\mathrm{val}_p(f(Y))$ is bounded from below for every $f \in L[X_\alpha]_{\alpha \in R_N}$. It obviously suffices to check this for the coordinate functions. Thus Y is BT-bounded if and only if $Y \subset (p^{-k} O_L)^{R_N}$ for some integer k. But this is equivalent to Y being bounded by the description given in Remark 13.1. □

13.3. **Admissible subsets of $N(0)$ and the p-adic loop group.** As a first step we now define the numerical dimension for a family of subsets of $N(0)$. We will extend this definition to certain subsets of $N(L)$ (called "admissible") is the next subsection. Before that we give a remainder on p-adic loop groups.

Definition 13.4. Let H be a linear algebraic group over $\mathbb{Z}_p$.

(1) The p-adic loop group is the ind-affine ind-scheme over $\mathbb{F}_p$ representing the functor
$$L_p H(R) = H(W(R)_{\mathbb{Q}}).$$
(2) The positive p-adic loop group is the affine scheme over $\mathbb{F}_p$ with R-valued points
$$L_p^+ H(R) = H(W(R)).$$
(3) Let j be a positive integer. We define the positive p-adic loop group truncated at level j as the affine scheme of finite type over $\mathbb{F}_p$ representing the functor
$$L_p^{+,j} H(R) = H(W_j(R)).$$

For the proof that the functors above can be represented as claimed, we redirect the reader to [Kre] ch. 4.

Lemma 13.5. *Let H be a linear algebraic group over $\mathbb{Z}_p$. The truncation maps $t_j : L_p^+ H \to L_p^{+,j} H$ are open and surjective.*

Proof. We have for any $\mathbb{F}_p$-algebra R

$$\begin{aligned} L_p^+ H(R) &= \mathrm{Mor}(\mathrm{Spec} \varprojlim W_j(R), H) \\ &= \varinjlim \mathrm{Mor}(\mathrm{Spec}\, W_j(R), H) \\ &= \varinjlim L_p^{+,j} H(R). \end{aligned}$$

Thus $L_p^+ H$ is the projective limit of the $L_p^{+,j} H$ and the truncation maps are the canonical projections. In particular we have an homeomorphism of the underlying topological spaces $L_p^+ H \cong \varprojlim L_p^{+,j} H$ by [EGA4], Cor. 8.2.10. Thus it remains to show that the transition maps are surjective. But this follows from the infinitesimal lifting property, as H is smooth. □

We note that $L_pH(k) = H(L)$, $L_p^+H(k) = H(O_L)$ and that we have a canonical isomorphism of Γ-groups

$$L_p^{+,j}H(k) \cong H_0/H_j.$$

In our future considerations we equip the quotient N_0/N_j with the structure of a variety over $\mathbb{F}_p$ via this identification. We denote

$$d_j := \dim L_p^{+,j}N$$

Now our considerations in subsection 13.1 motivate the following definition.

Definition 13.6. (1) A subset $Y \subset N_0$ is called admissible if there exists a positive integer j such that Y is the pre-image of a locally closed subset $\overline{Y} \subset N_0/N_j$.

(2) We define the numerical dimension of an admissible subset as

$$\operatorname{ndim} Y := \dim \bar{Y} - d_j$$

with $\bar{Y}$ and j as above.

Note that definition of the numerical dimension is certainly independent of the choice of j.

13.4. **Admissible and ind-admissible subsets of** $N(L)$**.** It is a straight forward idea to define admissibility of bounded subsets of $N(L)$ by checking admissibility for a $\lambda(p^i)$-conjugate which is a subset of N_0. For this definition to make sense, we have to check that any $\lambda(p^i)$-conjugate of an admissible subset of N_0 is again admissible.

Lemma 13.7. *Let $Y \subset N_0$ and i be a positive integer.*

(1) $Y(i)$ is admissible if and only if Y is admissible.
(2) If Y is admissible then $Y \cap N(i)$ is admissible.

Proof. (1) Assume first that Y is admissible. Let $j, \overline{Y}$ as in Definition 13.6 and let $\mathscr{Y} := t_j^{-1}(\overline{Y})$ where $\overline{Y}$ is regarded as sub*scheme* of $L_p^{+,j}N$. Then $\mathscr{Y}$ is a locally closed subset of L_p^+N, which we equip with the reduced subscheme structure. Since L_p^+N is a closed subfunctor of L_pN, the functor $\mathscr{Y}$ is also locally closed in L_pN. Now conjugation with $\lambda(p^i)$ defines an automorphism of L_pN, hence $\mathscr{Y}' := \lambda(p^i)\mathscr{Y}\lambda(p^{-i})$ is again a locally closed subfunctor of L_pN and thus a locally closed subscheme of L_p^+N. We note that $Y(i) = \mathscr{Y}'(\operatorname{Spec} k)$ and that $\mathscr{Y}'$ is the pre-image of a subset $\overline{Y}' \subset L_p^{+,j'}N$ for j' big enough such that $N_{j'} \subset \lambda(p^i)N_j\lambda(p^i)$ (One checks the second assertion on geometric points). Thus $\mathscr{Y}' = t_{j'}^{-1}(t_{j'}\mathscr{Y}')$, hence we have $t_{j'}(\overline{\mathscr{Y}'}) = \overline{t_{j'}(\ Yscr')}$ as $t_{j'}$ is open and surjective. So the restriction $t_{j'} : \overline{\mathscr{Y}'} \to \overline{t_{j'}(\mathscr{Y}')}$ is again open, in particular $\bar{Y}' = t_{j'}(\mathscr{Y})$ is open in its closure. Hence Y' is admissible. The other direction has the same proof.

(2) We have seen that $N(i)$ is admissible and obviously the intersection of two admissible sets is again admissible, proving the claim. □

Definition 13.8. (1) A subset $Y \subset N(L)$ is called admissible, if $Y \subset N(-k)$ for some non-negative integer k and $Y(k)$ is admissible in the sense of Definition 13.6

(2) A subset $Y \subset N(L)$ is called ind-admissible if $Y \cap N(-k)$ is admissible for all non-negative integers k.

We note that by Lemma 13.7 (1), the definition of admissibility does not depend on the choice of the integer k and by Lemma 13.7 (2) that every admissible subset is also ind-admissible.

Before we can introduce our new notion of dimension, we have to show a few auxiliary results to ensure that our definition will be well-defined. First, we recall a result of Lang and Tate which reduces the calculation of the dimension of certain schemes to the counting of points.

Definition 13.9. Let q be a p-power and let f, g be two functions defined on the set of q-powers with values in the non-negative integers. Then we write $f \sim g$ if there exist positive real constants C_1, C_2 with $C_1 f(q^n) \leq g(q^n) \leq C_2 f(q^n)$ for $n \gg 0$.

Proposition 13.10. *Let V be a scheme of finite type over $\mathbb{F}_q$. Then*

$$\# V(\mathbb{F}_{q^n}) \sim q^{n \dim V}.$$

Proof. This is an easy consequence of [LW54], Thm. 1. □

Lemma 13.11. *Let $Y \subset N(0)$ be admissible and let i be a positive integer. Then*

$$\operatorname{ndim} Y(i) = \operatorname{ndim} Y - 2\langle \rho_N, i\lambda \rangle.$$

Proof. Choose j such that $Y(i)$ (and thus Y) is N_j-stable and denote by $\bar{Y}$ (resp. $\bar{Y}(i)$) their images in N_0/N_j. Let $\bar{c}_i : N_0/N_j \to N_0/N_j$ be the morphism induced by conjugation with $\lambda(p^i)$. Then $\bar{Y}$ is the full pre-image of $\bar{Y}(i)$ w.r.t. $\bar{c}_i$ and the restriction $\bar{c}_{i|\bar{Y}} : \bar{Y} \to \bar{Y}(i)$ is surjective. Thus

$$\dim \bar{Y} = \dim \bar{Y}(i) + \dim \ker \bar{c}_i$$

Now by Proposition 13.10 we have

$$q_F^{(\dim \ker \bar{c}_j)\cdot s} \sim \prod_{\alpha \in R_N} \# \ker(W_j(\mathbb{F}_{q_F^s}) \stackrel{\cdot p^{\langle \alpha, i\lambda \rangle}}{\longrightarrow} W_j(\mathbb{F}_{q_F^s})) = \prod_{\alpha \in R_N} p^{(\min\{j, \langle \alpha, i\lambda \rangle\})\cdot s}.$$

Thus for j big enough, which we may assume (and actually is automatic),

$$\dim \ker \bar{c}_i = \sum_{\alpha \in R_N} \langle \alpha, i\lambda \rangle = 2\langle \rho_N, i\lambda \rangle.$$

□

We denote

$$d(i) = 2\langle \rho_N, i\lambda \rangle.$$

Definition 13.12. (1) Let $Y \subset N(-k)$ be admissible. We define the numerical dimension of Y as

$$\operatorname{ndim} Y := \operatorname{ndim} Y(k) + d(k).$$

(2) The numerical dimension of an ind-admissible subset $Y \subset N(L)$ is defined as

$$\operatorname{ndim} Y = \sup_{k > 0} \operatorname{ndim}(Y \cap N(-k)).$$

Corollary 13.13. *Let $Y \subset N(L)$ be admissible and $m \in M(L)$. Then mYm^{-1} is admissible and has numerical dimension* $\operatorname{ndim} Y - 2\langle \rho_N, \nu_M(m) \rangle$.

Proof. By Cartan decomposition it suffices to consider the following two cases.

If $m = \lambda'(p)$ for a coweight λ' which is dominant w.r.t. $T_{O_L} \subset B_{O_L} \subset M_{O_L}$ then the claim follows as in Lemma 13.11.

If $m \in M(O_L)$ then conjugation with m stabilizes the N_j, thus the admissibility and the numerical dimension do not change. □

13.5. **Equality of** $\dim \mathscr{F}$ **and** $\operatorname{ndim} \tilde{\mathscr{F}}$**.** By Proposition 13.10 the dimension of $\mathscr{F}$ can be determined by knowing the cardinality of $\mathscr{F}(k)^{\sigma_E^s}$ for any positive integer s. In order to relate $\dim \mathscr{F}$ to $\operatorname{ndim} \tilde{\mathscr{F}}$ (under the assumption that the latter is well-defined) we need a similar assertion for the numerical dimension. For this we need to do bit of preparatory work.

Lemma 13.14. *Let $F \subset \mathbb{Q}_{p^s}$. We have short exact sequences*

$$0 \longrightarrow N[i+1](L)^{\sigma^s} \longrightarrow N[i](L)^{\sigma^s} \longrightarrow N\langle i\rangle(L)^{\sigma^s} \longrightarrow 0$$

and

$$0 \longrightarrow N[i+1](O_L)^{\sigma^s} \longrightarrow N[i](O_L)^{\sigma^s} \longrightarrow N\langle i\rangle(O_L)^{\sigma^s} \longrightarrow 0.$$

Proof. This is an easy consequence of the description of $N[i]$ resp. $N\langle i\rangle$ as product of root groups. ☐

Lemma 13.15. *Let $F \subset \mathbb{Q}_{p^s}$. For any i, the map $N[i](L)^{\sigma^s} \to (N[i](L)/N[i](O_L))^{\sigma^s}$ is surjective.*

Proof. We prove the claim by descending induction on i. For the induction beginning choose $i \gg 0$ such that $N[i] = 0$. Then the claim is certainly true. Now assume that our assertion is true for $i+1$. Then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N[i+1](O_L)^{\sigma^s} \arrow[r] \arrow[d] & N[i](O_L)^{\sigma^s} \arrow[r] \arrow[d] & N\langle i\rangle(O_L)^{\sigma^s} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N[i+1](L)^{\sigma^s} \arrow[r, "\phi"] \arrow[d, two heads] & N[i](L)^{\sigma^s} \arrow[r, "\psi"] \arrow[d] & N\langle i\rangle(L)^{\sigma^s} \arrow[r] \arrow[d, two heads] & 0 \\
& \left(N[i+1](L)/N[i+1](O_L)\right)^{\sigma^s} \arrow[r, "\overline{\phi}"] & \left(N[i](L)/N[i](O_L)\right)^{\sigma^s} \arrow[r, "\overline{\psi}"] & \left(N\langle i\rangle(L)/N\langle i\rangle(O_L)\right)^{\sigma^s}. &
\end{tikzcd}$$

One easily checks that $\overline{\phi}$ is injective, $\overline{\psi}$ surjective and that $\operatorname{im}\overline{\phi} = \ker\overline{\psi}$ in the category of pointed sets.

We choose an element $\overline{n} \in (N[i](L)/N[i](O_L))^{\sigma^s}$. By diagram chasing we find an $n \in N(L)^{\sigma^s}$ such that n and $\overline{n}$ have the same image in $(N\langle i\rangle(L)/N\langle i\rangle(O_L))^{\sigma^s}$. Thus $n^{-1} \cdot \overline{n} \in \ker\overline{\psi}$ and we find $n' \in N[i+1](L)^{\sigma^s}$ such that n' is mapped to $n^{-1} \cdot \overline{n}$. Hence $n\phi(n') \in N[i](L)^{\sigma^s}$ is mapped to $\overline{n}$, finishing the proof.

☐

Corollary 13.16. *Let $F \subset \mathbb{Q}_{p^s}$. For any integer j, the map $N(L)^{\sigma^s} \to (N(L)/N(j))^{\sigma^s}$ is surjective.*

Proof. By conjugating with $\lambda(p)$, we see that it suffices to prove the claim for $j = 0$. Now our assertion coincides with the assertion of the previous lemma for $i = 0$. ☐

Lemma 13.17. *Let $F \subset \mathbb{Q}_{p^s}$. The canonical map $N_0^{\sigma_F} \to (N_0/N_j)^{\sigma_F}$ is surjective.*

Proof. (1) We show that $H^1_{\mathrm{cont}}(\Gamma_{\mathbb{Q}_{p^s}}, N[i]_j) = 0$ by descending induction on i. In particular for $i = 0$ we get $H^1_{\mathrm{cont}}(\Gamma_{\mathbb{Q}_{p^s}}, N_j) = 0$ and the claim of the lemma follows. For $i \gg 0$ we have $N[i]_j = 0$ and thus $H^1_{\mathrm{cont}}(\Gamma_{\mathbb{Q}_{p^s}}, N[i]) = 0$. Now the short exact sequence

$$0 \longrightarrow N[i+1]_j \longrightarrow N[i]_j \longrightarrow N\langle i\rangle_j \longrightarrow 0$$

gives the exact sequence

$$H^1_{\mathrm{cont}}(\Gamma, N[i+1]_j) \longrightarrow H^1_{\mathrm{cont}}(\Gamma, N[i]_j) \longrightarrow H^1_{\mathrm{cont}}(\Gamma, N\langle i\rangle_j).$$

Under the identification $N\langle i\rangle(L) \cong \prod L$ the group $N\langle i\rangle_j$ is identified with $\prod p^j O_L$. So N_j is isomorphic to a finite product of copies of O_L and thus has trivial cohomology. We may assume that $H^1_{\mathrm{cont}}(\Gamma_{\mathbb{Q}_{p^s}}, N[i+1]) = 0$ by induction assumption, then the sequence above implies that also $H^1_{\mathrm{cont}}(\Gamma_{\mathbb{Q}_{p^s}}, N[i]) = 0$. □

Proposition 13.18. *Let $Y \subset N(L)$ be admissible and $N(i)$-stable, assume that F is big enough such that Y is σ_F-stable. Then*

$$(Y/N(i))^{\sigma_F^s} \sim q^{(\mathrm{ndim}\, Y + d(i))\cdot s}.$$

Proof. Choose k such that $Y \subset N(-k)$. Then conjugation with $\lambda(p^k)$ induces an isomorphism of Γ_F-sets

$$Y/N(i) \cong Y(k)/N(k+i)$$

We choose j such that $N_j \subset N(k+i)$. Let $\pi^{(s)} : (Y(k)/N_j)^{\sigma_F^s} \to (Y(k)/N(k+i))^{\sigma_F^s}$ be the map induced by the canonical projection. We get a commutative diagram.

$$\begin{tikzcd} & Y(k)^{\sigma_F^s} \arrow[dl, two heads] \arrow[dr, two heads] & \\ (Y(k)/N_j)^{\sigma_F^s} \arrow[rr, "\pi^{(s)}"] & & (Y(k)/N(k+i))^{\sigma_F^s} \end{tikzcd}$$

Thus $\pi^{(s)}$ is surjective. Every of its fibres is canonically bijective to

$$N(k+i)^{\sigma_F^s}/N_j^{\sigma_F^s} \cong \prod_{\alpha \in R_N} (p^{(i+k)\cdot\langle \delta_N, \alpha\rangle} O_{F_s})/p^j O_{F_s}$$

where F_s denotes the (unique) unramified extension of F of degree s. In particular every fibre has $q^{(d_j - d(i+k))\cdot s}$ elements. Altogether,

$$\begin{aligned} \#(Y/N(k))^{\sigma_F^s} &= \#(Y(i)/N(k+i))^{\sigma_F^s} \\ &= \#(Y(k)/N_j)^{\sigma_F^s} \cdot q_F^{(d(k+i)-d_j)\cdot s} \\ &\sim q_F^{(\mathrm{ndim}\, Y(k) + d_j)\cdot s} \cdot q_F^{(d(k+i)-d_j)\cdot s} \\ &= q_F^{(\mathrm{ndim}\, Y + d(i))\cdot s}. \end{aligned}$$

□

Proposition 13.19. *Assume that $\tilde{\mathscr{F}}$ is ind-admissible. Then*

$$\dim \mathscr{F} = \mathrm{ndim}\, \tilde{\mathscr{F}}.$$

Proof. The obstacle that prevents us from applying Proposition 13.10 directly to $\mathscr{F}$ is the fact that $\mathscr{F}$ is not quasi-compact in general. Thus our method of proof is to find a filtration of $\mathscr{F}$ by quasi-compact subschemes and compare it to the filtration of $\tilde{\mathscr{F}}$ by $\tilde{\mathscr{F}} \cap N(-k)$.

Now as $\mathrm{ht}\, \rho_{|\mathbb{X}_i}$ is constant on $\mathscr{F}$ for every i, the the restriction of i_P defines an isomorphism of $\mathscr{F}$ onto its image in $\mathscr{M}_G(b,\mu)$. Thus by [RZ96], Cor. 2.31 such a filtration is given by

$$\mathscr{F}_{-k} := \{(X,\rho) \in \mathscr{F} \mid p^k\rho \text{ and } p^k\rho^{-1} \text{ are isogenies}\}.$$

Its pre-image in $N(L)$ is

$$\tilde{\mathscr{F}}_{-k} := \{n \in N(L); p^k\Lambda_0 \subset n\Lambda_0 \subset p^{-k}\Lambda_0\}.$$

Now by Proposition 13.10 and 13.18 we have

$$\dim \mathscr{F} = \sup \dim \mathscr{F}_{-k} = \sup \operatorname{ndim} \tilde{\mathscr{F}}_{-k}.$$

It remains to show that the filtrations $(\tilde{\mathscr{F}}_{-k})_{k\in\mathbb{N}}$ and $(\tilde{\mathscr{F}} \cap N(-k))_{k\in\mathbb{N}}$ are refinements of each other. Indeed, by Bruhat-Tits theory the sets $\tilde{\mathscr{F}}_{-k}$ are bounded and any bounded subset of $\tilde{\mathscr{F}}$ is contained in one of the sets $\tilde{\mathscr{F}}_{-k}$. □

14. Calculation of the fibre dimension

Let $K := G(O_L)$ and $K_M := M(O_L)$. Analogous to [GHKR06] we define the function f_{m_1,m_2} for $m_1, m_2 \in M(L)$ by

$$f_{m_1,m_2} : N(L) \to N(L), n \mapsto m_1 n^{-1} m_1^{-1} \cdot m_2 \sigma(n) m_2^{-1}.$$

Then we have

$$\begin{aligned} \tilde{\mathscr{F}} &= \{n \in N(L) \mid n^{-1} b\sigma(n) \in K\mu(p)K\} \\ &= \{n \in N(L) \mid n^{-1} b\sigma(n) b^{-1} \in K\mu(p)Kb^{-1} \cap N(L)\} \\ &= f_{1,b}^{-1}(K\mu(p)Kb^{-1} \cap N(L)). \end{aligned}$$

Hence we divide the computation of $\operatorname{ndim} \tilde{\mathscr{F}}$ into two steps:

- We have to show that $K\mu(p)Kb^{-1} \cap N(L)$ is admissible and compute its dimension.
- We have to calculate the difference $\operatorname{ndim} f_{m_1,m_2}^{-1} Y - \operatorname{ndim} Y$ for admissible $Y \subset N(L)$.

The maps f_{m_1,m_2} are defined in greater generality than the functions we actually need, but this will turn out to be an advantage in the second step.

14.1. Admissibility and dimension of $K\mu(p)Kb^{-1} \cap N(L)$.

We note that we have two notions of dominant elements in $X_*(T)$, one coming from the Killing pair $T \subset B$ in G and one coming from $T \subset B \cap M$ in M. Let $X_*(T)_{\mathrm{dom}}$ resp. $X_*(T)_{M-\mathrm{dom}}$ denote the set of cocharacters that are dominant in G resp. M.

For $\mu_M \in X_*(T), \mu \in X_*(T)_{\mathrm{dom}}$, we denote

$$C(\mu, \mu_M) := (N(L)\mu_M(p)K \cap K\mu(p)K)/K$$

considered as Γ_F-set. In order to calculate the numerical dimension of $K\mu(p)Kb^{-1} \cap N(L)$, we first study the sets $C(\mu, \mu_M)$.

We denote for μ, μ_M as above and $\kappa \in \pi_1(M)_\Gamma$

$$\begin{aligned} S_M(\mu) &:= \{\mu_M \in X_*(T)_{M-\mathrm{dom}} \mid C(\mu, \mu_M) \neq \emptyset\} \\ S_M(\mu, \kappa) &:= \{\mu_M \in S_M(\mu, \kappa) \mid \kappa_M(\mu_M) = \kappa\}. \end{aligned}$$

We will compare these sets to

$$\begin{aligned} \Sigma(\mu) &:= \{\mu' \in X_*(T) \mid \mu'_{\mathrm{dom}} \leq \mu\} \\ \Sigma(\mu)_{M-\mathrm{dom}} &:= \{\mu' \in \Sigma(\mu) \mid \mu' \in X_*(T)_{M-\mathrm{dom}}\} \\ \Sigma(\mu)_{M-\max} &:= \{\mu' \in \Sigma(\mu)_{M-\mathrm{dom}} \mid \mu' \text{ is maximal w.r.t. the Bruhat-order of } M\}. \end{aligned}$$

Lemma 14.1. *There are inclusions*

$$\Sigma(\mu)_{M-\max} \subset S_M(\mu) \subset \Sigma(\mu)_{M-\mathrm{dom}}.$$

Proof. This is the p-adic analogue of Lemma 5.4.1 in [GHKR06]. The proof is literally the same. □

Remark 14.2. Assume that we are in the situation of Prop. 12.5. Then

$$I_{M,\mu,b} = \{\mu_M \in \Sigma(\mu)_{M-\mathrm{dom}} \mid \kappa_M(\mu_M) = \kappa_M(b)\} = \{\mu \in \Sigma(\mu)_{M-\max} \mid \kappa_M(\mu_M) = \kappa_M(b)\},$$

where the first equality follows from the fact that b is superbasic in $M(L)$ and the second equality holds because μ is minuscule. Now the above lemma implies that $I_{M,\mu,b} = S_M(\mu, \kappa_M(b))$.

Proposition 14.3. *(1) Let μ be a dominant coweight. Then for every $\mu_M \in S(\mu)$ there exists an integer $d(\mu, \mu_M)$ such that*

$$C(\mu, \mu_M)^{\sigma_F^s} \sim q_F^{d(\mu,\mu_M)\cdot s}.$$

(2) We have an inequality

$$d(\mu, \mu_M) \leq \langle \rho, \mu + \mu_M \rangle - 2\langle \rho_M, \mu_M \rangle. \tag{14.1}$$

(3) If $\mu_M \in \Sigma(\mu)_{M-\max}$, then the inequality (14.1) is an equality.

Proof. The function field analogue of (1) and (2) are proven in [GHKR06], Prop. 5.4.2. Its proof determines the number of points

$$\# \left(N(\mathbb{F}_q(\!(t)\!))\mu_M(t)G(\mathbb{F}_q[\![t]\!]) \cap G(\mathbb{F}_q[\![t]\!])\mu(t)G(\mathbb{F}_q[\![t]\!]) \big/ G(\mathbb{F}_q[\![t]\!]) \right),$$

which still works in the p-adic case. Thus (1) and (2) follow and (3) is the analogue of Corollary 5.4.4 in [GHKR06]. □

Proposition 14.4. *(1) The set $K\mu(p)Kb \cap N(L)$ is admissible.*
(2) Let $b \in K_M\mu_M(p)K_M$. Then

$$\mathrm{ndim}\, K\mu(p)Kb \cap N(L) = d(\mu, \mu_M) - 2\langle \rho_N, \nu_M(b) \rangle$$

Proof. The set $K\mu(p)Kb \cap N(L)$ is bounded by Lemma 13.3. Choose k such that it is contained in $N(-k)$. We denote $Y' := \lambda(p^k)(K\mu(p)Kb \cap N(L))\lambda(p^{-k})$. Let j be big enough such that $N_j \subset \lambda(p^k)b^{-1}N_0 b\lambda(p^{-k})$. Then Y' is right-N_j-stable.

For every $\overline{\mathbb{F}}_p$-algebra R and $g \in L_pG(R)$ the subset

$$\{s \in \operatorname{Spec} R \mid g_{\overline{k(s)}} \in L_p^+(\overline{k(s)})\mu(p)L_p^+(\overline{k(s)})\}$$

is locally closed in $\operatorname{Spec} R$ (cf. [CKV] Lemma 2.1.6). Here $k(s)$ denotes the fraction field at s. Thus the set of all $s \in L_p^+N$ whose geometric points are an element of

$$\lambda(p^k)(L_p^+(\overline{k(s)})\mu(p)L_p^+(\overline{k(s)})b \cap L_pN(\overline{k(s)}))\lambda(p^{-k})$$

form a locally closed subset $\mathscr{Y}'$ of L_p^+N with $\mathscr{Y}'(k) = Y'$. Furthermore, $\mathscr{Y}'$ is the pre-image of some subset of $L_p^{+,j}N$ w.r.t. t_j. As we have seen in the proof of Lemma 13.7, this implies that Y' is admissible. Thus $K\mu(p)Kb \cap N(L)$ is admissible.

Now the map $x \mapsto xb^{-1}$ induces an isomorphism of Γ_F-sets

$$N(L)bK \cap K\mu_M(p)K/_K \xrightarrow{\sim} N(L) \cap K\mu(p)Kb^{-1} \big/ bN(0)b^{-1}.$$

Now choose $k_M \in K_M$ such that $bK_M = k_M\mu_M(p)K_M$. Then multiplication by k_M defines an isomorphism of Γ_F-sets

$$N(L)\mu_M(p)K \cap K\mu_M(p)K/_K \xrightarrow{\sim} N(L)bK \cap K\mu_M(p)K/_K.$$

Altogether, we get

$$\begin{aligned} \mathrm{ndim}\, K\mu Kb \cap N(L) &= d(\mu, \mu_M) + \mathrm{ndim}\, bN(0)b^{-1} \\ &= d(\mu, \mu_M) - \langle 2\rho_N, \nu_M(b) \rangle. \end{aligned}$$

The last equality is an easy consequence of Corollary 13.13. □

14.2. **Relative dimension of certain morphisms $f : L^n \to L^n$.** Before we can continue with the second step of our proof, we need to explain how the analogue of section three and four of [GHKR06] works in the p-adic case.

Let V be a finite dimensional vector space over L and $\Lambda_2 \subset \Lambda_1$ be two lattices in V. We define the structure of a variety on Λ_1/Λ_2 as follows. By the elementary divisor theorem, we find a basis $v_1, \dots, v_n$ of Λ_1 such that Λ_2 has a basis of the form $p^{\alpha_1} v_1, \dots, p^{\alpha_n} v_n$. This induces an isomorphism

$$\Lambda_1/\Lambda_2 \xrightarrow{\sim} \prod W_{\alpha_i}(k), \quad \sum \beta_i v_i \mod \Lambda_2 \mapsto (\beta_i \mod p^{\alpha_i})_i.$$

As W_{α_i} is represented by the scheme $\mathbb{A}^{\alpha_i}$, this defines the structure of an affine space on Λ_1/Λ_2. The variety structure does not depend on the choice of $v_1, \dots, v_n$: Let $w_1, \dots, w_n$ another basis as above. Define ϕ such that the diagram

$$\begin{tikzcd} \Lambda_1/\Lambda_2 \arrow[r, "\mathrm{id}"] \arrow[d, "{\sum \beta_i v_i \mod \Lambda_2 \mapsto \atop \beta_i \mod p^{\alpha_i}}"'] & \Lambda_1/\Lambda_2 \arrow[d, "{\sum \beta_i w_i \mod \Lambda_2 \mapsto \atop \beta_i \mod p^{\alpha_i}}"] \\ \prod W_{\alpha_i}(k) \arrow[r, "\phi"] & \prod W_{\alpha_i}(k) \end{tikzcd}$$

commutes. Now ϕ is $W(k)$-linear and hence can be expressed as family of polynomials in the coordinates of the truncated Witt vectors. So ϕ is a morphism of varieties. The same argument shows that ϕ^{-1} is also a morphism of varieties, thus ϕ is an isomorphism and the structure of an affine space on Λ_1/Λ_2 given by the bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are the same.

Now one can can define admissible resp. ind-admissible subsets of V and their dimension literally as in [GHKR06]. Also, the p-adic analogue of the statements and proofs of section 4 in [GHKR06] hold. Thereof we will need the following notations and results.

Definition 14.5. Let (V, Φ) be an isocrystal. We define

$$d(V, \Phi) = \sum_{\lambda < 0} \lambda \dim V_\lambda,$$

where V_λ is the isoclinic component of (V, Φ) of slope λ.

Proposition 14.6. *Let V, V' be two finite dimensional L-vector spaces of the same dimension. Let $\phi : V \to V'$ be an L-linear isomorphism and $\psi : V \to V'$ be a σ-linear bijection. We define $f : V \to V'$ by $f := \psi - \phi$. Then for any lattice Λ in V there exists a lattice Λ' in V' and a non-negative integer j such that*

$$p^j \Lambda' \subset f\Lambda \subset \Lambda'.$$

For any such triple j, Λ, Λ' and $l \geq j$, consider the induced morphism

$$\overline{f} : \Lambda/p^l\Lambda \to \Lambda'/p^l\Lambda'.$$

Then

(1) $\operatorname{im} \overline{f} \supset p^j \Lambda'/p^l \Lambda'$.
(2) $\dim \ker \overline{f} = d(V, \phi^{-1}\psi) + [\Lambda : \phi^{-1}\Lambda']$.
(3) $(\ker \overline{f})^0 \subset p^{l-j}\Lambda/p^l\Lambda$

Proof. The proposition is the p-adic analogue of Prop. 4.2.2 in [GHKR06]. Its proof is literally the same. □

14.3. **Dimension of the pre-image under $f^{-1}_{m_1,m_2}$.**

Proposition 14.7. *The map f_{m_1,m_2} is surjective. Moreover, for any admissible subset $Y \subset N(L)$ the inverse image is ind-admissible and*

$$\operatorname{ndim} f^{-1}_{m_1,m_2} Y - \operatorname{ndim} Y = d(\mathfrak{n}(L), \operatorname{Ad}_{\mathfrak{n}}(m_1)^{-1} \operatorname{Ad}_{\mathfrak{n}}(m_2\sigma)) + \operatorname{val} \det \operatorname{Ad}_{\mathfrak{n}}(m_1). \tag{14.2}$$

We denote the right hand side of (14.2) by $d(m_1, m_2)$ for convenience.

Proof. This is the analogue of Prop. 5.3.2 of [GHKR06]. Its proof is almost literally the same. As this is the part where the main part of the calculation of $\dim \mathscr{F}$ is done, we give a brief outline of the proof and explain why the arguments carry over.

By multiplying m_1 and m_2 by a suitable power of $\lambda(p)$ we may assume that the N_j are stable under conjugation with m_1 and m_2, in particular f maps N_j into N_j. One easily checks by replacing m_1 and m_2 by their $\lambda(p^k)$-multiple increases both sides of (14.2) by $d(k)$, thus the assertion of the proposition does not change.

Now we consider the maps $f\langle i\rangle : N\langle i\rangle \to N\langle i\rangle$ induced by f. By choosing an isomorphism of the root subgroups with their Lie algebra, we identify $N\langle i\rangle \cong \mathrm{Lie}\, N\langle i\rangle$. Under this identification $f\langle i\rangle$ is identified with $\mathrm{Ad}_{\mathrm{Lie}\, N\langle i\rangle}(m_2)\sigma - \mathrm{Ad}_{\mathrm{Lie}\, N\langle i\rangle}(m_1)$. Thus we are in the situation considered in the previous subsection.

The following two claims are obtained from Proposition 14.6 (resp. Prop. 4.2.2 in the proof in [GHKR06]) using purely group theoretic arguments. Therefore the proofs given in [GHKR06] also work in the p-adic case.

Claim 1. There exists an integer k such that for any $i \geq 1$

$$\lambda(p^k)N[i]_0\lambda(p^{-k}) \subset f(N[i]_0).$$

Claim 2. Choose positive integers j, k, l such that $f\langle i\rangle N\langle i\rangle_0 \supset p^j N\langle i\rangle$ for any i, k as above and $N[i]_{l-j} \subset \lambda(p^k)N[i]_0\lambda(p^{-k})$. We denote by $H = N_0/N_l$ and by $\overline{f} : H \to H$ the morphism induced by f. Then

$$\dim \overline{f}^{-1}(1) = d(m_1, m_2).$$

We will give the rest of the proof in greater detail, as this is the part where we have to work with the notion of admissibility, which is slightly different from the one in [GHKR06]. However the concept is still the same as in their proof.

We denote by f_0 the restriction of f to N_0.

Claim 3. Assume that Y is an admissible subset of $N(k)$ with k as in claim 2 (ensuring that Y is contained in the image of f_0). Then $f_0^{-1}Y$ is admissible and

$$\mathrm{ndim}\, f_0^{-1}Y - \mathrm{ndim}\, Y = d(m_1, m_2).$$

To prove claim 3, we choose $l \gg 0$ such that Claim 2 holds and such Y is the pre-image of a locally closed subset $\overline{Y}$ in $H = N_0/N_l$. Then all non-empty (reduced) fibres of $\overline{f}$ are isomorphic to each other. Indeed, if $n \in \mathrm{im}(f)$ and n_0 is a pre-image of n then

$$\overline{f}^{-1}(n) = \overline{f}^{-1}(1)n_0.$$

Now $\overline{Y}$ is contained in the image of $\overline{f}$ by Claim 1 and hence every fibre of $\overline{f}$ has dimension $d(m_1, m_2)$ by claim 2, so

$$\dim f_0^{-1}\overline{Y} - \dim \overline{Y} = d(m_1, m_2). \tag{14.3}$$

As $f_0^{-1}Y = t_l^{-1}(\overline{f}^{-1}\overline{Y})(k)$, we see that it is admissible that the equation (14.3) implies

$$\mathrm{ndim}\, f_0^{-1}Y - \mathrm{ndim}\, Y = d(m_1, m_2)$$

This finishes the proof of the claim.

Now let $Y \subset N(L)$ be admissible and j big enough such that $Y \subset N(-j)$. Now f_{m_1,m_2} and conjugation with (a power of) $\lambda(p)$ commute. Hence

$$f_{m_1,m_2}^{-1}(Y) \cap N(-j-k) = (f_0^{-1}(Y(j+k)))(-j-k),$$

where k is chosen as above. Hence $f^{-1}_{m_1,m_2}(Y) \cap N(-j-k)$ is admissible by Claim 3 and has numerical dimension $\operatorname{ndim} Y + d(m_1, m_2)$, proving the ind-admissibility of $f^{-1}_{m_1,m_2}(Y)$ and the dimension formula (14.2).

For the surjectivity of f, note that by Claim 1 there exists an integer k such that $N(k)$ is contained in the image of f. As f commutes with conjugation with $\lambda(p)$ this implies that $N(j)$ is contained in the image of f_{m_1,m_2} for every j. As the $N(j)$ exhaust $N(L)$, the assertion follows. □

Proof of Proposition 12.5. Altogether, we get

$$\begin{aligned}
\dim \mathscr{F} &\overset{\text{Prop. 13.19}}{=} \operatorname{ndim} \tilde{\mathscr{F}} \\
&\overset{\text{Prop. 14.7}}{=} \operatorname{ndim}(K\mu(p)Kb^{-1} \cap N(L)) + \langle \rho, \nu_M(b) - \nu_M(b)_{\mathrm{dom}} \rangle \\
&\overset{\text{Prop. 14.4}}{=} d(\mu, \mu_M) - 2\langle \rho_N, \nu_M(b) \rangle + \langle \rho, \nu_M(b) - \nu_M(b)_{\mathrm{dom}} \rangle \\
&\overset{\text{Prop. 14.3}}{=} \langle \rho, \mu + \mu_M \rangle - 2\langle \rho_M, \mu_M \rangle - 2\langle \rho_N, \nu_M(b) \rangle + \langle \rho, \nu_M(b) - \nu_M(b)_{\mathrm{dom}} \rangle \\
&= \langle \rho, \mu - \nu_M(b)_{\mathrm{dom}} \rangle - \langle \rho_M, \mu_M \rangle \\
&\quad + \underbrace{\langle \rho_N, \mu_M \rangle - \langle \rho_N, \nu_M(b) \rangle}_{=0} + \underbrace{\langle \rho, \nu_M(b) \rangle - \langle \rho_N, \nu_M(b) \rangle}_{=0} \\
&= \langle \rho, \mu - \nu_G(b) \rangle - \langle \rho_M, \mu_M \rangle.
\end{aligned}$$

□

15. Reduction to the dimension formula for superbasic Rapoport-Zink spaces of EL-type

We first reduce to the case of a superbasic Rapoport-Zink datum which still might be of PEL type.

Proposition 15.1. *Assume Proposition 11.1 is true for every simple superbasic Rapoport-Zink datum. Then it is also true in general.*

Proof. We use the notation of Prop. 12.5 and choose b and M such that $b \in M(L)$ is superbasic. If Proposition 11.1 is true in the superbasic case, we have

$$\dim \mathscr{M}_{M,b,\mu_M} \leq \langle \rho_M, \mu_M \rangle - \frac{1}{2} \operatorname{def}_M(b).$$

By Proposition 12.5 we get for $\mu_M \in I_{\mu,b,M}$

$$\dim p_M^{-1} \mathscr{M}_{M,b,\mu_M} \quad \leq \quad \langle \rho, \mu - \nu_G(b) \rangle - \frac{1}{2} \operatorname{def}_M(b).$$

Since M is a Levi subgroup of M_b, the group $J_{M,b}$ is a Levi subgroup of $J_{G,b}$. As the $\mathbb{Q}_p$-rank of a linear algebraic group is the same as the $\mathbb{Q}_p$-rank of its Levi subgroups, this implies $\operatorname{def}_G(b) = \operatorname{def}_M(b)$. Altogether,

$$\dim \mathscr{M}_G(b, \mu) = \dim \mathscr{M}_P(b, \mu) = \max \dim p_M^{-1} \mathscr{M}_{M,b,\mu_M} \leq \langle \rho, \mu - \nu_G(b) \rangle.$$

□

In the superbasic case we can (and will) prove Theorem 1.2 directly, i.e. without the detour via Proposition 11.1.

Lemma 15.2. *Assume that Theorem 1.2 holds for every superbasic simple Rapoport-Zink datum of EL type. Then it is also true for every for every superbasic Rapoport-Zink datum of PEL type.*

Proof. Let $\hat{\mathscr{D}}$ be a superbasic unramified RZ-datum of PEL type. Then by [CKV] Lemma 3.1.1 the adjoint group G^{ad} is isomorphic to a product of $\mathrm{Res}_{F_i/\mathbb{Q}_p} \mathrm{PGL}_{h_i}$. This leaves two cases:

1. $G \cong \mathrm{GSp}_{F,2}$.
2. $G \cong \mathrm{GU}_{F,2}$.

Let $\hat{\mathscr{D}}'$ be the Rapoport-Zink datum of EL type one gets by forgetting the polarisation and G' the associated linear algebraic group. We get a canonical closed embedding

$$\mathscr{M}_G(b,\mu) \hookrightarrow \mathscr{M}_{G'}(b,\mu).$$

In the first case this is an isomorphism since $\mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GSp}_2 = \mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}_2$, so Theorem 1.2 is true for $\mathscr{D}$.

Now assume that $G \cong \mathrm{GU}_{F,2}$. We first show that $\hat{\mathscr{D}}'$ is also superbasic. Using the explicit description of $X_*(T)$ given after Proposition 7.13, we see that the Newton point of the σ-conjugacy class associated to $\hat{\mathscr{D}}'$ is of the form $(\alpha, 1-\alpha)$. This is central in $\mathrm{GU}_{F,2}$ if and only $\alpha = \frac{1}{2}$, thus $\hat{\mathscr{D}}'$ is also superbasic. Now each connected component of $\mathscr{M}_G(b,\mu)$ is isomorphic to a closed subset of a connected component of $\mathscr{M}_{G'}(b,\mu)$ and thus projective.

By [CKV] Thm. 1.3, all connected components of a Rapoport-Zink space are isomorphic, so we can determine their dimension by counting points of some connected component. For any reductive group H over $\mathbb{Z}_p$, $b' \in H(E)$, $\mu' \in X_*(H_{O_L})$ let

$$X_H(b',\mu') := \{h \cdot H(O_L) \in H(L)\big/H(O_L) \mid hb'\sigma(h)^{-1} \in H(O_L)p^{\mu'}H(O_L),\}$$

denote the affine Deligne-Lusztig set. Let $\eta_H : H(L) \to \pi_1(H)$ denote the unique $H(O_L)$-bi-invariant map which maps $\mu_H(p)$ to the image of μ_H in $\pi_1(H)$ for every dominant cocharacter μ_H. We denote for $\omega' \in \pi_1(H)$

$$X_H(b',\mu')^{\omega'} := X_H(b',\mu') \cap \eta_H^{-1}(\{\omega'\}).$$

By [CKV] Thm. 1.1 the connected components of $\mathscr{M}_G(b,\mu)(k)$ are precisely the subspaces $X_G(b',\mu)^{\omega}$ which are non-empty.

As $\mathrm{GL}_{F',2}$ and $\mathrm{GU}_{F,2}$ are not isomorphic, we compare their affine Deligne-Lusztig sets via their (isomorphic) adjoint groups. For $\omega \in \pi_1(G)$ with $X_G(b,\mu)_\omega \neq \emptyset$ we have

$$X_G(b,\mu)_\omega \cong X_{G^{\mathrm{ad}}}(b^{\mathrm{ad}},\mu^{\mathrm{ad}})_{\omega^{\mathrm{ad}}}$$

as Γ_E-sets, where $b^{\mathrm{ad}}, \mu^{\mathrm{ad}}, \omega^{\mathrm{ad}}$ denote the images of b,μ,ω in $G^{\mathrm{ad}}(L), X_*(G^{\mathrm{ad}}), \pi_1(G^{\mathrm{ad}})$ respectively. Using the same argument for $G'' = \mathrm{GL}_{F',2}$ and suitable b'',μ'',ω'' we get

$$X_G(b,\mu)_\omega \cong X_{G''}(b'',\mu'')_{\omega''}$$

as Γ_E-sets and thus $\dim \mathscr{M}_G(b,\mu) = \dim \mathscr{M}_{G''}(b'',\mu'')$ by Proposition 13.10. As the value of the right hand side of (1.2) only depends on the images in $G^{\mathrm{ad}}(L)$ resp. $X_*(T^{\mathrm{ad}})^{\Gamma}_{\mathbb{Q}}$ this proves Theorem 1.2 for $\hat{\mathscr{D}}$. □

So Theorem 1.2 is reduced to the claim that it holds in the case of a simple superbasic Rapoport-Zink datum of EL type. This is proved in the next section, see Corollary 16.8 and Proposition 16.9.

16. The dimension of superbasic Rapoport-Zink spaces of EL-type

16.1. **Notation and conventions.** From now on we restrict to the EL-case with $[b]$ superbasic. Fixing a basis e_i of Λ_0 (as $O_F \otimes O_L$-module), we get an identification $G = \mathrm{Res}_{O_F/\mathbb{Z}_p} \mathrm{GL}_h$. We proceed as in the case of affine Deligne Lusztig varieties (cf. section 4 for details).

Let d be the degree of the unramified field extension $F/\mathbb{Q}_p$, then $I := \mathrm{Gal}(F/\mathbb{Q}_p) \cong \mathrm{Gal}(k_F/\mathbb{F}_p) \cong \mathbb{Z}/d{\cdot}\mathbb{Z}$. We choose the isomorphism such that the Frobenius σ is mapped to 1. Let $T \subset B \subset G$

where T is the diagonal (maximal) torus and B is the Borel subgroup of lower triangular matrices in G.

We fix a superbasic element $b \in G(L)$ with Newton point $\nu \in X_*(T)_{\mathbb{Q}}^{\Gamma}$ and a dominant cocharacter $\mu \in X_*(T)$ such that $b \in B(G,\mu)$. We have to show that

$$\dim \mathscr{M}_G(b,\mu) = \langle \rho, \mu - \nu\rangle - \frac{1}{2}\operatorname{def}_G(b).$$

By Corollary 2.12 this is equivalent to

$$\dim \mathscr{M}_G(b,\mu) = \sum_{i=1}^{h-1} \lfloor \langle \underline{\omega}_i, \mu - \nu \rangle \rfloor. \tag{16.1}$$

We denote by $(N,F) \cong ((L \otimes_{\mathbb{Q}_p} F)^h, b\sigma)$ the $\mathscr{B}$-isocrystal associated to our Rapoport-Zink datum. We decompose $N = \prod_{\tau \in I} N_\tau$ according to the F-action. Denote by $e_{\tau,i}$ the image of e_i in N_τ, then $\{e_{\tau,i}\}_{i=1}^h$ is a basis of the N_τ and $\varsigma(e_{\tau,i}) = e_{\varsigma\tau,i}$ for all $\varsigma \in I$. For $\tau \in I, l \in \mathbb{Z}, i = 1,\dots,h$ denote $e_{\tau,i+l\cdot h} := p^l \cdot e_{\tau,i}$. Then each $v \in N_\tau$ can be written uniquely as infinite sum

$$v = \sum_{n \gg -\infty} [a_n] \cdot e_{\tau,n}$$

with $a_n \in k$.

Using Dieudonné theory, we get an identification

$$\mathscr{M}_G(b,\mu)(k) = \{(M_\tau \subset N_\tau \text{ lattice})_{\tau\in I} \mid \operatorname{inv}(M_\tau, b\sigma(M_{\tau-1})) = \mu_\tau\}.$$

Definition 16.1. (1) We call a tuple of lattices $(M_\tau \subset N_\tau)_{\tau \in I}$ a G-lattice.
(2) We define the volume of a G-lattice $M = gM^0$ to be the tuple

$$\operatorname{vol}(M) = (\operatorname{val}\det g_\tau)_{\tau\in I}.$$

Similarly, we define the volume of M_τ to be $\operatorname{val}\det g_\tau$. We call M special if $\operatorname{vol}(M) = (0)_{\tau \in I}$.

As $[b]$ is superbasic, ν is of the form $(\frac{m}{d\cdot h}, \frac{m}{d\cdot h}, \dots, \frac{m}{d\cdot h})$ with $(m,h) = 1$. Now the condition $[b] \in B(G,\mu)$ translates to $\sum_{\tau\in I, i=1,\dots h} \mu_{\tau,i} = m$. Replacing b by a σ-conjugate if necessary, we assume that b is the form $b(e_{\tau,i}) = e_{\tau,i+m_\tau}$ where $m_\tau = \sum_{i=1}^h \mu_{\tau,i}$.

In the cases $\nu = (0)$ and $\nu = (1)$ the moduli space $\mathscr{M}_G(b,\mu)$ is isomorphic to $\operatorname{End}_{\mathbb{Q}}(\underline{\mathbb{X}})/\operatorname{End}(\underline{\mathbb{X}}) \cong \mathbb{Z}$, considered as discrete union of points ([CKV] Thm. 1.1). In this case we have $\mu = \nu$ thus the right hand side of (16.1) is also zero and Theorem 1.2 holds. We assume $\nu \neq (0),(1)$ from now on.

Then the connected components of $\mathscr{M}_G(b,\mu)$ are

$$\mathscr{M}_G(b,\mu)^i = \{M \in \mathscr{M}_G(b,\mu)(k); \operatorname{vol} M = (i)_{\tau\in I}\}.$$

where i ranges over the integers (use [CKV] Thm. 1.1). In particular, we have

$$\dim \mathscr{M}_G(b,\mu) = \dim \mathscr{M}_G(b,\mu)^0.$$

16.2. **A decomposition of $\mathscr{M}_G(b,\mu)$.** In order to calculate the dimension of $\mathscr{M}_G(b,\mu)$, we decompose $\mathscr{M}_G(b,\mu)^0$ into locally closed sets, whose dimension is given by a purely combinatorial formula. Let

$$\begin{array}{rcl} \mathscr{I}_\tau : N_\tau \setminus \{0\} & \to & \mathbb{Z} \\ \sum_{n\gg-\infty} [a_n]\cdot e_{\tau,n} & \mapsto & \min\{n \in \mathbb{Z}; a_n \neq 0\}. \end{array}$$

Note that $\mathscr{I}_\tau$ satisfies the strong triangle inequality for every τ. We denote $N_{hom} := \coprod_{\tau\in I}(N_\tau \setminus \{0\})$, analogously M_{hom}. For $M \in \mathscr{M}_G(b,\mu)^0(k)$, we define

$$A(M) := \mathscr{I}(M_{hom})$$

where $\mathscr{I} = \sqcup \mathscr{I}_\tau : N_\tau \to \coprod_{\tau\in I}\mathbb{Z}$. For a subset A of $\coprod_{\tau\in I}\mathbb{Z}$ we denote $\mathscr{S}_A$ the subset of all G-lattices in $\mathscr{M}_G(b,\mu)^0$ whose image under $\mathscr{I}$ equals A.

Remark 16.2. This decomposition is the same as the decomposition according to the invariant $(A(M),\varphi(M))$, where $\varphi(M)$ is defined as in section 4.2 (replacing t by p). As the proof of Lemma 4.3 also works in the unequal characteristic case, we would obtain that $(A(M),\varphi(M))$ is an extended EL-chart for μ and thus cyclic by Corollary 5.17. In other words, $\varphi(M)$ is already uniquely determined by $A(M)$.

As we work with EL-charts rather than with extended EL-charts, we may simplify the notions introduced in section 5 as follows.

Definition 16.3. (1) An EL-chart A is called small if $f(A)\subset A+h$, i.e. if all coordinates of the type of A are either 0 or 1.

(2) The Hodge-point of a small EL-chart A is defined as μ'_{dom} where μ' denotes the type of A. We also say that A is a small EL-chart for μ'_{dom}

(3) Let A and μ' be as above, $B := A\setminus(A+h) = \{b_i \mid 0\le i< d\cdot h\}$ as in section 5.2. Let

$$\begin{aligned} B^+ &:= \{b_i\in B\mid \mu'_{i+1}=0\} = \{b\in B\mid f(b)\in B\}\\ B^- &:= \{b_i\in B\mid \mu'_{i+1}=1\} = \{b\in B\mid f(b)-h\in B\}. \end{aligned}$$

We define

$$\mathscr{V}_A := \{(j,i)\in(\mathbb{Z}/dh\mathbb{Z})^2 \mid b_j\in B^-, b_i\in B^+, b_j<b_i\}.$$

Now the combinatorics introduced in section 5 may be applied to small EL-charts because of the following lemma.

Lemma 16.4. *The map*

$$\begin{aligned} \{\text{extended EL-charts for } \mu\} &\longrightarrow \{\text{small EL-chart for } \mu\}\\ (A,\varphi) &\longmapsto A \end{aligned}$$

is well-defined and a bijection. Moreover, we have

$$\mathscr{V}(A,\varphi) = \{(b_j,b_i)\mid (j,i)\in\mathscr{V}_A\}.$$

In particular, $\#\mathscr{V}_{(A,\varphi)} = \#\mathscr{V}_A$.

Proof. The first assertion follows from Corollary 5.17 and the second assertion is a direct consequence of the definitions using that (A,φ) is cyclic. □

Proposition 16.5. *Let $M\in\mathscr{M}_G(b,\mu)^0(k)$. Then $A=A(M)$ is a normalized small EL-chart with Hodge-point μ.*

Proof. $A(M)$ is stable under f and addition with h since

$$\begin{aligned} \mathscr{I}(Fv) &= f(\mathscr{I}(v))\\ \mathscr{I}(p\cdot v) &= \mathscr{I}(v)+h. \end{aligned}$$

Furthermore, we have

$$A(M)+h = \mathscr{I}(pM)\subset \mathscr{I}(FM) = f(A(M)).$$

The fact that $A(M)$ is bounded from below is obvious. Let $M=gM^0$. As $\{g(e_{0,i})\mid 1\le i\le h\}$ is a basis of M_0 we have

$$B_{(0)} = \{\mathscr{I}(g(e_{0,i}\mid 1\le i\le h\},$$

hence

$$0 = \operatorname{val}\det g_0 = \sum_{i=1}^h \left\lfloor \frac{\mathscr{I}(g(e_{0,i}))}{h}\right\rfloor = \sum_{b_{(0)}\in B_{(0)}} \left\lfloor\frac{b}{h}\right\rfloor = h\cdot(\sum_{b_{(0)}\in B_{(0)}} b - \frac{h(h-1)}{2})$$

and thus $A(M)$ is indeed a normalized small EL-chart. We have for every $\tau \in I$

$$\#\{i \mid \mu_{\tau,i} = 1\} = \dim_{k_0} M_\tau \big/ b\sigma(M)_\tau = \#B^-_{(\tau-1)},$$

thus the Hodge point of $A(M)$ is μ. □

Corollary 16.6. *The $\mathscr{S}_A$ define a decomposition of $\mathscr{M}_G(b,\mu)^0$ into finitely many locally closed subsets. In particular,* $\dim \mathscr{M}_G(b,\mu)^0 = \max_A \dim \mathscr{S}_A$.

Proof. By Proposition 16.5, $\mathscr{M}_G(b,\mu)$ is the (disjoint) union of the $\mathscr{S}_A$ with A being a small EL-chart with Hodge-point μ. By Corollary 5.18 this union is finite. It remains to show that $\mathscr{S}_A$ is locally closed. One shows that the condition $A(M)_{(\tau)} = A_{(\tau)}$ is locally closed analogously to the proof of Prop. 5.1 in [Vie08a]. Then $\mathscr{S}_A$ is locally closed as it is the intersection of finitely many locally closed subsets. □

Proposition 16.7. *Let A be an EL-chart with Hodge-point μ. Then $\mathscr{S}_A \cong \mathbb{A}^{\mathscr{V}_A}$.*

Proof. This proposition is proven for the case $G = \mathrm{GL}_h$ in [Vie08a], §5. The construction of a morphism $f : \mathbb{A}^{\mathscr{V}_A} \to \mathscr{S}_A$ is very similar to that in [Vie08a] and the proof that it is well-defined and an isomorphism is the same. Therefore we only explain the construction of f.

We denote $R = k[t_{j,i} \mid (j,i) \in \mathscr{V}_A]$. The morphism $f : \mathbb{A}^{\mathscr{V}_A} \to \mathscr{S}_A$ corresponds to a quasi-isogeny $X \mapsto \mathbb{X}_{\mathbb{A}^{\mathscr{V}_A}}$, which we will describe by the construction a subdisplay of the isodisplay $N_{W(R)_\mathbb{Q}}$ of $\mathbb{X}_{\mathbb{A}^{\mathscr{V}_A}}$. There exists a unique family $\{v_i \mid 0 \leq i < dh\} \subset N_{W(R)_\mathbb{Q}}$ which satisfies the following relations:

$$\begin{array}{rcl} v_0 & = & e_{b_0} \\ v_{i+1} & = & \begin{cases} Fv_i & \text{if } b_i, b_{i+1} \in B^+ \\ Fv_i + \sum_{(j,i)\in\mathscr{V}_A} [t_{j,i}] v_i & \text{if } b_i \in B^+, b_{i+1} \in B^- \\ \frac{F(v_i)}{p} & \text{if } b_i \in B^-, b_{i+1} \in B^+ \\ \frac{F(v_i)}{p} + \sum_{(j,i)\in\mathscr{V}_A} [t_{j,i}] v_i & \text{if } b_i, b_{i+1} \in B^- \end{cases} \end{array}$$

The proof that (v_i) exists and is unique is literally the same as in [Vie08a]. Let

$$\begin{array}{rcl} L & = & \mathrm{span}_{W(R)}(v_i \mid b_i \in B^-) \\ T & = & \mathrm{span}_{W(R)}(v_i \mid b_i \in B^+) \\ P & = & L \oplus T \\ Q & = & L \oplus I_R T. \end{array}$$

Then $(P, Q, F, \frac{F}{p})$ is a subdisplay of $N_{W(R)_\mathbb{Q}}$, which yields a quasi-isogeny $X \mapsto \mathbb{X}_{\mathbb{A}^{\mathscr{V}_A}}$ corresponding to a point $f \in \mathscr{S}_A(\mathbb{A}^{\mathscr{V}_A})$. □

Corollary 16.8. *$\mathscr{M}_G(b,\mu)^0$ is projective*

Proof. By above proposition and Corollary 16.6 the underlying topological space of $\mathscr{M}_G(b,\mu)^0$ has a decomposition in finitely many quasi-compact subspaces. Thus it quasi-compact. Now [RZ96] Cor. 2.31 and Prop. 2.32 imply that $\mathscr{M}_G(b,\mu)^0$ is quasi-projective with projective irreducible components. Thus it is projective. □

Proposition 16.9. *The dimension formula (1.2) holds for superbasic Rapoport-Zink data of EL type.*

Proof. As we remarked at the beginning of this section, we have to show that

$$\dim \mathscr{M}_G(b,\mu) = \sum_{i=1}^{h-1} \lfloor \langle \omega_i, \nu - \mu \rangle \rfloor.$$

Now by Proposition 16.7 we get

$$\dim \mathscr{M}_G(b,\mu) = \max\{\#\mathscr{V}_A \mid A \text{ is a small EL-chart with Hodge point } \mu\}.$$

By Proposition 5.20 and Theorem 5.21 the right hand side equals $\sum_{i=1}^{h-1}\lfloor\langle\omega_i,\nu-\mu\rangle\rfloor$, finishing the proof. □

17. Irreducible components

We now consider the $J_b(F_0)$-action on the irreducible components of a superbasic Rapoport-Zink space $\mathscr{M}_G(b,\mu)$ of EL-type resp. of an affine Deligne-Lusztig variety $X_\mu(b)$ for superbasic b and arbitrary G.

We first consider the case of an affine Deligne-Lusztig variety. Recall that the canonical projection $G \twoheadrightarrow G^{\mathrm{ad}}$ induces isomorphisms

$$X_\mu(b)^\omega \cong X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^{\omega_{\mathrm{ad}}}$$

for all $X_\mu(b)^\omega \neq \emptyset$. As $J_b(F_0)$ acts transitively on the set of non-empty $X_\mu(b)^\omega$ (cf. [CKV] sect.3.3), this implies that the induced map on $J_b(F_0)$-orbits to $J_b^{\mathrm{ad}}(F_0)$-orbits of the respective irreducible components is bijective. Using the same argument as in the end of section 3, the investigation of the set of irreducible components of $X_\mu(b)^\omega$ (resp. the set of $J_b(F_0)$-orbits of irreducible components of $X_\mu(b)$) can be reduced to the case $G = \mathrm{Res}_{l/l_0}\mathrm{GL}_h$.

Lemma 17.1. *In the case of affine Deligne-Lusztig varieties let $b \in G(L)$ be superbasic and $\omega \in \pi_1(G)$ such that $X_\mu(b)^\omega$ is non-empty. Then every $J_b(F_0)$-orbit of irreducible components of $X_\mu(b)$ contains a unique irreducible component of $X_\mu(b)^\omega$.*

Similarly, every $J_b(\mathbb{Q}_p)$-orbit of irreducible components of a superbasic Rapoport-Zink space $\mathscr{M}_G(b,\mu)$ of EL-type contains a unique irreducible component of each connected component of $\mathscr{M}_G(b,\mu)$.

Proof. We only prove the first statement. The proof for Rapoport-Zink spaces is the same.

Denote by $J_b(F_0)^0$ the stabiliser of $X_\mu(b)^\omega$. Using the argument above, it suffices to show that $J_b(F_0)^0$ stabilises the irreducible components of $X_\mu(b)$ in the case $G = \mathrm{Res}_{l/l_0}\mathrm{GL}_h$.

For this we consider the action of $J_b(F_0)$ on N_{hom}. Let $g \in J_b(F_0)$ and let $v_0 := g(e_{0,1}), c(g) := \mathscr{I}(v_0)$. Now every element $e_{\tau,i}$ can be written in the form $e_{\tau,i} = \frac{1}{t^j}(b\sigma)^k(e_{0,1})$ for some integers j,k; then

$$g(e_{\tau,i}) = \frac{1}{t^j}(b\sigma)^k(v_0).$$

Hence $\mathrm{val}\det(g) = (c(g))_{\tau\in I}$, in particular we have $g \in J_b(F_0)^0$ if and only if $c(g) = 0$. In this case the above formula implies $\mathscr{I}(g(e_{\tau,i})) = i_{(\tau)}$ and thus $\mathscr{I}(g(v)) = \mathscr{I}(v)$ for all $v \in N_{\mathrm{hom}}$.

We obtain $A(M) = A(g(M))$ and $\varphi(M) = \varphi(g(M))$ for all $M \in X_\mu(b)^0$. Thus $\mathscr{S}_{A,\varphi}$ is $J_b(F_0)^0$-stable for every extended EL-chart (A,φ) for μ. As the $\mathscr{S}_{A,\varphi}$ are irreducible, every irreducible component of $X_\mu(b)^0$ is of the form $\overline{\mathscr{S}_{A,\varphi}}$ and thus $J_b(F)^0$-stable □

We denote by Max_μ the set of extended EL-Charts (A,φ) for μ for which $\#\mathscr{V}(A,\varphi)$ is maximal. As noted above we have a bijection

$$\begin{array}{ccl}\mathrm{Max}_\mu & \longleftrightarrow & \{\text{top-dimensional irreducible components of } X_\mu(b) \text{ resp. } \mathscr{M}_G(b,\mu)\} \\ (A,\varphi) & \longmapsto & \overline{\mathscr{S}_{(A,\varphi)}}.\end{array}$$

Rapoport conjectured in [Rap05] that $X_\mu(b)$ and $\mathscr{M}_G(b,\mu)$ are equidimensional. If this holds true, the above bijection becomes

$$\begin{array}{ccl}\mathrm{Max}_\mu & \longleftrightarrow & \{\text{irreducible components of } X_\mu(b) \text{ resp. } \mathscr{M}_G(b,\mu)\} \\ & \longleftrightarrow & \{J_b(F)-\text{orbits of irreducible components of } X_\mu(b) \text{ resp. } \mathscr{M}_G(b,\mu)\}.\end{array}$$

It is known in the case of affine Deligne-Lusztig varieties that $J_b(F_0)$ does not act transitively on the irreducible components of $X_\mu(b)$ in general, even in the case $G = \mathrm{GL}_h$ ([Vie06] Ex. 6.2). The following lemma shows that we have transitive action of $J_b(F_0)$ only in a few degenerate cases.

Lemma 17.2. *(1) Assume there exist $\tau_1, \tau_2 \in I$ such that μ_{τ_1} and μ_{τ_2} are not of the form $(a, a, \ldots, a)$ for some integer a. Then Max_μ contains more than one element.*

(2) On the contrary, if there exists $\tau \in I$ such that $\mu_\varsigma = (a_\varsigma, a_\varsigma, \ldots, a_\varsigma)$ for some integer a_ς for all $\varsigma \neq \tau$ then

$$\#\mathrm{Max}_\mu = \#\mathrm{Max}_{\mu_\tau}$$

Proof. *(1)* We assume without loss of generality that $\tau_1 \in [0, \tau_2)$. By Proposition 5.20 the cyclic EL-chart of type μ is contained in Max_μ. The same reasoning shows that Max_μ also contains the cyclic EL-chart of type $(\mu'_0, \ldots, \mu'_{\tau_2 - 1}, \mu_{\tau_2}, \ldots, \mu_{d-1})$ with $\mu'_\varsigma = (\mu_{\varsigma,h}, \mu_{\varsigma,1}, \ldots, \mu_{\varsigma,h-1})$. Note that our condition on μ_{τ_1} implies $\mu_{\tau_1} \neq \mu'_{\tau_1}$ and thus $\mu \neq (\mu'_0, \ldots, \mu'_{\tau_2-1}, \mu_{\tau_2}, \ldots, \mu_{d-1})$.

(2) The claim holds as the bijection

$$\begin{array}{rcl} \{\text{extended EL-charts for } \mu\} & \longleftrightarrow & \{\text{extended semi-modules for } \mu_\tau\} \\ (A, \varphi) & \longmapsto & (A_\tau, \varphi_\tau) \\ (\sqcup_{\tau \in I} B, \sqcup \varphi) & \longleftarrow\!\shortmid & (B, \varphi) \end{array}$$

preserves $\#\mathscr{V}(A, \varphi)$. □

We now consider the case μ minuscule. De Jong and Oort showed that in this case, $\#\mathrm{Max}_\mu = 1$ for (extended) semi-modules (i.e. the case $G = \mathrm{GL}_h$). For EL-charts we have the following conjecture, which generalises [dJO00] Rem. 6.16.

Conjecture 17.3. *Let μ be minuscule. Then the construction of $\tilde{\mu}$ in the proof of Thm. 5.21 induces a bijection*

$$\{(\text{extended) EL-charts for } \mu\} \to \{\tilde{\mu} \in W.\sigma^{-1}(\mu) \mid \nu \preceq \bar{\tilde{\mu}}\}$$

where $W = (S_n)^I$ denotes the absolute Weyl group of G. In particular

$$\begin{array}{rcl} \#\mathrm{Max}_\mu & = & \#\{\tilde{\mu} \in W.\mu \mid \nu \preceq \bar{\tilde{\mu}}, \delta_G(\nu, \tilde{\mu}) = 0\} \\ & = & \#\{\tilde{\mu} \in W.\mu \mid \underline{\tilde{\mu}} = (\underbrace{\lfloor \frac{m}{h} \rfloor, \ldots, \lfloor \frac{m}{h} \rfloor}_{h \cdot (1 - \{m/h\})}, \underbrace{\lceil \frac{m}{h} \rceil, \ldots, \lceil \frac{m}{h} \rceil}_{h \cdot \{m/h\}})\}, \end{array}$$

Here m is defined as above, i.e. $\nu = (\frac{m}{dh}, \ldots, \frac{m}{dh})$.

This conjecture has been verified for $d = 1, h \leq 15$; $d = 2, h \leq 10$; $d = 3, h \leq 7$ and $d = 4, h \leq 6$ using MAGMA (see appendix).

Appendix A. Root data of some reductive group schemes

In this section we calculate the root data of G and the relative root data of $G_{\mathbb{Q}_p}$ where G is either $\mathrm{GL}_{O_F,n}, \mathrm{GSp}_{O_F,n}$ or $\mathrm{GU}_{O_F,n}$. The absolute root data can be easily deduced from the root datum of GL_n and GSp_n. Then relative root data can be obtained form the absolute root datum together with the Galois action by a straight forward calculation.

We use the notation of section 2.1. Furthermore, we denote all relative root data of $G_{\mathbb{Q}_p}$ by a subscript $\mathbb{Q}_p$. Let I denote the Galois group of O_F over $\mathbb{Z}_p$.

A.1. $\mathrm{GL}_{O_F,n}$.
We have $\mathrm{GL}_{O_F,n} \otimes O_F \cong \prod_{\tau\in I} \mathrm{GL}_n$ with the Galois action cyclically permuting the GL_n-factors. We choose $T_1 \subset B_1 \subset \mathrm{GL}_{O_F,n}$ to be the diagonal torus torus resp. the Borel subgroup of upper triangular matrices. Furthermore, let

$$\begin{array}{lcl} e_{\varsigma,i} : T_1 \to \mathbb{G}_m & , & (\mathrm{diag}(t_{\tau,1},\ldots,t_{\tau,n}))_{\tau\in I} \mapsto t_{\varsigma,i} \\ e^\vee_{\varsigma,i} : \mathbb{G}_m \to T_1 & , & x \mapsto (\mathrm{diag}(1,\ldots,1),\ldots,\mathrm{diag}(1,\ldots,1,x,1,\ldots,1),\ldots,\mathrm{diag}(1,\ldots,1)) \end{array}$$

where the entry x is the $(\varsigma,i)^{\text{th}}$ entry. The $e_{\varsigma,i}$ resp. $e^\vee_{\varsigma,i}$ form a basis of $X^*(T)$ resp. $X_*(T)$, thus

$$\begin{array}{rcl} X^*(T_1) & \cong & \prod_{\tau\in I} \mathbb{Z}^n \\ X_*(T_1) & \cong & \prod_{\tau\in I} \mathbb{Z}^n \\ R & = & \{e_{\tau,i} - e_{\tau,j} \mid \tau\in I, i\neq j\in\{1,\ldots,n\}\} \\ R^\vee & = & \{e^\vee_{\tau,i} - e^\vee_{\tau,j} \mid \tau\in I, i\neq j\in\{1,\ldots,n\}\} \\ R^+ & = & \{e_{\tau,i} - e_{\tau,j} \mid \tau\in I, i< j\in\{1,\ldots,n\}\} \\ R^{\vee,+} & = & \{e_{\tau,i} - e_{\tau,j} \mid \tau\in I, i< j\in\{1,\ldots,n\}\} \\ \Delta^+ & = & \{e_{\tau,i} - e_{\tau,i+1} \mid \tau\in I, i\in\{1,\ldots,n-1\}\} \\ \Delta^{\vee,+} & = & \{e^\vee_{\tau,i} - e^\vee_{\tau,i+1} \mid \tau\in I, i\in\{1,\ldots,n-1\}\}. \end{array}$$

Hence

$$X_*(T_1)_{\mathrm{dom}} = \{\mu\in \prod_{\tau\in I}\mathbb{Z}^n \mid \mu_{\tau,1}\geq \ldots \mu_{\tau,n} \text{ for all } \tau\}.$$

Now the maximal split torus $S\subset T_{1,\mathbb{Q}_p}$ is given by

$$S_1 = \{(\mathrm{diag}(t_1,\ldots,t_n))_{\tau\in I}\in T_{\mathbb{Q}_p}\}.$$

We define

$$\begin{array}{rcl} e_{\mathbb{Q}_p,i} & := & e_{\tau,i|S} \text{ for some} \tau\in I \\ e^\vee_{\mathbb{Q}_p,i} & := & \sum_{\tau\in I} e^\vee_{\tau,i}. \end{array}$$

The $e_{\mathbb{Q}_p,i}$ resp. $e^\vee_{\mathbb{Q}_p,i}$ form a basis of $X^*(S_1)$ resp. $X_*(S_1)$, thus

$$\begin{array}{rcl} X^*(S_1) & \cong & \mathbb{Z}^n \\ X_*(S_1) & \cong & \mathbb{Z}^n \\ R_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i} - e_{\mathbb{Q}_p,j} \mid i\neq j\in\{1,\ldots,n\}\} \\ R^\vee_{\mathbb{Q}_p} & = & \{e^\vee_{\mathbb{Q}_p,i} - e^\vee_{\mathbb{Q}_p,j} \mid i\neq j\in\{1,\ldots,n\}\} \\ R^+_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i} - e_{\mathbb{Q}_p,j} \mid i< j\in\{1,\ldots,n\}\} \\ R^{\vee,+}_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i} - e_{\mathbb{Q}_p,j} \mid i< j\in\{1,\ldots,n\}\} \\ \Delta^+_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i} - e_{\mathbb{Q}_p,i+1} \mid i\in\{1,\ldots,n-1\}\} \\ \Delta^{\vee,+}_{\mathbb{Q}_p} & = & \{e^\vee_{\mathbb{Q}_p,i} - e^\vee_{\mathbb{Q}_p,i+1} \mid \tau\in I, i\in\{1,\ldots,n-1\}\}. \end{array}$$

As a consequence

$$X_*(S_1)_{\mathbb{Q},\mathrm{dom}} = \{\nu\in\mathbb{Z}^n \mid \nu_1\geq\ldots\geq\nu_n\}$$

A.2. $\mathrm{GSp}_{O_F,n}$.
We have

$$\mathrm{GSp}_{O_F,n}\otimes O_F \cong (\prod_{\tau\in I}\mathrm{GSp}_n)^1 = \{(g_\tau)\in\prod_{\tau\in I}\mathrm{GSp}_n \mid c(g_\tau) = c \text{ does not depend on } \tau\}$$

with the Galois action cyclically permuting the factors. Let $T_2 \subset B_2 \subset \mathrm{GSp}_{O_F,n}$ denote the maximal torus resp. the Borel subgroup of upper triangular matrices. We denote by

$$c : T_2 \to \mathbb{G}_m$$

The similitude factor. Then

$$\begin{array}{rcl}
X^*(T_2) & \cong & X_*(T_1)\Big/\langle (e_{\tau,i} + e_{\tau,n+1-i}) - (e_{\varsigma,j} + e_{\varsigma,n+1-j}) \rangle_{\tau,\varsigma,i,j} \\
X_*(T_2) & = & \{(\mu \in X_*(T) \mid \mu_{\tau,i} + \mu_{\tau,n+1-i} = c(\mu) \text{ for some integer } c(\mu)\} \\
R & = & \{e_{\tau,i|T_2} - e_{\tau,j|T_2} \mid \tau \in I, i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm(e_{\tau,i|T_2} + e_{\tau,j|T_2} - c) \mid \tau \in I i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm(2e_{\tau,i|T_2} - c) \mid \tau \in I, i \in \{1,\dots,n/2\}\} \\
 & = & \{e_{\tau,i|T_2} - e_{\tau,j|T_2} \mid \tau \in I, i \neq j \in \{1,\dots,n\}\} \\
R^\vee & = & \{e^\vee_{\tau,i} - e^\vee_{\tau,j} + e^\vee_{\tau,n+1-j} - e^\vee_{\tau,n+1-i} \mid \tau \in I, i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm(e^\vee_{\tau,i} + e^\vee_{\tau,j} - e^\vee_{\tau,n+1-i} - e^\vee_{\tau,n+1-j}) \mid \tau \in I, i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm e^\vee_{\tau,i} \mid \tau \in I, i \in \{1,\dots,n/2\} \\
R^+ & = & \{e_{\tau,i|T_2} - e_{\tau,j|T_2} \mid \tau \in I, i < j \in \{1,\dots,n/2\}\} \\
 & & \cup\{e_{\tau,i|T_2} + e_{\tau,j|T_2} - c \mid i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{2e_{\tau,i|T_2} - c \mid \tau \in I, i \in \{1,\dots,n/2\}\} \\
 & = & \{e_{\tau,i|T_2} - e_{\tau,j|T_2} \mid \tau \in I, i \neq j \in \{1,\dots,n\}\} \\
R^{\vee,+} & = & \{e^\vee_{\tau,i} - e^\vee_{\tau,j} + e^\vee_{\tau,n+1-j} - e^\vee_{\tau,n+1-i} \mid \tau \in I, i < j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm(e^\vee_{\tau,i} + e^\vee_{\tau,j} - e^\vee_{\tau,n+1-i} - e^\vee_{\tau,n+1-j} \mid \tau \in I, i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm(e^\vee_{\tau,i} \mid \tau \in I, i \in \{1,\dots,n/2\} \\
\Delta^+ & = & \{e_{\tau,i|T_2} - e_{\tau,i+1|T_2} \mid \tau \in I, i \in \{1,\dots,n/2-1\}\} \cup \{2e_{\tau,n/2|T_2} - c \mid \tau \in I\} \\
 & = & \{e_{\tau,i|T_2} - e_{\tau,i+1|T_2} \mid \tau \in I, i \in \{1,\dots,n-1\}\} \\
\Delta^{\vee,+} & = & \{e^\vee_{\tau,i} - e^\vee_{\tau,i+1} + e^\vee_{\tau,n+1-i} - e^\vee_{\tau,n-i} \mid \tau \in I, i \in \{1,\dots,n/2-1\}\} \cup \{e^\vee_{\tau,n/2} \mid \tau \in I\}
\end{array}$$

Hence

$$X_*(T_2)_{\mathrm{dom}} = \{\mu \in \prod_{\tau \in I} \mathbb{Z}^n \mid \mu_{\tau,1} \geq \dots \mu_{\tau,n} \text{ for all } \tau, \mu_{\tau,i} + \mu_{\tau,n+1-i} = c(\mu) \text{ for some integer } c\}.$$

Denote by S_2 the maximal split torus of $T_{2,\mathbb{Q}_p}$. Now

$$\begin{array}{rcl}
X^*(S_2) & \cong & X_*(S_1)\Big/\langle (e_{\mathbb{Q}_p,i} + e_{\mathbb{Q}_p,n+1-i}) - (e_{\mathbb{Q}_p,j} + e_{\mathbb{Q}_p,n+1-j}) \rangle_{i,j} \\
X_*(S_2) & = & \{(\nu \in X_*(S) \mid \nu_i + \nu_{n+1-i} = c(\nu) \text{ for some integer } c(\nu)\} \\
R_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i|S_2} - e_{\mathbb{Q}_p,j|S_2} \mid i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm(e_{\mathbb{Q}_p,i|S_2} + e_{\mathbb{Q}_p,j|S_2} - c) \mid i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm(2e_{\mathbb{Q}_p,i|S_2} - c) \mid i \in \{1,\dots,n/2\}\} \\
 & = & \{e_{\mathbb{Q}_p,i|S_2} - e_{\mathbb{Q}_p,j|S_2} \mid i \neq j \in \{1,\dots,n\}\} \\
R^\vee_{\mathbb{Q}_p} & = & \{e^\vee_{\mathbb{Q}_p,i} - e^\vee_{\mathbb{Q}_p,j} + e^\vee_{\mathbb{Q}_p,n+1-j} - e^\vee_{\mathbb{Q}_p,n+1-i} \mid i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm(e^\vee_{\mathbb{Q}_p,i} + e^\vee_{\mathbb{Q}_p,j} - e^\vee_{\mathbb{Q}_p,n+1-i} - e^\vee_{\mathbb{Q}_p,n+1-j}) \mid i \neq j \in \{1,\dots,n/2\}\} \\
 & & \cup\{\pm e^\vee_{\mathbb{Q}_p,i} \mid i \in \{1,\dots,n/2\}
\end{array}$$

$$\begin{array}{rcl}
R^+_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i|S_2} - e_{\mathbb{Q}_p,j|S_2} \mid i < j \in \{1,\dots,n/2\}\} \\
& & \cup\{e_{\mathbb{Q}_p,i|S_2} + e_{\mathbb{Q}_p,j|S_2} - c \mid i \neq j \in \{1,\dots,n/2\}\} \\
& & \cup\{2e_{\mathbb{Q}_p,i|S_2} - c \mid i \in \{1,\dots,n/2\}\} \\
& = & \{e_{\mathbb{Q}_p,i|S_2} - e_{\mathbb{Q}_p,j|S_2} \mid i \neq j \in \{1,\dots,n\}\} \\
R^{\vee,+}_{\mathbb{Q}_p} & = & \{e^\vee_{\mathbb{Q}_p,i} - e^\vee_{\mathbb{Q}_p,j} + e^\vee_{\mathbb{Q}_p,n+1-j} - e^\vee_{\mathbb{Q}_p,n+1-i} \mid i < j \in \{1,\dots,n/2\}\} \\
& & \cup\{\pm(e^\vee_{\mathbb{Q}_p,i} + e^\vee_{\mathbb{Q}_p,j} - e^\vee_{\mathbb{Q}_p,n+1-i} - e^\vee_{\mathbb{Q}_p,n+1-j} \mid i \neq j \in \{1,\dots,n/2\}\} \\
& & \cup\{\pm(e^\vee_{\mathbb{Q}_p,i} \mid i \in \{1,\dots,n/2\} \\
\Delta^+ & = & \{e_{\mathbb{Q}_p,i|S_2} - e_{\mathbb{Q}_p,i+1|S_2} \mid i \in \{1,\dots,n/2-1\}\} \cup \{2e_{\mathbb{Q}_p,n/2|S_2} - c\} \\
& = & \{e_{\mathbb{Q}_p,i|S_2} - e_{\mathbb{Q}_p,i+1|S_2} \mid i \in \{1,\dots,n-1\}\} \\
\Delta^{\vee,+} & = & \{e^\vee_{\mathbb{Q}_p,i} - e^\vee_{\mathbb{Q}_p,i+1} + e^\vee_{\mathbb{Q}_p,n+1-i} - e^\vee_{\mathbb{Q}_p,n-i} \mid i \in \{1,\dots,n/2-1\}\} \cup \{e^\vee_{\mathbb{Q}_p,n/2}\}.
\end{array}$$

Thus

$$X_*(S_2)_{\mathbb{Q},\mathrm{dom}} = \{\nu \in \mathbb{Q}^n \mid \nu_1 \geq \ldots \geq \nu_n, \nu_i + \nu_{n+1-i} = c(\nu) \text{ for some integer } c(\nu)\}$$

A.3. $\mathrm{GU}_{O_F,n}$.
We have

$$\mathrm{GU}_{O_F,n} \otimes O_F = \{(g_\tau) \in \prod_{\tau\in I} GL_n \mid g_\tau J g_{\sigma_{F'}\circ\tau} = c(g)J\}$$

With I cyclically permuting the factors. et $T_3 \subset B_3 \subset \mathrm{GU}_{O_F,n}$ denote the maximal torus resp. the Borel subgroup of upper triangular matrices. We denote by

$$c : T_3 \to \mathbb{G}_m$$

the similitude factor. We fix a system of representatives $I' \subset I$ of $I/\sigma_{F'}$ Now

$$\begin{array}{rcl}
X^*(T_3) & \cong & X_*(T_1)\Big/\langle (e_{\tau,i} + e_{\sigma_{F'}+\tau,n+1-i}) - (e_{\varsigma,j} + e_{\sigma_{F'}+\varsigma,n+1-j})\rangle_{\tau,\varsigma,i,j} \\
X_*(T_3) & = & \{(\mu \in X_*(T) \mid \mu_{\tau,i} + \mu_{\sigma_{F'}+\tau,n+1-i} = c(\mu) \text{ for some integer } c(\mu)\} \\
R & = & \{e_{\tau,i|T_3} - e_{\tau,j|T_3} \mid \tau \in I', i \neq j \in \{1,\dots,n\}\} \\
& = & \{e_{\tau,i|T_3} - e_{\tau,j|T_3} \mid \tau \in I, i \neq j \in \{1,\dots,n\}\} \\
R^\vee & = & \{e^\vee_{\tau,i} - e^\vee_{\tau,j} + e_{\sigma_{F'}+\tau,n+1-j} - e_{\sigma_{F'}+\tau,n+1-i} \mid \tau \in I', i \neq j \in \{1,\dots,n\}\} \\
R^+ & = & \{e_{\tau,i|T_3} - e_{\tau,j|T_3} \mid \tau \in I', i < j \in \{1,\dots,n\}\} \\
& = & \{e_{\tau,i|T_3} - e_{\tau,j|T_3} \mid \tau \in I, i < j \in \{1,\dots,n\}\} \\
R^{\vee,+} & = & \{e^\vee_{\tau,i} - e^\vee_{\tau,j} + e_{\sigma_{F'}+\tau,n+1-j} - e_{\sigma_{F'}+\tau,n+1-i} \mid \tau \in I, i < j \in \{1,\dots,n\}\} \\
\Delta^+ & = & \{e_{\tau,i|T_3} - e_{\tau,i+1|T_3} \mid \tau \in I', i \in \{1,\dots,n-1\}\} \\
& = & \{e_{\tau,i|T_3} - e_{\tau,i+1|T_3} \mid \tau \in I, i \in \{1,\dots,n-1\}\} \\
\Delta^{\vee,+} & = & \{e^\vee_{\tau,i} - e^\vee_{\tau,i+1} + e_{\sigma_{F'}+\tau,n-i} - e_{\sigma_{F'}+\tau,n+1-i} \mid \tau \in I', i \in \{1,\dots,n-1\}\}.
\end{array}$$

In particular,

$$X_*(T_3)_{\mathrm{dom}} = \{(\mu \in X_*(T) \mid \mu_{\tau,1} \geq \ldots \geq \mu_{\tau,n}, \mu_{\tau,i} + \mu_{\sigma_{F'}+\tau,n+1-i} = c(\mu) \text{ for some integer } c(\mu)\}.$$

We denote by S_3 the maximal split torus of $T_{3,\mathbb{Q}_p}$. If n is even, then

$$\begin{array}{rcl}
X^*(S_3) & \cong & X_*(S_1)\Big/\langle (e_{\mathbb{Q}_p,i}+e_{\mathbb{Q}_p,n+1-i})-(e_{\mathbb{Q}_p,j}+e_{\mathbb{Q}_p,n+1-j})\rangle_{i,j} \\
X_*(S_3) & = & \{(\nu\in X_*(S)\mid \nu_i+\nu_{n+1-i}=c(\nu) \text{ for some integer } c(\nu)\} \\
R_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i|S_2}-e_{\mathbb{Q}_p,j|S_2}\mid i\neq j\in\{1,\ldots,n/2\}\} \\
& & \cup\{\pm(e_{\mathbb{Q}_p,i|S_2}+e_{\mathbb{Q}_p,j|S_2}-c)\mid i\neq j\in\{1,\ldots,n/2\}\} \\
& & \cup\{\pm(2e_{\mathbb{Q}_p,i|S_2}-c)\mid i\in\{1,\ldots,n/2\}\} \\
& = & \{e_{\mathbb{Q}_p,i|S_2}-e_{\mathbb{Q}_p,j|S_2}\mid i\neq j\in\{1,\ldots,n\}\} \\
R^\vee_{\mathbb{Q}_p} & = & \{e^\vee_{\mathbb{Q}_p,i}-e^\vee_{\mathbb{Q}_p,j}+e^\vee_{\mathbb{Q}_p,n+1-j}-e^\vee_{\mathbb{Q}_p,n+1-i}\mid i\neq j\in\{1,\ldots,n/2\}\} \\
& & \cup\{\pm(e^\vee_{\mathbb{Q}_p,i}+e^\vee_{\mathbb{Q}_p,j}-e^\vee_{\mathbb{Q}_p,n+1-i}-e^\vee_{\mathbb{Q}_p,n+1-j})\mid i\neq j\in\{1,\ldots,n/2\}\} \\
& & \cup\{\pm e^\vee_{\mathbb{Q}_p,i}\mid i\in\{1,\ldots,n/2\} \\
R^+_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i|S_2}-e_{\mathbb{Q}_p,j|S_2}\mid i<j\in\{1,\ldots,n/2\}\} \\
& & \cup\{e_{\mathbb{Q}_p,i|S_2}+e_{\mathbb{Q}_p,j|S_2}-c\mid i\neq j\in\{1,\ldots,n/2\}\} \\
& & \cup\{2e_{\mathbb{Q}_p,i|S_2}-c\mid i\in\{1,\ldots,n/2\}\} \\
& = & \{e_{\mathbb{Q}_p,i|S_2}-e_{\mathbb{Q}_p,j|S_2}\mid i\neq j\in\{1,\ldots,n\}\} \\
R^{\vee,+}_{\mathbb{Q}_p} & = & \{e^\vee_{\mathbb{Q}_p,i}-e^\vee_{\mathbb{Q}_p,j}+e^\vee_{\mathbb{Q}_p,n+1-j}-e^\vee_{\mathbb{Q}_p,n+1-i}\mid i<j\in\{1,\ldots,n/2\}\} \\
& & \cup\{\pm(e^\vee_{\mathbb{Q}_p,i}+e^\vee_{\mathbb{Q}_p,j}-e^\vee_{\mathbb{Q}_p,n+1-i}-e^\vee_{\mathbb{Q}_p,n+1-j}\mid i\neq j\in\{1,\ldots,n/2\}\} \\
& & \cup\{\pm(e^\vee_{\mathbb{Q}_p,i}\mid i\in\{1,\ldots,n/2\} \\
\Delta^+ & = & \{e_{\mathbb{Q}_p,i|S_2}-e_{\mathbb{Q}_p,i+1|S_2}\mid i\in\{1,\ldots,n/2-1\}\}\cup\{2e_{\mathbb{Q}_p,n/2|S_2}-c\} \\
& = & \{e_{\mathbb{Q}_p,i|S_2}-e_{\mathbb{Q}_p,i+1|S_2}\mid i\in\{1,\ldots,n-1\}\} \\
\Delta^{\vee,+} & = & \{e^\vee_{\mathbb{Q}_p,i}-e^\vee_{\mathbb{Q}_p,i+1}+e^\vee_{\mathbb{Q}_p,n+1-i}-e^\vee_{\mathbb{Q}_p,n-i}\mid i\in\{1,\ldots,n/2-1\}\}\cup\{e^\vee_{\mathbb{Q}_p,n/2}\}.
\end{array}$$

If n is odd, then

$$\begin{array}{rcl}
X^*(S_3) & \cong & X_*(S_1)\Big/\langle (e_{\mathbb{Q}_p,i}+e_{\mathbb{Q}_p,n+1-i})-(e_{\mathbb{Q}_p,j}+e_{\mathbb{Q}_p,n+1-j})\rangle_{i,j} \\
X_*(S_3) & = & \{(\nu\in X_*(S_1)\mid \nu_i+\nu_{n+1-i}=c(\nu) \text{ for some integer } c(\nu)\} \\
R_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i|S_3}-e_{\mathbb{Q}_p,j|S_3}\mid i\neq j\in\{1,\ldots,(n-1)/2\}\} \\
& & \cup\{\pm(e_{\mathbb{Q}_p,i|S_3}+e_{\mathbb{Q}_p,j|S_3}-c)\mid i\neq j\in\{1,\ldots(n-1)/2\}\} \\
& & \cup\{\pm(2e_{\mathbb{Q}_p,i|S_3}-c)\mid i\in\{1,\ldots,(n-1)/2\}\} \\
& & \cup\{\pm(e_{\mathbb{Q}_p,i|S_3}-c)\mid i\in\{1,\ldots,(n-1)/2\}\} \\
& = & \{e_{\mathbb{Q}_p,i|S_2}-e_{\mathbb{Q}_p,j|S_2}\mid i\neq j\in\{1,\ldots,n\}\} \\
R^\vee_{\mathbb{Q}_p} & = & \{e^\vee_{\mathbb{Q}_p,i}-e^\vee_{\mathbb{Q}_p,j}+e^\vee_{\mathbb{Q}_p,n+1-j}-e^\vee_{\mathbb{Q}_p,n+1-i}\mid i\neq j\in\{1,\ldots,(n-1)/2\}\} \\
& & \cup\{\pm(e^\vee_{\mathbb{Q}_p,i}+e^\vee_{\mathbb{Q}_p,j}-e^\vee_{\mathbb{Q}_p,n+1-i}-e^\vee_{\mathbb{Q}_p,n+1-j})\mid i\neq j\in\{1,\ldots,(n-1)/2\}\} \\
& & \cup\{\pm e^\vee_{\mathbb{Q}_p,i}\mid i\in\{1,\ldots,(n-1)/2\} \\
& & \cup\{\pm 2e^\vee_{\mathbb{Q}_p,i}\mid i\in\{1,\ldots,(n-1)/2\} \\
R^+_{\mathbb{Q}_p} & = & \{e_{\mathbb{Q}_p,i|S_3}-e_{\mathbb{Q}_p,j|S_3}\mid i<j\in\{1,\ldots,(n-1)/2\}\} \\
& & \cup\{e_{\mathbb{Q}_p,i|S_3}+e_{\mathbb{Q}_p,j|S_3}-c\mid i\neq j\in\{1,\ldots,(n-1)/2\}\} \\
& & \cup\{2e_{\mathbb{Q}_p,i|S_3}-c\mid i\in\{1,\ldots,(n-1)/2\}\} \\
& & \cup\{e_{\mathbb{Q}_p,i|S_3}\mid i\in\{1,\ldots,(n-1)/2\}\} \\
& = & \{e_{\mathbb{Q}_p,i|S_2}-e_{\mathbb{Q}_p,j|S_2}\mid i\neq j\in\{1,\ldots,n\}\}
\end{array}$$

$$\begin{aligned}
R^{\vee,+}_{\mathbb{Q}_p} &= \{e^\vee_{\mathbb{Q}_p,i} - e^\vee_{\mathbb{Q}_p,j} + e^\vee_{\mathbb{Q}_p,n+1-j} - e^\vee_{\mathbb{Q}_p,n+1-i} \mid i < j \in \{1,\ldots,(n-1)/2\}\} \\
&\quad \cup \{e^\vee_{\mathbb{Q}_p,i} + e^\vee_{\mathbb{Q}_p,j} - e^\vee_{\mathbb{Q}_p,n+1-i} - e^\vee_{\mathbb{Q}_p,n+1-j} \mid i \neq j \in \{1,\ldots,(n-1)/2\}\} \\
&\quad \cup \{e^\vee_{\mathbb{Q}_p,i} \mid i \in \{1,\ldots,(n-1)/2\} \cup \{2e^\vee_{\mathbb{Q}_p,i} \mid i \in \{1,\ldots,(n-1)/2\} \\
\Delta^+ &= \{e_{\mathbb{Q}_p,i|S_3} - e_{\mathbb{Q}_p,i+1|S_3} \mid i \in \{1,\ldots,(n-1)/2-1\}\} \cup \{e_{\mathbb{Q}_p,(n-1)/2|S_2} - c\} \\
&= \{e_{\mathbb{Q}_p,i|S_3} - e_{\mathbb{Q}_p,i+1|S_3} \mid i \in \{1,\ldots,n-1\}\} \\
\Delta^{\vee,+} &= \{e^\vee_{\mathbb{Q}_p,i} - e^\vee_{\mathbb{Q}_p,i+1} + e^\vee_{\mathbb{Q}_p,n+1-i} - e^\vee_{\mathbb{Q}_p,n-i} \mid i \in \{1,\ldots,(n-1)/2-1\}\} \cup \{e^\vee_{\mathbb{Q}_p,n/2}\}.
\end{aligned}$$

In any case we get

$$X_*(S_3)_{\mathbb{Q},\mathrm{dom}} = \{\nu \in \mathbb{Q}^n \mid \nu_1 \geq \ldots \geq \nu_n, \nu_i + \nu_{n+1-i} = c(\nu) \text{ for some integer } c(\nu)\}$$

Appendix B. A Magma program to test Conjecture 17.3

The code below test Conjecture 17.3 for given values d and h. Please note that the code is not meant for everyday use, i.e. it does not check for errors in the input data except for some very few arithmetic dependencies and there are still many possibilities for runtime optimisation.

In Magma code, we represent a normalized small EL-chart A by a quadruple (d,h,m,μ) where the d,h are integers, m is a tuple of integers $m[\tau] = m_\tau$ which are chosen as in the definition of a small EL-chart and μ denotes its type. Note that m and μ have a different meaning in the previous sections and that we have $\tau \in \{1,\ldots,d\}$ instead of $\tau \in \{0,\ldots,d-1\}$ due to the Magma syntax. The sum of all $m[\tau]$, which is denoted m in section 5, is calculated and denoted by det in this program. Furthermore, we use the following record to represent an element of $B = A \setminus (A+h)$ together with an integer μ such that $f(b) - h\cdot\mu$ lies in B.

```
GEN := recformat<b : Integers(), μ: Integers()> ;
```

Before we define the functions which will test Conjecture 17.3, we introduce some auxiliary functions. The following function determines all possible types of a small EL-chart whose Hodge-point is $((0,\ldots,0,1,\ldots,1))_\tau$ with precisely $m[\tau]$ ones in the τ-th component. Note that the conditions which determine whether an element of $\prod_{\tau=1}^d \mathbb{Z}^h$ is a type are given by Lemma 5.8 (2) and Lemma 5.16.

```
GetELtypes := function(d,h,m)
  error if #m ne d,
    "The last variable has to be a tuple of precisely", d,
    "integers." ;
  det := &+[ m[τ] : τ in [1..d]] ;
  error if Gcd(h,det) ne 1,
    "The variables m and h have to be coprime" ;
  mTuples := < {v : v in CartesianPower({0,1},h) |&+[ v[i] : i in [1..h] ] eq m[τ]}
              : τ in [1..d] > ;
  return {μ : μ in CartesianProduct(mTuples) |
            &and[ &+[ μ[τ][i] : τ in [1..d], i in [1..j] ] le det*j/h : j in [1..h] ] } ;
end function ;
```

This program also ensures that m has size d and that det and h are coprime and returns an error message otherwise.

We note that when we check the hypothesis for different values of m the instances of the above function would do the same time-consuming calculations multiple times when defining $mTuples$. In order to reduce the runtime when testing the hypothesis for different values of m, we introduce the following variant of above function, which has the additional input det and

$S =< S[1], \ldots, S[h-1] >$ where

$$\begin{aligned} det &= \sum_{\tau=1}^{d} m[\tau], \\ S[i] &= \{\mu \in \{0,1\}^h \mid \sum_{j=1}^{h} \mu[j] = i\}. \end{aligned}$$

These values will be determined once and for all before executing this function. This prevents us from repeating the same calculations in different instances of this function.

```
GetELtypesEfficiently := function(d, h, m, det, S)
return {μ : μ in CartesianProduct(<S[m[τ]]: τ in [1..d]>) |
            &and[ &+[ μ[τ][i] : τ in [1..d], i in [1..j] ] le det*j/h : j in [1..h] ] } ;
end function ;
```

The following auxiliary function calculates $\sigma(\tilde{\mu})$ for a given normalised EL-chart A represented by a tuple (d, h, m, μ). The calculation consists of two steps. First, we determine $B := ((b_{\tau,i}, \mu_{\tau+1,i}))_{\tau,i}$ using the recursion formula given before Lemma 5.8. To be precise, we will not have $\{b_{\tau,i}\}_{\tau,i} = A \setminus (A+h)$, but $\{b_{\tau,i}\}_{\tau,i} = (A \setminus (A+h)) + k$ for some integer k, as we do not normalise after constructing B. However, this will not interfere with the calculation of $\sigma(\tilde{\mu})$. Afterwards we permute the components of μ as described in the proof of Theorem 5.21 to determine $\tilde{\mu}$ and perform an index shift to obtain $\sigma(\tilde{\mu})$.

```
ComputeAltType := function(d, h, m, μ)
  B := <[ rec<GEN | > : i in [1..h]] : τ in [1..d]> ;
  (B[1][1])`b := 0 ;
  (B[d][h])`μ := μ[1][h] ;
  for i := 1 to h do
    for τ := 2 to d do
      (B[τ][i])`b := (B[τ−1][i])`b + m[τ] −h*μ[τ][i] ;
      (B[τ−1][i])`μ := μ[τ][i] ;
    end for ;
    if i ne h then
      (B[1][i+1])`b := (B[d][i])`b +m[1] − h*μ[1][i] ;
      (B[d][i])`μ := μ[1][i] ;
    end if ;
  end for ;
  for τ := 1 to d do
    Sort(~B[τ], func<x, y | x`b−y`b>) ;
  end for ;
  return << (B[((τ−2) mod d)+1][i])`μ : i in [1..h] > : τ in [1..d] > ;
end function ;
```

In order to test Conjecture 17.3 for given values d, h and m, we first determine the set $types$ of possible types of EL-Charts and then check whether we have an equality

$$\{\sigma(\tilde{\mu}) \mid \mu \in types\} = types.$$

This is done by the following function.

```
KleinerHypothesenTest := function(d, h, m)
  types := GetELtypes(d, h, m) ;
  altTypes := { } ;
  for μ in types do
    Include(~altTypes, ComputeAltType(d, h, m, μ)) ;
  end for ;
  return (types eq altTypes) ;
end function;
```

The following function tests the conjecture for a given $G = \mathrm{Res}_{l/l_0} \mathrm{GL}_h$, i.e. for fixed d and h but arbitrary m. Of course this could be done by repeating the above function for different values of m. But in order to improve the runtime, we perform the following optimisations.

- As mentioned above, we determine *det* and the set S before calculating the set *types* for different values of m.
- We only consider Hodge points where none of the components consist only of zeroes or only of ones, i.e. we assume $m[\tau] \neq 0$ and $m[\tau] \neq h$ for all τ. By the virtue of the proof of Lemma 17.2 (2) the assertion that Conjecture 17.3 holds true for those Hodge points with $m[\tau] = 0$ or $m[\tau] = h$ for one τ is equivalent to the assertion that the conjecture holds true for $G' = \mathrm{Res}_{l'/l_0} \mathrm{GL}_h$ with $[l' : l_0] = d - 1$, *which we assume to have verified beforehand.*

```
HypothesenTest := function(d, h)
  hypothesis := true;
  S := < {v : v in CartesianPower({0,1}, h) | &+[ v[i] : i in [1..h] ] eq m}
          : m in [1..h−1] > ;
  for det := 1 to d∗(h−1) do
    if Gcd(det, h) eq 1 then
      for m in RestrictedPartitions(det, d, {1..h−1}) do
        types := GetELtypesEfficiently(d, h, m, det, S) ;
        altTypes := { } ;
        for μ in types do
          Include(~altELtypes, ComputeAltType(d, h, m, μ)) ;
        end for ;
        hypothesis := hypothesis and (types eq altTypes);
      end for;
    end if;
  end for;
  return hypothesis ;
end function ;
```

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