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Approximation with respect to disc polynomials

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Contents

1 Introduction

Fourier series are a well-known and powerful tool in mathematics. Jean Baptiste Joseph Fourier discovered the principle of expanding functions into Fourier series more than two hundred years ago and in his honour they have been named after him. In nowadays literature, classical Fourier series are mostly given in their complex version, where the formal Fourier expansion of a 2π-periodic function f is given by

$$S\big(f\big)(x) = \sum_{k=-\infty}^{\infty} c_n e^{inx} \qquad \text{where} \qquad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt \ .$$

However, Fourier discovered this principle with the two trigonometric functions sine and cosine. The equivalent formula with respect to the functions sine and cosine has the form $S\big(f\big)(x) = \frac{a_0}{2} + \sum_{k=0}^{\infty}\Big(a_k\cos(kx) + b_k\sin(kx)\Big)$ while the corresponding coefficients are given by $a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)dt$ and $b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt)dt$. The immediate questions that arise are the following: for which values of x does the series converge, and does $S\big(f\big)$ converge to f in a certain norm?

The attempt to answer the questions after the different sorts of convergence of this series is called classical Fourier analysis. Due to the long history of classical Fourier analysis the theory is developed quite well and there are many books where the theory is focused on, like for example [6] and [23]. There, we learn for example that we have norm convergence i.e. $S(f) = f$ almost everywhere in general L^p-spaces for $1 < p < \infty$, a result that is refered to M. Riesz. Already earlier the case $p = 1$ was proved to be wrong almost everywhere by Kolmogorov.

By looking at the classical Fourier analysis from another point of view one sees that the functions $e_n(x) = e^{inx}$ constitute an orthogonal basis in the $L^2([0, 2\pi))$ space, a Hilbert space with inner product, i.e. we have $\langle e_n, e_m\rangle = \frac{1}{2\pi}\int_0^{2\pi} e_n(x)\overline{e_m(x)}dx = \delta_{n,m}$. Furthermore, the Fourier coefficients c_n are defined by $c_n = \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{e_n(t)}dt$. In terms of the inner product in $L^2([0, 2\pi))$ this means we can write the formal Fourier series of a function f now via

$$S(f)(x) = \sum_{n=-\infty}^{\infty} \langle f, e_n\rangle \ e_n(x) \quad .$$

From this point of view one is led to the question what happens if the orthogonal system is changed, together with the domain on which the system is orthogonal. Can we drop the periodicity condition? Of course, the same convergence question is: for which real values p with $1 \le p \le \infty$ does $S(f)$ converge towards f in the L^p norm? Can we extend this to the general setting of locally compact groups? It is well established that we can change the orthogonal system. This way of generalizing classical Fourier analysis is a branch of so called harmonic analysis.

Famous and important orthogonal systems are the so called orthogonal polynomial systems. There are various families of orthogonal polynomials and one important class are the Jacobi polynomials. They are orthogonal on the interval $[-1, 1]$ with respect to a certain weight function. Indeed, the Fourier analysis is successfully taken over to the Jacobi setting and similar questions are investigated leading to similar results. Norm convergence of the formal Fourier series is not given in general. But as in classical Fourier analysis with similar modifications this goal can be reached. This generalized Fourier analysis has also been well developed for years and many results and applications are known, cf. for example [2, 4, 29, 30, 31].

The disc polynomials are an orthogonal polynomial system on the unit disc. Thus we are now working in two dimensions and generalized Fourier analysis with respect to the disc polynomials can be investigated. The disc polynomials are orthogonal on the unit disc with respect to the weight function $\mu_\alpha(z) = (1 - |z|^2)^\alpha$ for $\alpha > -1$ so we see that a parameter α is included in the investigations. Indeed, many concepts from the one dimensional case can be adopted but one is confronted with new problems the two dimensionality brings about.

The disc polynomials are not as popular as other one dimensional orthogonal polynomial systems. However, in the last years they have been rediscovered as they can be applied in quantum optics mentioned in [34] and [36]. In fact, a special case of disc polynomials -the Zernike polynomials- primarily have been developed to describe optical properties. The Nobel prize winner Frits Zernike found the disc polynomials for $\alpha = 0$ and used them in his research on optics and microscopy. Later a generalization led to the nowadays called disc polynomials. They are defined with the help of the Jacobi polynomials and use their orthogonality relation. But nevertheless they need an independent analysis.

One of the main tasks of applied mathematics is the approximation of functions. There are many methods and ways to do so. Of course, there are different problems of approximation. One problem is the question for norm convergence with different kernels together with the question for the speed of convergence. In addition, it is questioned if there are explicit constants provided in the convergence estimation. This is desirable for practical reasons.

To put this thesis in a nutshell we will work on generalized Fourier analysis with disc polynomials. The concepts of classical Fourier analysis that have been succesfully taken over to the setting with Jacobi polynomials, for example, will be illustrated in the disc polynomial case on the unit disc. The aim is to find kernels that approximate functions so that we have norm convergence in appropriate Lebesgue spaces. Recent results will be transformed from the Jacobi into the disc polynomial setting. This way we derive a generalized de-la-Vallée-Poussin kernel V_n with the help of disc polynomials as it has been done in [10] for Jacobi polynomials. This kernel V_n provides norm convergence $\|V_n f - f\| \to 0$ for $n \to \infty$ for all functions f from certain Lebesgue spaces on the unit disc. Furthermore, we know explicitly a constant C which this kernel is bounded with via $\|V_n f\| \leq C\|f\|$. Then we take a look at general classes of kernels with certain localisation properties. Again, we take the cases in the Jacobi setting as inspiration and try to derive analogue results with disc polynomials. For the result on almost exponential localisation we followed the example of Petrushev and Xu [27] and for exponentially localised kernels we followed [11]. The de-la-Vallée-Poussin kernel derived in this thesis could possibly be employed as a concrete kernel in these localisation problems. It has the requested properties in the derived theorems.

These results are new as well as another result: An explicit formula for the linearisation coefficients for the disc polynomials for the parameter $\alpha = 0$. Linearisation coefficients of orthogonal polynomial systems are interesting for hypergroup theoretical questions if they are all non-negative. In case of the disc polynomials, linearisation coefficients are the particular coefficients of the disc polynomials that are needed to represent the product of two disc polynomials as a linear combination of disc polynomials:

$$P_{m,n}P_{k,l} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g(m,n;k,l;i,j)P_{i,j} \ .$$

$P_{i,j}$ are the disc polynomials and $g(m,n;k,l;i,j)$ are the specific linearisation coefficients for the product of the (m,n)-th disc polynomial with the (k,l)-th.

Outline of the thesis

First, the classical Fourier Analysis with focus on approximation kernels is presented in chapter 2. In many following chapters kernels with a reference to well known classical kernels are presented. That's why an introducing chapter describes where all kind of approximation theory with kernels come from.

Chapter 3 gives a short compendium on Jacobi polynomials and generalised Fourier analysis with respect to the Jacobi polynomials. Besides Laguerre- and Hermite-Polynomials, the Jacobi polynomials are the best known orthogonal polynomials. Nevertheless, we introduce them formally and show some basic properties. On the one hand they are needed to define the disc polynomials. On the other hand we want to introduce a well-known system of orthogonal polynomials to indicate the way we want to continue with the disc polynomials. We will keep it very short there and show how everything is formally done.

Finally, we present recent results on approximation with respect to the Jacobi polynomials that have been made in the last years by different people. We want to transfer these results to our disc polynomial setting.

In chapter 4 we introduce the disc polynomials and their natural environment, the unit disc. Interesting properties and simple corollaries of the disc polynomials are given, as for example differential equations the disc polynomials satisfy and special values they have. If similar properties exist for the Jacobi polynomials, cross references to chapter 2 are given. A proof different from the classical is given for the property that the maximal absolute value of every disc polynomial on the disc equals one. Essential structures we work with like the Lebesgue spaces, the generalised translation and the resulting convolution are introduced. They are of major importance for our purposes. In addition, simple properties are collected which have no direct bearing on the approximational aspects of this work.

Chapter 5 comes up with a new result, an explicit formula for the linearisation coefficients of the disc polynomials for the parameter $\alpha = 0$. The proof is given explicitly in detail because the idea of the proof has no analogon in other proofs of linearisation coefficients for orthogonal polynomials.

Finally chapter 6 concludes with the results on approximation on the unit disc. The formal tools for Fourier series with respect to disc polynomials are introduced. After that the promised uniformly bounded generalised de-la-Vallée-Poussin kernel is derived. The result on almost exponentially localised kernels with respect to disc polynomials is not satisfying as we can not differentiate two values that have the same radial part but are not equal. Therefore it is pleasing to present the result on exponentially localised kernels with disc polynomials. Exponential localisation for sure is a stronger result than almost exponential localisation. Together with our generalised de-la-Vallée-Poussin kernel an example for an exponentially localised kernel is given. An upper bound for the operator is explicitly given in this case.

Theorems containing new results are marked by (*) in the thesis.

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2 Classical Fourier analysis approximation

Fourier series are well known in many branches of mathematics. Questions on convergence are mostly treated only shortly or superficially, except in the domain of harmonic analysis. In this chapter we will give a short overview of the most important concepts of Fourier analysis. Functions with certain properties, so called kernels, help to guarantee that all functions from certain function spaces can be approximated (norm convergence). Some of these summability kernels are introduced here in their classical appearance. When classical Fourier analysis is carried over to other settings the design of these classical kernels serve as a model for analogue kernels in those settings. Therefore, a slight idea of those classical kernels will be given here.

Interested readers can find valuable information on the whole wide theory of harmonic analysis in several introductive works such as "Introduction to Fourier analysis" (Lasser) [23], "An introduction to harmonic analysis" (Katznelson) [17] or "Fourier analysis and approximation" (Butzer and Nessel) [6].

2.1 Fourier series on $\mathbb{T}$

Classical Fourier analysis takes place on the one dimensional torus $\mathbb{T}$. This means we consider 2π-periodic functions ($f(x) = f(x + k \cdot 2\pi), k \in \mathbb{Z}$) and the interval $[0\ , 2\pi)$ is a model of $\mathbb{T}$. Thus, we have:

$$\int_{\mathbb{T}} f(x)\ dx = \int_0^{2\pi} f(x)\ dx\ .$$

Shifting a function on the torus is a fundamental process playing a major role in the theory of Fourier analysis approximation.

Definition 2.1 (translation operator). *For a real number $y \in [0, 2\pi)$ the translation operator T_y acts on the argument of a function $f \in L^1(\mathbb{T})$ and is given by*

$$T_y f(x) = f(x - y)\ .$$

The 2π-periodicity of functions $f \in L^1(\mathbb{T})$ gives the following translation invariance of a function that is integrated over $\mathbb{T}$:

$$\int_{\mathbb{T}} T_y f(x)\ dx = \int_{\mathbb{T}} f(x-y)\ dx = \int_{\mathbb{T}} f(x)\ dx\ .$$

The function spaces we are working on are of course the usual Lebesgue-spaces on $[\ 0\ , 2\pi)$ with the usual measure and the familiar standard norms. In the sequel we will use the notation

$$X^p(\mathbb{T}) = \begin{cases} L^p(\mathbb{T}), & \text{if } 1 \le p < \infty \\ C(\mathbb{T}), & \text{if } p = \infty \ . \end{cases}$$

A non-trivial result is the fact that the series of exponential functions $e_n(x)$ given in the next definition constitute a basis on the spaces $X^p(\mathbb{T}), (1 \le p \le \infty)$. The proof will be omitted.

Definition 2.2 (orthogonal basis on $\mathbb{T}$). *The so called basis functions for $\mathbb{T}$ are*

$$e_n(x) = e^{inx} \ , \ \textit{with } x \in \mathbb{T}, n \in \mathbb{Z} \ .$$

It is easy to see that in the Hilbert space $L^2(\mathbb{T})$ with the usual inner product, given by

$$< f \ , \ g > = \int_{\mathbb{T}} f(x)\overline{g(x)} \ dx$$

holds the following L^2-orthogonality relation:

$$< e_n \ , \ e_m > = 2\pi \cdot \delta_{m,n} \ .$$

Another basic property of the basis functions is the fact that a *product formula* exists with respect to the translation operator, i.e. the translation of a basis function can alternatively be written as a product of two basis functions. It is easy to see that the following formula holds:

$$T_y e_n(x) = e_n(x-y) = e_n(x) \cdot \overline{e_n(y)} \ .$$

Taking the role of monomials for defining usual polynomials the basis functions are taken to define trigonometric polynomials.

Definition 2.3 (trigonometric polynomial). *For $N \in \mathbb{N}$ a trigonometric polynomial on $\mathbb{T}$ is an expression of the form*

$$P(x) = \sum_{n=-N}^{N} a_n e_n(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

with $0 \ne a_n \in \mathbb{C}$. P has degree N.

One can easily see with the orthogonality relation of the basis functions that for a given trigonometric polynomial $P(x)$ the coefficients a_n can be computed by

$$a_n = \frac{1}{2\pi} \int_{\mathbb{T}} P(x) \ e_{-n}(x) \ dx = \frac{1}{2\pi} < P, e_n > \ .$$

This way of computing the coefficients of a trigonometric polynomial serves as idea of expanding a general function $f \in L^1(\mathbb{T})$ into a Fourier series or a partial Fourier series (where the result is a trigonometric polynomial).

Definition 2.4 (Fourier coefficient). *Let $f \in L^1(\mathbb{T})$. For $n \in \mathbb{Z}$ the n-th Fourier coefficent of f is given by the formula*

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x)\ e^{-inx}\ dx = \frac{1}{2\pi} < f, e_n > \quad .$$

Definition 2.5 (Fourier series). *Let $f \in L^1(\mathbb{T})$. The formal Fourier expansion of the function f is given by*

$$S(f)(x) := \sum_{n=-\infty}^{\infty} c_n e_n(x) \qquad \textit{with } c_n = \frac{1}{2\pi} \int_{\mathbb{T}} f(x)\ e_{-n}(x)\ dx = \hat{f}(n) \ .$$

The N-th partial Fourier series is given by the formula

$$S_N(f)(x) := \sum_{n=-N}^{N} c_n e_n(x) = \sum_{n=-N}^{N} \hat{f}(n) e_n(x) \ .$$

With the help of the translation operator and the product formula for the basis functions $e_n(x)$ the partial Fourier series can be written with respect to these two concepts as

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n) e_n(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{\mathbb{T}} f(y)\ \overline{e_n(y)}\ e_n(x) dy \\ &= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{\mathbb{T}} f(y)\ e_n(x-y) dy = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{\mathbb{T}} f(y)\ T_y e_n(x) dy \ . \end{aligned}$$

The first integral in the second line may look familiar to the experienced eye and indeed, this is the usual definition of the convolution between the function f and the basis function e_n:

Definition 2.6 (convolution). *Let $f, g \in L^1(\mathbb{T})$. The convolution $*$ of the function f with the function g is given by*

$$(f * g)(x) := \frac{1}{2\pi} \int_{\mathbb{T}} f(y) g(x-y) dy = \frac{1}{2\pi} \int_{\mathbb{T}} f(y)\ T_y g(x) dy \quad .$$

An important fact for our purposes is the commutativity of the translation:

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(y)g(x-y)dy = \frac{1}{2\pi}\int_{\mathbb{T}} f(x-y)g(y)dy \ .$$

Further easy results on the convolution are the associativity, linearity with respect to f and g, continuity on L^p and the identity $\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$. The proofs and further properties of the convolution can be found in every standard analysis book.
Our purpose with respect to the convolution is using it to rewrite the partial Fourier series:

$$S_N(f)(x) = \frac{1}{2\pi}\sum_{n=-N}^{N} \int_{\mathbb{T}} f(y)\ e_n(x-y)dy = \sum_{n=-N}^{N} (f*e_n)(x) \ .$$

Finally, we define the so called N-th Dirichlet kernel by $D_N(x) := \sum_{n=-N}^{N} e_n(x)$ and write partial Fourier series in a compact way as the convolution of the function f with the N-th Dirichlet kernel D_N:

$$S_N(f)(x) = (D_N * f)(x)$$

which leads to the definition of the Dirichlet kernel.

Definition 2.7 (Dirichlet kernel). *For $N \in \mathbb{N}$ the N-th Dirichlet kernel D_N is given by the summation of the basis functions e_n:*

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) \ .$$

With the help of simple calculations, rearrangements and the Euler's formula one can find an explicit closed formula for the Dirichlet kernel:

$$D_N(x) = \frac{\sin\left((N+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)} \ .$$

Proof:

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{inx} = e^{-iNx}\sum_{n=0}^{2N}(e^{ix})^n = e^{-iNx}\left(\frac{e^{i(2N+1)x}-1}{e^{ix}-1}\right) = \frac{e^{i(N+1)x}-e^{-iNx}}{e^{ix}-1} \\ &= \frac{e^{i(N+\frac{1}{2})x}-e^{-i(N+\frac{1}{2})x}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}} = \frac{e^{i(N+\frac{1}{2})x}-e^{-i(N+\frac{1}{2})x}}{2i}\frac{2i}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}} = \frac{\sin\left((N+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)} \end{aligned}$$

Thus, we can represent the N-th partial Fourier series of a function f as an integral equation by using the explicit formula for the Dirichlet kernel:

$$S_N\left(f\right)\left(x\right) = \frac{1}{2\pi}\int_{\mathbb{T}} D_N(x-y)f(y)dy = \frac{1}{2\pi}\int_{\mathbb{T}} D_N(y)f(x-y)dy$$

Next, the question that is of interest is whether norm convergence is achieved with Fourier series. In the following chapter we will give a criterion for summability kernels such that $k_n * f \to f$ in norm. This is necessary as the Fourier series of an L^1-function does not converge in general.

2.2 Approximation with summability kernels

For each $N \in \mathbb{N}$ the convolution of a function $f \in X^p(\mathbb{T})$ $(1 \leq p \leq \infty)$ with the N-th element of the sequence of Dirichlet kernels $(D_N)_{N\in\mathbb{N}}$ returns a trigonometric polynomial $(D_N * f)(x)$. From the previous section we know that the trigonometric polynomials are dense in $X^p(\mathbb{T})$ ($(e_n)_{n\in\mathbb{N}}$ is a basis). Thus, the question if the limit of the sequence $(D_N * f)(x)$ converges to f arises naturally. One can furthermore ask for pointwise or norm convergence. But our interest will be the norm convergence: does the following limiting process hold: $\|D_N * f - f\|_{X^p(\mathbb{T})} \longrightarrow 0, \quad (N \longrightarrow \infty)$?

It turned out however, that this is only true for values $1 < p < \infty$. The proof is beyond the scope of this short introduction. Modifications of the Dirichlet kernel D_N may in some cases now lead to a so called *summability kernel*, such that norm convergence for approximation with those modified kernels is established for $1 \leq p \leq \infty$. Most of these modifications work (mathematically incorrect speaking) the way, that a bit "less" of most basis functions e_n is taken: The Dirichlet kernel sums up the basis functions $D_N = \sum_{i=-N}^{N} a_i e_i$ with $a_i = 1$, in contrast for summability kernels "most" $a_i < 1$.

A whole class of kernels with which norm convergence is achieved is known as summability kernels. A summability kernel needs to fulfill three criteria, which are concretised in the following.

Definition 2.8 (summability kernel). *A sequence $\big(k_n(x)\big)_{n\in\mathbb{N}} \in L^1(\mathbb{T})$ is a summability kernel if it satisfies the following three properties:*

$$(S1) \qquad \frac{1}{2\pi}\int_{\mathbb{T}} k_n(x)\ dx = 1$$

$$(S2) \qquad \frac{1}{2\pi}\int_{\mathbb{T}} |k_n(x)|\ dx = \|k_n\|_{L^1(\mathbb{T})} \leq const$$

$$(S3) \qquad \textit{For all } 0 < \delta < \pi: \quad \lim_{n\to\infty}\int_{\delta}^{2\pi-\delta} |k_n(x)| dx = 0\ .$$

Property $(S2)$ is already fulfilled by $(S1)$ for positive kernels and we have an equality with $const = 1$. Furthermore we sometimes consider summability kernels depending on a continuous parameter $r \in (0, 1)$. The limit in $(S3)$ is then replaced by letting either $r \to 1$ or in some cases $r \to 0$.

As mentioned before the convolution with the Dirichlet kernel fails to approximate some L^1-functions. Regarding the three properties a summability kernel needs to fulfill, the Dirichlet kernel D_N does neither fulfill $(S2)$ nor $(S3)$. But of course these are the properties that finally guarantee the norm convergence of the sequence $(k_n * f)$ towards f. And this is the content of the following theorem: the convolution of a summability kernel with a function $f \in X^p(\mathbb{T})$ converging to f in norm with $n \longrightarrow \infty$.

Theorem 2.9 (summability in X^p). *Let k_n be a summability kernel and $f \in X^p(\mathbb{T})$ for $1 \leq p \leq \infty$. Then $k_n * f$ converges to f in norm, i.e.*

$$\left\|k_n * f - f\right\|_{X^p(\mathbb{T})} \to 0 \quad , \quad \textit{for} \quad n \to \infty \ .$$

Proof (sketch):
In the following proof we need two well-known facts we don't proof:
1. Minkowski inequality:

$$\left\| \int_Y f(x,y) d\nu(y) \right\|_{L^p(X)} \leq \int_Y \left\|f(x,y)\right\|_{L^p(X)} d\nu(y) \ .$$

2. Continuity of the translation T_y: For $1 \leq p \leq \infty$ and $f \in X^p(\mathbb{T})$:

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x, y \in \mathbb{T} \ \text{ with } \ |x-y| < \delta \text{ holds } \left\|T_y f - T_x f\right\|_{X^p(\mathbb{T})} < \varepsilon \ .$$

Then we can start to estimate the norm:

$$\left\|k_n * f - f\right\|_{L^p(\mathbb{T})} = \left(\frac{1}{2\pi} \int_{\mathbb{T}} |k_n * f(x) - f(x)|^p dx \right)^{\frac{1}{p}}$$

with property $(S1)$ of k_n we can write $f(x) = \int_{\mathbb{T}} k_n(y) f(x) \ dy$ and continue

$$= \left(\frac{1}{2\pi} \int_{\mathbb{T}} \left(\frac{1}{2\pi} \int_{\mathbb{T}} k_n(y)(f(x-y) - f(x)) dy \right)^p dx \right)^{\frac{1}{p}}$$

$$\leq \left\| \frac{1}{2\pi} \int_{\mathbb{T}} |k_n(y)||f(\cdot - y) - f(\cdot)| dy \right\|_{L^p(\mathbb{T})}$$

$$\overset{Minkowski}{\leq} \frac{1}{2\pi} \int_{\mathbb{T}} |k_n(y)| \ \left\|T_y f - f\right\|_{L^p(\mathbb{T})} \ dy \ .$$

At that point we have to split the integral in the following way:

$$\int_{\mathbb{T}} = \int_0^{2\pi} = \int_{-\pi}^{\pi} = \int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} = \int_{-\delta}^{\delta} + \int_{\delta}^{2\pi-\delta} \ .$$

Then we have

$$\frac{1}{2\pi} \int_{-\delta}^{\delta} |k_n(y)| \ \left\|T_y f - f\right\|_{L^p(\mathbb{T})} \ dy \leq const \cdot \varepsilon$$

by continuity of T_y and $(S2)$ of k_n as well as

$$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |k_n(y)| \left\|T_y f - f\right\|_{L^p(\mathbb{T})} dy \leq \frac{1}{\pi} \left\|f\right\|_{L^p(\mathbb{T})} \cdot \varepsilon$$

by property $(S3)$ for summability kernels k_n.

Many different summability kernels are known and have been intensively studied in Fourier analysis literature. In the next chapter we will give a few examples of summability kernels with a variety of designs.

2.3 Examples of summability kernels

In the introduction of this work it has already been said that the Fourier analysis approximation on the torus has been adapted in many other settings by taking an appropriate basis for the domain of the alternative setting. The focus of this work is on the domain of the unit disc. For better understanding what is happening here it is useful to know the standard case. Right here we try to adopt the standard approximation kernels.

This chapter aims at giving an overview of the design of different types of summability kernels. In this section we will not go to deep into details, not prove every mentioned fact and keep proofs short. The main focus is on the architecture of the presented kernel.

The first summability kernel we will take a closer look at is the Fejér kernel, surely one of the best known summability kernels in Fourier approximation.

Definition 2.10 (Fejér kernel). *The Fejér kernel is given by taking the arithmetic mean of the the Dirichlet kernels $D_0, ...D_N$:*

$$F_N(x) = \frac{1}{N+1}\sum_{j=0}^{N} D_j(x) = \sum_{j=-N}^{N}\left(1 - \frac{|j|}{N+1}\right) e_n(x)\ .$$

Like the Dirichlet kernel, the Fejér kernel has a simple closed formula:

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin\left(\frac{N+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}\right)^2\ .$$

Proof:

With the closed formula for the Dirichlet kernel we can write

$$F_N(x) = \frac{1}{N+1}\sum_{j=0}^{N} D_j(x) = \frac{1}{N+1}\sum_{j=0}^{N}\frac{\sin\left((j+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)}$$

$$= \frac{1}{N+1}\sum_{j=0}^{N}\left(\frac{1}{2}\cdot\frac{\sin\left((j+\frac{1}{2})x\right)\sin\left(\frac{x}{2}\right)}{\left(\sin\left(\frac{x}{2}\right)\right)^2} + \frac{1}{2}\cdot\frac{\sin\left((j+\frac{1}{2})x\right)\sin\left(\frac{x}{2}\right)}{\left(\sin\left(\frac{x}{2}\right)\right)^2}\right)\ .$$

Then the addition theorem for arguments of the cosine is employed:
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$ and we can go on like

$$= \frac{1}{N+1} \sum_{j=0}^{N} \frac{\cos(jx) - \cos\left((j+1)x\right)}{2\left(\sin\left(\frac{x}{2}\right)\right)^2} = \frac{1}{N+1}\left(\frac{1-\cos((N+1)x)}{2\left(\sin\left(\frac{x}{2}\right)\right)^2}\right)$$

$$= \frac{1}{N+1}\left(\frac{1-\cos\left(\frac{N+1}{2}x + \frac{N+1}{2}x\right)}{2\left(\sin\left(\frac{x}{2}\right)\right)^2}\right) = \frac{1}{N+1}\left(\frac{1-\cos\left(\left(\frac{N+1}{2}x\right)\right)^2 + \left(\sin\left(\frac{N+1}{2}x\right)\right)^2}{2\left(\sin\left(\frac{x}{2}\right)\right)^2}\right)$$

$$= \frac{1}{N+1}\left(\frac{2\left(\sin\left(\frac{N+1}{2}x\right)\right)^2}{2\left(\sin\left(\frac{x}{2}\right)\right)^2}\right) = \frac{1}{N+1}\left(\frac{\sin\left(\frac{N+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}\right)^2 .$$

With this formula one can easily see that the Fejér kernel is positive and the critera $(S1)$ and $(S3)$ for summability kernels are given for the Fejér kernel. The resulting sequence of the convolution with a function f converges in norm towards f and looks as follows:

$$(F_N * f)(x) = \frac{1}{N+1} \sum_{j=-N}^{N} S_j(f)(x) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j) e_j(x) .$$

The classical de-la-Vallée-Poussin kernel $V_N(x)$ is a summability kernel which is a composition of Fejér kernels. It has the property to be able to reproduce trigonometric polynomials of degrees not exceeding N and the kernel itself is a trigonometric polynomial of degree $2N+1$. Generalisations of the de-la-Vallée-Poussin kernel exist and are introduced in chapter 3.3.

Definition 2.11 (de-la-Vallée-Poussin kernel). *The classical de-la-Vallée-Poussin kernel is given by*

$$V_N(x) = 2 \cdot F_{2N+1}(x) - F_N(x) = \frac{1}{N+1} \sum_{j=N+1}^{2N+1} D_j(x) .$$

The summability kernel properties $(S1) - (S3)$ for the de-la-Vallée-Poussin kernel are easy to proof by using the known closed formulas and properties of the Fejér and Dirichlet kernel. The resulting sequence has the form:

$$(V_N * f)(x) = \sum_{j=-N}^{N} \hat{f}(j) e_j(x) + \sum_{|j|=N+1}^{2N+1} \left(1 - \frac{|j| - N - 1}{N+1}\right) \hat{f}(j) e_j(x) .$$

The next two kernels are examples of summability kernels where the approximation parameter r is continous. The easiest and best known is the Poisson kernel.

Definition 2.12 (Poisson kernel). *For $0 < r < 1$ the Poisson kernel is given by*

$$P_r(x) = \sum_{j=-\infty}^{\infty} r^{|j|} e_j(x) \ .$$

The Poisson kernel has the closed formula

$$P_r(x) = \frac{1-r^2}{1-2r\cos(x)+r^2} \ .$$

Proof:

$$P_r(x) = \sum_{j=-\infty}^{\infty} r^{|j|} e_j(x) = \sum_{j=0}^{\infty} \left(re^{ix}\right)^j + \sum_{j=0}^{\infty} \left(re^{-ix}\right)^j - 1 = \frac{1}{1-re^{ix}} + \frac{1}{1-re^{-ix}} - 1$$

by the limit of the geometric series; further using the identities:

$$(1-re^{ix})(1-re^{-ix}) = |1-re^{ix}|^2 = 1 - r(e^{ix}+e^{-ix}) + r^2$$

$$\text{and} \quad 2\cos(x) = e^{ix}+e^{-ix} \quad \text{we can conclude}$$

$$= \frac{1-re^{-ix}}{|1-re^{ix}|^2} + \frac{1-re^{ix}}{|1-re^{ix}|^2} - \frac{1-r(e^{ix}+e^{-ix})+r^2}{|1-re^{ix}|^2} = \frac{1-r^2}{1-2r\cos(x)+r^2} \ .$$

The properties $(S1)-(S3)$ of summability kernels can be seen here without any problem, too. With the closed formula one can see that the Poisson kernel is positive. And $\sup\limits_{\delta<x<\pi} |P_r(x)| \leq \frac{1-r^2}{1-2r\cos(\delta)+r^2} \longrightarrow 0$ as $r \to 1$. The sequence that approximates a function f is of the form:

$$\left(P_r * f\right)(x) = \sum_{j=-\infty}^{\infty} r^{|j|} \ \hat{f}(j) \ e_j(x) \ .$$

In this formula one sees that a limiting process of the parameter r towards 1 would end up in the usual Fourier series. Thus, parameter values r close to 1 are interesting for approximation.

The second example of a summability kernel with a continuous parameter s is the Gauss kernel. The Gauss kernel became rather popular in statistics. But in an appropriate form it can be used for approximation. The resulting approximation of a function f would turn into the Fourier series $S(f)$ for $s = 0$. Therefore, small values for s are interesting for approximation.

Definition 2.13 (Gauss kernel). *For $s > 0$ the Gauss kernel is given by*

$$G_s(x) = \sum_{j=-\infty}^{\infty} e^{-j^2 s} \; e_j(x) \; .$$

Without proof we give another representation of the Gauss kernel:

$$G_s(x) = \frac{2\pi}{\sqrt{4\pi s}} \sum_{j=-\infty}^{\infty} e^{-\frac{(x+2\pi j)^2}{4s}} \; .$$

The Gauss kernel is also a positive kernel and the summability kernel properties are similar to see as it is the case for the Poisson kernel. For approximating a suitable function f with the help of the Gauss kernel we form the convolution:

$$\left(G_s * f\right)(x) = \sum_{j=-\infty}^{\infty} e^{-j^2 s} \; \hat{f}(j) \;\; e_j(x) \; .$$

Finally we give the example of a summability kernel that has again a discrete paramater n. A composition of Dirichlet kernels together with the discrete manipulation of the arguments lead to the so called Rogosinski kernel.

Definition 2.14 (Rogosinski kernel). *The Rogosinski kernel is given by*

$$R_n(x) = \frac{1}{2}\left(D_n\left(x + \frac{\pi}{2n+1}\right) + D_n\left(x - \frac{\pi}{2n+1}\right)\right) \; .$$

Without proofs we state that the Rogosinski kernel is a summability kernel and alternative representations are:

$$R_n(x) = 1 + 2\sum_{j=1}^{n} \cos\left(\frac{j\;\pi}{2n+1}\right)\cos(jx) = \sum_{j=-n}^{n} \cos\left(\frac{j\;\pi}{2n+1}\right) e_j(x) \; .$$

The resulting approximation of a function f is given by

$$\left(R_n * f\right)(x) = \sum_{j=-n}^{n} \cos\left(\frac{j\;\pi}{2n+1}\right) \hat{f}(j) \; e_j(x) \; .$$

The following figures show the plots of the discussed kernels as well as their way of approximating a simple function. For the kernels that are trigonometric polynomials we have chosen the parameter n such that all the kernels have degree 21 (pay attention to the de-la-Vallée-Poussin kernel).

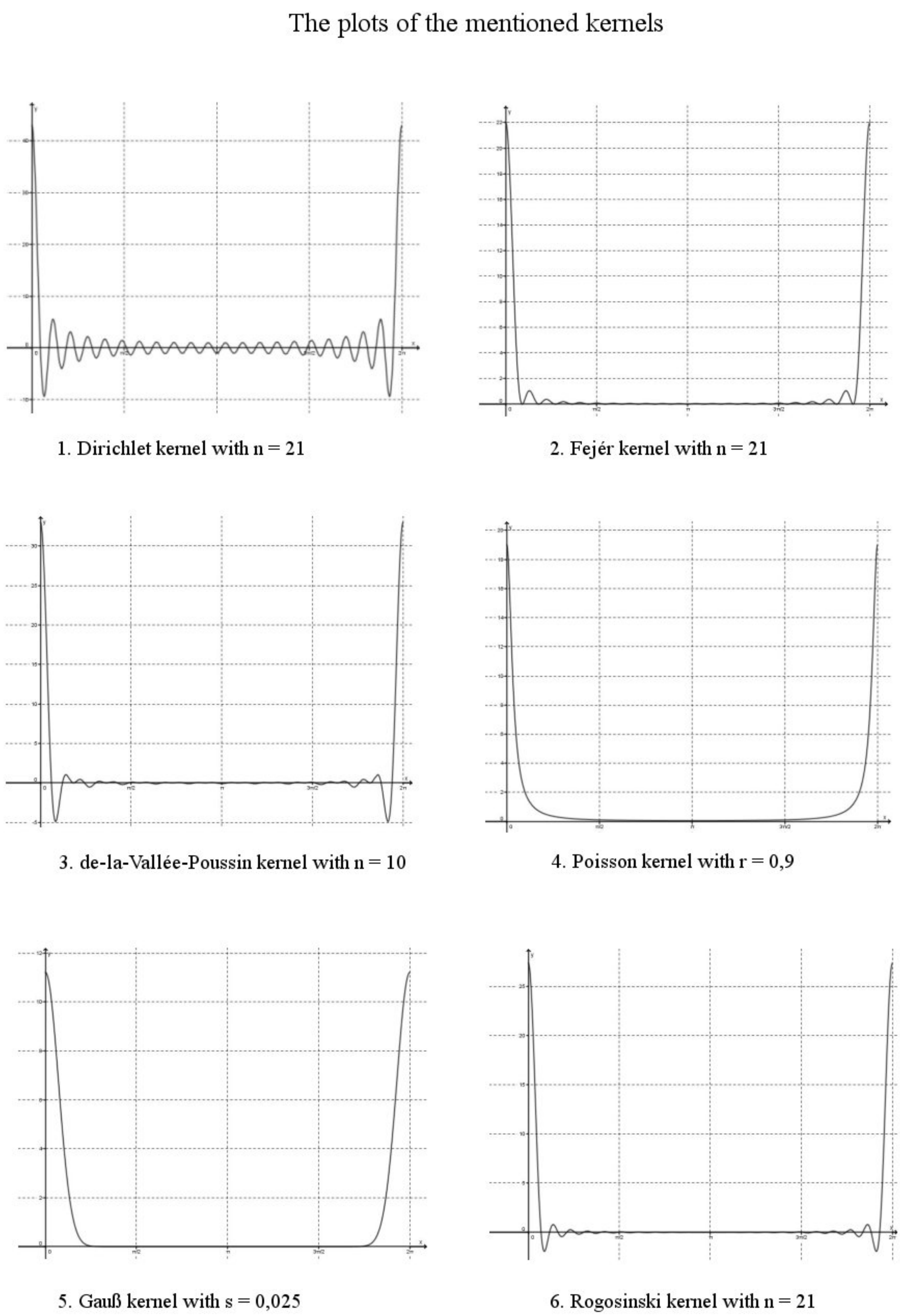

Figure 1: the kernels

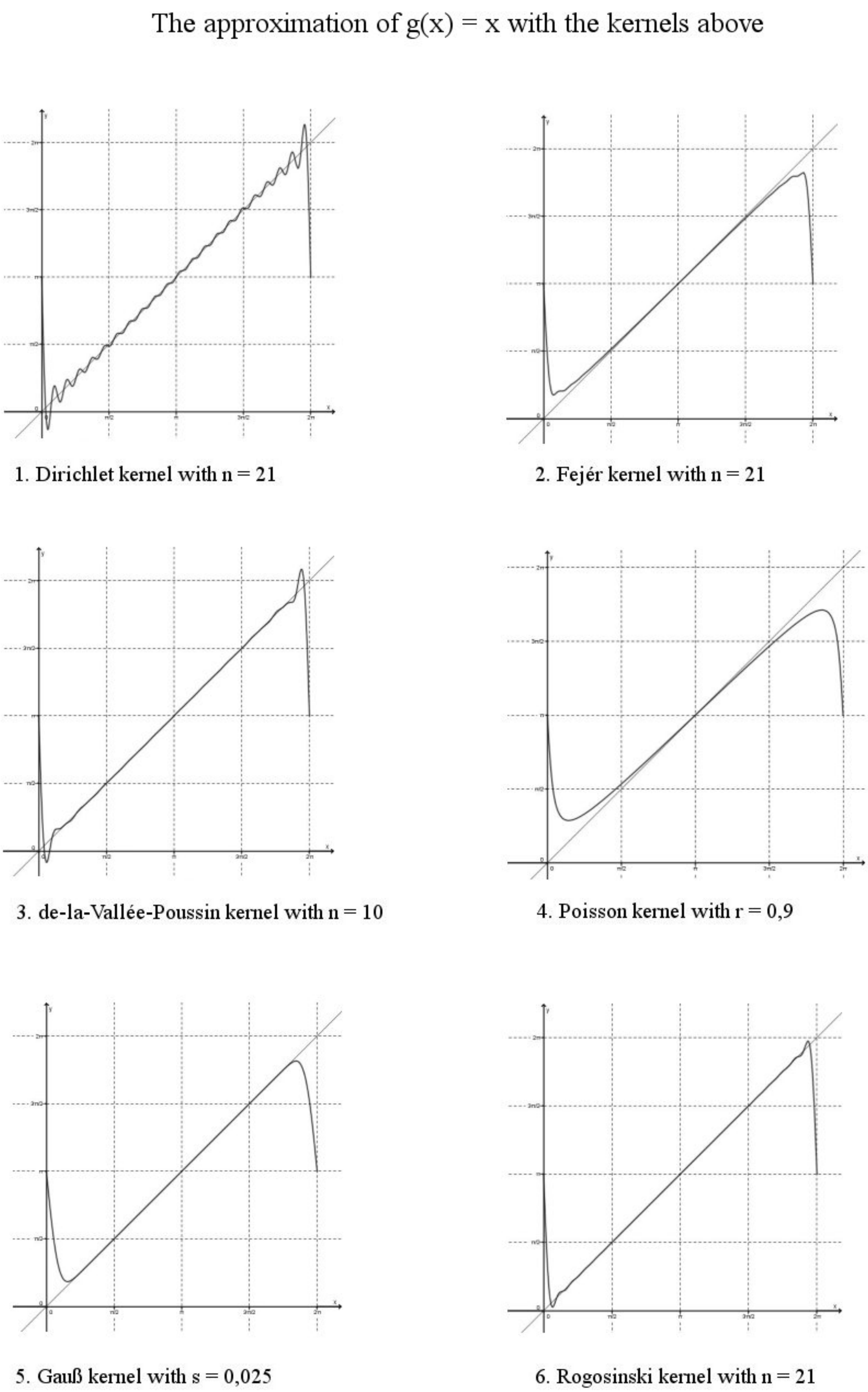

Figure 2: the approximation of the linear function

3 Fourier analysis and approximation with Jacobi polynomials

3.1 Notations and preliminaries

In this section we give some basic definitions of functions and notations essential for this work. Often we will give them in an extended or generalized manner in order to answer non-obvious notational questions and to use them in the following work without any further explanation. Also, we avoid generalizations if they are not needed for our purposes.

The well-known **Gamma function** is given by the formula $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$, in which $x \in \mathbb{R}_+$. The functional equation $\Gamma(x+1) = x \cdot \Gamma(x)$ together with the value of the Gamma function at 1, $\Gamma(1) = 1$, gives a connection to the **factorial**, so that for $n \in \mathbb{N}$ we have $\Gamma(n+1) = n!$. This concept is extended and we define factorials of $\alpha > -1$:

Definition 3.1 (generalized factorial). *For $\alpha \in \mathbb{R}$, $\alpha > -1$, the generalized factorial $\alpha!$ is given by*

$$\alpha! := \Gamma(\alpha + 1) \ .$$

This notation will make the formulas that occur in this work more intuitive and easier to read. Another function in that context is the **Pochhammer symbol**. We give a definition for real numbers.

Definition 3.2 (generalized Pochhammer symbol). *For $\alpha \in \mathbb{R}$, and $k \in \mathbb{N}$ the generalized Pochhammer symbol is given by*

$$(\alpha)_k := \alpha \cdot (\alpha - 1) \cdot (\alpha - 2) \cdots (\alpha - k + 1) \ .$$

For illustration purposes we look at some special cases in the following

Corollary 3.3. *For $n, k \in \mathbb{N}, \alpha \in \mathbb{R}_+, \beta \in \mathbb{R} \setminus (-\mathbb{N}_0)$ holds:*

1) $(\alpha)_k = \dfrac{\alpha!}{(\alpha - k)!}$, *for* $\alpha > k - 1$
2) $(n)_k = 0$ *for* $n < k$
3) $(0)_k = 0$
4) $(-\beta)_k = (-1)^k (\beta + k - 1)_k$.

The **binomial coefficient** is usually defined with the help of the factorial, i.e. with our generalized factorial we would be able to define a generalized binomial coefficient for real numbers greater than -1. But with the help of the generalized Pochhammer symbol we are even able to define it for real numbers of any size.

Definition 3.4 (generalized binomial coefficient). *For $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$, the generalized binomial coefficient is given by*

$$\binom{\alpha}{k} := \frac{(\alpha)_k}{k!} \ .$$

In contrast to the usual binomial coefficient the generalized one can be zero and have negative values. Again we want to give a few examples.

Corollary 3.5. *For $n, k \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus (-\mathbb{N}_0)$ holds*

$$1) \qquad \binom{-\alpha}{k} = (-1)^k \binom{\alpha + k - 1}{k}$$

$$2) \qquad \binom{n}{k} = \begin{cases} \frac{n!}{(n-k)!k!} & \textit{for } n \geq k \\ 0 & \textit{for } n < k \ . \end{cases}$$

We will see in the sequel that these notations and functions are often used in the work with Jacobi and especially with disc polynomials.

3.2 Jacobi polynomials

Orthogonal polynomials play an important role in various fields of mathematics. The Jacobi polynomials are some of the most important orthogonal polynomials in one dimension and therefore intensively studied. Szegö's book "Orthogonal Polynomials" [33] is the standard work for the studies of one dimensional orthogonal polynomials and in the chapter on Jacobi polynomials the most common and basic properties are discussed. Although several excellent introductions on Jacobi polynomials exist we give here their most important properties and those we work with later on. We will not give any proofs in this section but only state the definitions and theorems.

Several equivalent formulas for the Jacobi polynomials exist and we want to give the three most common ones, especially showing later on that for the disc polynomials there are corresponding forms. In the first and the second line of the subsequent definition two different explicit forms are given and in the third line the so called "Rodrigues' formula" is stated.

Definition 3.6. *For $n \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta > -1$ the Jacobi polynomials are given by*

$$\begin{aligned} J_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+n)!}{n!(\alpha+\beta+n)!} \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha+\beta+n+k)!}{(\alpha+k)!} \left(\frac{x-1}{2}\right)^k \\ &= 2^{-n} \sum_{k=0}^{n} \binom{\alpha+n}{k} \binom{\beta+n}{n-k} (-1)^{n-k} (1-x)^{n-k} (1+x)^k \\ &= \frac{(-1)^n}{n!} 2^{-n} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left((1-x)^{\alpha+n} (1+x)^{\beta+n} \right) . \end{aligned}$$

This definition leads to a normalization such that $J_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n}$. In the interval $[-1, 1]$ this is the maximal absolute value in case $\alpha \geq \beta \geq -\frac{1}{2}$. For special values of α and β the Jacobi polynomials are known under different names. For example with $\alpha = \beta = 0$ they are called the **Legendre Polynomials**, for $\alpha = \beta = -\frac{1}{2}$ the **Chebyshev Polynomials of the first kind**, for $\alpha = \beta = \frac{1}{2}$ the **Chebyshev Polynomials of the second kind** and for $\alpha = \beta$ in general they are the so called **Gegenbauer** or **Ultraspherical Polynomials**.

They are of such importantance because they satisfy an **orthogonality relation** on the interval $[-1, 1]$ with respect to the weight function $w_{\alpha,\beta}(x) := (1-x)^\alpha (1+x)^\beta$, which implies that they constitute an orthogonal basis in the Hilbert space $L^2([-1, 1], w_{\alpha,\beta})$.

As the weight function for $\alpha, \beta > -1$ is positive on the interval $[-1, 1]$, define a Borel measure $\mu_{\alpha,\beta}(x) := w_{\alpha,\beta}(x)dx$ and one can formulate the orthogonality relation for two Jacobi polynomials via:

$$\int_{-1}^{1} J_k^{(\alpha,\beta)}(x) J_l^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) = h^{(\alpha,\beta)}(k)\ \delta_{k,l}$$

while $h^{(\alpha,\beta)}(k)$ is the so called **Haar function** which explicitly written looks like

$$h^{(\alpha,\beta)}(k) = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1} \frac{(k+\alpha)!(k+\beta)!}{k!(\alpha+\beta+k)!} \qquad \text{for } k \in \mathbb{N}$$

$$\text{and for } k = 0: \qquad h^{(\alpha,\beta)}(0) = 2^{\alpha+\beta+1} \frac{\alpha!\ \beta!}{(\alpha+\beta+1)!} .$$

Another property which has an analogon for the disc polynomials is that each Jacobi polynomial is a solution to a second-order differential equation. It is easy to check that $J_n^{(\alpha,\beta)}$ solves

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0 . \tag{3.1}$$

Furthermore the Jacobi polynomials satisfy a recurrence relation such that with our normalization, i.e. $J_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n}$, it holds

$$J_{n+1}^{(\alpha,\beta)}(x) = (A_n x + B_n) J_n^{(\alpha,\beta)}(x) - C_n J_{n-1}^{(\alpha,\beta)}(x) \tag{3.2}$$

with

$$\begin{aligned}
A_n &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)} \\
B_n &= \frac{(2n+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \\
C_n &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \ .
\end{aligned}$$

These basic properties of the Jacobi polynomials will be sufficient for our purposes.

3.3 Fourier approximation

The well known theory of Fourier Analysis on the interval $[0, 2\pi)$ with trigonometric functions is carried over to other settings. Now we are interested in the Fourier Analysis on the interval $[-1, 1]$ with respect to Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$. And so we will give an introduction to that situation and at the end of the chapter focus on kernels that are suitable for approximation in norm.

Due to the orthogonality weight of the Jacobi polynomials suitable **Lebesgue spaces** L^p are defined by:

For $1 \le p < \infty$ and $\alpha, \beta > -1$ we have

$$L^p([-1,1], w_{\alpha,\beta}) := \left\{ f : [-1,1] \to \mathbb{R} \;\middle|\; \|f\|_{p,\alpha,\beta} < \infty \right\}$$

$$\text{with} \quad \|f\|_{p,\alpha,\beta} := \left(\int_{-1}^{1} |f(x)|^p \; w_{\alpha,\beta}(x) \; dx \right)^{\frac{1}{p}} \ .$$

Now the function spaces X^p we will be working with are defined by

$$X^p = \begin{cases} L^p([-1,1], w_{\alpha,\beta}) \ , & \text{for } 1 \le p < \infty \\ C([-1,1]) \qquad\qquad , & \text{for } p = \infty \ . \end{cases}$$

And the $C([-1, 1])$-norm is given by the supremum. Furthermore for brevity we want to think of the values of α and β as fixed and $\alpha, \beta > -1$ and omit them in the formulas of this chapter except in cases where necessary.

We define the k-th **Fourier coefficient** for a function f on $[-1, 1]$:

$$\text{for } \ k \in \mathbb{N}_0 \ \text{ and } f \in L^1 \text{ we have } \ \hat{f}(k) := \int_{-1}^{1} f(x) J_n(x) w(x) dx$$

so that the n-th partial **Fourier series** of f is given by

$$S_n f(x) := \sum_{i=0}^{n-1} \hat{f}(i) \ J_i(x) \ \big(h(i)\big)^{-1}$$

while $h(i)$ is the Haar function corresponding to the Jacobi polynomials that was introduced in the last chapter.

In order to stay in analogy with the trigonometric case we need a translation operator with certain properties. The property which will be the most difficult one to get is the so called **product formula** the Jacobi polynomials need to fulfill. This means we want a translation operator T_x with which we have $T_x J_n(y) = J_n(y) J_n(x)$. Indeed for $\alpha + \beta > -1$ there exists a Borel measure $\mu_{x,y}$ on $[-1, 1]$ such that we have for $x, y \in [-1, 1]$ and $\forall n \in \mathbb{N}_0$ the following identity

$$J_n(x) J_n(y) = \int_{-1}^{1} J_n(z) d\mu_{x,y}(z) \ .$$

In [12] this formula is proved. There is a difference to our formula concerning the normalization of the Jacobi polynomials. Gasper used a normalization such that the Jacobi polynomials have the value 1 at $x = 1$:

$$R_n(x) := \frac{J_n(x)}{J_n(1)} \ .$$

For the Jacobi polynomials $\big(R_n(x)\big)_{n \in \mathbb{N}}$ the Haar function is different to the one we use in this thesis. And in addition, the measure for the translation for $R_n(x)$ is even a probability measure. But for our purposes the Jacobi polynomials $J_n(x)$ are more suitable. This is the reason why we will continue working with this normalization. Furthermore in [12] one can find the proof for the boundedness of this measure for various $\alpha, \beta > -1$ and several formulas for the measure itself.

With this measure we can define the **translation operator** T_x for $x, y \in [-1, 1]$ in order to fulfill the product formula for Jacobi polynomials:

$$T_x f(y) := \int_{-1}^{1} f(z) d\mu_{x,y}(z) \ .$$

As in the trigonometric setting we define a **convolution** $*$ between two functions with the help of the achieved translation. It is given by

$$\big(f * g\big)(x) := \int_{-1}^{1} f(y) \ T_x g(y) \ w(y) \ dy \ .$$

Indeed one can check that the following four properties that are required for a translation are satisfied:

1. $\forall n \in \mathbb{N}_0,\ x, y \in [-1,1]: \quad T_x J_n(y) = J_n(y) J_n(x)$ (product formula)
2. $\forall a, b \in \mathbb{R},\ x, y \in [-1,1]: \quad T_x\big(af + bg\big)(y) = aT_x f(y) + bT_x g(y)$ (linearity)
3. $\forall x \in [-1,1]: \quad \big(f * g\big)(x) = \big(g * f\big)(x)$ (commutativity)
4. $\forall x \in [-1,1],\ f \in L^p,\ 1 \le p \le \infty: \ \|T_x f\|_p \le C\|f\|_p$ (boundedness) and C does not depend on f or x. (with the normalization in [12] it holds $C = 1$)

Now we can write the n-th partial Fourier series of f as a convolution of f itself with a function D_n that is called the **Dirichlet kernel**. So if we want

$$\big(D_n * f\big)(x) = \int_{-1}^{1} f(y) T_x D_n(y) w(y) dy \overset{!}{=} \sum_{i=0}^{n-1} \hat{f}(i)\ J_i(x)\ \big(h(i)\big)^{-1}$$

one easily sees with the definition of the Fourier coefficient and the product formula for Jacobi polynomials that the Dirichlet kernel must be

$$D_n(x) := \sum_{i=0}^{n-1} J_i(x) \big(h(i)\big)^{-1} .$$

Now the approximation aspects of the Dirichlet kernel ($D_n * f \to f$ for $n \to \infty$) are not satified in all cases. The norm convergence of the Fourier series with respect to Jacobi polynomials has been thoroughly studied. First Pollard achieved a certain convergence result for $\alpha, \beta \ge -\frac{1}{2}$ [31], nonconvergence results can be found in [31] and [9]. Finally Muckenhoupt completely solved this convergence problem [26]. The result states that for $\alpha, \beta > -1$ and $\gamma := \max(\alpha, \beta, -\frac{1}{2})$ convergence depends on the value p of the Lebesgue space L^p: for $1 < p < \infty$ it holds:

$$\text{if } p \in \left(\frac{4(\gamma+1)}{2\gamma+3}, \frac{4(\gamma+1)}{2\gamma+1} \right): \quad \|D_n^{(\alpha,\beta)} * f - f\| \to 0 \quad \forall\ f \in L^p([-1;1], w_{\alpha,\beta}) .$$

For other values of p $\exists$ $f \in L^p([-1;1], w_{\alpha,\beta})$ such that $D_n^{(\alpha,\beta)} * f$ does not converge.

The proof of that convergence result is done by showing that the inequality $\|D_n * f\| \le C\|f\|$ holds with a constant C independent of f and n. This already suffices for the mean convergence result. Because convergence respectively nonconvergence of approximation kernels are mostly proved using a fundamental theorem from Banach and Steinhaus, which states that uniform boundedness is equivalent with convergence. The proof of this theorem using the uniform boundedness principle can be found in [23] (Theorem 5.1). From there we also cite the theorem:

Theorem 3.7. *Let B be a Banach space with norm $\|\cdot\|_B$. Also, let $(T_n)_{n\in\mathbb{N}}$ be a family of continuous linear operators of B into itself. For a dense subset M of B we assume that $\lim_{n\to\infty}\|T_n g - g\|_B = 0$ for all $g \in M$.*
Then the following two statements are equivalent:

(1) *For all $f \in B$ we have $\lim_{n\to\infty}\|T_n f - f\|_B = 0$*

(2) $\|T_n f\|_B \le C\|f\|_B$ *for all $f \in B, n \in \mathbb{N}$, where C is a constant independent of f and n .*

The way this theorem can be used in approximation theory is the following. If it should be checked whether a kernel K_n can be used for approximation or not, the statement (2) about the uniform boundedness will be inspected. If a uniform bound is found the kernel K_n is suitable for approximation. If an unbounded function $C(n)$ and a function f is found with which we can show the inequality $\|K_n * f\| > C(n)\|f\|$ then functions f exist with which $K_n * f$ does not converge.

Often the set of polynomials is taken as the dense set M in the function spaces L^p. And kernels K_n are designed in that way that they reproduce polynomials up to a certain degree depending on n. Then the assumptions for Theorem 3.7 are fulfilled.

In the sequel we often prove norm estimates on $L^p(\Omega,\mu)$-spaces only for the values $p = 1$ and $p = \infty$ and interpolate the cases in between, i.e. for $1 < p < \infty$. The background is the following theorem that justifies this approach. We cite it from [23]:

Theorem 3.8 (Riesz-Thorin). *Let (X_i, γ_i) for $i = 1, 2$ be two measure spaces, and let S be a linear operator defined on all γ_1-simple functions and taking values in the space of γ_2-measurable functions on X_2. Given points (α_1, β_1) and (α_2, β_2) of the square $[0, 1] \times [0, 1]$ assume that simultaneously hold*

$$\|Sf\|_{\frac{1}{\beta_1},\gamma_2} \le M_1\|f\|_{\frac{1}{\alpha_1},\gamma_1}$$

and

$$\|Sf\|_{\frac{1}{\beta_2},\gamma_2} \le M_2\|f\|_{\frac{1}{\alpha_2},\gamma_1}$$

then for all interpolating values $\alpha = (1-\lambda)\alpha_1 + \lambda\alpha_2$, $\beta = (1-\lambda)\beta_1 + \lambda\beta_2$ with $0 < \lambda < 1$ we have

$$\|Sf\|_{\frac{1}{\beta},\gamma_2} \le M_1^{1-\lambda}M_2^{\lambda}\|f\|_{\frac{1}{\alpha},\gamma_1} \ .$$

The norms in the Riesz-Thorin theorem are given by $\|f\|_{p,\gamma} := \left(\int_X |f(x)|^p d\gamma(x)\right)^{\frac{1}{p}}$.

In classical introductions on Fourier Analysis on the torus after the treatment of the Dirichlet kernel one is automatically led to the **Fejér kernel** and the **de-la-Vallée-Poussin kernel** (besides many others). These two kernels in contrast to the Dirichlet kernel converge to a function f in norm ($\|K_n * f - f\| \to 0$) in the X^p-spaces for all values of p which fulfill $1 \leq p \leq \infty$ and therefore are important for approximation theory.

In the setting with Jacobi polynomials analogues of the Fejér (F_n) and de-la-Vallée-Poussin kernels (V_n) (for a $n \in \mathbb{N}_0$) are given by:

$$F_n(x) := \sum_{i=0}^{n-1} \left(1 - \frac{i}{n} \right) \left(h(i)\right)^{-1} J_i(x) = \frac{1}{n} \sum_{i=1}^{n} D_i(x)$$

$$V_n(x) := 2F_{2n}(x) - F_n(x) = \frac{1}{n} \sum_{i=n+1}^{2n} D_i(x) = D_{n+1}(x) + \sum_{i=n+2}^{2n} \left(2 - \frac{i}{n} \right) \left(h(i)\right)^{-1} J_i(x) \ .$$

Now the norm convergence of these two kernels in the Jacobi case is not given for every α, β and $1 \leq p \leq \infty$ but they can be modified in a way that norm convergence is achieved. For the Fejér kernel one works with one of the so called generalized **Cesàro means** C_α. Fejér-type kernels for Jacobi polynomials, considered from a different point of view can be found in [22]. In [22] the Fejér-type kernels are $D_n^2(x)$. This fits also in the concept of the de-la-Vallée-Poussin kernels, see below. But as our focus is on the de-la-Vallée-Poussin kernels we will concentrate on these. In [7] a de-la-Vallée-Poussin kernel with respect to Jacobi polynomials is introduced. It has a more general form than the classical trigonometric de-la-Vallée-Poussin kernel. A theorem concerning the norm convergence states the following:

For a de-la-Vallée-Poussin kernel of the form

$$V_m^n(x) := \frac{1}{2m+1} \sum_{i=n-m}^{n+m} D_i(x) \ , \quad n > m$$

with $-1 < \alpha, \beta < \frac{1}{2}$ and $n = cm$ where $n, m \in \mathbb{N}$ and a fixed $c > 1$ (often written as $n \sim m$) the boundedness of the kernel is given by

$$\|V_m^n * f\|_{p,\alpha,\beta} \leq C \|f\|_{p,\alpha,\beta} \ , \qquad 1 \leq p \leq \infty$$

with a constant C independent of n, m and f. But as one can see there is still a limitation towards α and β.

A different version of a de-la-Vallée-Poussin type kernel which provides norm convergence for any $\alpha, \beta > -1$ and $\alpha + \beta > -1$ and $1 \leq p \leq \infty$ is presented in [10]. Thus, the range for α and β is considerably bigger. The idea for the definition of this kernel comes from the trigonometric case. It is a well-known fact that in $L^1([0, 2\pi))$ we have for the Dirichlet kernel $D_n^t(x)$ the identity

$$D_n^t(x) = \frac{\sin\left((2n+1)\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \quad .$$

However, with help of induction and the trigonometric addition identity for the sine function $\big(\ \sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)\ \big)$ the following identity for Dirichlet kernels can be proved:

$$\sum_{i=n-m}^{n+m} D_i^t(x) = D_n^t(x) D_m^t(x) \ .$$

So now there is a change in view by stating that in the trigonometric case the de-la-Vallée-Poussin kernel is given by

$$\mathcal{V}_m^{t,n}(x) = \frac{D_n^t(x) D_m^t(x)}{2m+1} \quad \text{instead of} \quad V_m^{n,t}(x) = \frac{1}{2m+1} \sum_{i=n-m}^{n+m} D_i^t(x) \quad .$$

Of course that makes no difference in the trigonometric case, but in the Jacobi case this change of view makes a big difference. Together with the observation concerning the maximal value of the trigonometric Dirichlet kernel, that $\max\limits_{x\in[0,2\pi)} |D_m^t(x)| = 2m+1$, one carries this over to the Jacobi case, where the maximal value of the Dirichlet kernel is taken at $x = 1$. If we define a generalized de-la-Vallée-Poussin kernel for Jacobi polynomials by

$$\mathcal{V}_m^n(x) := \frac{D_n(x) D_m(x)}{D_m(1)} \ , \qquad n = cm; \ n, m \in \mathbb{N}; \ c > 1$$

we get the boundedness of this kernel for $\alpha, \beta > -1$ and $\alpha + \beta > -1$ with an explicit constant C, thus $\|\mathcal{V}_m^n * f\|_p \leq C\|f\|_p$, for $1 \leq p \leq \infty$.

Therefore it is clear that such a kernel is very powerful for approximation and it is worth trying to construct kernels of that type in other settings. This will be done for the disc and the disc polynomials in a later chapter of this work.

3.4 Localized Jacobi kernels

3.4.1 Almost exponentially localized kernels

A general form for almost exponentially localized kernels on the interval $[-1, 1]$ with respect to the weight function $w_{\alpha,\beta}(x)$ and the Jacobi polynomials is presented in [5, 27]. Kernels of the type

$$L_n^{\alpha,\beta}(x,y) = \sum_{i=0}^{\infty} \hat{a}\left(\frac{i}{n}\right) \left(h^{\alpha,\beta}(i)\right)^{-1} J_i^{(\alpha,\beta)}(x) J_i^{(\alpha,\beta)}(y)$$

with a function $\hat{a} \in C^\infty\big([0,\infty)\big)$ such that $\operatorname{supp} \hat{a} \subseteq [0, 2]$, $\hat{a} = 1$ on $[0, 1]$ and $0 \leq \hat{a} \leq 1$ on the interval $[1, 2]$ are analysed. The main result is the almost exponential localisation of this kernel which means that the following inequality is derived for $\alpha, \beta > -\frac{1}{2}$:

$$|L_n^{\alpha,\beta}(\cos\theta, \cos\phi)| \leq c_k \frac{n}{\sqrt{w_{\alpha,\beta}(n; \cos\theta)}\sqrt{w_{\alpha,\beta}(n; \cos\phi)}(1 + n|\theta - \phi|)^k}$$

for arbitrary $k \in \mathbb{N}$, for $0 \le \theta, \phi \le \pi$ and $w_{\alpha,\beta}(n;x) = (1 - x + n^{-2})^{\alpha+\frac{1}{2}}(1 + x + n^{-2})^{\beta+\frac{1}{2}}$ while there exists a constant c_k independent of n but depending on α, β and k. With the translation and the product formula for Jacobi polynomials introduced in the previous chapter one can write

$$L_n^{\alpha,\beta}(x,y) = T_y L_n^{\alpha,\beta}(x) := \sum_{i=0}^{n} \hat{a}\left(\frac{i}{n}\right)\left(h^{\alpha,\beta}(i)\right)^{-1} T_y J_i^{(\alpha,\beta)}(x)$$

and it is clear that polynomials of degree less or equal to n are reproduced. In fact the almost exponential localisation stems from the smoothness of the function $\hat{a}$. The idea of the proof is, that first it is shown that for any $k \in \mathbb{N}$

$$|L_n^{\alpha,\beta}(\cos\theta)| \le c_k \frac{n^{2\alpha+2}}{(1+n\theta)^k}, \quad \text{for} \quad 0 \le \theta \le \pi$$

and secondly the translation is applied on $L_n^{\alpha,\beta}(x)$. With the explicit formula for the translation the upper localisation of $L_n^{\alpha,\beta}(x,y)$ can be proved.

The next step, in achieving norm convergence of this kernel towards a function is the theorem that states that

$$\int_{-1}^{1} |L_n^{\alpha,\beta}(x,y)|^p w_{\alpha,\beta}(y) dy \le c_k \; \frac{n^{p-1}}{\left(w_{\alpha,\beta}(n;x)\right)^{p-1}}$$

for $k \ge 3\alpha + 3\beta + 5$, arguments $-1 \le x \le 1$ and exponents $1 \le p < \infty$. The constant c_k of course is independent of n. Now one can choose $p = 1$ and with this choice the kernel in the L^1-norm is uniformly bounded. But then it is easy to prove that

$$\|L_n^{\alpha,\beta} * f\|_{p,\alpha,\beta} \le c(\alpha,\beta)\|f\|_{p,\alpha,\beta}, \quad \text{for } 1 \le p \le \infty$$

with $c(\alpha,\beta)$ independent of n and with $\left(L_n^{\alpha,\beta} * f\right)(x) = \int\limits_{-1}^{1} L_n^{\alpha,\beta}(x,y) f(y) w_{\alpha,\beta}(y) dy$. With theorem (3.7) and the reproducing property for polynomials of degree n of the kernel the norm convergence $\|L_n^{\alpha,\beta} * f - f\| \to 0$ is satisfied.

3.4.2 Exponentially localized kernels

In [25, 11] a method is presented showing how to construct exponentially localized Jacobi kernels, which are kernels $K_n(x,y)$ bounded in the following way:

$$|K_n(x,y)| \le \tilde{c}\; n^{2\max(\alpha,\beta)+2} \exp(-c\; n\; \mathrm{dist}(x,y))$$

while the constants $c, \tilde{c}$ are independent of n and $\mathrm{dist}(x,y)$ is a suitable distance function. As in the proofs in which the translation T_x is employed, the range of α and β is limited to $\alpha, \beta > -\frac{1}{2}$. The idea of how to construct such a kernel is to take any reproducing summability kernel K_{2n} that reproduces polynomials of degree $2n$ and multiply it with a polynomial S_n of degree n that satisfies the condition $S_n(1) = 1$ and the inequality $|S_n(x)| \leq c_2\ \exp(-n\phi(1-x))$ for $|x| \leq 1$ and a nondecreasing function ϕ with certain properties. Then it is clear that the new kernel $\Phi_n(x) := S_n(x)K_{2n}(x)$ reproduces polynomials of degree $\leq n$. With the explicit formula for the translation in the Jacobi case the following two estimates can be made: For x $= \cos\theta$, $y = \cos\varphi\ \in [-1,1]$ it holds

$$|T_y\Phi_n(x)| \leq c_2\|K_{2n}\|_\infty \exp\big(-n\phi(1-\cos(\theta-\varphi))\big)$$

and

$$\sup_{x\in[-1,1]} \int_{-1}^{1} |T_y\Phi_n(x)|w_{\alpha,\beta}(y)dy \leq c_2 \int_{-1}^{1} |K_{2n}(x)|w_{\alpha,\beta}(x)dx\ .$$

As S_n is uniformly bounded and K_{2n} is a reproducing summability kernel and therefore also uniformly bounded as a direct result of Theorem 3.7, the boundedness of Φ_n follows together with the most important property: the norm convergence. This results in a theorem that states that for $1 \leq p \leq \infty$ and a function $f \in X^p([-1,1], w_{\alpha,\beta})$ it holds:

$$\min_{P\in\Pi_{5n}} \|f-P\|_{p,\alpha,\beta} \leq \|f-\Phi_n * f\|_{p,\alpha,\beta} \leq c_2 \min_{P\in\Pi_n} \|f-P\|_{p,\alpha,\beta}$$

and Π_n is the set of all polynomials of degree $\leq n$. Since the set of polynomials is dense in each L^p-space this result tells something about the speed of the convergence in relation to an approximation with polynomials of a certain degree.

4 Disc polynomials and structures on the unit disc

4.1 Disc polynomials

Now we study the family of functions which shall be in the focus of our investigations. The **disc polynomials** are a series of two dimensional polynomials that are orthogonal on the unit disc with respect to a certain weight function.

The first investigations on orthogonal polynomials on the unit disc were made by Frits Zernike, the Dutch Nobel prize winner, who developed the so called **Zernike polynomials**. They are related to the disc polynomials, which in some documents are even called **generalized Zernike polynomials**. Zernike used them not only because of their orthogonality but also because of their properties in optics, namely modelling optical abberations like defocus, coma or astigmatism for example.

A good overview of the disc polynomials gave A. Wünsche in his work "Generalized Zernike or disc polynomials" [36]. Most of the definitions and properties we give in this section can also be found there. Wünsche gives analogons of many properties of classical one dimensional orthogonal polynomials.

Before we give the general definition of the disc polynomials we introduce some notations that shall be used in the rest of the work. Let z be a complex variable with its complex conjugate $\bar{z}$ and let the unit disc be given by $D := \{z = x + iy \in \mathbb{C} \mid |z| \leq 1\}$. Recall that $J_n^{(\alpha,\beta)}(x)$ is the n-th Jacobi polynomial.

Definition 4.1 (disc polynomials). *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}, \alpha > -1$ the (n, m)-th disc polynomial is given by*

$$P_{m,n}^{\alpha}(z, \bar{z}) = \begin{cases} \frac{n!\alpha!}{(n+\alpha)!} \; z^{m-n} \; J_n^{(\alpha,m-n)}(2z\bar{z} - 1) & \text{for } n \leq m \\ \frac{m!\alpha!}{(m+\alpha)!} \; \bar{z}^{n-m} \; J_m^{(\alpha,n-m)}(2z\bar{z} - 1) & \text{for } n > m \; . \end{cases}$$

This is the definition found in most of the literature on disc polynomials. One can also write them in terms of the hypergeometric function:

$$P_{m,n}^{\alpha}(z, \bar{z}) = z^m \bar{z}^n \; {}_2F_1\left(-m, -n; \alpha + 1; 1 - \frac{1}{z\bar{z}}\right) \; , \; \forall \; m, n \in \mathbb{N}_0.$$

We are also interested in the form how the disc polynomials explicitly look like. With the help of explicit formulas for the Jacobi polynomials one can easily check the following very important two formulas for the disc polynomials:

$$P_{m,n}^{\alpha}(z, \bar{z}) = \frac{m! \; n! \; \alpha!}{(m + \alpha)!(n + \alpha)!} \sum_{k=0}^{\min(m,n)} \frac{(-1)^k \; (m + n + \alpha - k)!}{k!(m - k)!(n - k)!} \; z^{m-k} \; \bar{z}^{n-k} \tag{4.1}$$

$$P_{m,n}^{\alpha}(z, \bar{z}) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k \; m! \; n! \; \alpha!}{k!(m - k)!(n - k)!(k + \alpha)!} \; (1 - z\bar{z})^k \; z^{m-k} \; \bar{z}^{n-k} \; . \tag{4.2}$$

As for the Jacobi polynomials one can give a Rodrigues' formula for the disc polynomials, which also is easy to verify by simple differential calculus rules and subsequent comparison with (4.2):

$$P^{\alpha}_{m,n}(z,\bar{z}) = \frac{(-1)^{m+n}\ \alpha!}{(m+n+\alpha)!}\ \frac{1}{(1-z\bar{z})^{\alpha}}\ \frac{\partial^{m+n}}{\partial\bar{z}^{m}\partial z^{n}}\ \big(1-z\bar{z}\big)^{m+n+\alpha}\ .$$

Remark: The so called Zernike polynomials $Z^n_m(r,\varphi)$ already mentioned before, have the following relation to the disc polynomials. They are given for $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ with $0 \le |m| \le n$ and $(n+|m|) \in 2\mathbb{N}_0$ in $[0,1] \times [0,2\pi)$ by

$$Z^n_m(r,\varphi) = \begin{cases} P^0_{(n+|m|)/2,(n-|m|)/2}(r,r)\cos(m\varphi) & \text{for } m \ge 0 \\ P^0_{(n+|m|)/2,(n-|m|)/2}(r,r)\sin(m\varphi) & \text{for } m < 0\ . \end{cases}$$

In the sequel we will often switch between the following two notations, used identically:

$$P^{\alpha}_{m,n}(z,\bar{z}) = P^{\alpha}_{m,n}(z) \text{ as well as } f(z,\bar{z}) = f(z)\ .$$

The first just merely intends to emphasize that we are talking about two dimensional functions. One should bear in mind, that we are dealing with polynomials of the kind $\sum_{n,m} \alpha_{n,m} z^n \bar{z}^m$. Of course polynomials $\sum_n \alpha_n z^n \in \mathbb{C}[z]$ are included in our set of polynomials, but indeed it is a much bigger class we are talking about.

With the defintion of the disc polynomials we can already find some interesting properties. In particular the following property concerning complex conjugation will be refered to several times in the sequel:

$$\overline{P^{\alpha}_{m,n}(z,\bar{z})} = P^{\alpha}_{m,n}(\bar{z},z) = P^{\alpha}_{n,m}(z,\bar{z})\ . \tag{4.3}$$

Concerning parity we see that $P^{\alpha}_{m,n}(-z,-\bar{z}) = (-1)^{m+n} P^{\alpha}_{m,n}(z,\bar{z})$. We also want to mention some special values that are easy to check:

$$P^{\alpha}_{m,n}(0,0) = \frac{(-1)^n\ n!\ \alpha!}{(n+\alpha)!}\ \delta_{m,n}\quad , \qquad \text{as well as} \qquad P^{\alpha}_{m,n}(1,1) = 1\ .$$

A very natural and important way to write functions on balls and especially on the disc is to transform them into polar coordinates. So for $z = re^{i\varphi}$ the disc polynomials are given by

$$P^{\alpha}_{m,n}(re^{i\varphi}, re^{-i\varphi}) = e^{i(m-n)\varphi} P^{\alpha}_{m,n}(r,r) \tag{4.4}$$

and we easily see that the disc polynomials on the boundary of the disc have the values $P^{\alpha}_{m,n}(e^{i\varphi}, e^{-i\varphi}) = e^{i(m-n)\varphi}$.

We want to give the first few disc polynomials in order to get a slight idea how they look like. The first thing that one notices is the fact that whenever one of the indices is zero the corresponding disc polynomial is a monomial in z or $\bar{z}$ depending on whether the first or the second index is zero. This means that we have:

$$\forall n \in \mathbb{N}_0 : P_{n,0}^{\alpha}(z, \bar{z}) = z^n \text{ and } P_{0,n}^{\alpha}(z, \bar{z}) = \bar{z}^n \ .$$

By looking at formula (4.1) it becomes clear that the highest exponent of the disc polynomial $P_{m,n}^{\alpha}(z, \bar{z})$ is (m, n) and that the subsequent exponents decrease by j for $j = 1$ up to $\min(m, n)$ via $(m - j, n - j)$. It makes sense to order the disc polynomials by the sum of their indices, later we will establish a two dimensional order on $\mathbb{N}_0 \times \mathbb{N}_0$. So the first few disc polynomials $P_{m,n}^{\alpha}(z, \bar{z})$ are:

$$\begin{aligned}
&\text{for } \ m+n=0: && P_{0,0}^{\alpha}(z, \bar{z}) = 1 \\
&\text{for } \ m+n=1: && P_{1,0}^{\alpha}(z, \bar{z}) = z \ , \quad P_{0,1}^{\alpha}(z, \bar{z}) = \bar{z} \\
&\text{for } \ m+n=2: && P_{2,0}^{\alpha}(z, \bar{z}) = z^2 \ , \quad P_{1,1}^{\alpha}(z, \bar{z}) = \frac{\alpha+2}{\alpha+1} z\bar{z} - \frac{1}{\alpha+1} \ , \quad P_{0,2}^{\alpha}(z, \bar{z}) = \bar{z}^2 \\
&\text{for } \ m+n=3: && P_{3,0}^{\alpha}(z, \bar{z}) = z^3 \ , \quad P_{2,1}^{\alpha}(z, \bar{z}) = \frac{\alpha+3}{\alpha+1} z^2\bar{z} - \frac{2}{\alpha+1} z \\
& && P_{1,2}^{\alpha}(z, \bar{z}) = \frac{\alpha+3}{\alpha+1} z\bar{z}^2 - \frac{2}{\alpha+1} \bar{z} \ , \quad P_{0,3}^{\alpha}(z, \bar{z}) = \bar{z}^3 \\
&\text{for } \ m+n=4: && P_{4,0}^{\alpha}(z, \bar{z}) = z^4 \ , \quad P_{3,1}^{\alpha}(z, \bar{z}) = \frac{\alpha+4}{\alpha+1} z^3\bar{z} - \frac{3}{\alpha+1} z^2 \\
& && P_{2,2}^{\alpha}(z, \bar{z}) = \frac{(\alpha+4)(\alpha+3)}{(\alpha+2)(\alpha+1)} z^2\bar{z}^2 - \frac{4(\alpha+3)}{(\alpha+2)(\alpha+1)} z\bar{z} + \frac{2}{(\alpha+2)(\alpha+1)} \\
& && P_{3,1}^{\alpha}(z, \bar{z}) = \frac{\alpha+4}{\alpha+1} z\bar{z}^3 - \frac{3}{\alpha+1} \bar{z}^2 \ , \quad P_{0,4}^{\alpha}(z, \bar{z}) = \bar{z}^4 \ .
\end{aligned}$$

The basic properties given in this chapter already tell much about the character of the disc polynomials. Further properties which are not that easy to acquire will be given in another section.

4.2 Lebesgue spaces and orthogonality on the disc

The orthogonality of the disc polynomials is certainly the most important property. We will give a detailed proof here. The orthogonality relation will open the door to a generalized Fourier Analysis on the disc and an associated approximation theory with respect to the disc polynomials. The natural function spaces for this theory are of course suitable Lebesgue spaces which we will define in this chapter.

As mentioned before the orthogonality relation holds with respect to a certain weight function which we will call $\omega_\alpha(z)$ and which is given for real α with $\alpha > -1$ by:

$$\omega_\alpha(z) := \frac{\alpha+1}{\pi}(1-z\bar{z})^\alpha, \text{ for } z \in D \ .$$

Because of its nonnegativity it induces a Borel measure on D and we define:

$$d\mu_\alpha(z) := \omega_\alpha(z)dz = \frac{\alpha+1}{\pi}\left(1-x^2-y^2\right)^\alpha dxdy \ , \quad \text{for } z = x+iy \ .$$

By a simple calculation we can see that $\mu_\alpha(z)$ is even a probability measure on D. Thus, the integral over the domain D of the identity function is equal to 1:

$$\begin{aligned}\int_D 1 \ d\mu_\alpha(z) &= \frac{\alpha+1}{\pi}\int_D \left(1-z\bar{z}\right)^\alpha dz = \frac{\alpha+1}{\pi}\int_0^1\int_0^{2\pi} r\left(1-r^2\right)^\alpha d\varphi dr \\ &= 2(\alpha+1)\left[\frac{-1}{2(\alpha+1)}\left(1-r^2\right)^{\alpha+1}\right]_0^1 = 1\end{aligned}$$

The next step is to define function spaces taking into account this weight function. They are given in

Definition 4.2 (Lebesgue space). *For $1 \le p \le \infty$ and $\alpha > -1$ the Lebesgue space $L^p_\alpha(D)$ is given by*

$$L^p_\alpha(D) := \left\{ f(z,\bar{z}) \to \mathbb{C} \ \middle| \ z \in D \text{ and } \|f\|_{L^p_\alpha} < \infty \right\}$$

where the norm is given by

$$\|f\|_{L^p_\alpha} := \begin{cases} \left(\int_D |f(z,\bar{z})|^p d\mu_\alpha(z)\right)^{\frac{1}{p}} & \text{for } 1 \le p < \infty \\ \sup_{x\in D} |f(x,\bar{x})| & \text{for } p = \infty \ . \end{cases}$$

The function space $X^p_\alpha(D)$ is given for $1 \le p \le \infty$ with the same norm as $L^p_\alpha(D)$ by

$$X^p_\alpha(D) := \begin{cases} L^p_\alpha(D) & \text{for } 1 \le p < \infty \\ C(D) & \text{for } p = \infty \ . \end{cases}$$

With Hölder's inequality we see that for $q > p$ we have $L^q_\alpha(D) \subseteq L^p_\alpha(D)$. And it is clear that the $L^p_\alpha(D)$ spaces are Banach spaces and $L^2_\alpha(D)$ is a Hilbert space. The corresponding inner product is given by

$$\langle f, g\rangle_\alpha := \int_D \overline{f(z)}g(z)d\mu_\alpha(z)$$

for $f, g \in L^2_\alpha(D)$.

Now we have all prerequisites to study orthogonality. In a Hilbert space two elements are orthogonal if their inner product is equal to zero. So what we will prove in order to show the orthogonality of the disc polynomials is the following:

$$\left\langle P^\alpha_{k,l}, P^\alpha_{m,n} \right\rangle_\alpha = \frac{\delta_{k,m}\ \delta_{l,n}}{h^\alpha(m,n)}$$

while $h^\alpha(m,n)$ is the Haar function for the disc polynomials given by the formula

$$\begin{aligned} h^\alpha(m,n) &= \frac{m+n+\alpha+1}{\alpha+1}\binom{m+\alpha}{m}\binom{n+\alpha}{n} \\ &= \frac{m+n+\alpha+1}{\alpha+1}\ \frac{(m+\alpha)!}{m!\ \alpha!}\ \frac{(n+\alpha)!}{n!\ \alpha!}\ . \end{aligned}$$

As one can see the Haar function is symmetric in m and n which will be of importance later. So now we come to the actual proof. Recall the properties of the disc polynomials from the previous section and the orthogonality relation of the Jacobi polynomials which will be used in the last step.

$$\begin{aligned} \left\langle P^\alpha_{k,l}, P^\alpha_{m,n} \right\rangle_\alpha &= \int_D \overline{P^\alpha_{k,l}(z,\bar z)}\ P^\alpha_{m,n}(z,\bar z) d\mu_\alpha(z) \overset{(4.3)}{=} \int_D P^\alpha_{l,k}(z,\bar z)\ P^\alpha_{m,n}(z,\bar z) d\mu_\alpha(z) \\ &= \frac{\alpha+1}{\pi}\int_0^{2\pi}\int_0^1 P^\alpha_{l,k}(re^{i\varphi}, re^{-i\varphi})\ P^\alpha_{m,n}(re^{i\varphi}, re^{-i\varphi})\ \left(1-r^2\right)^\alpha r\ dr d\varphi \\ &\overset{(4.4)}{=} \frac{\alpha+1}{\pi}\int_0^{2\pi} e^{i(l-k)\varphi} e^{i(m-n)\varphi} d\varphi \int_0^1 P^\alpha_{l,k}(r,r)\ P^\alpha_{m,n}(r,r)\ \left(1-r^2\right)^\alpha r\ dr \end{aligned}$$

now let w.l.o.g. $n \le m$ s.th. in case $l+m = k+n$ we can replace k by $l+m-n$

$$\begin{aligned} &= \frac{\alpha+1}{\pi}\ 2\pi\ \delta_{l+m,k+n}\int_0^1 P^\alpha_{l,l+m-n}(r,r)\ P^\alpha_{m,n}(r,r)\ \left(1-r^2\right)^\alpha r\ dr \\ &= 2(\alpha+1)\delta_{l+m,k+n}\int_0^1 \binom{l+\alpha}{l}^{-1} r^{m-n} J_l^{(\alpha,m-n)}(2r^2-1)\times \\ &\qquad \times \binom{n+\alpha}{n}^{-1} r^{m-n} J_n^{(\alpha,m-n)}(2r^2-1)\left(1-r^2\right)^\alpha r\ dr \end{aligned}$$

now a substitution of $x = 2r^2-1$ gives

$$\begin{aligned} &= 2(\alpha+1)\ \delta_{l+m,k+n}\binom{n+\alpha}{n}^{-1}\binom{l+\alpha}{l}^{-1}\times \\ &\qquad \times \int_{-1}^1 \frac{1}{4}\ J_l^{(\alpha,m-n)}(x) J_n^{(\alpha,m-n)}(x)\left(\frac{1+x}{2}\right)^{m-n}\left(\frac{1-x}{2}\right)^\alpha dx \\ &= 2^{-\alpha-(m-n)-1}\ (\alpha+1)\ \delta_{l+m,k+n}\binom{n+\alpha}{n}^{-1}\binom{l+\alpha}{l}^{-1}\times \\ &\qquad \times \int_{-1}^1 J_l^{(\alpha,m-n)}(x) J_n^{(\alpha,m-n)}(x)\left(1+x\right)^{m-n}\left(1-x\right)^\alpha dx \end{aligned}$$

$$= \binom{n+\alpha}{n}^{-1} \binom{l+\alpha}{l}^{-1} \frac{\alpha+1}{m+n+\alpha+1} \, \frac{(n+\alpha)!\ m!}{n!\ (m+\alpha)!} \, \delta_{l,n} \ \delta_{l+m,k+n}$$
$$= \frac{\delta_{l,n}\ \delta_{k,m}}{h^\alpha(m,n)} \ .$$

4.3 Generalized translation and convolution

In the chapters on Fourier Analysis with respect to Jacobi polynomials we already mentioned the relevance of a translation operator for Fourier analysis and approximation theory. We are aiming at doing approximation theory on the unit disc with respect to the disc polynomials so the question for a generalized translation is of great interest. Indeed a translation for our purposes exists and the associated convolution. In [1] and [15] the definition and some properties of the generalized translation can be found together with further considerations. Here we will just concentrate on the definition and some basic corollaries, applying them in later chapters.

Definition 4.3 (generalized translation). *For a function $f \in L^1_\alpha(D)$, $\alpha > 0$ and a point $z \in D$ the operator T^α_z, called generalized translation, is defined by*

$$T^\alpha_z f(\xi) := \frac{\alpha}{\alpha+1} \int_D f(\bar{z}\xi + \sqrt{1-|z|^2}\sqrt{1-|\xi|^2}\ y) \frac{d\mu_\alpha(y)}{1-|y|^2} \ , \quad \forall \xi \in D$$
$$= \int_D f(\bar{z}\xi + \sqrt{1-|z|^2}\sqrt{1-|\xi|^2}\ y) d\mu_{\alpha-1}(y) \ .$$

In this definition it becomes clear, that we can no longer maintain $\alpha > -1$ and have to change it towards $\alpha > 0$, which will be our assumption in the sequel.

It is worth taking a closer look at the argument of the integrand. We see that we actually integrate over the set $\left\{\bar{z}\xi + \sqrt{1-|z|^2}\sqrt{1-|\xi|^2}\ y \mid y \in D\right\}$, i.e. a disc of radius $r = \sqrt{(1-|z|^2)(1-|\xi|^2)}$ with center $\bar{z}\xi$. This disc will be called in the sequel and in the following figure $B_r(\bar{z}\xi)$. It is clear, that for $z, \xi \in D$ also the product $\bar{z}\xi \in D$. Then we ascertain that the translation operator is well defined by checking that $B_r(\bar{z}\xi)$ lies entirely in D. Therefore we switch to polar coordinates and let $\bar{z} := (1-\varepsilon)e^{i\varphi}$ and $\xi := (1-\delta)e^{i\theta}$ with $0 \le \varepsilon, \delta \le 1$; $\varphi, \theta \in [0, 2\pi)$:

Then the center $\bar{z}\xi$ is given by $\bar{z}\xi = (1-\varepsilon)(1-\delta)e^{i(\varphi+\theta)}$ and its distance from the boundary is $\operatorname{dist}(\bar{z}\xi, \partial D) = \varepsilon(1-\delta) + \delta$.

The radius of $B_r(\bar{z}\xi)$, $r = \sqrt{(1-|z|^2)(1-|\xi|^2)}$ then, in terms of ε, δ, is given by

$$0 \le r = \sqrt{(1-(1-\varepsilon)^2)(1-(1-\delta)^2)} = \sqrt{4\varepsilon\delta - 2\varepsilon\delta^2 - 2\varepsilon^2\delta + \varepsilon^2\delta^2} \ .$$

With $(\operatorname{dist}(\bar{z}\xi, \partial D))^2 = \varepsilon^2 + 2\varepsilon\delta + \delta^2 - 2\varepsilon^2\delta - 2\varepsilon\delta^2 + \varepsilon^2\delta^2$ we can conclude

$$\begin{aligned}0 \le r &= \sqrt{(\mathrm{dist}(\bar{z}\xi, \partial D))^2 - \varepsilon^2 + 2\varepsilon\delta - \delta^2} \\ &= \sqrt{(\mathrm{dist}(\bar{z}\xi, \partial D))^2 - (\varepsilon - \delta)^2} \le \mathrm{dist}(\bar{z}\xi, \partial D)\ .\end{aligned}$$

For illustration different cases of that are given in the following figure:

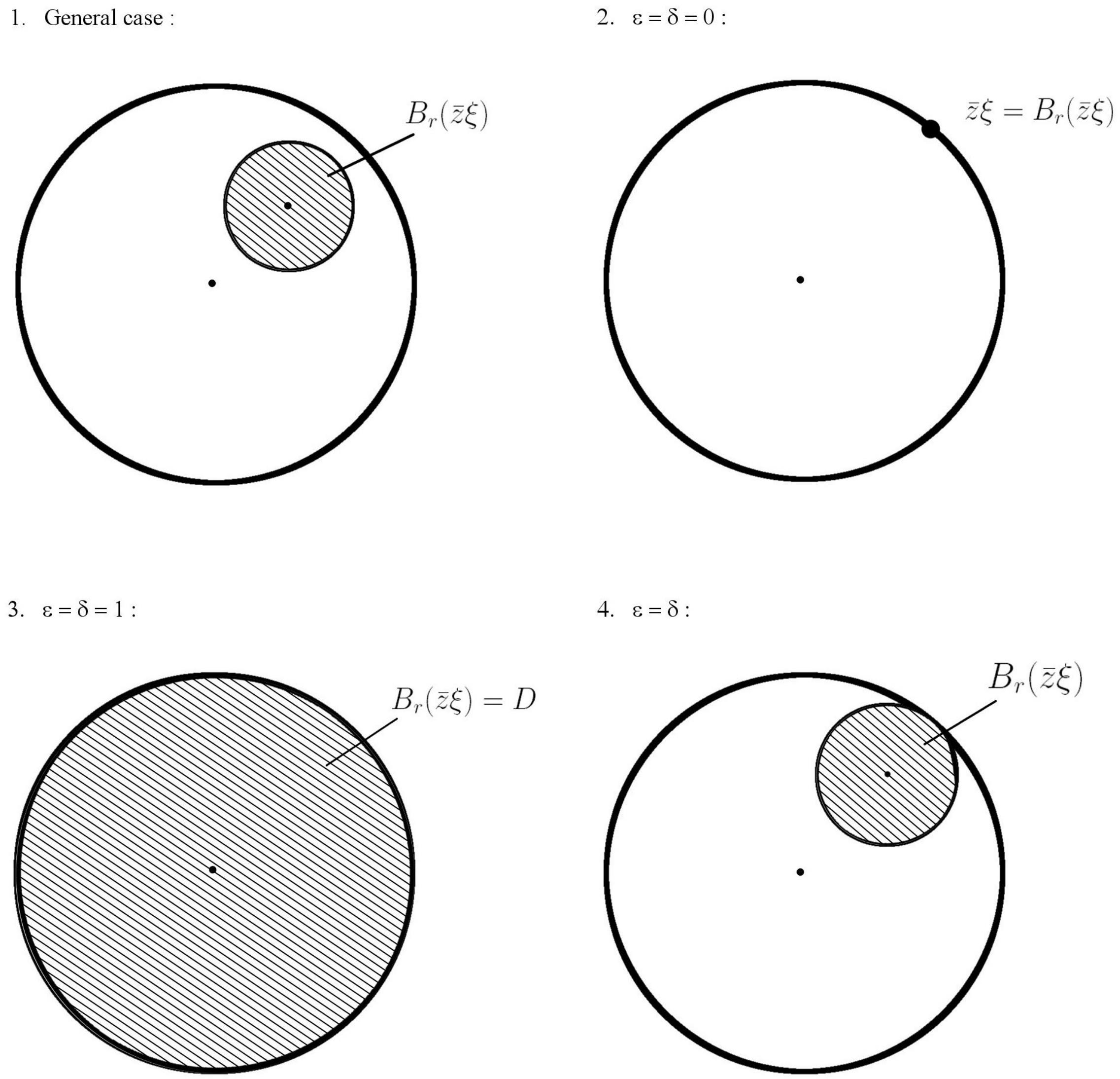

Figure 3: the domain of integration of the translation for different values of $\bar{z} = (1-\varepsilon)e^{i\varphi}$ and $\xi = (1-\delta)e^{i\theta}$

If we perform a variable transformation such that the integration domain is $B_r(\bar{z}\xi)$ the concept of the translation changes in the following way:

Lemma 4.4. *For $z, \xi, y \in D$ we have:*

$$(i) \qquad T_z^\alpha f(\xi) = \int_D f(y) E_\alpha(z,\xi,y) d\mu_\alpha(y)$$

and $E_\alpha(z,\xi,y)$ is given by

$$E_\alpha(z,\xi,y) = \begin{cases} \frac{\alpha}{\alpha+1} \frac{\left(1-|z|^2-|\xi|^2-|y|^2+2Re(\bar{z}\xi\bar{y})\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha(1-|y|^2)^\alpha} & \text{for } y \in B_r(\bar{z}\xi) \\ 0 & \text{else} \end{cases}$$

$$(ii) \qquad E_\alpha(z,\xi,y) \geq 0$$

$$(iii) \qquad \int_D E_\alpha(z,\xi,y) d\mu_\alpha(y) = 1$$

$$(iv) \qquad E_\alpha(z,\xi,y) = E_\alpha(z,\bar{y},\bar{\xi}) \quad \text{as well as} \quad E_\alpha(z,\xi,y) = E_\alpha(y,\xi,z) \ .$$

Proof (i):

$$\begin{aligned} T_z^\alpha f(\xi) &= \frac{\alpha}{\alpha+1} \int_D f(\bar{z}\xi + \sqrt{1-|z|^2}\sqrt{1-|\xi|^2}\ y) \frac{d\mu_\alpha(y)}{1-|y|^2} \\ &= \frac{\alpha}{\alpha+1} \int_D f(\bar{z}\xi + \sqrt{1-|z|^2}\sqrt{1-|\xi|^2}\ y)(1-|y|^2)^{\alpha-1} \frac{\alpha+1}{\pi} dy_1 dy_2 \ . \end{aligned}$$

Now we perform a substitution by $x = \bar{z}\xi + \sqrt{1-|z|^2}\sqrt{1-|\xi|^2}\ y$

which leads to $y = \dfrac{x-\bar{z}\xi}{\sqrt{1-|z|^2}\sqrt{1-|\xi|^2}}$,

$$dy_1 dy_2 = (1-|z|^2)^{-1}(1-|\xi|^2)^{-1} dx_1 dx_2$$

and D transforms to $B_r(\bar{z}\xi)$

$$\begin{aligned} &= \frac{\alpha}{\alpha+1} \int_{B_r(\bar{z}\xi)} f(x) \left(1 - \frac{(x-\bar{z}\xi)(\bar{x}-z\bar{\xi})}{(1-|z|^2)(1-|\xi|^2)}\right)^{\alpha-1} \frac{\alpha+1}{\pi(1-|z|^2)(1-|\xi|^2)}\ dx_1 dx_2 \\ &= \frac{\alpha}{\alpha+1} \int_{B_r(\bar{z}\xi)} f(x) \frac{\left(1-|z|^2-|\xi|^2-|x|^2+2Re(\bar{z}\xi\bar{x})\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha(1-|x|^2)^\alpha}\ d\mu_\alpha(x) \\ &= \int_D f(x) E_\alpha(z,\xi,x) d\mu_\alpha(x) \ . \end{aligned}$$

Proof (ii):

For $y \in B_r(\bar{z}\xi)$ we have : $E_\alpha(z,\xi,y) = \dfrac{\alpha}{\alpha+1} \dfrac{\left(1-|z|^2-|\xi|^2-|y|^2+2Re(\bar{z}\xi\bar{y})\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha(1-|y|^2)^\alpha}$

and so if we take $y = x + \bar{z}\xi,\ x \in B_{\sqrt{(1-|z|^2)(1-|\xi|^2)}}(0)$, $E_\alpha(z,\xi,y)$ transforms to

$$
\begin{aligned}
E_\alpha(z,\xi,\bar z\xi + x) &= \frac{\alpha}{\alpha+1}\frac{\left(1-|z|^2-|\xi|^2-|\bar z\xi+x|^2+2Re(\bar z\xi(\overline{\bar z\xi+x}))\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha(1-|\bar z\xi+x|^2)^\alpha} \\
&= \frac{\alpha}{\alpha+1}\frac{\left(1-|z|^2-|\xi|^2+|z|^2|\xi|^2-|x|^2\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha(1-|\bar z\xi+x|^2)^\alpha} \\
&= \frac{\alpha}{\alpha+1}\frac{\left((1-|z|^2)(1-|\xi|^2)-|x|^2\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha(1-|\bar z\xi+x|^2)^\alpha} \quad ,
\end{aligned}
$$

and as $|x|^2 \le (1-|z|^2)(1-|\xi|^2)$ the assertion that $E_\alpha(z,\xi,y) = E_\alpha(z,\xi,\bar z\xi+x) \ge 0$ follows as all terms in the last line are ≥ 0, in fact:

$$
\begin{aligned}
\left((1-|z|^2)(1-|\xi|^2)-|x|^2\right)^{\alpha-1} &\ge 0 \\
(1-|z|^2)^\alpha &\ge 0 \\
(1-|\xi|^2)^\alpha &\ge 0 \\
(1-|\bar z\xi+x|^2)^\alpha &\ge 0 \ .
\end{aligned}
$$

Proof (iii):

$$
\int_D E_\alpha(z,\xi,y)d\mu_\alpha(y) = \frac{\alpha}{\pi}\int_{B_r(\bar z\xi)} \frac{\left(1-|z|^2-|\xi|^2-|y|^2+2Re(\bar z\xi\bar y)\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha}\, dy_1dy_2
$$

$$
= \frac{\alpha}{\pi}\frac{1}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha}\int_{B_{\sqrt{(1-|z|^2)(1-|\xi|^2)}}(0)} \left((1-|z|^2)(1-|\xi|^2)-|y|^2\right)^{\alpha-1}dy_1dy_2
$$

$$
= \frac{\alpha}{\pi}\frac{1}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha}\int_0^{\sqrt{(1-|z|^2)(1-|\xi|^2)}}\int_0^{2\pi} \left((1-|z|^2)(1-|\xi|^2)-r^2\right)^{\alpha-1}\, r\; d\varphi dr
$$

$$
\left[-\frac{((1-|z|^2)(1-|\xi|^2)-r^2)^\alpha}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha}\right]_0^{\sqrt{(1-|z|^2)(1-|\xi|^2)}} = 1 \ .
$$

Proof (iv):

It is clear, that $\frac{\alpha}{\alpha+1}\frac{\left(1-|z|^2-|\xi|^2-|y|^2+2Re(\bar z\xi\bar y)\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha(1-|y|^2)^\alpha} = \frac{\alpha}{\alpha+1}\frac{\left(1-|z|^2-|\xi|^2-|y|^2+2Re(\bar z\bar y\xi)\right)^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\xi|^2)^\alpha(1-|y|^2)^\alpha}$, because of the multiplicative commutativity of the complex numbers, so it suffices to show

$$
y \in B_r(\bar z\xi) \iff \bar\xi \in B_{r_1}(\bar z\bar y)
$$

where $r = \sqrt{1-|z|^2}\sqrt{1-|\xi|^2}$ and $r_1 = \sqrt{1-|z|^2}\sqrt{1-|y|^2}$. So

$$
y \in B_r(\bar z\xi) \iff y \in B_{\sqrt{(1-|z|^2)(1-|\xi|^2)}}(\bar z\xi) \iff |y-\bar z\xi|^2 < (1-|z|^2)(1-|\xi|^2)
$$

$$\begin{aligned}
&\iff |y|^2 - 2Re(\bar{y}\bar{z}\xi) + |z|^2|\xi|^2 < 1 - |z|^2 - |\xi|^2 + |z|^2|\xi|^2 \\
&\iff |\xi|^2 - 2Re(\bar{y}\bar{z}\xi) + |z|^2|y|^2 < 1 - |z|^2 - |y|^2 + |z|^2|y|^2 \\
&\iff |\bar{\xi} - \bar{y}\bar{z}|^2 < (1-|z|^2)(1-|y|^2) \iff \bar{\xi} \in B_{r_1}(\bar{z}\bar{y}) \ .
\end{aligned}$$

The second equality is proved in the same way.

Now we are in the situation to prove the first two important properties a translation has to fulfill, namely linearity and boundedness:

Lemma 4.5 (linearity and boundedness of the translation). *For $y, z \in D$ and $a, b \in \mathbb{C}$, $f, g \in L^1_\alpha(D)$ for $\alpha > 0$ it holds*

$$\begin{aligned}
&(i) \qquad T_z^\alpha(af+bg)(y) = a\ T_z^\alpha f(y) + b\ T_z^\alpha g(y) \\
&(ii) \qquad \|T_z^\alpha h\|_{X_\alpha^q} \le \|h\|_{X_\alpha^q} \text{ for } h \in X_\alpha^q(D) \text{ for } 1 \le q \le \infty \ .
\end{aligned}$$

Proof (i): The linearity follows by the linearity of the integral:

$$T_z^\alpha(af+bg)(y) = \int_D (af+bg)(\xi)E_\alpha(z,y,\xi)d\mu_\alpha(\xi) = a\ T_z^\alpha f(y) + b\ T_z^\alpha g(y) \ .$$

Proof (ii): case 1): $q=1$:

$$\begin{aligned}
\|T_z^\alpha f\|_{X_\alpha^1} &= \int_D |T_z^\alpha f(y)| d\mu_\alpha(y) \le \int_D\int_D |f(\xi)|E_\alpha(z,y,\xi)d\mu_\alpha(\xi)d\mu_\alpha(y) \\
&\overset{4.4\ iv)}{=} \int_D |f(\xi)| \underbrace{\int_D E_\alpha(z,\bar{\xi},\bar{y})d\mu_\alpha(y)}_{=1} d\mu_\alpha(\xi) = \|f\|_{X_\alpha^1}
\end{aligned}$$

case 2): $q=\infty$:

$$\begin{aligned}
\|T_z^\alpha f\|_{X_\alpha^\infty} &= \sup_{y\in D} |T_z^\alpha f(y)| \le \sup_{y\in D} \int_D |f(\xi)|E_\alpha(z,y,\xi)d\mu_\alpha(\xi) \\
&\le \sup_{x\in D}|f(x)| \sup_{y\in D}\int_D E_\alpha(z,y,\xi)d\mu_\alpha(\xi) = \|f\|_{X_\alpha^\infty} \ .
\end{aligned}$$

Based on the generalized translation we can now define the generalized convolution of two functions on the unit disc in the expected manner:

Definition 4.6 (generalized convolution). *For two functions $f, g \in L^1_\alpha(D)$, and $\alpha > 0, \xi \in D$ the generalized convolution $\underset{\alpha}{*}$ is given by*

$$\big(f \underset{\alpha}{*} g\big)(\xi) := \int_D T_z^\alpha f(\xi)g(z)d\mu_\alpha(z) = \int_D\int_D f(y)E_\alpha(z,\xi,y)d\mu_\alpha(y)g(z)d\mu_\alpha(z) \ .$$

The next important property being required from the translation is the product formula for disc polynomials. Koornwinder proved it in [19] as a corollary of the product formula for Jacobi polynomials.

Theorem 4.7 (product formula for disc polynomials). *For the disc polynomials we have the following product formula, that is associated with the generalized translation:* $\forall\ \alpha > 0; m, n \in \mathbb{N}_0$

$$T_z^\alpha P_{m,n}^\alpha(\xi, \bar{\xi}) = P_{m,n}^\alpha(\bar{z}, z) P_{m,n}^\alpha(\xi, \bar{\xi}) \ .$$

We do not give the proof of this property. It is a consequence of the product formula for Jacobi polynomials, which is rather complicated.

The final property we need to show in order to have a valid translation on the unit disc is the commutativity of the generalized convolution. This will be done in a lemma concluding this chapter.

Lemma 4.8. *For* $f, g \in L_\alpha^1(D),\ \alpha > 0$ *it holds*

$$\begin{array}{ll} i) & f \underset{\alpha}{*} g \in L_\alpha^1(D) \\ ii) & f \underset{\alpha}{*} g = g \underset{\alpha}{*} f \end{array} \ .$$

Proof i): Recall the properties of $E_\alpha(z, \xi, y)$, (4.4 (iii) and (iv)) and then we see that

$$\begin{aligned} \|f \underset{\alpha}{*} g\|_{L_\alpha^1} &= \int_D \left| \int_D \int_D f(y) g(z) E_\alpha(z, \xi, y) d\mu_\alpha(y) d\mu_\alpha(z) \right| d\mu_\alpha(\xi) \\ &\le \int_D \int_D \int_D |f(y)||g(z)| E_\alpha(z, \bar{y}, \bar{\xi}) d\mu_\alpha(y) d\mu_\alpha(z) d\mu_\alpha(\xi) \\ &= \int_D \int_D |f(y)||g(z)| \underbrace{\int_D E_\alpha(z, \bar{y}, \bar{\xi}) d\mu_\alpha(\bar{\xi})}_{=1} \ d\mu_\alpha(y) d\mu_\alpha(z) \\ &= \|f\|_{L_\alpha^1} \|g\|_{L_\alpha^1} \ . \end{aligned}$$

Proof ii): Recalling 4.4 (iv) we obtain

$$\begin{aligned} \left(f \underset{\alpha}{*} g\right)(\xi) &= \int_D \int_D f(y) g(z) E_\alpha(z, \xi, y) d\mu_\alpha(y) d\mu_\alpha(z) \\ &= \int_D \int_D f(y) g(z) E_\alpha(y, \xi, z) d\mu_\alpha(y) d\mu_\alpha(z) \\ &= \left(g \underset{\alpha}{*} f\right)(\xi) \ . \end{aligned}$$

4.4 Further properties of the disc polynomials

In the introductory chapter on Jacobi polynomials was mentioned that they satisfy a certain differential equation. As we want to look at the disc polynomials in a similar way one can ask the question for differential equations they satisfy. The two common coordinate systems the disc polynomials are given in are the polar coordinates and the one with a complex coordinate and its conjugate. These can be transformed into each other such that a differential equation for the disc polynomials can always be given in those two forms.

A very easy differential equation can be found by differentiation of a disc polynomial in polar coordinates by its angular variable:

$$\frac{\partial}{\partial\varphi}P^{\alpha}_{m,n}(re^{i\varphi},re^{-i\varphi}) = i(m-n)P^{\alpha}_{m,n}(re^{i\varphi},re^{-i\varphi}) \ . \tag{4.5}$$

With the relations $\frac{\partial z}{\partial\varphi} = iz$ and $\frac{\partial\bar{z}}{\partial\varphi} = -i\bar{z}$ this differential equation can be transformed into

$$\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\right)P^{\alpha}_{m,n}(z,\bar{z}) = (m-n)P^{\alpha}_{m,n}(z,\bar{z}) \ .$$

With the help of the differential equation for the Jacobi polynomials (3.1) one can derive a second differential equation for the disc polynomials, which we will again give in the two different coordinate systems:

$$\left((1-r^2)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right) - 2(1+\alpha)r\frac{\partial}{\partial r} + 4mn + 2(1+\alpha)(m+n)\right)\times$$
$$\times P^{\alpha}_{m,n}(re^{i\varphi},re^{-i\varphi}) = 0$$

or alternatively

$$\left(2\left[(1-z\bar{z})\frac{\partial^2}{\partial z\partial\bar{z}} + \frac{\partial^2}{\partial z\partial\bar{z}}(1-z\bar{z})\right] - \alpha\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial\bar{z}} + \frac{\partial}{\partial z}z + \frac{\partial}{\partial\bar{z}}\bar{z}\right) + 4mn+\right.$$
$$\left. +2(1+\alpha)(m+n+1)\right)P^{\alpha}_{m,n}(z,\bar{z}) = 0 \ .$$

We will sketch here the way this is to prove. In equation (3.1) we insert $\beta = m-n$ (w.l.o.g. $m > n$) and perform a subsequent substitution $x = 2r^2 - 1$. Therefore we get for the differential operators $\frac{\partial}{\partial x} = \frac{1}{4r}\frac{\partial}{\partial r}$ and $\frac{\partial^2}{\partial x^2} = \frac{1}{16r^2}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)$. This leads to

$$\left((1-r^2)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + 2(m-n)(1-r^2)\frac{1}{r}\frac{\partial}{\partial r} - 2(1+\alpha)\frac{1}{r}\frac{\partial}{\partial r} + 4mn + 4n(1+\alpha)\right)\times$$
$$\times J_n^{(\alpha,m-n)}(2r^2-1) = 0 \ .$$

Now we plug into this equation the (m,n)-th disc polynomial $P^\alpha_{m,n}(re^{i\varphi}, re^{-i\varphi}) = \frac{n!\alpha!}{(n+\alpha)!}$ $e^{i(m-n)\varphi}\ r^{m-n} J_n^{(\alpha,m-n)}(2r^2-1)$ and adjust the terms that occur by the product rule. Finally we need the help of equation (4.5) by using $\frac{\partial^2}{\partial\varphi^2} P^\alpha_{m,n} = -(m-n)^2 P^\alpha_{m,n}$ to get the desired result.

The differential equation in the variables $(z,\bar z)$ can be derived with the following two transformations of differential operators: $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} = 4\frac{\partial^2}{\partial z \partial \bar z}$ as well as $r\frac{\partial}{\partial r} = z\frac{\partial}{\partial z} + \bar z \frac{\partial}{\partial \bar z}$.

Similar to the recurrence relation for the Jacobi polynomials (3.2) there exist recurrence relations for the disc polynomials. The most easy one which can be checked with the explicit formula (4.2) is given by

$$(m-n)P^\alpha_{m,n}(z,\bar z) = mzP^\alpha_{m-1,n}(z,\bar z) - n\bar z P^\alpha_{m,n-1}(z,\bar z) \ .$$

Two other recurrence relations are connected to each other due to the complex conjugation property (4.3). They can be checked by simple calculations with formula (4.1) and comparison of the coefficients of the monomials $z^{m+1-k}\bar z^{n-k}$.

$$(m+n+\alpha+1)zP^\alpha_{m,n}(z,\bar z) = (m+\alpha+1)P^\alpha_{m+1,n}(z,\bar z) + nP^\alpha_{m,n-1}(z,\bar z)$$

$$(m+n+\alpha+1)\bar zP^\alpha_{m,n}(z,\bar z) = (n+\alpha+1)P^\alpha_{m,n+1}(z,\bar z) + mP^\alpha_{m-1,n}(z,\bar z) \ .$$

By looking at these two recurrence relations from a different point of view and the fact that $z = P^\alpha_{1,0}(z,\bar z)$ one sees, that here are given the linearisation coefficients $g^\alpha(1,0;m,n;0)$ and $g^\alpha(1,0;m,n;1)$. (Further informations and results on linearisation coefficients are given in the next chapter.) So we have in terms of linearisation coefficients

$$P^\alpha_{1,0}(z,\bar z)P^\alpha_{m,n}(z,\bar z) = g^\alpha(1,0;m,n;0)P^\alpha_{m+1,n}(z,\bar z) + g^\alpha(1,0;m,n;1)P^\alpha_{m,n-1}(z,\bar z)$$

(and for the complex conjugate case respectively).

Another interesting question is, whether we can determine the maximal value of the disc polynomials on the unit disc D. The answer is yes and it is known, that it is equal to one in case $\alpha \geq 0$:

$$\text{For every } \alpha \geq 0 \text{ and every } (m,n) \in \mathbb{N}^2, \text{ we have } \sup_{z\in D} |P^\alpha_{m,n}(z)| = 1 \ .$$

In [3] Askey gives the following proof:

We can write the product of two disc polynomials as a sum of several disc polynomials multiplied with their corresponding linearization coefficient, namely

$$P^\alpha_{m,n}(z)P^\alpha_{k,l}(z) = \sum_{i,j} g^\alpha(m,n;k,l;i,j)P^\alpha_{i,j}(z) \ .$$

Koornwinder proved that the linearisation coefficients $g^\alpha(m,n;k,l;i,j)$ are non-negative for every $(m,n),(k,l)$ and (i,j) in case $\alpha \geq 0$ which is our general assumption here.

Since for all disc polynomials we have $P^\alpha_{m,n}(1) = 1$ it is clear, that firstly we have $\sup_{z\in D} |P^\alpha_{m,n}(z)| \geq 1$ and secondly for the sum of the linearisation coefficients we have $\sum_{i,j} g^\alpha(m,n;k,l;i,j) = 1$. With the notation that $M(z) := \sup_{(m,n)} |P^\alpha_{m,n}(z)|$ and $M := \sup_{z\in D} M(z)$ we get from the equation above $M(z)^2 \leq M$ and therefore $M^2 \leq M$. But then it must be $M = 1$.

Subsequently I will give another proof of this result. I don't know whether the following proof is already known in literature. In contrast to the upper proof it is only valid for $\alpha > 0$:

i) For every $z_0 = r_0 e^{i\varphi_0}$ we have the following relation for the absolute value of a disc polynomial, namely that it only depends on the radial part of the variable:

$$|P^\alpha_{m,n}(z_0, \overline{z_0})| = |P^\alpha_{m,n}(r_0 e^{i\varphi_0}, r_0 e^{-i\varphi_0})| = |e^{i(m-n)\varphi_0} P^\alpha_{m,n}(r_0, r_0)| = |P^\alpha_{m,n}(r_0, r_0)| \ .$$

ii) This little consideration we need later: Every $r_0 \in [0,1]$ is contained in the disc of radius $(1 - r_0^2)$ with center r_0^2 :

$$\forall r_0 \in [0,1] : r_0 \in B_{(1-r_0^2)}(r_0^2) \subset D$$

We see that the disc $B_{(1-r_0^2)}(r_0^2)$ contains the real interval $[2r_0^2 - 1, 1]$ thus we need to check if everything works with the left end of the interval. But we have

$$2r_0^2 - 1 \leq r_0^2 \leq r_0 \ ; \ \forall r_0 \in [0,1] \ .$$

iii) Assume there is a $z_0 = r_0 e^{i\varphi_0} \in D$ such that

$$\sup_{z\in D} |P^\alpha_{m,n}(z)| = |P^\alpha_{m,n}(z_0)| \overset{i)}{=} |P^\alpha_{m,n}(r_0)| > 1.$$

With the generalized translation

$$T^\alpha_z f(y) := \int_D f(\bar{z}y + \sqrt{1-|z|^2}\sqrt{1-|y|^2}\ \xi) d\mu_{\alpha-1}(\xi)$$

and the product formula for disc polynomials

$$T^\alpha_z P^\alpha_{m,n}(y) = P^\alpha_{m,n}(\bar{z}) P^\alpha_{m,n}(y)$$

given in an earlier chapter (both only given for $\alpha > 0$) we have the following inequality:

$$\begin{aligned}
\left|P^{\alpha}_{m,n}(r_0)\right|^2 &\overset{i)}{=} \left|P^{\alpha}_{m,n}(\overline{z_0})P^{\alpha}_{m,n}(z_0)\right| = \left|T^{\alpha}_{z_0}P^{\alpha}_{m,n}(z_0)\right| \\
&= \left|\int_D P^{\alpha}_{m,n}(\overline{z_0}z_0 + \sqrt{1-|z_0|^2}\sqrt{1-|z_0|^2}\xi)d\mu_{\alpha-1}(\xi)\right| \\
&= \left|\int_D P^{\alpha}_{m,n}(|z_0|^2 + (1-|z_0|^2)\xi)d\mu_{\alpha-1}(\xi)\right| \\
&\leq \int_D \left|P^{\alpha}_{m,n}(|z_0|^2 + (1-|z_0|^2)\xi)\right| d\mu_{\alpha-1}(\xi) \\
&= \int_D \left|P^{\alpha}_{m,n}(r_0^2 + (1-r_0^2)\xi)\right| d\mu_{\alpha-1}(\xi) \\
&\overset{ii)}{\leq} \left|P^{\alpha}_{m,n}(r_0)\right| \int_D 1\ d\mu_{\alpha-1}(\xi) = \left|P^{\alpha}_{m,n}(r_0)\right| \ .
\end{aligned}$$

In the last inequality we have used the fact, that the point r_0 on the real axis at which $P^{\alpha}_{m,n}$ takes its maximal value is contained in the domain over which effectively is integrated which was the consideration in ii), and in the last equality that the measure $\mu_{\alpha-1}$ is a probability measure over the disc.

But then we have the contradiction that $\left|P^{\alpha}_{m,n}(r_0)\right|^2 \leq \left|P^{\alpha}_{m,n}(r_0)\right|$ which doesn't fit to the assumption $|P^{\alpha}_{m,n}(r_0)| > 1$.

5 Linearization coefficients of the disc polynomials

5.1 Linearization coefficients

By looking at the disc polynomials it gets clear that they constitute a basis of the set of polynomials of the form $P(z) = \sum\limits_{i=0}^{m} \sum\limits_{j=0}^{n} a_{i,j}\; z^i \bar{z}^j$ on $\mathbb{C}$. Hence every such polynomial can be represented by a linear combination of disc polynomials. A question that arises in hypergroup theory (cf [8] and [14]) is how do the so called **linearization coefficients** look like for the product of two orthogonal polynomials, are they non-negative?

Definition 5.1 (Linearization coefficient). *For $n, m, k, l \in \mathbb{N}_0$ the product of the two disc polynomials $P^\alpha_{m,n}(z)$ and $P^\alpha_{k,l}(z)$ can be written as a finite linear combination of disc polynomials:*

$$P^\alpha_{m,n}(z) P^\alpha_{k,l}(z) = \sum_{i,j} g^\alpha(m, n; k, l; i, j) P^\alpha_{i,j}(z) \ .$$

$g^\alpha(m, n; k, l; i, j)$ is called the (i, j)-th linearization coefficient of $P^\alpha_{m,n}(z)P^\alpha_{k,l}(z)$.

A first result on the form of the sum and the linearization coefficients of the disc polynomials is given in the following

Theorem 5.2 (Linearization coefficients). *For $\alpha > -1$ and for all $m, n, k, l \in \mathbb{N}_0$ $\exists M \in \mathbb{N}_0$, such that*

$$P^\alpha_{m,n} P^\alpha_{k,l} = \sum_{i=0}^{M} g^\alpha(m, n; k, l; i) P^\alpha_{m+k-i,n+l-i}$$

where $M \leq \min\{m + k, n + l, m + n, k + l\}$ and $g^\alpha(m, n; k, l; i) \in \mathbb{R}\ \forall i \in \mathbb{N}_0$.

Remarks: a) We changed the order of summation in this formula from ascending to descending beginnig with the highest linearization coefficient that is not zero and named the $g^\alpha(m, n; k, l; m + k - i, n + l - i) =: g^\alpha(m, n; k, l; i)$ for $0 \leq i \leq M$. All the other linearization coefficients are zero so we didn't list them. In our eyes this enhances the readability and understanding.

b) Koornwinder proved in [21] the linearization coefficients $g^\alpha(m, n; k, l; i, j)$ for the disc polynomials to be non-negative for $\alpha \geq 0$.

c) For the special case $\alpha = 0$ we were able to calculate the linearization coefficients $g^0(m, n; k, l; i)$ explicitely. This calculation is very tedious based on a two-dimensional induction.
We have included this result since the explicit form of the coefficients $g^0(m, n; k, l; i)$ -if neccessary- can be used for the analysis of properties of the disc polynomials, e.g. generalized translation on the disc in case $\alpha = 0$.

Proof of Theorem 5.2:

i) At first, we prove the fact that we need for the representation of the product $P^\alpha_{m,n}P^\alpha_{k,l}$ in terms of a linear combination of disc polynomials only those of the form $P^\alpha_{m+k-i,n+l-i}$, while i is in the range of $0 \le i \le \min\{m+k, n+l\}$:

By the explicit formula for the disc polynomials (4.1) we can write the product in the following way:

$$\begin{aligned}
P^\alpha_{m,n}(z)P^\alpha_{k,l}(z) &= \frac{m!\; n!\; \alpha!}{(m+\alpha)!(n+\alpha)!} \sum_{i=0}^{\min(m,n)} \frac{(-1)^i(m+n+\alpha-i)!}{i!(m-i)!(n-i)!} z^{m-i}\bar{z}^{n-i} \\
&\quad \times \frac{k!\; l!\; \alpha!}{(k+\alpha)!(l+\alpha)!} \sum_{j=0}^{\min(k,l)} \frac{(-1)^j(k+l+\alpha-j)!}{j!(k-j)!(l-j)!} z^{k-j}\bar{z}^{l-j} \\
&= \frac{m!\; n!\; k!\; l!\; \alpha!\; \alpha!}{(m+\alpha)!(n+\alpha)!(k+\alpha)!(l+\alpha)!} \\
&\quad \times \sum_{i=0}^{\min(m,n)} \sum_{j=0}^{\min(k,l)} (-1)^{i+j} \frac{(m+n+\alpha-i)!}{i!(m-i)!(n-i)!} \frac{(k+l+\alpha-j)!}{j!(k-j)!(l-j)!} z^{m+k-(i+j)}\bar{z}^{n+l-(i+j)} \\
&= \frac{m!\; n!\; k!\; l!\; \alpha!\; \alpha!}{(m+\alpha)!(n+\alpha)!(k+\alpha)!(l+\alpha)!} \\
&\quad \times \sum_{s=0}^{T} \left[\sum_{t=0}^{s} \frac{(-1)^s(m+n+\alpha-s+t)!}{(s-t)!(m-s+t)!(n-s+t)!} \frac{(k+l+\alpha-t)!}{t!(k-t)!(l-t)!} \right] z^{m+k-s}\bar{z}^{n+l-s} \qquad (5.1)
\end{aligned}$$

where $T = \min(m,n) + \min(k,l)$. We want to remind that in the inner sum of the last line (5.1) only those terms are nonzero, where in the denominator are no negative factorials. Or in other words: the inner summation should look like

$$\sum_{t=\max\{s-m,s-n,0\}}^{\min\{k,l,s\}} \cdots$$

but this is too clumsy and doesn't help the readability so we keep the upper notation.

Now we list some observations that can be made by looking at this formula and the form of the disc polynomials.

(P1) One sees in formula (5.1) that the exponents of z and $\bar{z}$ in the product of $P^\alpha_{m,n}(z)P^\alpha_{k,l}(z)$ reduce by the factor one simultaneously, i.e. only exponents of the form $(m+k-s, n+l-s)$ for $s = 0, ..., \min(m,n) + \min(k,l)$ can occur. Moreover all coefficients are real numbers.

(P2) If we look at the exponents of the disc polynomials $P^\alpha_{m+k-i,n+l-i}(z)$ with which we want to make the linear combination of the product $P^\alpha_{m,n}(z)P^\alpha_{k,l}(z)$ one also sees that for fixed i we only have exponents of z and $\bar{z}$ of the form $(m+k-i-s, n+l-i-s)$ for $s = 0, ..., \min(m+k-i, n+l-i)$. And the coefficients of the disc polynomials are real numbers.

(P3) Now we can procede successively and at first choose $g^\alpha(m,n;k,l;0) \in \mathbb{R}$ in such a way that the difference $\big(P^\alpha_{m,n}(z)P^\alpha_{k,l}(z) - g^\alpha(m,n;k,l;0)P^\alpha_{m+k,n+l}(z)\big)$ is a polynomial where the exponents of z and $\bar{z}$ are only of the form $(m+k-1-j, n+l-1-j)$ for $0 \le j \le \min(m+k-1, n+l-1)$. That works because of the two facts (P1) and (P2).

Then we choose $g^\alpha(m,n;k,l;1) \in \mathbb{R}$ in such a way that the difference $\big((P^\alpha_{m,n}(z)P^\alpha_{k,l}(z) - g^\alpha(m,n;k,l;0)P^\alpha_{m+k,n+l}(z)) - g^\alpha(m,n;k,l;1)P^\alpha_{m+k-1,n+l-1}(z)\big)$ is a polynomial where the exponents of z and $\bar{z}$ are only of the form $(m+k-2-j, n+l-2-j)$ for $0 \le j \le \min(m+k-2, n+l-2)$ what again works because of (P1) and (P2).

And so forth and so on. There are two possible ends of this procedure: either one $g^\alpha(m,n;k,l;j)$ for $0 \le j \le (\min(m+k, n+l) - 1)$ fully regulates the difference, i.e.

$$P^\alpha_{m,n}P^\alpha_{k,l} - \sum_{i=0}^{j} g^\alpha(m,n;k,l;i)P^\alpha_{m+k-i,n+l-i} = 0$$

or we come to the point where the difference $\big(P^\alpha_{m,n}P^\alpha_{k,l} - \sum\limits_{i=0}^{j} g^\alpha(m,n;k,l;i)P^\alpha_{m+k-i,n+l-i}\big)$ is a polynomial of degree $(0, n+l-m-k)$ or $(m+k-n-l, 0)$ what means that $j = \big(\min(m+k, n+l) - 1\big)$. But then $\exists b_{n+l-m-k} \in \mathbb{R}$, $(b_{m+k-n-l})$ respectively such that we can regulate the rest by putting

$$\begin{aligned} P^\alpha_{m,n}P^\alpha_{k,l} - \textstyle\sum_{i=0}^{m+k-1} g^\alpha(m,n;k,l;i)P^\alpha_{m+k-i,n+l-i} &= b_{n+l-m-k}\bar{z}^{n+l-m-k} \\ := g^\alpha(m,n;k,l;m+k)\bar{z}^{n+l-m-k} &= g^\alpha(m,n;k,l;m+k)P^\alpha_{0,n+l-m-k} \end{aligned}$$

or respectively

$$\begin{aligned} P^\alpha_{m,n}P^\alpha_{k,l} - \textstyle\sum_{i=0}^{n+l-1} g^\alpha(m,n;k,l;i)P^\alpha_{m+k-i,n+l-i} &= b_{m+k-n-l}z^{m+k-n-l} \\ := g^\alpha(m,n;k,l;n+l)z^{m+k-n-l} &= g^\alpha(m,n;k,l;n+l)P^\alpha_{m+k-n-l,0}\ . \end{aligned}$$

Thus, we proved already that

$$\exists g^\alpha(m,n;k,l;i) \in \mathbb{R}\ :\ P^\alpha_{m,n}P^\alpha_{k,l} = \sum_{i=0}^{M} g^\alpha(m,n;k,l;i)P^\alpha_{m+k-i,n+l-i}$$

and $M \le \min(m+k, n+l)$.

It remains to show:

ii) $M \le \min\{m+k, n+l, m+n, k+l\}$:

The case that the minimum is $m+k$ or $n+l$ is clear from part i) of the proof, so let $m+n = \min\{m+k, n+l, m+n, k+l\}$ and w.l.o.g. $m+n < k+l, m+k, n+l$ from what we get $\Longrightarrow k-n > 0$, $l-m > 0$ and let w.l.o.g. $n < m$:

Suppose $\exists d_0 \in \mathbb{N}$; $1 \leq d_0 \leq \min(k-n, l-m)$ so that

$$P^\alpha_{m,n} P^\alpha_{k,l} = \sum_{i=0}^{m+n+d_0} g^\alpha(m,n;k,l;i) P^\alpha_{m+k-i,n+l-i} \text{ , and}$$

$$g^\alpha(m,n;k,l;i) \neq 0 \text{ for } m+n+1 \leq i \leq m+n+d_0 \ .$$

Recall that $\big(h^\alpha(m,n)\big)^{-1} = \int_D P^\alpha_{m,n}(z) P^\alpha_{m,n}(\bar{z}) d\mu_\alpha(z) = \int_D P^\alpha_{m,n}(z) \overline{P^\alpha_{m,n}(z)} d\mu_\alpha(z)$ and the orthogonality relation:

$$\int_D P^\alpha_{m,n}(z) \overline{P^\alpha_{k,l}(z)} d\mu_\alpha(z) = \delta_{m,k}\delta_{n,l}\big(h^\alpha(m,n)\big)^{-1} \ .$$

Then for all $1 \leq d \leq d_0$ we have:

$$\begin{aligned}
g^\alpha(m,n;&k,l;m+n+d) \\
&= h^\alpha\big(m+k-(m+n+d), n+l-(m+k+d)\big)\times \\
&\quad \times \int_D \sum_{i=0}^{m+n+d_0} g^\alpha(m,n;k,l;i) P^\alpha_{m+k-i,n+l-i}(z) \overline{P^\alpha_{m+k-(m+n+d),n+l-(m+n+d)}(z)} d\mu_\alpha \\
&= h^\alpha(k-n-d, l-m-d) \int_D P^\alpha_{m,n}(z) P^\alpha_{k,l}(z) \overline{P^\alpha_{k-n-d,l-m-d}(z)} d\mu_\alpha \\
&= h^\alpha(k-n-d, l-m-d) \int_D \overline{P^\alpha_{l,k}(z)} P^\alpha_{m,n}(z) P^\alpha_{l-m-d,k-n-d}(z) d\mu_\alpha \\
&= h^\alpha(k-n-d, l-m-d)\times \\
&\quad \times \int_D \overline{P^\alpha_{l,k}(z)} \sum_{i=0}^{M} g^\alpha(m,n;l-m-d,k-n-d;i) P^\alpha_{l-d-i,k-d-i}(z) d\mu_\alpha \\
&= 0
\end{aligned}$$

because of the orthogonality of $P^\alpha_{l,k}(z) \perp P^\alpha_{l-d-i,k-d-i}(z)$ as $d \geq 1$, $i \geq 0 \Rightarrow l-d-i < l$ and $k-d-i < k$. But this is a contradiction. So we see that indeed

$$g(m,n;k,l;i) = 0 \quad \text{for} \quad m+n+1 \leq i \leq m+n+d_0 \ .$$

The case that $k+l = \min\{m+k, n+l, m+n, k+l\}$ follows identically.

One easy consequence for the sum of the linearization coefficients is the following fact.

Corollary 5.3. *For all $m,n,k,l \in \mathbb{N}_0$ with $M = \min\{m+k, n+l, m+n, k+l\}$ it holds*

$$\sum_{i=0}^{M} g^\alpha(m,n;k,l;i) = 1 \ .$$

Proof:
In the chapter on disc polynomials we saw that for $\alpha > -1$ and $\forall m, n \in \mathbb{N}_0$ we have $P^{\alpha}_{m,n}(1,1) = 1$.

$$\Rightarrow 1 = P^{\alpha}_{m,n}(1,1)P^{\alpha}_{k,l}(1,1) = \sum_{i=0}^{M} g^{\alpha}(m,n;k,l;i)P^{\alpha}_{m+k-i,n+l-i}(1,1) = \sum_{i=0}^{M} g^{\alpha}(m,n;k,l;i)\ .$$

5.2 Explicit formula for $\alpha = 0$

In this chapter we focus on the disc polynomials for the parameter $\alpha = 0$ and derive an explicit formula for their linearization coefficients. So first of all we will introduce notations that should be valid only in this chapter. We don't want to carry along $\alpha = 0$ all the time and that is the reason why we want to fix $\alpha = 0$ here.

The notation for the disc polynomials for $\alpha = 0$ is given by the following explicit formula:

$$\forall m, n \in \mathbb{N}_0, z \in D : P^{0}_{m,n}(z) = \sum_{k=0}^{\min(m,n)} (-1)^k \frac{(m+n-k)!}{k!(m-k)!(n-k)!}\ z^{m-k}\bar{z}^{n-k}\ .$$

So let in the sequel but only in this chapter $\alpha = 0$ and $P_{m,n} := P^{0}_{m,n}(z)$. As the formulas here hold for all $z \in D$ we do not write z explicitely. Furthermore for the linearization coefficients let $g(m,n;k,l;i) := g^{0}(m,n;k,l;i)$.

A second thing we need to talk about here is the generalized Pochhammer symbol. Recall the property of the generalized Pochhammer symbol of Corollary 3.3, 2):

$$(m)_j = \frac{m!}{(m-j)!} = 0 \qquad \text{for } j, m \in \mathbb{N},\ j > m\ .$$

In the sequel of this chapter there will appear terms of the form $\frac{m!}{(m-j)!}$ possibly for $j > m$. Of course they should be zero. So whenever we have a negative integer factorial in the denominator of a term, then the whole term should be zero.

As a preparation for the proof of the formula for the explicit linearization coefficients we provide the following lemma.

Lemma 5.4. *For $k, n, m, l \in \mathbb{N}_0$ and $N \in \mathbb{N}$ the following equation holds:*

$$\sum_{i=0}^{N}(-1)^i \frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{m!}{(m-i)!}\binom{N}{i}$$
$$= \sum_{i=0}^{N-1}(-1)^i \frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{m!}{(m-i)!}\binom{N-1}{i}$$

$$-\sum_{i=0}^{N-1}(-1)^i \frac{k!}{(k+1-N+i)!}\frac{n!}{(n+1-N+i)!}\frac{l!}{(l-1-i)!}\frac{m!}{(m-1-i)!}\binom{N-1}{i}$$

what written in terms of the Pochhammer symbol looks like

$$\sum_{i=0}^{N}(-1)^i (k)_{N-i}(n)_{N-i}(l)_i(m)_i\binom{N}{i} = \sum_{i=0}^{N-1}(-1)^i (k)_{N-i}(n)_{N-i}(l)_i(m)_i\binom{N-1}{i}$$
$$-\sum_{i=0}^{N-1}(-1)^i (k)_{N-1-i}(n)_{N-1-i}(l)_{i+1}(m)_{i+1}\binom{N-1}{i}\ .$$

Remark 1: Here we see that the situation with the Pochhammer symbol described above may arise, namely when $N > k$, $N > n$, $N > l$ or $N > m$. So the actual summation in the first sum according to N is

$$\sum_{i=\max\{N-k,N-n,0\}}^{\min\{l,m,N\}} \dots$$

but that is more confusing than helpful, so we will write it the easy way.

Remark 2: In our opinion for the understanding of the subsequent proofs in this chapter it is more intuitive to write the Pochhammer terms in the form $\frac{m!}{(m-j)!}$ so we will do that.

Proof of Lemma 5.4: We use the wellknown binomial identity $\binom{N}{i} = \binom{N-1}{i} + \binom{N-1}{i-1}$ and an index shifting to perform the proof.

Due to the long terms that occur, for better readability we write the equations in small font:

$$\sum_{i=0}^{N}(-1)^i \frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{m!}{(m-i)!}\binom{N}{i}$$
$$= \sum_{i=1}^{N-1}(-1)^i \frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{m!}{(m-i)!}\left[\binom{N-1}{i-1}+\binom{N-1}{i}\right] + \frac{k!}{(k-N)!}\frac{n!}{(n-N)!} + (-1)^N\frac{l!}{(l-N)!}\frac{m!}{(m-N)!}$$
$$= \sum_{i=1}^{N-1}(-1)^{(i-1)+1} \frac{k!}{(k+1-N+(i-1))!}\frac{n!}{(n+1-N+(i-1))!}\frac{l!}{(l-1-(i-1))!}\frac{m!}{(m-1-(i-1))!}\binom{N-1}{i-1}$$
$$+(-1)^N\frac{l!}{(l-N)!}\frac{m!}{(m-N)!} + \sum_{i=0}^{N-1}(-1)^i \frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{m!}{(m-i)!}\binom{N-1}{i}$$
$$= \sum_{i=0}^{N-1}(-1)^i \frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{m!}{(m-i)!}\binom{N-1}{i}$$
$$-\sum_{i=0}^{N-1}(-1)^i \frac{k!}{(k+1-N+i)!}\frac{n!}{(n+1-N+i)!}\frac{l!}{(l-1-i)!}\frac{m!}{(m-1-i)!}\binom{N-1}{i}\ .$$

Now we have all prerequisits to prove the formula for the linearization coefficients.

Theorem 5.5 ((*) Explicit formula for the linearization coefficients of the disc polynomials for $\alpha = 0$). *$\forall m, n, k, l, N \in \mathbb{N}_0$ the linearization coefficients of the disc polynomials for $\alpha = 0$ are*

$$g(m,n;k,l;N) = \frac{(m+n-N)!(k+l-N)!(m+k-N)!(n+l-N)!}{m!\ n!\ k!\ l!\ N!\ (m+n+k+l+1-N)!} \times$$
$$\times (m+n+k+l+1-2N)\left(\sum_{i=0}^{N}(-1)^i \frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{m!}{(m-i)!}\binom{N}{i}\right)^2 .$$

Remarks: i) One sees that M in Theorem 5.2 for $\alpha = 0$ is not only $\leq \min\{m+n, k+l, m+k, n+l\}$ but "$=$".

ii) All $g(m,n;k,l;i)$ with $i \notin \{0,...,M\}$ are zero.

iii) With this formula the non-negativity of the linearization coefficients for $\alpha = 0$ is clear to see.

iv) $g(m,n;k,l;i) \in \mathbb{Q}$

Proof of Theorem 5.5: We prove the theorem in several steps. The principle is an induction in two dimensions.

a) let $k, l \in \mathbb{N}_0$ arbitrary. Then one easily checks, as $P_{0,0} = 1$:

$$P_{0,0}P_{k,l} = g(0,0;k,l;0)P_{k,l} = P_{k,l}$$

so we need to show, that $g(0,0;k,l;0) = 1$, which can be seen here:

$$\frac{(0+0-0)!(k+l-0)!(0+k-0)!(0+l-0)!}{0!\ 0!\ k!\ l!\ 0!\ (0+0+k+l+1-0)!}(0+0+k+l+1-2\cdot 0)\left(\sum_{i=0}^{0}\frac{k!}{(k-0+i)!}\frac{0!}{(0-0+i)!}\frac{l!}{(l-i)!}\frac{0!}{(0-i)!}\binom{N}{i}\right)^2$$

$$= \frac{(k+l)!\ k!\ l!}{k!\ l!\ (k+l+1)!}(k+l+1) = 1 \ . \qquad \checkmark$$

b) In the second step, we prove for arbitrary $k, l \in \mathbb{N}_0$:

$$i)\quad P_{1,0}P_{k,l} = \frac{k+1}{k+l+1}P_{k+1,l} + \frac{l}{k+l+1}P_{k,l-1}$$

$$ii)\quad g(1,0;k,l;0) = \frac{k+1}{k+l+1}, \qquad g(1,0;k,l;1) = \frac{l}{k+l+1} \ .$$

regarding $i)$:

$$P_{1,0}P_{k,l} = z\sum_{i=0}^{\min(k,l)}\frac{(-1)^i(k+l-i)!}{i!(k-i)!(l-i)!}z^{k-i}\bar{z}^{l-i} = \sum_{i=0}^{\min(k,l)}\frac{(-1)^i(k+l-i)!}{i!(k-i)!(l-i)!}z^{k+1-i}\bar{z}^{l-i}$$

on the other hand we calculate:

$$\frac{k+1}{k+l+1}P_{k+1,l} + \frac{l}{k+l+1}P_{k,l-1} = \frac{k+1}{k+l+1}\sum_{i=0}^{\min(k+1,l)}(-1)^i\frac{(k+1+l-i)!}{i!(k+1-i)!(l-i)!}z^{k+1-i}\bar{z}^{l-i}$$

$$+ \frac{l}{k+l+1}\sum_{i=0}^{\min(k,l-1)}\frac{(-1)^i(k+l-1-i)!}{i!(k-i)!(l-1-i)!}z^{k-i}\bar{z}^{l-1-i} = \frac{k+1}{k+l+1}\sum_{i=1}^{\min(k+1,l)}\frac{(-1)^i(k+1+l-i)!}{i!(k+1-i)!(l-i)!}z^{k+1-i}\bar{z}^{l-i}$$

$$+\ \tfrac{k+1}{k+l+1}\tfrac{(k+1+l)!}{(k+1)!\ l!}z^{k+1}\bar{z}^{l} + \ \tfrac{l}{k+l+1}\sum_{i=0}^{\min(k,l-1)}(-1)^i\tfrac{(k+l-1-i)!}{i!(k-i)!(l-1-i)!}z^{k-i}\bar{z}^{l-1-i}$$
$$= \tfrac{(k+l)!}{k!\ l!}z^{k+1}\bar{z}^{l}\ +\ \tfrac{k+1}{k+l+1}\sum_{i=0}^{\min(k,l-1)}(-1)^{i+1}\tfrac{(k+l-i)!}{(i+1)!(k-i)!(l-1-i)!}z^{k-i}\bar{z}^{l-1-i}$$
$$+\ \tfrac{l}{k+l+1}\sum_{i=0}^{\min(k,l-1)}(-1)^i\tfrac{(k+l-1-i)!}{i!(k-i)!(l-1-i)!}z^{k-i}\bar{z}^{l-1-i}$$
$$= \sum_{i=0}^{\min(k,l-1)}(-1)^i\tfrac{(k+l-1-i)!}{(i+1)!(k-i)!(l-1-i)!(k+l+1)}\big(l(i+1)-(k+1)(k+l-i)\big)z^{k-i}\bar{z}^{l-1-i}$$
$$+\ \tfrac{(k+l)!}{k!\ l!}z^{k+1}\bar{z}^{l} = \sum_{i=0}^{\min(k-1,l-1)}\tfrac{(-1)^{i+1}(k+l-1-i)!}{(i+1)!(k-i)!(l-1-i)!(k+l+1)}(k-i)(k+l+1)z^{k-i}\bar{z}^{l-1-i}$$
$$+\ \tfrac{(k+l)!}{k!\ l!}z^{k+1}\bar{z}^{l} = \sum_{i=1}^{\min(k,l)}(-1)^i\tfrac{(k+l-i)!}{i!(k-i)!(l-i)!}z^{k+1-i}\bar{z}^{l-i}\ +\ \tfrac{(k+l)!}{k!\ l!}z^{k+1}\bar{z}^{l} = P_{1,0}P_{k,l}\ .$$

concerning ii):

$$g(1,0;k,l;0) = \tfrac{(1+0-0)!(k+l-0)!(1+k-0)!(0+l-0)!}{0!\ 1!\ k!\ l!\ 0!\ (0+1+k+l+1-0)!}(1+0+k+l+1-2\cdot 0)\left(\tfrac{k!}{k!}\tfrac{0!}{0!}\tfrac{l!}{l!}\tfrac{1!}{1!}\tbinom{0}{0}\right)^2$$
$$= \frac{1!(k+l)!(k+1)!l!}{1!\ k!\ l!\ (k+l+2)!}(k+l+2) = \frac{k+1}{k+l+1} \qquad \checkmark$$
$$g(1,0;k,l;1) = \tfrac{(1+0-1)!(k+l-1)!(1+k-1)!(0+l-1)!}{0!\ 1!\ k!\ l!\ 1!\ (0+1+k+l+1-1)!}(1+0+k+l+1-2)\left((-1)\tfrac{l!}{(l-1)!}\tfrac{1!}{0!}\right)^2$$
$$= \frac{(k+l-1)!k!(l-1)!}{k!\ l!\ (k+l+1)!}(k+l)l^2 = \frac{l}{k+l+1}\ . \qquad \checkmark$$

c) In the same manner as b) one checks:

$$P_{0,1}P_{k,l} = \frac{l+1}{k+l+1}P_{k,l+1} + \frac{k}{k+l+1}P_{k-1,l}$$

what means that

$$g(0,1;k,l;0) = \frac{l+1}{k+l+1}, \qquad g(0,1;k,l;1) = \frac{k}{k+l+1}\ .$$

d) $\forall n,k,l \in \mathbb{N}_0$ holds:

$$P_{0,n}P_{k,l} = \sum_{i=0}^{M} g(0,n;k,l;i)P_{k-i,n+l-i}, \quad M = \min\{n, k+l, k, n+l\}\ .$$

Therefore we look at the following equation:

$$P_{0,n}P_{k,l} = P_{0,1}P_{0,n-1}P_{k,l} = P_{0,1}\sum_{i=0}^{\tilde{M}} g(0,n-1;k,l;i)P_{k-i,n+l-1-i}$$

$$\stackrel{c)}{=} \sum_{i=0}^{\tilde{M}} g(0,n-1;k,l;i) \frac{n+l-i}{n+k+l-2i} P_{k-i,n+l-i}$$

$$+ \sum_{i=0}^{\tilde{M}} g(0,n-1;k,l;i) \frac{k-i}{n+k+l-2i} P_{k-1-i,n+l-1-i}$$

with $\tilde{M} = \min\{n-1, k+l, k, n-1-l\}$.
Hence it holds:

$$\sum_{i=0}^{M} g(0,n;k,l;i) P_{k-i,n+l-i} = \sum_{i=0}^{\tilde{M}} g(0,n-1;k,l;i) \frac{n+l-i}{n+k+l-2i} P_{k-i,n+l-i}$$

$$+ \sum_{i=0}^{\tilde{M}} g(0,n-1;k,l;i) \frac{k-i}{n+k+l-2i} P_{k-1-i,n+l-1-i} \; .$$

One sees that the linearization coefficients $g(0,n;k,l;i)$ can, by induction, be won by known coefficients, namely $g(0,n-1;k,l;i)$, i.e. induction over n

We have to distinguish between four cases, depending on M :
1. $M = n \le k+l, k, n+l \Longrightarrow \tilde{M} = n-1$
2. $M = k < n; k \le k+l, n+l \Longrightarrow \tilde{M} = k$
3. $M = n+l < n;$ is not possible and $n+l = n$ is case 1 .
4. $M = k+l < k;$ is not possible and $k+l = k$ is case 2 .

By comparison of the coefficients of the disc polynomials it needs to hold:

In case 1: i) $g(0,n;k,l;0) \stackrel{!}{=} \frac{n+l}{n+k+l}\; g(0,n-1;k,l;0)$

ii) $g(0,n;k,l;n) \stackrel{!}{=} \frac{k-(n-1)}{n+k+l-2(n-1)}\; g(0,n-1;k,l;n-1)$

iii) $g(0,n;k,l;N) \stackrel{!}{=} \frac{n+l-N}{n+k+l-2N}\; g(0,n-1;k,l;N)$
$+\frac{k-(N-1)}{n+k+l-2(N-1)}\; g(0,n-1;k,l;N-1)$, for $1 \le N \le n-1$

In case 2: i) $g(0,n;k,l;0) \stackrel{!}{=} \frac{n+l}{n+k+l}\; g(0,n-1;k,l;0)$

ii) $g(0,n;k,l;N) \stackrel{!}{=} \frac{n+l-N}{n+k+l-2N}\; g(0,n-1;k,l;N)$
$+\frac{k-(N-1)}{n+k+l-2(N-1)}\; g(0,n-1;k,l;N-1)$, for $1 \le N \le k$

Case 1,i) and case 2,i) are identical:

$$\frac{(0+n-0)!(k+l-0)!(0+k-0)!(n+l-0)!}{0!\; n!\; k!\; l!\; 0!\; (0+n+k+l+1-0)!}(0+n+k+l+1-2\cdot 0) \left(\sum_{i=0}^{0} (-1)^i \frac{k!}{(k-0+i)!} \frac{n!}{(n-0+i)!} \frac{l!}{(l-i)!} \frac{0!}{(0-i)!} \binom{0}{i} \right)^2$$

$$\stackrel{!}{=} \frac{n+l}{n+k+l} \frac{(0+n-1-0)!(k+l-0)!(0+k-0)!(n+l-1-0)!}{0!\; (n-1)!\; k!\; l!\; 0!\; (0+n-1+k+l+1-0)!}(0+n-1+k+l+1-2\cdot 0)\big(1\big)^2$$

$$\Longleftrightarrow \frac{n!(k+l)!k!(n+l)!}{n!\; k!\; l!\; (n+k+l+1)!}(n+k+l+1) = \frac{(n-1)!(k+l)!k!(n+l)!}{(n-1)!\; k!\; l!\; (n+k+l)!} \; . \qquad \checkmark$$

Concerning case 1,ii):

$$\tfrac{(0+n-n)!(k+l-n)!(0+k-n)!(n+l-n)!}{0!\ n!\ k!\ l!\ n!\ (0+n+k+l+1-n)!}(0+n+k+l+1-2n)\left(\sum_{i=0}^{n}(-1)^i\tfrac{k!}{(k-n+i)!}\tfrac{n!}{(n-n+i)!}\tfrac{l!}{(l-i)!}\tfrac{0!}{(0-i)!}\tbinom{n}{i}\right)^2$$

$$\overset{!}{=}\tfrac{k+1-n}{k+l+2-2n}\tfrac{(k+l+1-n)!(k+1-n)!\ l!}{(n-1)!\ k!\ l!\ (n-1)!(k+l+1)!}(k+l+2-2n)\left(\sum_{i=0}^{n-1}(-1)^i\tfrac{k!}{(k+1-n+i)!}\tfrac{n!}{i!}\tfrac{l!}{(l-i)!}\tfrac{0!}{(-i)!}\tbinom{n-1}{i}\right)^2$$

$$\Longleftrightarrow\ \tfrac{(k+l-n)!(k-n)!\ l!\ (k+l+1-n)}{n!\ k!\ l!\ n!\ (k+l+1)!}\left(\tfrac{k!}{(k-n)!}\tfrac{n!}{0!}\right)^2=\tfrac{(k+l+1-n)!(k+1-n)!\ l!\ (k+1-n)}{(n-1)!\ k!\ l!\ (n-1)!(k+l+1)!}\left(\tfrac{k!(n-1)!}{(k+1-n)!}\right)^2$$

$$\Longleftrightarrow\ \tfrac{k!}{(k-n)!}=\tfrac{(k+1-n)k!}{(k+1-n)!}\ . \qquad \checkmark$$

Cases 1,iii) and 2,ii):

$$\tfrac{(n-N)!(k+l-N)!(k-N)!(n+l-N)!}{n!\ k!\ l!\ N!\ (n+k+l+1-N)!}(n+k+l+1-2N)\left(\sum_{i=0}^{N}(-1)^i\tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{0!}{(0-i)!}\tbinom{N}{i}\right)^2$$

$$\overset{!}{=}\tfrac{(n-1-N)!(k+l-N)!(k-N)!(n+l-1-N)!(n+l-N)}{(n-1)!\ k!\ l!\ N!\ (n+k+l-N)!}\left(\sum_{i=0}^{N}(-1)^i\tfrac{k!}{(k-N+i)!}\tfrac{(n-1)!}{(n-1-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{0!}{(-i)!}\tbinom{N}{i}\right)^2$$

$$+\tfrac{(n-N)!(k+l+1-N)!(k+1-N)!(n+l-N)!(k+1-N)}{(n-1)!\ k!\ l!\ (N-1)!(n+k+l+1-N)!}\left(\sum_{i=0}^{N-1}(-1)^i\tfrac{k!}{(k+1-N+i)!}\tfrac{(n-1)!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{0!}{(-i)!}\tbinom{N-1}{i}\right)^2$$

$$\Longleftrightarrow\ \tfrac{(n-N)!(k+l-N)!(k-N)!(n+l-N)!}{n!\ k!\ l!\ N!\ (n+k+l+1-N)!}(n+k+l+1-2N)\left(\tfrac{k!}{(k-N)!}\tfrac{n!}{(n-N)!}\right)^2$$

$$=\tfrac{(n-1-N)!(k+l-N)!(k-N)!(n+l-N)!}{(n-1)!\ k!\ l!\ N!\ (n+k+l-N)!}\left(\tfrac{k!}{(k-N)!}\tfrac{(n-1)!}{(n-1-N)!}\right)^2$$

$$+\tfrac{(n-N)!(k+l+1-N)!(k+1-N)!(n+l-N)!}{(n-1)!\ k!\ l!\ (N-1)!(n+k+l+1-N)!}(k+1-N)\left(\tfrac{k!}{(k+1-N)!}\tfrac{(n-1)!}{(n-N)!}\right)^2$$

$$\Longleftrightarrow\ \tfrac{(k+l-N)!(n+l-N)!\ k!\ n!}{l!\ N!\ (n+k+l+1-N)!(k-N)!(n-N)!}(n+k+l+1-2N)$$

$$=\tfrac{(k+l-N)!(n+l-N)!\ k!\ (n-1)!}{l!\ N!\ (n+k+l-N)!(k-N)!(n-1-N)!}+\tfrac{(k+l-N)!(n+l-N)!\ k!\ (n-1)!(k+l+1-N)}{l!\ (N-1)!\ (n+k+l+1-N)!(k-N)!(n-N)!}$$

$$\Longleftrightarrow\ (n+k+l+1-2N)n=(n+k+l+1-N)(n-N)+N(k+l+1-N)$$

$$\Longleftrightarrow\ (n+k+l+1-2N)n=(n+k+l+1-N)n-nN-N(k+l+1-N)+N(k+l+1-N)$$

$$\Longleftrightarrow\ (n+k+l+1-2N)n=(n+k+l+1-N)n-nN$$

$$\Longleftrightarrow\ (n+k+l+1-2N)n=(n+k+l+1-2N)n\ . \qquad \checkmark$$

e) $\forall m,k,l\in\mathbb{N}_0$ holds

$$P_{m,0}P_{k,l}=\sum_{i=0}^{M}g(m,0;k,l;i)P_{m+k-i,l-i}\ .$$

Therefore we use the fact that $\overline{P_{m,n}}=P_{n,m}$, i.e. complex conjugation of the disc polynomials leads to a change of the two indices:

$$\overline{P_{m,0}P_{k,l}}=P_{0,m}P_{l,k}\overset{d)}{=}\sum_{i=0}^{M}g(0,m;l,k;i)P_{l-i,m+k-i}=\sum_{i=0}^{M}g(0,m;l,k;i)\overline{P_{m+k-i,l-i}}\ .$$

On the other hand by Theorem 5.2 we see $\exists g(m,0;k,l;i) \in \mathbb{R}$:

$$\overline{P_{m,0}P_{k,l}} = \sum_{i=0}^{M} g(m,0;k,l;i)\overline{P_{m+k-i,l-i}} \ .$$

So by comparison of the coefficients of the disc polynomials it holds:

$$g(m,0;k,l;i) \overset{!}{=} g(0,m;l,k;i) \quad \forall i: 0 \le i \le M \quad :$$

$$\tfrac{(m+0-i)!(k+l-i)!(m+k-i)!(0+l-i)!}{m!\ 0!\ k!\ l!\ i!\ (m+0+k+l+1-i)!}(m+k+l+1-2i)\left(\sum_{j=0}^{i}(-1)^j\tfrac{k!}{(k-i+j)!}\tfrac{0!}{(-i+j)!}\tfrac{l!}{(l-j)!}\tfrac{m!}{(m-j)!}\tbinom{i}{j}\right)^2$$

$$\overset{!}{=} \tfrac{(0+m-i)!(l+k-i)!(0+l-i)!(m+k-i)!}{0!\ m!\ l!\ k!\ i!\ (0+m+l+k+1-i)!}(m+l+k+1-2i)\left(\sum_{j=0}^{i}(-1)^j\tfrac{l!}{(l-i+j)!}\tfrac{m!}{(m-i+j)!}\tfrac{k!}{(k-j)!}\tfrac{0!}{(0-j)!}\tbinom{i}{j}\right)^2$$

$$\iff \tfrac{(m-i)!(k+l-i)!(m+k-i)!(l-i)!}{m!\ k!\ l!\ i!\ (m+k+l+1-i)!}(m+k+l+1-2i)\left(\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\right)^2$$

$$= \tfrac{(m-i)!(l+k-i)!(l-i)!(m+k-i)!}{m!\ l!\ k!\ i!\ (m+l+k+1-i)!}(m+l+k+1-2i)\left(\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\right)^2 \ . \qquad \checkmark$$

That means for now we have proved that for arbitrary $m,k,l,N \in \mathbb{N}_0$ the linearization coefficients of the disc polynomials for the parameter $\alpha = 0$ of the form $g(m,0;k,l;N)$ and $g(0,m;k,l;N)$ are given by the formula in the theorem.

f) now we prove the correctness of the general cases. With part b) of the proof we see that:

$$\begin{aligned}
P_{1,0}P_{m-1,n}P_{k,l} =& P_{1,0}\sum_{i=0}^{M} g(m-1,n,k,l,i)P_{m+k-1-i,n+l-i} \\
\overset{b)}{=}& \sum_{i=0}^{M_1} g(m-1,n;k,l;i)\frac{m+k-i}{m+n+k+l-2i}P_{m+k-i,n+l-i} \\
&+ \sum_{i=0}^{M_2} g(m-1,n;k,l;i)\frac{n+l-i}{m+n+k+l-2i}P_{m+k-1-i,n+l-1-i} \ .
\end{aligned}$$

On the other hand one has

$$\begin{aligned}
P_{1,0}P_{m-1,n}P_{k,l} \overset{b)}{=}& \left(\frac{m}{m+n}P_{m,n} + \frac{n}{m+n}P_{m-1,n-1}\right)P_{k,l} \\
=& \frac{m}{m+n}\sum_{i=0}^{M_3} g(m,n;k,l;i)P_{m+k-i,n+l-i} \\
&+ \frac{n}{m+n}\sum_{i=0}^{M_4} g(m-1,n-1;k,l;i)P_{m+k-1-i,n+l-1-i} \ .
\end{aligned}$$

Like in case d) one can get the unknown linearization coefficients $g(m,n;k,l;i)$ from known ones, namely, $g(m-1,n-1;k,l;i)$ and $g(m-1,n;k,l;i)$ via

$$\begin{aligned}\tfrac{m}{m+n}\sum_{i=0}^{M_3} g(m,n;k,l;i)P_{m+k-i,n+l-i} \overset{!}{=} &\sum_{i=0}^{M_1} g(m-1,n;k,l;i)\tfrac{m+k-i}{m+n+k+l-2i}P_{m+k-i,n+l-i}\\ &+\sum_{i=0}^{M_2} g(m-1,n;k,l;i)\tfrac{n+l-i}{m+n+k+l-2i}P_{m+k-1-i,n+l-1-i}\\ &-\tfrac{n}{m+n}\sum_{i=0}^{M_4} g(m-1,n-1;k,l;i)P_{m+k-1-i,n+l-1-i}\ .\end{aligned}$$

So again we use induction, this time over m. Let $k,l \in \mathbb{N}_0,\ n \in \mathbb{N}$ be arbitrary and the beginning cases of the induction are given in a),b),c),d),e).

Here, for the terms at the end and the beginning of the left hand side sum, we have to distinguish between six cases, depending on M_3.

1. case: $M_3 = m+n \le k+l, m+k, n+l$:
$\Rightarrow M_1 = m+n-1\ , M_2 = m+n-1\ , M_4 = m+n-2$ and we have to prove:

i) (Rel1) $\frac{m}{m+n}\ g(m,n;k,l;0) = \frac{m+k}{m+n+k+l}\ g(m-1,n;k,l;0)$

ii) $\frac{m}{m+n}\ g(m,n;k,l;m+n) = \frac{n+l-(m+n-1)}{m+n+k+l-2(m+n-1)}\ g(m-1,n;k,l;m+n-1)$

iii) (Rel2) $\frac{m}{m+n}\ g(m,n;k,l;N) = \frac{m+k-N}{m+n+k+l-2N}\ g(m-1,n;k,l;N)$
$+\frac{n+l-(N-1)}{m+n+k+l-2(N-1)}\ g(m-1,n;k,l;N-1) - \frac{n}{m+n}\ g(m-1,n-1;k,l;N-1)\ ,$
for $1 \le N \le m+n-1$

2. case: $M_3 = n+l \le k+l\ ;\ n+l < m+n\ ,\ m+k$
$\Rightarrow M_1 = n+l\ , M_2 = n+l-1\ , M_4 = n+l-1$ and we have to prove:
i) (Rel1)
ii) (Rel2) for $1 \le N \le n+l$

3. case: $M_3 = n+l = m+k \le k+l\ ;\ n+l = m+k < m+n$
is an impossible case, because:
$m+k < m+n \Rightarrow k < n \Rightarrow k+l < n+l$ which is a contradition to $k+l \ge n+l$.

4. case: $M_3 = m+k \le k+l\ ;\ m+k < m+n\ ,\ n+l$
$\Rightarrow M_1 = m+k-1\ , M_2 = m+k-1\ , M_4 = m+k-1$ and we have to prove:
i) (Rel1)
ii) (Rel2) for $1 \le N \le m+k-1$

iii) $\frac{m}{m+n}\ g(m,n;k,l;m+k) = \frac{n+l-(m+k-1)}{m+n+k+l-2(m+k-1)}\ g(m-1,n;k,l;m+k-1)$
$-\frac{n}{m+n}\ g(m-1,n-1;k,l;m+k-1)$

5. case: $M_3 = k+l < m+k \ , \ n+l \ , \ m+n-1$
$\Rightarrow M_1 = k+l \ , M_2 = k+l \ , M_4 = k+l$ and we have to prove:
i) (Rel1)
ii) (Rel2) for $1 \leq N \leq k+l$
iii) $\frac{n+l-(k+l)}{m+n+k+l-2(k+l)} \ g(m-1,n;k,l;k+l) = \frac{n}{m+n} \ g(m-1,n-1;k,l;k+l)$

6. case: $M_3 = k+l = m+n-1 \ ; \ k+l < m+k \ , \ n+l$
also is an impossible case, because:
$k+l < m+k \Rightarrow l < m \ ; \ k+l < n+l \Rightarrow k < n \Rightarrow l+1 \leq m \ , \ k < n \Rightarrow k+l+1 < m+n$ which is a contradition to $k+l = m+n-1$.

Proof of (Rel1):

$$\frac{m}{m+n} \ g(m,n;k,l;0) \stackrel{!}{=} \frac{m+k}{m+n+k+l} \ g(m-1,n;k,l;0)$$

$$\Leftrightarrow \frac{m}{m+n} \frac{(m+n)!(k+l)!(m+k)!(n+l)!(m+n+k+l+1)}{m! \ n! \ k! \ l! \ (m+n+k+l+1)!} = \frac{m+k}{m+n+k+l} \frac{(m+n-1)!(k+l)!(m+k-1)!(n+l)!(m+n+k+l)}{(m-1)! \ n! \ k! \ l! \ (m+n+k+l)!}$$

$$\Longleftrightarrow \frac{(m+n-1)!(k+l)!(m+k)!(n+l)!}{(m-1)! \ n! \ k! \ l! \ (m+n+k+l)!} = \frac{(m+n-1)!(k+l)!(m+k-1)!(n+l)!}{(m-1)! \ n! \ k! \ l! \ (m+n+k+l)!} \ . \qquad \checkmark$$

Proof of case 1,ii)

$$\frac{m}{m+n} \ g(m,n;k,l;m+n) = \frac{l+1-m}{m+n+k+l-2(m+n-1)} \ g(m-1,n;k,l;m+n-1)$$

$$\Leftrightarrow \frac{m}{m+n} \frac{(m+n-(m+n))!(k+l-(m+n))!(m+k-(m+n))!(n+l-(m+n))!(m+n+k+l+1-2(m+n))}{m! \ n! \ k! \ l! \ (m+n)!(m+n+k+l+1-(m+n))!}$$
$$\left(\sum_{i=0}^{m+n} (-1)^i \frac{k!}{(k-(m+n)+i)!} \frac{n!}{(n-(m+n)+i)!} \frac{l!}{(l-i)!} \frac{m!}{(m-i)!} \binom{m+n}{i}\right)^2$$
$$= \frac{(m+n-1-(m+n-1))!(k+l-(m+n-1))!(m+k-1-(m+n-1))!(n+l-(m+n-1))!(l+1-m)}{(m-1)! \ n! \ k! \ l! \ (m+n-1)!(m+n+k+l-(m+n-1))!}$$
$$\left(\sum_{i=0}^{m+n-1} (-1)^i \frac{k!}{(k-(m+n-1)+i)!} \frac{n!}{(n-(m+n-1)+i)!} \frac{l!}{(l-i)!} \frac{(m-1)!}{(m-1-i)!} \binom{m+n-1}{i}\right)^2$$
$$\Leftrightarrow \frac{(k+l-m-n)!(k-n)!(l-m)!}{(m-1)! \ n! \ k! \ l! \ (m+n-1)!(k+l+1)!} \frac{k+l+1-m-n}{(m+n)^2} \left(\frac{k!}{(k-n)!} \frac{n!}{0!} \frac{l!}{(l-m)!} \frac{m!}{0!} \frac{(m+n)!}{m! \ n!}\right)^2$$
$$= \frac{(k+l+1-m-n)!(k-n)!(l-m)!}{(m-1)! \ n! \ k! \ l! \ (m+n-1)!(k+l+1)!} (l+1-m)^2 \left(\frac{k!}{(k-n)!} \frac{n!}{0!} \frac{l!}{(l+1-m)!} \frac{(m-1)!}{0!} \frac{(m+n-1)!}{(m-1)! \ n!}\right)^2$$
$$\Leftrightarrow \frac{1}{(m+n)^2} \left(\frac{k!}{(k-n)!} \frac{l!}{(l-m)!} (m+n)!\right)^2 = (l+1-m)^2 \left(\frac{k!}{(k-n)!} \frac{l!}{(l+1-m)!} (m+n-1)!\right)^2 \ . \quad \checkmark$$

Proof of case 4,iii)

$$\frac{m}{m+n} \ g(m,n;k,l;m+k) = \frac{n+l+1-m-k}{m+n+k+l-2(m+k-1)} \ g(m-1,n;k,l;m+k-1)$$
$$- \frac{n}{m+n} \ g(m-1,n-1;k,l;m+k-1)$$

$$\Leftrightarrow \frac{m}{m+n} \frac{(m+n-(m+k))!(k+l-(m+k))!(m+k-(m+k))!(n+l-(m+k))!(m+n+k+l+1-2(m+k))}{m! \ n! \ k! \ l! \ (m+k)!(m+n+k+l+1-(m+k))!}$$

$$\left(\sum_{i=0}^{m+k}(-1)^i \frac{k!}{(k-(m+k)+i)!}\frac{n!}{(n-(m+k)+i)!}\frac{l!}{(l-i)!}\frac{m!}{(m-i)!}\binom{m+k}{i}\right)^2$$

$$= \frac{(m+n-1-(m+k-1))!(k+l-(m+k-1))!(m+k-1-(m+k-1))!(n+l-(m+k-1))!(n+l+1-m-k)}{(m-1)!\ n!\ k!\ l!\ (m+k-1)!(m+n+k+l-(m+k-1))!}$$

$$\left(\sum_{i=0}^{m+k-1}(-1)^i \frac{k!}{(k-(m+k-1)+i)!}\frac{n!}{(n-(m+k-1)+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{m+k-1}{i}\right)^2$$

$$-\frac{n}{m+n}\frac{(m+n-2-(m+k-1))!(k+l-(m+k-1))!(m+k-1-(m+k-1))!(n+l-(m+k-1))!(m+n+k+l-1-2(m+k-1))}{(m-1)!\ (n-1)!\ k!\ l!\ (m+k-1)!(m+n+k+l-1-(m+k-1))!}$$

$$\left(\sum_{i=0}^{m+k-1}(-1)^i \frac{k!}{(k-(m+k-1)+i)!}\frac{(n-1)!}{(n-1-(m+k-1)+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{m+k-1}{i}\right)^2$$

$$\Leftrightarrow \frac{(n-k)!(l-m)!(n+l-m-k)!}{(m-1)!\ n!\ k!\ l!\ (m+k)!(n+l+1)!}\frac{n+l+1-m-k}{m+n}\left(k!\frac{n!}{(n-k)!}\frac{l!}{(l-m)!}m!\frac{(m+k)!}{m!\ k!}\right)^2$$

$$= \frac{(n-k)!(l+1-m)!(n+l+1-m-k)!}{(m-1)!\ n!\ k!\ l!\ (m+k-1)!(n+l+1)!}(n+l+1-m-k)\left(k!\frac{n!}{(n-k)!}\frac{l!}{(l+1-m)!}(m-1)!\frac{(m+k-1)!}{(m-1)!\ k!}\right)^2$$

$$-\frac{(n-1-k)!(l+1-m)!(n+l+1-m-k)!}{(m-1)!\ n!\ k!\ l!\ (m+k-1)!(n+l)!}\frac{n^2}{m+n}\left(k!\frac{(n-1)!}{(n-1-k)!}\frac{l!}{(l+1-m)!}(m-1)!\frac{(m+k-1)!}{(m-1)!\ k!}\right)^2$$

$$\begin{aligned}\Leftrightarrow (l+1-m)(m+k) &= (n+l+1-m-k)(m+n)-(n+l+1)(n-k)\\ &= (l+1-m)(m+n)+(n-k)(m+n)-(n+l+1)(n-k)\\ &= (l+1-m)(m+n)-(n-k)(l+1-m)\\ &= (l+1-m)(m+k)\ . \qquad \checkmark\end{aligned}$$

Proof of case 5, iii)

$$\frac{n+l-(k+l)}{m+n+k+l-2(k+l)}\ g(m-1,n;k,l;k+l) \overset{!}{=} \frac{n}{m+n}\ g(m-1,n-1;k,l;k+l)$$

$$\Leftrightarrow \frac{n-k}{m+n+k+l-2(k+l)}\frac{(m+n-1-(k+l))!(k+l-(k+l))!(m+k-1-(k+l))!(n+l-(k+l))!(m+n+k+l-2(k+l))}{(m-1)!\ n!\ k!\ l!\ (k+l)!(m+n+k+l-(k+l))!}$$

$$\left(\sum_{i=0}^{k+l}(-1)^i \frac{k!}{(k-(k+l)+i)!}\frac{n!}{(n-(k+l)+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{k+l}{i}\right)^2$$

$$= \frac{n}{m+n}\frac{(m+n-2-(k+l))!(k+l-(k+l))!(m+k-1-(k+l))!(n+l-1-(k+l))!(m+n+k+l-1-2(k+l))}{(m-1)!\ (n-1)!\ k!\ l!\ (k+l)!(m+n+k+l-1-(k+l))!}$$

$$\left(\sum_{i=0}^{k+l}(-1)^i \frac{k!}{(k-(k+l)+i)!}\frac{(n-1)!}{(n-1-(k+l)+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{k+l}{i}\right)^2$$

$$\Leftrightarrow (n-k)\frac{(m+n-1-k-l)!(m-1-l)!(n-k)!}{(m-1)!\ n!\ k!\ l!\ (m+n)!}\left(k!\frac{n!}{(n-k)!}\ l!\frac{(m-1)!}{(m-1-l)!}\frac{(k+l)!}{k!\ l!}\right)^2$$

$$= \frac{(m+n-2-k-l)!(m-1-l)!(n-1-k)!}{(m-1)!\ n!\ k!\ l!\ (m+n-1)!}\frac{(m+n-1-k-l)n^2}{m+n}\left(k!\frac{(n-1)!}{(n-1-k)!}\ l!\frac{(m-1)!}{(m-1-l)!}\frac{(k+l)!}{k!\ l!}\right)^2$$

$$\Leftrightarrow (n-k)^2\left(\frac{n!}{(n-k)!}\frac{(m-1)!}{(m-1-l)!}(k+l)!\right)^2 = n^2\left(\frac{(n-1)!}{(n-1-k)!}\frac{(m-1)!}{(m-1-l)!}(k+l)!\right)^2\ . \qquad \checkmark$$

Finally, the last step is to prove (Rel2), what is equivalent to:

$$\text{(Eq1)}: \qquad m\ g(m,n;k,l;N) - \frac{(m+n)(m+k-N)}{m+n+k+l-2N} g(m-1,n;k,l;N)$$
$$\stackrel{!}{=} \frac{(m+n)(n+l+1-N)}{m+n+k+l-2(N-1)} g(m-1,n;k,l;N-1) - n\ g(m-1,n-1;k,l;N-1)\ .$$

At first we treat the right hand side and the left hand side separately. The left hand side is:

$$\begin{aligned} LHS &:= m \tfrac{(m+n-N)!(k+l-N)!(m+k-N)!(n+l-N)!}{m!\ n!\ k!\ l!\ N!\ (m+n+k+l+1-N)!} (m+n+k+l+1-2N) \\ &\qquad \left(\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N}{i} \right)^2 \\ &\quad -(m+n) \tfrac{m+k-N}{m+n+k+l-2N} \tfrac{(m+n-1-N)!(k+l-N)!(m+k-1-N)!(n+l-N)!}{(m-1)!\ n!\ k!\ l!\ N!\ (m+n+k+l-N)!} (m+n+k+l-2N) \\ &\qquad \left(\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N}{i} \right)^2 \end{aligned}$$

$$\Rightarrow \tfrac{(m-1)!\ n!\ k!\ l!\ N!\ (m+n+k+l+1-N)!}{(m+n-1-N)!(k+l-N)!(m+k-N)!(n+l-N)!} \cdot LHS$$
$$\begin{aligned} &= (m+n-N)(m+n+k+l+1-2N) \left(\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N}{i} \right)^2 \\ &\qquad -(m+n)(m+n+k+l+1-N) \left(\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N}{i} \right)^2 \\ &= [(m+n)(m+n+k+l+1-N) - N(m+n+k+l+1-2N) - N(m+n)] \times \\ &\qquad \left(\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N}{i} \right)^2 \\ &\quad -(m+n)(m+n+k+l+1-N) \left(\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N}{i} \right)^2 \ . \end{aligned}$$

With the binomial theorem via $(S_1+S_2)(S_1-S_2) = S_1^2 - S_2^2$ we can go on

$$\begin{aligned} &= (m+n)(m+n+k+l+1-N) \times \\ &\left[\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N}{i} - \sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N}{i} \right] \times \\ &\left[\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N}{i} + \sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N}{i} \right] \\ &-N\big(2(m+n-N) + (k+l+1)\big) \left(\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N}{i} \right)^2 \end{aligned}$$

$$= (m+n)(m+n+k+l+1-N)\times$$
$$\Bigg[\sum_{i=0}^{N}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\big(m-(m-i)\big)\tbinom{N}{i} \times$$
$$\sum_{i=0}^{N}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\big(m+(m-i)\big)\tbinom{N}{i}\Bigg]$$
$$-N\big(2(m+n-N)+(k+l+1)\big)\left(\sum_{i=0}^{N}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N}{i}\right)^2$$
$$= (m+n)(m+n+k+l+1-N)\left[\sum_{i=0}^{N}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\ i\ \tbinom{N}{i} \times\right.$$
$$\left.\left\{2\sum_{i=0}^{N}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N}{i} - \sum_{i=0}^{N}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\ i\ \tbinom{N}{i}\right\}\right]$$
$$-N\big(2(m+n-N)+(k+l+1)\big)\left(\sum_{i=0}^{N}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N}{i}\right)^2$$

and together with the wellknown fact $i\binom{N}{i} = N\binom{N-1}{i-1}$ and that the sum starts at $i=1$, as for $i=0$ the summand is zero

$$= (m+n)(m+n+k+l+1-N)\left[\sum_{i=1}^{N}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\ N\ \tbinom{N-1}{i-1}\right.$$
$$\left.\left\{2\sum_{i=0}^{N}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N}{i} - \sum_{i=1}^{N}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\ N\ \tbinom{N-1}{i-1}\right\}\right]$$
$$-N\big(2(m+n-N)+(k+l+1)\big)\left(\sum_{i=0}^{N}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N}{i}\right)^2$$

index shifting and Lemma 5.4 give

$$= -(m+n)(m+n+k+l+1-N)N\left[\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\times\right.$$
$$\left\{\sum_{i=0}^{N-1}\tfrac{2(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i} - \sum_{i=0}^{N-1}\tfrac{2(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{m!}{(m-1-i)!}\tbinom{N-1}{i}\right.$$
$$\left.\left.+N\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right\}\right]$$
$$-N\big(2(m+n-N)+(k+l+1)\big)\left(\sum_{i=0}^{N}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N}{i}\right)^2$$
$$= -(m+n)(m+n+k+l+1-N)N\Big[$$
$$2\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}$$

$$-2m\left(\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$\left.+N\left(\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2\right]$$

$$-N\Big(2(m+n-N)+(k+l+1)\Big)\Big\{$$

$$\left(\sum_{i=0}^{N-1}(-1)^i\tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}\right)^2$$

$$-2\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{m!}{(m-1-i)!}\tbinom{N-1}{i}$$

$$\left.+\left(\sum_{i=0}^{N-1}(-1)^i\tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{m!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2\right\}$$

$$= N\Big(2m(m+n)(m+n+k+l+1-N)-N(m+n)(m+n+k+l+1-N)$$

$$-m^2[2(m+n-N)+(k+l+1)]\Big)\left(\sum_{i=0}^{N-1}(-1)^i\tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$-N\Big(2(m+n-N)+(k+l+1)\Big)\left(\sum_{i=0}^{N-1}(-1)^i\tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}\right)^2$$

$$+N\Big(2m\big(2(m+n-N)+(k+l+1)\big)-2(m+n)(m+n+k+l+1-N)\Big)\times$$

$$\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}$$

$$= N\Big((m+n-N)(m+n)(2n+k+l+1-N)-n^2\big(2(m+n-N)+(k+l+1)\big)\Big)\times$$

$$\left(\sum_{i=0}^{N-1}(-1)^i\tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$-N\Big(2(m+n-N)+(k+l+1)\Big)\left(\sum_{i=0}^{N-1}(-1)^i\tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}\right)^2$$

$$+N\Big(2(m+n)(m+n-N)-2n\big(2(m+n-N)+(k+l+1)\big)\Big)\times$$

$$\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}$$

and again with the binomial theorem via $(nS_1+S_2)^2 = n^2S_1^2 + 2nS_1S_2 + S_2^2$

$$= N(m+n-N)(m+n)(2n+k+l+1-N)\times$$

$$\left(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$+2N(m+n)(m+n-N)\times$$

$$\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}$$

$$-N\Big[2(m+n-N)+(k+l+1)\Big]\left(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{m!}{(m-i)!}\tbinom{N-1}{i}\right.$$

$$\left.+n\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2 .$$

Now we consider the right hand side of (Eq1):

$$RHS := \tfrac{(m+n-1-N)!(k+l-N)!(m+k-N)!(n+l-N)}{(m-1)!\ n!\ k!\ l!\ (N-1)!\ (m+n+k+l+1-N)!}\Big[(m+n)(m+n-N)(k+l+1-N)\times$$

$$(n+l+1-N)^2\left(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$-n^2(k+l+1-N)(m+n+k+l+1-2N)(m+n+k+l+1-N)\times$$

$$\left(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{(n-1)!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2\Big] .$$

We divide by the common factor, and split the term $(n+l+1-N) = (n+1-N+i)$ $+(l-i)$:

$$\tfrac{(m-1)!\ n!\ k!\ l!\ (N-1)!\ (m+n+k+l+1-N)!}{(m+n-1-N)!(k+l-N)!(m+k-N)!(n+l-N)}\cdot RHS = (m+n)(m+n-N)(k+l+1-N)\times$$

$$\left(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}(n+1-N+i)\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right.$$

$$\left.+\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-i)!}(l-i)\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$-(k+l+1-N)^2(m+n+k+l+1-N)\left(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$-(k+l+1-N)(m+n-N)(m+n+k+l+1-N)\left(\sum_{i=0}^{N-1}\tfrac{(-1)^i k!}{(k+1-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$= (m+n)(m+n-N)(k+l+1-N)\left(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right.$$

$$\left.+\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\right)^2$$

$$-(k+l+1-N)^2(m+n+k+l+1-N)\left(\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2$$

$$-(k+l+1-N)^2(m+n-N)\left(\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2$$

$$-(k+l+1-N)(m+n-N)(m+n)\left(\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2$$

$$= (m+n)(m+n-N)(k+l+1-N)\left(\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2$$

$$+2(m+n)(m+n-N)(k+l+1-N)\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\times$$

$$\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n+1-N+i)!}\frac{l!}{(l-1-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}$$

$$+(m+n)(m+n-N)(k+l+1-N)\left(\sum_{i=0}^{N-1}\frac{(-1)^ik!}{(k+1-N+i)!}\frac{n!}{(n+1-N+i)!}\frac{l!}{(l-1-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2$$

$$-(k+l+1-N)^2\big(2(m+n-N)+(k+l+1)\big)\left(\sum_{i=0}^{N-1}\frac{(-1)^ik!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2$$

$$-(k+l+1-N)(m+n-N)(m+n)\left(\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2.$$

The first and the last term sum up to zero, and in the second and fourth term we split up the term $(k+l+1-N) = (k+1-N+i)+(l-i)$

$$= 2(m+n)(m+n-N)\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\times$$

$$\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n+1-N+i)!}\frac{l!}{(l-1-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}$$

$$+2(m+n)(m+n-N)\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-1-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\times$$

$$\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n+1-N+i)!}\frac{l!}{(l-1-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}$$

$$+(m+n)(m+n-N)(k+l+1-N)\left(\sum_{i=0}^{N-1}\frac{(-1)^ik!}{(k+1-N+i)!}\frac{n!}{(n+1-N+i)!}\frac{l!}{(l-1-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2$$

$$-\big(2(m+n-N)+(k+l+1)\big)\left(\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right.$$

$$\left.+\sum_{i=0}^{N-1}(-1)^i\frac{k!}{(k+1-N+i)!}\frac{n!}{(n-N+i)!}\frac{l!}{(l-1-i)!}\frac{(m-1)!}{(m-1-i)!}\binom{N-1}{i}\right)^2.$$

The factors by which we divided the RHS and the LHS are equal up to N, so if we bring together the two sides again, divide by the common factors, and subtract the RHS from the LHS, and simplify common sums, we arrive at

$$
\begin{aligned}
0 \stackrel{!}{=} {} & 2n(m+n)(m+n-N) \left(\sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n+1-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \right)^2 \\
& +2(m+n)(m+n-N) \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N-1}{i} \times \\
& \qquad \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n+1-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \\
& -2(m+n)(m+n-N) \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \times \\
& \qquad \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n+1-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \\
& -2(m+n)(m+n-N) \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \times \\
& \qquad \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n+1-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \\
& -\big[2(m+n-N) + (k+l+1)\big] \Bigg\{ \Bigg(\sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N-1}{i} \\
& \qquad +n \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n+1-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \Bigg)^2 \\
& \qquad - \Bigg(\sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \\
& \qquad\qquad + \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \Bigg)^2 \Bigg\} .
\end{aligned}
$$

Now we add the first with the fourth sum, the second with the third, and again use the binomial theorem in the last term via $a^2 - b^2 = (a+b)(a-b)$

$$
\begin{aligned}
= {} & 2(m+n)(m+n-N) \Bigg[\sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n+1-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \times \\
& \qquad \sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n+1-N+i)!} \big(n - (n+1-N+i)\big) \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \Bigg] \\
& +2(m+n)(m+n-N) \Bigg[\sum_{i=0}^{N-1} (-1)^i \tfrac{k!}{(k+1-N+i)!} \tfrac{n!}{(n+1-N+i)!} \tfrac{l!}{(l-1-i)!} \tfrac{(m-1)!}{(m-1-i)!} \tbinom{N-1}{i} \times
\end{aligned}
$$

$$\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\big(m-(m-i)\big)\tbinom{N-1}{i}\Big]$$

$$-\big(2(m+n-N)+(k+l+1)\big)\Big\{$$

$$\Big(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\big(n+(n+1-N+i)\big)\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}$$

$$+\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\big(m+(m-i)\big)\tbinom{N-1}{i}\Big)$$

$$\Big(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\big(n-(n+1-N+i)\big)\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}$$

$$+\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\big(m-(m-i)\big)\tbinom{N-1}{i}\Big)\quad\Big\}$$

$$=2(m+n)(m+n-N)\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}\tbinom{N-1}{i}\times$$

$$\Big(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}(N-1-i)\tfrac{(N-1)!}{i!(N-1-i)!}$$

$$+\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\ i\ \tfrac{(N-1)!}{i!(N-1-i)!}\Big)$$

$$-\big(2(m+n-N)+(k+l+1)\big)\Big\{$$

$$\Big(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}(2n+1-N+i)\tbinom{N-1}{i}$$

$$+\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}(2m-i)\tbinom{N-1}{i}\Big)$$

$$\Big(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}(N-1-i)\tfrac{(N-1)!}{i!(N-1-i)!}$$

$$+\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\ i\ \tfrac{(N-1)!}{i!(N-1-i)!}\Big)\quad\Big\}\,.$$

Finally the proof is complete as we can show that one factor of each product is zero, namely:

$$\Big(\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k+1-N+i)!}\tfrac{n!}{(n+1-N+i)!}\tfrac{l!}{(l-1-i)!}\tfrac{(m-1)!}{(m-1-i)!}(N-1-i)\tfrac{(N-1)!}{i!(N-1-i)!}$$

$$+\sum_{i=0}^{N-1}(-1)^i \tfrac{k!}{(k-N+i)!}\tfrac{n!}{(n-N+i)!}\tfrac{l!}{(l-i)!}\tfrac{(m-1)!}{(m-i)!}\ i\ \tfrac{(N-1)!}{i!(N-1-i)!}\Big)$$

$$= \left(\sum_{i=0}^{N-2} (-1)^i \frac{k!}{(k+1-N+i)!} \frac{n!}{(n+1-N+i)!} \frac{l!}{(l-1-i)!} \frac{(m-1)!}{(m-1-i)!} \frac{(N-1)!}{i!(N-2-i)!} \right.$$

$$\left. + \sum_{i=1}^{N-1} (-1)^i \frac{k!}{(k-N+i)!} \frac{n!}{(n-N+i)!} \frac{l!}{(l-i)!} \frac{(m-1)!}{(m-i)!} \frac{(N-1)!}{(i-1)!(N-1-i)!} \right) .$$

An index shift in the second sum gives

$$= \left(\sum_{i=0}^{N-2} (-1)^i \frac{k!}{(k+1-N+i)!} \frac{n!}{(n+1-N+i)!} \frac{l!}{(l-1-i)!} \frac{(m-1)!}{(m-1-i)!} \frac{(N-1)!}{i!(N-2-i)!} \right.$$

$$\left. - \sum_{i=0}^{N-2} (-1)^i \frac{k!}{(k+1-N+i)!} \frac{n!}{(n+1-N+i)!} \frac{l!}{(l-1-i)!} \frac{(m-1)!}{(m-1-i)!} \frac{(N-1)!}{i!(N-2-i)!} \right) = 0 .$$

■

Corollary 5.6. *For $m, n, k, l, N \in \mathbb{N}_0$ and $0 \le N \le \min\{m+n, k+l, m+k, n+l\}$ the following identity holds:*

$$\sum_{i=0}^{N} \frac{(m+n-N+i)!}{(N-i)!(m-N+i)!(n-N+i)!} \frac{(k+l-i)!}{i!(k-i)!(l-i)!}$$

$$= \sum_{i=0}^{N} (-1)^i \frac{(m+n-i)!(k+l-i)!(m+k-i)!(n+l-i)!}{m!\ n!\ k!\ l!\ i!\ (m+n+k+l+1-i)!} (m+n+k+l+1-2i) \times$$

$$\frac{(m+n+k+l-N-i)!}{(N-i)!(m+k-N)!(n+l-N)!} \left(\sum_{j=0}^{i} (-1)^j \frac{k!}{(k-i+j)!} \frac{n!}{(n-i+j)!} \frac{l!}{(l-j)!} \frac{m!}{(m-j)!} \right)^2 .$$

Proof: i)

$$\begin{aligned} P_{m,n} P_{k,l} &= \sum_{j=0}^{\min\{m,n\}} (-1)^j \frac{(m+n-j)!}{j!(m-j)!(n-j)!} z^{m-j} \bar{z}^{n-j} \sum_{i=0}^{\min\{k,l\}} (-1)^i \frac{(k+l-i)!}{i!(k-i)!(l-i)!} z^{k-i} \bar{z}^{l-i} \\ &= \sum_{j=0}^{\min\{m,n\}} \sum_{i=0}^{\min\{k,l\}} (-1)^{i+j} \frac{(m+n-j)!}{j!(m-j)!(n-j)!} \frac{(k+l-i)!}{i!(k-i)!(l-i)!} z^{m+k-(i+j)} \bar{z}^{n+l-(i+j)} \\ &= \sum_{j=0}^{\min\{m+k,n+l\}} (-1)^j z^{m+k-j} \bar{z}^{n+l-j} \sum_{i=0}^{j} \frac{(m+n-j+i)!}{(j-i)!(m-j+i)!(n-j+i)!} \frac{(k+l-i)!}{i!(k-i)!(l-i)!} \end{aligned}$$

ii)

$$\begin{aligned} P_{m,n} P_{k,l} &= \sum_{i=0}^{M} g(m,n;k,l;i) P_{m+k-i,n+l-i} \\ &= \sum_{i=0}^{M} g(m,n;k,l;i) \sum_{j=0}^{\min\{m+k-i,n+l-i\}} \frac{(-1)^j (m+n+k+l-2i-j)!}{j!(m+k-i-j)!(n+l-i-j)!} z^{m+k-i-j} \bar{z}^{n+l-i-j} \\ &= \sum_{j=0}^{\min\{m+k,n+l\}} (-1)^j z^{m+k-j} \bar{z}^{n+l-j} \sum_{i=0}^{j} g(m,n;k,l;i)(-1)^i \frac{(m+n+k+l-i-j)!}{(j-i)!(m+k-j)!(n+l-j)!} \end{aligned}$$

By comparing the coefficients of $z^{m+k-j} \bar{z}^{n+l-j}$ the identity follows. ■

Corollary 5.7. *An alternative representation of the linearization coefficients of the disc polynomials for $\alpha = 0$ is given by*

$$g(m,n;k,l;N) = \frac{m!\ n!\ k!\ l!\ (m+n+k+l+1-2N)}{N!\ (m+n+k+l+1-N)!} \times$$
$$\sum_{i=0}^{N} (-1)^i \binom{m+k-N}{m-i} \binom{n+l-N}{l-i} \sum_{j=0}^{N} (-1)^j \binom{m+n-N}{m-j} \binom{k+l-N}{l-j} .$$

Proof:

$$g(m,n;k,l;N) = \frac{(m+n-N)!(k+l-N)!(m+k-N)!(n+l-N)!}{m!\ n!\ k!\ l!\ N!\ (m+n+k+l+1-N)!} \times$$
$$\times (m+n+k+l+1-2N) \left(\sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N}{i} \right)^2$$
$$= \tfrac{(m+n-N)!(k+l-N)!(m+k-N)!(n+l-N)!}{m!\ n!\ k!\ l!\ N!\ (m+n+k+l+1-N)!} \sum_{i=0}^{N} (-1)^i \tfrac{k!}{(k-N+i)!} \tfrac{n!}{(n-N+i)!} \tfrac{l!}{(l-i)!} \tfrac{m!}{(m-i)!} \tbinom{N}{i} \times$$
$$\times (m+n+k+l+1-2N) \sum_{j=0}^{N} (-1)^j \tfrac{k!}{(k-N+j)!} \tfrac{n!}{(n-N+j)!} \tfrac{l!}{(l-j)!} \tfrac{m!}{(m-j)!} \tbinom{N}{j}$$
$$= \tfrac{m!\ n!\ k!\ l!\ (m+n+k+l+1-2N)}{N!\ (m+n+k+l+1-N)!} \sum_{i=0}^{N} (-1)^i \tfrac{(m+k-N)!}{(k-N+i)!(m-i)} \tfrac{(n+l-N)!}{(n-N+i)!(l-i)!} \tbinom{N}{i} \times$$
$$\times \sum_{j=0}^{N} (-1)^j \tfrac{(k+l-N)!}{(k-N+j)!(l-j)!} \tfrac{(n+m-N)!}{(n-N+j)!(m-j)!} \tbinom{N}{j}$$

which gives the result.

With the result in Corollary 5.3 one sees that the sum of the explicit linearization coefficients sums up to one in Theorem 5.5 and Corollary 5.7.

Corollary 5.8. *For $m,n,k,l \in \mathbb{N}_0$ and $0 \le N \le \min\{m+k, n+l\}$ it holds*

$$g(m,n;k,l;N) = h^0(m+k-N, n+l-N) \int_D P_{m,n}(z) P_{k,l}(z) P_{m+k-N,n+l-N}(\bar{z}) d\mu_0(z)$$
$$= (m+n+k+l+1-2N) \int_D P_{m,n}(z) P_{k,l}(z) P_{m+k-N,n+l-N}(\bar{z}) d\mu_0(z) .$$

Proof:

$$P_{m,n}(z) P_{k,l}(z) = \sum_{i=0}^{M} g(m,n;k,l;i) P_{m+k-i,n+l-i}(z)$$
$$\Longleftrightarrow \int_D P_{m,n}(z) P_{k,l}(z) P_{m+k-N,n+l-N}(\bar{z}) d\mu_0(z)$$

$$= \sum_{i=0}^{M} g(m,n;k,l;i) \int_D P_{m+k-i,n+l-i}(z) P_{m+k-N,n+l-N}(\bar{z}) d\mu_0(z)$$

$$\Longleftrightarrow \int_D P_{m,n}(z) P_{k,l}(z) P_{m+k-N,n+l-N}(\bar{z}) d\mu_0(z) = \frac{g(m,n;k,l;N)}{h^0(m+k-N,n+l-N)}$$

where we used the orthogonality relation. With the explicit formula for the Haar function the second form follows.

5.3 On a general formula

Of course it would be desirable to have an explicit formula for the linearization coefficients for general $\alpha \geq 0$. The first two coefficients $g^\alpha(m,n;k,l;0)$ and $g^\alpha(m,n;k,l;1)$ look promising that a certain regularity might be in the formula but already the third coefficient $g^\alpha(m,n;k,l;2)$ kills this hope. As the amount of terms kind of *explodes* with increasing "i" and time for investigations was running out we give here only those first three general linearization coefficients for $2 < m,n,k,l \in \mathbb{N}$ in a closed form which we could determine explicitely.

1. $$g^\alpha(m,n;k,l;0) = \frac{(m+n+\alpha)!(k+l+\alpha)!(m+k+\alpha)!(n+l+\alpha)!\alpha!}{(n+\alpha)!(m+\alpha)!(k+\alpha)!(l+\alpha)!(m+n+k+l+\alpha)!}$$

2. $$g^\alpha(m,n;k,l;1) = \frac{(m+n+\alpha-1)!(k+l+\alpha-1)!(m+k+\alpha-1)!}{(n+\alpha)!(m+\alpha)!(k+\alpha)!(l+\alpha)!(m+n+k+l+\alpha-2)!}\times$$

$$\times\frac{(n+l+\alpha-1)!\alpha!}{(m+n+k+l+\alpha)}\Big(k(k+\alpha)n(n+\alpha) - 2\ k\ n\ m\ l + m(m+\alpha)l(l+\alpha)\Big)\ .$$

Recall here that the term in the last parenthesis resembles the corresponding term for $\alpha = 0$: $(kn - ml)^2$ in a certain way.

3. $$g^\alpha(m,n;k,l;2) = \frac{(m+n+\alpha-2)!(k+l+\alpha-2)!(m+k+\alpha-2)!}{(n+\alpha)!(m+\alpha)!(k+\alpha)!(l+\alpha)!\ 2!}\times$$

$$\times\frac{(n+l+\alpha-2)!(m+n+k+l+\alpha-3)\alpha!}{(m+n+k+l+\alpha-1)!}\times$$

$$\begin{aligned}\Big(&k(k+\alpha)(k-1)(k+\alpha-1)n(n+\alpha)(n-1)(n+\alpha-1)\\ &-4(k+\alpha)(k-1)(n+\alpha)(n-1)k\ n\ m\ l\\ &+2k(k-1)n(n-1)m(m-1)l(2l-1)\\ &+2(k+\alpha)(l+\alpha)(n+\alpha)(m+\alpha+1)k\ n\ m\ l\end{aligned}$$

$$\begin{aligned}&+2(m-1)(n+k+\alpha-1)(l+\alpha)k\ n\ m\ l\\&-4(m+\alpha)(m-1)(l+\alpha)(l-1)k\ n\ m\ l\\&m(m+\alpha)(m-1)(m+\alpha-1)l(l+\alpha)(l-1)(l+\alpha-1)\Big)\end{aligned}$$

If we compare this last term with the result for $\alpha = 0$:

$$\big(k(k-1)n(n-1) - 2knml - m(m-1)l(l-1)\big)^2$$

one sees that the terms with coefficient "2" don't fit any scheme given from $\alpha = 0$.

Anyway we believe that even a small part of the formula for $\alpha = 0$ can be taken over to a general formula. We believe that a general linearization coefficient for the disc polynomials must look like

$$g^\alpha(m,n;k,l;i) = \frac{(m+n+\alpha-i)!(k+l+\alpha-i)!(m+k+\alpha-i)!}{(n+\alpha)!(m+\alpha)!(k+\alpha)!(l+\alpha)!\ i!}\times$$

$$\times\frac{(n+l+\alpha-i)!\alpha!}{(m+n+k+l+\alpha+1-i)!}(m+n+k+l+\alpha+1-2i)\tilde{g}^\alpha(m,n;k,l;i)$$

with a suitable but hard to find $\tilde{g}^\alpha(m,n;k,l;i)$.

Exactly as in Corollary 5.8 it follows that we have for general linearization coefficients:

$$g^\alpha(m,n;k,l;N) = h^\alpha(m+k-N, n+l-N)\int_D P^\alpha_{m,n}(z)P^\alpha_{k,l}(z)P^\alpha_{m+k-N,n+l-N}(\bar{z})d\mu_\alpha(z)\ .$$

6 Approximation on the unit disc with disc polynomials

6.1 Fourier analysis on the unit disc

In chapter 2 we have seen how classical Fourier analysis is carried over to a setting where Jacobi polynomials play the role of the trigonometric polynomials in the classical Fourier analysis. It is clear that it can be tried to adopt this principle for other orthogonal polynomial systems.

In this chapter we want to set up the basic modules for Fourier analysis with disc polynomials on the unit disc. One of the main differences to the Jacobian case is the two dimensionality. At first we want to establish an order on the set of disc polynomials $\{P^{\alpha}_{m,n} \mid m, n \in \mathbb{N}_0\}$. In one dimensional cases the natural ordering of an orthogonal polynomial system $(P_n)_{n\in\mathbb{N}_0}$ works with the degree of each polynomial. As P_n has exact degree n it is clear that $P_{n+i},\ i \in \mathbb{N}$ comes on a later position than P_n. But the total degree of the following two disc polynomial is the same: $P^{\alpha}_{m-i,n}$ and $P^{\alpha}_{m,n-i}$, $0 < i \le m, n$, namely $n + m - i$, even though the polynomials are not identical. We see that with the classical term of total degree we cannot say which of these two polynomials comes first. In order to be able to distinguish the position of disc polynomials of that kind we introduce a two dimensional order on the set $\mathbb{N}_0^2$ in order to be able to compare two elements of that set.

Definition 6.1 (two dimensional order, degree). *For tuples (m, n), $(k, l)\ \in \mathbb{N}_0^2$ we say that*

$$(k, l) < (m, n) \iff k + l < m + n \ \vee\ \big(k + l = m + n\ \wedge\ l < n\big)\ .$$

We say a monomial $m(z, \bar{z}) = z^k \bar{z}^l$ has degree $(k, l)$and write degree$\big(m(z, \bar{z})\big) := (k, l)$. For any polynomial $p(z, \bar{z})$ that is a linear combination of monomials we say it has degree (k, l) if the monomial with the highest order has degree (k, l) in the sense of the introduced order, and write degree$\big(p(z, \bar{z})\big) = (k, l)$.

Together with this definition we can introduce sets of polynomials that do not exceed a certain degree.

Definition 6.2. *For $m, n \in \mathbb{N}_0$ the set of two dimensional polynomials with degree not exceeding (n, m) is given by*

$$\Pi_{n,m} := \{p(z, \bar{z}) \mid p \text{ is a polynomial in } z, \bar{z},\ \text{degree}(p) \le (n, m)\}\ .$$

For brevity we put $\Pi_n := \Pi_{0,n}$ which corresponds to the set of all polynomials with total degree $\le n$.

The set of monomials of exact degree n has dimension $n+1$, and so we see that Π_n has dimension $\frac{1}{2}(n+1)(n+2)$ and $\Pi_{n,m}$ has dimension $\frac{1}{2}(n+m)(n+m+1)+m+1$.

Now it is clear, that for the disc polynomials we have $degree\big(P_{m,n}(z,\bar{z})\big) = (m,n)$. As well as the fact that the set $\{P^{\alpha}_{k,l}(z) |\ (k,l) \leq (m,n)\}$ is a basis of $\Pi_{n,m}$. This ordering will be important later when we are looking at partial Fourier series.

Another point we need to mention here is the following: From now on for brevity we want to leave out the α in our notations. That means we fix $\alpha > 0$ and handle it like a constant and keep in mind that our formulas indeed depend on α. The reason why we chose $\alpha > 0$ is that we want to make use of the generalized translation that is only valid for that range. Some of the following definitions and theorems can also be given for $-1 < \alpha \leq 0$ but as soon as we work with the generalized translation we need $\alpha > 0$. And as our focus is on actions that work with the translation we chose only the reduced range.

Now we can start to define the required instruments for doing Fourier analysis on the unit disc.

Definition 6.3 (Fourier coefficient). *For a function $f \in L^1(D)$ and natural numbers $m,n \in \mathbb{N}_0$ the (n,m)-th Fourier coefficient is defined as*

$$\widehat{f}(m,n) := \int_D f(z)P_{m,n}(\bar{z})d\mu(z) = \int_D f(z)P_{n,m}(z)d\mu(z) \ .$$

In the next lemma we see that this definition is compatible with the generalized translation definied in chapter 3. We get a property for the Fourier coefficient well-known from classical theory.

Lemma 6.4. *For functions $f,g \in L^1(D)$ and $m,n \in \mathbb{N}_0$ it holds*

$$\widehat{f*g}(m,n) = \widehat{f}(m,n)\widehat{g}(m,n) \ .$$

Proof: Recall the definition of the translation and the function $E(x,y,z)$ of Lemma 4.4

$$\begin{aligned}
\widehat{f}(m,n)\widehat{g}(m,n) &= \int_D f(z)P_{m,n}(\bar{z},z)d\mu(z)\int_D g(\bar{y})P_{m,n}(y,\bar{y})d\mu(\bar{y}) \\
&= \int_D\int_D f(z)g(\bar{y})T_zP_{m,n}(y,\bar{y})d\mu(z)d\mu(\bar{y}) \\
&= \int_D\int_D\int_D f(z)g(\bar{y})P_{m,n}(\xi,\bar{\xi})E(z,y,\xi)d\mu(\xi)d\mu(z)d\mu(\bar{y}) \\
&= \int_D\int_D\int_D f(z)g(\bar{y})E(z,\bar{\xi},\bar{y})d\mu(\bar{y})d\mu(z)P_{m,n}(\xi,\bar{\xi})d\mu(\xi) \\
&= \int_D (g*f)\,(\bar{\xi})P_{m,n}(\xi,\bar{\xi})d\mu(\xi)
\end{aligned}$$

$$= \int_D (f * g)(\bar{\xi}) P_{m,n}(\xi, \bar{\xi}) d\mu(\xi) = \widehat{f * g}(m,n) \ .$$

With this definition of the Fourier coefficient the formal Fourier expansion and the partial Fourier series of a function $f \in L^1(D)$ can be given:

Definition 6.5 (Fourier series). *For a function $f \in L^1(D)$ and $n, m \in \mathbb{N}_0$ we define the formal Fourier expansion as*

$$f(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \widehat{f}(i,j) h(i,j) P_{i,j}(z)$$

and the (n,m)-th partial Fourier series as

$$\begin{aligned} S_{n,m} f(z) := & \sum_{i=0}^{n+m-1} \sum_{j=0}^{i} \widehat{f}(i-j,j) h(i-j,j) P_{i-j,j}(z) \\ & + \sum_{j=0}^{m} \widehat{f}(n+m-j,j) h(n+m-j,j) P_{n+m-j,j}(z) \ . \end{aligned}$$

The (n,m)-th partial Fourier series follows the described ordering of the disc polynomials which means that we sum up increasing coefficients until (n,m) appears. Of course this ordering is special for the polynomials of total degree $m+n$ and one could have chosen a different ordering. The way we sum up should illustrate that we sum up successively the product of the Fourier coefficients the Haar functions and the disc polynomials in their ordering.

For our convenience we define the following notation, for a $\beta(i,j)$ of any kind:

$$\sum_{i=0}^{n+m-1} \sum_{j=0}^{i} \beta(i-j,j) + \sum_{j=0}^{m} \beta(m+n-j,j) := \sum_{(i,j)=(0,0)}^{(n,m)} \beta(i,j)$$

such that we can write our (n,m)-th partial Fourier series as

$$S_{n,m} f(z) = \sum_{(i,j)=(0,0)}^{(n,m)} \widehat{f}(i,j) h(i,j) P_{i,j}(z) \ .$$

Of course for polynomials the formal Fourier expansion is finite and provides a linear combination of disc polynomials. As an example we will give here the expansion of a monomial:

$$z^m \bar{z}^n = \frac{m! \ n!}{\alpha!} \sum_{i=0}^{\min(m,n)} \frac{(m+\alpha-i)!(n+\alpha-i)!(m+n+\alpha+1-2i)}{i!(m-i)!(n-i)!(m+n+\alpha+1-i)!} P_{m-i,n-i}(z, \bar{z}) \ .$$

Proof: Let $g(z,\bar z) := z^m \bar z^n$. Then for $k,l \in \mathbb{N}_0$ we have

$$
\begin{aligned}
\widehat{g}(l,k) &= \int_D z^m \bar z^n P_{l,k}(\bar z, z) d\mu(z) = \int_D z^m \bar z^n P_{k,l}(z,\bar z) d\mu(z) \\
&= \int_0^{2\pi}\int_0^1 r^m e^{im\varphi} r^n e^{-in\varphi} P_{k,l}(re^{i\varphi}, re^{-i\varphi}) \frac{\alpha+1}{\pi}(1-r^2)^\alpha \; r\; dr d\varphi \\
&\text{let w.l.o.g. } l \geq k \\
&= \frac{\alpha+1}{\pi}\frac{k!\ \alpha!}{(k+\alpha)!}\int_0^{2\pi} e^{i(m+k-n-l)\varphi} d\varphi \int_0^1 r^{m+n+1+l-k} J_k^{(\alpha,l-k)}(2r^2-1)\ (1-r^2)^\alpha dr \\
&= 2(\alpha+1)\frac{k!\ \alpha!}{(k+\alpha)!}\ \delta_{m+k,n+l} \int_0^1 r^{2m+1} J_k^{(\alpha,m-n)}(2r^2-1)\ (1-r^2)^\alpha dr \\
&\text{we perform a substitution } x = 2r^2-1 \\
&= 2(\alpha+1)\frac{k!\ \alpha!}{(k+\alpha)!}\ \delta_{m+k,n+l} \int_{-1}^1 \left(\frac{x+1}{2}\right)^m J_k^{(\alpha,m-n)}(x)\ \left(\frac{1-x}{2}\right)^\alpha \frac{1}{4}\ dx \\
&= 2^{-m-\alpha-1}(\alpha+1)\frac{k!\ \alpha!}{(k+\alpha)!}\ \delta_{m+k,n+l} \int_{-1}^1 (x+1)^m J_k^{(\alpha,m-n)}(x)\ (1-x)^\alpha dx \\
&\text{now we take the Jacobi polynomial in the Rodrigues formula} \\
&= 2^{-m-k-\alpha-1}(-1)^k \frac{(\alpha+1)!}{(k+\alpha)!}\ \delta_{m+k,n+l} \int_{-1}^1 (x+1)^n \frac{d^k}{dx^k}\big((1+x)^{m-n+k}(1-x)^{k+\alpha}\big) dx \\
&\text{and } k \text{ times an integration by parts gives} \\
&= 2^{-m-k-\alpha-1}\frac{(\alpha+1)!}{(k+\alpha)!}\ \delta_{m+k,n+l}\ \frac{n!}{(n-k)!} \int_{-1}^1 (x+1)^m (1-x)^{\alpha+k} dx \\
&\text{another substitution via } x = 2r-1 \text{ gives} \\
&= 2^{-m-k-\alpha-1}\frac{(\alpha+1)!}{(k+\alpha)!}\ \delta_{m+k,n+l}\ \frac{n!}{(n-k)!} \int_0^1 (2x)^m (2-2x)^{\alpha+k} 2\ dx \\
&= \frac{(\alpha+1)!}{(k+\alpha)!}\ \delta_{m+k,n+l}\ \frac{n!}{(n-k)!} \int_0^1 x^m (1-x)^{\alpha+k} dx \\
&\text{the integral is just the beta function which provides the value} \\
&= \frac{(\alpha+1)!}{(k+\alpha)!}\ \delta_{m+k,n+l}\ \frac{n!}{(n-k)!}\ \frac{m!(k+\alpha)!}{(m+k+\alpha+1)!}\ .
\end{aligned}
$$

Altogether we arrive at

$$
\begin{aligned}
\widehat{g}(l,k)h(l,k) &= \frac{(\alpha+1)!}{(m+k+\alpha+1)!}\frac{n!m!}{(n-k)!}\delta_{m+k,n+l}\frac{(k+l+\alpha+1)}{\alpha+1}\frac{(k+\alpha)!}{k!\ \alpha!}\frac{(l+\alpha)!}{l!\ \alpha!} \\
&= \frac{n!\ m!}{\alpha!}\frac{(k+\alpha)!(l+\alpha)!(k+l+\alpha+1)}{(n-k)!\ k!\ l!\ (m+k+\alpha+1)!}\ \delta_{m+k,n+l}\ .
\end{aligned}
$$

Our assumption was $l \geq k$ from which follows together with $\delta_{m+k,n+l}$ that our monomial should have fulfilled the condition $m \leq n$ again w.l.o.g. But then we can see that only those Fourier coefficients are non-zero which fulfill $k = n - m + l$:

$$\widehat{g}(l, n-m+l)h(l, n-m+l) = \frac{n!\ m!}{\alpha!} \frac{(n-m+l+\alpha)!(l+\alpha)!(n-m+2l+\alpha+1)}{(m-l)!\ (n-m+l)!\ l!\ (n+l+\alpha+1)!} \ .$$

From the monomial $z^m \bar{z}^n$ it is clear that the range of l is $l = 0, ..., m$. We want to run through it starting from m therefore we set for $i = 0, ..., m : \ l = m - i$

$$\widehat{g}(m-i, n-i)h(m-i, n-i) = \frac{n!\ m!}{\alpha!} \frac{(n+\alpha-i)!(m+\alpha-i)!(n+m+\alpha+1-2i)}{i!\ (n-i)!\ (m-i)!\ (n+m+\alpha+1-i)!} \ .$$

Recall that we have $m = \min(m, n)$ by assumption. For the inverse case change the roles of k and l. Then the example is proved.

Like we have seen in the chapter on Jacobi polynomials and we know from classical Fourier analysis the (n, m)-th Dirichlet kernel should be the kernel whose convolution with a function gives the (n, m)-th partial Fourier series of that function. So the Dirichlet kernel is the sum of the first (n, m) disc polynomials each multiplied with the corresponding Haar function.

Definition 6.6 (Dirichlet kernel). *For disc polynomials and $n, m \in \mathbb{N}_0$ the (m, n)-th Dirichlet kernel is defined by:*

$$D_{m,n}(z) := \sum_{(k,l)=(0,0)}^{(m,n)} h(k,l) P_{k,l}(z)$$

and we want to write for brevity

$$D_n(z) := \sum_{(k,l)=(0,0)}^{(0,n)} h(k,l) P_{k,l}(z) \ .$$

So here we see that $D_{n,m} * f$ is indeed the (n, m)-th partial Fourier series of f:

$$(D_{m,n} * f)(z) = \int_D T_y D_{m,n}(z) f(y) d\mu(y) = \int_D \sum_{(k,l)=(0,0)}^{(m,n)} h(k,l) P_{k,l}(z) P_{k,l}(\bar{y}) f(y) d\mu(y)$$

$$= \sum_{(k,l)=(0,0)}^{(m,n)} h(k,l) P_{k,l}(z) \int_D P_{k,l}(\bar{y}) f(y) d\mu(y) = \sum_{(k,l)=(0,0)}^{(m,n)} h(k,l) P_{k,l}(z) \hat{f}(k,l) = S_{n,m} f(z) \ .$$

Another clear property is the fact, that the (n, m)-th Dirichlet kernel reproduces polynomials of degree (n, m) which means that the convolution with a polynomial of degree $\leq (n, m)$ gives back the original polynomial. As the set of disc polynomials $\{P_{k,l}(z) \mid (k, l) \leq (n, m)\}$ is a basis of $\Pi_{n,m}$ we can write a polynomial $q(z) \in \Pi_{n,m}$ in the following way:

$$q(z) = \sum_{(k,l)=(0,0)}^{(n,m)} \beta_{k,l} P_{k,l}(z) \ .$$

And therefore we have:

$$(D_{n,m} * q)(z) = \int_D T_y D_{n,m}(z) \sum_{(k,l)=(0,0)}^{(n,m)} \beta_{k,l} P_{k,l}(y) d\mu(y)$$

$$= \sum_{(k,l)=(0,0)}^{(n,m)} \beta_{k,l} \int_D \sum_{(i,j)=(0,0)}^{(n,m)} P_{i,j}(\bar{y}) P_{i,j}(z) h(i,j) P_{k,l}(y) d\mu(y)$$

$$= \sum_{(k,l)=(0,0)}^{(n,m)} \sum_{(i,j)=(0,0)}^{(n,m)} \beta_{k,l} P_{i,j}(z) \delta_{k,i} \delta_{l,j} = q(z) \ .$$

Another observation is the fact, that the sum of $P_{k,l}(z)$ and $P_{l,k}(z)$ is two times the real part of one of the polynomials, as they are just the complex conjugate of each other.

$$P_{k,l}(z) + P_{l,k}(z) = 2 \ \mathrm{Re}\big(P_{k,l}(z)\big) \ .$$

As $h(m, n)$ is symmetric in m and n and real and for $k \in \mathbb{N}_0$ the $P_{k,k}(z)$ are also real, we can at once conclude, that $D_N(z) \in \mathbb{R} \ \forall z \in \mathbb{C}, \ N \in \mathbb{N}_0$.

To give an illustration, we give the first three Dirichlet-Kernels of the form $D_N(z)$ for $N \in \mathbb{N}_0$ in polar coordinates where we can see $D_N(z) \in \mathbb{R}$ immediately:

$$\begin{aligned}
D_1(r,\varphi) &= 1 + (\alpha+2)(e^{i\varphi} + e^{-i\varphi})r = 1 + 2(\alpha+2)cos(\varphi)r \\
D_2(r,\varphi) &= -(\alpha+2) + (\alpha+2)(e^{i\varphi} + e^{-i\varphi})r + (\alpha+2)(\alpha+3)\left(\frac{1}{2}(e^{2i\varphi} + e^{-2i\varphi}) + 1\right) r^2 \\
&= -(\alpha+2) + 2(\alpha+2)cos(\varphi)r + (\alpha+2)(\alpha+3)\left(cos(2\varphi) + 1\right) r^2 \\
D_3(r,\varphi) &= -(\alpha+2) - 2(\alpha+3)(\alpha+2)cos(\varphi)r + (\alpha+2)(\alpha+3)\left(cos(2\varphi) + 1\right) r^2 \\
&\quad + (\alpha+4)(\alpha+3)(\alpha+2)\left(\frac{1}{3}cos(3\varphi) + cos(\varphi)\right) r^3 \ .
\end{aligned}$$

This leads to an explicit formula for Dirichlet kernels of the form $D_N(r, \varphi)$ in polar coordinates, if we order the terms in the degree of the radius.

$$D_N(r,\varphi) = \sum_{j=0}^{N}(-1)^{\lfloor\frac{j}{2}\rfloor}\frac{(\alpha+N+1-\lfloor\frac{j+1}{2}\rfloor)!}{(\alpha+1)!(\lfloor\frac{j}{2}\rfloor)!}\sum_{k=0}^{N-j}\frac{1}{k!(N-j-k)!}\; e^{i(N-j-2k)\varphi}\; r^{N-j}$$

$$= \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}(-1)^{j}\frac{(\alpha+N+1-j)!}{(\alpha+1)!j!}\sum_{k=0}^{N-2j}\frac{1}{k!(N-2j-k)!}\; e^{i(N-2j-2k)\varphi}\; r^{N-2j}$$

$$+ \sum_{j=0}^{\lfloor\frac{N-1}{2}\rfloor}(-1)^{j}\frac{(\alpha+N-j)!}{(\alpha+1)!j!}\sum_{k=0}^{N-1-2j}\frac{1}{k!(N-1-2j-k)!}\; e^{i(N-1-2j-2k)\varphi}\; r^{N-1-2j}$$

This formula can be proven by a simple induction.

6.2 A generalized de-la-Vallée-Poussin kernel

We are interested in the question whether the convolution of a kernel with a function converges to that function in norm. Like in settings with other orthogonal polynomial systems one can expect that the Dirichlet kernel for disc polynomials does not provide norm convergence in general, i.e. in L^p spaces with $p \neq 2$. In [15] there is introduced a Poisson kernel on the unit disc of the form

$$POI_s^{\alpha}(z) := \sum_{i,j} s^{\max(i,j)} h^{\alpha}(i,j) P_{i,j}^{\alpha}(z)$$

that provides norm convergence in the L^p spaces $1 \leq p \leq \infty$ in case $\alpha > 0$ and $0 \leq s < 1$. With the value $s = 1$ this Poisson kernel would just be the Dirichlet kernel.

In this chapter we define a generalized de-la-Vallée-Poussin kernel with disc polynomials that follows the example given in the introduction on approximation in the Jacobi setting. We can prove that with this kernel we get norm convergence because of the fact that its norm is uniformly bounded. Together with its reproducing property of polynomials we can use Theorem 3.7 to get this result.

But we want to go through it step by step and start by defining this generalized de-la-Vallée-Poussin kernel.

Definition 6.7 ((*) Generalized de-la-Vallée-Poussin kernel). *For $\alpha \in \mathbb{R}_+$ and two tuples $(k,l) < (m,n) \in \mathbb{N}_0^2$ the generalized de-la-Vallée-Poussin kernel is defined in the following way:*

$${}_{\alpha}\mathcal{V}_{(k,l)}^{(m,n)}(z) := \frac{D_{m,n}^{\alpha}(\bar{z})D_{k,l}^{\alpha}(z)}{D_{k,l}^{\alpha}(1)} \;.$$

The generalized de-la-Vallée-Poussin Operator ${}_{\alpha}V_{(k,l)}^{(m,n)} : L_{\alpha}^p \to L_{\alpha}^p$ for $1 \leq p \leq \infty$ is given by:

$$\left({}_{\alpha}V_{(k,l)}^{(m,n)} f\right) := \left({}_{\alpha}\mathcal{V}_{(k,l)}^{(m,n)} \underset{\alpha}{*} f\right) \;.$$

And for convinience we will later work with the following operator and kernel:
($M < N \in \mathbb{N}_0$)

$$\mathcal{V}_M^N := {}_\alpha\mathcal{V}_{(0,M)}^{(0,N)} \text{ and similarly } V_M^N := {}_\alpha V_{(0,M)}^{(0,N)} \ .$$

At first we will check the reproducing property of the general de-la-Vallée-Poussin kernel for polynomials of a certain degree, but later on we will only work with the kernels of the form $\mathcal{V}_M^N$ because we find out that their reproducing properties are optimal for our purposes.

Theorem 6.8 ((*) reproducing property). *For the generalized de-la-Vallée-Poussin kernel $\mathcal{V}_{(k,l)}^{(m,n)}$ with $(k,l) < (m,n)$ we have the following reproducing property for polynomials:*

$$\left(\mathcal{V}_{(k,l)}^{(m,n)} * P\right)(z) = P(z) \text{ or alternatively } \left(V_{(k,l)}^{(m,n)} P\right)(z) = P(z)$$

$$a) \text{ if } k \geq m \ : \ \forall P \in \Pi_{m+n-k-l}$$

$$b) \text{ if } k < m \ : \ \forall P \in \Pi_{m+n-k-l-1} \ .$$

Proof:
Step 1) $\forall\ Q \in \Pi_{m,n}$ holds $\int_D D_{m,n}(\bar{z})Q(z)d\mu(z) = Q(1)$. That can be checked in the following way:
We can write $Q \in \Pi_{m,n}$ as a linear combination of disc polynomials with suitable $\beta_{i,j} \in \mathbb{C}$:

$$Q(z) = \sum_{(i,j)=(0,0)}^{(m,n)} \beta_{i,j} P_{i,j}(z) \ , \text{ but then we have:}$$

$$\int_D D_{m,n}(\bar{z})Q(z)d\mu(z) = \int_D \sum_{(k,l)=(0,0)}^{(m,n)} h(k,l)\overline{P_{k,l}(z)} \sum_{(i,j)=(0,0)}^{(m,n)} \beta_{i,j} P_{i,j}(z)\ d\mu(z)$$

$$= \sum_{(k,l)=(0,0)}^{(m,n)} \sum_{(i,j)=(0,0)}^{(m,n)} \delta_{k,i}\ \delta_{l,j}\ \beta_{i,j} = \sum_{(i,j)=(0,0)}^{(m,n)} \beta_{i,j} = Q(1) \ .$$

Step 2) If $P(y) \in \Pi_{i,j} \ \Rightarrow\ T_z P(y) \in \Pi_{i+j}$ w.r.t. z (we can't get a better result, like e.g. $\Pi_{j,i}$, and that is the limiting factor for our setting), because:

$$\text{Let } \ P(y) = \sum_{(k,l)=(0,0)}^{(i,j)} \beta_{k,l} P_{k,l}(z) \ , \quad \text{for suitable } \beta_{k,l}, \text{ then}$$

$$\begin{aligned} T_z P(y) &= \sum_{k=0}^{i+j-1} \sum_{l=0}^{k} \beta_{k-l,l} T_z P_{k-l,l}(y) + \sum_{l=0}^{j} \beta_{i+j-l,l} T_z P_{i+j-l,l}(y) \\ &= \underbrace{\sum_{k=0}^{i+j-1} \sum_{l=0}^{k} \beta_{k-l,l} P_{l,k-l}(z) P_{k-l,l}(y)}_{\in\ \Pi_{i+j-1} \text{ w.r.t. } z} + \underbrace{\sum_{l=1}^{j} \beta_{i+j-l,l} P_{l,i+j-l}(z) P_{i+j-l,l}(y)}_{\in\ \Pi_{1,i+j-1} \text{ w.r.t. } z} \end{aligned}$$

$$+ \underbrace{\beta_{i+j,0} P_{0,i+j}(z) P_{i+j,0}(y)}_{\in \Pi_{i+j} \text{ w.r.t. } z} \quad \Rightarrow T_z P \in \Pi_{i+j} \quad \text{w.r.t. } z \ .$$

Step 3) $D_{k,l}(z)T_z P \in \Pi_{k,i+j+l}$ w.r.t. z, $\forall\ P \in \Pi_{i,j}$, because:

$$D_{k,l} \in \Pi_{k,l} \quad \text{and} \quad T_z P \in \Pi_{0,i+j} \ .$$

Step 4) $D_{k,l}(z)T_z P \in \Pi_{m,n}$ w.r.t. z

$$i) \text{ if } k \geq m : \forall P \in \Pi_{m+n-l-k}$$

$$ii) \text{ if } k < m : \forall P \in \Pi_{m+n-l-k-1}$$

because: in case $i)$ $k \geq m$:

$$P \in \Pi_{m+n-l-k} \overset{Step\ 2)}{\Longrightarrow} T_z P \in \Pi_{m+n-l-k} \quad \text{w.r.t.} \quad z$$

$$\overset{Step\ 3)}{\Longrightarrow} D_{k,l}(z)T_z P \in \Pi_{k,m+n-k} \quad \text{w.r.t.} \quad z \text{ and } \quad \Pi_{k,m+n-k} \subseteq \Pi_{m,n}$$

and in case $ii)$ $k < m$:

$$\text{as } k < m \ \text{ and } \ (k,l) \leq (m,n) \Rightarrow k+l < m+n \Rightarrow m+n-l-k-1 \geq 0$$

$$P \in \Pi_{m+n-l-k-1} \overset{Step\ 2)}{\Longrightarrow} T_z P \in \Pi_{m+n-l-k-1} \quad \text{w.r.t.} \quad z$$

$$\overset{Step\ 3)}{\Longrightarrow} D_{k,l}(z)T_z P \in \Pi_{k,m+n-k-1} \quad \text{w.r.t.} \quad z \text{ and } \quad \Pi_{k,m+n-k-1} \subseteq \Pi_{m,n}$$

So finally we can conclude for a polynomial P as in the theorem, we have:

$$D_{k,l}(z)T_z P(y) \in \Pi_{m,n} \quad \text{w.r.t. } z$$

$$\Rightarrow \int_D D_{m,n}(\bar{z})\big(D_{k,l}(z)T_z P(y)\big) d\mu(z) \overset{Step\ 1)}{=} D_{k,l}(1)T_1 P(y) = D_{k,l}(1)P(y)$$

$$\Rightarrow \left(V_{(k,l)}^{(m,n)} P\right)(y) = P(y) \ .$$

With this theorem we see that generalized de-la-Vallée-Poussin operators of the type V_M^N reproduce the biggest set of polynomials in relation to $(n,m),(k,l)$. (So we choose $N = n+m$ and $M = k+l$). They reproduce all polynomials of total degree $N-M$, i.e. from the set Π_{N-M}. That is the reason why we will concentrate in the sequel only on generalized de-la-Vallée-Poussin operators of this type.

The next property which we need for norm convergence $\|V_M^N f - f\| \to 0$ is the uniform boundedness of the de-la-Vallée-Poussin operator. Therefore it will suffice to have the boundedness of the kernel in the L^1 norm. That is the assertion in the next theorem.

Recall that for convenience we left out the α. This we will still do so, but in the sequel it can reappear in the constants which will not affect the uniformity as in our setting we work with fixed α.

Theorem 6.9 ((*) boundedness of the de-la-Vallée-Poussin kernel). *The generalized de-la-Vallée-Poussin kernel $\mathcal{V}_M^N$ is uniformly bounded in L^1, if N and M are related $N \sim M$.*

For $N = CM$; $N, M \in \mathbb{N}, C > 1$ we have:

$$\|\mathcal{V}_M^N\|_{L^1} \leq C^{\lceil\alpha\rceil+\frac{3}{2}} \ .$$

We prove this theorem in several steps. Therefore we provide some lemmata.

Lemma 6.10.

(*i*) *The L^1-norm of $\mathcal{V}_{(k,l)}^{(m,n)}$ is bounded in the following way:*

$$\|\mathcal{V}_{(k,l)}^{(m,n)}\|_{L^1} = \int_D |\mathcal{V}_{(k,l)}^{(m,n)}(z)| d\mu(z) \leq \sqrt{\frac{D_{m,n}(1)}{D_{k,l}(1)}} \ .$$

(*ii*) *For $p = 1, \infty$ and $f \in X^p$ the X^p-norm of $\left(V_{(k,l)}^{(m,n)} f\right)$ is bounded by:*

$$\|V_{(k,l)}^{(m,n)} f\|_{X^p} \leq \sqrt{\frac{D_{m,n}(1)}{D_{k,l}(1)}} \ \ \|f\|_{X^p} \ .$$

Proof (i):

a)

$$\int_D |D_{m,n}(z)|^2 d\mu(z) = \int_D |D_{m,n}(\bar{z})|^2 d\mu(z) = \int_D D_{m,n}(z) D_{m,n}(\bar{z}) d\mu(z)$$

$$= \sum_{(i,j)=(0,0)}^{(m,n)} \sum_{(k,l)=(0,0)}^{(m,n)} h(i,j)h(k,l) \int_D P_{i,j}(z)\overline{P_{k,l}(z)} d\mu(z)$$

$$= \sum_{(i,j)=(0,0)}^{(m,n)} \sum_{(k,l)=(0,0)}^{(m,n)} h(i,j)h(k,l) \frac{\delta_{i,k}\delta_{j,l}}{h(k,l)} = \sum_{(i,j)=(0,0)}^{(m,n)} h(i,j) = D_{m,n}(1) \ .$$

b) With Cauchy-Schwarz inequality we get

$$\int_D | \ \mathcal{V}_{(k,l)}^{(m,n)}(z)| d\mu(z) = \frac{1}{D_{k,l}(1)} \int_D |D_{m,n}(\bar{z}) D_{k,l}(z)| \ (1-|z|^2)^\alpha \ dz$$

$$\leq \frac{1}{D_{k,l}(1)} \left(\int_D |D_{m,n}(\bar{z})|^2 (1-|z|^2)^\alpha dz \right)^{\frac{1}{2}} \left(\int_D |D_{k,l}(z)|^2 (1-|z|^2)^\alpha dz \right)^{\frac{1}{2}}$$

$$\overset{a)}{=} \frac{1}{D_{k,l}(1)} \big(D_{m,n}(1)\big)^{\frac{1}{2}} \big(D_{k,l}(1)\big)^{\frac{1}{2}} = \sqrt{\frac{D_{m,n}(1)}{D_{k,l}(1)}} \qquad .$$

Proof (ii):
case $p = 1$:

$$\begin{aligned}
\|V_{(k,l)}^{(m,n)} f\|_{L^1} &= \int_D |(\mathcal{V}_{(k,l)}^{(m,n)} * f)(x)| d\mu(x) \\
&\leq \int_D \int_D \int_D |\mathcal{V}_{(k,l)}^{(m,n)}(y)||f(\xi)| E(y,x,\xi) d\mu(\xi) d\mu(y) d\mu(x) \\
&\leq \int_D \int_D |\mathcal{V}_{(k,l)}^{(m,n)}(y)||f(\xi)| \underbrace{\int_D E(y,\bar{\xi},\bar{x}) d\mu(x)}_{=1} d\mu(\xi) d\mu(y) \\
&= \|\mathcal{V}_{(k,l)}^{(m,n)}\|_{L^1} \|f\|_{L^1} \overset{(i)}{\leq} \sqrt{\frac{D_{m,n}(1)}{D_{k,l}(1)}} \ \|f\|_{L^1} \ .
\end{aligned}$$

case $p = \infty$:

$$\begin{aligned}
\|V_{(k,l)}^{(m,n)} f\|_{X^\infty} &= \sup_{x \in D} \left| \int_D \int_D \mathcal{V}_{(k,l)}^{(m,n)}(y) f(\xi) E(y,x,\xi) d\mu(\xi) d\mu(y) \right| \\
&\leq \sup_{z \in D} |f(z)| \sup_{x \in D} \int_D |\mathcal{V}_{(k,l)}^{(m,n)}(y)| \int_D E(y,x,\xi) d\mu(\xi) d\mu(y) \\
&= \sup_{z \in D} |f(z)| \ \|\mathcal{V}_{(k,l)}^{(m,n)}\|_{L^1} \leq \sqrt{\frac{D_{m,n}(1)}{D_{k,l}(1)}} \ \|f\|_{X^\infty} \ .
\end{aligned}$$

Now with Lemma 6.10 we see, that we need to uniformly bound $\sqrt{\frac{D_{m,n}(1)}{D_{k,l}(1)}}$, in order to get the assertion in Theorem 6.9. Indeed we can find a closed formula for $D_{m,n}(1)$, which is the crucial step. To be able to prove that closed formula we need some information about generalized binomial coefficients, which we provide in the following lemma. Properties i) and ii) are certainly known.

Lemma 6.11 (properties of generalized binomial coefficients).

$$\begin{aligned}
&(i) && \forall \alpha \in \mathbb{R}, \forall k \in \mathbb{N}: && \binom{-\alpha}{k} = (-1)^k \binom{\alpha + k - 1}{k} \\
&(ii) && \forall \alpha, \beta \in \mathbb{R}, \forall k \in \mathbb{N}: && \sum_{j=0}^{k} \binom{\alpha}{j} \binom{\beta}{k-j} = \binom{\alpha + \beta}{k} \\
&(iii) && \forall \alpha \in \mathbb{R}, \forall k \in \mathbb{N}: && \sum_{j=0}^{k} \binom{k - j + \alpha}{k - j} \binom{j + \alpha}{j} = \binom{k + 2\alpha + 1}{k} \\
&(iv) && \forall \alpha \in \mathbb{R}, \forall k \in \mathbb{N}: && \sum_{j=0}^{k} \binom{\alpha + j}{j} = \binom{\alpha + k + 1}{k} \ .
\end{aligned}$$

Proof:

(i)

$$\binom{-\alpha}{k} = \frac{(-\alpha)_k}{k!} = \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-k+1)}{k!}$$
$$= (-1)^k \, \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} = (-1)^k \, \frac{(\alpha+k-1)_k}{k!} = (-1)^k \binom{\alpha+k-1}{k} \; .$$

(ii) By the generalized binomial theorem one knows that $\forall \gamma, x \in \mathbb{R}, |x| < 1$ we can expand the following function in a Taylor series:

$$f(x) := (1+x)^\gamma = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = \sum_{i=0}^{\infty} \binom{\gamma}{i} x^i \; .$$

But then we know by setting $\gamma = \alpha, \beta, \alpha+\beta$ respectively:

$$(1+x)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} x^i, \quad (1+x)^\beta = \sum_{i=0}^{\infty} \binom{\beta}{i} x^i, \quad (1+x)^{\alpha+\beta} = \sum_{k=0}^{\infty} \binom{\alpha+\beta}{k} x^k \; .$$

Multiplying the first two formulas with $\gamma = \alpha$ and $\gamma = \beta$ gives:

$$(1+x)^{\alpha+\beta} = \sum_{i=0}^{\infty} \binom{\alpha}{i} x^i \sum_{l=0}^{\infty} \binom{\beta}{l} x^l = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{\alpha}{j} \binom{\beta}{k-j} x^k \; .$$

By comparing these binomial coefficients with those of the third formula with $\gamma = \alpha+\beta$, one sees:

$$\sum_{j=0}^{k} \binom{\alpha}{j} \binom{\beta}{k-j} = \binom{\alpha+\beta}{k} \; .$$

(iii)

$$\sum_{j=0}^{k} \binom{k-j+\alpha}{k-j} \binom{j+\alpha}{j} \overset{(i)}{=} \sum_{j=0}^{k} (-1)^{k-j} \binom{-\alpha-1}{k-j} (-1)^j \binom{-\alpha-1}{j}$$
$$= (-1)^k \sum_{j=0}^{k} \binom{-\alpha-1}{k-j} \binom{-\alpha-1}{j} \overset{(ii)}{=} (-1)^k \binom{-2\alpha-2}{k} \overset{(i)}{=} \binom{k+2\alpha+1}{k} \; .$$

(iv) This identity we prove by induction over k. So for $k=1$ we get

$$\binom{\alpha}{0} + \binom{\alpha+1}{1} = 1 + \alpha + 1 = \binom{\alpha+2}{1}$$

$k \to k+1$:

$$\sum_{j=0}^{k+1} \binom{\alpha+j}{j} = \sum_{j=0}^{k} \binom{\alpha+j}{j} + \binom{\alpha+k+1}{k+1}$$

$$= \binom{\alpha+k+1}{k} + \binom{\alpha+k+1}{k+1} = \frac{(\alpha+k+1)_k}{k!} + \frac{(\alpha+k+1)_{k+1}}{(k+1)!}$$
$$= \frac{(k+1)(\alpha+k+1)_k}{(k+1)!} + \frac{(\alpha+k+1)_k(\alpha+1)}{(k+1)!}$$
$$= \frac{(\alpha+k+1)_k(\alpha+k+2)}{(k+1)!} = \frac{(\alpha+k+2)_{k+1}}{(k+1)!} = \binom{\alpha+k+2}{\alpha+1} .$$

Now we can prove the closed formula for $D_{m,n}(1)$. Furthermore it is clear that this is also the maximal value the (n,m)-th Dirichlet kernel takes on on the unit disc, as the maximal value of each disc polynomial is taken on there and the Haar function is positive.

Theorem 6.12 ((*)). *We have* $\forall N \in \mathbb{N}$ *and* $\forall \alpha \in \mathbb{R}_+$*:*

$$D_N^\alpha(1) = \frac{2N+2\alpha+3}{2\alpha+3}\binom{2\alpha+N+2}{N} .$$

Proof:

$$D_N^\alpha(1) = \sum_{k=0}^{N}\sum_{l=0}^{k} h^\alpha(k-l,l) = \sum_{k=0}^{N}\sum_{l=0}^{k} \frac{k+\alpha+1}{\alpha+1}\ \frac{(k-l+\alpha)!}{(k-l)!\ \alpha!}\ \frac{(l+\alpha)!}{l!\ \alpha!}$$
$$= \sum_{k=0}^{N}\left(\frac{k}{\alpha+1}+1\right)\sum_{l=0}^{k}\binom{k-l+\alpha}{k-l}\binom{l+\alpha}{l} \overset{6.11(iii)}{=} \sum_{k=0}^{N}\left(\frac{k}{\alpha+1}+1\right)\binom{2\alpha+k+1}{k}$$
$$= \sum_{k=0}^{N}\frac{k}{\alpha+1}\binom{2\alpha+k+1}{k} + \sum_{k=0}^{N}\binom{2\alpha+k+1}{k}$$
$$\overset{6.11(iv)}{=} \sum_{k=0}^{N}\frac{2k}{2\alpha+2}\frac{(2\alpha+k+1)!}{k!\ (2\alpha+1)!} + \binom{2\alpha+N+2}{N}$$

as each term in the sum is multiplied by "k" the sum starts at $k=1$

$$= 2\sum_{k=1}^{N}\frac{(2\alpha+k+1)!}{(k-1)!(2\alpha+2)!} + \binom{2\alpha+N+2}{N} \quad \text{and an index shifting leads to}$$
$$= 2\sum_{k=0}^{N-1}\frac{(2\alpha+k+2)!}{k!\ (2\alpha+2)!} + \binom{2\alpha+N+2}{N} = 2\sum_{k=0}^{N-1}\binom{2\alpha+k+2}{k} + \binom{2\alpha+N+2}{N}$$
$$\overset{6.11(iv)}{=} 2\binom{2\alpha+3+N-1}{N-1} + \binom{2\alpha+N+2}{N} = 2\frac{(2\alpha+N+2)!}{(2\alpha+3)!(N-1)!} + \frac{(2\alpha+N+2)!}{(2\alpha+2)!N!}$$
$$= (2N+2\alpha+3)\frac{(2\alpha+N+2)!}{(2\alpha+3)!N!} = \frac{2N+2\alpha+3}{2\alpha+3}\binom{2\alpha+N+2}{N} .$$

But still there is some work to be done. To get the constant from $\sqrt{\frac{D_N(1)}{D_M(1)}}$ we have to consider the Dirichlet kernel at 1 as a polynomial in the variable N and M respectively. This is done in the next lemma where one should be aware that in that lemma the values of α are explicitly given in each numbering.

Lemma 6.13 ((*) properties of $D_N(1)$). *Let $N, M \in \mathbb{N}$ so that $N = CM, 1 < C \in \mathbb{R}$. Then we have:*

$$(i) \qquad \forall k \in \mathbb{N} : \exists \beta_i \in \mathbb{R}_+ : D_N^k(1) = \sum_{i=0}^{2k+3} \beta_i N^i$$

$$(ii) \qquad \forall k \in \mathbb{N} : \frac{D_N^k(1)}{D_M^k(1)} \leq C^{2k+3}$$

$$(iii) \qquad \forall \alpha, \delta \in \mathbb{R}_+ : \frac{D_N^\alpha(1)}{D_M^\alpha(1)} \leq \frac{D_N^{\alpha+\delta}(1)}{D_M^{\alpha+\delta}(1)} \ .$$

Proof:
(i)

$$D_N^k(1) \overset{6.12}{=} \frac{(2N+2k+3)}{(2k+3)!}(N+(2k+2))(N+(2k+1))\cdots(N+1)$$

and so it is clear to see that these are $2k+3$ terms of the form $(aN+b)$ with $a, b \in \mathbb{R}_+$.
(ii)

$$D_N^k(1) \overset{(i)}{=} \sum_{i=0}^{2k+3} \beta_i N^i = \sum_{i=0}^{2k+3} \beta_i (CM)^i \overset{\beta_i \geq 0, (i)}{\leq} C^{2k+3} \sum_{i=0}^{2k+3} \beta_i M^i = C^{2k+3} D_M^k(1) \ .$$

(iii) The proof of this property is quite clumsy. We have to distinguish some cases, because for small values of M and N the general proof does not work.
case 1: $M = 1, N \geq 2$:

$$D_N^\alpha(1) D_1^{\alpha+\delta}(1) \leq D_N^{\alpha+\delta}(1) D_1^\alpha(1) \overset{6.12}{\Longleftrightarrow}$$

$$\frac{2N+2\alpha+3}{2\alpha+3}\binom{2\alpha+N+2}{N}(2\alpha+2\delta+5) \leq \frac{2N+2\alpha+2\delta+3}{2\alpha+2\delta+3}\binom{2\alpha+2\delta+N+2}{N}(2\alpha+5)$$

$$\Longleftrightarrow \frac{(2\alpha+2N+3)}{N!}\frac{(2\alpha+N+2)!}{(2\alpha+3)!}(2\alpha+2\delta+5)$$

$$\leq \frac{(2\alpha+2\delta+2N+3)}{N!}\frac{(2\alpha+2\delta+N+2)!}{(2\alpha+2\delta+3)!}(2\alpha+5) \ .$$

case1.1: $M = 1, N \geq 3$: in that case we have the terms $\frac{1}{N!}$, $(2\alpha+2\delta+5)$ and $(2\alpha+5)$ on each side of the equality, so it reduces to:

$$\Longleftrightarrow (2\alpha+2N+3)(2\alpha+N+2)\cdots(2\alpha+6)(2\alpha+4) \leq$$

$$\leq (2\alpha+2\delta+2N+3)(2\alpha+2\delta+N+2)\cdots(2\alpha+2\delta+6)(2\alpha+2\delta+4)$$

and this is obvious by comparing the terms in their order, because α and δ are greater than zero.

case 1.2: $M=1, N=2$:

$$\iff \frac{1}{2}(2\alpha+7)(2\alpha+4)(2\alpha+2\delta+5) \leq \frac{1}{2}(2\alpha+2\delta+7)(2\alpha+2\delta+4)(2\alpha+5)$$

$$\iff 8\alpha^2\delta + 8\alpha\delta^2 + 40\alpha\delta + 20\delta^2 + 54\delta \geq 0 \quad \text{which is clear.}$$

case2: $N > M > 1$:

$$D_N^{\alpha}(1)D_M^{\alpha+\delta}(1) \leq D_N^{\alpha+\delta}(1)D_M^{\alpha}(1) \overset{6.12}{\iff}$$

$$\frac{(2N+2\alpha+3)}{N!}\,(N+2\alpha+2)_{N-1}\,\frac{(2M+2\alpha+2\delta+3)}{M!}\,(M+2\alpha+2\delta+2)_{M-1} \leq$$
$$\leq \frac{(2N+2\alpha+2\delta+3)}{N!}\,(N+2\alpha+2\delta+2)_{N-1}\,\frac{(2M+2\alpha+3)}{M!}(M+2\alpha+2)_{M-1}\;.$$

As $N > M$ the following terms appear on both sides of the inequality:
$(M+2\alpha+2)\cdots(2\alpha+4)$ and $(M+2\alpha+2\delta+2)\cdots(2\alpha+2\delta+4)$
as well as $\frac{1}{M!\;N!}$. Division by these terms provides

$$\iff (2N+2\alpha+3)(N+2\alpha+2)\cdots(M+2\alpha+3)(2M+2\alpha+2\delta+3) \leq$$

$$\leq (2N+2\alpha+2\delta+3)(N+2\alpha+2\delta+2)\cdots(M+2\alpha+2\delta+3)(2M+2\alpha+3)$$

picking on each side those three terms, that appear $\forall N > M > 1$ and splitting them from the rest leads to $\iff$

$$(2N+2\alpha+3)(M+2\alpha+3)(2M+2\alpha+2\delta+3) \leq (2N+2\alpha+2\delta+3)(2\alpha+2\delta+M+3)(2M+2\alpha+3)$$

$$\wedge \quad (N+2\alpha+2)\cdots(M+2\alpha+4) \leq (N+2\alpha+2\delta+2)\cdots(M+2\alpha+2\delta+4)$$

so the second term again is clear by comparison term by term and the first leads to the condition:

$$8\alpha^2\delta + 8\alpha\delta^2 + 16\alpha\delta M + 24\alpha\delta + 8\delta^2 M + 12\delta^2 + 4\delta M^2 + 4\delta MN + 24\delta M + 18\delta \geq 0$$

which is truely fulfilled and that completes the proof.

Now we are finally able to prove Theorem 6.9:

Proof of Theorem 6.9:
First of all let $N, M \in \mathbb{N}$ with $N \sim M$ in a way that $N = cM$ with $c > 1$. From Lemma 6.10 (i) we know that:

$$\|\mathcal{V}_M^N\|_{L^1} = \int_D |\mathcal{V}_M^N(z)|d\mu(z) \leq \sqrt{\frac{D_N(1)}{D_M(1)}}\;.$$

Keep in mind that $\lceil\alpha\rceil := \min_{k\in\mathbb{Z}}\{k|k \geq \alpha\}$ and that in our setting $\alpha > 0$, hence it follows that $\lceil\alpha\rceil \in \mathbb{N}$.

Withs $N = cM$ we can use the results of Lemma 6.13 to further estimate:

$$\left(\frac{D_N^\alpha(1)}{D_M^\alpha(1)}\right)^{\frac{1}{2}} \overset{6.13(iii)}{\leq} \left(\frac{D_N^{\lceil\alpha\rceil}(1)}{D_M^{\lceil\alpha\rceil}(1)}\right)^{\frac{1}{2}} \overset{6.13(ii)}{\leq} c^{\lceil\alpha\rceil+\frac{3}{2}} .$$

But this is the conclusion in Theorem 6.9.

The approximation properties of this generalized de-la-Vallée-Poussin operator we will formulate in a theorem concluding this section.

Theorem 6.14 ((*) Norm convergence). *Let $\alpha > 0$ and let $V_M^N := V_M^{cM}$ with $M \in \mathbb{N}$ and $1 > c \in \mathbb{N}$. For $1 \leq p \leq \infty$ and functions $f \in X^p(D)$ we have*

1) $\|V_M^N f - f\|_{X^p(D)} \longrightarrow 0, \quad$ *for* $M \longrightarrow \infty$
2) $\|V_M^N f - f\|_{X^p(D)} \leq (1 + c^{\lceil\alpha\rceil+\frac{3}{2}}) \min\limits_{Q\in\Pi_{(c-1)M}} \|f - Q\|_{X^p(D)}$.

Proof:

We want to use Theorem 3.7 to prove statement 1). In place of the family of continuous linear operators let us choose $T_M := V_M^N : X^p(D) \to X^p(D)$. The role of the dense subset is played by the set of all polynomials Π. It is clear that they are dense in all $X^p(D)$ spaces. With the reproducing property of T_M given in Theorem 6.8 it follows

$$\lim_{M\to\infty} \|T_M P - P\|_{X^p(D)} = 0 \ \forall P \in \Pi$$

as T_M reproduces polynomials of total degree $(c-1)M$ and $c > 1$. But then the precondition for Theorem 3.7 is fulfilled. From the two equivalent statements there we choose the 2)nd to prove true, i.e.

$$\|T_M f\|_{L^p(D)} \leq C\|f\|_{X^p(D)} \quad \forall f \in X^p(D), \quad M \in \mathbb{N}$$

with C to be independent of M and f. But this is done in Lemma 6.10 (ii) (recall that we use for interpolating the intermediate cases the Riesz-Thorin theorem) with which we have

$$\|T_M f\|_{X^p(D)} = \|V_M^{cM} f\|_{X^p(D)} \overset{6.10(ii)}{\leq} \sqrt{\tfrac{D_{cM}(1)}{D_M(1)}}\|f\|_{X^p(D)} \leq c^{\lceil\alpha\rceil+\frac{3}{2}} \ \|f\|_{X^p(D)}$$

and c is the constant giving the proportion of N to M: $N = cM$. But then Theorem 3.7 provides the statemanet in 1).

To prove statement 2) let $Q \in \Pi_{(c-1)M}$. Then it holds:

$$\begin{aligned}\|V_M^N f - f\|_{X^p(D)} &\leq \|V_M^N f - Q\|_{X^p(D)} + \|Q - f\|_{X^p(D)} \\ &= \|f - Q\|_{X^p(D)} + \|V_M^N (f - Q)\|_{X^p(D)} \leq (1 + c^{\lceil\alpha\rceil+\frac{3}{2}})\|f - Q\|_{X^p(D)}\end{aligned}$$

and as this inequality holds for any $Q \in \Pi_{(c-1)M}$ we can choose the one that minimizes the norm on the right hand side of the inequality. This is statement 2).

One interesting fact about this generalized de-la-Vallée-Poussin kernel is that the constant with which its norm is bounded is explicitly known. That may be quite useful if in some settings or situations one wants to know something about the speed of convergence or error estimation. Then such a kernel can be helpfull.

6.3 Radially almost exponentially localized kernels

The method explained in the chapter on almost exponentially localized kernels in [28] in the Jacobi setting is taken over to the unit ball B^d in $\mathbb{R}^d$ ($d > 1$) with weights $w_\alpha(x) = (1 - |x|^2)^\alpha$. We see that this is exactly the same weight as in our setting with disc polynomials but in [28] are not considered disc polynomials but functions that are related to the Gegenbauer polynomials. And indeed there is derived a result saying that the developed kernel $L_n^\alpha(x, y)$ has almost exponential localisation:

$$|L_n^\alpha(x, y)| \leq c_k \frac{n^d}{\sqrt{W_\alpha(n; x)}\sqrt{W_\alpha(n; y)}(1 + nd(x, y))^k}$$

for any $k \in \mathbb{N}$, for $x, y \in B^d$ and a constant c_k that is independent of n, while

$$W_\alpha(n; x) = \left(\sqrt{1 - |x|^2} + n^{-1}\right)^{2\alpha} .$$

Especially for $d = 2$ this result is for the unit disc. We want to try to achieve a similar result for kernels based on disc polynomials. But unfortunately the generalized translation in our setting is not friendly enough to provide such a strong result.

Nevertheless we want to take a look at the construction of kernels with respect to disc polynomials that have in other settings almost exponential localisation. We will see that the problem lies in the estimate of the Jacobi polynomial, given in the definition of the disc polynomial, that only works with radial properties of the argument and therefore information about the exact location on the disc gets lost. So in the end we cannot differentiate two points on the disc having the same radius but different angles.

Definition 6.15. *(admissible function) A function $\widehat{a} \in C^\infty[0, \infty)$ is called admissible if $\widehat{a}(t) \geq 0$ and*

$$\widehat{a}(t) = \begin{cases} 1, & \text{if} \quad 0 \leq t \leq 1 \\ \in [0, 1], & \text{if} \quad 1 < t \leq 2 \\ 0, & \text{if} \quad t > 2 \ . \end{cases}$$

And we define a function $\widehat{a} : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$\widehat{a}(x, y) := \widehat{a}(x + y) \ .$$

Then it is clear that $\widehat{a}(x,y)$ is admissible if $\widehat{a}(t)$ is admissible and we have

$$\widehat{a}(x,y) = \begin{cases} 1, & \text{if} \quad x+y \leq 1 \\ \in [0,1], & \text{if} \quad 1 < x+y \leq 2 \\ 0, & \text{if} \quad x+y > 2 \ . \end{cases}$$

Definition 6.16 ((*)). *For $n \in \mathbb{N}_0$ and $\alpha > 0$ we define the following reproducing kernel for an admissible function $\widehat{a}$:*

$$L_n^\alpha(z) := \sum_{(i,j)=(0,0)} \widehat{a}\Big(\frac{i}{n}, \frac{j}{n}\Big) h(i,j) P_{i,j}(z) \ .$$

Note that the support of an admissible function $\widehat{a}(t)$ is only $[0,2]$. This allows us to let the indices go to infinity in Definition 6.16.

Then we provide here two fundamental theorems on Jacobi polynomials that are very important for the upcoming proofs. We cite them from Szegö's book on orthogonal polynomials [33]:

Theorem 6.17 (Identity for Jacobi polynomials). *[33], (4.5.3)*

$$\sum_{i=0}^{n} \frac{(2i+\alpha+\beta+k+1)(\alpha+\beta+k+i)!}{(\beta+i)!} J_i^{(\alpha+k,\beta)}(x) = \frac{(\alpha+\beta+n+k+1)!}{(\beta+n)!} J_n^{(\alpha+k+1,\beta)}(x) \ .$$

Theorem 6.18 (Estimate for Jacobi polynomials). *[33], (7.32.6)*

$$|J_n^{(\alpha,\beta)}(\cos\theta)| \leq c(\alpha,\beta) \begin{cases} \min\{n^\alpha, n^{-\frac{1}{2}}\theta^{-\alpha-\frac{1}{2}}\} & \text{if} \ \ 0 \leq \theta \leq \frac{\pi}{2} \\ \min\{n^\beta, n^{-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\} & \text{if} \ \ \frac{\pi}{2} \leq \theta \leq \pi \ . \end{cases}$$

With Theorem 6.17 we are able to write our new kernel L_n^α in a different way, which will help us estimate it later on.

Theorem 6.19 ((*)). *For $k \in \mathbb{N}$*

$$L_n^\alpha(z) = \sum_{\beta=0}^{2n} \sum_{\nu=0}^{n-\lceil\frac{\beta}{2}\rceil} \left(\widehat{a}\Big(\frac{\beta+\nu}{n}, \frac{\nu}{n}\Big) z^\beta + \widehat{a}\Big(\frac{\nu}{n}, \frac{\beta+\nu}{n}\Big)(1-\delta_{\beta,0})\bar{z}^\beta \right) \times$$

$$(\alpha+2\nu+\beta+1)\frac{(\alpha+\beta+\nu)!}{(\beta+\nu)!(\alpha+1)!} \ J_\nu^{(\alpha,\beta)}(2|z|^2-1)$$

$$= \sum_{\beta=0}^{2n}\sum_{\nu=0}^{\infty}\Big(A_k(\beta+\nu,\nu)z^\beta + A_k(\nu,\beta+\nu)(1-\delta_{\beta,0})\bar{z}^\beta\Big)\times$$
$$\frac{(\alpha+\beta+\nu+k)!}{(\beta+\nu)!(\alpha+1)!}\; J_\nu^{(\alpha+k,\beta)}(2|z|^2-1)$$

with

$$A_1(\beta+\nu,\nu) := \widehat{a}\Big(\frac{\beta+\nu}{n}\;,\frac{\nu}{n}\Big) - \widehat{a}\Big(\frac{\beta+\nu+1}{n}\;,\frac{\nu+1}{n}\Big)$$

and for $k \geq 1$

$$A_{k+1}(\beta+\nu,\nu) := \frac{A_k(\beta+\nu,\nu)}{2\nu+\alpha+\beta+k+1} - \frac{A_k(\beta+\nu+1,\nu+1)}{2\nu+\alpha+\beta+k+3}\;.$$

Proof: The first equality is just a re-ordering of the former definition of L_n^α given in 6.16. This formula can be proved by induction principle. The second equality can be proved with the result of Theorem 6.17. We will give the proof for $k=1$: We will prove it bottom up beginning with

$$L_n^\alpha(z) = \sum_{\beta=0}^{2n}\sum_{\nu=0}^{\infty}\Big(A_1(\beta+\nu,\nu)z^\beta + A_1(\nu,\beta+\nu)(1-\delta_{\beta,0})\bar{z}^\beta\Big)\times$$
$$\frac{(\alpha+\beta+\nu+1)!}{(\beta+\nu)!(\alpha+1)!}\; J_\nu^{(\alpha+1,\beta)}(2|z|^2-1)$$

and with theorem 6.17 and the definition of A_1

$$= \sum_{\beta=0}^{2n}\sum_{\nu=0}^{\infty}\Big(\big(\widehat{a}\big(\frac{\beta+\nu}{n}\;,\frac{\nu}{n}\big)z^\beta + \widehat{a}\big(\frac{\nu}{n}\;,\frac{\beta+\nu}{n}\big)(1-\delta_{\beta,0})\bar{z}^\beta\big)$$
$$-\big(\widehat{a}\big(\frac{\beta+\nu+1}{n}\;,\frac{\nu+1}{n}\big)z^\beta + \widehat{a}\big(\frac{\nu+1}{n}\;,\frac{\beta+\nu+1}{n}\big)(1-\delta_{\beta,0})\bar{z}^\beta\big)\Big)\times$$
$$\sum_{j=0}^{\nu}\frac{(2j+\alpha+\beta+1)(\alpha+\beta+j)!}{(\alpha+1)!(\beta+j)!}J_j^{(\alpha,\beta)}(2|z|^2-1)$$
$$= \sum_{\beta=0}^{2n}\sum_{\nu=0}^{\infty}\sum_{j=0}^{\nu}\Big(\big(\widehat{a}\big(\frac{\beta+\nu}{n}\;,\frac{\nu}{n}\big)z^\beta + \widehat{a}\big(\frac{\nu}{n}\;,\frac{\beta+\nu}{n}\big)(1-\delta_{\beta,0})\bar{z}^\beta\big)\Big)\times$$
$$\frac{(2j+\alpha+\beta+1)(\alpha+\beta+j)!}{(\alpha+1)!(\beta+j)!}J_j^{(\alpha,\beta)}(2|z|^2-1)$$
$$-\sum_{\beta=0}^{2n}\sum_{\nu=1}^{\infty}\sum_{j=0}^{\nu-1}\Big(\big(\widehat{a}\big(\frac{\beta+\nu}{n}\;,\frac{\nu}{n}\big)z^\beta + \widehat{a}\big(\frac{\nu}{n}\;,\frac{\beta+\nu}{n}\big)(1-\delta_{\beta,0})\bar{z}^\beta\big)\Big)\times$$
$$\frac{(2j+\alpha+\beta+1)(\alpha+\beta+j)!}{(\alpha+1)!(\beta+j)!}J_j^{(\alpha,\beta)}(2|z|^2-1)$$

$$= \sum_{\beta=0}^{2n}\sum_{\nu=0}^{\infty}\left(\widehat{a}\Big(\frac{\beta+\nu}{n}\,,\frac{\nu}{n}\Big)z^{\beta}+\widehat{a}\Big(\frac{\nu}{n}\,,\frac{\beta+\nu}{n}\Big)(1-\delta_{\beta,0})\bar{z}^{\beta}\right)\times$$
$$(2\nu+\alpha+\beta+1)\frac{(\alpha+\beta+\nu)!}{(\alpha+1)!(\beta+\nu)!}\;J_{\nu}^{(\alpha,\beta)}(2|z|^2-1)\;.$$

But that was the assumption. The cases with $k>1$ follow similarly.

Lemma 6.20. *For $k\in\mathbb{N},\ k\geq 2$ we have*

$$\|A_k\|_{\infty}\leq c\;n^{-2k+1}\;\max_{0\leq\nu\leq k}\|\widehat{a}^{(\nu)}(\cdot)\|_{\infty}$$

with $c=c(\alpha,k)$.

Proof: The proof can be done inductively and it can be reduced to the case of an admissible function $\widehat{a}(t)$ in the one dimensional case, which is given in [28]. Note that our constant is of the factor 2^{2k-1} greater than the constant in [28], but still independent of n.

Now we can formulate the result that is the crucial theorem in [28] and in the one dimensional cases. An estimate where only one variable is involved, with an kernel on which later on the translation should be applied. Also in the disc polynomial case this result can be achieved, but as mentioned before the translation in our case together with the definition of the disc polynomials don't provide the result of the almost exponential localisation in case the two variables x,y have the same radial part.

Theorem 6.21 ((*)). *Let $\widehat{a}$ be admissible and $\alpha>0$. Then for every $k\in\mathbb{N}$ and $z\in D$ we have*

$$|L_n^{\alpha}(z)|\leq c_k\;\frac{n^{2\alpha+3}}{(1+n\arccos\big(2|z|^2-1\big))^k}$$

with a constant c_k independent of n.

Already here we see that the angular part of the complex variable got lost. But the angular part gives the position on the disc apart from the radial part. That will lead to the situation that for $T_yL_n^{\alpha}(z)$ we don't get almost exponential localisation in case that y and z have the same radius.

Proof of Theorem 6.21: Of course we want to use the estimate of Theorem 6.18. Therefore we make the substitution $\cos\theta=2|z|^2-1$ for $\theta\in[0,\pi]$ from which we can conclude $|z|=\cos\left(\frac{\theta}{2}\right)$ for $\theta\in[0,\pi]$ with the addition formula for trigonometric functions. First we see that we want to estimate

$$|L_n(z)| = \Big|\sum_{\beta=0}^{2n}\sum_{j=0}^{\infty}\Big(A_k(\beta+j,j)z^\beta + A_k(j,\beta+j)(1-\delta_{\beta,0})\bar{z}^\beta\Big)\times$$
$$\frac{(\alpha+\beta+j+k)!}{(\alpha+1)!(\beta+j)!}J_j^{(\alpha+k,\beta)}(2|z|^2-1)\Big|$$
$$\leq \sum_{\beta=0}^{2n}\sum_{j=0}^{\infty}\frac{2}{(\alpha+1)!}\|A_k\|_\infty\Big|\cos\left(\frac{\theta}{2}\right)\Big|^\beta|\beta+j|^{\alpha+k}\big|J_j^{(\alpha+k,\beta)}(\cos\theta)\big|$$
$$\leq C\sum_{\beta=0}^{2n}\sum_{j=0}^{\infty} n^{-2k+1}\Big|\cos\left(\frac{\theta}{2}\right)\Big|^\beta n^{\alpha+k}\ \big|J_j^{(\alpha+k,\beta)}(\cos\theta)\big|$$

where in the last inequality we have used the result from Lemma 6.20. Now with Theorem 6.18 it is clear that we have to split into four cases.

1.) For $\theta\in[0,\frac{\pi}{2}]$ we have

$$|J_n^{(\alpha+k,\beta)}(\cos\theta)| \leq c(\alpha,\beta)\min\{n^{\alpha+k}, n^{-\frac{1}{2}}\theta^{-\alpha-k-\frac{1}{2}}\}\ .$$

1.a) Let $\min\{n^{\alpha+k}, n^{-\frac{1}{2}}\theta^{-\alpha-k-\frac{1}{2}}\} = n^{\alpha+k} \implies \theta\leq\frac{1}{n}$ and we can further estimate:

$$|L_n(z)| \leq C\ n^{2\alpha+3} = C\ \frac{n^{2\alpha+3}}{(1+n\theta)^k}(1+n\theta)^k \overset{\theta\leq\frac{1}{n}}{\leq} \tilde{C}\ \frac{n^{2\alpha+3}}{(1+n\theta)^k}\ .$$

1.b) Now let $\min\{n^{\alpha+k}, n^{-\frac{1}{2}}\theta^{-\alpha-k-\frac{1}{2}}\} = n^{-\frac{1}{2}}\theta^{-\alpha-k-\frac{1}{2}} \implies \theta\geq\frac{1}{n}$ and:

$$|L_n(z)| \leq C\ \frac{n^{2\alpha+3}}{(n\theta)^{k+\alpha+\frac{1}{2}}} \leq C\ \frac{n^{2\alpha+3}}{(1+n\theta)^{k+\alpha+\frac{1}{2}}}\frac{(1+n\theta)^{k+\alpha+\frac{1}{2}}}{(n\theta)^{k+\alpha+\frac{1}{2}}}$$
$$\overset{1\leq n\theta}{\leq} C\ \frac{n^{+2\alpha+3}}{(1+n\theta)^k}\frac{(2n\theta)^{k+\alpha+\frac{1}{2}}}{(n\theta)^{k+\alpha+\frac{1}{2}}} \leq \tilde{C}\frac{n^{2\alpha+3}}{(1+n\theta)^k}\ .$$

2.) The second case is for $\theta\in[\frac{\pi}{2},\pi]$ and we have

$$|J_n^{(\alpha+k,\beta)}(\cos\theta)| \leq c(\alpha,\beta)\min\{n^\beta, n^{-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\}\ .$$

2.a) First let $\min\{n^\beta, n^{-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\} = n^\beta \implies\ \theta\geq\pi-\frac{1}{n}$. So

$$|L_n^\alpha(z)| \leq C\ \frac{n^{\alpha+3}}{n^k}\ \Big|\cos\left(\frac{\theta}{2}\right)\Big|^\beta n^\beta \leq C\ \frac{n^{\alpha+3}}{n^k}\ \Big|\cos\left(\frac{\pi-\frac{1}{n}}{2}\right)\Big|^\beta n^\beta$$
$$= C\ \frac{n^{\alpha+3}}{n^k}\ \Big|\cos\left(\frac{\pi}{2}+\frac{1}{2n}\right)\Big|^\beta n^\beta = C\ \frac{n^{\alpha+3}}{n^k}\ \Big|\sin\left(\frac{1}{2n}\right)\Big|^\beta n^\beta$$

but the sine we can estimate in the following way:

$$\sin\left(\frac{1}{2n}\right) = \sum_{j=0}^{\infty}(-1)^j\left(\frac{1}{2n}\right)^{2j+1}\frac{1}{(2j+1)!} = \frac{1}{2n} + \underbrace{\sum_{j=0}^{\infty}(-1)^j\left(\frac{1}{2n}\right)^{2j+1}\frac{1}{(2j+1)!}}_{\leq 0} \leq \frac{1}{2n}$$

thus we can go on

$$\leq C\ \frac{n^{\alpha+3}}{n^k}\ \left(\frac{1}{2n}\right)^{\beta} n^{\beta} \leq \tilde{C}\ \frac{n^{2\alpha+3}}{(1+n\theta)^k}\ \frac{(1+n\theta)^k}{n^k} \leq C\ \frac{n^{2\alpha+3}}{(1+n\theta)^k}\ \frac{(2\pi n)^k}{n^k} \leq \tilde{C}\ \frac{n^{2\alpha+3}}{(1+n\theta)^k}\ .$$

2.b) Finally let $\min\{n^{\beta}, n^{-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\} = n^{-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}} \implies \theta \leq \pi - \frac{1}{n}$:

$$\begin{aligned}
|L_n(z)| &\leq C\ n^{-k+\alpha+3-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\left|\cos\left(\frac{\theta-\pi}{2}+\frac{\pi}{2}\right)\right|^{\beta} \\
&= C\ n^{-k+\alpha+3-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\left|\sin\left(\frac{\theta-\pi}{2}\right)\right|^{\beta} \\
&= C\ n^{-k+\alpha+3-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\left|\sin\left(\frac{\pi-\theta}{2}\right)\right|^{\beta} \\
&\leq C\ n^{-k+\alpha+3-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\left(\frac{\pi-\theta}{2}\right)^{\beta}
\end{aligned}$$

$$\leq \tilde{C}\ \frac{n^{2\alpha+3}}{n^{k+1}} \leq \tilde{C}\ \frac{n^{2\alpha+3}}{(1+n\theta)^k}\frac{(1+n\theta)^{k+1}}{n^{k+1}} \leq \overline{C}\ \frac{n^{2\alpha+3}}{(1+n\theta)^k}\ .$$

A final resubstitution via $\theta = \arccos(2|z|^2-1)$ gives the result.

Now we are applying the generalized translation on this kernel:

$$|T_y L_n^{\alpha}(z)| = \left|\int_D L_n^{\alpha}(z\bar{y} + \sqrt{1-|z|^2}\sqrt{1-|y|^2}\xi)d\mu_{\alpha-1}(\xi)\right|$$

and the simply use of Theorem 6.21 delivers

$$\leq C\ \int_D \frac{n^{2\alpha+3}}{\left(1+n\arccos\left(2\left|z\bar{y}+\sqrt{1-|z|^2}\sqrt{1-|y|^2}\ \xi\ \right|^2-1\right)\right)^k}\ d\mu_{\alpha-1}(\xi)\ .$$

Recall the properties of the generalized translation and the situations in figure 3. As soon as z and y have the same radial part the domain over which we integrate (in figure 3: $B_r(z\bar{y})$) includes a point with absolute value 1. And then we have for our estimate:

$$\leq C\ \int_D \frac{n^{2\alpha+3}}{\left(1+n\arccos\left(1\right)\right)^k}\ d\mu_{\alpha-1}(\xi) \leq C\ n^{2\alpha+3}$$

even though the points z and y may lie diametrically opposed but on a circle with the same radius, e.g. $z = re^{i\varphi}$ and $y = re^{i(\varphi+\pi)}$.

On the other hand, if the two numbers have different radial parts the domain $B_r(z\bar{y})$ has a positive distance to the boundary of the unit disc and

$$\arccos\left(2\left|z\bar{y} + \sqrt{1-|z|^2}\sqrt{1-|y|^2}\ \xi\ \right|^2 - 1\right) > 0$$

such that the almost exponential localisation principle t, albeit only very weak. It is easy to show that for $z = r_1 e^{i\varphi_1}$ and $z = r_2 e^{i\varphi_2}$ we can estimate

$$|T_y L_n^\alpha(z)| \leq \frac{n^{2\alpha+3}}{\left(1 + n\arccos\left(2\left|r_1 r_2 + \sqrt{1-r_1^2}\sqrt{1-r_2^2}\right|^2 \ - 1\right)\right)^k}$$

and $\arccos\left(2\left|r_1 r_2 + \sqrt{1-r_1^2}\sqrt{1-r_2^2}\right|^2 \ - 1\right)$ behaves like a metric for r_1 and r_2.

Clearly this result is not satisfying for the reason that two elements of the unit disc with maximal distance (e.g. $z_1 = 1$ and $z_2 = -1$) show no localisation effect. Therefore we will no longer study this type of kernel and besides we studied kernels that are exponentially localised. With these kernels we will deal in the next section.

6.4 Exponentially localized kernels

Here we will discuss the construction of kernels $\tilde{K}_n$ with respect to disc polynomials that are exponentially localized, which means that they can be estimated in the following way:

$$\left|\tilde{K}_n(x,y)\right| \leq c\ n^{2\alpha+3}\ \exp\left(-n\phi(1-\cos(d(x,y)))\right)$$

with an increasing function ϕ and an appropriate distance $d(\cdot,\cdot)$. We follow the way we described in section 2.4.2 for Jacobi polynomials.

We are dealing with distances and with the generalized translation in the disc polynomial setting. So the first point is to find a distance function on the unit disc which is suitable for our purposes. And the second point is the question how our translation is connected to this distance. So here is the first point:

Lemma 6.22 ((*)). *For $x, y \in D$ with $x = |x|\ (\cos(\theta) + i\sin(\theta))$, $y = |y|\ (\cos(\varphi) + i\sin(\varphi))$, $\theta, \varphi \in [0, 2\pi)$*

$$d(x,y) = \arccos\left(|xy|\cos(\theta-\varphi) + \sqrt{1-|x|^2}\sqrt{1-|y|^2}\right)$$

is a metric.

Proof: It is a well-known fact that on the unit sphere for two points a, b we have a metric by putting $d_S(a,b) := \arccos(a \cdot b)$ the so called spherical distance. This means for our points $x, y \in D$ the following: we project them orthogonally w.r.t the disc (here for exemplification x) via

$$x = \big(|x|\cos\theta\ ,|x|\sin\theta\big) \mapsto \big(|x|\cos\theta\ ,|x|\sin\theta, \sqrt{1-|x|^2}\big)$$

on the unit sphere and use the metric d_S there. It is clear that this projection preserves the relation between two points on the disc. Important is, that the euclidean distance of $x \neq y$ on the disc is smaller than the spherical distance of the projections on the sphere, and equal when $x = y$, namely zero. So for $x = |x|\ (\cos(\theta) + i\sin(\theta))$ and $y = |y|\ (\cos(\varphi) + i\sin(\varphi))$, $\theta, \varphi \in [0, 2\pi)$ we have:

$$\begin{aligned}
d(x,y) := d_S\Big(&\big(|x|\cos(\theta)\ ,|x|\sin(\theta), \sqrt{1-|x|^2}\big), \big(|y|\cos(\varphi)\ ,|y|\sin(\varphi), \sqrt{1-|y|^2}\big)\Big)\\
&= \arccos\big(|xy|\big(\cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)\big) + \sqrt{1-|x|^2}\sqrt{1-|y|^2}\big)\\
&= \arccos\big(|xy|\cos(\theta-\varphi) + \sqrt{1-|x|^2}\sqrt{1-|y|^2}\big)\ .
\end{aligned}$$

Of course our new distance on the disc inherits the properties of a metric from the spherical distance and the proof is done.

Now we want to take a look at the relation between our new distance on the unit disc $d(\cdot,\cdot)$ and the argument of the generalized translation $x\bar{y} + \sqrt{1-|x|^2}\sqrt{1-|y|^2}z$ with $x, y, z \in D$:

Lemma 6.23 ((*)). *$\forall x, y, z \in D$ with $x = |x|\ (\cos(\theta) + i\sin(\theta))$, $y = |y|\ (\cos(\varphi) + i\sin(\varphi))$, $\theta, \varphi \in [0, 2\pi)$, $z = z_1 + iz_2$ we have:*

$$\big|\ 1 - x\bar{y} - \sqrt{1-|x|^2}\sqrt{1-|y|^2}z\ \big| \geq 1 - |xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ .$$

Proof: We see that the right hand side of the inequality is equal to $1 - \cos d(x, y)$. A first step towards this inequality is

a)

$$\begin{aligned}
1 - |xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z_1 &\geq 0\\
\Leftrightarrow |xy|\cos(\theta-\varphi) + \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z_1 &\leq 1\\
\Leftrightarrow Re\left(x\bar{y} + \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z\right) &\leq 1
\end{aligned}$$

But the last line is a property of the argument of the translation which we already proved after the definition of the generalized translation. Also confer figure 3 for that fact. This inequality we need in b)

$$
\begin{aligned}
&\left|1 - x\bar{y} - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z\ \right| \\
&\quad = \left|1 - |xy|e^{i(\theta-\varphi)} - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ (z_1 + iz_2)\right| \\
&\quad = \Big|\big(1 - |xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z_1\big) \\
&\qquad\qquad + i\big(|xy|\sin(\theta-\varphi) + \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z_2\big)\Big| \\
&\quad = \Big\{\big(1 - |xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z_1\big)^2 \\
&\qquad\quad + \underbrace{\big(|xy|\sin(\theta-\varphi) + \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z_2\big)^2}_{\geq 0}\ \Big\}^{\frac{1}{2}} \\
&\quad \geq \Big\{\big(\underbrace{1 - |xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z_1}_{\geq 0 \text{ because of a)}}\ \big)^2\Big\}^{\frac{1}{2}} \\
&\quad = 1 - |xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\ z_1 \\
&\quad \geq 1 - |xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2}
\end{aligned}
$$

where the last inequality holds as $z_1 \in [0,1]$ by definition. So the lemma is proved.

Before we come to the construction of exponentially localized kernels we prove a technical lemma. Then we want to prove a general theorem on exponentially localized kernels and after that take a look at a concrete kernel where our generalized de-la-Vallée-Poussin kernel could possibly be applied. But at first the technical lemma:

Lemma 6.24 ((*)). *Let $F \in L^1(D); \phi^\diamond : [0,2] \to [0,\infty]$ a nonincreasing function, and for some constant $A > 0 : |F(z)| \leq A\phi^\diamond(|1-z|), \forall z \in D$. Then we have for almost all $z, y \in D$ with $z = |z|e^{i\theta}, y = |y|e^{i\varphi}; \theta, \varphi \in [0, 2\pi)$:*

$$
|T_yF(z)| \leq A\phi^\diamond\big(1 - |zy|cos(\theta-\varphi) - \sqrt{1-|z|^2}\sqrt{1-|y|^2}\big)\ .
$$

Proof:

$$
\begin{aligned}
|T_yF(z)| &\overset{\text{by definition}}{=} \left|\int_D F(z\bar{y} + \sqrt{1-|z|^2}\sqrt{1-|y|^2}\ x)d\mu_{\alpha-1}(x)\right| \\
&\leq \int_D |F(z\bar{y} + \sqrt{1-|z|^2}\sqrt{1-|y|^2}\ x)|d\mu_{\alpha-1}(x) \\
&\overset{\text{precondition}}{\leq} A\int_D \phi^\diamond\big(|1 - z\bar{y} - \sqrt{1-|z|^2}\sqrt{1-|y|^2}\ x|\big)d\mu_{\alpha-1}(x)
\end{aligned}
$$

and with Lemma 6.23 and the precondition that $\phi^\diamond$ is nonincreasing we conclude

$$
\leq A\int_D \phi^\diamond\big(1 - |zy|cos(\theta-\varphi) - \sqrt{1-|z|^2}\sqrt{1-|y|^2}\big)d\mu_{\alpha-1}(x)
$$

$$= A\phi^{\Diamond}\big(1 - |zy|cos(\theta-\varphi) - \sqrt{1-|z|^2}\sqrt{1-|y|^2}\big)$$

where in the last equality we used that $\mu_{\alpha-1}$ is a probability measure and the argument of the function $\phi^{\Diamond}$ is independent of the integration variable.

We see that the argument on the right hand side of the inequality in Lemma 6.24 can be written in terms of our distance function via $1 - \cos\big(d(x,y)\big)$.

In the following definition we give the form of the kernels we want to allow:

Definition 6.25. *For $n \in \mathbb{N}$ let $\{b_{k,l;n}\}_{k,l=0,1,2,\dots}$ be a sequence with $b_{k,l;n} \in \mathbb{R}$. Then we define for $\alpha > 0$ the kernel $\tilde{\Phi}_n(z)$ in the following way:*

$$\tilde{\Phi}_n(z) := \sum_{(k,l)=(0,0)}^{(0,4n)} b_{k,l;n} h^{\alpha}(k,l) P^{\alpha}_{k,l}(\bar{z}) \ .$$

The next theorem is a general result on a special class of exponentially localized kernels with respect to disc polynomials. In the theorem we can take any kernel of the form $\tilde{\Phi}_n$ given in Definition 6.25 but in practice one naturally chooses a summation kernel that is uniformly norm-bounded. To find a suitable function $S_n \in \Pi_n$ (compare 3.4.2) is also a task that needs some reflections but first of all we want to give the theorem:

Theorem 6.26 ((*)). *Let $\phi : [0,2] \to [0,\infty]$ a nondecreasing continuous function. For $n \in \mathbb{N}$ let $\tilde{\Phi}_n$ as in Definition 6.25 and $S_n \in \Pi_n$ with:*

$$S_n(1) = 1; \quad |S_n(z)| \le c_1 \exp\big(-n\phi(|1-z|)\big); \ \forall z \in D \ .$$

Then with the same c_1 as above and $x = |x|e^{i\theta}, y = |y|e^{i\varphi}; \theta, \varphi \in [0,2\pi)$ we have:

1. $|T_y(S_n\tilde{\Phi}_n)(x)| \le c_1\|\tilde{\Phi}_n\|_\infty \exp\big(-n\phi(1-|xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2})\big)$
2. $\sup\limits_{x\in D}\displaystyle\int_D |T_y(S_n\tilde{\Phi}_n)(x)| d\mu_\alpha(y) \le c_1 \int_D |\tilde{\Phi}_n(z)| d\mu_\alpha(z)$
3. *if* $b_{k,l;n} = 1$ *for* $k+l = 0,1,\dots,2n;\ 0 \le k,l \le 2n$*:*

$$\big((S_n\tilde{\Phi}_n) * P\big)(x) = P(x) \ \ \forall P \in \Pi_n \text{ and } x \in D.$$

Proof:

1.) $\phi(t)$ is nondecreasing $\Rightarrow \exp\big(-n\phi(t)\big)$ is nonincreasing $\Rightarrow$ we can use Lemma 6.24 with $\phi^{\Diamond} = \exp(-n\phi)$ which means that we have:

$$|T_y\big(S_n\tilde{\Phi}_n\big)(x)| \le \|\tilde{\Phi}_n\|_\infty |T_y S_n(x)|$$

$$\le c_1\|\tilde{\Phi}_n\|_\infty \exp\big(-n\phi(1-|xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2})\big) \ .$$

2.) $S_n \in \Pi_n$, $\tilde{\Phi}_n \in \Pi_{4n} \Rightarrow (S_n\tilde{\Phi}_n) \in \Pi_{5n}$ and $\exists \beta_{k,l} : \big(S_n\tilde{\Phi}_n\big)(z) = \sum_{(k,l)=(0,0)}^{(0,5n)} \beta_{k,l} P_{k,l}^{\alpha}(z)$.

Hence

$$\sup_{x\in D} \int_D |T_y(S_n\tilde{\Phi}_n)(x)| d\mu_\alpha(y) = \sup_{x\in D} \int_D |T_y \left(\sum_{(k,l)=(0,0)}^{(0,5n)} \beta_{k,l} P_{k,l}^{\alpha}(x) \right) |d\mu_\alpha(y) \ .$$

The product formula for disc polynomials gives the following two equations

$$= \sup_{x\in D} \int_D |\sum \beta_{k,l} P_{k,l}^{\alpha}(\bar{y}) P_{k,l}^{\alpha}(x)| d\mu_\alpha(y)$$

$$= \sup_{x\in D} \int_D |T_{\bar{x}} \left(\sum \beta_{k,l} P_{k,l}^{\alpha}(\bar{y}) \right) |d\mu_\alpha(y)$$

the boundedness of the translation, Lemma 4.5 ii) gives

$$\leq \int_D |\sum \beta_{k,l} P_{k,l}^{\alpha}(\bar{y})| d\mu_\alpha(y)$$

$$= \int_D |S_n(\bar{y})\tilde{\Phi}_n(\bar{y})| d\mu_\alpha(y)$$

a simple substitution of the radial part of y shows

$$= \int_D |S_n(y)\tilde{\Phi}_n(y)| d\mu_\alpha(y)$$

$$\leq \|S_n\|_\infty \int_D |\tilde{\Phi}_n(y)| d\mu_\alpha(y)$$

and finally the condition of the exponential boundedness of S_n gives

$$\leq c_1 \int_D |\tilde{\Phi}_n(y)| d\mu_\alpha(y)$$

3.) $Q \in \Pi_{2n} \Rightarrow Q(y) = \sum_{(k,l)=(0,0)}^{(0,2n)} \hat{Q}(k,l) P_{k,l}^{\alpha}(y)$. If $b_{k,l;n} = 1$ for $k + l = 0, 1, \dots, 2n$; with $0 \leq k, l \leq 2n$:

$$\int_D \tilde{\Phi}_n(y) Q(y) d\mu(y) = \int_D \left(\sum_{(k,l)=(0,0)}^{(0,4n)} b_{k,l;n} h(k,l) P_{k,l}(\bar{y}) \sum_{(i,j)=(0,0)}^{(0,2n)} \hat{Q}(i,j) P_{i,j}(y) \right) d\mu(y)$$

$$= \sum_{(k,l)=(0,0)}^{(0,4n)} \sum_{(i,j)=(0,0)}^{(0,2n)} b_{k,l;n}\ \delta_{k,i}\ \delta_{l,j}\ \hat{Q}(i,j) = \sum_{(i,j)=(0,0)}^{(0,2n)} \hat{Q}(i,j) = Q(1)$$

Hence with a polynomial $P \in \Pi_n$ and $S_n \in \Pi_n \Rightarrow (PS_n) \in \Pi_{2n}$, and as $S_n(1) = 1$:

$$\Rightarrow \int_D \tilde{\Phi}_n(y) S_n(y) P(y) d\mu(y) = P(1)$$

$$\Rightarrow \Big((S_n\tilde{\Phi}_n) * P\Big)(x) = \int_D T_y(S_n\tilde{\Phi}_n)(x)P(y)d\mu(y)$$
$$= \int_D S_n(y)\tilde{\Phi}_n(y)T_yP(x)d\mu(y) = T_1P(x) = P(x)$$

For technical reasons in a proof below we give the definition of a function that always gives back a value in $[-\pi, \pi]$. Because of their periodicity we can avoid values outside of the inteval $[-\pi, \pi]$ for trigonometric functions.

Definition 6.27. *For $\theta, \varphi \in [0, 2\pi)$ holds: $\exists a \in [-2, 2] : (\theta - \varphi) = a\pi$. Then let*

$$|\theta - \varphi|_{min} := \begin{cases} a\pi, & -1 \le a \le 1 \\ (a+2)\pi, & -2 \le a < -1 \\ (a-2)\pi, & 1 < a \le 2\ . \end{cases}$$

If we think of it geometrically in the setting of complex numbers this function has the effect, that we take the smaller angle between the angular parts of two complex numbers.

Finally we want to give a concrete example for the function S_n.

Definition 6.28. *For $n \in \mathbb{N}$ let*

$$S_n(z) := \left(\frac{1+z}{2}\right)^n$$
$$\tilde{\Phi}_n^*(z) := \left(\frac{1+z}{2}\right)^n \tilde{\Phi}_n(z)$$
$$\Phi_n^*(z, y) := T_y\tilde{\Phi}_n^*(z)\ .$$

Theorem 6.29 ((*)). *Let $n \in \mathbb{N}$*

1. *For $x = |x|e^{i\theta}; y = |y|e^{i\varphi}$ holds:*

$$|\Phi_n^*(x, y)| \le \left[\frac{4 - \Big(1 - \cos\big(d(x, y)\big)\Big)^2}{4}\right]^{\frac{n}{2}} \|\tilde{\Phi}_n\|_\infty$$
$$\le \exp\left(-\frac{n}{2\pi^4}|xy|^2|\theta - \varphi|_{min}^4\right) \|\tilde{\Phi}_n\|_\infty$$

2. $$\sup_{x\in D}\int_D |\Phi_n^*(x, y)|d\mu(y) \le \int_D |\tilde{\Phi}_n(z)|d\mu(z)$$

3. *If* $b_{k,l;n} = 1$ *for* $0 \le k+l \le 2n$, $0 \le k,l \le n$ *:*

$$\int_D \Phi_n^*(x,y)P(y)d\mu(y) = P(x) \quad \forall P \in \Pi_n, x \in D \ .$$

Proof: As we want to apply Theorem 6.26, we have to check the following conditions: $S_n(1) = 1$ and $S_n \in \Pi_n$ are clear so it remains to show:

$$|S_n(z)| \le c_1 \exp\big(-n\phi(|1-z|)\big),\ z \in D, \text{ with } \ c_1 = 1 \text{ and } \phi(t) = \log\left[\left(\frac{4}{4-t^2}\right)^{\frac{1}{2}}\right]$$

$\phi : [0,2] \to [0,\infty]$, ϕ is continuous and nondecreasing. So:

$$|S_n(z)| \le c_1 \exp\big(-n\phi(|1-z|)\big)$$

$$\Leftrightarrow \left|\left(\frac{1+z}{2}\right)^n\right| = \left(\frac{|1+z|}{2}\right)^n \le \exp\left(-n\log\left(\frac{4}{4-|1-z|^2}\right)^{\frac{1}{2}}\right)$$

$$\Leftrightarrow \log\left(\frac{|1+z|}{2}\right)^n = n\log\left(\frac{|1+z|}{2}\right) \le -n\log\left(\frac{4}{4-|1-z|^2}\right)^{\frac{1}{2}} = n\log\left(\frac{4-|1-z|^2}{4}\right)^{\frac{1}{2}}$$

$$\overset{\text{log is nondecreasing}}{\Longleftrightarrow} \quad \frac{|1+z|}{2} \le \left(\frac{4-|1-z|^2}{4}\right)^{\frac{1}{2}} \Longleftrightarrow \frac{|1+z|^2}{4} \le \frac{4-|1-z|^2}{4}$$

$$\Leftrightarrow \quad \frac{|1+z|^2 + |1-z|^2}{4} \le 1 \quad \overset{z:=z_1+iz_2}{\Longleftrightarrow} \quad \frac{(1+z_1)^2 + z_2^2 + (1-z_1)^2 + z_2^2}{4} \le 1$$

$$\Leftrightarrow \frac{1 + 2z_1 + z_1^2 + z_2^2 + 1 - 2z_1 + z_1^2 + z_2^2}{4} = \frac{1+z_1^2+z_2^2}{2} \le 1 \ \Leftrightarrow |z|^2 \le 1$$

which is clear from the fact that $z \in D$. With Theorem 6.26, 1.) follows:

$$\begin{aligned}|\Phi_n^*(x,y)| &= |T_y\tilde{\Phi}_n^*(x)| = |T_y\big(S_n\tilde{\Phi}_n\big)(x)| \\ &\le \|\tilde{\Phi}_n\|_\infty \exp\left(-n\phi\big(1-|xy|\cos(\theta-\varphi) - \sqrt{1-|x|^2}\sqrt{1-|y|^2}\big)\right) \\ &\le \left(\frac{4-\big(1-|xy|\cos(\theta-\varphi)-\sqrt{1-|x|^2}\sqrt{1-|y|^2}\big)^2}{4}\right)^{\frac{n}{2}} \|\tilde{\Phi}_n\|_\infty \quad .\end{aligned}$$

Parts 2.) and 3.) follow directly from Theorem 6.26, 2.) and 3.) respectively.
So it remains to show the following result from part a):

$$\left(\frac{4-\big(1-|xy|\cos(\theta-\varphi)-\sqrt{1-|x|^2}\sqrt{1-|y|^2}\big)^2}{4}\right)^{\frac{n}{2}} \le \exp\left(-\frac{n}{2\pi^4}|xy|^2|\theta-\varphi|^4_{\min}\right) \ .$$

So:

$$\left(\frac{4-\big(1-|xy|\cos(\theta-\varphi)-\sqrt{1-|x|^2}\sqrt{1-|y|^2}\big)^2}{4}\right)^{\frac{n}{2}}$$

$$=\left(\frac{4-\big(1-\sqrt{1-|x|^2}\sqrt{1-|y|^2}-2|xy|\frac{\cos(\theta-\varphi)}{2}-|xy|+|xy|\big)^2}{4}\right)^{\frac{n}{2}}$$

$$=\left(\frac{4-\big(1+|xy|-\sqrt{1-|x|^2}\sqrt{1-|y|^2}-2|xy|\frac{1+\cos(\theta-\varphi)}{2}\big)^2}{4}\right)^{\frac{n}{2}}$$

$$=\left(\frac{4-\big(1+|xy|-\sqrt{1-|x|^2}\sqrt{1-|y|^2}-2|xy|\big(1-\sin^2(\frac{\theta-\varphi}{2})\big)\ \big)^2}{4}\right)^{\frac{n}{2}}$$

$$=\left(\frac{4-\big(1-|xy|-\sqrt{1-|x|^2}\sqrt{1-|y|^2}+2|xy|\sin^2(\frac{\theta-\varphi}{2})\ \big)^2}{4}\right)^{\frac{n}{2}}$$

$$=\left(\frac{4-[\big(\overbrace{1-|xy|-\sqrt{1-|x|^2}\sqrt{1-|y|^2}}^{\geq 0\ \Leftrightarrow\ (*)}\big)+\big(\overbrace{2|xy|\sin^2(\frac{\theta-\varphi}{2})}^{\geq 0}\big)\ ]^2}{4}\right)^{\frac{n}{2}}$$

$$(*)\qquad 1-|xy|-\sqrt{1-|x|^2}\sqrt{1-|y|^2}\geq 0\ \Leftrightarrow\ (1-|xy|)^2\geq(1-|x|^2)(1-|y|^2)$$
$$1-2|xy|+|xy|^2\geq 1-|x|^2-|y|^2+|xy|^2\iff(|x|-|y|)^2\geq 0$$

so we can further estimate

$$\leq\left(\frac{4-\big(2|xy|\sin^2(\frac{\theta-\varphi}{2})\ \big)^2}{4}\right)^{\frac{n}{2}}=\left(1-|xy|^2\sin^4\Big(\frac{\theta-\varphi}{2}\Big)\right)^{\frac{n}{2}}$$

$$=\exp\left(\frac{n}{2}\log\Big(1-|xy|^2\sin^4\Big(\frac{\theta-\varphi}{2}\Big)\Big)\right)\ .$$

Now, because of the 2π peridiocity of the sine it holds: $\sin^4\left(\frac{\theta-\varphi}{2}\right)=\sin^4\left(\frac{|\theta-\varphi|_{\min}}{2}\right)$

$$=\exp\left(\frac{n}{2}\log\Big(\underbrace{1-|xy|^2\sin^4\Big(\frac{|\theta-\varphi|_{\min}}{2}\Big)}_{\leq 1\quad\&\quad\log(1-t)\leq -t,\ t\in[0,1]}\Big)\right)$$

$$\leq\exp\left(-\frac{n}{2}|xy|^2\sin^4\big(\frac{|\theta-\varphi|_{\min}}{2}\big)\right)\ .$$

By the definition of $|\theta-\varphi|_{\min}$ follows: $\frac{|\theta-\varphi|_{\min}}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ but then for the sine term we have:
$\sin^4\left(\frac{|\theta-\varphi|_{\min}}{2}\right) \geq \left(\frac{|\theta-\varphi|_{\min}}{\pi}\right)^4$, because for $\tilde{x} \in [-\frac{\pi}{2}, \frac{\pi}{2}] : \sin^2(\tilde{x}) \geq \left(\frac{2\tilde{x}}{\pi}\right)^2$
$\Leftrightarrow x \in [0, \frac{\pi}{2}] : \sin(x) \geq \frac{2x}{\pi} \Leftrightarrow g(x) := \sin(x) - \frac{2x}{\pi} \geq 0$. Indeed for g holds:
$g(0) = g(\frac{\pi}{2}) = 0;\ g''(x) = -\sin(x) \leq 0 \in [0, \frac{\pi}{2}] \Rightarrow g$ is nonnegative in $[0, \frac{\pi}{2}]$
and we get our final estimation

$$\leq \exp\left(-\frac{n}{2\pi^4}|xy|^2|\theta-\varphi|^4_{\min}\right)$$

and the theorem is proved.

Again we can conclude this section with a final theorem on the approximation properties of a special kernel of that kind, namely that this kernels provides norm convergence:

Theorem 6.30 ((*)). *Let S_n like in Theorem 6.26 and $\tilde{\Phi}_n(z) := \mathcal{V}^{4n}_{2n}(z)$. With $\tilde{\Phi}^*_n(z) = S_n(z)\tilde{\Phi}_n(z)$ we have for $1 \leq p \leq \infty$ and $\alpha > 0$*

1) $\|\tilde{\Phi}^*_n * f - f\|_{X^p(D)} \longrightarrow 0$ *for $n \to \infty$*

2) $\|\tilde{\Phi}^*_n * f - f\|_{X^p(D)} \leq c \min\limits_{Q\in\Pi_n} \|f-Q\|_{X^p(D)}$.

Proof: Because of the similarity of this proof to the proof of Theorem 6.14 we will sketch it here only:
i) We have $\tilde{\Phi}^*_n * P = P$ for polynomials $P \in \Pi_n$.
ii) The L^1 norm of $\tilde{\Phi}^*_n$ is uniformly bounded via:

$$\|\tilde{\Phi}^*_n\|_{L^1(D)} = \int_D |S_n(z)\tilde{\Phi}_n(z)|d\mu(z) \leq c_1 \int_D |\mathcal{V}^{4n}_{2n}(z)|d\mu(z) \leq 2^{\lceil\alpha\rceil+\frac{3}{2}}\ c_1 \ .$$

iii) For $p = 1, \infty$ and $f \in L^p(D)$ we have

$$\|\tilde{\Phi}^*_n * f\|_{L^p(D)} \leq \|\tilde{\Phi}^*_n\|_{L^1(D)}\|f\|_{L^p(D)} \leq 2^{\lceil\alpha\rceil+\frac{3}{2}}\ c_1\ \|f\|_{L^p(D)} \ .$$

iv) With Theorem 3.7 follows statement 1).
v) For polynomials $Q \in \Pi_n$ holds:

$$\|\tilde{\Phi}^*_n * f - f\|_{L^p(D)} \leq (1 + 2^{\lceil\alpha\rceil+\frac{3}{2}}\ c_1)\|f-Q\|_{L^p(D)} \ .$$

So we even see that the constant in 2) is explicitly known.

References

[1] H. Annabi, K. Trimèche, *Convolution généralisée sur le disque unité*, C.R. Acad. Sci. Paris Sér. A 278 (1974), 21-28

[2] R. Askey and I. I. Hirschman, JR., *Mean summability for ultraspherical polynomials*, Math. Scand. 12 (1963), 167-177.

[3] R. Askey, *Orthogonal polynomials and positivity*. Studies in Applied Math. 6, Special functions and wave propagation. SIAM Philadelphia (1970).

[4] H. Bavinck, *Approximation processes for Fourier-Jacobi expansions*, Applicable Analysis 5 (1976), 293-312.

[5] G. Brown, F. Dai, *Approximation of smooth functions on compact two-point homogeneous spaces*, J. Funct. Anal. 220 (2005), no. 2, 401-423

[6] P. L. Butzer, R. J. Nessel, *Fourier analysis and approximation. Vol. I. One-dimensional theory*, Basel-Stuttgart, Birkhäuser Verlag, 1971

[7] M. R. Capobianco, W. Themistoclakis, *On the boundedness of some de la Vallée Poussin operators*, East J. Approx., 7, No.4 (2001), 417-444

[8] C.F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. 179 (1973), 331-348

[9] B. Dreseler, P. M. Soardi, *A Cohen-type inequality for Jacobi expansions and divergence of Fourier series on compact symmetric spaces*, J. Approx. Theory, 35 (1982), 214-221

[10] F. Filbir, W. Themistoclakis, *Generalized de la Vallée Poussin Operators for Jacobi weights*, Proceedings of the international conferende on numerical analysis and approximation theory Naat2006, Cluj-Napoca (Romania), July 5-8, 2006, pp. 25-34

[11] F. Filbir, H. N. Mhaskar, J. Prestin, *On a filter for exponentially localized kernels based on Jacobi polynomials*, Journal of Approximation Theory Volume 160, (2009), 256-280

[12] G. Gasper, *Banach algebras for Jacobi series and positivity of a kernel*, Ann. Math. 95, No.2 (1972), 261-280

[13] H. Heyer and S. Koshi, *Harmonic Analysis on the Disk Hypergroup*, Mathematical Seminar Notes, Tokyo Metropolitan University, 1993

[14] R.I. Jewett, *Spaces with an abstract convolution of measures*, Adv. in Math. 18 (1975), 1-101

[15] Y. Kanjin, *A convolution measure algebra on the unit disc*, Tôhoku Math. Journ. 28 (1976), 105-115

[16] Y. Kanjin, *Banach algebra related to disk polynomials*, Tôhoku Math. Journ. 37 (1985), 395-404

[17] Y. Katznelson, *An Introduction to Harmonic Analysis*, Cambridge Univ. Press, Cambridge, 3rd edition, 2004.

[18] T. Koornwinder, *The addition formula for Jacobi polynomials, I. Summery of results*, Indag. Math. 34 (1972), 188-191

[19] T. H. Koornwinder, *The addition formula for Jacobi polynomials II. The Laplace type integral and the product formula*, Report TW 133/72. Mathematisch Centrum, Amsterdam, 1972, 2+29 pp.

[20] T. Koornwinder, *Two-varible analogues of the classical orthogonal polynomials*, theory and application of special functions, Proceedings of Advanced Seminar, Mathematics Research Center, University Wisconsin, Madison, WI, 1975, Publ. No. 35, Academic Press, New York, 1975, 435-495

[21] T. H. Koornwinder, *Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an additional formula*, J. London Math. Soc. 28, 101-114,1978

[22] R. Lasser, J. Obermaier, *On the convergence of weighted Fourier expansions with respect to orthogonal polynomials*, Acta Sci. Math. 61, 345-355, 1995

[23] R. Lasser, *Introduction to Fourier Series*, Marcel Dekker, New York, 1996

[24] R. Lasser, *Discrete Commutative Hypergroups*, Inzell Lectures on Orthogonal Polynomials, Advances in the Theory of Special Functions and Orthogonal Polynomials, Nova Science Publishers, Eds. W. zu Castell, F. Filbir, B. Forster, 55-102, 2003

[25] H. N. Mhaskar, J. Prestin, *Polynomial operators for spectral approximation of piecewise analytic functions*, Appl. Comput. Harmon. Anal. 26 (2009), 121-142

[26] B. Muckenhoupt, *Mean convergence of Jacobi series*, Proc. Amer. Math. Soc., 23 (1969), 306-310

[27] P. Petrushev, Yuan Xu, *Localized polynomial frames on the interval with Jacobi weights*, J. Fourier Anal. and Appl. 5 (2005), 557-575

[28] P. Petrushev, Yuan Xu, *Localized polynomial frames on the ball*, Constr. Approx. 27 (2008), pp. 121-148

[29] H. Pollard, *The mean convergence of orthogonal series I*, Trans. Amer. Math. Soc. 62 (1947), 387-403.

[30] H. Pollard, *The mean convergence of orthogonal series II*, Trans. Amer. Math. Soc. 63 (1948), 355-367.

[31] H. Pollard, *The mean convergence of orthogonal series III*, Duke Math. J. 16 (1949), 189-191.

[32] R. Spector, *Aperçu de la théorie des hypergroups*, in: Analys harmonique sur les groupes de Lie. Séminaire Nancy-Strasbourg 1973-57, P. Eymard, J. Faraut, G. Schiffmann & R. Takahshi (eds), Lecture Notes in Math. 497, Springer, Berlin, 1975, 643-673

[33] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Vol.23, Providence, 4th edition, 1975

[34] A. Torre, *Generalized Zernike or disc polynomials: An application in quantum optics* , Journal of Computational and Applied Mathematics 222, Issue 2, (2008), 622-644

[35] G. M. Wing, *The mean convergence of orthogonal series*, Amer. J. Math. 72 (1950), 792-807.

[36] Alfred Wünsche, *Generalized Zernike or disc polynomials*, Journal of Computational and Applied Mathematics 174 (2005) 135 - 163