Iterative Network and Channel Decoding on a Tanner Graph*

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Abstract

We propose a joint network-channel coding scheme for the multiple-access relay channel. In this scenario, two users use a common relay which performs network coding. We show how a distributed Low-Density Parity-Check (LDPC) code can be applied as a joint network-channel code. The network-channel code is described by one single regular Tanner graph and is decoded with the iterative message-passing algorithm. A numerical comparison with reference systems for block fading channels confirms the diversity and code length gain which is provided by iterative network and channel decoding.

1 Introduction

1.1 Network Coding in Networks with noisy Channels

Ahlswede, Cai, Li, and Yeung proposed in [1] a new concept termed network coding that allows to increase the achievable throughput in a network. The basic idea is that intermediate nodes in a network are allowed not only to route but also to perform operations on the incoming data. Moreover, the authors of [1] proved that in multicast transmission with one source it is possible to achieve the min-cut capacity between the source and the sinks with network coding. The topic is discussed in detail in [2].

Whereas the authors in [1] assumed a network with error-free transmission, the benefit of network coding in networks with noisy channels was considered in [3]. Fig. 1 depicts an example of [3], where network coding at a relay is used to produce diversity and redundancy which results in a lower error probability of the transmission. The example in Fig. 1 can be seen as a variation of the multiple-access relay channel [4] whereas the different point-to-point channels are assumed to be orthogonal. In the example, two bits

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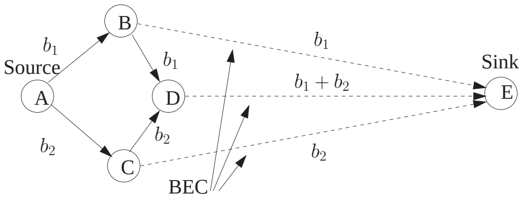

Figure 1: Example of [3]: Node D performs network coding. The dashed lines depict independent binary erasure channels (BEC).

b_1 and b_2 are sent over a network from the source at node A to the sink at node B. It is assumed that no transmission errors occur at the links A-B, A-C, B-D and C-D and thus the bits are available at the nodes B, C and D. In contrast, the links B-E, C-E and D-E are assumed to be binary erasure channels (BEC). The bits b_1 and b_2 are sent from B and C to E. Node D performs network coding[1] and sends $b_1 + b_2$ to E, where '+' denotes modulo 2 addition. As only two out of the three incoming bits $\{b_1, b_2, b_1 + b_2\}$ have to be transmitted correctly to reconstruct b_1 and b_2 at the sink, with the use of network coding a lower bit erasure probability is obtained compared with the case that the relay would just forward one of the two bits b_1 and b_2 or with the case that we would not use a relay at all.

1.2 Main Contributions and Organization of the Paper

Motivated by the example of [3], we show in this work how the redundancy provided by network coding can be used to support the channel code to protect the information transmitted over a noisy channel. We consider the multi-access relay channel [4] where two transmitters use a common relay which performs network coding. A possible application would be the uplink in a cellular based mobile communication system. As the channels to the base station are assumed to be independent block fading channels, the diversity provided by network coding improves the system performance. The transmitters perform channel coding with a low-density parity-check (LDPC) code [5, 6]. We show that the channel codes of the two transmitters and the network code can be described by one Tanner graph [7] on which the decoder performs iterative message-passing to jointly decode the network and the channel code. The proposed network-channel coding scheme extends the concept of a distributed channel code [8] and allows to increase cooperative diversity [9].

We start by explaining the assumed system model. Then, we describe how to construct a Tanner graph which describes the joint network-channel code. Numerical simulation results demonstrate that the use of a common relay which performs network coding results in a significant SNR-gain.

2 System Model

2.1 System Setup

In a cellular based mobile communication system (Fig. 2) two users MS1 and MS2 want to transmit statistically independent data which is segmented in blocks $\mathbf{u}_1$ and $\mathbf{u}_2$ of

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[1] Coding operations which combine data which is transmitted from different nodes are termed network coding.

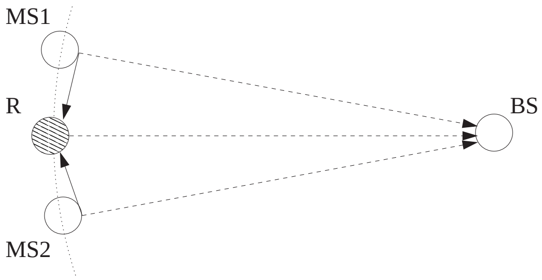


Figure 2: System model: The dashed lines depict independent block fading channels.

length K to the base station BS. A block diagram of the system is depicted in Fig. 3.

The information bits $\mathbf{u}_1$ and $\mathbf{u}_2$ are protected against transmission errors with channel encoders which output the code bits $\mathbf{x}_1$ and $\mathbf{x}_2$ with the block length N. The relay R receives the disturbed versions of the code bits $\mathbf{x}_1$ and $\mathbf{x}_2$ to obtain the estimates $\hat{\mathbf{u}}_{14}$ and $\hat{\mathbf{u}}_{24}$. Assuming correct decoding at the relay, these estimates of the information bits of MS1 and MS2 are linearly jointly combined to the network code bits $\mathbf{x}_4$ and sent to the base station to provide additional error protection. The network code bits $\mathbf{x}_4$ have the block length N_R.

2.2 Channel Model

The channels from the mobile stations and from the relay to the base station BS are assumed to be block Rayleigh fading channels and thus, the received samples after the matched filter are

$$\mathbf{y}_{i3} = a_i \cdot \mathbf{x}_i + \mathbf{n}_i, \tag{1}$$

for $i \in \{1, 2, 4\}$ where the noise $\mathbf{n}_i$ is zero mean and Gaussian with variance σ^2 and the elements of the code blocks $\mathbf{x}_i$ are either -1 or $+1$. The channel factor a_i, which is constrained by $E[{a_i}^2] = 1$, is Rayleigh distributed and represents the fading due to multipath propagation and the motion of the transmitter. The fading factors a_i ($i \in \{1, 2, 4\}$) of the three channels are statistically independent and constant over one block.

As fading only appears at the channel from the relay R to the base station BS, if the relay is mobile (e.g. another user is the relay), we also consider a second model for this channel. If the relay is stationary (e.g. installed at a traffic light) we assume this channel to be an additive white Gaussian noise (AWGN) channel without fading ($a_4 = 1$).

We make the following two assumptions to simplify the system model:

1. The relay R and the two users MS1 and MS2 have the same distances to the base station BS. Thus, the signal to noise ratio (SNR) is equal for the three channels to the base station. This is a pessimistic assumption, because in most scenarios several potential relays would be available and a closer relay to the base station could provide more gain.

2. The relay R is close to the two users MS1 and MS2. Thus, the channels to the relay have a very high SNR and the relay can decode without any error ($\hat{\mathbf{u}}_{14} = \mathbf{u}_1$ and $\hat{\mathbf{u}}_{24} = \mathbf{u}_2$). This is an optimistic assumption and it should be a topic of further research how to deal with decoding errors at the relay. However, as the error detection is very simple with LDPC codes, the relay could decide autonomously the correctness of the decoding result and forbear to send anything to the base station.

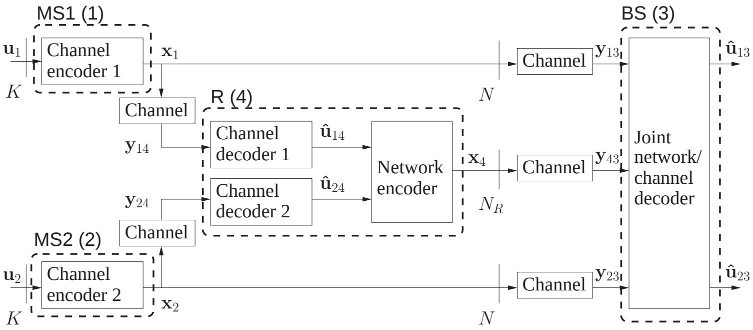


Figure 3: Block diagram of the system.

2.3 Reference Systems

As we want to know the benefit provided by network coding, we compare our system with two reference systems.

The first reference system contains the relay as well. The relay is shared by MS1 and MS2 but the data of MS1 and MS2 is not jointly processed and no network coding is applied. Fig. 4 depicts the block diagram of the relay of this reference system. This reference system allows us to analyze the benefit of network coding. The physical hardware of our system and the reference system is the same and the benefit is achieved only by coding.

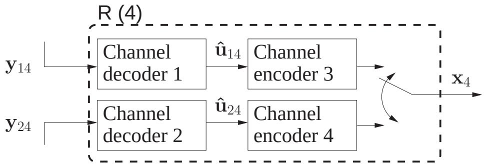
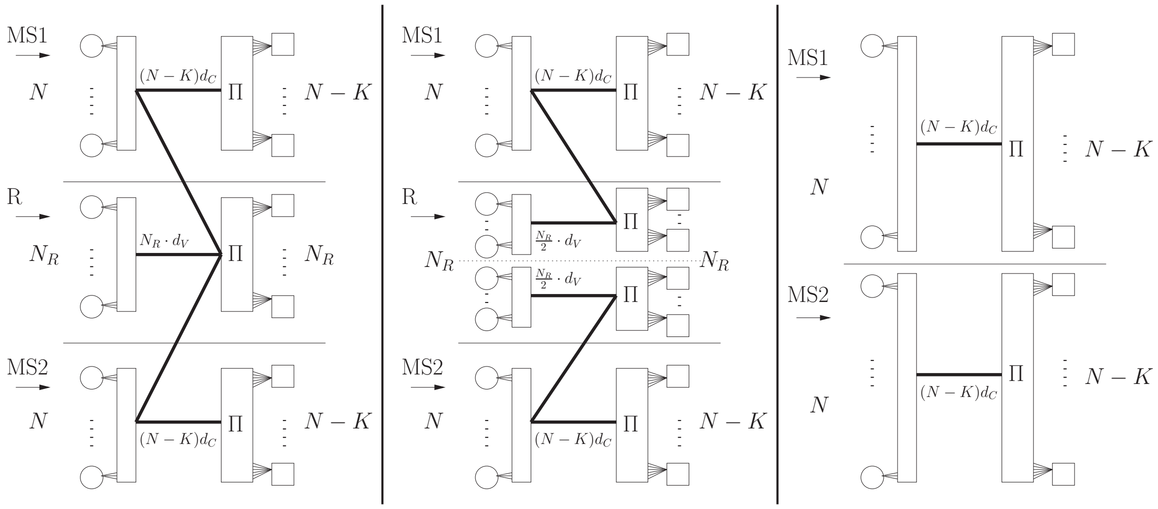


Figure 4: Block diagram of the relay of the first reference system without network coding: The relay is shared by MS1 and MS2.

The second reference system does not contain the relay. Every user transmits over one point-to-point communication to the base station. This reference system is less complex and we are able to analyze the benefit of the relay and of network coding. However, we will compare the systems in such a way that they require the same bandwidth and transmission energy.

3 Channel and Network Coding

Iterative decoding of low-density parity-check (LDPC) codes is a powerful method for approaching capacity on AWGN channels [10]. An LDPC code can be either characterized through the parity-check matrix **H** or the corresponding Tanner graph [7]. In this section, we will show that the Tanner graph provides a framework to describe the channel code and the network code and allows to decode them jointly. This idea extends the work in [11] where the joint linear design of source, channel and network code was proposed.



(a) (b) (c)

Figure 5: **(a):** Structure of the Tanner graph corresponding to the joint network-channel code. Upper and lower part represent the channel codes of the two users, the middle part the network code. As the network code connects the two channel codes the diversity provided by network coding can be exploited. **(b):** Structure of the Tanner graph of the first reference system with relay without network coding. **(c):** Structure of the Tanner graph of the second reference system without relay. To get a fair comparison the sum of the code bits in the system $2 \cdot N + N_R$ stays constant.

3.1 Channel Coding

We use LDPC codes for the channel coding at MS1 and MS2. The two LDPC codes with rate $R = K/N$ are linear block codes specified by sparse parity-check matrices $\mathbf{H}_i$ ($i \in \{1, 2\}$) with N columns and $N - K$ rows. The Tanner graph of an LDPC code consists of N variable nodes on one side and $N - K$ check nodes on the other side. Each variable node represents a code bit, each check node a parity-check equation which corresponds to one row of $\mathbf{H}_i$. The code bits $\mathbf{x}_i = \mathbf{u}_i\mathbf{G}_i$ ($i \in \{1, 2\}$) are generated from the information bits by multiplication with the generator matrix $\mathbf{G}_i$, which has to fulfill the condition $\mathbf{G}_i\mathbf{H}_i^{\mathrm{T}} = \mathbf{0}$.

3.2 Network Coding

The network encoder linearly combines the decoded information bits $\hat{\mathbf{u}}_{14} = \mathbf{u}_1$ and $\hat{\mathbf{u}}_{24} = \mathbf{u}_2$, which are assumed to be recovered correctly according to the assumptions made in Section 2.2, to generate the network code bits

$$\mathbf{x}_4 = \mathbf{u}_1\mathbf{G}_{41} + \mathbf{u}_2\mathbf{G}_{42}. \tag{2}$$

The network code of rate $R_R = (2 \cdot K)/N_R$ provides N_R additional parity-check equations that support the decoding at the base station. In contrast to the channel codes the network code combines the information bits of MS1 and MS2. Therefore, we describe the encoding operations at MS1, MS2 and at the relay jointly:

$$\mathbf{x} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_4] = [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \mathbf{G}_{41} \\ \mathbf{0} & \mathbf{G}_2 & \mathbf{G}_{42} \end{bmatrix} = \mathbf{u}\mathbf{G} \tag{3}$$

Although the different coding operations are processed spatially distributed, we will treat them as one network-channel code with the system rate

$$R_S = \frac{2 \cdot K}{2 \cdot N + N_R} = \frac{1}{\frac{1}{R} + \frac{1}{R_R}}. \tag{4}$$

The parity-check matrix $\mathbf{H}$ of the network-channel code contains $2 \cdot N + N_R$ columns and $2 \cdot (N - K) + N_R$ rows and has to fulfill $\mathbf{G}\mathbf{H}^{\mathrm{T}} = \mathbf{0}$. The decoder at the base station uses the message-passing algorithm to decode the LDPC network-channel code with parity-check matrix $\mathbf{H}$ on the Tanner graph and thus, exploit the diversity provided by the network coding scheme.

3.3 Network-Channel Code Construction

The design of the network-channel code will be done jointly, even if the encoders are spatially distributed. We will use a regular LDPC code where each code symbol participates in $d_V = 3$ equations, so that there are 3 branches leaving each variable node and where each parity-check equation contains $d_C = 6$ symbols, so that there are 6 branches leaving each check node. Thus, the rate of the network-channel code is given by $R_S = 1 - d_V/d_C = 1/2$. As we choose the network code length N_R to be equal to the channel code length ($N_R = N$), we get a channel code rate $R = K/N = (3 \cdot R_R)/2 = 3/4$ using (4). Due to the distributed network-channel encoding, parity-check equations belonging to a channel code are only allowed to contain code bits of the same channel code, whereas there are no restrictions for the parity-check equations of the network code. Instead of assigning the code randomly under these restrictions, we propose the following improved design rule for the network code which allows to exploit the diversity more efficiently. We ensure that a check node of the network code (middle part) connects all three groups of variable nodes. Three out of six edges of one check node are connected with the variable nodes of the middle part, one edge is connected alternately with a variable node of the upper or lower part and the other two edges are connected with the remaining part. Keeping these restrictions we will assign the code randomly. The network-channel code can be regarded as a distributed randomized network code which was investigated in [12].

The structure of the Tanner graph corresponding to the network-channel code is depicted in Fig. 5(a). The circles depict the variable nodes and the squares the check nodes. The random assignment of variable nodes and check nodes is depicted by Π. The graph consists of three parts for the two channel codes (upper and lower part) and the network code (middle part). Each part gets the information which initializes the message-passing algorithm from a different channel, whose fading factors are statistically independent. As the network code combines information of MS1 and MS2, its check nodes connect variable nodes of all three code parts and thus, the received information of all channels can be used to decode the information bits of one user. This allows to exploit the diversity provided by the three independent fading channels. For example, if the transmission from MS1 to the base station has very strong fading ($a_1 = 0$) it could be possible to reconstruct the information bits of MS1 only from the received information from MS2 and the relay. This is the same principle to use network coding like in the example in Fig. 1 for long block lengths.

Moreover, network coding is combined with channel coding. As the performance of LDPC codes depends strongly on the block length, the joint decoding of the network and the channel code has the advantage that the Tanner graph which is used for decoding has a larger block length, as the information of two users is jointly decoded. This positive effect could be used for AWGN channels as well and would be even more significant, if network coding would be applied to more than two users (cp. section 5.1).

3.4 Reference Systems

Let us describe how the two reference systems decode the information sent by MS1 and MS2.

In the first reference system the relay is shared by the two users. The Tanner graph which is used for the decoding in this system is depicted in Fig. 5(b). Again, we ensure that the diversity is exploited. As the information bits of MS1 and MS2 are processed separately at the relay, the Tanner graphs of the two channel codes are not connected. One half of the information sent from the relay supports one channel code. Again the LDPC code is regular where the variable nodes have degree $d_V = 3$ and the check nodes degree $d_C = 6$ and the code is assigned randomly under these restrictions. The code is a special case of the network-channel code where the Tanner graph is split into two parts where each tries to decode the information of one user separately. In [8] a similar concept was presented, whereas a distributed turbo code was used instead of a distributed LDPC code.

The second reference system does not contain a relay. The users use regular LDPC codes with degree-3 variable and degree-6 check nodes, which are depicted in Fig. 5(c). As we want to use the same bandwidth and transmission energy like in the case with relay, the channel code has a lower rate R. For the simulation in the next section, we choose the parameters given in Table 1.
In all systems MS1 an MS2 have $K = 1500$ information bits which they have to transmit to BS. The comparison is fair because in all systems $2 \cdot N + N_R = 6000$ code bits are used in the complete system. Thus, the system code rate is always $R_S = 0.5$.

System	K	N	N_R	R	R_R	R_S
Relay, network cod.	1500	2000	2000	0.75	1.5	0.5
Relay, no network cod.	1500	2000	2000	0.75	1.5	0.5
No relay	1500	3000	0	0.5	∞	0.5

Table 1: Parameters which are chosen for the simulation: K: block length (bl.) of information bits; N: bl. of channel code bits; N_R: bl. of network code bits; R: channel code rate; R_R: network code rate; R_S: system code rate

4 Numerical Results

In this section we consider simulation results for the system setup with two users and one relay which is depicted in Fig. 2. We compare the system applying the iterative joint network and channel decoding with the two reference systems. We use a regular LDPC code with $d_C = 6$ and $d_V = 3$ and the parameters which are denoted in Table 1. We use either a network code which is assigned randomly or the improved network code

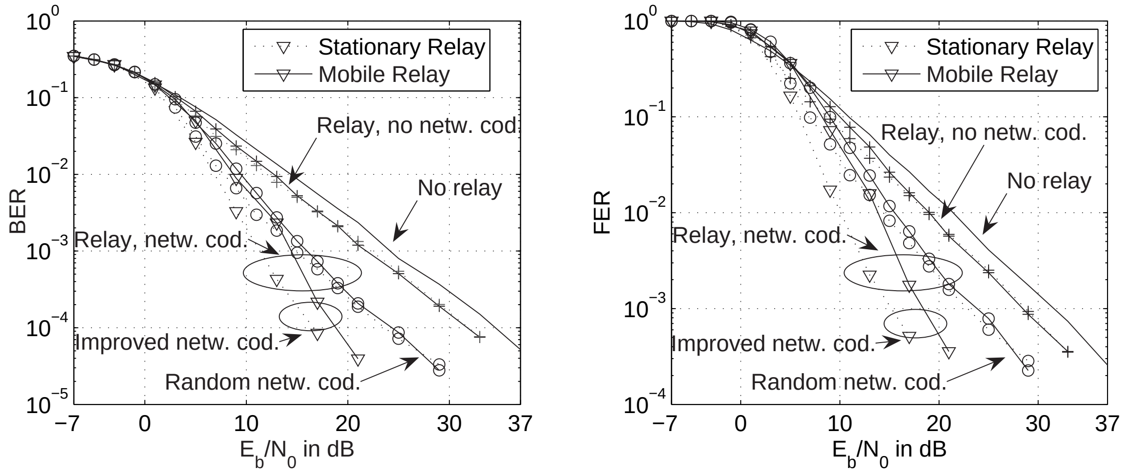


Figure 6: Simulation results: Bit error rate (BER) and frame error rate (FER) of the system applying network coding at a stationary (dashed lines) or a mobile (solid lines) relay.

design which was explained in Section 3.3. We decode the network-channel code with the message-passing algorithm with 30 iterations. Fig. 6 depicts the bit error rate (BER) and the frame error rate (FER) over the signal to noise ratio (SNR) $E_b/N_0 = 1/(R_S \cdot 2 \cdot \sigma^2)$ in dB. The system applying the iterative joint network and channel decoding achieves a significant gain compared with the reference systems. From the slope of the error rates curves we can recognize the diversity which is provided by the use of the relay and the use of network coding. If we use the improved network code design, we can exploit the diversity more efficiently. Moreover, the AWGN channel of the stationary relay only shows a significant benefit, if we use the improved network code. As the system with the relay but without network coding gains much less, we know that the application of network coding is mainly responsible for the gain.

5 Outlook on Further Work

5.1 Application for more than two Users

The application of joint network and channel decoding for more than two users could be done in several ways. For example, several users could use one relay (Fig. 7(a)). Another interesting setup would be, if one relay is only used by two users but one users uses several relays.

Fig. 7(b) depicts a possible setup with 8 users and 8 relays. The data from all users would be decoded on one graph. For both cases the length of the graph could be enlarged by a high factor. This would improve the performance especially in applications where short block lengths are used due to delay constraints.

For applications with correlated sources, such as reachback communication in large-scale sensor networks, joint network-channel decoding on a Tanner graph and joint source-channel decoding on a factor graph [13] could be combined to joint source-network-channel decoding.

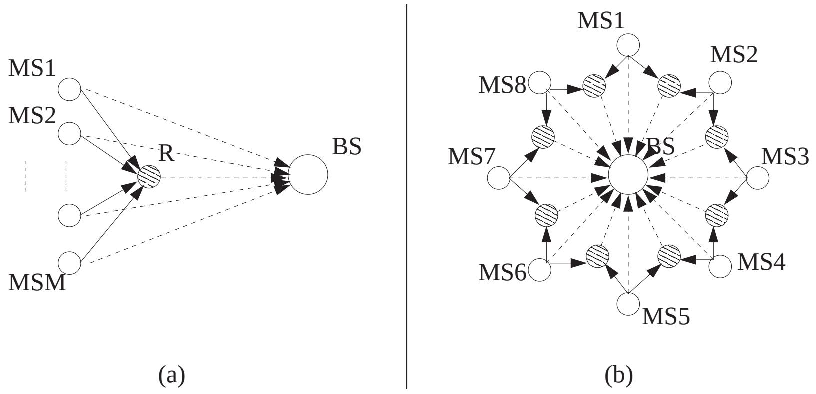


Figure 7: Two possible system setups with more than two users where joint network-channel decoding can be applied. The information of all users are decoded jointly on one Tanner graph.

5.2 Network-Channel Code Construction

We used a regular LDPC code to demonstrate the basic principle. However, in channel coding much better performances were achieved with irregular codes [10]. Due to the assignment constraints in the graph (cp. Section 3.3) a first approach would be to assign the variable nodes with the smallest degree and the check nodes with the highest degree to the network code. Then the assignment constraints could be fulfilled in any case because only the check nodes of the network code and the variable nodes of the channel nodes are allowed to be assigned to all variable nodes and all check nodes, respectively (cp. Fig. 5(a)).

Moreover, the network-channel code should be constructed such that the channel code alone provides a good performance as well, because the relay has to decode the channel code and the transmission should also work if no relay is available.

6 Conclusion

We considered the uplink in a cellular based mobile communication system of two users that share one relay. For this system setup we showed how to jointly decode the channel code and network code on a single Tanner graph with the message-passing algorithm. We studied the performance improvement of this method with simulations, where we used for the network-channel code a distributed regular LDPC code. Moreover, we presented possibilities how to apply iterative network and channel decoding for more than two users.

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