Uncontrollability of Controlled Consensus Networks characterized by Faria Vectors

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Abstract— **In this paper we investigate the controllability of a controlled agreement problem where the interaction dynamics is given by a nearest-neighbor averaging. A single agent or a group of agents is selected to be the leader(s) and act(s) as control input to all other nodes. This opens the question, where the leader(s) should be placed such that arbitrary configurations of the nodes can be achieved. Based on the observation that a zero-entry in the Laplacian eigenvector at the position of a leader affects an uncontrollable subspace we study the characterization of the uncontrollable subspace by means of a generalized version of Faria vectors. Faria vectors are eigenvectors of a Laplacian which have two entries unequal to zero, $+1, -1$. This leads to a novel topological characterization of the uncontrollable subspace. The results are valid not only for the single leader but also the multi-leader case. Numerical investigations show the advantages of the proposed approach using Faria vectors to characterize the uncontrollable subspace under certain conditions.**

I. Introduction

Large-scale networked systems have moved into the focus of current research activities in the control community due to their many societally relevant applications such as environmental monitoring by mobile sensor nodes and vehicle/robot coordination in production, logistics and transport systems. Due to their scalability distributed control approaches are often preferable in such settings. The consensus problem is a widely studied canonical problem of distributed decision making, see e.g. [1]. The controlled consensus problem is a modification of the original setting in the sense that agents are selected to be controlled by an exogenous control input. The selected agents can be interpreted as leader nodes, while all others are follower nodes. Besides investigating the system properties like convergence of the system under the regime of such controlled agreement protocols, one specific research direction focuses on the controllability of such controlled agreement protocols. Controllability indicates if it is possible to drive the system states to any configuration [2]–[4]. The identification of the uncontrollable subspace depending on the placement of the input nodes in the graph is an important problem which is studied within this work.

The controllability properties can be investigated based on algebraic graph properties [2]. If the eigenvector of the Laplacian has a zero-entry at the position of the leader the leader consensus network becomes uncontrollable [3], [5]. The topological characterization in terms of graph clustering is used to describe to controllable subspace in terms of leader-invariant external equitable partitions (LEP) [6], [7]. LEPs provide an upper bound on the dimension of the controllable subspace for single leader networks. In general the *complete* controllable subspace is not characterized by LEPs. Describing the complete controllable subspace of controlled consensus networks based on topological properties is an open issue [8], [9].

The main goal of this paper is to narrow the gap between the algebraic and the topological characterization of the controllable subspace. We make use of the fact that a $0-$entry in an eigenvector of the Laplacian L leads to an uncontrollable subspace. Under specific neighboring circumstances two vertex sets with equal cardinality lead to Laplacian eigenvectors with entries $+1, -1, 0$, called Faria vector [10], [11]. In this paper we augment the class of Faria vectors in order to apply it to the controlled consensus problem. By doing so we can characterize an uncontrollable subspace of the controlled consensus problem and its corresponding eigenvalue. Based on these findings, we discuss under which conditions the selection of multiple leaders in a network leads to an uncontrollable subspace. Due to the duality of controllability and observability, the obtained results translate to the corresponding observability problem. The numerical examples show that the proposed Faria vector based condition can topologically characterize a different part of uncontrollable subspace than LEP condition under certain conditions. The combination of both conditions provides a more complete picture of the uncontrollable subspace. Still, a gap remains to the full characterization as indicated by a counter example.

The remainder of this paper is organized as follows. Section II describes formally the controlled agreement problem considered in this work. The influence of Faria vectors to the controllability of the system is presented in Section III.

Notation: The identity matrix is $I_n \in \mathbf{R}^n$. The zero matrix with appropriate dimension is denoted by 0. span(A) denotes the span of a matrix A. The range of A is denoted by $\mathrm{range}(A)$. $\Gamma(v)$ denotes the set of neighbors of vertex v. $\Gamma_{V_1}(v)$ denotes the set of neighbors of vertex v within the vertex set V_1. Here, $|\cdot|$ denotes the cardinality of a set.

II. The controlled consensus problem

For the following analysis we consider a multi-agent system where each agent is labeled as $1, \ldots, N+1$ and the set of all agents is denoted by $\mathcal{V}$. The state of agent i is identified by $x_i \in \mathbb{R}^d$ where d denotes its dimension. An agent j is called a neighbor $\mathcal{N}_i$ of agent i if agent i has knowledge about the state $x_j(t)$. The dynamics of each agent

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evolve according to the consensus equation

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i). \tag{1}$$

This system can also be characterized as a graph where each agent is represented by a vertex in $\mathcal{V} = \{v_1, \ldots, v_{N+1}\}$. The neighboring relation between two agents i and j is represented by an edge in $\mathcal{E}_{ij}$ and the set of all edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is assumed to be undirected and static. By concatenating the agent states in $\bar{x} = [x_1, \ldots, x_{N+1}]$ and assuming $d = 1$ for simplicity of notation in this paper, (1) can be compactly rewritten as

$$\dot{\bar{x}} = -L\bar{x},$$

where L denotes the graph Laplacian. We introduce a set $\mathcal{V}_l$ of the leader node and distinguish it from the set $\mathcal{V}_f$ of follower nodes with $\mathcal{V} = \mathcal{V}_f \bigcup \mathcal{V}_l$. We assume that the last node v_{N+1} corresponds to the leader and the first N nodes are the follower nodes (the labels can always be re-indexed such that this assumption is satisfied). Under this convention the graph Laplacian L is decomposed as

$$L = -\left[\begin{array}{c|c} A & B \\ \hline B^T & \gamma \end{array}\right], \tag{2}$$

where $A = A^T \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^N$, and $\gamma \in \mathbb{R}$. By collecting all follower states into $x = [x_1, \ldots, x_N]$ and using the leader node as input $u = [x_{N+1}]$, we formulate a standard LTI-system for the controlled system

$$\dot{x} = Ax + Bu. \tag{3}$$

Note that the edges belonging to the leader are directed ones here and hence the follower have no influence on the leader.

Remark 1: For simplicity of exposition and since LEPs are only valid for the single leader problem [2], [3], [6]–[9] we restrict ourselves us to a single leader node in the problem formulation and the description of LEPs. Note, however, that the developed Faria vector condition for uncontrollability also extends to the multi-leader case as will be discussed later on.

A. Controllability Problem and Kalman Decomposition

The controllability matrix C of the controlled consensus system (3) is given by

$$C = \begin{bmatrix} B & AB & A^2B & \ldots & A^{N-1}B \end{bmatrix}. \tag{4}$$

The question to be answered is whether the leader can drive the follower states to any arbitrary final configuration x^f, which is true if and only if $\operatorname{rank} C = N$. For $\operatorname{rank} C < N$ the uncontrollable subspace needs to be determined.

An exact characterization of the controllable and uncontrollable subspace can be given based on algebraic properties. If C is rank deficient the LTI-system (3) can be decomposed into its controllable and uncontrollable part by the Kalman decomposition [6]. The similarity transformation is given by $T = \begin{bmatrix} C^{\parallel} \mid C^{\perp} \end{bmatrix}$, where $C^{\parallel} = \operatorname{span}(C) \in \mathbb{R}^{N \times \operatorname{rank} C}$ indicates the range of the controllable subspace and $C^{\perp} = \operatorname{null}(C)$ the range of the uncontrollable subspace. This similarity transformation results in

$$T^T A T = \begin{bmatrix} A_c & 0 \\ 0 & A_{\bar{c}} \end{bmatrix}, \quad T^T B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}, \text{ and}$$
$$\begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} = T^T x,$$

where c and $\bar{c}$ corresponds to the controllable and uncontrollable part and results in two decoupled subsystems

$$\dot{x}_c = A_c x_c + B_c u_c \text{ and} \tag{5}$$
$$\dot{x}_{\bar{c}} = A_{\bar{c}} x_{\bar{c}}. \tag{6}$$

By definition the lower left block of $T^T A T$ must result in $0 \in \mathbb{R}^{\operatorname{rank} C \times (N - \operatorname{rank} C)}$ as x_c has no influence on $x_{\bar{c}}$. Since $A = A^\top$, the upper right block also results as $0 \in \mathbb{R}^{(N - \operatorname{rank} C) \times \operatorname{rank} C}$ and hence x_c is unaffected by $x_{\bar{c}}$. The uncontrollable state vector is concatenated as follows: $x_{\bar{c}} = \begin{bmatrix} x_{\bar{c},1}, \ldots, x_{\bar{c},(N-\operatorname{rank}(C))} \end{bmatrix}$.

Remark 2: The uncontrollable part (6) is asymptotically stable in case of the controlled consensus problem. Since T represents a similarity transformation, the eigenvalues of A and $\begin{bmatrix} A_c & 0 \\ 0 & A_{\bar{c}} \end{bmatrix}$ remain the same. The eigenvalues of A are negative, which is a consequence of the spectra of the Laplacian L and that A is a principal submatrix of $-L$. Note that the spectra of L and $-L$ are $\sigma(-L) = -\sigma(L)$. See [12] for a detailed proof of this fact.

B. External Equitable Partitions

In large networks it is desirable to characterize the controllable and uncontrollable subspace by topological properties such as LEPs [6], [8]. LEPs exploit the fact that certain follower agents tend to cluster and in general only the average of these agents can be controlled. Here, the clustering of nodes of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined by a partition map $\pi : \mathcal{V} \to \{C_1, \ldots, C_k\}$, where $\pi(i)$ is the assigned cell for node i and k denotes the number of cells under the partition π. Consequently, an inverse operation $\pi^{-1}(C_i) = \{j \in \mathcal{V} | \pi(j) = C_i\}$ indicates the set of nodes belonging to cell C_i. The set of all cluster is defined as $\operatorname{range}(\pi) = \{C_1, \ldots, C_k\}$. The *node-to-cell degree* $\deg_\pi(i, C_j)$ indicates how many neighbors agent i has in cell C_j regarding the partition π. A clustering π of the nodes is called *external equitable partition* (EEP) if, for all C_i, C_j, where $i \neq j$

$$\deg_\pi(k, C_j) = \deg_\pi(l, C_j), \text{ for all } k, l \in \pi^{-1}(C_i). \tag{7}$$

The *leader-invariant* EEP is defined as a clustering where the leader node is in a trivial cell, i.e. $\pi^{-1}(\pi(N+1)) = \{N+1\}$. Such maximal, leader-invariant EEPs (LEP) are denoted as $\pi^\star$, where *maximal* refers to the smallest possible number of cells. For $V_{\pi^*} = \operatorname{range}(\pi^*)$ we obtain a weighted and directed graph that reflects the controllable subspace of the system (3), see [8] for a detailed description. A key result from [7] states a necessary condition for the controllability of single-leader networks based on the topological property:

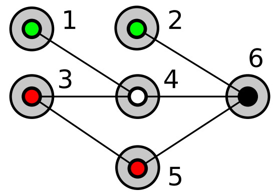

Fig. 1. The leader is denoted as • and the partition π^* is denoted by the gray enclosures around the nodes. Here, $\pi^\star = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$ is trivial, but $x_{\bar{c}} = -x_5 - x_3 + x_1 + x_2$ is uncontrollable. The signs of $x_{\bar{c}}$ is denoted by •/•

Proposition 1 ([7]): A single leader network (3) is completely controllable only if $\mathcal{G}$ is connected and π^* is trivial, i.e. $\pi^{\star-1}(\pi^\star(i)) = \{i\}, \forall i \in \mathcal{V}$.

Consequently, the upper bound for the dimension of the controllable subspace of (3) is characterized in terms of an inequality as $\text{rank}(C) \leq |\pi^\star| - 1$. Since this proposition lacks sufficiency, the complete controllable subspace is not completely described by LEPs. As stated in [8] and [9], there exist trivial LEP for which the complete controllable subspace is still unknown. For illustration we consider the following well-known example from the literature [9].

Example 1: We consider the graph of the controlled consensus problem as illustrated in Fig. 1. Node 6 is the leader node, all other nodes are followers. The partition $\pi^\star$ is given by $\pi^\star = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$, which is trivial. According to Proposition 1 we know that the controllable subspace is $\text{rank}(C) \leq 5$. However, from the algebraic condition (4) we know that $\text{rank}(C) = 4$ and from (6) we know that $x_{\bar{c}} = -x_5 - x_3 + x_1 + x_2$.

A novel method is required to characterize the uncontrollable subsystem as in Fig. 1.

III. Uncontrollable subspace by Faria Vectors

A. Uncontrollability by 0 entries in Laplacian eigenvectors

For an undirected Graph $\mathcal{G}$, a necessary and sufficient condition for the controllability of (3) is derived based on the eigenvalues of $-L$ and A [3], [13]. It is shown that system (3) is uncontrollable if a Laplacian eigenvector has a 0-entry at the $N+1$th element, where $N+1$ denotes the position of the leader. Due to the 0-entry at the leader position, the leader has no impact on this eigenmode.

Proposition 2 ([5]): Assume the system (3) to be uncontrollable. Then there exists an eigenvector of L that has a zero component on the index that corresponds to the input.

Here, the Hautus criterion requires that $A\nu_A = \lambda\nu_A$ and $B^T\nu_A = 0$ are satisfied for each uncontrollable eigenmode. It is shown in [5] that ν_A is a left eigenvector of A and $\left[\nu_A^T,\ 0\right]$ is the left eigenvector of $-L$ associated with the common eigenvalue λ. Due to the symmetry of $-L$ and A, the left eigenvectors are equal to the right eigenvectors here. A conclusion of Proposition 2 is to investigate the appearance of 0 elements in the eigenvectors of Laplacian and relate this to the uncontrollability of controlled agreement systems.

B. General Faria Vectors in Laplacians

A particular structure of Laplacian eigenvectors is called *Faria* vectors. A Faria vector has only zero entries except for two which are $+1$ and -1, see e.g. [10]. Hence, they can lead to an uncontrollable subspace. A general version of these vectors is first introduced in [11], where eigenvectors are investigated corresponding to an integer eigenvalue. Faria vectors are examined in the context of the multiplicity of integer roots of the characteristic polynomial of the Laplacian L. A Faria vector occurs in a graph if there exists a subset of vertices with the same degree p which have particular neighbors within the subset and particular edges to the remaining nodes. This is formally stated in the following.

Proposition 3 ([11]): Let $\mathcal{G}$ be a graph on N vertices and V_p the set of vertices of $\mathcal{G}$ of degree p. If there are vertices $v_1, v_2, \ldots, v_{2r} \in V_p$, such that $\Gamma(v_j) \cap \Gamma(v_k) = \emptyset, 1 \leq j < k \leq r, \Gamma(v_j) \cap \Gamma(v_k) = \emptyset, r+1 \leq j < k \leq 2r$, and $\Gamma(\{v_1, \ldots, v_r\}) = \Gamma(\{v_{r+1}, \ldots, v_{2r}\})$, then p is an eigenvalue of L with the corresponding Faria eigenvector $\nu = [\nu_i]$, where $\nu_i = 1, i = 1 \ldots r, \nu_i = -1, i = r+1 \ldots 2r, \nu_i = 0, i = 2r+1 \ldots N$.

In consequence there are Laplacian eigenvectors with 0 entries depending on topological conditions, which can lead to an uncontrollable subspace. However, this theorem is very strict since all vertices within the set V_p are required to have the same degree p. To obtain a more general version of the previous theorem, we relax the condition that each vertex of V_p requires the same degree p. Therefore we divide the considered vertex set into two equal subsets. Here, we require that each vertex inside one subset is associated with a specific value with respect to its own degree, its neighbors in its own subset and the neighbors in the other subset. This condition is less restrictive in finding Faria eigenvectors.

Theorem 1: Let $\mathcal{G}$ be a graph of N vertices. If there exists a partition π consisting of the three vertex sets $V_1 = \{v_1, \ldots, v_r\}$, $V_2 = \{v_{r+1}, \ldots, v_{2r}\}$ for some r and $V_\sigma = \mathcal{V} \backslash \{V_1, V_2\} = \{v_{2r+1}, \ldots, v_N\}$ and if now

$$\begin{aligned} \Gamma_{V_\sigma}(v_j) \cap \Gamma_{V_\sigma}(v_k) = \emptyset \qquad & 1 \leq j < k \leq r, \\ \Gamma_{V_\sigma}(v_j) \cap \Gamma_{V_\sigma}(v_k) = \emptyset \qquad & r+1 \leq j < k \leq 2r \end{aligned} \tag{8}$$

and

$$\Gamma_{V_\sigma}(V_1) = \Gamma_{V_\sigma}(V_2) \tag{9}$$

and

$$\begin{aligned} &\exists p \in \mathbb{N}, \\ &p = \deg(v_j) - \deg_\pi(v_j, V_1) + \deg_\pi(v_j, V_2) \\ &\qquad\qquad\qquad 1 \leq j \leq r, \\ &p = \deg(v_j) - \deg_\pi(v_j, V_2) + \deg_\pi(v_j, V_1) \\ &\qquad\qquad\qquad r+1 \leq j \leq 2r, \end{aligned} \tag{10}$$

then p is an integer eigenvalue of L with the associated eigenvector $\nu = [\nu_i]$, where $\nu_i = 1$ for $i \in \{1, \ldots, r\}, \nu_i = -1$ for $i \in \{r+1, \ldots, 2r\}, \nu_i = 0$ for $i \in \{2r+1, \ldots, N\}$.

Proof: For a given r the vertices of $\mathcal{G}$ can be relabeled in the following way. Let $V_1 = \{v_1, \ldots, v_r\}$ be the first r

vertices and $V_2 = \{v_{r+1}, \dots, v_{2r}\}$ be the next r vertices. The adjacent vertices of both V_1 and V_2 are ordered afterward. The remaining nodes are arbitrarily and thus the Laplacian L is given by:

$$L = \left[\begin{array}{ccc|ccc|c|c}
\deg(v_1) & \Psi_1 & & & & & & \\
& \ddots & & & \Omega & & \Xi & 0 \\
\Psi_1^T & & \deg(v_r) & & & & & \\ \hline
& & & \deg(v_{r+1}) & \Psi_2 & & & \\
& \Omega^T & & & \ddots & & \Pi & 0 \\
& & & \Psi_2^T & & \deg(v_{2r}) & & \\ \hline
& \Xi^T & & & \Pi^T & & \star & \star \\ \hline
& 0 & & & 0 & & \star & \star
\end{array}\right],$$

where due to (8) the rows of Π are linearly independent. The same fact is applicable for Ξ. However, since both sets V_1 and V_2 have the same set of neighbors due to (9), the row space of Π is equivalent to Ξ. The sum of each row in the strictly upper triangular matrices Ψ_1 and Ψ_2 is $\deg_\pi(v_j, V_1)$ and $\deg_\pi(v_j, V_2)$, respectively. The sum of each row in Ω and Ω^T is $\deg_\pi(v_j, V_2)$ and $\deg_\pi(v_j, V_1)$, respectively. Since p is equal for vertices in V_1, V_2, p is an eigenvalue of L with the corresponding eigenvector $\nu = [\nu_i]$, $\nu_i = 1, i = 1 \dots r, \nu_i = -1, i = r+1 \dots 2r, \nu_i = 0, i = 2r+1 \dots n$. ■

Theorem 1 provides a characterization of Laplacian eigenvectors with $0, -1, +1$ entries based on the topological conditions (8)-(10). It should be noted that Faria eigenvectors can only occur if the Laplacian has at least one integer eigenvalue constraining the class of graphs for which the method is suitable. For the controlled consensus problem we arrive at the following observations. If the leader node is selected among the nodes of $V_\sigma = \{v_{2r+1}, \dots, v_N\}$, the Faria eigenvector of the Laplacian has a zero entry at the leader position. It follows from Proposition 2, that under this leader the eigenmode associated with this Faria vector is uncontrollable. Note that analogously to Proposition 2 the Faria eigenvector is given as $\nu = \left[\nu_A^\top, 0\right]^\top$ where ν_A is also an eigenvector of A. Hence, the Faria vector ν describes one uncontrollable direction $x_{\bar{c}} = \nu_A^\top x$ which is asymptotically stable, $\lim_{t\to\infty} x_{\bar{c}}(t) = 0$. Since the Faria vector is constructed as $\nu = [\nu_i]$, $\nu_i = 1, i \in V_1, \nu_i = -1, i \in V_2$ an equality constraint appears for $t \to \infty$ as

$$\lim_{t\to\infty} \sum_{i\in V_1} x_i(t) - \sum_{i\in V_2} x_i(t) = 0.$$

If Faria vectors exist in a controlled agreement problem, they lead to equality conditions for the final states x^f of the leader-follower network. Besides characterizing the uncontrollable subspace, condition (10) provides us with the corresponding eigenvalue p. Note that through the decomposition of (A, B) from $-L$ in (2), the eigenvalue of A and $-L$ is $-p$. Hence, the equation of motion of the uncontrollable state characterized by a Faria vector ν is given by $e^{-pt}\nu_A^T x_0$ where p and ν_A arise from topological properties. If we know all eigenvalues of the uncontrollable system, we can further conclude about the convergence rate within the uncontrollable subspace.

Corollary 1: If the uncontrollable subspace is completely characterized by Faria vectors, the rate of convergence within the uncontrollable subspace is bounded by the smallest integer eigenvalue p_{min} associated with the characterizing Faria eigenvector since $|x_{\bar{c}}(t)| \le e^{-p_{\text{min}} t}|x_{\bar{c}}(t_0)|$.

Proof: The dynamics of the uncontrollable subsystems are given as $\dot{x}_{\bar{c}} = A_{\bar{c}} x_{\bar{c}}$ and we use the positive definite $V = \frac{1}{2} x_{\bar{c}}^\top x_{\bar{c}}$ as Lyapunov function candidate for the uncontrollable dynamics. Hence, $\dot{V} = \frac{1}{2}(\dot{x}_{\bar{c}}^\top x_{\bar{c}} + x_{\bar{c}}^\top \dot{x}_{\bar{c}}) = \frac{1}{2} x_{\bar{c}}^\top (A_{\bar{c}}^\top + A_{\bar{c}}) x_{\bar{c}} \le \lambda_{\max}(A_{\bar{c}}) x_{\bar{c}}^\top x_{\bar{c}} < 0 \quad \forall x_{\bar{c}} \neq 0$, where $\lambda_{\max}(A_{\bar{c}}) < 0$ is the largest eigenvalue of $A_{\bar{c}}$. The spectra of $A_{\bar{c}}$ and $A_{\bar{c}}^\top$ are equal, $\sigma(A_{\bar{c}}) = \sigma(A_{\bar{c}}^\top)$, and the inequality $\frac{1}{2} x_{\bar{c}}^\top (A_{\bar{c}}^\top + A_{\bar{c}}) x_{\bar{c}} \le \lambda_{\max}(A_{\bar{c}}) x_{\bar{c}}^\top x_{\bar{c}}$ follows from the Rayleigh quotient. This proves that V is a valid Lyapunonv function. By assumption the uncontrollable subspace is completely characterized by Faria vectors which have associated eigenvalues p_i, of which the minimal eigenvalue is denoted as p_{min}. These eigenvalues p_i are also the spectra of $-A_{\bar{c}}$ and $\lambda_{\max}(A_{\bar{c}}) = -p_{\text{min}}$. Hence, the uncontrollable states vanish exponentially as $|x_{\bar{c}}(t)| \le e^{-p_{\text{min}} t}|x_{\bar{c}}(t_0)|$. ■

Remark 3: The conditions (8)- (10) from Theorem 1 need to be tested as follows. For testing the conditions all possible vertex sets V_1, V_2 for all set cardinalities $r = 1 \dots \text{floor}\left(\frac{N}{2}\right)$ have to be created with $\text{floor}\left(\frac{N}{2}\right)$ being the maximal cardinality to create two vertex sets with equal cardinality. The remaining nodes are collected in V_σ. For the simplest cardinality, $r = 1$, condition (8) is always true. In order to find a Faria eigenvector, condition (8)- (10) have to be evaluated for each vertex set V_1, V_2.

Remark 4: Due to the duality of controllability and observability, the obtained results can be applied to the output node. Consider the consensus equation (2) and select an output node y resulting in the decomposition

$$\begin{aligned} \dot{x} &= Ax, \\ y &= B^T x, \end{aligned} \tag{11}$$

where y represent the output nodes. Then the question is, whether the full system state can be reconstructed based on the observations at the output node. As a consequence on our results on the controllability we arrive at the following consideration here. The system (11) is unobservable if L has a Faria eigenvector and an output node belongs to the set V_σ of this Faria eigenvector.

Faria Vectors and Multiple Leaders

LEPs are only valid for single-leader consensus networks. An uncontrollable subspace characterized by Faria vectors can also occur for multiple leaders. The argumentation straightforwardly extends the single leader problem. In case of M leaders, the LTI system (2) is resized as $A \in \mathbb{R}^{(N+1-M)\times(N+1-M)}$ and $B \in \mathbb{R}^{(N+1-M)\times M}$. The description of uncontrollable subspaces described by Faria vectors builds on Proposition 2 and this theorem is also valid for multiple leaders and enhances as follows: If the $N-M$th entries are 0s in any eigenvector of L the associated eigenmode is uncontrollable. Consequently, the satisfaction of the conditions within the follower nodes in Theorem 1 leads to an uncontrollable subspace characterized by Faria vectors. Based on the previous characterization we can provide a design guideline for selecting the leaders.

Design Guideline for Multiple Leaders

As 0 entries in the Laplacian eigenvector at the leader index lead to an uncontrollable subspace and Faria eigenvector do only have $0, -1, +1$ entries, we are also able to interpret this in the context of leader-selection.

Corollary 2: If a Faria vector is present as an eigenvector of the Laplacian L at least one input of (3) has to be chosen among the set V_1 and V_2 to avoid an uncontrollable subspace.

Proof: We know from Proposition 2 that the corresponding system motion is uncontrollable if the Laplacian eigenvector has a 0 entry at the leader position From Theorem 1 we know that the vertices of V_1, V_2 have $+1/-1$ entries in the eigenvector. So when picking one leader among the two sets V_1, V_2 we avoid the 0 entry in the Laplacian eigenvector at the position of the leader. ■

Example 2: Consider the graph in Fig. 1. If we choose the leader nodes as $\mathcal{V}_l = \{4, 6\}$ then the uncontrollable subspace $x_{\bar{c}} = x_5 + x_3 - x_1 - x_2$ remains uncontrollable despite more inputs. In contrast by choosing the leader set as $\mathcal{V}_l = \{1, 6\}$ we have direct access to $x_{\bar{c}}$ by agent 1 and the system is then completely controllable.

Faria Vectors and the Edge Principle

Since the entries of Faria vectors can only take the values $-1, +1, 0$ the edge principle is of interest here

Theorem 2 ([10]): Let λ be an eigenvalue of L associated with the eigenvector ν. If $\nu[i] = \nu[j]$, then λ is an eigenvalue of $L(\mathcal{G}^\star)$ associated with ν, where $\mathcal{G}^\star$ is the graph obtained from $\mathcal{G}$ by deleting or adding $e = \{i, j\}$, depending on whether or not it is an edge of $\mathcal{G}$.

Here it follows that we can add or remove edges connecting nodes within the sets V_1 and V_2, respectively, from the graph without effecting the uncontrollable subspace $x_{\bar{c}}$. Note that adding or removing edges can alter the LEP $\pi^\star$ and is hence not straightforwardly applicable.

Example 3: Note the graph in Fig. 1 here as example. Adding the edge $e = \{1, 2\}$ or removing $e = \{3, 5\}$ preserves the eigenvector $\nu = [-1, -1, 1, 0, 1, 0]$ and its corresponding uncontrollable space.

C. Counterexample for Uncontrollable Subspaces

The topological characterization of uncontrollable subspaces can now be described by vertex clustering (LEPs) and equality conditions (Faria vectors). However, the topological condition for controllability remains necessary until now. We now give a counterexample to demonstrate this. Consider the graph in Fig. 2 which has a trivial LEP and there are no two sets with equal cardinality who satisfy the conditions for Faria vectors. However, the uncontrollable subspace can be computed as $x_{\bar{c}} = 2x_5 + x_2 - x_3 - x_4 - x_1$ where the 5th vertex is double weighted here. Doubling a weight in the uncontrollable part has no similar correspondence in both

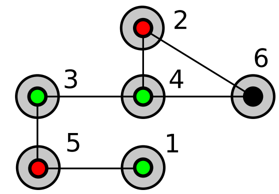

Fig. 2. The leader is denoted as ● and the partition π^* is denoted by the gray enclosures around the nodes. Here, $\pi^\star = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ is trivial, but $x_{\bar{c}} = 2x_5 + x_2 - x_3 - x_4 - x_1$ is uncontrollable. The signs of the elements in $x_{\bar{c}}$ are denoted by ●/●

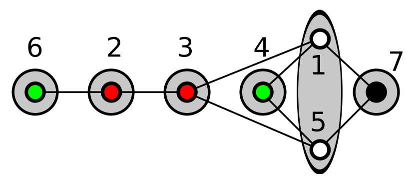

Fig. 4. The leader is denoted as ● and the partition π^* is denoted by the gray enclosures around the nodes. We obtain that $\pi^\star = \{\{1, 5\}, \{2\}, \{3\}, \{4\}, \{6\}, \{7\}\}$ is not trivial, but $|\pi^\star| - 1 \neq \mathrm{rank}(C)$. Here, $x_{\bar{c},1} = x_6 + x_4 - x_2 - x_3$ is uncontrollable, which cannot be described by Faria vectors due to unequal partitions V_1, V_2. The signs of the elements in $x_{\bar{c}}$ are denoted by ●/●

LEPs and Faria vectors and is thus a novel phenomena of controlled single leader networks. The full essence of the relationship between algebraic and topological characterization is still an open problem.

IV. Numerical Investigations

To validate the statements about Faria vectors and LEP numerically we consider the graph depicted in Fig. 4. However, the structure of this graph also reveals two sets with equal cardinality which satisfy the Faria condition in Theorem 1. Here, $V_1 = \{2, 3\}, V_2 = \{4, 6\}, p = 2$ results in an equality constraint for the states and gives $x_{\bar{c},1} = x_6 + x_4 - x_2 - x_3$. Furthermore $V_1 = \{1\}, V_2 = \{5\}, p = 3$ satisfies the Faria conditions. Here, the uncontrollable subspace $x_{\bar{c},2} = x_1 - x_5$ can also be characterized by the LEP. Based on numerical investigations we observe the following.

Remark 5: Uncontrollable subspaces characterized by Faria vectors and uncontrollable subspaces by the clustering of an LEP can be equal and are no distinct sets. This coincidence of Faria vectors and LEPs needs further investigation.

We excite the system (3) with a sinusoidal signal $u(t) = 5\sin(0.2{\cdot}\pi t)$. We observe that both uncontrollable subspaces converge as illustrated in in Fig 5: $\lim_{t\to\infty} (x_1(t) - x_2(t)) = 0$ and $\lim_{t\to\infty} (x_6(t) + x_4(t) - x_2(t) - x_3(t)) = 0$.

Often it is necessary to drive the controllable system states from an initial to a final configuration with an open-loop control input. To drive the controllable states from an initial x_c^0 to a final configuration x_c^f within a finite time horizon t_f, we can directly apply the open-loop input

$$u_{[0,t_f]}(t) = -B_c^T e^{A_c^T(t_f - t)} W_s^{-1} \left(e^{A_c t_f} x_c^0 - x_c^f\right), \quad (12)$$

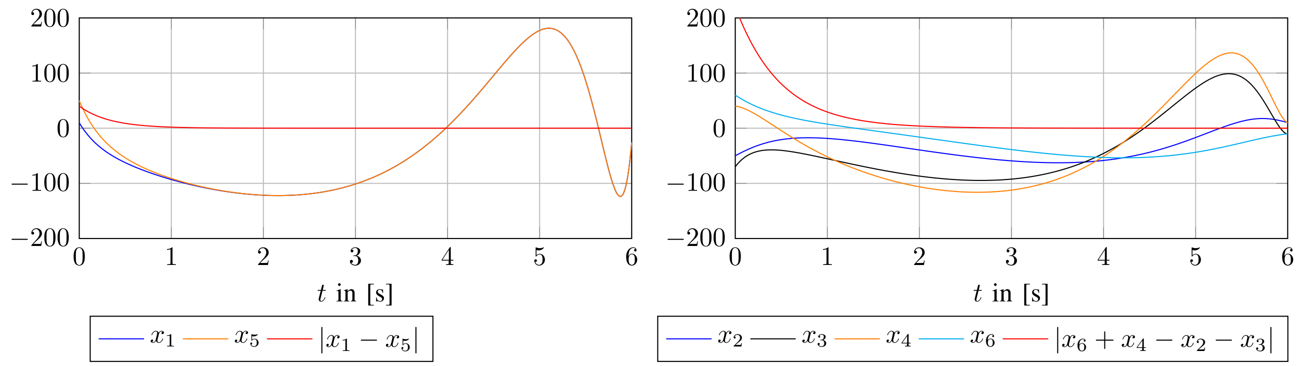

Fig. 3. Both uncontrollable subspaces $|x_6 + x_4 - x_2 - x_3|$ and $|x_1 - x_5|$ are asymptotically stable. Both x_1 and x_5 become a single controlled system on the left. The states x_2, x_3, x_4, x_5 remain different in the transition phase for the second uncontrollable subspace

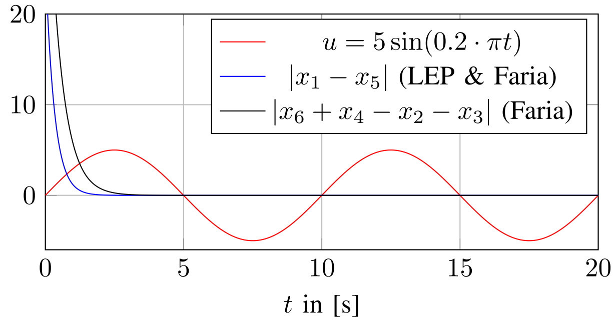

Fig. 5. Though the system is excited, the uncontrollable subspace characterized by LEP & Faria vanishes.

with the Grammian matrix $W_s = \int_0^{t_f} e^{A_c \tau} B_c B_c^T e^{A_c^T \tau} d\tau$. As the uncontrollable subspace is asymptotically stable, the uncontrollable states $x_{\bar{c}}$ are located inside an ϵ-ball after t_f

$$|x_{\bar{c}}(t)| = |e^{A_{\bar{c}} t_f}||x_{\bar{c}}(t_0)| \leq e^{\lambda_{\max}(A_{\bar{c}})}|x_{\bar{c}}(t_0)| \leq \epsilon.$$

Hence, the minimal time horizon results in

$$t_f \leq \frac{1}{\lambda_{\max}(A_{\bar{c}})} \log\left(\frac{\epsilon}{|x_{\bar{c}}(t_0)|}\right). \tag{13}$$

We want to drive the controllable states of graph in Fig. 4 from an initial configuration $x(0) = [10, -50, -70, 40, 50, 60]$ into a final configuration $x(t_f) = [-20, 10, -10, 10, -20, -10]$. The final configuration satisfies both uncontrollable subspaces since $\lim_{t\to\infty} (x_1(t) - x_5(t)) = 0$ and $\lim_{t\to\infty} (x_6(t) + x_4(t) - x_2(t) - x_3(t)) = 0$. Here, $|x_{\bar{c}}(t_0)| = 223.6$. Although $\{1, 5\}$ can be specified by LEPs, it can also be characterized by Faria vectors. A side effect of Faria vectors is that we can derive the eigenvalues, here $\lambda(A_{\bar{c}}) = (-2, -3)$ and so $\lambda_{\max}(A_{\bar{c}}) = -2$. Due to (13) the minimal time t_f to drive the system to a final configuration results as $t_f \leq \frac{1}{\lambda_{\max}(A_{\bar{c}})} \log(\frac{\Delta\epsilon}{|x_{\bar{c}}(t_0)|}) = -\frac{1}{2}\log(\frac{\Delta\epsilon}{220})$. We assume $\Delta\epsilon = 1e^{-2}$ and thus $t_f \leq 4.66$. Fig. 3 shows the states that are driven from $x(0)$ to $x(t_f)$

V. Conclusions

In this paper we provide a novel approach to characterize the uncontrollable subspace of a controlled consensus problem based on topological properties. It is based on the knowledge that zero entries in the eigenvectors of the Laplacian of the standard consensus results in an uncontrollable subspace. These eigenvectors with 0s are characterized by Faria vectors which arise from particular adjacency relations of two equal vertex sets and lead to equality constraints. The proposed approach complements the existing one based on leader-invariant external equitable partitions (LEPs). We verify these uncontrollable subspaces for counterexamples of graphs that are previously defined in the literature.

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