Analyzing the Space of Functions Analog-Computable via Wireless Multiple-Access Channels

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***Abstract*—To efficiently compute linear functions of the measurements in sensor networks, it was recently shown that the superposition property of the wireless multiple-access channel can profitably be exploited. Using suitable pre- and post-processing functions operating on real sensor readings and on the superimposed signal received by a fusion center, respectively, the natural computation property of the wireless channel can be adapted such that a much larger class of functions is efficiently computable as well. In this paper, we analyze the corresponding space of functions which are in principle computable, or at least approximable, in an analog fashion via a wireless multiple-access channel and show to what extend this impacts the communication pattern and the complexity of nodes. Finally, we change the transmission scenario to a sequence of successively received multiple-access channel output-signals and observe that the resulting questions on the computability of functions are related to the famous 13th Hilbert problem.**

I. Introduction

The efficient computation of *functions* of the measurements is one of the key problems in many wireless sensor network applications such as environmental monitoring [1]. In this context, "efficiency" refers to the need of (i) low-priced sensor nodes with strongly reduced hardware complexity and (ii) simple communication strategies requiring less wireless resources and allowing a longer operational lifetime due to the reduced overall energy consumption.

In current networks deployed to compute functions of sensor readings, the nodes convey their observations to a fusion center by using their integrated radio interfaces. To avoid interference due to concurrent transmissions, the medium access is generally coordinated by a standard medium-access protocol such as time-division multiple access (TDMA) or carrier-sense multiple access (CSMA), which separate the communication links in time by assigning each sensor node its own time slot. The fusion center reconstructs the sensor readings from the separately received transmit signals and subsequently computes the function.

This approach strictly separates the processes of communication and computation, and can be therefore highly suboptimal since the fusion center is not interested in individual raw sensor readings but rather in some function of these readings called the desired function. In order to merge the processes, Nazer and Gastpar introduced a novel information theoretic concept in [2] which directly exploits the superposition property of the underlying multiple-access channel (MAC). Similarly, in [3], Stańczak et al. used the wireless MAC to distributively compute functions of signal-to-interference ratios to determine optimal resource allocations. This view leads to more efficient computations provided that the desired function matches the natural computation property of the MAC.

Motivated by this abstraction of the wireless channel as a *computer*, the authors of [4], [5] proposed a novel analog (i.e., non-digital/uncoded) computation scheme that outperforms TDMA in many relevant wireless sensor network scenarios of interest. The scheme employs appropriate pre-processing functions, operating on real sensor readings prior to transmissions, and a post-processing function, operating on the real superimposed signal received by the fusion center, to match the resulting overall channel to a nonlinear desired function. In principle, the approach allows the analog computation of every real function f of sensor readings $x_1, \ldots, x_n$ which is representable in the form $f(x_1, \ldots, x_n) = \psi\big(\sum_{i=1}^{n} \varphi_i(x_i)\big)$, where n denotes the number of nodes and $\varphi_1, \ldots, \varphi_n, \psi$ are real univariate pre- and post-processing functions, respectively.

In previous work we presented several examples of desired functions of practical relevance that possess such representations without specifying an exact characterization of the corresponding function space. A more detailed analysis presented in this paper may help a system designer to decide whether or not a given desired function of sensor readings can be computed, or at least approximated, via a wireless multiple-access channel by using our previously proposed analog computation scheme. The analysis in this paper is carried out under the assumption of an ideal MAC or a time series of ideal MAC outputs.

In this context, we focus on dynamic computation networks in which desired functions may change and study the problem of identifying *communication patterns* that are sufficient in the sense of feedback between the fusion center and the nodes. This is necessary since the amount of feedback information about the function the fusion center wants to compute crucially determines the hardware complexity, and thus the intelligence,

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of sensor nodes (i.e., smart nodes vs. dumb nodes).

More precisely, the paper addresses the following two problems.

Problem 1 (Computation): Which functions are computable or approximable via wireless multiple-access channels?[1]

Problem 2 (Communication): How much *knowledge* about the desired function is necessary at sensor nodes?

In Problem 2, we distinguish between 1-way and 2-way computation networks, where the latter refers to the case with feedback channels between the fusion center and the sensor nodes (see Section II-B).

A. Contributions

The contributions of the paper are summarized as follows:

- Theorem 1 states if no additional restrictions are imposed, then *any real function* of n variables is computable via the wireless channel. Moreover, no information about the desired function is necessary at sensor nodes (i.e., no feedback).
- If pre- and post-processing functions are required to be continuous, Theorem 2 states that only a sparse subset of continuous real functions of n variables is computable.
- We show that certain desired functions of practical relevance that are not included in the sparse subset can be computed or approximated via the wireless MAC by simple relaxations.
- Then, we change the transmission scenario by using a time series of successively received MAC output signals for computations and realize that now the question regarding computable desired functions is closely related to the 13[th] Hilbert problem. This insight leads to Theorem 3 which states that each continuous function of n variables is computable with universal continuous pre- and post-processing functions and $2n+1$ successively received MAC output signals.
- From Theorems 1 and 3 we conclude that it is sufficient to employ *dumb sensor nodes* with hardware which is independent of the desired function, such that the complexity of nodes can drastically be reduced.

B. Notational Remarks

The k-times cartesian product $\mathbb{A} \times \cdots \times \mathbb{A}$ of a space $\mathbb{A}$ is written as $\mathbb{A}^k$. The natural and real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, respectively, and $\mathbb{E} := [0,1] \subset \mathbb{R}$ defines the unit interval. Let $\mathbb{A}^k$ be a compact Hausdorff topological space, then $\mathcal{C}[\mathbb{A}^k]$ denotes with the infinity norm $\|\cdot\|_\infty$ the Banach space of real-valued continuous functions of $k \in \mathbb{N}$ variables, defined on $\mathbb{A}^k$. Furthermore, $\mathcal{F}[\mathbb{B}^\ell]$ denotes the space of any function $g : \mathbb{B}^\ell \subseteq \mathbb{R}^\ell \to \mathbb{R}$, $\ell \in \mathbb{N}$.

II. System Model

Consider a wireless sensor network consisting of $n \geq 2$ spatially distributed, simultaneously active nodes and a designated fusion center. The nodes jointly observe a certain physical phenomenon resulting in uncorrupted sensor readings $x_i \in \mathbb{E}$, $i = 1, \ldots, n$.

A. Computation

We view a sensor network as a collection of distributed computation devices that aim at efficiently computing a given *desired function* $f \in \mathcal{F}[\mathbb{E}^n]$ of sensor readings at a fusion center (i.e., $f(x_1, \ldots, x_n)$). Some desired function examples of high practical relevance are the following [4], [5].

Example 1 (Desired Functions): (i) *Arithmetic Mean:* $f(x_1, \ldots, x_n) = \frac{1}{n}\sum_i x_i$. (ii) *Geometric Mean:* $f(x_1, \ldots, x_n) = \left(\prod_i x_i\right)^{\frac{1}{n}}$. (iii) *Maximum Value:* $f(x_1, \ldots, x_n) = \max_{1 \leq i \leq n}\{x_i\}$. (iv) *Number of Active Nodes*: $f(x_1, \ldots, x_n) = n$.

Definition 1 (Wireless MAC): Let $s_i(x_i) \in \mathbb{R}$ be a transmit signal of node i depending on sensed value $x_i \in \mathbb{E}$, $i = 1, \ldots, n$, let $h_i \in \mathbb{R}$ be a fading gain between node i and the fusion center, and let $n_i \in \mathbb{R}$ be receiver noise. Then, we refer to the standard linear model

$$(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} h_i s_i(x_i) + n_i \tag{1}$$

of a wireless multiple-access channel as the *wireless MAC* and to

$$(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} s_i(x_i) \tag{2}$$

as the *ideal MAC*, respectively.

Equation (2) reveals that the natural mathematical characteristic of the wireless MAC is simply *summation* which can immediately be used to compute some linear desired functions in an efficient way. To enable the computation of certain nonlinear desired functions by means of the wireless MAC as well, we introduce the notions of pre- and post-processing functions [4], [5].

Definition 2 (Pre-Processing Functions): We define the n real univariate functions $\varphi_i \in \mathcal{F}[\mathbb{E}]$, operating on the sensor readings $x_i \in \mathbb{E}$ (i.e., $\varphi_i(x_i)$, $i = 1, \ldots, n$), to be the *pre-processing functions*.

Definition 3 (Post-Processing Function): Let $y \in \mathbb{R}$ be the output of the wireless MAC according to Definition 1. Then, we define the univariate function $\psi \in \mathcal{F}[\mathbb{R}]$, operating on y (i.e., $\psi(y)$), to be the *post-processing function*.

Remark 1: The pre- and post-processing functions transform the ideal MAC such that the resulting overall channel, which we denote in the following as the f-MAC, *matches* the structure of the desired function (see Fig. 1).

It follows that the space of desired functions which can be computed by an f-MAC consists of any function having a representation indicated in Fig. 1, that is

$$\left\{ f \in \mathcal{F}[\mathbb{E}^n] \,\middle|\, \exists(\varphi_1, \ldots, \varphi_n, \psi) \in \mathcal{F}[\mathbb{E}] \times \cdots \right.$$
$$\left. \cdots \times \mathcal{F}[\mathbb{E}] \times \mathcal{F}[\mathbb{R}] : f(x_1, \ldots, x_n) = \psi\left(\sum_{i=1}^{n} \varphi_i(x_i)\right) \right\}. \tag{3}$$

[1]To avoid confusion with information-theoretic settings please note that whenever we write in this paper "computable" or "approximable", we exclusively mean analog-computable and analog-approximable.

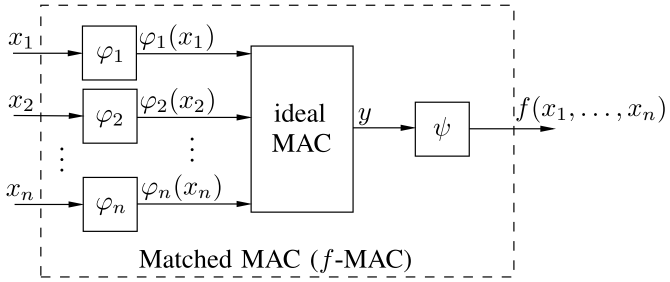

Fig. 1. The ideal MAC *matched* to the desired function f by pre-processing functions $\varphi_1, \ldots, \varphi_n$ and post-processing function ψ, which we denote as the f-MAC.

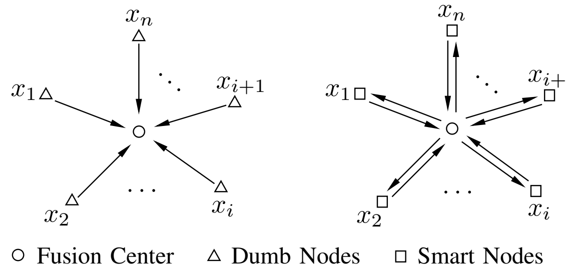

Fig. 2. (left) 1-way computation network, (right) 2-way computation network (i.e., with feedback channels between nodes and the fusion center).

In order to solve Problems 1 and 2, we analyze in this paper the function space (3).

Remark 2: Although practical computation schemes suffer from limitations such as power constraints, fading, receiver noise, synchronization issues, we focus in this paper on the function space (3) based on the f-MAC. The extension to realistic MACs (1) follows along the same lines as in [5], [6].

B. Communication

In what follows we allow that desired functions may change over time and distinguish therefore between the following communication patterns to determine which one is necessary or sufficient for appropriate computations (see Fig. 2).

Definition 4 (Communication Patterns):

(i) *1-way computation network*: A computation network consisting of nodes which do not require knowledge about the desired function is called a *1-way computation network*.

(ii) *2-way computation network*: A computation network is said to be a *2-way computation network* if it contains feedback channels between the fusion center and the nodes to inform about the desired function.

Remark 3: Note that the communication pattern immediately affects the complexity of nodes since a 1-way computation network only needs dumb sensors because the entire intelligence (i.e., the knowledge about the function to be computed) is concentrated in the fusion center, while a 2-way computation network requires smart nodes such that the intelligence is in the entire network (see Fig. 2). In this context Problem 2 poses the question which desired functions are computable by 1-way or 2-way computation networks.

III. Computation via Ideal Multiple-Access Channels

In this section we analyze the function space (3) with and without restrictions on pre- and post-processing functions. It turns out in Section III-B that continuity plays a crucial role.

A. Results

The following theorem as the main theorem of this paper solves both Problems 1 and 2. The proof can be found in Section III-B.

Theorem 1: Any desired function is computable via an f-MAC using a 1-way-computation network.

Remark 4: It is quite surprising that any desired function of sensor readings can be *universally* computed by a wireless MAC such that no feedback about the desired function (i.e., no 2-way computation network) is required. The fusion center decides by an appropriate choice of ψ which f shall be computed (i.e., the entire intelligence is in the fusion center).

Observation 1: The hard- and software of sensor nodes is independent of the desired function to be computed at the fusion center.

B. Proofs, Observations and Relaxations

It is an interesting coincidence that our practically motivated function space (3) to be further characterized is in mathematics known as the space of *nomographic functions* [7], which we denote in the following by $\mathcal{N}[\mathbb{E}^n]$. The functions are called nomographic functions since they are the basis of *nomographs*. Nomographs are graphical representations which are useful for solving certain types of equations [8]. A popular example is the Smith Chart, often used in microwave engineering.

Example 2 (Nomographic Functions): (i) *Arithmetic Mean:* $f(x_1, \ldots, x_n) = \frac{1}{n}\sum_i x_i$, with $\varphi_i(x) = x$, for all $i = 1, \ldots, n$, and $\psi(y) = y/n$. (ii) *Euclidean Norm:* $f(x_1, \ldots, x_n) = \sqrt{x_1^2 + \cdots + x_n^2}$, with $\varphi_i(x) = x^2$, for all $i = 1, \ldots, n$, and $\psi(y) = \sqrt{y}$. (iii) *Number of Active Nodes:* $f(x_1, \ldots, x_n) = n$, with $\varphi_i(x) = 1$, for all $x \in \mathbb{E}$ and $i = 1, \ldots, n$, and $\psi(y) = y$.

1) Nomographic Functions with Arbitrary Pre- and Post-Processing Functions: The observation that (3) is exactly the space of nomographic functions immediately leads us to the proof of Theorem 1.

proof of Theorem 1: The proof results from [7] where it was shown that any $f \in \mathcal{F}[\mathbb{E}^n]$ has a nomographic representation

$$f(x_1, \ldots, x_n) = \psi\left(\sum_{i=1}^{n} \underbrace{2^{i-1}\varphi(x_i)}_{=:\varphi_i(x_i)}\right), \qquad (4)$$

with an increasing $\varphi : \mathbb{E} \to \mathbb{E}$ which is independent of f (i.e., universal). As a consequence $\mathcal{N}[\mathbb{E}^n] = \mathcal{F}[\mathbb{E}^n]$. ■

Remark 5: A similar theorem which is limited to continuous desired functions can be found in [9].

Remark 6: Note that in Theorem 1 there are no restrictions on the spaces of pre- and post-processing functions. Moreover

ψ and $g(x_1,\ldots,x_n) := \sum_i \varphi_i(x_i)$ are discontinuous in general. However, each function in Example 2 has a continuous representation in the sense that the pre- and post-processing functions are continuous (but not universal). Therefore, we introduce in the following subsection the requirement of continuity in the hope that this does not affect the statement of Theorem 1 since continuity can be advantageous for practical implementations.

2) Nomographic Functions with Continuous Pre- and Post-Processing Functions: In the following, $\mathcal{N}_0[\mathbb{E}^n]$ denotes in contrast to $\mathcal{N}[\mathbb{E}^n]$ the space of nomographic functions with the additional property $(\varphi_1,\ldots,\varphi_n,\psi) \in \mathcal{C}[\mathbb{E}]\times\cdots\times\mathcal{C}[\mathbb{E}]\times\mathcal{C}[\mathbb{R}]$. How crucial this continuity property is, is mercilessly exposed by the next theorem.

Theorem 2: The space of nomographic functions with continuous pre- and post-processing functions is *nowhere dense* in the space of continuous functions, that is $\mathcal{N}_0[\mathbb{E}^n]$ nowhere dense in $\mathcal{C}[\mathbb{E}^n]$.

Proof: The constructive proof for arbitrary n is given by Buck in [10]. However, for the special case $n = 2$ the theorem was previously proven by Arnol'd [11]. ■

Although the continuity of pre- and post-processing functions reduces the amount of computable desired functions, we show in the remaining section that different small relaxations can lead to desired functions that are computable via a wireless MAC which are not nomographic according to $\mathcal{N}_0[\mathbb{E}^n]$.

Let us consider the desired function "geometric mean" from Example 1(ii). Even though it is the uniform limit of the sequence of nomographic functions

$$\Big\{f_m(x_1,\ldots,x_n) = \psi\Big(\sum_i \varphi_{i_m}(x_i)\Big) = \mathrm{e}^{\frac{1}{n}\sum_i \log_{\mathrm{e}}(x_i+\frac{1}{m})}\Big\}\,,$$

$(x_1,\ldots,x_n) \in \mathbb{E}^n$, as $m \in \mathbb{N}$ tends to infinity, the "geometric mean" is not in the space $\mathcal{N}_0[\mathbb{E}^n]$. Would indeed $\psi(\sum_i \varphi_i(x_i)) = (\prod_i x_i)^{\frac{1}{n}}$ everywhere in $\mathbb{E}^n$, then $\varphi_1(0)+\cdots+\varphi_n(0) = \varphi_1(1)+\cdots+\varphi_n(1)$ would apply [11] and therefore $\psi\big(\varphi_1(0)+\cdots+\varphi_n(0)\big) = \psi\big(\varphi_1(1)+\cdots+\varphi_n(1)\big)$. However, this leads to a contradiction because of $0 = f(0,\ldots,0) \neq f(1,\ldots,1) = 1$ and we conclude that there do not exist continuous functions $\varphi_1,\ldots,\varphi_n,\psi$ which could ensure that "geometric mean" is in $\mathcal{N}_0[\mathbb{E}^n]$.

Apparently the problem lies in the behavior of f on the boundary of $\mathbb{E}^n$ (i.e., 0 belongs to the domain) such that an appropriate restriction can lead to a nomographic representation for some $f \notin \mathcal{N}_0[\mathbb{E}^n]$, illustrated in the following examples.

Example 3: Let $(x_1,\ldots,x_n) \in \mathbb{X}^n := [x_{\min},1]^n$, with $0 < x_{\min} < 1$, such that $\mathbb{X}^n \subset \mathbb{E}^n$. (i) *Geometric Mean:* $f(x_1,\ldots,x_n) = (\prod_i x_i)^{1/n} \in \mathcal{N}_0[\mathbb{X}^n]$, with $\varphi_i : \mathbb{X} \to \mathbb{R}$, $\varphi_i(x) = \log_{\mathrm{e}}(x)$, for all $i = 1,\ldots,n$, and $\psi(y) = \mathrm{e}^{y/n}$. (ii) *Cosine of the Product:* $f(x_1,\ldots,x_n) = \cos(\prod_i x_i) \in \mathcal{N}_0[\mathbb{X}^n]$, with $\varphi_i(x) = \log_{\mathrm{e}}(x)$, for all $i = 1,\ldots,n$, and $\psi(y) = (\cos\circ\exp)(y) = \cos(\mathrm{e}^y)$.

An alternative approach to enlarge the space of desired functions computable via wireless MACs is to allow appropriate approximations of desired functions by nomographic representations. This results in a multivariate approximation problem, specified more precisely in the following definition.

Definition 5 (Approximable Nomographic Functions): Let $\varepsilon > 0$ be arbitrary but fixed. Then, we define

$$\mathcal{N}_0^{\varepsilon}[\mathbb{E}^n] := \Big\{ f \in \mathcal{F}[\mathbb{E}^n] \,\Big|\, \exists(\varphi_1,\ldots,\varphi_n,\psi) \in \mathcal{C}[\mathbb{E}]\times\cdots \\ \cdots\times\mathcal{C}[\mathbb{E}]\times\mathcal{C}[\mathbb{R}] : \Big\| f - \psi\Big(\sum_i \varphi_i(x_i)\Big)\Big\|_\infty \le \varepsilon \Big\}\,,$$

as the space of *approximable nomographic functions* with respect to precision ε. If $f \in \mathcal{N}_0^{\varepsilon}[\mathbb{E}^n]$, we write $f(x_1,\ldots,x_n) \approx \psi\big(\sum_i \varphi_i(x_i)\big)$.

An adequate characterization of any of the spaces $\mathcal{N}_0^{\varepsilon}[\mathbb{E}^n]$ is currently a serious problem, but because of the well known fact that for every $\varepsilon > 0$ there exists a $p_0 = p_0(\varepsilon)$ such that

$$\forall p \ge p_0 \forall (x_1,\ldots,x_n) \in \mathbb{E}^n : \left| \max_{1\le i\le n}\{x_i\} - \Big(\sum_{i=1}^n x_i^p\Big)^{\frac{1}{p}} \right| \le \varepsilon,$$

the following two examples, which are very important desired functions for alarm-driven sensor network applications, justify future investigations of the spaces from Definition 5.

Example 4 (Approximable Nomographic Functions): Let $\varepsilon > 0$ be arbitrary but fixed. (i) *Maximum Value:* $\max_{1\le i\le n}\{x_i\} \approx \psi\big(\sum_i \varphi_i(x_i)\big)$, with $\varphi_i(x) = x^{p_0(\varepsilon)}$, for all $i = 1,\ldots,n$, and $\psi(y) = y^{\frac{1}{p_0(\varepsilon)}}$. (ii) *Minimum Value:* $\min_{1\le i\le n}\{x_i\} \approx \psi\big(\sum_i \varphi_i(x_i)\big)$, with $\varphi_i(x) = \frac{1}{x^{p_0(\varepsilon)}}$, for all $i = 1,\ldots,n$, and $\psi(y) = y^{-\frac{1}{p_0(\varepsilon)}}$.

IV. Computation via Ideal Time Division Multiple-Access Channels

In the previous section we have seen that the continuity of pre- and post-processing functions crucially impacts the space of computable functions. To get a more complete understanding of this relationship we extend in Definition 6 the communication scenario to a time series of f-MAC outputs, since we want further exploit the natural computation property of wireless MACs.

Definition 6 (Extended f-MAC): Let $\ell \in \mathbb{N}$ be a fixed time instance and $\{\varphi_{ji}\}_{1\le j\le n, 1\le i\le \ell}$ be a collection of pre-processing functions. Then, we define an *Extended f-MAC of order ℓ* as the time series $\big\{\sum_{j=1}^n \varphi_{j1}(x_j),\ldots,\sum_{j=1}^n \varphi_{j\ell}(x_j)\big\}$ of ℓ successively received f-MAC outputs at the fusion center.

It is interesting to realize that the question regarding desired functions which are computable by means of an Extended f-MAC of order ℓ is closely related to the famous 13th problem formulated by David Hilbert in 1900 [12]. The original problem involves the study of solutions of algebraic equations and Hilbert conjectured that a solution of the general equation of degree seven cannot be represented as a superposition of continuous functions of two variables.

Because of Theorem 2 we already know that the output of an Extended f-MAC of order 1 can not be sufficient to compute any $f \in \mathcal{C}[\mathbb{E}^n]$. Implicitly Hilbert conjectured that on the basis of an Extended f-MAC of finite order, the computation of any continuous desired function is not possible which was disproven by Kolmogorov in [13]. We use a remarkable refinement of Kolmogorov's result to state the following.

Theorem 3: Each continuous desired function of n variables is computable by an Extended f-MAC of order $2n+1$ with continuous pre-processing functions using a 1-way computation network.

Proof: The proof follows from [14, Ch. 11, Thm. 1] where it was shown that each $f \in \mathcal{C}[\mathbb{E}^n]$ is representable as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{2n+1} f_i(x_1, \ldots, x_n) \tag{5}$$

with $f_i(x_1, \ldots, x_n) = \psi\big(\sum_{j=1}^{n} \xi_j \varphi_i(x_j)\big) \in \mathcal{N}_0[\mathbb{E}^n]$ and constants $\xi_j \in (0,1]$, $j = 1, \ldots, n$, where only $\psi \in \mathcal{C}[\mathbb{R}^n]$ depends on f but the $n(2n+1)$ continuous pre-processing functions $\xi_j \varphi_i$ do not. ■

Observation 2: As in Observation 1, the hard- and software of sensor nodes is completely independent of the desired function the fusion center intends to compute, such that no kind of a priori or feedback information is necessary (i.e., no 2-way computation network) to inform the nodes whether the desired function has changed.

Remark 7: In [15] it was shown that in Theorem 3 the order $2n+1$ can not be reduced.

Corollary 1: The Banach space of continuous functions on $\mathbb{E}^n$ is the algebraic sum of $2n+1$ nowhere dense subspaces.

Theorem 3 leads with Corollary 1 to further general questions. In particular Theorem 3 means that with continuous pre- and post-processing functions at least $2n+1$ time slots are necessary to compute any continuous desired function by means of the wireless MAC which can be regarded as collecting topological dimensions over time. However, the desired functions from Examples 2, 3 and 4 are obviously already satisfied with an Extended f-MAC of order 1 such that a naturally arising task for future work is to characterize which desired functions are computable by an Extended f-MAC of order ℓ, $2 \leq \ell < 2n+1$.

Although for $\ell \geq n$ there exist alternative approaches to appropriately compute desired functions over sensor networks (e.g., in an ideal TDMA protocol the entire analog sensor readings are conveyed to the fusion center), the investigation of Extended f-MACs of order $\ell > n$ can be relevant since practical TDMA protocols induce a lot of overhead and process quantized sensor data only, which significantly increases the transmission time by a decreased computation accuracy.

V. Conclusion

In the present paper we were interested in analyzing the space of functions analog-computable via wireless multiple-access channels and in the classification of corresponding communication patterns. Since quantization significantly reduces the computation accuracy and an attempt to control the corresponding quantization error may fail already for linear systems [16], [17], analog schemes are feasible for a much larger space of desired functions. In particular we have found that indeed any real function of n variables can be universally computed via a wireless MAC as long as there are no restrictions regarding the choice of pre- and post-processing functions. In this context universality means that the pre-processing functions and thus the hard- and software of nodes are completely independent of the desired function which is equivalent to the fact that feedback channels are unnecessary.

The situation changes if continuous pre- and post-processing functions are required since then the space of computable desired functions reduces to a sparse subset of $\mathcal{C}[\mathbb{E}^n]$. However, we demonstrated that some functions can be saved if one restricts the domain of pre-processing functions to appropriate subsets, if one uses approximable nomographic representations or if one exploits a time series of MAC outputs, where the latter is strongly related to Hilbert's 13^{th} problem and results in universal computation schemes as well.

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