A Probabilistic Downlink Beamforming Approach with Multiplicative and Additive Channel Errors

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***Abstract*—We study an outage constrained beamformer design in the multi-user vector downlink. The transmitter's statistical information about the fading channels is subject to an additive and a multiplicative random channel error. We split the outage probabilities and the optimization into two parts for the two types of channel errors. An inner problem takes the additive channel errors into account, assuming prior information of the multiplicative errors, and is solved in terms of convex conservative approximations. The outer optimization considers the multiplicative channel errors and adjusts the information for the inner problem. For the outer search, we compare an equal design with a locally optimal iterative search.**

***Index Terms*—chance-constrained optimization; beamformer design; rate balancing; outage constraints; SatCom channel**

I. Introduction

Rate maximization under outage probability constraints has recently become a key problem for physical layer designs [1], [2] with only imperfect knowledge of the fading channels. We consider linear beamforming in the downlink of a multi-antenna multi-user setup in this context. The common channel for the intended and the interfering signals to a user results in correlations of the intended useful received signal power and the experienced interference power, when only a statistical channel model can be acquired at the transmitter.[1]

Due to this correlation, the outage probability computation for an additive Gaussian channel estimation error requires a numerical integration (e.g., [4]) and the direct beamformer optimizations with probabilistic constraints becomes intractable. The literature focuses on conservative approximations of the chance constraints to optimize the beamformers [1], [5]–[8] and maximize the reliably achievable rates. All these approximations assume that the additive channel error is small.

Similarly, we focused on a balancing formulation in [9], i.e., the minimum of the reliably achievable rates is maximized, which is also the optimization task for this work. Therein, we assumed a multiplicative random factor for the vector channels, which leads to familiar optimizations. This multiplicative channel error model may serve as an approximation of the additive error model [10], when the channel covariance matrix is close to rank-one or the channel error is only minor.

The contribution of this work is the combination of both channel error models, that is, we consider an additive error for the known channel estimate and a multiplicative error. The motivation stems from mobile *satellite communications* (SatCom), where the channel model is based on Rician fading, which is distorted by the beam gains and rain attenuation (shadow fading) [11]. The attenuation is essentially an uncertainty about the noise power in the downlink (see Section V).

Under the assumption that these two types of errors are statistically independent, we propose to maximize a lower bound on the outage constrained achievable rates in the downlink. The idea is to virtually split the probability of outage into the outage probabilities due to the multiplicative error and the additive error. Therewith, we can make assumptions on the multiplicative error and apply known conservative approaches to approximate the problem with additive errors. How to choose this splitting becomes a new degree of freedom for the rate maximization. An equal and an iterative optimization for this probability split are proposed in Section V and a comparison of these optimizations is given in Section VI.

II. System Model

We consider a Gaussian vector downlink model, where K receivers are served in the same frequency band by an N-antenna transmitter and are subject to additive Gaussian noise $n_k \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_k^2)$. The transmitter applies linear beamforming on the modulated i.i.d. Gaussian data signals and transmits the superimposed outcome over the frequency flat fading channels $\boldsymbol{h}_k^{\mathrm{H}} \in \mathbb{C}^{1\times N}$, $k = 1, \ldots, K$, to the K mobiles. If $\boldsymbol{t}_k$, $k = 1, \ldots, K$, denote the beamforming vectors, the achievable rate for receiver k in this system reads as

$$r_k = \log_2(1 + \mathrm{SINR}_k) \tag{1}$$

with the *signal-to-interference-plus-noise-ratio* (SINR)

$$\mathrm{SINR}_k = \frac{|\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{t}_k|^2}{\sigma_k^2 + \sum_{i\neq k} |\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{t}_i|^2}. \tag{2}$$

The transmit performance is limited by the convex (conic) set of feasible beamformers $\boldsymbol{t} = [\boldsymbol{t}_1^{\mathrm{T}}, \ldots, \boldsymbol{t}_K^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{Q}$, where

$$\mathcal{Q} = \Big\{ \boldsymbol{t} \in \mathbb{C}^{KN\times 1} \Big| \sum_{k=1}^{K} \boldsymbol{t}_k^{\mathrm{H}} \boldsymbol{A}_\ell \boldsymbol{t}_k \leq P_\ell, \ell = 1, \ldots, L \Big\}. \tag{3}$$

For per-antenna constraints, which were used for the simulations in this work, $\boldsymbol{A}_\ell$ has only a one at the ℓ-th diagonal entry and zeros elsewhere. The constraint set is then formed by

$$\sum_{k=1}^{K} |[\boldsymbol{t}_k]_\ell|^2 \leq P_\ell, \quad \ell = 1, \ldots, N \tag{4}$$

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[1]In contrast, for the probabilistically constrained uplink equalizer design, the SINR's nominator and denominator are statistically independent [3].

where $[\boldsymbol{t}_k]_\ell$ denotes the ℓ-th entry of the vector $\boldsymbol{t}_k$.

III. Channel Error Model

While the receiver is implicitely assumed to have perfect access to the channel states in this rate expression, the transmitter only knows the statistics of the fading channel, e.g., it can obtain some characteristic estimate $\bar{\boldsymbol{h}}_k$. Furthermore, it knows that the (estimation) contains two errors, an additive error $\boldsymbol{e}_k \sim \mathcal{N}_\mathbb{C}(\boldsymbol{0}, \boldsymbol{C}_k)$, and a multiplicative error ξ_k, i.e.,

$$\boldsymbol{h}_k = (1+\xi_k)(\bar{\boldsymbol{h}}_k + \boldsymbol{e}_k), \quad k \in \{1,\dots,K\}. \tag{5}$$

The inverse multiplicative factor $(1+\xi_k)^{-1}$ models the attenuation, with a log-normal distributed value in *decibel* (dB) for the simulations. We remark that we consider the errors ξ_k and $\boldsymbol{e}_k$ to be mutually independent and also for indices $i \neq k$.

Above channel error model stems from SatCom (e.g., see [11]), where the strong channel mean is due to the *line-of-sight* (LoS) component of the channel that is estimated in the return link. The unknown additive error results from scattering, receiver mobility, and estimation errors in the training phase of the return link—from the mobile to the satellite. The multiplicative error is due to changing shadowing and atmospheric attenuation effects on the path from the satellite to the earth.

Alternatively, the satellites CSI could be based on feedback information from the mobiles. However, even though a sufficiently accurate channel estimation at the receivers is possible in the SatCom forward link, e.g., via pilot aided channel estimation, the feedback to the gateway is limited and subject to delays [12], which also results in imperfect transmitter CSI.

For the considered multi-spotbeam mobile SatCom scenario in the simulations, we used an S-band (2–4 GHz) channel model (e.g., see [13] and [11]). The basis is Rician fading and can be written as

$$\boldsymbol{z}_k = \sqrt{\frac{\kappa}{\kappa+1}}\bar{\boldsymbol{z}}_k + \sqrt{\frac{1}{\kappa+1}}\tilde{\boldsymbol{z}}_k \tag{6}$$

with factor κ, line-of-sight component $\bar{\boldsymbol{z}} = \boldsymbol{1}_N$ and complex random $\tilde{\boldsymbol{z}}_k$. Since scatterers are mainly around the receivers and far from the transmitting satellite, we assumed a fully correlated random vector $\tilde{\boldsymbol{z}}_k = w_k \bar{\boldsymbol{z}}_k$ with $w_k \sim \mathcal{N}_\mathbb{C}(0, \sigma^2_{w_k})$ in [14]. Such a restriction is not imposed in this work. The random vector $\tilde{\boldsymbol{z}}_k \sim \mathcal{N}_\mathbb{C}(\boldsymbol{0}, \boldsymbol{C}_{\boldsymbol{z}})$ may have a full rank covariance matrix, e.g., $\boldsymbol{C}_{\boldsymbol{z}} = \mathbf{I}_N$, in the worst case.

The Rician fading vector in (6) is distorted by the beam gain characteristic $\boldsymbol{G}_k = \operatorname{diag}(g_{1,k},\dots,g_{N,k})$ of the horn antennas at the satellite. This characteristic depends on the relative position of the receivers to the spotbeam centers. Here, $g_{i,k} = |g_{i,k}|\,\mathrm{e}^{-\mathrm{j}\psi_{i,k}}$ contains the tapered-aperture antenna gain [15] from antenna i to user k with

$$|g_{i,k}|^2 = \left(\tfrac{J_1(u_{i,k})}{2u_{i,k}} + 36\tfrac{J_3(u_{i,k})}{u_{i,k}^3}\right)^2,$$

where $J_1(\cdot)$ and $J_3(\cdot)$ are the first kind Bessel functions of order one and three, respectively, of $u_{i,k} = 2.07123\frac{\sin(\theta_{i,k})}{\sin(\theta_{3\mathrm{dB}})}$. The angle $\theta_{i,k}$ is between beamcenter i and user k as seen from the satellite and $\theta_{3\mathrm{dB}}$ is the one-sided half-power beamwidth. For an approximation of the small phase shifts $\psi_{i,k}$, we assumed that the antennas are in a plane.

With above beam gain characteristic $\boldsymbol{G}_k$, the channel to receiver k is written as (cf. [11])

$$\boldsymbol{h}_k = g_{\mathrm{FSL},k}\zeta_k \boldsymbol{G}_k \boldsymbol{z}_k \tag{7}$$

where $g_{\mathrm{FSL},k} = \frac{\lambda}{4\pi\sqrt{d_k}}$ is the *free space loss* (FSL) coefficient with wavelength λ and altitude d_k. The multiplicative factor ζ_k stems from a log-normally distributed attenuation that comprises the losses due to the rain-fading, i.e., $a_{\mathrm{rain},k} = \zeta_k^{-2}$ in dB is distributed as $\ln(a^{(\mathrm{dB})}_{\mathrm{rain},k}) \sim \mathcal{N}(m_{\mathrm{rain},k}, \sigma^2_{\mathrm{rain},k})$.

The two channel models in (5) and (7) are equivalent to each other if we substitute $\xi_k = \zeta_k - 1$ and

$$\begin{aligned}\bar{\boldsymbol{h}}_k &= g_{\mathrm{FSL},k}\sqrt{\frac{\kappa}{\kappa+1}}\boldsymbol{G}_k\bar{\boldsymbol{z}}_k, \\ \boldsymbol{C}_k &= \frac{g^2_{\mathrm{FSL},k}}{\kappa+1}\boldsymbol{G}_k\boldsymbol{G}_k^{\mathrm{H}}.\end{aligned} \tag{8}$$

This equivalence is the basis for our SatCom simulations, even though the generic model in (5) reflects other scenarios as well. For example, the multiplicative error may be seen as an effective noise uncertainty (see Section IV).

IV. Outage Based Max-Min Rate Optimization

Max-min (rate) optimizations for perfect CSI (e.g., see [16], [17]) can be solved very efficiently, but lead to many outages when based only on the channel estimates. Here, we restrict the outage probabilities to lie below ε_k when takeing the fading into account. Thereby, the resulting *chance-constrained* problem reads as

$$\max_{\rho_0, \boldsymbol{t}\in\mathcal{Q}} \rho_0 \quad \text{s. t. } \Pr(r_k \geq \rho_0\rho_k) \geq 1-\varepsilon_k,\ k=1,\dots,K \tag{9}$$

for balancing the reliably achievable rates $\rho_0\rho_k$, where the $\rho_k \in \mathbb{R}_+$ weight the receivers based on their importance.

The difficulty of solving (9) lies in the probabilistic requirement which we can rewrite as

$$\Pr(r_k \geq \rho_0\rho_k) = \Pr\!\big(\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{B}_k\boldsymbol{h}_k \geq \sigma_k^2\big) \geq 1-\varepsilon_k \tag{10}$$

where $\boldsymbol{B}_k = \frac{1}{2^{\rho_0\rho_k}-1}\boldsymbol{t}_k\boldsymbol{t}_k^{\mathrm{H}} - \sum_{i\neq k}\boldsymbol{t}_i\boldsymbol{t}_i^{\mathrm{H}}$ is indefinite with at most one positive eigenvalue. The probability computation of this indefinite quadratic form in $\boldsymbol{h}_k$ requires a nested numerical integration for the additive Gaussian errors [4], [18] and the multiplicative errors (cf. Appendix).

To find a conservative approximation of the probabilistic constraints, we first insert (5) into (10), i.e.,

$$\Pr\!\left(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq \frac{\sigma_k^2}{|1+\xi_k|^2}\right) \geq 1-\varepsilon_k \tag{11}$$

where we substituted the indefinite quadratic form in the Gaussian channel errors as

$$b_k(\boldsymbol{t}, \boldsymbol{e}_k) = (\bar{\boldsymbol{h}}_k + \boldsymbol{e}_k)^{\mathrm{H}}\boldsymbol{B}_k(\bar{\boldsymbol{h}}_k + \boldsymbol{e}_k). \tag{12}$$

In (11), we separated the optimization variables and the additive error $\boldsymbol{e}_k$ from the multiplicative channel variable ξ_k, which forms the effective random noise power $\tilde{\sigma}_k^2 = \frac{\sigma_k^2}{|1+\xi_k|^2}$.

Hence, if we were aware of $\boldsymbol{e}_k$, the probability in (11) would be the CDF of $\tilde{\sigma}_k^2$ evaluated at $b_k(\boldsymbol{t}, \boldsymbol{e}_k)$. Vice versa, if we were aware of $\tilde{\sigma}_k^2$, the randomness in the stochastic constraint would only be due to the additive error $\boldsymbol{e}_k$.

In this work, we follow the second idea and introduce an allowed uncertainty area for the effective noise power, i.e., $\tilde{\sigma}_k^2 \leq a_k$ with $\Pr(\tilde{\sigma}_k^2 \leq a_k) = \mathrm{F}_{\tilde{\sigma}_k^2}(a_k) = \alpha_k$. The inverse CDF for $\tilde{\sigma}_k^2$ may be analytically available or easy to compute to obtain the bound $a_k(\alpha_k) = \mathrm{F}_{\tilde{\sigma}_k}^{-1}(\alpha_k)$ as a function of $\alpha_k \in [1-\varepsilon_k, 1]$. With this uncertainty bound, we approximate the left-hand-side of the chance-constraint in (11) as

$$\begin{aligned}&\Pr\big(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq \tilde{\sigma}_k^2\big)\\ &\quad > \Pr\big(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq a_k(\alpha_k)\big|a_k(\alpha_k) \geq \tilde{\sigma}_k^2\big)\alpha_k.\end{aligned} \tag{13}$$

The approximation in (13) is conservative, but provides the additional degree of freedom to choose α_k. Therefore, we add an optimization over $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_K]^{\mathrm{T}}$ if we use (13) instead of the original constraints (10) for the beamformer design, i.e.,

$$\begin{aligned}&\max_{\boldsymbol{\alpha}, \rho_0, \boldsymbol{t}\in\mathcal{Q}} \rho_0 \quad \text{s.t.} \quad \mathbf{1}-\boldsymbol{\varepsilon} \leq \boldsymbol{\alpha} \leq \mathbf{1}\\ &\Pr\big(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq a_k\big|a_k \geq \tilde{\sigma}_k^2\big) \geq \frac{1-\varepsilon_k}{\alpha_k}, \quad k=1,\ldots,K\end{aligned} \tag{14}$$

where the vector inequalities for $\boldsymbol{\alpha}$ are elementwise.

V. Conservative Lower Bound Maximization

In what follows, we divide the joint optimization in (14) into an inner optimization w.r.t. the beamformers $\boldsymbol{t}$ for fixed prior probabilities and an outer optimization w.r.t. the prior probabilities $\boldsymbol{\alpha}$. Therewith, we maximize a lower bound on the reliably achievable rates.

A. Conservative Inner Optimization

The inner beamformer optimization with fixed prior probabilities $\boldsymbol{\alpha}$ is difficult to solve itself, i.e., the outage probability is non-convex in $\boldsymbol{t}$ and its computation involves a numerical integration [18]. Therefore, we restrain to one of the available conservative (convex) approximations in [1], [5], [6]. For example, we replace the chance constraints with deterministic uncertainty constraints and apply a *semidefinite relaxation* (SDR) approach for the resulting problem.

For this purpose, we additionally assume that the additive channel error $\boldsymbol{e}_k$ in (5) lies in a predefined set $\mathcal{S}_k$, e.g., the sphere $\mathcal{S}_k = \big\{\boldsymbol{e} \in \mathbb{C}^N \big| \|\boldsymbol{C}_k^{-\frac{1}{2}}\boldsymbol{e}\|_2^2 \leq d_k\big\}$, with probability $\beta_k = \Pr(\boldsymbol{e}_k \in \mathcal{S}_k)$ for the probabilistic constraint in (14). This results in the probability lower bound approximation

$$\begin{aligned}&\Pr\big(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq a_k\big|a_k \geq \tilde{\sigma}_k^2\big)\\ &\quad > \Pr\big(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq a_k\big|a_k \geq \tilde{\sigma}_k^2, \boldsymbol{e}_k \in \mathcal{S}_k\big)\beta_k.\end{aligned} \tag{15}$$

It must be larger than $\frac{1-\varepsilon_k}{\alpha_k}$ to assure the probabilistic constraint in (14).

If we fix $\beta_k = \frac{1-\varepsilon_k}{\alpha_k}$, then, the remaining conditional probability in the second line of (15) must be equal to one. In other words, the quadratic constraint $b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq a_k$ has to be fulfilled for all $\boldsymbol{e}_k \in \mathcal{S}_k$. The corresponding sphere bound is

$$d_k = \mathrm{F}_{\chi_{2N}^2}^{-1}(\beta_k)/2, \tag{16}$$

which is the inverse CDF of the χ^2-distributed random variable $2\|\boldsymbol{C}_k^{-\frac{1}{2}}\boldsymbol{e}\|_2^2$ that is of degree $2N$ and evaluated at β_k.

Replacing all probabilistic terms by their conservative worst-case counterparts, the beamformer design reads as

$$\begin{aligned}\max_{\rho_0, \boldsymbol{t}\in\mathcal{Q}} \rho_0 \quad \text{s.t.} \quad & b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq a_k,\\ & \forall \boldsymbol{e}_k: \|\boldsymbol{C}_k^{-\frac{1}{2}}\boldsymbol{e}_k\|_2^2 \leq d_k, \ k=1,\ldots,K.\end{aligned} \tag{17}$$

One may apply an iterative alternating beamformer design and worst-case channel search with local convergence for this problem (e.g., [6]). However, we require a tractable (quasi-convex) global solution approach for this inner problem in order to optimize the priors in an outer search. Recently, an approximation of the constraints in (17) with Lorentz positive maps and the corresponding linear matrix inequality reformulations were presented in [19]. This transformation to Lorentz positive maps of the uncertainty constraints requires an additional approximation of $b_k(\boldsymbol{t}, \boldsymbol{e}_k)$, which would result in an lower bound for the maximum balancing value in (17). Therefore, we apply the common SDR from [1] instead.

Let $\boldsymbol{W}_k = \boldsymbol{t}_k\boldsymbol{t}_k^{\mathrm{H}}$ with $\operatorname{rank}\{\boldsymbol{W}_k\} = 1$, then the per-antenna requirements simplify to the linear constraints

$$\sum_{k=1}^K [\boldsymbol{W}_k]_{\ell,\ell} \leq P_\ell, \quad \ell = 1,\ldots,L, \tag{18}$$

where $[\boldsymbol{W}_k]_{\ell,\ell}$ is the ℓ-th diagonal entry of the matrix $\boldsymbol{W}_k$. As is shown in [1], the S-Lemma can be applied to reformulate the uncertainty constraint in (17) to the semidefiniteness constraint

$$\boldsymbol{\Psi}_k(\rho_0, \boldsymbol{W}, \lambda_k) \succeq \mathbf{0} \tag{19}$$

where $\lambda_k \geq 0$ and the matrix function is defined as

$$\begin{aligned}\boldsymbol{\Psi}_k(\rho_0, \boldsymbol{W}, \lambda_k) = {} & \big[\boldsymbol{C}_k^{\frac{1}{2}}, \bar{\boldsymbol{h}}_k\big]^{\mathrm{H}}\boldsymbol{B}_k\big[\boldsymbol{C}_k^{\frac{1}{2}}, \bar{\boldsymbol{h}}_k\big]\\ & + \lambda_k \operatorname{bdiag}(\mathbf{I}_N, -d_k) - \operatorname{bdiag}(\mathbf{0}, a_k),\end{aligned} \tag{20}$$

where now $\boldsymbol{B}_k = \frac{1}{2^{\rho_0\rho_k}-1}\boldsymbol{W}_k - \sum_{i\neq k}\boldsymbol{W}_i$ and $\operatorname{bdiag}(\cdot)$ defines a block-diagonal matrix structure. Hence, the relaxed formulation of (17) with dropped rank constraints reads as

$$\max_{\rho_0, \boldsymbol{W}\in\mathcal{W}, \boldsymbol{\lambda}\geq\mathbf{0}} \rho_0 \ \text{s.t.} \ \boldsymbol{\Psi}_k(\rho_0, \boldsymbol{W}, \boldsymbol{\lambda}) \succeq \mathbf{0}, \ k=1,\ldots,K. \tag{21}$$

An alternative conservative formulation of (14) is obtained by applying a Bernstein's type inequality to bound the probability that the quadratic term in (12) deviates from its mean [1, Lemma 1]. The resulting covex transmit covaraince optimization problem reads as (cf. [1, Table 1])

$$\begin{aligned}&\max_{\rho_0, \boldsymbol{W}\in\mathcal{W}, \boldsymbol{y}\geq\mathbf{0}, \boldsymbol{x}} \rho_0 \quad \text{s.t.}\\ &\operatorname{tr}\big(\boldsymbol{\Psi}_k(\rho_0, \boldsymbol{W}, 0)\big) - \sqrt{2\ln\big(\tfrac{1}{1-\beta_k}\big)}x_k - \ln(1-\beta_k)y_k \geq 0,\\ &\big\|\operatorname{vec}\big(\boldsymbol{C}_k^{\frac{\mathrm{H}}{2}}\boldsymbol{B}_k\big[\boldsymbol{C}_k^{\frac{1}{2}}, \sqrt{2}\bar{\boldsymbol{h}}_k\big]\big)\big\|_2 \leq x_k,\\ &y_k\mathbf{I} + \boldsymbol{C}_k^{\frac{\mathrm{H}}{2}}\boldsymbol{B}_k\boldsymbol{C}_k^{\frac{1}{2}} \succeq \mathbf{0}, \qquad k=1,\ldots,K.\end{aligned} \tag{22}$$

This approximative problem is expected to be less conservative than the uncertainty approximation (21) for high transmit power. Therefore, the solution of (22) is expected to outperform that of (21) and both are feasible for (14).

Note that the constraints in (21) and (22) are still non-convex in ρ_0, but convex in all the other variables. Therefore, we can solve these problems via a bisection over ρ_0, for example. In each bisection step it is tested whether there exists a feasible $\boldsymbol{W}$, that satisfies the constraints for given ρ_0'. If a feasible $\boldsymbol{W}$ is found, e.g., with the disciplined convex programming toolbox CVX [20], ρ_0' is a lower bound for the optimal balancing factor and upper bound otherwise.

B. Outer Optimization of Priors

The remaining outer optimization is over the prior probabilities. We denote this optimization as

$$\max_{\boldsymbol{\alpha}} \rho_0(\boldsymbol{\alpha}) \quad \text{s.t.} \quad \mathbf{1} - \boldsymbol{\varepsilon} \leq \boldsymbol{\alpha} \leq \mathbf{1} \tag{23}$$

where $\rho_0(\boldsymbol{\alpha})$ denotes the optimum of either of the conservatively approximated chance-constrained beamformer designs with known priors. Since this optimization is non-convex w.r.t. $\boldsymbol{\alpha}$, one may choose the priors to be equal, i.e.,

$$\boldsymbol{\alpha} = \alpha_0 \mathbf{1} \tag{24}$$

and optimize over α_0 with $1 - \min\{\varepsilon_k\} \leq \alpha_0 \leq 1$ instead. The simplified prior optimization is solved via a line search over α_0, e.g., a golden section, as $\rho_0(\alpha_0 \mathbf{1})$ is quasiconcave in α_0 within its bounds. This equal prior optimization is expected to provide only slight losses in the achievable rate compared to the next detailed iterative outer optimization, when the required outage probabilities, rate targets, and the channel's fading parameters are equal as well.

We compare above equal prior optimization with an alternating optimization of the vector's entries $\boldsymbol{\alpha}$. Therein, the α_k's are sequentially updated for fixed α_i, $i \neq k$, in each iteration. In the k-th step of the i-th iteration, we find

$$\begin{aligned} \rho_0^{(i,k)} = \max_{\alpha_k} \quad & \rho_0\left(\left[\boldsymbol{\alpha}_{\underline{k}}^{(i+1),\mathrm{T}}, \alpha_k, \boldsymbol{\alpha}_{\bar{k}}^{(i),\mathrm{T}}\right]^{\mathrm{T}}\right) \\ \text{s.t.} \quad & 1 - \varepsilon_k \leq \alpha_k \leq 1 \end{aligned} \tag{25}$$

with the above mentioned golden section search, where $\boldsymbol{\alpha}_{\underline{k}} = [\alpha_1, \ldots, \alpha_{k-1}]^{\mathrm{T}}$ and $\boldsymbol{\alpha}_{\bar{k}} = [\alpha_{k+1}, \ldots, \alpha_K]^{\mathrm{T}}$.

The sequence $\{\rho_0^{(i,k)}\}_{i,k}$ is non-decreasing when starting from an initial feasible $\boldsymbol{\alpha}^{(0)}$, e.g., the solution of the equal prior optimization. Moreover, the sequence is bounded above by the transmit power limitations and the approximated chance-constraints. Hence, the proposed iteration converges in the objective. Convergence in the priors is expected for sufficiently small steps in $\boldsymbol{\alpha}^{(i)}$ as $\rho_0(\boldsymbol{\alpha})$ is continuous.

C. Beamformer Reconstruction and Power Allocation

After the iterative prior optimization converged, the transmit strategy design requires a beamformer reconstruction from the computed transmit covariances. Since the rank-one condition was dropped in favor of an efficiently solvable SDR formulation, a lossless reconstruction according to $\boldsymbol{W}_k = \boldsymbol{t}_k \boldsymbol{t}_k^{\mathrm{H}}$ is only possible if the obtained $\boldsymbol{W}_k$'s have rank one.

So far known necessary requirements for obtaining rank-one solutions from (21) (e.g., see [21], [22]), consider a sum transmit power constraint. For per-antenna constraints, we only know that we likely obtain rank-one solutions if the transmit power and d_k are sufficiently small (cf. [23]).

Parameter	Mobile Terminals
satellite configuration	GEO; S-band
beamwidth $\theta_{3\mathrm{dB}}$ (in degree)	0.2
number of beams; frequency reuse	3; 1
max satellite/user antenna gain	52 dBi/3 dBi
approximate FSL	190 dB
base receive noise power	-133 dBW
Rician fading factor κ	$10, 15$ dB
log-normal fading $m_{\mathrm{rain},k}/\sigma^2_{\mathrm{rain},k}$ [11]	$-2.62/1.63, 3.26$ dB
SNR P/σ_k^2	$0, \ldots, 30$ dB

Table I: Link budget parameters for the SatCom scenario

When non-rank-one transmit covariances were obtained, we would use Gaussian randomization together with a subsequent power allocation (cf. [23]). Let $\boldsymbol{\tau}_k$, $k = 1, \ldots, K$ denote a beamformer realization of the randomization procedure. Then, we optimize the power allocations $p_k > 0$, $k = 1, \ldots, K$ to obtain $\boldsymbol{t}_k = \sqrt{p_k}\boldsymbol{\tau}_k$ according to (21) or (22), where we replace $\boldsymbol{W}_k$ with $p_k \boldsymbol{\tau}_k \boldsymbol{\tau}_k^{\mathrm{H}}$, $k = 1, \ldots, K$. Therewith, we decrease ρ_0 until the beamformers satisfies the constraints.

If we additionally allowed for an accurate numerical calculation of the outage probabilities within the optimization, we would be able to increase the balanced rates until the outage requirements are met exactly [10]. However, this rate refinement and power adaption is not in the scope of this work.

VI. Numerical Results

The numerical results are for a SatCom system with $N = K = 3$ antennas and users, $\rho_k = 1$, $\varepsilon_k = 0.1$, per-antenna constraints with $P_\ell = P/3$, and the link parameters in Table I.

We realized 10 random positions of the users within the beams and, therewith, 10 realizations of the beamgain matrices $\boldsymbol{G}_k$ and FSLs $g_{\mathrm{FSL},k}$ according to the model in Section III. For the Rician fading in SatCom, $\kappa = 10, 15$ dB are considered as two typical values and we used $\sigma_{\mathrm{rain},k} = 1.63, 3.26$ dB for the log-normally distributed multiplicative channel error to differentiate between low and strong rain-fading, respectively (e.g., see [11]). For the 10 randomly created setups, we performed the proposed rate balancing optimization with equal prior probabilities and for iteratively determined priors.

A. Results for Equal Priors

The achievable rate vs. the per-antenna transmit power is presented in Fig. 1 in terms of averaging the results of the 10 channel realizations. As expected, the smaller κ and the larger $\sigma^2_{\mathrm{rain},k}$ is, the stronger is the influence of the additive and the multiplicative channel error, respectively, and the smaller is the reliably achievable balanced rate.

We also observe from these simulations, that the Bernstein's inequality method (22) outperforms the bounded additive channel error method (21) for the inner transmit covariance optimization, which was expected from [1]. Only for stronger rain and Rician fading parameters, i.e., $\sigma^2_{\mathrm{rain},k} = 3.26$ dB and $\kappa = 10$ dB, there is a slight loss from the Bernstein's type inequality to the bounded channel error method.

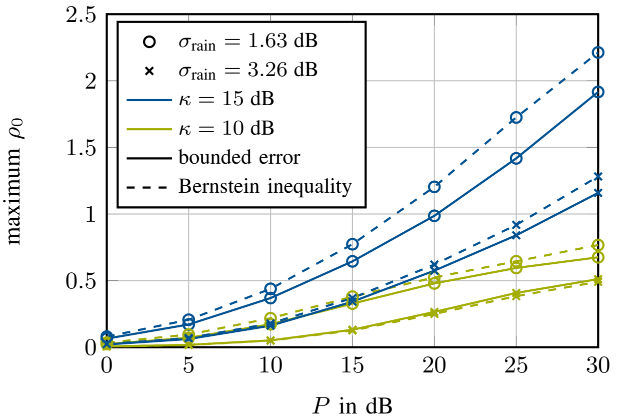

Figure 1: maximum ρ_0 vs. P for a $N = K = 3$ system with equal targets $\rho_k = 1$, requirements $\varepsilon_k = 0.1$, and priors $\alpha_k = \alpha_0$

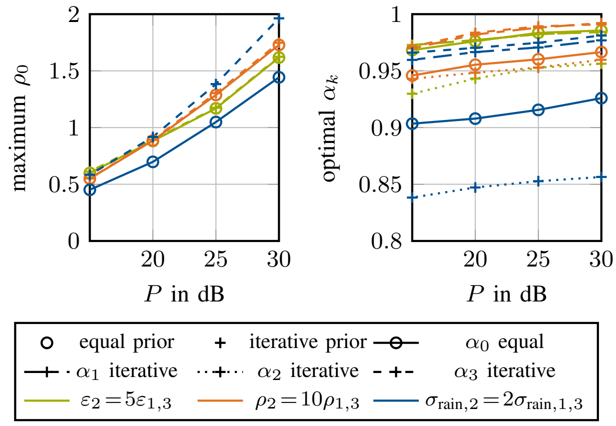

Figure 3: comparison of iterative and equal prior optimization for a $N = K = 3$ system with equal $\rho_k = 1$ or equal $\varepsilon_k = 0.1$

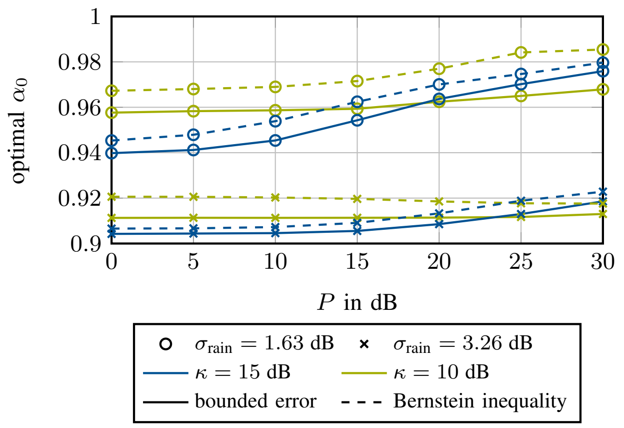

Figure 2: maximizing α_0 vs. P for a $N = K = 3$ system with equal targets $\rho_k = 1$, requirements $\varepsilon_k = 0.1$, and priors $\alpha_k = \alpha_0$

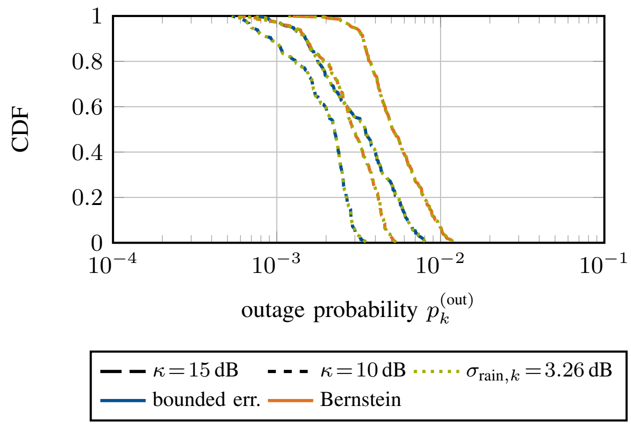

Figure 4: actual outage probabilities for a $N = K = 3$ system with $\rho_k = 1$, $\varepsilon_k = 0.1$, $\sigma_{\text{rain},k} = 1.63$ dB, and equal priors

In Fig. 2, we sketched the rate maximizing α_0 vs. P in dB. For strong rain fading, i.e., $\sigma^2_{\text{rain},k} = 3.26$ dB, the optimal prior α_0 remains almost constant between 91 and 92 percent if $\kappa = 10$ dB and increases only slightly with P from about 90.5% to about 92% if $\kappa = 15$ dB. For weak rain fading, i.e., $\sigma^2_{\text{rain},k} = 1.63$ dB, the optimal common prior α_0 is larger than for strong rain fading. It increases from about 94% to 98% in the considered interval for P when $\kappa = 15$ dB. For $\kappa = 10$ dB, only a slight increase is visible.

We remark that the larger the prior probability α_0 is, the smaller the posterior probabilities $\frac{1-\varepsilon_k}{\alpha_0}$ will be, and therefore, the uncertainty radii d_k in (17) when bounding the channel error by a sphere. Hence, the uncertainty radii d_k become smaller the larger the transmit power P is. In other words, the method shifts some of the rate limiting influence of the additive channel error to the multiplicative error in order to maximize the achievable rate.

B. Comparison to Iteratively Computed Priors

To compare the optimized ρ_0 and α_k's of the equal and the iterative prior optimization (see Figure 3), we considered three variations of the weak rain fading scenario.

- S.1 First, we varied the rate targets to $\rho_1 = \rho_3 = 0.25$ and $\rho_2 = 2.5$ (orange lines).
- S.2 Alternatively, we changed the outage requirements to $\varepsilon_1 = \varepsilon_3 = 0.05$ and $\varepsilon_2 = 0.25$ (green lines).
- S.3 Finally, only receiver 2 experiences strong rain fading with $\sigma_{\text{rain},2} = 3.26$ dB. Therefore, we allow for an outage requirement $\varepsilon_2 = 0.20$, while receivers 1 and 3 have low rain fading and the requirements $\varepsilon_1 = \varepsilon_3 = 0.10$.

We see that the maximum ρ_0 of the iterative prior optimization does not significantly outperform that of the equal prior optimization (see Fig. 3) for the Scenarios S.1 and S.2, even though the iteratively obtained α_2 lie about 3% below α_3 for these scenarios. In contrast, if we assume strong rain fading and allow for more outages only for receiver 2, as in Scenario S.3, the balanced rates benefit from an iterative prior optimization (cf. Fig. 3). In this case, the difference between the optimized α_2 and α_1, α_3 is more than 10%, while the equally optimization α_0 is in the middle of these bounds.

Therefore, an adaptive prior optimization needs to be implemented only if the statistics of the multiplicative channel errors strongly differ for the served receivers. An equal prior choice performs sufficiently well for channels with similar multiplicative channel errors, even though the receivers have different QoS requirements, i.e., rate or outages.

C. Actually Achieved Outage Probabilities

The actually achieved outage probabilities

$$p_k^{(\mathrm{out})} = \Pr(r_k \leq \rho_0 \rho_k) \tag{26}$$

can be computed for given beamformers in $\boldsymbol{t}$ as detailed in the Appendix. The probability $p_k^{(\mathrm{out})}$ of the proposed conservative approach, with an equal outer prior restriction and an inner conservative beamformer design, lies strictly below the requirements ε_k. This is also seen in Fig. 4, where we plotted the empirical CDF of the actually achieved outage probabilities based on the simulations for Fig. 1.

In accordance with [1], the inner Bernstein's type inequality approach is slightly less conservative than the bounded channel error approach. Moreover, we see that the smaller the Rician factor κ is, the smaller the obtained outage probabilities are. In other words, the inner optimizations are statistically more conservative when the likelihood of a large additive channel error increases. In contrast, changes in the rain fading parameter $\sigma^2_{\mathrm{rain},k}$ do not influence the obtained outage probability. We conclude therefore that the prior approximation approach takes the multiplicative channel errors sufficiently into account.

VII. Conclusion

We studied the outage constrained downlink beamforming problem with multiplicative and additive channel errors. We used a conservative inner beamformer design to be robust against the additive channel errors and take care about multiplicative channel errors with an outer prior optimization strategy to determine the users' effective noise levels. The simulations showed that an equal effective noise adaption is sufficient as long as the general fading parameters, i.e., the Rician factor and the rain fading characteristic, differ only slightly. However, there is still room for improving the inner optimization in terms of computational complexity and the degree of conservatism compared to the state-of-the-art error bounding and Bernstein's inequality approximation.

Appendix

The computation of the k'th user's actual outage probability

$$p_k^{(\mathrm{out})} = 1 - \Pr\left(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq \tilde{\sigma}_k^2\right)$$

requires numerical integration for given ρ_0 and $\boldsymbol{t}$. To this end, we rewrite above right-hand-side probability as

$$\begin{aligned} &\Pr\big(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq \tilde{\sigma}_k^2\big) = \\ &\int_0^\infty \mathrm{f}_z(z) \Pr\left(b_k(\boldsymbol{t}, \boldsymbol{e}_k) \geq \sigma_k^2 10^{\frac{z}{10}} \,\middle|\, z\right) \mathrm{d}\, z \end{aligned} \tag{27}$$

with the log-normal *probability density function* (PDF)

$$\mathrm{f}_z(z) = \frac{1}{z\sigma_{\mathrm{rain}}\sqrt{2\pi}} \, \mathrm{e}^{-\frac{(\ln(z)-\mu_{\mathrm{rain}})^2}{2\sigma^2_{\mathrm{rain}}}} \tag{28}$$

for the rain attenuation in dB. The inner conditional probability in (27) is computed with Imhof's method [18] for all required values z to numerically evaluate the outer integration, e.g., with a standard quadrature integration method.

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