Bounds on the Outage Constrained Capacity of the Single-Antenna Gaussian Relay Channel

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***Abstract*—We study the probabilistically constrained capacity of a Gaussian relay channel in terms of the corresponding outage constrained *decode-and-forward* (DF) achievable rate and the *cut-set bound* (CSB). The probabilistic outage constraint is necessary due to Rayleigh fading and the absence of *channel state information* (CSI) at the transmitting nodes. The DF outage probability can be calculated in closed form, which allows a direct optimization of the joint source and relay transmit strategy. In contrast, the CSB probability can only be calculated in closed form for noncoherent transmission from the source and relay to the destination. For coherent transmission, we propose two upper bounds based on a genie-aided transmission and Markov's inequality. The numerical results verify the quality of the bounds.**

***Index Terms*—outage constrained DF rate; CSB probability; chance-constrained robust relaying; imperfect CSI**

I. Introduction

The capacity of the standard Gaussian relay channel—the transmission from the source to the destination with the help of a relay—is only known for special cases. For example, the *decode-and-forward* (DF) scheme achieves the capacity if the relay is close to the source [1]. In general, the capacity can only be bounded from above and below, e.g., with the *cut-set bound* (CSB) and the DF scheme, respectively. These bounds are important for the physical layer design with multiple antennas [2]. We use the bounds for a probabilistic physical layer design with only channel *distribution* information at the transmitting nodes, while the receiving nodes are aware of the channels' states that are required for reliable decoding.

While the ergodic capacity, i.e., the maximum average achievable rate, is the most important metric for fast fading channels, in case of slow fading channels, which is addressed in this work, the appropriate figure-of-merit is the outage probability [1]. An outage occurs when the channel is so poor that no error-free communication from the source to the destination is possible at the desired rate.

In [1], the minimal outage probability for a given data rate was described, for phase fading only, via an upper bound based on DF and a lower bound obtained via the CSB. These outage probability bounds were also extended to the half-duplex case by Høst-Madsen and Zhang [3], who used Monte-Carlo simulations for their numerical evaluations. Closed-form solutions for the outage probability of half-duplex DF relaying with Rayleigh fading channels between the single-antenna source, relay(s), and destination were presented in [4], [5]. Other works considered the low- or high-SNR regime to derive bounds on the outage probabilities of relay systems (e.g., see [6]–[10]) and to simplify the analysis.

Instead of minimizing the outage probability for a fixed rate, we are interested in the inverse problem, i.e., maximizing the achievable rate at a certain outage probability. Note that the solution, i.e., the outage capacity, is unknown, even for Rayleigh fading channels. In this work, we present outage capacity bounds based on the probabilistically constrained DF achievable rate and the CSB. These bounds are analyzed via numerical simulations for a line network.

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II. Three-Node Relay Model

We consider a standard single-antenna relay scenario, where the source conveys information to the destination with the help of a full-duplex operating relay. The capacity of this system is unknown in general, but can be bounded. Two important capacity bounds for perfect CSI with similar structure are the the achievable DF rate and the CSB. The achievable DF rate with, respectively, source and relay channel inputs X_{S} and X_{R} and relay and destination outputs Y_{R} and Y_{D} reads as [1], [3]

$$R_{\mathrm{DF}} = \max_{p(x_{\mathrm{S}},x_{\mathrm{R}})} \min\left\{ I(X_{\mathrm{S}};Y_{\mathrm{R}}|X_{\mathrm{R}}), I(X_{\mathrm{S}},X_{\mathrm{R}};Y_{\mathrm{D}})\right\}. \quad (1)$$

Note that the DF strategy requires that the relay to completely decodes the received message before sending the re-encoded data jointly (coherently) with the source to the destination. The general form of the CSB can be expressed as [1]

$$C_{\mathrm{CSB}} = \max_{p(x_{\mathrm{S}},x_{\mathrm{R}})} \min\left\{ I(X_{\mathrm{S}};Y_{\mathrm{R}},Y_{\mathrm{D}}|X_{\mathrm{R}}), I(X_{\mathrm{S}},X_{\mathrm{R}};Y_{\mathrm{D}})\right\} \quad (2)$$

and only differs in the first mutual information argument from R_{DF}, where C_{CSB} additionally takes into account Y_{D}.

In the corresponding Gaussian relay model, the received signals at the relay and the destination can be written as

$$y_{\mathrm{R}} = h_{\mathrm{SR}}x_{\mathrm{S}} + \eta_{\mathrm{R}},$$
$$y_{\mathrm{D}} = h_{\mathrm{SD}}x_{\mathrm{S}} + h_{\mathrm{RD}}x_{\mathrm{R}} + \eta_{\mathrm{D}},$$

where $\eta_{\mathrm{R}} \sim \mathcal{N}_{\mathbb{C}}(0,1)$ and $\eta_{\mathrm{D}} \sim \mathcal{N}_{\mathbb{C}}(0,1)$ are the additive white Gaussian noise components and h_{SR}, h_{SD}, and h_{RD} are the complex channels between source, relay, and destination. Note that a joint zero-mean circularly symmetric complex Gaussian signaling with full transmit power at the source and relay maximizes the DF rate and the CSB, i.e.,

$x_S \sim \mathcal{N}_{\mathbb{C}}(0, P_S)$, $x_R \sim \mathcal{N}_{\mathbb{C}}(0, P_R)$, and $\mathrm{E}[x_S x_R{}^*] = \beta\sqrt{P_S P_R}$. This result is independent of the realization of the channels and the actual choice of $\beta \in \mathbb{C}$ with $|\beta|^2 \le 1$. Therefore, (1) and (2) can be rewritten as

$$R_{\mathrm{DF}}(\boldsymbol{h}) = \max_{|\beta|^2 \le 1} \min\{R_{\mathrm{DF}}^{(1)}(\beta, \boldsymbol{h}), R_{\mathrm{DF}}^{(2)}(\beta, \boldsymbol{h})\} \tag{3}$$

$$C_{\mathrm{CSB}}(\boldsymbol{h}) = \max_{|\beta|^2 \le 1} \min\{C_{\mathrm{CSB}}^{(1)}(\beta, \boldsymbol{h}), C_{\mathrm{CSB}}^{(2)}(\beta, \boldsymbol{h})\} \tag{4}$$

where $\boldsymbol{h} = [h_{\mathrm{SR}}, h_{\mathrm{RD}}, h_{\mathrm{SD}}]^{\mathrm{T}}$ comprises the channel coefficients. The first mutual information expressions inside the minimum operator of (3) and (4), respectively, read as

$$R_{\mathrm{DF}}^{(1)}(\beta, \boldsymbol{h}) = \log_2\left(1 + (1 - |\beta|^2)|h_{\mathrm{SR}}|^2 P_S\right), \tag{5}$$

$$C_{\mathrm{CSB}}^{(1)}(\beta, \boldsymbol{h}) = \log_2\left(1 + (1 - |\beta|^2)(|h_{\mathrm{SR}}|^2 + |h_{\mathrm{SD}}|^2) P_S\right). \tag{6}$$

The second mutual information expressions inside the minimum operator of (3) and (4) are equal [cf. (1) and (2)] and depicted at the top of the next page [see (7)].

III. Capacity Bounds For Perfect CSI

For perfect *channel state information* (CSI), the remaining optimization in (3) and (4) over $\beta = |\beta|\,\mathrm{e}^{\mathrm{j}\phi_\beta}$ takes into account the realizations of $\boldsymbol{h}$. The maximum is obtained by [1]

$$\mathrm{e}^{\mathrm{j}\phi_\beta} = \frac{h_{\mathrm{SD}}^* h_{\mathrm{RD}}}{|h_{\mathrm{SD}}||h_{\mathrm{RD}}|}, \tag{9}$$

with the resulting second DF and CSB expression in (8). The positive scalar $|\beta|$ equalizes the rates inside $\min\{\cdot,\cdot\}$ (if possible), i.e., $R_{\mathrm{DF}}^{(1)} = R_{\mathrm{DF}}^{(2)}$ and $C_{\mathrm{CSB}}^{(1)} = C_{\mathrm{CSB}}^{(2)}$ if

$$|h_{\mathrm{SR}}|^2 P_S \ge |h_{\mathrm{SD}}|^2 P_S + |h_{\mathrm{RD}}|^2 P_R, \tag{10}$$

$$|h_{\mathrm{SR}}|^2 P_S \ge |h_{\mathrm{RD}}|^2 P_R \tag{11}$$

for the DF rate (3) and the CSB (4), respectively. Otherwise noncoherent transmission is optimal, i.e., $|\beta| = 0$.

IV. Outage Constrained Capacity Bounds

Now, consider that the channels are slowly Rayleigh-fading

$$h_{(\cdot)} \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_{(\cdot)}^2) \tag{12}$$

where $(\cdot)$ stands for the indices SR, RD, and SD of the corresponding channels. Furthermore, let only the statistics of the channels in (12) be available at the transmitting nodes, that is, the source has statistical information of h_{SR} and h_{SD}, the relay has statistical information of h_{RD}, but perfect CSI for h_{SR}, and the destination has perfect CSI for h_{SD} and h_{RD}. Then, the transmission from the source to the destination with the help of a relay might fail if the channels are so poor that the desired data rate ρ is not reached—an outage might occur.

For a reliable transmission and in accordance to above assumption of slow fading, the probability of an outage is limited to lie below $\epsilon \in (0, 1)$. Under this condition, we aim at a maximization of the rate ρ. In other words, we aim at finding the outage constrained capacity $C^{(\mathrm{out})}$ which is unfortunately unknown. Instead, a lower and an upper bound for the outage constrained capacity are found. For example, we obtain these bounds via the maximization of the DF rate and CSB, i.e.,

$$R_{\mathrm{DF}}^{(\mathrm{out})} = \max_{\rho, |\beta|^2 \le 1} \left\{\rho \in \mathbb{R} : p_{\mathrm{DF}}(\rho, \beta) \ge 1 - \epsilon\right\}, \tag{13}$$

$$C_{\mathrm{CSB}}^{(\mathrm{out})} = \max_{\rho, |\beta|^2 \le 1} \left\{\rho \in \mathbb{R} : p_{\mathrm{CSB}}(\rho, \beta) \ge 1 - \epsilon\right\}. \tag{14}$$

The probabilities in (13) and (14) are functions of the joint transmit parameter β and the imposed data rate ρ and read as

$$p_{\mathrm{DF}}(\rho, \beta) = \Pr\Big[\min_{i=1,2}\{R_{\mathrm{DF}}^{(i)}(\beta, \boldsymbol{h})\} \ge \rho\Big], \tag{15}$$

$$p_{\mathrm{CSB}}(\rho, \beta) = \Pr\Big[\min_{i=1,2}\{C_{\mathrm{CSB}}^{(i)}(\beta, \boldsymbol{h})\} \ge \rho\Big], \tag{16}$$

respectively, where the randomness in the stochastic constraint is due to the random channels.

A solution of either of the remaining optimization for the DF rate and the CSB is a two dimensional search over $\beta \in \mathbb{C}$ and $\rho \in \mathbb{R}$. Next, we refine this search and provide a generic solution approach to analyze the outage constrained capacity via its upper and lower bounds in the numerical results, i.e.,

$$R_{\mathrm{DF}}^{(\mathrm{out})} \le C^{(\mathrm{out})} \le C_{\mathrm{CSB}}^{(\mathrm{out})}.$$

V. Probabilistic Optimization for Bounds

We can establish the general procedure via two properties that hold also for the given formulations in (13) and (14).

Lemma 1. The probability $p_{(\cdot)}(\rho, \beta)$, with $p_{(\cdot)}(0, \beta) = 1$ and $\lim_{\rho\to\infty} p_{(\cdot)}(\rho, \beta) = 0$, is decreasing in ρ for fixed β. It is continuous if the channels' *probability density functions* (PDFs) are continuous as well.

Lemma 2. The joint transmit parameter β can without loss of optimality be restricted to be real and nonnegative if any of the channels to the destination, h_{SD} and h_{RD}, has a uniformly distributed phase in $[0, 2\pi)$.

The proofs are provided in Appendices A and B. The case with an independent zero-mean Gaussian distributed $\boldsymbol{h}$ [cf. (12)] is only one example for a fading environment with channels that have a continuous PDF and uniformly distributed phase-fading. We remark that Lemma 2 would not hold if both channels h_{SD} and h_{RD} were nonzero-mean Gaussian.

From Lemma 1, it follows that the optimizer $\rho^\star$ of either of the optimizations in (13) and (14) satisfies the chance-constraint—the outage requirement—with equality, i.e.,

$$p_{(\cdot)}(\rho^\star, \beta) = 1 - \epsilon. \tag{17}$$

Moreover, we can find the optimizer $\beta^\star$ for a given rate target ρ as (cf. Lemma 2)

$$\beta^\star = \operatorname*{argmax}_{0 \le \beta \le 1} p_{(\cdot)}(\rho, \beta). \tag{18}$$

This motivates a two-fold general optimization approach, where in an outer loop the optimal rate $\rho^\star$ is searched via a bisection. The optimal $\beta^\star$ according to (18) is found in each bisection step via a line search, e.g., the golden section method [11], for given $\rho^{(i)}$. Depending on whether the resulting probability $p_{(\cdot)}(\rho^{(i)}, \beta^\star)$ is smaller or larger than $1 - \varepsilon$,

$$C^{(2)}_{\text{CSB}}(\beta,\boldsymbol{h}) = R^{(2)}_{\text{DF}}(\beta,\boldsymbol{h}) = \log_2\left(1+|h_{\text{SD}}|^2P_{\text{S}}+|h_{\text{RD}}|^2P_{\text{R}}+2\,\text{Re}\{\beta h_{\text{SD}}h^*_{\text{RD}}\}\sqrt{P_{\text{S}}P_{\text{R}}}\right) \tag{7}$$

$$C^{(2)}_{\text{CSB}}\left(|\beta|\tfrac{h^*_{\text{SD}}h_{\text{RD}}}{|h_{\text{SD}}||h_{\text{RD}}|},\boldsymbol{h}\right) = R^{(2)}_{\text{DF}}\left(|\beta|\tfrac{h^*_{\text{SD}}h_{\text{RD}}}{|h_{\text{SD}}||h_{\text{RD}}|},\boldsymbol{h}\right) = \log_2\left(1+|h_{\text{SD}}|^2P_{\text{S}}+|h_{\text{RD}}|^2P_{\text{R}}+2|\beta||h_{\text{SD}}||h_{\text{RD}}|\sqrt{P_{\text{S}}P_{\text{R}}}\right) \tag{8}$$

$\rho^{(i)}$, respectively, $\rho^{(i)}$ serves as a new lower or upper bound for the uncertainty interval of the maximum rate $\rho^\star \in [\underline{\rho},\bar{\rho}]$. The initial lower bound is $\underline{\rho}=0$ and we may use (37) for the initial upper bound $\bar{\rho}$, which we obtain via approximation of the CSB probability in (16) with Markov's inequality.

Alternatively, the outer bisection and inner optimization may be reversed. Any of the two versions for the probabilistically constrained DF rate and CSB maximization requires a large number of probability computations since the inner problem has only linear convergence. For an efficient computation, we require either closed-form formulations for the probabilities or tight probabilistic constraint approximations that are used if no closed-form probability expressions exist. We also present computationally more efficient upper (optimistic) and lower (conservative) bounds for the DF and CSB version, respectively, for noncoherent transmission, i.e., $\beta=0$.

A. Decode-and-Forward Success Probability

Note that the first and the second argument within the $\min\{\cdot,\cdot\}$ operation in (15) are mutually independent.[1] Therefore, this probability can be rewritten as the product

$$p_{\text{DF}}(\rho,\beta) = p^{(1)}_{\text{DF}}(\rho,\beta)p^{(2)}_{\text{DF}}(\rho,\beta), \tag{19}$$

where the two multiplied probabilities are defined as

$$p^{(i)}_{\text{DF}}(\rho,\beta) = \Pr\left[R^{(i)}_{\text{DF}}(\beta,\boldsymbol{h}) \geq \rho\right]. \tag{20}$$

The stochastic constraint within the probability of $p^{(1)}_{\text{DF}}(\rho,\beta)$ is separable, i.e., the optimization variables ρ and β can be separated from the stochastic channel by the inequality sign. Moreover, since $2|h_{\text{SR}}|^2/\sigma^2_{\text{SR}}$ is standard χ^2-distributed with two degrees of freedom, the first probability is (cf. [12])

$$p^{(1)}_{\text{DF}}(\rho,\beta) = \Pr\left[\frac{|h_{\text{SR}}|^2}{\sigma^2_{\text{SR}}} \geq \frac{\gamma(\rho)}{(1-\beta^2)P_{\text{SR}}}\right] = \mathrm{e}^{\frac{-\gamma(\rho)}{(1-\beta^2)P_{\text{SR}}}} \tag{21}$$

where we substituted $P_{\text{SR}} = P_{\text{S}}\sigma^2_{\text{SR}}$ and $\gamma(\rho) = 2^\rho - 1$.

In contrast, the optimization variables in $p^{(2)}_{\text{DF}}(\rho,\beta)$ are non-separable from the random channels with respect to the inequality sign. However, we can rewrite $p^{(2)}_{\text{DF}}(\rho,\beta)$ as

$$p^{(2)}_{\text{DF}}(\rho,\beta) = \Pr\left[|w_1|^2\lambda_{\text{DF},1}+|w_2|^2\lambda_{\text{DF},2} \geq \gamma(\rho)\right] \tag{22}$$

where $w_i \sim \mathcal{N}(0,1)$ and $\lambda_{\text{DF},i}$, $i=1,2$ are the eigenvalues of the positive semidefinite matrix[2]

$$\boldsymbol{C} = \begin{bmatrix} P_{\text{RD}} & \beta\sqrt{P_{\text{RD}}P_{\text{SD}}} \\ \beta\sqrt{P_{\text{SD}}P_{\text{RD}}} & P_{\text{SD}} \end{bmatrix}, \tag{23}$$

with the substitutes $P_{\text{RD}} = P_{\text{R}}\sigma^2_{\text{RD}}$ and $P_{\text{SD}} = P_{\text{S}}\sigma^2_{\text{SD}}$. If we had $\beta=0$ for example, $\lambda_{\text{DF},1}$ and $\lambda_{\text{DF},2}$ would be the maximum and the minimum, respectively, of the two effective powers P_{RD} and P_{SD} for the transmission to the destination.

Note that $\lambda_{\text{DF},1}$ and $\lambda_{\text{DF},2}$, respectively, are increasing and decreasing in β and $\lambda_{\text{DF},1}+\lambda_{\text{DF},2} = P_{\text{RD}}+P_{\text{SD}}$ is independent of β.[3] Furthermore, $|w_1|^2\lambda_{\text{DF},1}+|w_2|^2\lambda_{\text{DF},2}$ is the (weighted) sum of two independent standard χ^2-distributed random variables with degree 2 if $\lambda_{\text{DF},1} > \lambda_{\text{DF},2}$ and χ^2-distributed with degree 4 if $\lambda_{\text{DF},1} = \lambda_{\text{DF},2}$. With the corresponding PDFs that are provided in [13], or with (41) and (44) from Appendix C, the probability in (22) may be formulated as

$$p^{(2)}_{\text{DF}}(\rho,\beta) = \begin{cases} \frac{f(\rho,\lambda_{\text{DF},1})-f(\rho,\lambda_{\text{DF},2})}{\lambda_{\text{DF},1}-\lambda_{\text{DF},2}} & \text{if } \lambda_{\text{DF},1} > \lambda_{\text{DF},2}, \\ \left(1+\frac{\gamma(\rho)}{\lambda_{\text{DF},1}}\right)\frac{f(\rho,\lambda_{\text{DF},1})}{\lambda_{\text{DF},1}} & \text{if } \lambda_{\text{DF},1} = \lambda_{\text{DF},2}, \end{cases} \tag{24}$$

where we substituted the function $f(\rho,\lambda_{\text{DF},i}) = \lambda_{\text{DF},i}\,\mathrm{e}^{-\frac{\gamma(\rho)}{\lambda_{\text{DF},i}}}$.

With the closed-form probability expressions in (21) and (24), we may calculate the DF success probability in (19) for many values β (and ρ) to find $R^{(\text{out})}_{\text{DF}}$. However, a closer analysis of the formulations in (21) and (24) leads us to the observation that noncoherent transmission is optimal for reasonable scenarios when employing the DF scheme and the channels are Rayleigh fading as in (12).

Observation 1. Noncoherent transmission, i.e., $\beta=0$, maximizes the probabilistically constrained DF achievable rate $R^{(\text{out})}_{\text{DF}}$ in (13) if the outage probability ϵ is sufficiently small.

To confirm Observation 1, we remark that [cf. (17) and (19)]

$$p^{(1)}_{\text{DF}}(\rho^\star,\beta)p^{(2)}_{\text{DF}}(\rho^\star,\beta) = 1-\epsilon \tag{25}$$

holds at the optimum of (13). Therefore, $\beta=0$ is optimal if it maximizes both probabilities, $p^{(1)}_{\text{DF}}(\rho^\star,\beta)$ and $p^{(2)}_{\text{DF}}(\rho^\star,\beta)$.

Since $p^{(1)}_{\text{DF}}(\rho^\star,\beta)$ in (21) is decreasing in β, $\beta=0$ maximizes this probability. To show a similar behavior for the latter probability, we assume the outage threshold ϵ_2 for

$$p^{(2)}_{\text{DF}}(\rho^\star,\beta) = 1-\epsilon_2. \tag{26}$$

Note that $\epsilon_2 \leq \epsilon$ due to (25). Inserting (22), we can equivalently rewrite (26) into the following equality condition for a generic *cumulative distribution function* (CDF):

$$\mathrm{F}(\gamma^\star) = \Pr\left[|w_1|^2\lambda+|w_2|^2(1-\lambda) \leq \gamma^\star\right] = \epsilon_2 \tag{27}$$

where we substituted $\gamma^\star = \frac{\gamma(\rho^\star)}{P_{\text{RD}}+P_{\text{SD}}}$ and

$$\lambda = \frac{\lambda_{\text{DF},1}}{P_{\text{RD}}+P_{\text{SD}}} = 1-\frac{\lambda_{\text{DF},2}}{P_{\text{RD}}+P_{\text{SD}}}. \tag{28}$$

[1]This is in contrast to the outage capacity upper bound based on CSB, where both arguments of the $\min\{\cdot,\cdot\}$ operation depend on h_{SD}.

[2]The matrix $\boldsymbol{C}$ can be seen as the joint source and relay transmit covariance matrix of an equivalent system with i.i.d. channels with variance 1.

[3]Here, we assume without loss of generality that $\lambda_{\text{DF},1} \geq \lambda_{\text{DF},2}$

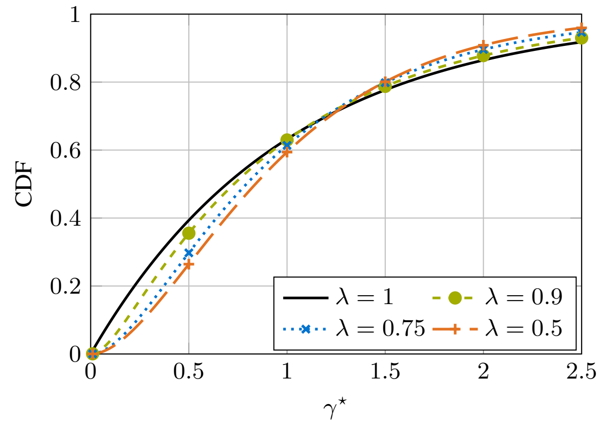

Figure 1. CDF $\mathrm{F}(\gamma^\star)$ of the weighted χ^2-distributed random variable $|w_1|^2\lambda + |w_2|^2(1-\lambda)$ with complex normal $w_i \sim \mathcal{N}_\mathbb{C}(0,1)$, $i=1,2$.

and where we note that λ is increasing in β as it is $\lambda_{\mathrm{DF},1}$. This generic CDF is that of a positively weighted sum of i.i.d. χ^2-distributed variables with degree two, where the weights sum up to one. The closed form of $\mathrm{F}(\gamma^\star)$ reads as [cf. (24)]

$$\mathrm{F}(\gamma^\star) = \begin{cases} 1 - \frac{\lambda\,\mathrm{e}^{-\frac{\gamma^\star}{\lambda}} - (1-\lambda)\,\mathrm{e}^{-\frac{\gamma^\star}{1-\lambda}}}{2\lambda-1} & \text{if } \lambda > 0.5, \\ 1 - \left(1 + \frac{\gamma^\star}{\lambda}\right)\mathrm{e}^{-\frac{\gamma^\star}{\lambda}} & \text{if } \lambda = 0.5 \end{cases} \tag{29}$$

and is plotted for various parameters $\lambda \in [0.5, 1]$ in Fig. 1. For the figure, we varied $\gamma^\star$ in steps of 0.01 and λ in steps of 0.1. Note that the restriction to $\lambda \in [0.5, 1]$ is without loss of generality since the CDF is symmetric to $\lambda = 0.5$, that is, we obtain the same CDF curves for $\lambda = 0.5 + x$ and $\lambda = 0.5 - x$.

From Fig. 1, we see that the CDF curves are increasing in λ in the lower left and the CDF curves are decreasing in λ in the upper right area of the figure. Therefore, $p^{(2)}_{\mathrm{DF}}(\rho^\star, \beta) = 1 - \mathrm{F}(\gamma^\star)$ is decreasing in β for a sufficiently small $\gamma^\star$, respectively, a sufficiently small $\rho^\star$. Hence, given a small $\rho^\star$, the probability $p^{(2)}_{\mathrm{DF}}(\rho^\star, \beta)$ is maximized for $\beta = 0$.

Similarly, we may have given a certain ϵ_2 and now search for the maximum $\rho^\star$ with respect to β, i.e.,

$$\rho^\star = \max\left\{\rho \in \mathbb{R}_+ : p^{(2)}_{\mathrm{DF}}(\rho, \beta) = 1 - \epsilon_2,\ 0 \le \beta \le 1\right\}. \tag{30}$$

This optimization formulation is equivalent to the problem where we search for the maximum achievable $\gamma^\star$ such that $\mathrm{F}(\gamma^\star) = \epsilon_2$. Graphically, we find this solution as the rightmost point of the ϵ_2-quantiles of the CDF curves in Fig 1, i.e., $\gamma^\star = \mathrm{F}^{-1}(\epsilon_2)$. In other words, we cut the plots in Fig. 1 parallel to the $\gamma^\star$-axis at ϵ_2 and search the rightmost $\gamma^\star$ along this cut. This rightmost point is defined by the minimum achievable λ, which corresponds to $\beta = 0$, if ϵ_2 is sufficiently small because the CDF curves are increasing in λ in this case. This confirms the observation as a small ϵ implies a small ϵ_2.

B. Cut-Set Bound Probability

The CSB probability has a different structure than the DF rate probability. The terms $C^{(1)}_{\mathrm{CSB}}(\beta, \boldsymbol{h})$ and $C^{(2)}_{\mathrm{CSB}}(\beta, \boldsymbol{h})$ are correlated due to their dependence on h_{SD} [see (6) and (7)] and the probability calculation requires a numerical integration when we generally allow for a coherent transmission of the relay and source to the destination. Therefore, we cannot extend the observation of the DF scheme to the CSB. However, a restriction to noncoherent transmission may fail to provide an upper bound for the outage constrained capacity. For these reasons, we present bounds for the outage constrained CSB optimization. First, we use a genie-aided approximation for the CSB and calculate the resulting probability. Alternatively, we upper bound the CSB probability with Markov's inequality.

Genie-Aided Upper Bound: For the first bound, we approximate $\mathrm{Re}\{\beta h_{\mathrm{SD}} h^*_{\mathrm{RD}}\} \le \beta |h_{\mathrm{SD}}||h_{\mathrm{RD}}|$. This approach is genie-aided in the sense that the same approximation is obtained if $\mathrm{e}^{\mathrm{j}\phi_\beta}$ were designed according to perfect CSI [see (9)]. Besides maximizing $C^{(2)}_{\mathrm{CSB}}(\beta, \boldsymbol{h})$ which is shown in (8), this approximation allows an outage probability computation with a single numerical integration. To this end, we rewrite the CSB probability in (16) as the sum of three probabilities

$$p_{\mathrm{CSB}}\left(\rho, \beta \frac{h^*_{\mathrm{SD}} h_{\mathrm{RD}}}{|h_{\mathrm{SD}}||h_{\mathrm{RD}}|}\right) = p^{(1)}_{\mathrm{CSB}} + p^{(2)}_{\mathrm{CSB}} + p^{(3)}_{\mathrm{CSB}}. \tag{31}$$

The probabilities $p^{(i)}_{\mathrm{CSB}}$, $i = 1,2,3$ correspond to three disjoint ranges for $z_{\mathrm{SD}} = \frac{|h_{\mathrm{SD}}|^2}{\sigma^2_{\mathrm{SD}}} \in [0, \infty)$, that are identified for satisfying $C_{\mathrm{CSB}}(\beta, \boldsymbol{h}) \ge \rho$. Namely,

$$z_{\mathrm{SD}} \ge \frac{\gamma(\rho)}{(1-\beta^2)P_{\mathrm{SD}}} \tag{32a}$$

$$\frac{\gamma(\rho)}{P_{\mathrm{SD}}} \le z_{\mathrm{SD}} \le \frac{\gamma(\rho)}{(1-\beta^2)P_{\mathrm{SD}}} \tag{32b}$$

$$z_{\mathrm{SD}} \le \frac{\gamma(\rho)}{P_{\mathrm{SD}}}. \tag{32c}$$

With (32a), $C^{(1)}_{\mathrm{CSB}}(\beta, \boldsymbol{h}) \ge \rho$ and $C^{(2)}_{\mathrm{CSB}}(\beta, \boldsymbol{h}) \ge \rho$ hold independent of the realizations of $z_{\mathrm{SR}} = \frac{|h_{\mathrm{SR}}|^2}{\sigma^2_{\mathrm{SR}}} \in [0, \infty)$ and $z_{\mathrm{RD}} = \frac{|h_{\mathrm{RD}}|^2}{\sigma^2_{\mathrm{RD}}} \in [0, \infty)$. In case of (32b),

$$P_{\mathrm{SR}} z_{\mathrm{SR}} \ge \gamma(\rho) - P_{\mathrm{SD}} z_{\mathrm{SD}}$$

is additionally required to fulfill $C^{(1)}_{\mathrm{CSB}}(\beta, \boldsymbol{h}) \ge \rho$, and for event (32c), we need moreover

$$P_{\mathrm{RD}} z_{\mathrm{RD}} \ge \left(\sqrt{\gamma(\rho) - (1 - \tfrac{\beta^2}{4})P_{\mathrm{SD}} z_{\mathrm{SD}}} - \sqrt{\tfrac{\beta^2}{4} P_{\mathrm{SD}} z_{\mathrm{SD}}}\right)^2$$

to satisfy $C^{(2)}_{\mathrm{CSB}}(\beta, \boldsymbol{h}) \ge \rho$. The first two probabilities read as

$$p^{(1)}_{\mathrm{CSB}} = \mathrm{e}^{-\frac{\gamma(\rho)}{(1-\beta^2)P_{\mathrm{SD}}}}, \tag{33a}$$

$$p^{(2)}_{\mathrm{CSB}} = \frac{P^{-1}_{\mathrm{SD}}}{P^{-1}_{\mathrm{SD}} - P^{-1}_{\mathrm{SR}}}\left(\mathrm{e}^{-\frac{\gamma(\rho)}{P_{\mathrm{SD}}} - \frac{\gamma(\rho)\beta^2}{(1-\beta^2)P_{\mathrm{SR}}}} - \mathrm{e}^{-\frac{\gamma(\rho)}{(1-\beta^2)P_{\mathrm{SD}}}}\right), \tag{33b}$$

and the third probability expression is given at the top of the next page. The computation of $p^{(3)}_{\mathrm{CSB}}$ requires a numerical integration to the best of our knowledge.

$$p_{\mathrm{CSB}}^{(3)} = \int_0^{\frac{\gamma(\rho)}{P_{\mathrm{SD}}}} \mathrm{e}^{-z_{\mathrm{SD}}}\, \mathrm{e}^{-\frac{\gamma(\rho)-(1-\beta^2)P_{\mathrm{SD}}z_{\mathrm{SD}}}{(1-\beta^2)P_{\mathrm{SR}}}}\, \mathrm{e}^{-\frac{1}{P_{\mathrm{RD}}}\left(\sqrt{\gamma(\rho)-(1-\frac{\beta^2}{4})P_{\mathrm{SD}}z_{\mathrm{SD}}}-\sqrt{\frac{\beta^2}{4}P_{\mathrm{SD}}z_{\mathrm{SD}}}\right)^2}\, \mathrm{d}z_{\mathrm{SD}} \tag{28c}$$

Markov Based Upper Bound: Another upper bound approximation is obtained when non-conservatively approximating the probabilistic constraint of (14) with Markov's inequality,

$$1-\epsilon \leq p_{\mathrm{CSB}}(\rho,\beta) \leq \frac{1}{\rho}\min_{i=1,2}\left\{\mathrm{E}\left[C_{\mathrm{CSB}}^{(i)}(\beta,\boldsymbol{h})\right]\right\} \tag{34}$$

where the second inequality additionally exploits $\mathrm{E}[\min\{x,y\}] \leq \min\{\mathrm{E}[x],\mathrm{E}[y]\}$. Therewith, the probabilistic measure is transformed to an average rate, where $\beta = 0$ is optimal due to the uniformly distributed phase of $h_{\mathrm{SD}}h_{\mathrm{RD}}^*$ [1]. The resulting average rate expressions are given by

$$\mathrm{E}[C_{\mathrm{CSB}}^{(1)}(0,\boldsymbol{h})] = \frac{\log_2(\mathrm{e})}{P_{\mathrm{SR}}P_{\mathrm{SD}}}\frac{g(P_{\mathrm{SR}})-g(P_{\mathrm{SD}})}{P_{\mathrm{SD}}^{-1}-P_{\mathrm{SR}}^{-1}} \tag{35}$$

$$\mathrm{E}[C_{\mathrm{CSB}}^{(2)}(0,\boldsymbol{h})] = \frac{\log_2(\mathrm{e})}{P_{\mathrm{RD}}P_{\mathrm{SD}}}\frac{g(P_{\mathrm{RD}})-g(P_{\mathrm{SD}})}{P_{\mathrm{SD}}^{-1}-P_{\mathrm{RD}}^{-1}} \tag{36}$$

where $g(x) = x\,\mathrm{e}^{\frac{1}{x}}\,\mathrm{E}_1\left(\frac{1}{x}\right)$ and $\mathrm{E}_1(\cdot)$ is the exponential integral function [14]. This result also leads to a closed-form solution for the approximated probabilistic CSB problem, i.e.,

$$\rho = \frac{\min_{i=1,2}\{\mathrm{E}[C_{\mathrm{CSB}}^{(i)}(0,\boldsymbol{h})]\}}{1-\epsilon}. \tag{37}$$

C. Noncoherent Cut-Set Bound Probability

While we were able to observe that noncoherent transmission—no cooperation between the source and the relay—is optimal for maximizing the DF based lower bound on the outage constrained relay capacity, a restriction to noncoherent transmission for the CSB may fail to provide a true upper bound for the outage constrained capacity. Nevertheless, we provide the noncoherent CSB for bounding the theoretically achievable outage constrained rate with only separately transmitting relay and source. This is anyway preferred for practical implemented wireless systems because it avoids the synchronization overhead of the relay and source, which would have been required to set up the joint transmission.

Moreover, there are still arguments which indicate that $\beta = 0$ might be optimizer for the outage based CSB in some scenarios and otherwise provide a good approximation:

1) Besides the outage constrained achievable DF rate, $\beta = 0$ also maximizes (6).
2) It also maximizes the ergodic DF rate [1] and the ergodic CSB (cf. Section V-B).
3) The probability that $\beta > 0$ is harmful or helpful, (7) is reduced or increased, for a realization of the uniformly phase fading channels is exactly $1/2$.
4) From the perfect CSI case, it is known that $\beta > 0$ only improves the the CSB for cases where (11) is satisfied.
5) Finally, also if $\beta \neq 0$ were optimal for some outage requirements, its value would be small due to the previous reasons and due to the continuity of the CSB expressions (6) and (7) in β.

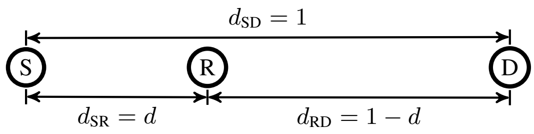

Figure 2. Line network model for the Gaussian relay channel with channel variances according to the path loss.

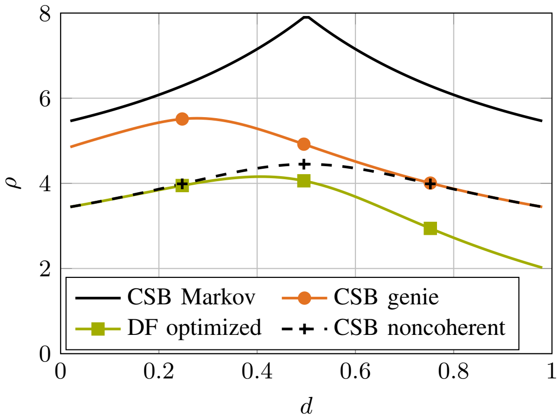

Figure 3. Outage constrained capacity bounds for a line network with zero-mean Gaussian channels, with variances $\sigma_{\mathrm{SD}}^2 = 1$, $\sigma_{\mathrm{SR}}^2 = d^{-\alpha}$, and $\sigma_{\mathrm{RD}}^2 = (1-d)^{-\alpha}$, $\alpha = 3$, and powers $P_{\mathrm{S}} = P_{\mathrm{R}} = 10\,\mathrm{dB}$.

The CSB probability can be calculated in closed form for $\beta = 0$ and $P_{\mathrm{SD}}^{-1} \neq P_{\mathrm{SR}}^{-1} + P_{\mathrm{RD}}^{-1}$. It reads as

$$p_{\mathrm{CSB}}(\rho,0) = \frac{\left(\frac{1}{P_{\mathrm{SR}}}+\frac{1}{P_{\mathrm{RD}}}\right)\mathrm{e}^{-\frac{\gamma(\rho)}{P_{\mathrm{SD}}}} - \frac{1}{P_{\mathrm{SD}}}\mathrm{e}^{-\frac{\gamma(\rho)}{P_{\mathrm{SR}}}-\frac{\gamma(\rho)}{P_{\mathrm{RD}}}}}{\frac{1}{P_{\mathrm{SR}}}+\frac{1}{P_{\mathrm{RD}}}-\frac{1}{P_{\mathrm{SD}}}}.$$

Similar to the DF case, it only remains to find the ρ that satisfies $p_{\mathrm{CSB}}(\rho,0) = 1-\epsilon$. Even though $\beta = 0$ generally lower bounds the CSB, i.e.,

$$C_{\mathrm{CSB}}^{(\mathrm{out})} \geq \underline{C}_{\mathrm{CSB}}^{(\mathrm{out})} = \max_{\rho}\left\{\rho \in \mathbb{R} : p_{\mathrm{CSB}}(\rho,0) \geq 1-\epsilon\right\}, \tag{38}$$

this solution serves as an indication on the accuracy of the CSB upper bound approximations in Section V-B.

VI. Numerical Results

The simulation setup consists of the three-node line network in Fig. 2, where the relay's location is between the source and the destination. The source destination distance is normalized to one and the source relay distance is d. The channels are independent zero-mean complex Gaussian with variances according to the path loss, i.e., $h_{\mathrm{SD}} \sim \mathcal{N}(0,1)$, $h_{\mathrm{SR}} \sim \mathcal{N}(0,d^{-\alpha})$, and $h_{\mathrm{RD}} \sim \mathcal{N}(0,(1-d)^{-\alpha})$, with $\alpha = 3$.

In Fig. 3, we depict the discussed outage constrained capacity bounds for $\epsilon = 0.25$, and $P_{\mathrm{S}} = P_{\mathrm{R}} = 10\,\mathrm{dB}$. Markov's inequality gives only a loose upper bound for the CSB. The genie-aided approximation gives a tighter upper bound than the CSB with Markov's inequality. It meets the CSB bound for noncoherent transmission if $d \approx 1$, i.e., when the relay is close to the destination, because $\beta \approx 0$ in this case. However, when the relay is close to the source, $\beta > 0$ is optimal for the genie-aided bound, which results in a gap of about 1.5 bits to the achievable DF rate curve. The gap between the genie-aided CSB and the DF rate reduces to around 1 bit for $d \approx 0.5$ and increases again when d approaches 1.

We remark that noncoherent transmission, i.e., $\beta = 0$, maximized the DF achievable rate for all values of $d \in [0, 1]$ that were used for the plot. This is a consequence of the sufficiently small $\epsilon = 0.25$ (cf. Fig. 1). Therefore, the CSB noncoherent bound directly meets the DF bounds for small d. The gap between the DF curve and the noncoherent CSB curve, which increases with d, is due to the necessity of the relay in the DF scheme to perfectly decode the complete message before forwarding it to the destination.

VII. Conclusion

We studied the achievable DF lower bound and the CSB for the probabilistically constrained capacity of a Gaussian relay channel with Rayleigh fading. For the chance constrained DF lower bound, we were able to calculate a closed form for the DF outage probability. We observed from the study of this probability that noncoherent transmission maximizes the outage constrained DF rate if the outage requirement is small. For the CSB, the genie-aided approximation provided the best upper bound for the chance-constrained capacity.

Appendix

A. Proof of Lemma 1

The probability bounds for $\rho = 0$ and $\rho \to \infty$ are obvious as the rate expressions $R_{\text{DF}}^{(i)}(\beta, \boldsymbol{h})$ and $C_{\text{CSB}}^{(i)}(\beta, \boldsymbol{h})$ are strictly positive and logarithmically increasing with the channels' absolute values. Moreover, $R_{\text{DF}}^{(i)}(\beta, \boldsymbol{h})$ and $C_{\text{CSB}}^{(i)}(\beta, \boldsymbol{h})$ are continuous in $\boldsymbol{h}$. Therefore, also $p_{(\cdot)}(\rho, \beta)$ is continuous and monotonic in ρ if the channels' PDF are continuous.

B. Proof of Lemma 2

The value of the mutual information terms in (5) and (6) are independent of the phase ϕ_β of β. In contrast, the value of (7) depends on ϕ_β (cf. Section III), but its distribution does not if at least one of the channels h_{SD} and h_{RD} has a uniformly distributed phase in $[0, 2\pi)$. Assume the phase of h_{SD} is uniformly distributed in $[0, 2\pi)$, then

$$R_{\text{DF}}^{(2)}(\beta, \boldsymbol{h}) = R_{\text{DF}}^{(2)}\left(|\beta|, [h_{\text{SR}}, h_{\text{RD}}, \mathrm{e}^{\mathrm{j}\phi_\beta}\, h_{\text{SD}}]^{\mathrm{T}}\right) \simeq R_{\text{DF}}^{(2)}(|\beta|, \boldsymbol{h})$$

as $h_{\text{SD}} \simeq \mathrm{e}^{\mathrm{j}\phi_\beta}\, h_{\text{SD}}$, and, hence, $p_{(\cdot)}(\rho, \beta) = p_{(\cdot)}(\rho, |\beta|)$. The proof with a uniformly distributed phase for h_{RD} is equivalent.

C. Calculation of $p_{\text{DF}}^{(1)}(\rho, \beta)$ and $p_{\text{DF}}^{(2)}(\rho, \beta)$

The PDF of the χ^2-distributed $z = |w|^2$, $w \sim \mathcal{N}_{\mathbb{C}}(0, 1)$, is $\mathrm{f}_z(z) = \mathrm{e}^{-z}$ and the probability of z exceeding t reads as

$$\Pr[z \geq t] = \int_t^\infty \mathrm{e}^{-z}\, \mathrm{d}z = \mathrm{e}^{-t}\,. \tag{39}$$

The probability $p_{\text{DF}}^{(1)}(\rho, \beta)$ in (21) directly follows from (39).

If the weighted sum of two independent χ^2-distributed random variables $az_1 + bz_2$ of degree 2, with $z_i = |w_i|^2$ and $w_i \sim \mathcal{N}_{\mathbb{C}}(0, 1)$, $i = 1, 2$, and $a \neq b$, shall exceed t, we can write the probability for this event as

$$\begin{aligned}\Pr[az_1 + bz_2 \geq t] = &\Pr[az_1 \geq t] \\ &+ \Pr[az_1 < t \wedge bz_2 \geq t - az_1].\end{aligned} \tag{40}$$

The previous of the right-hand-side probabilities is [cf. (39)]

$$\Pr\left[z_1 \geq \frac{t}{a}\right] = \mathrm{e}^{-\frac{t}{a}} \tag{41}$$

and the latter can be calculated as

$$\begin{aligned}\Pr\left[z_1 < \frac{t}{a} \wedge z_2 \geq \frac{t - az_1}{b}\right] &= \int_0^{\frac{t}{a}} \mathrm{e}^{-z_1} \int_{\frac{t-az_1}{b}}^\infty \mathrm{e}^{-z_2}\, \mathrm{d}z_2\, \mathrm{d}z_1 \\ &= \frac{b}{a-b}\left(\mathrm{e}^{-\frac{t}{a}} - \mathrm{e}^{-\frac{t}{b}}\right).\end{aligned} \tag{42}$$

The sum of (41) and (42) can be rewritten as

$$\Pr[az_1 + bz_2 \geq t] = \frac{a\,\mathrm{e}^{-\frac{t}{a}} - b\,\mathrm{e}^{-\frac{t}{b}}}{a-b}. \tag{43}$$

In contrast, if $a = b$, above sum is a central χ^2-distributed variable with degree 4, and the probability of exceeding t is obtained with [14, Chapter 26] as

$$\Pr[a(z_1 + z_2) \geq t] = \left(\frac{t}{a} + 1\right) \mathrm{e}^{-\frac{t}{a}}\,. \tag{44}$$

References

[1] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative Strategies and Capacity Theorems for Relay Networks," *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3037–3063, Sep. 2005.

[2] L. Gerdes and W. Utschick, "Optimized Capacity Bounds for the MIMO Relay Channel," in *Proc. ICASSP*, May 2011, pp. 3336–3339.

[3] A. Høst-Madsen and J. Zhang, "Capacity Bounds and Power Allocation for Wireless Relay Channels," *IEEE Trans. Inf. Theory*, vol. 51, no. 6, pp. 2020–2040, Jun. 2005.

[4] Yi Zhao, R. Adve, and Teng Joon Lim, "Outage Probability at Arbitrary SNR with Cooperative Diversity," *IEEE Com. Lett.*, vol. 9, no. 8, pp. 700–702, Aug. 2005.

[5] N. C. Beaulieu and J. Hu, "A Closed-Form Expression for the Outage Probability of Decode-and-Forward Relaying in Dissimilar Rayleigh Fading Channels," *IEEE Com. Lett.*, vol. 10, no. 12, pp. 813–815, 2006.

[6] J. Gómez-Vilardebó and A. I. Perez-Neira, "Upper Bound on Outage Capacity of Orthogonal Relay Networks," in *Proc. SPAWC*, Jul. 2006.

[7] A. S. Avestimehr and D. N. C. Tse, "Outage Capacity of the Fading Relay Channel in the Low-SNR Regime," *IEEE Trans. Inf. Theory*, vol. 53, no. 4, pp. 1401–1415, Apr. 2007.

[8] J. N. Laneman and G. W. Wornell, "Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2415–2425, Oct. 2003.

[9] R. U. Nabar, H. Bölcskei, and F. W. Kneubühler, "Fading Relay Channels: Performance Limits and Space–Time Signal Design," *IEEE J. Sel. Areas Commun.*, vol. 22, no. 6, pp. 1099–1109, Aug. 2004.

[10] M. Yüksel and E. Erkip, "Multiple-Antenna Cooperative Wireless Systems: A Diversity-Multiplexing Tradeoff Perspective," *IEEE Trans. Inf. Theory*, vol. 53, no. 10, pp. 3371–3393, Oct. 2007.

[11] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Convex Optimization*, Wiley-Interscience, 3rd edition, May 2006.

[12] E. Telatar, "Capacity of Multi-Antenna Gaussian Channels," *European Trans. Telecomm.*, vol. 10, no. 6, pp. 585–595, Jan. 1999.

[13] E. Bjornson, D. Hammarwall, and B. Ottersten, "Exploiting Quantized Channel Norm Feedback Through Conditional Statistics in Arbitrarily Correlated MIMO Systems," *IEEE Trans. Signal Process.*, vol. 57, no. 10, pp. 4027–4041, Oct. 2009.

[14] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications Inc., 1st edition, 1964.