Finite-Length Scaling of Convolutional LDPC Codes

Markus Stinner, Pablo Olmos

markus.stinner@tum.de, olmos@tsc.uc3m.es

Motivation

- Capacity-reaching LDPC codes exist
- The optimal parameters are known for long block lengths
- Finite-length scaling laws are conjectured for regular codes

Question:

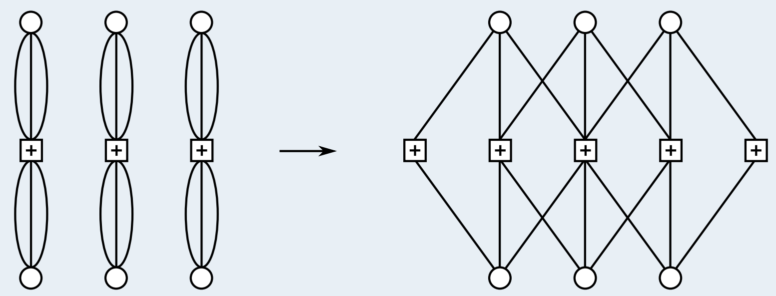
- Can we calculate the scaling laws for even more structured ensembles?
- Which design criteria hold for finite block lengths?

Spatially coupled $(l, r, L)_{\mathcal{P}}$ LDPC Codes [1]

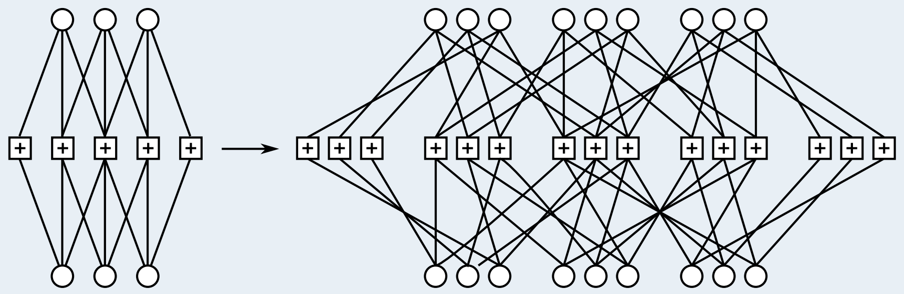
Definition: Low-density Parity-Check Code [2]

Low-density parity-check codes are codes specified by a matrix containing mostly 0's and only a small number of 1's.

Small Tanner graphs are used as a "blueprint" of the structure.



1. Choose a simple (l, r) protograph
2. Couple L protographs to a spatially coupled protograph



3. Lift the coupled protograph with the "copy-and-permute" operation (similar connections of several copies are randomly permuted to obtain larger girths)

The convolutional-like band matrix $\mathbf{H}$ consists of submatrices $\mathbf{H}_{i,j}$ which are permutation matrices for edge permutations:

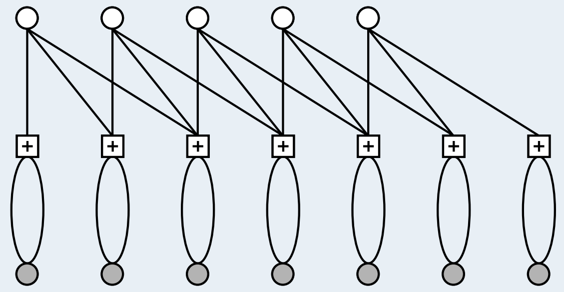
$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & & & & & \\ \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & & & \\ \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & \\ & & \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \\ & & & & \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \ddots \end{pmatrix}$$

Advantages

- Systematic encoding is possible
- **The MAP threshold can be reached with iterative belief propagation (BP) decoding [3]**

(q, a, L) SC-ARA Construction

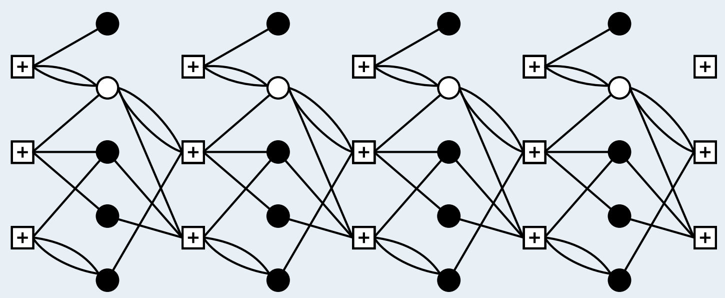
ARA graphs with message and accumulator nodes are coupled and terminated. The message node is connected to q check nodes.



- low error floor for the uncoupled graph
- linearly growing minimum Hamming distance

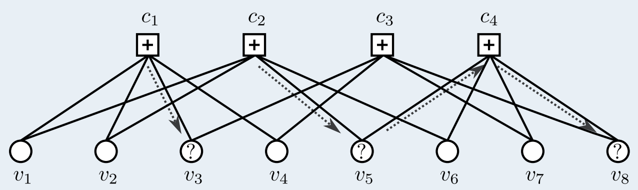
SC-TAR4JA Construction

TAR4JA structures are coupled by spreading some edges to the following block. This scheme includes punctured nodes.



- excellent decoding threshold

Peeling Decoding



If variable nodes are erased after the transmission over a binary erasure channels (BEC), they can be iteratively restored using the known part of the code graph. The decoding can only proceed as long as check nodes with only 1 unknown edge remain in the residual graph. This is used as stability criterion.

- τ: Decoding iterations normalized by M
- $\hat{c}_1(\tau)$: Sum of mean of deg-1 check nodes normalized by M

$$\hat{c}_1(\tau) \doteq \frac{1}{M} \sum_{i=1}^{m} \hat{R}(\mathbf{0}_{\sim i}, \tau)$$

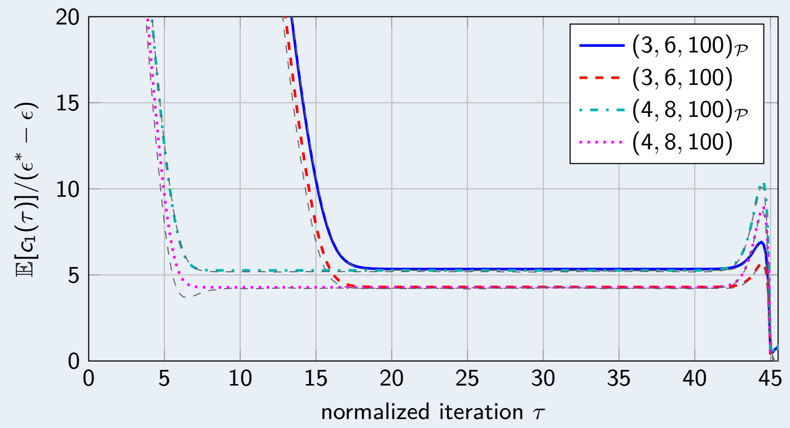
- $\delta_1(\tau)$: Variance of deg-1 check nodes of all processes

$$\mathrm{Var}[c_1(\tau)] = \frac{1}{M}\delta_1(\tau) = \frac{1}{M} \sum_{i=1}^{m} \sum_{b=1}^{m} \delta_{\mathbf{0}_{\sim i}, \mathbf{0}_{\sim b}}$$

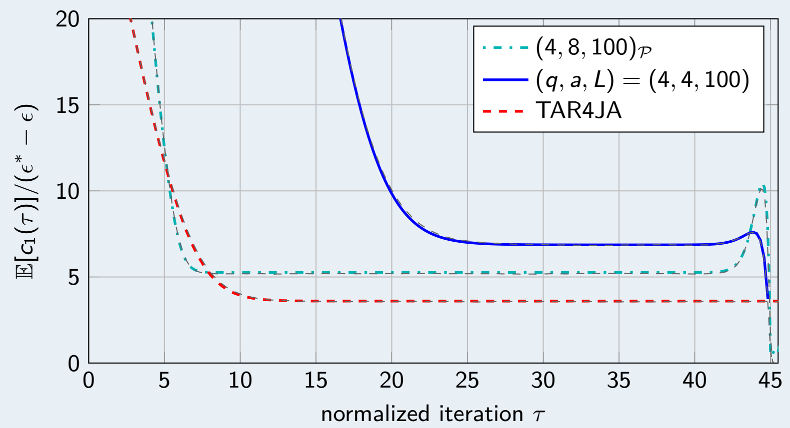
- $\phi_1(\tau, \zeta)$: process covariance with time

$$\phi_1(\tau, \zeta) \doteq \mathbb{E}[c_1(\tau)c_1(\zeta)] - \hat{c}_1(\tau)\hat{c}_1(\zeta)$$

Mean Evolution of Deg-1 Check Nodes



$\hat{c}_1(\tau)$ for the ensembles $(3,6,100)_{\mathcal{P}}$, $(3,6,100)$, $(4,8,100)_{\mathcal{P}}$ and $(4,8,100)$.



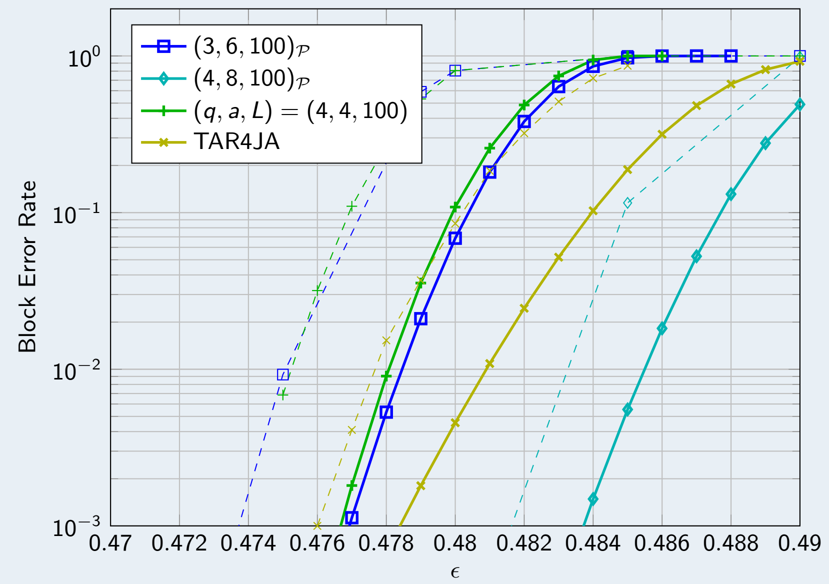
$\hat{c}_1(\tau)$ for the ensembles (q,a,L) SC-ARA, TAR4JA and $(4,8,100)_{\mathcal{P}}$.

Finite-length Scaling Conjecture [4]

Scaling law for LDPC codes using an iterative erasure decoder:

$$P^* \approx 1 - \exp\left(-\frac{(\epsilon L - \tau^*)}{\mu_0(M, \epsilon, l, r)}\right)$$

$(\epsilon L - \tau^*)$ is the duration of the steady-state phase. The average survival time μ_0 during the steady-state phase depends on $\hat{c}_1(\tau)$, $\delta_1(\tau)$.

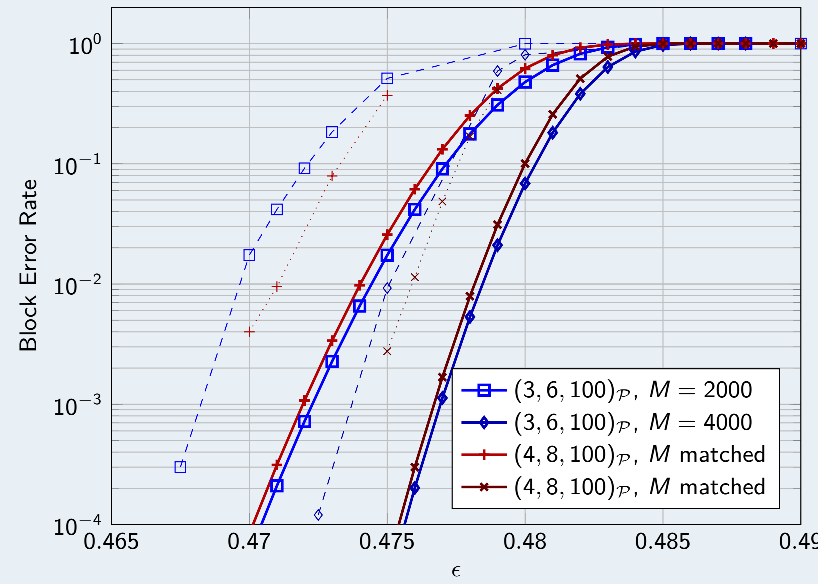


Finite length scaling predictions (solid lines) and simulated error rate (dashed lines) for different SC-LDPC codes with $L = 100$ and $M = 4000$.

- The $(4,8)_{\mathcal{P}}$ code outperforms the other structures.

Matching Codes

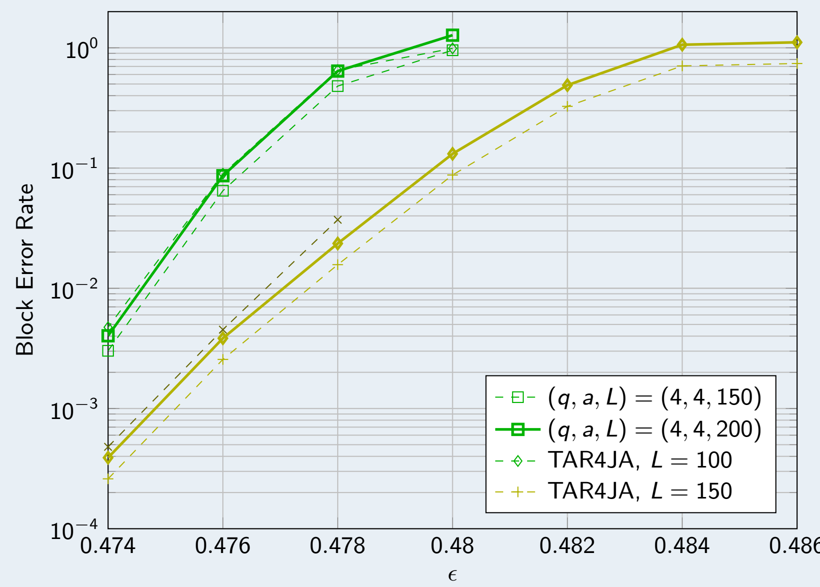
P^* is dominated from M and $\gamma/\sqrt{\delta_1^*}$ depending on the codes. We exploit this to match the performance of a code with another code ensemble.



$(4,8)_{\mathcal{P}}$ codes matched to $(3,6)_{\mathcal{P}}$ ensembles with $M = 2000$ and $M = 4000$.

Increasing Chain Lengths

For large L, P^* scales linearly with L which is exploited for the performance prediction.



Prediction for $L = 200$ by multiplying the $L = 150$ curve by 4/3.

References

[1] M. Lentmaier, G. P. Fettweis, K. S. Zigangirov, and D. J. Costello, "Approaching Capacity with Asymptotically Regular LDPC Codes," *Proc. Inf. Theory and Applicat. Workshop (ITA)*, pp. 173—177, 2009. [Online].

[2] R. Gallager, "Low-Density Parity-Check Codes," *IEEE Trans. Inf. Theory*, vol. 8, no. 1, pp. 21–28, Jan. 1962. [Online].

[3] S. Kudekar, T. J. Richardson, and R. L. Urbanke, "Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform so well over the BEC," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*. IEEE, 2011, pp. 684–688. [Online].

[4] M. Stinner and P. M. Olmos, "Analyzing Finite-length performance Protograph-based Spatially Coupled LDPC Codes," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2014.

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