Bounds on the Capacity of Wiener Phase Noise Channels

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Motivation

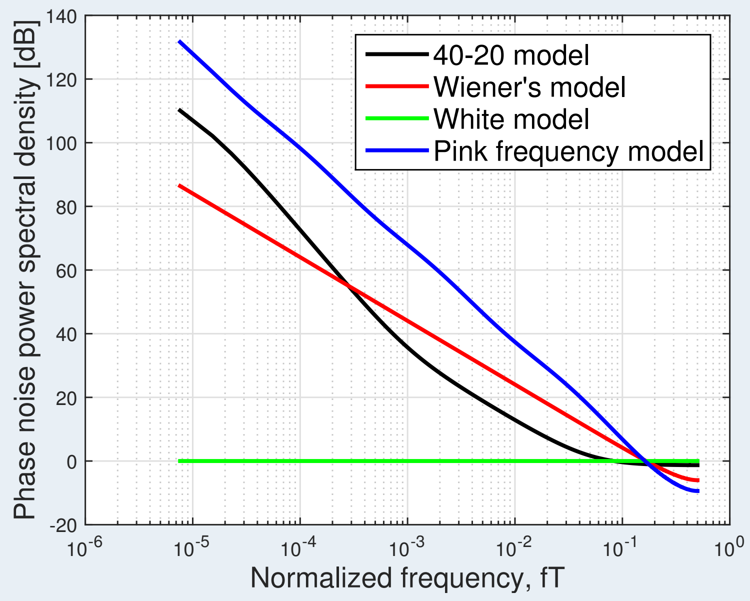
- Instabilities of the oscillators used for up- and down-conversion of signals in communication systems give rise to the phenomenon known as *phase noise*.
- The impairment on the system performance can be severe even for high-quality oscillators:
  - If the continuous-time waveform is processed by long filters at the receiver side;
  - When the symbol time is very long, as happens when using orthogonal frequency division multiplexing.
- Typically, the phase noise generated by oscillators is a random process with memory, and this makes the analysis of the capacity challenging.
- In the available literature, many papers do not consider the continuous-time nature of phase noise, thus overlooking the random amplitude fluctuations caused by filtering the phase noise process.

Contribution

- Analytical upper and lower bounds to the capacity of discrete-time Wiener phase noise channels
- Analytical lower bounds to the capacity of continuous-time Wiener phase noise channels

Phase Noise Models

- Wiener phase noise (random walk)
  - Lasers
  - Free-running oscillators
- 40-20 model, Pink frequency model
  - Phase-locked loop oscillators
- White model



Wiener Phase Noise

The phase process is given by

$$\Theta(t) = \Theta(0) + \gamma\sqrt{T}B(t/T), \qquad 0 \le t \le T, \qquad (1)$$

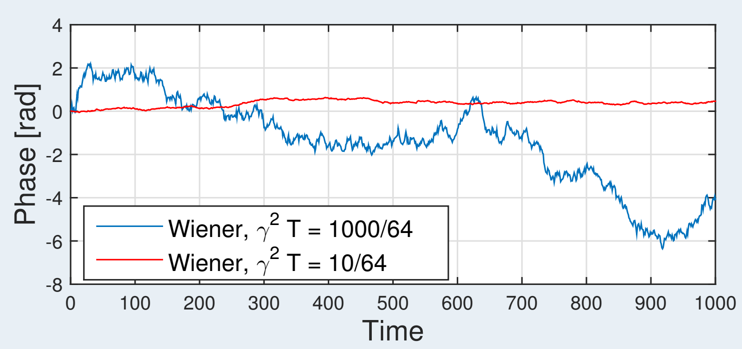
where $B(\cdot)$ is a standard Wiener process:

- $B(0) = 0$,
- for any $1 \ge t > s \ge 0$, $B(t) - B(s) \sim \mathcal{N}(0, t-s)$ is independent of the sigma algebra generated by $\{B(u) : u \le s\}$,
- $B(\cdot)$ has continuous sample paths.

One can think of the Wiener phase process as an accumulation of white noise:

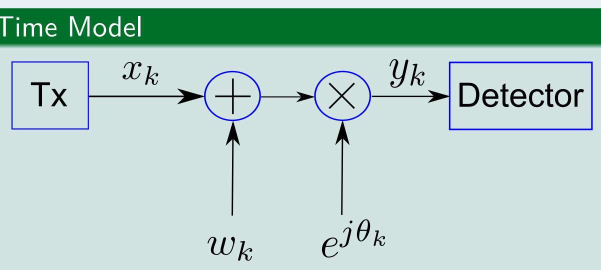
$$\Theta(t) = \Theta(0) + \gamma \int_0^t B'(\tau)\,\mathrm{d}\tau, \qquad 0 \le t \le T, \qquad (2)$$

where $B'(\cdot)$ is a standard white Gaussian noise process.



Wiener Phase Noise Channel

Discrete-Time Model



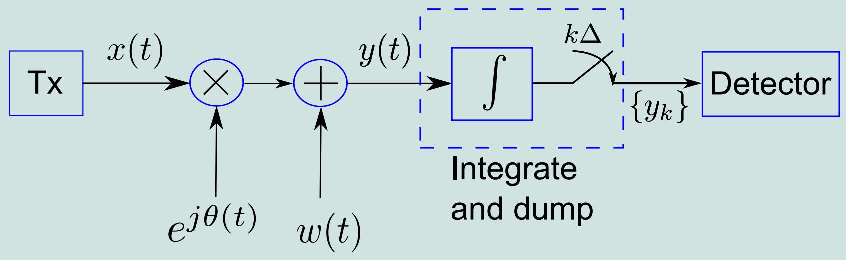
Denoting by Δ and T_{symb} the sampling and symbol time, the model obtained by sampling at time instants $t = k\Delta$ is

$$\Theta_k = \Theta_{k-1} + \gamma\sqrt{\Delta}N_k \qquad (3)$$

$$Y_k = X_{\lceil k\Delta/T_{\text{symb}}\rceil} e^{j\Theta_k} + W_k \qquad (4)$$

where the W_k's are independently and identically distributed (iid) random variables with $W_k \sim \mathcal{CN}(0,1)$.

Continuous-Time Model With Integrate-and-Dump Receiver



- Considering the filtering of $\Theta(\cdot)$ leads to the model [1]

$$\Theta_k = \Theta_{k-1} + \gamma\sqrt{\Delta}N_k \qquad (5)$$

$$Y_k = X_{\lceil k\Delta/T_{\text{symb}}\rceil} e^{j\Theta_k} \underbrace{\frac{1}{\Delta}\int_{(k-1)\Delta}^{k\Delta} e^{j(\Theta(t)-\Theta_k)}\,\mathrm{d}t}_{\triangleq F_k} + W_k \qquad (6)$$

- Determine the probability density function (pdf) of F_k is challenging [2], but some moments can be computed [1, 3].

Limits of Reliable Communication

- What is the best possible detector?
- Given input symbols $X_1^N = (X_1, X_2, \ldots, X_N)$ evaluate

$$I(X;Y) \triangleq \lim_{N\to\infty} \frac{1}{N} I\left(X_1^N; \mathbf{Y}_1^N\right) = \lim_{N\to\infty} \frac{1}{N} \mathrm{E}\left[\log_2 \frac{p(\mathbf{Y}_1^N|X_1^N)}{p(\mathbf{Y}_1^N)}\right] \qquad (7)$$

where $\mathbf{Y}_k = Y_{(k-1)L+1}^{(k-1)L+L}$ and $L = T_{\text{symb}}/\Delta$ is the oversampling factor.

- Under an average transmit power constraint and assuming iid input symbols, the best detector achieves the capacity

$$C(\text{SNR}) = \max_{\mathrm{E}[|X_k|^2] \le \text{SNR}} I(X;Y) \qquad (8)$$

Capacity pre-log

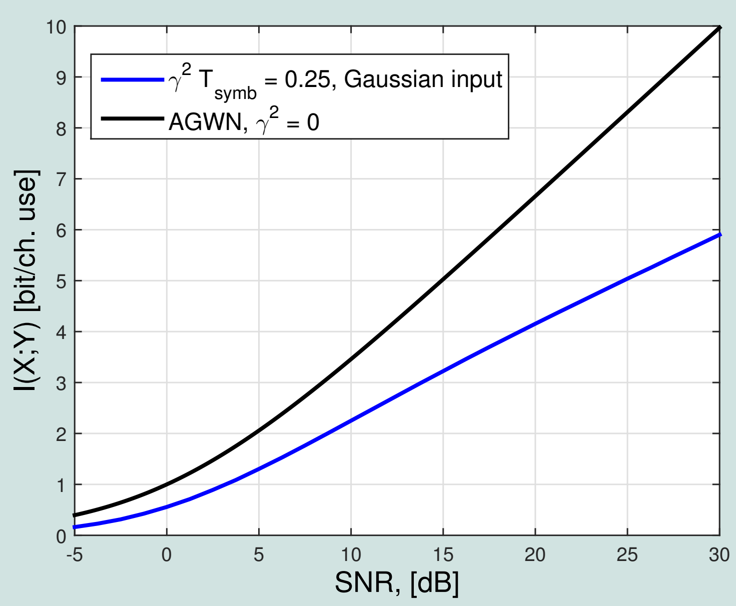
- We derived analytical results on the so-called capacity *prelog*:

$$\lim_{\text{SNR}\to\infty} \frac{C(\text{SNR})}{\log(\text{SNR})} \qquad (9)$$

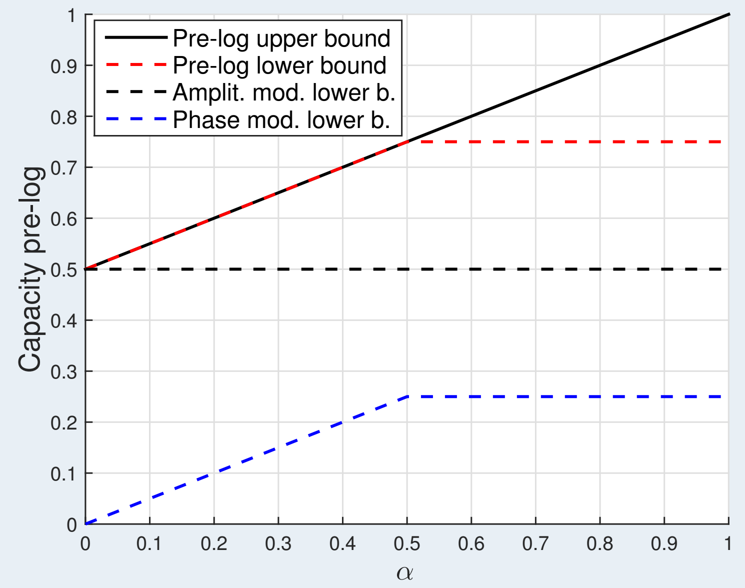
- Example: for an additive white Gaussian noise channel, $C(\text{SNR}) = \log(1 + \text{SNR})$, therefore the prelog is 1
- We let the sampling time Δ scale with the SNR as

$$\Delta = \frac{1}{\text{SNR}^{\alpha}}, \qquad 0 < \alpha < 1 \qquad (10)$$

Discrete-time Model With $L = 1$ ($\alpha = 0$) and $\gamma\sqrt{T_{\text{symb}}} = 0.5$ [4]



Discrete-Time Model: Analytical Bounds [1, 5, 6]



A capacity achieving scheme for $0 \le \alpha \le 1/2$ is the following:

- Choose a uniform pdf for $\angle X_k$ and

$$p_{|X_k|^2}(x) = \frac{\Delta}{\text{SNR}\Delta^2 - 1} \exp\left(-\frac{\Delta x - 1}{\text{SNR}\Delta^2 - 1}\right), \qquad x \ge 1/\Delta \qquad (11)$$

- Amplitude modulation: Use the statistic

$$V_k = \sum_{i=1}^{L} |Y_{(k-1)L+i}|^2 \qquad (12)$$

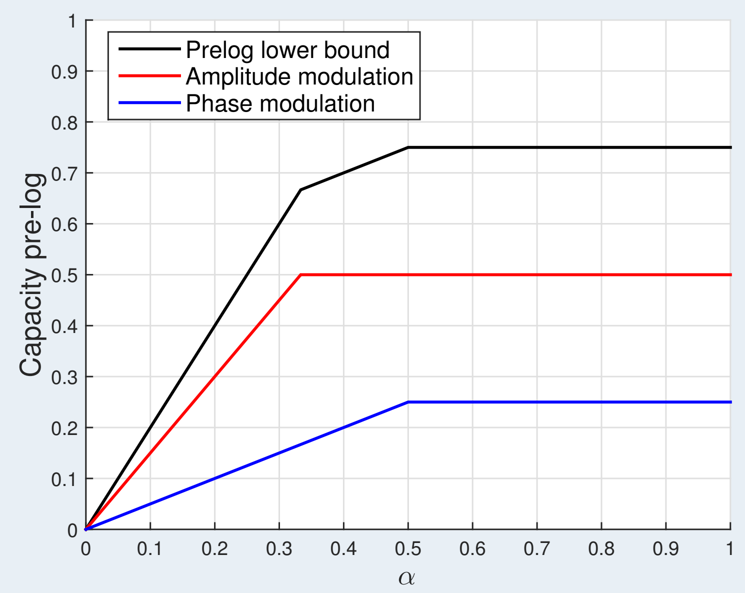
to detect $|X_k|$.

- Phase modulation: Use the statistic

$$\angle\widetilde{Y}_k = \angle\left(Y_{(k-1)L+1}\left(\frac{Y_{(k-1)L}}{X_{k-1}}\right)^*\right) \qquad (13)$$

to detect $\angle X_k$.

Continuous-Time Model: Analytical Lower Bound [1, 3]



- For $L = 1$ ($\alpha = 0$) the capacity is proportional to $\log\log(SNR)$ [7]
- The input distribution is chosen uniform in the phase and as

$$p_{|X_k|^2}(x) = \begin{cases} \frac{1}{\lambda}\exp\left(-\frac{x-\Delta^{-t}}{\lambda}\right) & x \ge \Delta^{-t} \\ 0 & \text{elsewhere} \end{cases} \qquad (14)$$

where $\lambda = \text{SNR}\Delta - \Delta^{-t} > 0$ with $0 < t < \alpha^{-1} - 1$.

- The detector is the same as in (12) and (13).

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