Capacity Regions of Two-User Broadcast Erasure Channels with Feedback and Hidden Memory

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***Abstract*—The two-receiver broadcast packet erasure channel with feedback and memory is studied. Memory is modelled using a finite-state Markov chain representing a channel state. The channel state is unknown at the transmitter, but observations of this hidden Markov chain are available at the transmitter through feedback. Matching outer and inner bounds are derived and the capacity region is determined. The capacity region does not have a single-letter characterization and is, in this sense, uncomputable. Approximations of the capacity region are provided and two optimal coding algorithms are outlined. The first algorithm is a probabilistic coding scheme that bases its decisions on the past L feedback sequences. Its achievable rate-region approaches the capacity region exponentially fast in L. The second algorithm is a backpressure-like algorithm that performs optimally in the long run.**

I. Introduction

In many communication protocols, information transmission is done in a packet by packet manner, and the receiving devices either correctly receive the transmitted packet or detect an erasure. Packet erasure channels (PECs) model such systems. Broadcast PECs (BPECs), in particular, are interesting models to study the broadcast nature of wireless systems.

The capacity region of the general broadcast channel (BC) remains unresolved both without and with feedback. It is known that feedback increases the capacity of general BCs and even partial feedback can help [1]–[3].

Remarkably, capacities of memoryless BPECs are known both with and without feedback. The former is a special case of the work in [4] which characterizes the capacity of degraded broadcast channels. The latter was derived in [5]. Generalizations of the idea presented in [5] have led to optimal coding schemes for BPECs with three receivers and near-optimal coding schemes for more receivers in [6]–[8].

This work studies BPECs with feedback and channel memory. Without feedback, one can use erasure correcting codes for memoryless channels in combination with interleavers to decorrelate the erasures. Interestingly, feedback enables more sophisticated coding techniques [9]. In a related work, [10] derived an infinite-letter capacity characterization for two special cases of the general BC with feedback, memory, and unidirectional receiver cooperation.

We model the memory of a channel by a finite state machine and a set of state-dependent erasure probabilities. This is a well-studied approach for wireless channels [11, Chapter 4.6].

In a previous work [12], we studied BPECs with feedback and memory under the assumption that the channel state is causally known at the transmitter. We derived close inner and outer bounds on the capacity region. This problem was also studied in parallel by Kuo and Wang in [13], [14]. They proposed a new coding strategy and characterized its corresponding rate region. The outer bound in [12] and the inner bound in [13], [14] match and thus characterize the capacity region.

This paper extends the previous results to the case where the channel state is no longer observable at the transmitter (except through the receivers' feedback) and thus evolves according to a hidden Markov model from the transmitter's point of view.

Our contribution in this paper is as follows. We propose an outer bound on the capacity region that has an n-letter characterization in terms of a feasibility problem. This region is not computable because its computation needs $n \to \infty$, and the number of parameters in the corresponding feasibility problem grows exponentially in n. Nevertheless, under mild conditions, we find a sequence of inner and outer approximations on this region for every $L \geq 1$. The L^{th} order approximation of the region approaches the outer bound exponentially fast in L. For every L, we propose a probabilistic encoding strategy at the transmitter that bases its decisions on the past L feedback symbols and achieves the L^{th} order approximation of the outer bound. Finally, a deterministic algorithm is outlined that bases its decisions on the entire past feedback symbols, is implementable, and optimal in the long run.

This paper is organized as follows. We introduce the system model in Sec. II. An outer bound on achievable rates is derived in Sec. III. Achievable schemes are discussed in Sec. IV before we conclude in Sec. V. Proofs and detailed derivations can be found in the extended version [15].

II. Notation and System Model

A. Notation

Random variables (RVs) are denoted by capital letters. A finite sequence (or string) of RVs $X_1, X_1, \ldots, X_n$ is denoted by X^n. Sequences can have subscripts, e.g. X_j^n is shorthand for $X_{j,1}, X_{j,2}, \ldots, X_{j,n}$. Vectors are written with underlined letters, e.g., $\underline{Z}_t = (Z_{1,t}, Z_{2,t})$. Sets are denoted by calligraphic letters, e.g., $\mathcal{X}$. The indicator function $\mathbb{1}\{\cdot\}$ takes on the value 1 if the event inside the brackets is true and 0 otherwise. The conditional probability is written equivalently as $\Pr[X = x|Y = y]$, $P_{X|Y}(x|y)$ or sometimes $P(x|y)$ if the involved RVs are clear from the context.

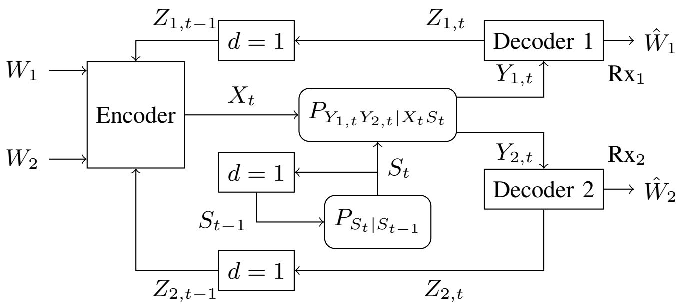

Fig. 1. Block diagram for the BPEC. The box marked with $d=1$ represents a delay of one time unit.

B. System Model

A transmitter wishes to communicate two independent messages W_1 and W_2 (of nR_1, nR_2 packets, respectively) to two receivers Rx_1 and Rx_2 over n channel uses. Communication takes place over a BPEC with memory and feedback as shown in Fig. 1 and described below.

The input to the BPEC at time t is denoted by $X_t \in \mathcal{X}$, $t = 1, \ldots, n$. The channel inputs correspond to packets of ℓ bits; i.e. $\mathcal{X} = \mathbb{F}_q$ with $q = 2^\ell$, and $\ell \gg 1$. All rates are measured in packets per slot and entropies and mutual information terms are with respect to logarithms to the base q.

The channel output at Rx_j at time t is written as $Y_{j,t} \in \mathcal{Y}$, $j \in \{1,2\}$, where $\mathcal{Y} = \mathcal{X} \cup \{E\}$. Each $Y_{j,t}$ is either X_t (received perfectly) or E (erased).

We define binary RVs $Z_{j,t}$, $j \in \{1,2\}$, $t = 1, \ldots, n$, to indicate erasure at Rx_j at time t; i.e. $Z_{j,t} = \mathbb{1}\{Y_{j,t} = E\}$. $Y_{j,t}$ can be expressed as a function of X_t and $Z_{j,t}$ but $Y_{j,t}$ also determines $Z_{j,t}$. We collect $(Z_{1,t}, Z_{2,t})$ into the vector $\underline{Z}_t$, with $\underline{Z}_t \in \mathcal{Z}$, $\mathcal{Z} = \{(0,0),(0,1),(1,0),(1,1)\}$.

The memory of the BPEC under study is modelled with a finite state machine with state S_t at time t. The underlying homogeneous finite state Markov chain is assumed to be irreducible and aperiodic with state space $\mathcal{S}$ and steady-state distribution π_s, $s \in \mathcal{S}$. The initial state S_0 is distributed according to the steady-state distribution, so the sequence S^n is stationary. The state-dependent erasure probabilities are specified through $P_{\underline{Z}_t|S_t}$ that does not depend on t. We permit arbitrary correlation between $(Z_{1,t}, Z_{2,t})$. The erasures $\underline{Z}^n$ are correlated in time in general, hence the channel has memory. The sequence $\underline{Z}^n$ is stationary due to stationarity of S^n.

After each transmission, an ACK or negative ACK (NACK) feedback is available at the encoder from both receivers. As opposed to our previous work in [12], we assume no separate feedback of the channel state; i.e. the channel state is *not* known at the transmitter. The channel input at time t may be written as

$$X_t = f_t(W_1, W_2, \underline{Z}^{t-1}). \tag{1}$$

The probability of erasure in each channel state is described based on the past channel state. Nevertheless, at the transmitter, the channel state is not available and thus the probabilities of erasure events depend on all the past observed feedback messages, $\underline{z}^{t-1}$, as follows:

$$\begin{aligned} \epsilon_{12}(\underline{z}^{t-1}) &= P_{\underline{Z}_t|\underline{Z}^{t-1}}(1,1|\underline{z}^{t-1}), \\ \epsilon_{\bar{1}2}(\underline{z}^{t-1}) &= P_{\underline{Z}_t|\underline{Z}^{t-1}}(0,1|\underline{z}^{t-1}), \\ \epsilon_{2}(\underline{z}^{t-1}) &= \epsilon_{12}(\underline{z}^{t-1}) + \epsilon_{\bar{1}2}(\underline{z}^{t-1}). \end{aligned} \tag{2}$$

$\epsilon_{1\bar{2}}(\underline{z}^{t-1})$ and $\epsilon_{1}(\underline{z}^{t-1})$ are defined similarly.

Each decoder Rx_j has to reliably estimate $\hat{W}_j$ from its received sequence Y_j^n. A rate-pair (R_1, R_2) is said to be achievable if the error probability $\Pr[\hat{W}_1 \neq W_1, \hat{W}_2 \neq W_2]$ can be made to approach zero as n gets large. The capacity region $\mathcal{C}_{\text{fb}}^{\text{mem}}$ is the closure of the achievable rate pairs.

III. The Outer Bound

Define $\bar{\mathcal{C}}_{n,\text{fb}}^{\text{mem}}$, for every integer n, as the closure of all rate pairs (R_1, R_2) for which there exist variables $x(\underline{z}^{t-1}), y(\underline{z}^{t-1})$, $\underline{z}^{t-1} \in \mathcal{Z}^{t-1}$, $t = 1, \ldots, n$, such that

$$0 \le x(\underline{z}^{t-1}),\ y(\underline{z}^{t-1}) \le 1,\ \forall t = 1, \ldots, n,\ \forall\, \underline{z}^{t-1} \in \mathcal{Z}^{t-1} \tag{3}$$

$$R_1 \le \frac{1}{n} \sum_{t=1}^{n} \sum_{\underline{z}^{t-1} \in \mathcal{Z}^{t-1}} P_{\underline{Z}^{t-1}}(\underline{z}^{t-1})(1-\epsilon_1(\underline{z}^{t-1}))x(\underline{z}^{t-1}) \tag{4}$$

$$R_1 \le \frac{1}{n} \sum_{t=1}^{n} \sum_{\underline{z}^{t-1} \in \mathcal{Z}^{t-1}} P_{\underline{Z}^{t-1}}(\underline{z}^{t-1})(1-\epsilon_{12}(\underline{z}^{t-1}))(1-y(\underline{z}^{t-1})) \tag{5}$$

$$R_2 \le \frac{1}{n} \sum_{t=1}^{n} \sum_{\underline{z}^{t-1} \in \mathcal{Z}^{t-1}} P_{\underline{Z}^{t-1}}(\underline{z}^{t-1})(1-\epsilon_2(\underline{z}^{t-1}))y(\underline{z}^{t-1}) \tag{6}$$

$$R_2 \le \frac{1}{n} \sum_{t=1}^{n} \sum_{\underline{z}^{t-1} \in \mathcal{Z}^{t-1}} P_{\underline{Z}^{t-1}}(\underline{z}^{t-1})(1-\epsilon_{12}(\underline{z}^{t-1}))(1-x(\underline{z}^{t-1})). \tag{7}$$

Define $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}$ as the limsup of $\bar{\mathcal{C}}_{n,\text{fb}}^{\text{mem}}$, when $n \to \infty$. With similar steps as in [12, Section IV], we have:

Theorem 1. Any achievable rate pair (R_1, R_2) is such that $(R_1 - \delta, R_2 - \delta) \in \bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}$, for $\delta > 0$.

The proof can be found in [15].

This result establishes an outer bound on the capacity region. The above characterization is, however, in an infinite-letter form and thus uncomputable. We find an approximation of the region next.

A. Approximation of the Outer Bound

The derived outer bound is an infinite-letter characterization because S_{t-1} is not fed back to the transmitter, but an estimate of it is available through the feedback symbols $\underline{Z}^{t-1}$. From the transmitter's perspective, the predicted erasure probabilities $P_{\underline{Z}_t|\underline{Z}^{t-1}}$ depend on all the past $\underline{Z}^{t-1}$. Intuitively, one expects that the effect of past feedback diminishes rapidly. We thus approximate the infinite-letter bound in (3) - (7) with a finite-letter bound where channel erasure events depend only on the past finite L feedback symbols, as follows: The predicted erasure probabilities for slot t are:

$$P(\underline{z}_t|\underline{z}^{t-1}) = \sum_{s \in \mathcal{S}} P_{\underline{Z}_t|S_t}(\underline{z}_t|s) P_{S_t|\underline{Z}^{t-1}}(s|\underline{z}^{t-1}). \tag{8}$$

$P_{\underline{Z}_t|S_t}$ does not depend on t, but $P_{S_t|\underline{Z}^{t-1}}$ does. In practice, one observes that the latter distribution is "close" to $P_{S_t|\underline{Z}_{t-L}^{t-1}}$

which predicts channel states from the past L feedback symbols if L is reasonably large. This is made precise by the adapted result from [16, Theorem 2.1]:

Theorem 2 (cf. [16])**.** Suppose that all entries of both the state transition matrix $P_{S_t|S_{t-1}}$ and the distribution matrix $P_{\underline{Z}_t|S_t}$ are strictly positive. For any observed sequence $\underline{z}^{t-1} \in \mathcal{Z}^{t-1}$, the variational distance is bounded by

$$\sum_{s\in\mathcal{S}} \left| P_{S_t|\underline{Z}^{t-1}}(s|\underline{z}^{t-1}) - P_{S_t|\underline{Z}_{t-L}^{t-1}}(s|\underline{z}_{t-L}^{t-1}) \right| \leq 2(1-\sigma)^L,$$

where $\sigma > 0$ depends on the smallest entry of matrix $P_{S_t|S_{t-1}}$ and the ratio of the largest and smallest values in matrix $P_{\underline{Z}_t|S_t}$. $\underline{Z}_{t-L}^{t-1}$ is shorthand for $\underline{Z}_{t-L}, \underline{Z}_{t-L+1}, \dots, \underline{Z}_{t-1}$.

Corollary 1. Under the conditions of Theorem 2, the total variation between $P_{\underline{Z}_t|\underline{Z}^{t-1}}$ and $P_{\underline{Z}_t|\underline{Z}_{t-L}^{t-1}}$ is bounded for any sequence $\underline{z}^{t-1} \in \mathcal{Z}^{t-1}$ by

$$\sum_{\underline{z}_t} \left| P(\underline{z}_t|\underline{z}^{t-1}) - P(\underline{z}_t|\underline{z}_{t-L}^{t-1}) \right| \leq 2(1-\sigma)^L. \tag{9}$$

Hence an approximate characterization of the outer bound $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}$ is given by the following feasibility problem.

$$0 \leq x(\underline{z}^L), y(\underline{z}^L) \leq 1, \qquad \forall\ \underline{z}^L \in \mathcal{Z}^L \tag{10}$$

$$R_1 \leq \sum_{\underline{z}^L \in \mathcal{Z}^L} P_{\underline{Z}^L}(\underline{z}^L)(1-\epsilon_1(\underline{z}^L))x(\underline{z}^L) + C_L \tag{11}$$

$$R_1 \leq \sum_{\underline{z}^L \in \mathcal{Z}^L} P_{\underline{Z}^L}(\underline{z}^L)(1-\epsilon_{12}(\underline{z}^L))(1-y(\underline{z}^L)) + C_L \tag{12}$$

$$R_2 \leq \sum_{\underline{z}^L \in \mathcal{Z}^L} P_{\underline{Z}^L}(\underline{z}^L)(1-\epsilon_2(\underline{z}^L))y(\underline{z}^L) + C_L \tag{13}$$

$$R_2 \leq \sum_{\underline{z}^L \in \mathcal{Z}^L} P_{\underline{Z}^L}(\underline{z}^L)(1-\epsilon_{12}(\underline{z}^L))(1-x(\underline{z}^L)) + C_L, \tag{14}$$

where $-2(1-\sigma)^L \leq C_L \leq 2(1-\sigma)^L$ and $\epsilon_j(\underline{z}^L)$, $\epsilon_{12}(\underline{z}^L)$ are computed via the distribution $P_{\underline{Z}_t|\underline{Z}_{t-L}^{t-1}}$. A detailed derivation can be found in [15].

By setting $C_L = 0$, we obtain what we call the L^{th} order approximation of the outer bound and we denote it by $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}(L)$. Clearly, $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}(L)$ approaches $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}$ exponentially fast in L. $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}(L)$ is of finite-letter form and thus computable. The number of parameters in the corresponding feasibility problem is exponential in L. Since the approximate rate-region approaches the outer bound exponentially fast in L, it can give good approximations for reasonable values of L.

IV. Achievability

We develop codes that achieve the outer bound in Sec. III. The coding strategy is based on the coding techniques discussed in [13]. More precisely, it uses network coding techniques of [5] and a proactive coding strategy that was proposed in [13] for the problem with causal channel state information. We discuss this coding scheme in our own framework as we believe our analysis is simpler than the one provided in [13].

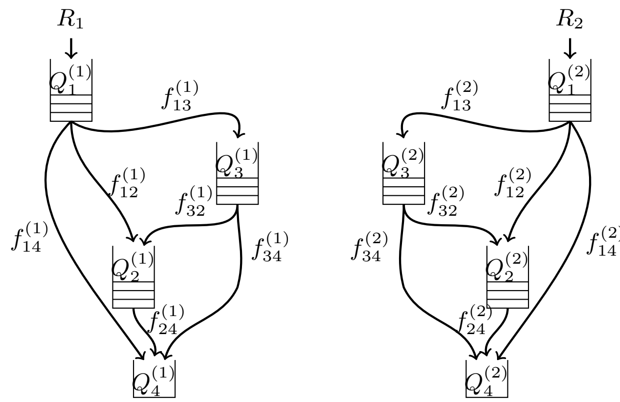

Fig. 2. Networked system of queues.

A. Queue Model

To analyze the coding scheme, we build on the idea of tracking packets through a network of queues, as done in [5], [6]. The transmitter has two buffers, $Q_1^{(1)}$ and $Q_1^{(2)}$, to store packets destined for Rx_1, Rx_2, respectively. We consider dynamic arrivals, where packets for Rx_1, Rx_2 arrive in each slot according to a Bernoulli process with probability R_1, R_2, respectively. These packets are called *original packets*. An analysis for more general arrival processes is possible. The transmitter maintains two additional buffers, $Q_2^{(1)}$ and $Q_2^{(2)}$, for packets that were received by the wrong receiver only. Buffer $Q_2^{(1)}$ contains packets that are destined for Rx_1 and were received at Rx_2 only, and vice versa for $Q_2^{(2)}$. If both $Q_2^{(1)}$ and $Q_2^{(2)}$ are nonempty, the transmitter can take a packet from both queues and send their XOR combination. Such coded packets are useful to both receivers, for each can decode a desired original packet upon reception of the coded packet.

Another coding operation that turned out to be necessary in [13, Example 2] is as follows. The transmitter takes one packet from $Q_1^{(1)}$, e.g. $p^{(1)}$, and one packet from $Q_1^{(2)}$, e.g. $p^{(2)}$, and sends the XOR combination of the two; i.e. $p^{(1)} + p^{(2)}$. The packets involved have not been transmitted before; hence, we call this action *proactive* coding or *poisoning*. A poisoned packet is not immediately useful for any receiver. However, it becomes useful together with a *remedy* packet that enables decoding of the original packet involved in the poisoned packet. For example, assume that the poisoned packet $p^{(1)} + p^{(2)}$ was received at Rx_2. If, at a later stage, the corresponding *remedy* $p^{(1)}$ is received at Rx_2, both $p^{(1)}$ and $p^{(2)}$ can be decoded at Rx_2. Therefore, upon arrival of the poisoned packet at Rx_2, $p^{(1)}$ becomes as useful to Rx_2 as $p^{(2)}$. Since $p^{(1)}$ is desired also at Rx_1, it is more efficient to send $p^{(1)}$ rather than $p^{(2)}$ in later channel uses; i.e., $p^{(2)}$ could, in principle, be replaced by the remedy packet $p^{(1)}$.

Remark 1. If the poisoned packet is received only at Rx_j, then the remedy packet is $p^{(\bar{j})}$, for $j, \bar{j} \in \{1, 2\}$, $j \neq \bar{j}$. If the poisoned packet is received at both receivers, the remedy is $p^{(1)}$ or $p^{(2)}$. We fix it to $p^{(1)}$.

Remedy packets are useful to both receivers. We put remedy packets into additional queues $Q_3^{(1)}$ and $Q_3^{(2)}$. These two queues are conceptually the same queue to track the remedy packets. We draw them separately to account for correct packet arrival for each receiver separately.

The system exit for Rx_j is represented by buffer $Q_4^{(j)}$. Once a packet reaches $Q_4^{(j)}$, it leaves the system. These buffers are always empty by definition.

The full networked queuing system is shown in Fig. 2. This queuing system is drawn as a graphical tool to track the packets that are sent and received. We restrict the set of actions at the transmitter to $\mathcal{A} = \{1, 2, 3, 4, 5\}$:

- $A_t = 1$: send an original packet for Rx_1 from $Q_1^{(1)}$
- $A_t = 2$: send an original packet for Rx_2 from $Q_1^{(2)}$
- $A_t = 3$: send a coded packet from $Q_2^{(1)}$ and $Q_2^{(2)}$
- $A_t = 4$: send a poisoned packet from $Q_1^{(1)}$ and $Q_1^{(2)}$
- $A_t = 5$: send a remedy packet from $Q_3^{(1)}$ or $Q_3^{(2)}$.

B. Packet Movement and Network Flow

As the algorithm evolves in time, packets flow on the network of Fig. 2. Packet movements that are due to $A_t = 1, 2, 3$ are based on [5], and are discussed in detail in [5], [6], [12]. Packet movements that are due to $A_t = 4, 5$ are based on [13]. We discuss them in our framework in the following.

Packets may move to $Q_3^{(1)}$ and $Q_3^{(2)}$ only if they are involved in a poisoned packet that is received by at least one of the receivers. To explain the packet movement from $Q_1^{(1)}$ to $Q_3^{(1)}$, for example, consider two cases:

(i) Packet $p^{(1)}$ is a remedy packet (i.e., the poisoned packet was received at Rx_2). In this case, $p^{(1)}$ is moved from $Q_1^{(1)}$ to $Q_3^{(1)}$ (to be transmitted in later channel uses).

(ii) Packet $p^{(1)}$ is not a remedy packet (i.e. the poisoned packet was received only at Rx_1). In this case, as discussed, $p^{(2)}$ is as useful as $p^{(1)}$ to Rx_1. So $p^{(1)}$ may be replaced by $p^{(2)}$ and moved from $Q_1^{(1)}$ to $Q_3^{(1)}$.

Remedy packets leave $Q_3^{(j)}$ when $A_t = 5$, according to Table I. This table appears in a similar form in [13].

For clarification, we elaborate the case where the remedy packet sent with $A_t = 5$ is received *only* at Rx_1. A remedy packet can only be sent if a poisoned packet, say $p^{(1)} + p^{(2)}$, was previously received at Rx_1 or Rx_2. The remedy packet is either $p^{(1)}$ or $p^{(2)}$. We consider the two cases separately:

1) If $p^{(1)}$ is the remedy packet, then its arrival at Rx_1 (and not at Rx_2) is captured by $Q_3^{(1)} \to Q_4^{(1)}$. Furthermore, since $p^{(1)}$ is a remedy packet, it is as useful to Rx_2 as $p^{(2)}$. So its arrival at Rx_1 lets us move the packet from $Q_3^{(2)}$ to $Q_2^{(2)}$, because Rx_1 knows $p^{(1)}$.
2) If $p^{(2)}$ is the remedy packet, then the poisoned packet $p^{(1)} + p^{(2)}$ must have been received only at Rx_1. Arrival of $p^{(2)}$ at Rx_1 (and not at Rx_2) lets Rx_1 decode $p^{(1)}$, hence $Q_3^{(1)} \to Q_4^{(1)}$. Also, since $p^{(2)}$ is received at Rx_1 and not at Rx_2, we have the packet movement $Q_3^{(2)} \to Q_2^{(2)}$.

Finally, we outline two algorithms that ensure network stability for rates in $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}$:

Action 5 (a remedy packet sent)	received only at Rx_1	received only at Rx_2	received at Rx_1, Rx_2
Packet movement	$Q_3^{(1)} \to Q_4^{(1)}$ $Q_3^{(2)} \to Q_2^{(2)}$	$Q_3^{(1)} \to Q_2^{(1)}$ $Q_3^{(2)} \to Q_4^{(2)}$	$Q_3^{(1)} \to Q_4^{(1)}$ $Q_3^{(2)} \to Q_4^{(2)}$

TABLE I
PACKET MOVEMENT FOR REMEDY PACKETS IN $Q_3^{(1)}$ AND $Q_3^{(2)}$.

Theorem 3. Rate-pairs (R_1, R_2) are achievable if $(R_1 + \delta, R_2 + \delta) \in \bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}$, for $\delta > 0$.

C. Probabilistic Scheme

The first scheme is a probabilistic scheme that bases its decisions on the past L feedback symbols. We prove achievability of $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}(L)$ for any integer $L > 0$.

Fix L to be an integer and consider an encoding strategy that bases its decisions on the past L feedback symbols $\underline{Z}_{t-L}^{t-1}$. The decisions are random and independent from previous decisions, according to a stationary probability distribution $P_{A_t|\underline{Z}_{t-L}^{t-1}}(a|\underline{z}^L)$, $a \in \{1, \ldots, 5\}$, $\underline{z}^L \in \mathcal{Z}^L$. To simplify notation here, we write $P(a|\underline{z}^L)$ for $P_{A_t|\underline{Z}_{t-L}^{t-1}}(a|\underline{z}^L)$ and $P(\underline{z}^L)$ for $P_{\underline{Z}^L}(\underline{z}^L)$. Based on the chosen probabilities, each link of the network in Fig. 2 has an effective capacity. This is the maximum number of packets that, on average, can go through each link. Denote the capacity of the link between $Q_r^{(j)}$ and $Q_l^{(j)}$ by $c_{rl}^{(j)}$, where $(r, l) \in \{(1, 2), (1, 3), (1, 4), (2, 4), (3, 2), (3, 4)\}$ and $j \in \{1, 2\}$. According to the packet movement in Sec. IV-B, we calculate the following link capacities:

$$\begin{aligned}
c_{12}^{(j)} &= \textstyle\sum_{\underline{z}^L \in \mathcal{Z}^L} P(\underline{z}^L)(\epsilon_j(\underline{z}^L) - \epsilon_{12}(\underline{z}^L))P(j|\underline{z}^L) \\
c_{13}^{(j)} &= \textstyle\sum_{\underline{z}^L \in \mathcal{Z}^L} P(\underline{z}^L)(1 - \epsilon_{12}(\underline{z}^L))P(4|\underline{z}^L) \\
c_{14}^{(j)} &= \textstyle\sum_{\underline{z}^L \in \mathcal{Z}^L} P(\underline{z}^L)(1 - \epsilon_j(\underline{z}^L))P(j|\underline{z}^L) \\
c_{24}^{(j)} &= \textstyle\sum_{\underline{z}^L \in \mathcal{Z}^L} P(\underline{z}^L)(1 - \epsilon_j(\underline{z}^L))P(3|\underline{z}^L) \\
c_{32}^{(j)} &= \textstyle\sum_{\underline{z}^L \in \mathcal{Z}^L} P(\underline{z}^L)(\epsilon_j(\underline{z}^L) - \epsilon_{12}(\underline{z}^L))P(5|\underline{z}^L) \\
c_{34}^{(j)} &= \textstyle\sum_{\underline{z}^L \in \mathcal{Z}^L} P(\underline{z}^L)(1 - \epsilon_j(\underline{z}^L))P(5|\underline{z}^L).
\end{aligned} \tag{15}$$

Using a max-flow analysis similar to [12], the rate pair (R_1, R_2) is achievable if there exists a distribution $P_{A_t|\underline{Z}_{t-L}^{t-1}}$ such that the following flow optimization problem is feasible:

$$\begin{aligned}
&R_j \le f_{12}^{(j)} + f_{13}^{(j)} + f_{14}^{(j)} \\
&f_{12}^{(j)} + f_{32}^{(j)} \le f_{24}^{(j)} \\
&f_{13}^{(j)} \le f_{32}^{(j)} + f_{34}^{(j)} \\
&f_{rl}^{(j)} \le c_{rl}^{(j)} \quad \forall (r,l) \in \begin{Bmatrix} (1,2), (1,3), (1,4) \\ (2,4), (3,2), (3,4) \end{Bmatrix}, \forall j \in \{1, 2\}.
\end{aligned} \tag{16}$$

This is a classical flow optimization problem where link capacities can be adjusted by $P_{A_t|\underline{Z}_{t-L}^{t-1}}$. Note that the flow networks for Rx_1 and Rx_2 are coupled only through $P_{A_t|\underline{Z}_{t-L}^{t-1}}$. Using the max-flow min-cut duality theorem, the above flow problem is equivalent to the following min-cut problem:

$$R_j \le c_{12}^{(j)} + c_{13}^{(j)} + c_{14}^{(j)}, \qquad \forall\, j \in \{1, 2\} \tag{17}$$

$$R_j \le c_{13}^{(j)} + c_{14}^{(j)} + c_{24}^{(j)}, \qquad \forall\, j \in \{1, 2\} \tag{18}$$

$$R_j \le c_{12}^{(j)} + c_{14}^{(j)} + c_{32}^{(j)} + c_{34}^{(j)}, \qquad \forall\, j \in \{1, 2\} \tag{19}$$

$$R_j \le c_{14}^{(j)} + c_{24}^{(j)} + c_{34}^{(j)}, \qquad \forall\, j \in \{1, 2\} \tag{20}$$

A_t	Weight depending on queue lengths and $\underline{Z}^{t-1} = \underline{z}^{t-1}$
1	$[1-\epsilon_1(\underline{z}^{t-1})]Q_{1,t}^{(1)} + \epsilon_{1\bar{2}}(\underline{z}^{t-1})(Q_{1,t}^{(1)} - Q_{2,t}^{(1)})$
2	$[1-\epsilon_2(\underline{z}^{t-1})]Q_{1,t}^{(2)} + \epsilon_{\bar{1}2}(\underline{z}^{t-1})(Q_{1,t}^{(2)} - Q_{2,t}^{(2)})$
3	$[1-\epsilon_1(\underline{z}^{t-1})]Q_{2,t}^{(1)} + [1-\epsilon_2(\underline{z}^{t-1})]Q_{2,t}^{(2)}$
4	$[1-\epsilon_{12}(\underline{z}^{t-1})]\left(Q_{1,t}^{(1)} - Q_{3,t}^{(1)} + Q_{1,t}^{(2)} - Q_{3,t}^{(2)}\right)$
5	$\epsilon_{1\bar{2}}(\underline{z}^{t-1})(Q_{3,t}^{(1)} - Q_{2,t}^{(1)}) + [1-\epsilon_1(\underline{z}^{t-1})]Q_{3,t}^{(1)}$
	$+\epsilon_{\bar{1}2}(\underline{z}^{t-1})(Q_{3,t}^{(2)} - Q_{2,t}^{(2)}) + [1-\epsilon_2(\underline{z}^{t-1})]Q_{3,t}^{(2)}$

TABLE II
DETERMINISTIC SCHEME WITH $A_t \in \mathcal{A}$. $Q_{l,t}^{(j)}$ DENOTES THE NUMBER OF PACKETS IN QUEUE $Q_l^{(j)}$ AT TIME t.

where (17) - (20) correspond to different cuts that separate $Q_1^{(j)}$ from $Q_4^{(j)}$, $j \in \{1,2\}$.

The achievable rate-region is thus given by the feasibility problem defined in (17) - (20) over the set of probabilities $P(1|\underline{z}^L), \ldots, P(5|\underline{z}^L)$, $\underline{z}^L \in \mathcal{Z}^L$. We show in [15] that, for the link capacities defined in (15), constraints (18), (19) are redundant and can be omitted. We are left with two bounds on R_j, $j \in \{1,2\}$:

$$R_j \leq \sum_{\underline{z}^L \in \mathcal{Z}^L} P(\underline{z}^L)(1-\epsilon_j(\underline{z}^L))\big[P(j|\underline{z}^L)+P(3|\underline{z}^L)+P(5|\underline{z}^L)\big] \quad (21)$$

$$R_j \leq \sum_{\underline{z}^L \in \mathcal{Z}^L} P(\underline{z}^L)(1-\epsilon_{12}(\underline{z}^L))\big[P(j|\underline{z}^L)+P(4|\underline{z}^L)\big]\,. \quad (22)$$

A feasible set of distributions $P(1|\underline{z}^L), \ldots, P(5|\underline{z}^L)$, $\underline{z}^L \in \mathcal{Z}^L$ can be found if $(R_1, R_2) \in \bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}(L)$.

One can verify that the region described by (21) - (22) is equivalent to $\bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}(L)$ by choosing a feasible set $P(1|\underline{z}^L), \ldots, P(5|\underline{z}^L)$, $\underline{z}^L \in \mathcal{Z}^L$ such that

$$\begin{aligned} x(\underline{z}^L) &= P(1|\underline{z}^L)+P(3|\underline{z}^L)+P(5|\underline{z}^L), \quad \forall \underline{z}^L \in \mathcal{Z}^L \\ y(\underline{z}^L) &= P(2|\underline{z}^L)+P(3|\underline{z}^L)+P(5|\underline{z}^L), \quad \forall \underline{z}^L \in \mathcal{Z}^L. \end{aligned} \quad (23)$$

D. Deterministic Scheme

For the probabilistic scheme, one must compute the optimal set of probability distributions in advance. This set depends on R_1, R_2 which might be unknown to the transmitter ahead of time. Furthermore, in the probabilistic approach, it might happen that an action is chosen but there is no packet in the corresponding queues to be transmitted. We avoid both drawbacks by a max-weight backpressure-like algorithm [17], [18] that bases its actions on both the queue state and the feedback. In each slot t, a weight function is computed for each action and the action with the higher weight is chosen. Table II lists the weights for each action depending on the current queue state and the previous feedback state $\underline{Z}^{t-1}$. Note that the values of $\epsilon_1(\underline{z}^{t-1})$, $\epsilon_2(\underline{z}^{t-1})$ and $\epsilon_{12}(\underline{z}^{t-1})$ can be computed recursively.

Proposition 1. The strategy in Table II stabilizes all queues in the network for every $(R_1+\delta, R_2+\delta) \in \bar{\mathcal{C}}_{\text{fb}}^{\text{mem}}$, $\delta > 0$.

Hence, the strategy in Table II provides an implicit way to compute the capacity region. The proof of Prop. 1 in [15] uses a finite-horizon Lyapunov drift analysis [18] that is adapted to account for *previous* observations (rather than the current channel state).

V. CONCLUSION

We studied the two-receiver broadcast packet erasure channel with feedback and hidden channel memory. The channel memory evolves according to a Markov chain that is not observable by the transmitter. We developed outer bounds that are achievable with two outlined coding schemes.

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