Finite-Length Scaling of Convolutional LDPC Codes Using Belief Propagation

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Motivation

- Finite-length scaling laws are based on the analysis of the Peeling Decoder (PD)
  - Very complex for spatially coupled LDPC codes
  - Only applicable for the Binary Erasure Channel (BEC)

Question:
Can we obtain scaling laws analyzing the less complex Belief Propagation (BP)?

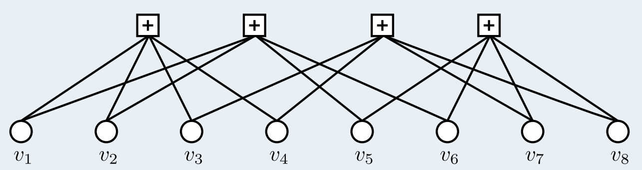
LDPC Codes [1]

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Example: $(2, 4)$ regular code

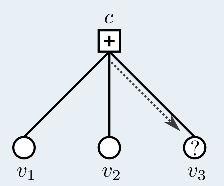
- **Regular** (l, r) codes:
  l ones in every column, respectively r ones in every row
- **Irregular** codes:
  Edge degree distributions described by polynomials

Graphical Representation as Tanner Graph (Tanner,1981):

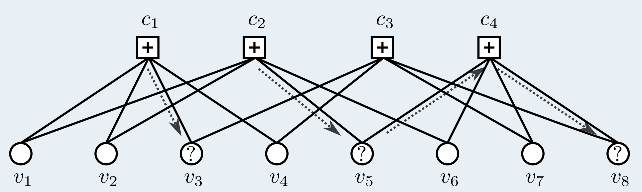


Properties
- **LDPC codes can reach capacity**
- **The decoding complexity stays linear**

Decoding

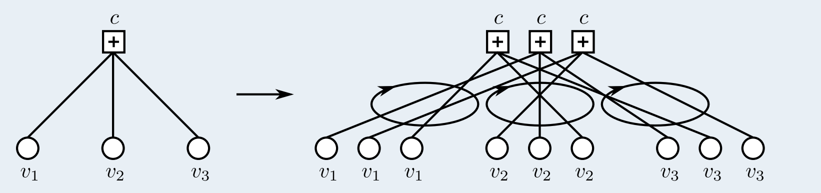


Check equation for check node c:
$$\sum_{k \in N(c)} v_k = 0 \mod 2$$



If variable nodes are erased due to the transmission over a binary erasure channels (BEC), they can be iteratively restored with the help of the knowledge of the rest of the graph of the code.

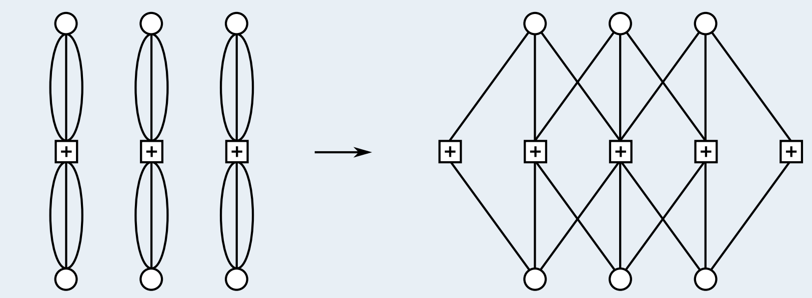
Protograph Based Construction [2]



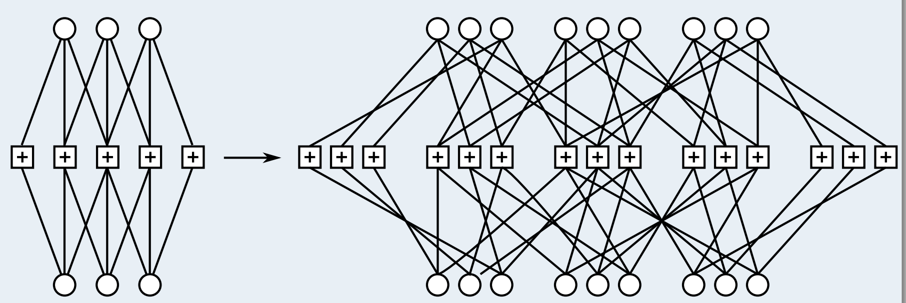
- Small Tanner graphs are used as a "blue print" of the structure
- This structure gets copied several times
- Similar connections are randomly permuted to obtain larger girths which avoids dependencies during the iterative decoding

Advantages
- The protograph representation can be used for analysis

$(l, r, L)_{\mathcal{P}}$ Codes Based on Coupled Protographs [3]



1. Choose a simple (l, r) protograph
2. Couple L protographs to a spatially coupled protograph



3. Lift the coupled protograph with the "copy-and-permute" operation

The convolutional-like band matrix $\mathbf{H}$ consists of submatrices $\mathbf{H}_{i,j}$ which are permutation matrices for edge permutations:

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & & & & & \\ \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & & & \\ \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & \\ & & \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \\ & & & & \mathbf{H}_{2,0} & \mathbf{H}_{2,1} & \ddots \end{pmatrix}$$

Advantages
- Systematic encoding is possible
- **The MAP threshold can be reached with iterative belief propagation (BP) decoding [4, 3]**

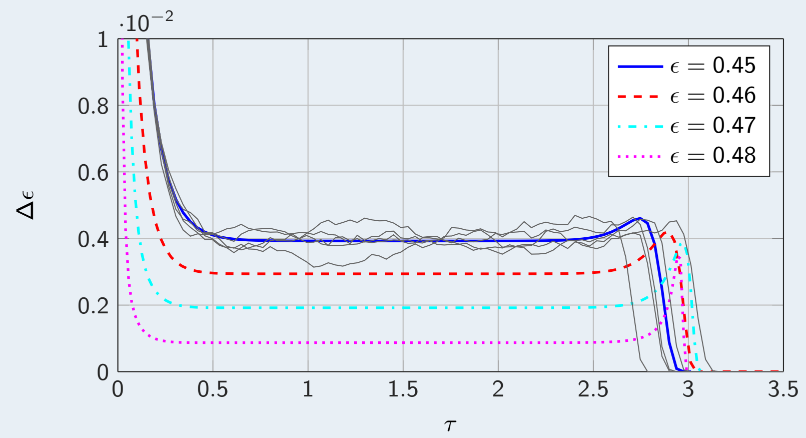
Belief Propagation Decoding

Messages are passed along the edges until the erasure probability does not decrease anymore. The decrease of erasure probability is used as stability criterion.

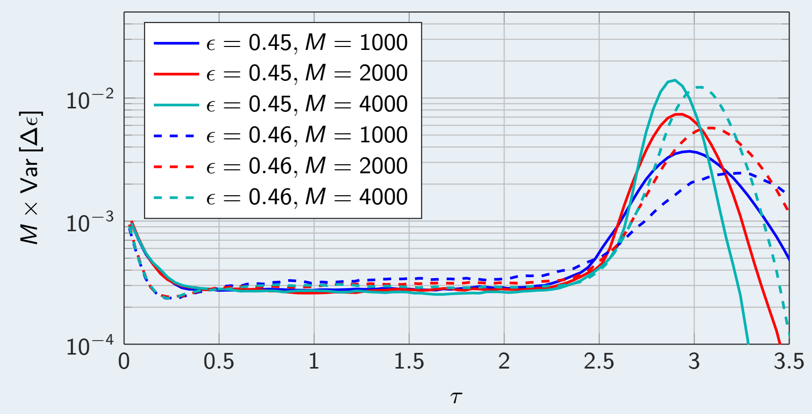
- τ: Decoding iterations normalized by $(\epsilon * -\epsilon)$
- $\Delta\hat{\epsilon}(\tau)$: Average decrease of erasure probability of variable nodes in iteration τ
- $\mathrm{Var}[\Delta\epsilon](\tau)$: Variance of $\Delta\epsilon(\tau)$ of all processes
- $\phi_1(\tau, \zeta)$: process covariance with time

$$\phi(\tau, \zeta) = \mathbb{E}[\Delta\epsilon(\tau)\Delta\epsilon(\zeta)] - \Delta\hat{\epsilon}(\tau)\Delta\hat{\epsilon}(\zeta)$$

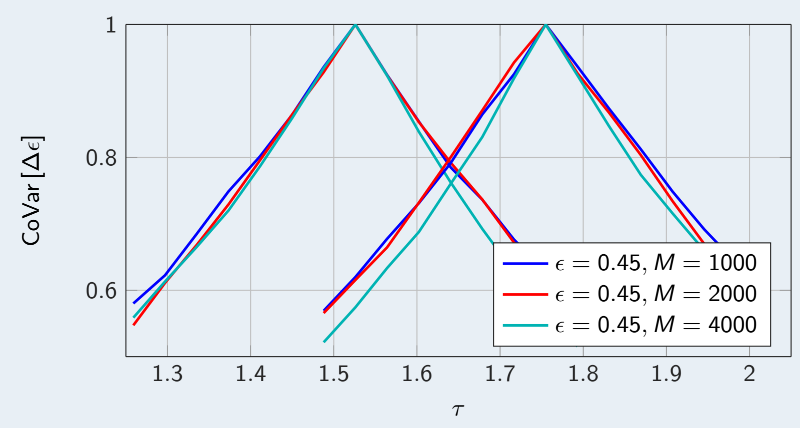
Analysis of the Belief Propagation



Calculated $\Delta\hat{\epsilon}(\tau)$ for the $(l, r, L)_{\mathcal{P}} = (3, 6, 50)_{\mathcal{P}}$ ensemble for a varying ϵ. For $\epsilon = 0.45$, the subplot includes actual decoding trajectories.



Simulated $\mathrm{Var}[\Delta\epsilon]\tau)$ for the $(3, 6, 50)_{\mathcal{P}}$ ensembles for various ϵ and M.



Estimation of the process covariance $\phi(\tau, \zeta)$ for the same ensembles for 2 points ζ. All results are computed for $\epsilon = 0.45$.

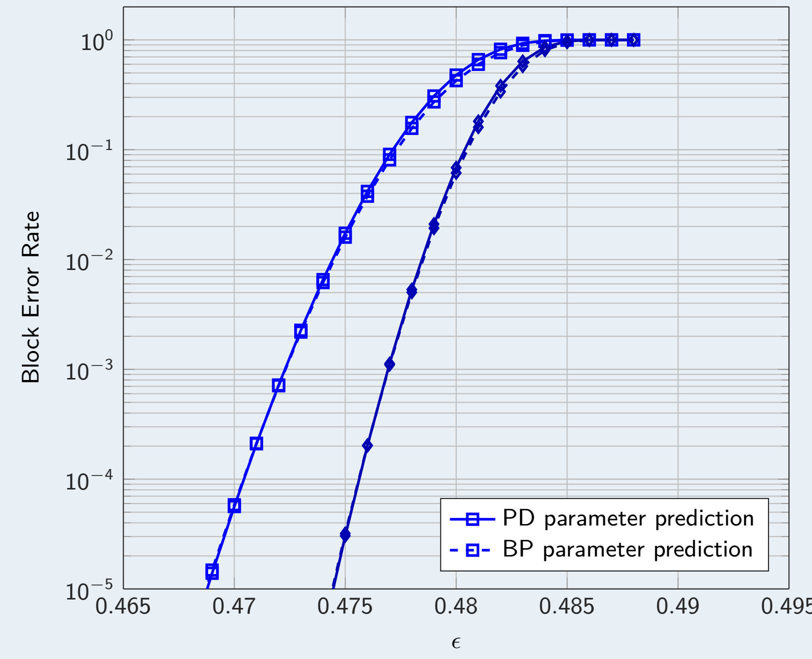
Conjecture of the Scaling Law [5]

Scaling laws stem from statistical physics where a system follows a control parameter in a very specific way around a phase transition. Around the threshold there holds a scaling law for LDPC codes using an iterative erasure decoder:

$$P^* \approx 1 - \exp\left(-\frac{(\epsilon L - \tau^*)}{\mu_0(M, \epsilon, l, r)}\right)$$

- $(\epsilon L - \tau^*)$ is the duration of the steady-state phase
- The average survival time μ_0 of $\Delta\epsilon$ during the steady-state phase is a function of $\Delta\hat{\epsilon}(\tau)$, $\mathrm{Var}[\Delta\epsilon(\tau)]$.

These parameters depend on the code ensemble and the process examined.



Word error rate on the BEC for the $(l, r, L)_{\mathcal{P}} = (3, 6, 50)_{\mathcal{P}}$ and the $(l, r, L)_{\mathcal{P}} = (3, 6, 100)_{\mathcal{P}}$ ensembles.

The resulting scaling law prediction using the BP process matches the slope of the PD prediction closely.

Outview and Future Tasks

- Can we analytically calculate variance and covariance?
- Can we use this analysis for the AWGN channel?

References

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[3] M. Lentmaier, G. P. Fettweis, K. S. Zigangirov, and D. J. Costello, "Approaching Capacity with Asymptotically Regular LDPC Codes," *Proc. Inf. Theory and Applicat. Workshop (ITA)*, pp. 173—177, 2009. [Online].

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