Ann Inst Stat Math
DOI 10.1007/s10463-017-0601-5

Asymptotic moving average representation of high-frequency sampled multivariate CARMA processes

Péter Kevei[1]

Received: 22 April 2016 / Revised: 8 December 2016

Abstract High-frequency sampled multivariate continuous time autoregressive moving average processes are investigated. We obtain asymptotic expansion for the spectral density of the sampled MCARMA process $(Y_{n\Delta})_{n\in\mathbb{Z}}$ as $\Delta \downarrow 0$, where $(Y_t)_{t\in\mathbb{R}}$ is an MCARMA process. We show that the properly filtered process is a vector moving average process, and determine the asymptotic moving average representation of it, thus generalizing the univariate results to the multivariate model. The determination of the moving average representation of the filtered process, important for the analysis of high-frequency data, is difficult for any fixed positive Δ. However, the results established here provide a useful and insightful approximation when Δ is very small.

Keywords Multivariate continuous time autoregressive moving average (CARMA) process · Spectral density · High-frequency sampling · Discretely sampled process

1 Introduction

The main object of this paper is a multivariate continuous time autoregressive moving average process (MCARMA) in d dimensions, which we define as follows. Let $p > q$ be nonnegative integers and let

This research was funded by a postdoctoral fellowship of the Alexander von Humboldt Foundation.

✉ Péter Kevei
peter.kevei@tum.de

[1] Center for Mathematical Sciences, Technische Universität München, Boltzmannstraße 3, 85748 Garching, Germany

Published online: 20 February 2017

$$P(z) = I_d z^p + A_1 z^{p-1} + \cdots + A_p,$$
$$Q(z) = B_0 z^q + B_1 z^{q-1} + \cdots + B_q$$

be the autoregressive and moving average polynomials, respectively, $A_i, B_j \in M_d$, and I_d is the d-dimensional identity matrix. The set of $m \times n$ real matrices is denoted by $M_{m,n}$, and M_n for $m = n$. The driving process is a two-sided d-dimensional Lévy process $(L_t)_{t\in\mathbb{R}}$, that is,

$$L_t = \begin{cases} L_1(t), & t \geq 0, \\ -L_2(-t-), & t < 0, \end{cases}$$

where $L_1(t), L_2(t), t \geq 0$, are iid (one-sided) d-dimensional Lévy processes, such that $\mathbf{E}L_1(1) = 0$ and $\mathbf{E}\|L_1(1)\|^2 < \infty$, with $\|\cdot\|$ being the usual Euclidean norm. The covariance matrix of $L_1(1)$ is Σ_L. For definition and properties of Lévy processes, we refer to Bertoin (1996).

The continuous time analogue of a discrete time ARMA equation is the differential equation

$$P(D)Y_t = Q(D)DL_t,$$

with D being the differential operator with respect to t. Since Lévy processes are not differentiable, this is understood as the following state space representation.

The d-dimensional stochastic process Y is an MCARMA process with autoregressive and moving average polynomial P and Q, respectively, if

$$\begin{aligned} \mathrm{d}G(t) &= \mathcal{A}G(t)\mathrm{d}t + \mathcal{B}\mathrm{d}L_t, \\ Y_t &= \mathcal{C}G(t), \quad t \in \mathbb{R}, \end{aligned} \tag{1}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & I_d & 0 & \dots & 0 \\ 0 & 0 & I_d & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & I_d \\ -A_p & -A_{p-1} & -A_{p-2} & \dots & -A_1 \end{pmatrix} \in M_{pd}, \quad \mathcal{C} = (I_d, 0, 0, \dots, 0) \in M_{d,pd},$$

and

$$\mathcal{B} = \left(\beta_1^\top, \beta_2^\top, \dots, \beta_p^\top\right)^\top \in M_{pd,d},$$
$$\beta_{p-j} = \begin{cases} -\sum_{i=1}^{p-j-1} A_i \beta_{p-j-i} + B_{q-j}, & 0 \leq j \leq q, \\ 0, & j > q. \end{cases}$$

Let $\lambda_1, \dots, \lambda_{pd}$ denote the eigenvalues of $\mathcal{A}$, which is the same as the zeros of $\det P(z)$; see Lemma 3.8 by Marquardt and Stelzer (2007). It is well known (Brockwell 2001a) that in one dimension a stationary causal solution exists if and only if the zeros of $\det P(z)$ have negative real parts. Under this condition a strictly stationary causal solution of the MCARMA Eq. (1) exists; see Marquardt and Stelzer (2007, Theorem 3.12 and Definition 3.20). Therefore, throughout the paper we assume that

the zeros of $\det P(z)$ have negative real part.

Under this assumption, the process Y can be represented as a continuous time moving average process (Marquardt and Stelzer 2007, (3.38)–(3.39))

$$Y_t = \int_{-\infty}^{t} g(t-s)\mathrm{d}L_s, \quad t \in \mathbb{R}, \tag{2}$$

where

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}tx} P(\mathrm{i}x)^{-1} Q(\mathrm{i}x)\mathrm{d}x.$$

By Lemma 3.24 in Marquardt and Stelzer (2007), the assumptions on the eigenvalues of $\mathcal{A}$ imply that g vanishes on the negative half-line; that is, our process is causal. In this paper, we use representation (2). Since we are only interested in second-order properties of the process Y, the integral in (2) is understood in the L^2-sense. For the same reason, our results remain valid in a more general setup, when the process $(L_t)_{t\in\mathbb{R}}$ has stationary orthogonal increments.

CARMA processes are natural continuous time analogues of discrete time ARMA processes, both in one and in several dimension. Gaussian CARMA processes date back to Doob (1944), while more general Lévy-driven CARMA models were introduced by Brockwell (2001b). The multivariate extension was introduced by Marquardt and Stelzer (2007). However, second-order stationary (also called weakly, or covariance stationary) continuous time processes have been long used in economic modeling. Already Phillips (1959) investigated a system of stochastic differential equations, which has a second-order stationary solution with rational spectral density, a property shared by MCARMA processes; see formula (3). Phillips also considered discrete time sampling of the continuous process and, based on a moving average representation of the sampled process, proposed an estimation method for the parameters. Robinson (1977, 1993) considered maximum likelihood type estimator of such models from discrete equidistant samples. The closest to our approach is Robinson (1993), where the spectral density matrix of the MCARMA process was determined by evaluation of contour integrals; see Propositions 2 and 3. Estimators and their asymptotics in the frequency domain were treated by Robinson (1976). For a more complete account on continuous time econometric models, we refer to Bergstrom (1988) and the references therein.

These models are important tools for stochastic modeling and have a wide range of applications, besides economics, in financial mathematics, electricity markets, and in turbulence. Andresen et al. (2014) used Gaussian CARMA processes to model the short and forward interest rate. It is worth mentioning that the Vasicek model corresponds to the Ornstein–Uhlenbeck process, which is the CARMA(1,0) process. Todorov and Tauchen (2006) applied Lévy-driven CARMA processes to model stochastic volatility in finance. For a review on applications of Lévy-driven time series models in finance, we refer to Brockwell and Lindner (2012). For spot prices in electricity markets, Benth et al. (2014) proposed a model with seasonality, consisting of a deterministic seasonality, a CARMA factor driven by a non-Gaussian stable Lévy process, and a nonstationary long-term factor given by a Lévy process. Brockwell et al. (2013) applied CARMA processes to model high-frequency sampled turbulence data.

Investigations of MCARMA models have become more active in the recent years. The state space representation of these models together with the ergodic and mixing properties of equidistantly sampled MCARMA processes were studied by Schlemm and Stelzer (2012). Fasen (2014) investigated asymptotic properties of high-frequency sampled models, and gave a parameter estimation, while Fasen and Kimmig (2017) studied information criteria for MCARMA models. Recovery of the driving Lévy process was treated by Brockwell and Schlemm (2013) and by Ferrazzano and Fuchs (2013).

In the present paper, we consider high-frequency sampling of an MCARMA process and investigate the characteristics of the resulting process $Y_n^\Delta = Y_{n\Delta}$. As $\Delta \downarrow 0$ we obtain the asymptotic moving average representation of the filtered process, thus extending the results in Brockwell et al. (2012), and partly in Brockwell et al. (2013) to the multivariate setup. The moving average representation of the sampled process was already noted by Phillips (1959, pp. 72–73) and proved later also by Schlemm and Stelzer (2012, Theorem 4.2). Phillips' estimation method is based on this representation; however, as Robinson (1977) noted, except in the simplest models, the relationship between the matrices in the moving average representation and the parameters of the continuous time model is very complicated. The determination of the moving average representation of the filtered process, important for the analysis of high-frequency data, is extremely difficult for any fixed positive Δ even in the one-dimensional case. Our main result Theorem 1 gives further information on the decomposition as $\Delta \downarrow 0$ and provides a useful and insightful approximation when Δ is very small.

In order to prove the asymptotic moving average representation, we determine the first-order behavior of the spectral density matrix of the properly filtered process. The technical tool of the spectral density analysis is computing residues, which was also used by Robinson (1993) and by Brockwell et al. (2012, 2013). Therefore, in the whole paper we consider the frequency domain approach.

In the next section, we give two representations of the spectral density of the sampled process. The first one in Proposition 1 is a Taylor expansion in Δ, which we use to prove the asymptotic moving average representation in Theorem 1. The second one in Propositions 2 and 3, which in a slightly different form was obtained by Robinson (1993), allows us to show that the filtered process is a moving average process. Section 3 contains the main result, the moving average representation of the filtered process. In Sect. 4, the simplest multivariate example is spelled out in detail, showing the difficulties to obtain higher-order approximations as is possible in the one-dimensional case. All the proofs are gathered in the last section.

2 Spectral density of the sampled process

It is well known that MCARMA processes have spectral density function, which is

$$f_Y(\lambda) = \frac{1}{2\pi} P(\mathrm{i}\lambda)^{-1} Q(\mathrm{i}\lambda) \Sigma_L Q(\mathrm{i}\lambda)^* (P(\mathrm{i}\lambda)^{-1})^*, \quad \lambda \in \mathbb{R}, \tag{3}$$

where A^* is the Hermite transpose of the complex matrix A (Marquardt and Stelzer 2007, (3.43)). Put

$$R(z) = P(z)^{-1} Q(z) \Sigma_L Q(-z)^\top (P(-z)^{-1})^\top, \; z \in \mathbb{C}. \tag{4}$$

Note that the components of the matrix $P(z)^{-1} Q(z)$ are meromorphic functions, which can have poles only at the zeros of $\det P(z)$. Due to our assumptions, the poles have negative real part.

We frequently use the following simple facts about residues. If $h(z)$ is a meromorphic function and ρ is a closed curve, which encircles the poles of h, then its residue at infinity is defined as

$$\mathrm{Res}(h(z), \infty) = -\frac{1}{2\pi \mathrm{i}} \int_\rho h(z) \mathrm{d}z;$$

moreover, it can be computed as

$$\mathrm{Res}(h(z), \infty) = -\mathrm{Res}\left(z^{-2} h\left(z^{-1}\right), 0\right).$$

For a matrix $M(z)$ with rational function entries, let $\deg M(z)_{i,j}$ denote the degree of the numerator minus the degree of the denominator in the (i, j)th element of M and put $\deg M(z) = \max_{i,j} M(z)_{i,j}$. Finally, for a square matrix A its adjugate is the transpose of its cofactor matrix, that is, $\mathrm{adj} A = C^\top$, where $C_{i,j} = (-1)^{i+j} A_{i,j}$, and $A_{i,j}$ is the determinant of the matrix that results from deleting row i and column j of A. When A is nonsingular then $A^{-1} = \mathrm{adj} A / \det A$.

From now on, let us fix a closed curve $\rho \subset (-\infty, 0) \times \mathrm{i}\mathbb{R}$ in the left half complex plane, which contains the zeros of $\det P(z)$. For any nonnegative integer k introduce the notation

$$\int_{\rho \cup -\rho} z^k R(z) \mathrm{d}z = -2\pi \mathrm{i}\, \mathrm{Res}\Big(z^k R(z), \infty\Big) =: 2\pi \mathrm{i} \Theta_k. \tag{5}$$

Since $P^{-1}(z) = \frac{\mathrm{adj} P(z)}{\det P(z)}$, we have

$$R(z) = \frac{1}{\det P(z) \det P(-z)} \mathrm{adj} P(z) Q(z) \Sigma_L (\mathrm{adj} P(-z) Q(-z))^\top.$$

Note that $\deg(\mathrm{adj}\, P(z)) \le (d-1)p$ and $\deg Q(z) \le q$; thus, we may write

$$(\mathrm{adj}\, P(z))Q(z) = \sum_{j=0}^{(d-1)p+q} S_j z^j, \qquad (6)$$

where $S_j \in M_d$, $j = 0, 1, \ldots, (d-1)p+q$. Since the polynomials on the main diagonal of $\mathrm{adj}\, P(z)$ are of degree $(d-1)p$, otherwise the degrees are strictly less than $(d-1)p$, we obtain that $S_{(d-1)p+q} = B_0$. Thus,

$$S(z) = \mathrm{adj}\, P(z)Q(z)\Sigma_L(\mathrm{adj}\, P(-z)Q(-z))^\top = \sum_{j=0}^{2[(d-1)p+q]} \widetilde{S}_j z^j, \qquad (7)$$

where the first coefficients are

$$\widetilde{S}_{2[(d-1)p+q]} = (-1)^{(d-1)p+q} B_0 \Sigma_L B_0^\top,$$
$$\widetilde{S}_{2[(d-1)p+q]-1} = (-1)^{(d-1)p+q-1} \left(B_0 \Sigma_L S_{(d-1)p+q-1}^\top - S_{(d-1)p+q-1} \Sigma_L B_0^\top \right).$$

From (7), we see that $S(z)^\top = S(-z)$; thus, for $j = 0, 1, \ldots$

$$\widetilde{S}_{2j}^\top = \widetilde{S}_{2j} \text{ and } \widetilde{S}_{2j+1}^\top = -\widetilde{S}_{2j+1}.$$

In the latter matrices, the main diagonal is 0. In particular, they are 0 in the one-dimensional case.

From (7), we see that each component of the matrix $z^k R(z)$ decreases at least as z^{-2} for $k \le 2(p-q)-2$; thus, from the definition of the residue at infinity follows $\Theta_k = 0$. For $k = 2(p-q)-1$ it is easy to see from the definition of $R(z)$ that $\Theta_{2(p-q)-1} = (-1)^{p-q} B_0 \Sigma_L B_0^\top$. To determine further coefficients, note that

$$\sum_{k=0}^\infty \Theta_k z^k = \frac{R(z^{-1})}{z}. \qquad (8)$$

As $\det P(z) = \prod_{j=1}^{pd} (z - \lambda_j)$ we obtain

$$z^{-1} R(z^{-1}) = \frac{(-1)^{pd} z^{2pd-1}}{\prod_{j=1}^{pd} \left(1 - \lambda_j^2 z^2\right)} \sum_{j=0}^{2[(d-1)p+q]} \widetilde{S}_j z^{-j}.$$

This formula implies a linear recursion for the coefficients Θ_k, which in a special case is spelled out in Sect. 4.

Let us define the coefficients $\widetilde{c}_k(\omega)$ for $\omega \ne 0, \omega \in (-\pi, \pi)$, via the series expansion

$$\frac{1}{1 - e^{z+\mathrm{i}\omega}} = \sum_{k=0}^\infty \widetilde{c}_k(\omega) z^k, \quad |z| < |\omega|. \qquad (9)$$

Let f_Δ denote the spectral density matrix of the sampled process $Y_n^\Delta = Y(n\Delta)$. In the following, we obtain a Taylor expansion in Δ for the spectral density matrix.

Proposition 1 *For any* $\omega \in (-\pi, \pi)\backslash\{0\}$ *if* $\Delta > 0$ *is small enough, then*

$$f_\Delta(\omega) = -\frac{1}{2\pi} \sum_{k=2(p-q)-1}^{\infty} (-\Delta)^k \Theta_k \widetilde{c}_k(\omega).$$

From the leading term, we obtain

Corollary 1 *As* $\Delta \downarrow 0$

$$f_\Delta(\omega) = (-1)^{p-q} \frac{\Delta^{2(p-q)-1}(1 + O(\Delta))}{2\pi} \widetilde{c}_{2(p-q)-1}(\omega) B_0 \Sigma_L B_0^\top.$$

By Lemma 1, $\widetilde{c}_{2(p-q)-1}(\omega)$ is real, which implies that the first-order approximation is real.

We also give another representation of the spectral density, from which the moving average representation (17) follows. In a slightly different form this representation was given by Robinson (1993, (20) and (26)). For the sake of completeness, we provide a short proof. First we state the result for different zeros.

Proposition 2 *Assume that* $\lambda_1, \ldots, \lambda_{pd}$ *are different zeros of* $\det P(z)$*; that is, each has multiplicity one. Then*

$$f_\Delta(\omega) = \frac{1}{4\pi} \sum_{\ell=1}^{pd} \frac{e^{-\mathrm{i}\omega}(\alpha(\ell) - \alpha(\ell)^\top) - e^{\Delta\lambda_\ell}\alpha(\ell) + e^{-\Delta\lambda_\ell}\alpha(\ell)^\top}{\cosh \Delta\lambda_\ell - \cos\omega}, \tag{10}$$

where the coefficient matrices $\alpha(\ell) \in M_d$ *come from the partial fraction decomposition of* $R(z)$*, i.e.,*

$$R(z) = \frac{S(z)}{\det P(z) \det P(-z)} = \sum_{\ell=1}^{pd} \left(\frac{\alpha(\ell)}{z - \lambda_\ell} + \frac{\beta(\ell)}{-z - \lambda_\ell} \right). \tag{11}$$

From $R(z)^\top = R(-z)$ it follows that $\alpha(\ell)^\top = \beta(\ell)$.

The further assumption on the multiplicity of the zeros is not necessary but it makes the formulas simpler. The following proposition gives the spectral density in the general case. Note that, as an abuse of notation, now $\lambda_1, \ldots, \lambda_m$ are the *different* zeros of $\det P(z)$.

Proposition 3 *Let* $\lambda_1, \ldots, \lambda_m$ *be the different zeros of* $\det P(z)$ *with multiplicity* $\nu_1, \ldots, \nu_m$*. Then*

$$f_\Delta(\omega) = \frac{1}{4\pi} \sum_{\ell=1}^{m} \frac{e^{-\mathrm{i}\omega}(\alpha(\ell,1) - \alpha(\ell,1)^\top) - e^{\Delta\lambda_\ell}\alpha(\ell,1) + e^{-\Delta\lambda_\ell}\alpha(\ell,1)^\top}{\cosh \Delta\lambda_\ell - \cos\omega}$$
$$+ \sum_{\ell=1}^{m} \sum_{j=2}^{\nu_\ell} \frac{s_j^\Delta(\omega,\lambda_\ell)\alpha(\ell,j) + s_j^\Delta(-\omega,\lambda_\ell)\alpha(\ell,j)^\top}{(\cosh \Delta\lambda_\ell - \cos\omega)^j}, \tag{12}$$

where $s_j^\Delta(\omega,\lambda_\ell)$ are trigonometric polynomials of ω of degree $j-1$, whose coefficients depend on Δ and λ_ℓ, and the coefficient matrices $\alpha(\ell,j) \in M_d$ come from the partial fraction decomposition of $R(z)$, i.e.,

$$R(z) = \frac{S(z)}{\det P(z) \det P(-z)} = \sum_{\ell=1}^{m} \sum_{j=1}^{\nu_\ell} \left(\frac{\alpha(\ell,j)}{(z-\lambda_\ell)^j} + \frac{\beta(\ell,j)}{(-z-\lambda_\ell)^j} \right). \tag{13}$$

3 Moving average representation

Recall that $\lambda_1, \ldots, \lambda_{pd}$ denote the zeros of $\det P(z)$. Define the polynomial

$$\Phi^\Delta(z) = \prod_{i=1}^{pd} \left(1 - e^{\Delta\lambda_i} z\right),$$

and consider the filtered process

$$X_n^\Delta = \Phi^\Delta(B) Y_n^\Delta, \tag{14}$$

where B is the backshift operator. This filter was already applied by Phillips (1959), who showed that the resulting process is a moving average process. Note that, whenever λ_i is complex, its complex conjugate is also a root of $\det P(z)$; thus, the polynomial Φ^Δ has real coefficients. The power transfer function (Brockwell and Davis 1987, Theorem 4.4.1) of the filter Φ^Δ is

$$\phi^\Delta(\omega) = \left| \prod_{i=1}^{pd} \left(1 - e^{\Delta\lambda_i + \mathrm{i}\omega}\right) \right|^2 = 2^{pd} e^{\Delta \sum_{j=1}^{pd} \lambda_j} \prod_{j=1}^{pd} (\cosh \Delta\lambda_j - \cos\omega). \tag{15}$$

When the zeros of $\det P(z)$ have multiplicity one, from (10) and (15) we see that the spectral density of the filtered process X^Δ is

$$\begin{aligned} f_{\mathrm{MA}}^\Delta(\omega) &= f_\Delta(\omega)\,\phi^\Delta(\omega) \\ &= \frac{2^{pd} e^{\Delta \sum_{j=1}^{pd} \lambda_j}}{4\pi} \prod_{j=1}^{pd} (\cosh \Delta\lambda_j - \cos\omega) \\ &\quad \times \sum_{\ell=1}^{pd} \frac{e^{-\mathrm{i}\omega}(\alpha(\ell) - \alpha(\ell)^\top) - e^{\Delta\lambda_\ell}\alpha(\ell) + e^{-\Delta\lambda_\ell}\alpha(\ell)^\top}{\cosh \Delta\lambda_\ell - \cos\omega} \end{aligned}$$

$$= \frac{2^{pd} e^{\Delta \sum_{j=1}^{pd} \lambda_j}}{4\pi} \sum_{\ell=1}^{pd} \prod_{j\neq\ell} (\cosh \Delta\lambda_j - \cos\omega)$$
$$\times \left[e^{-\mathrm{i}\omega}(\alpha(\ell) - \alpha(\ell)^\top) - e^{\Delta\lambda_\ell}\alpha(\ell) + e^{-\Delta\lambda_\ell}\alpha(\ell)^\top \right]. \quad (16)$$

This is clearly a trigonometric polynomial of degree less than or equal to pd. However, the coefficient of $(\cos\omega)^{pd}$ and of $(\cos\omega)^{pd-1}\sin\omega$ is a multiple of

$$\sum_{\ell=1}^{pd} (\alpha(\ell) - \alpha(\ell)^\top).$$

Since in the partial fraction decomposition in (11) $\deg S(z) \le 2((d-1)p+q)$, the sum above is necessarily 0. Therefore, $f_{\mathrm{MA}}^{\Delta}(\omega)$ is a trigonometric polynomial of degree strictly less than pd. In the general case, when the zeros are not necessarily different, by (12) we obtain a similar representation. Thus, we have shown the following [see also Schlemm and Stelzer (2012, Theorem 4.2) and Phillips (1959, pp. 72–73)].

Corollary 2 *For any $\Delta > 0$ the filtered process X^Δ is a d-dimensional moving average process of order less than pd; i.e., there exist a matrix polynomial Ψ^Δ with degree less than pd and a white noise sequence Z^Δ, such that*

$$X_n^\Delta = \Phi^\Delta(B) Y_n^\Delta = \Psi^\Delta(B) Z_n^\Delta. \quad (17)$$

The interesting feature of this long-known corollary is that apart from the simplest cases ($d = 1$, $p = 1, 2$) no explicit expression exists for the moving average polynomial Ψ^Δ. In Theorem 1, we determine the first-order asymptotic behavior of Ψ^Δ as $\Delta \downarrow 0$.

In order to state the asymptotic result for the moving average process we need a lemma about the coefficients $\widetilde{c}_k(\omega)$.

Lemma 1 *There exist polynomials q_{k-1}, r_{k-1} of degree $k-1$ with real coefficients such that*

$$\begin{aligned} (2k-1)!\,[2(1-\cos\omega)]^k\, \widetilde{c}_{2k-1}(\omega) &= (-1)^k q_{k-1}(\cos\omega), \\ (2k)!\,[2(1-\cos\omega)]^{k+1}\, \widetilde{c}_{2k}(\omega) &= (-1)^k\, \mathrm{i}\sin\omega \cdot r_{k-1}(\cos\omega). \end{aligned} \quad (18)$$

Moreover,

$$q_{k-1}(x) = (-1)^{k-1} 2^{k-1} \prod_{j=1}^{k-1} (1 - x - \xi_{2k-1,j}), \quad \prod_{j=1}^{k-1} \xi_{2k-1,j} = (2k-1)! 2^{-(k-1)},$$

and

$$r_{k-1}(x) = (-1)^{k-1} 2^k \prod_{j=1}^{k-1} (1 - x - \xi_{2k,j}), \quad \prod_{j=1}^{k-1} \xi_{2k,j} = (2k)! 2^{-k}.$$

Furthermore, $\xi_{2k-1,j}, \xi_{2k,j} \notin (0, 2)$*,* $j = 1, 2, \dots, k-1$.

The first few polynomials and the numerical values of the corresponding roots are

$$\begin{aligned}
&q_0(x) = 1,\\
&q_1(x) = 2(x+2), \{-2\},\\
&q_2(x) = 4(x^2+13x+16), \{-11.623, -1.377\},\\
&q_3(x) = 8(x^3+60x^2+297x+272), \{-54.657, -4.141, -1.202\},\\
&q_4(x) = 16(x^4+251x^3+3651x^2+10841x+7936),\\
&\qquad\quad \{-235.705, -11.59, -2.579, -1.126\},\\
&q_5(x) = 32(x^5+1018x^4+38158x^3+274418x^2+580013x+353792),\\
&\qquad\quad \{-979.322, -30.003, -5.615, -1.973, -1.087\},\\
r_0(x) &= 2,\\
r_1(x) &= 4(x+5), \{-5\},\\
r_2(x) &= 8(x^2+28x+61), \{-25.619, -2.381\},\\
r_3(x) &= 16(x^3+123x^2+1011x+1385), \{-114.258, -7.014, -1.728\},\\
r_4(x) &= 32(x^4+506x^3+11706x^2+50666x+50521),\\
&\quad \{-481.928, -18.784, -3.832, -1.457\},\\
r_5(x) &= 64(x^5+2041x^4+118546x^3+1212146x^2+3448901x+2702765),\\
&\quad \{-1981.48, -47.391, -8.116, -2.697, -1.315\}.
\end{aligned}$$

One sees that the polynomials have real roots; moreover, the roots have the interlacing property. However, we cannot prove this. Since the zeros of an orthogonal polynomial sequence have this property (Chihara 1978, Theorem I.5.3), it is tempting to think that the polynomial sequences above are orthogonal. However, it is easy to check that the recursion in Favard's Theorem (Chihara 1978, Theorem I.4.4) does not hold even for the first terms; therefore, neither of the two sequences of polynomials is orthogonal with any weight function.

For $\xi \in \mathbb{C}$, let us define $\eta(\xi) = 1 - \xi \pm \sqrt{\xi^2 - 2\xi}$, where the sign is chosen so that $|\eta(\xi)| < 1$. In the following, $(Z_n) \sim \mathrm{WN}(0, \Sigma)$ means that $(Z_n)_{n\in\mathbb{N}}$ is a sequence of uncorrelated random vectors, with mean zero, and covariance matrix Σ.

Now we can state the main result of the paper.

Theorem 1 *The moving average process* $X_n^\Delta = \Psi^\Delta(B) Z_n^\Delta$ *has the asymptotic form*

$$X_n^\Delta \sim (I_d - I_d B)^{p(d-1)+q} \prod_{j=1}^{p-q-1} (1 - \eta(\xi_{2(p-q)-1,j})B) Z_n^\Delta \quad \text{as } \Delta \downarrow 0,$$

where $(Z_n^\Delta) \sim \mathrm{WN}(0, \Sigma_Z^\Delta)$ *with*

$$\Sigma_Z^\Delta = \frac{\Delta^{2(p-q)-1}}{(2(p-q)-1)! \prod_{j=1}^{p-q-1} |\eta(\xi_{2(p-q)-1,j})|} B_0 \Sigma_L B_0^\top.$$

We note that $\eta(\xi_{2(p-q)-1,j})$ might be nonreal (although we conjecture that they are all real valued), in which case $\eta(\overline{\xi}_{2(p-q)-1,j})$ also appears in the product, which means that the coefficients in the moving average expansion are real.

It is interesting to observe that up to the first-order asymptotic the matrix moving average polynomial is in fact a scalar polynomial, and the covariance structure only appears in the covariance matrix Σ_Z^Δ of the white noise. Thus, Theorem 1 has the same form as the first-order version of Theorem 1 in Brockwell et al. (2013); see also Theorem 1 in Brockwell et al. (2012). Finally, we mention that the corresponding higher-order version of Theorem 1, the analogue of Theorem 1 in Brockwell et al. (2012), seems hopeless to prove. The proof breaks down on the factorization of the spectral density of the moving average process, since for matrix spectral density no factorization holds in general; compare Theorem 10 and 10' in Hannan (1970, Chapter II).

4 An example

Let us consider the simplest possible multivariate case. That is, $p = 1, q = 0, d = 2$. Then

$$P(z) = I_2 z + A_1, \;\; Q(z) = B_0, \quad A_1, B_0 \in M_2.$$

Moreover,

$$\mathrm{adj} P(z) Q(z) = z B_0 + \mathrm{adj} A_1 B_0,$$

that is in formula (6) $S_1 = B_0$, $S_0 = \mathrm{adj} A_1 B_0$. Furthermore, in (7) we have

$$\begin{aligned} R(z) &= \frac{1}{\det P(z) \det P(-z)} (z S_1 + S_0) \Sigma_L (-z S_1^\top + S_0^\top) \\ &= \frac{1}{\det P(z) \det P(-z)} \left(z^2 \widetilde{S}_2 + z \widetilde{S}_1 + \widetilde{S}_0 \right), \end{aligned} \tag{19}$$

with

$$\begin{aligned} &\widetilde{S}_2 = -B_0 \Sigma_L B_0^\top, \;\; \widetilde{S}_1 = B_0 \Sigma_L B_0^\top (\mathrm{adj} A_1)^\top - \mathrm{adj} A_1 B_0 \Sigma_L B_0^\top, \\ &\widetilde{S}_0 = \mathrm{adj} A_1 B_0 \Sigma_L B_0^\top (\mathrm{adj} A_1)^\top. \end{aligned} \tag{20}$$

Assume that the zeros of $\det P(z)$ are different. From (19), we can compute the matrices in the partial fraction decomposition. Standard calculation gives that the matrices in

Proposition 2 are

$$\alpha(1) = \frac{1}{2\lambda_1\left(\lambda_1^2 - \lambda_2^2\right)}\left(\widetilde{S}_2\lambda_1^2 + \widetilde{S}_1\lambda_1 + \widetilde{S}_0\right),$$
$$\alpha(2) = \frac{-1}{2\lambda_2\left(\lambda_1^2 - \lambda_2^2\right)}\left(\widetilde{S}_2\lambda_2^2 + \widetilde{S}_1\lambda_2 + \widetilde{S}_0\right).$$

Then using formula (16), lengthy but straightforward calculation gives

$$\begin{aligned} f_{\mathrm{MA}}^{\Delta}(\omega) =& \frac{2e^{\Delta(\lambda_1+\lambda_2)}}{2\pi(\lambda_1^2-\lambda_2^2)}\Bigg[\cos\omega\Bigg(\widetilde{S}_0\left(\frac{\sinh\lambda_1\Delta}{\lambda_1} - \frac{\sinh\lambda_2\Delta}{\lambda_2}\right) \\ &+ \widetilde{S}_2(\lambda_1\sinh\lambda_1\Delta - \lambda_2\sinh\lambda_2\Delta)\Bigg) \\ &+ \mathrm{i}\sin\omega\cdot\widetilde{S}_1(\cosh\lambda_1\Delta - \cosh\lambda_2\Delta) \\ &+ \widetilde{S}_0\left(\frac{\cosh\lambda_1\Delta\cdot\sinh\lambda_2\Delta}{\lambda_2} - \frac{\cosh\lambda_2\Delta\cdot\sinh\lambda_1\Delta}{\lambda_1}\right) \\ &+ \widetilde{S}_2(\lambda_2\cosh\lambda_1\Delta\cdot\sinh\lambda_2\Delta - \lambda_1\cosh\lambda_2\Delta\cdot\sinh\lambda_1\Delta)\Bigg]. \end{aligned}$$

The corresponding process is MA(1), and according to Theorem 10’ by Hannan (1970) there is a positive symmetric real matrix Ψ_0 and a real matrix Ψ_1, such that

$$f_{\mathrm{MA}}^{\Delta}(\omega) = \frac{1}{2\pi}(\Psi_0 + \Psi_1 e^{\mathrm{i}\omega})(\Psi_0 + \Psi_1^{\top} e^{-\mathrm{i}\omega}).$$

Short calculation gives that the first-order expansion is $X_n^{\Delta} \sim (I - B)Z_n^{\Delta}$, with covariance matrix $\Sigma_Z^{\Delta} = \Delta B_0 \Sigma_L B_0^{\top}$, as we have shown in Theorem 1. However, in general determining exactly the matrices Ψ_0, Ψ_1 is difficult.

We can also use Proposition 1. Combining (19) and (8), the Θ_k matrices can be calculated via the formula

$$\sum_{k=1}^{\infty}\Theta_k z^k = \frac{z^3}{\left(1-\lambda_1^2 z^2\right)\left(1-\lambda_2^2 z^2\right)}\left(z^{-2}\widetilde{S}_2 + z^{-1}\widetilde{S}_1 + \widetilde{S}_0\right).$$

Multiplying by $(1-\lambda_1^2 z^2)(1-\lambda_2^2 z^2)$ and equating the coefficients, we obtain

$$\begin{aligned} \widetilde{S}_2 &= \Theta_1 \\ \widetilde{S}_1 &= \Theta_2 \\ \widetilde{S}_0 &= \Theta_3 - (\lambda_1^2+\lambda_2^2)\Theta_1 \\ 0 &= \Theta_4 - (\lambda_1^2+\lambda_2^2)\Theta_2 \\ 0 &= \Theta_k - (\lambda_1^2+\lambda_2^2)\Theta_{k-2} + \lambda_1^2\lambda_2^2\Theta_{k-4},\ k \geq 5. \end{aligned} \tag{21}$$

We note that also in the general case there exists a (more complicated) linear recursion for the Θ_k matrices. Since $f_{\mathrm{MA}}^{\Delta}(\omega) = f_{\Delta}(\omega)\phi^{\Delta}(\omega)$, expanding $\cosh \Delta\lambda_i$ in Taylor series and combining with Proposition 1, we obtain

$$f_{\mathrm{MA}}^{\Delta}(\omega) = -\frac{4e^{\Delta(\lambda_1+\lambda_2)}}{2\pi}\sum_{k=1}^{\infty}\Delta^k C_k(\omega),$$

where $C_k(\omega)$ are trigonometric polynomials. Using the first few values of the coefficient functions $\widetilde{c}_k(\omega)$ and (21)

$$\begin{aligned}
C_1(\omega) &= (1-\cos\omega)\frac{\Theta_1}{2}\\
C_2(\omega) &= -\mathrm{i}\sin\omega\frac{\Theta_2}{4}\\
C_3(\omega) &= \frac{1}{4}\left(\Theta_1\left(\lambda_1^2+\lambda_2^2\right)-\Theta_3\frac{2+\cos\omega}{3}\right)\\
C_4(\omega) &= -\frac{\mathrm{i}\sin\omega}{48}\Theta_2\left(\lambda_1^2+\lambda_2^2\right).
\end{aligned}$$

Thus, we may obtain a higher-order expansion of the spectral density, e.g.,

$$\begin{aligned}
f_{\mathrm{MA}}^{\Delta}(\omega) = \frac{\Delta}{2\pi}\Bigg(&-2\widetilde{S}_2(1-\cos\omega)+\Delta\left[-2(\lambda_1+\lambda_2)\widetilde{S}_2(1-\cos\omega)+\mathrm{i}\sin\omega\,\widetilde{S}_1\right]\\
&-\Delta^2\Bigg((1-\cos\omega)\left[\frac{\widetilde{S}_0+\widetilde{S}_2(\lambda_1^2+\lambda_2^2)}{3}+\widetilde{S}_2(\lambda_1+\lambda_2)^2\right]\\
&-\widetilde{S}_0-\mathrm{i}\sin\omega\,\widetilde{S}_1(\lambda_1+\lambda_2)\Bigg)+O(\Delta^3)\Bigg),
\end{aligned}$$

from which the statement of Theorem 1 again follows. However, it is not clear how to obtain higher-order expansions for the process itself.

5 Proofs

Proof of Proposition 1 Let $\Gamma(t)$ denote the covariance matrix, i.e., $\Gamma(t) = \mathbf{E}Y_0Y_t^{\top}$. Then

$$\Gamma(t) = \int_{\mathbb{R}} e^{\mathrm{i}t\lambda} f_Y(\lambda)\mathrm{d}\lambda, \quad t\in\mathbb{R},$$

and applying Cauchy's theorem componentwise we have for $t > 0$

$$\Gamma(t) = \frac{1}{2\pi\mathrm{i}}\int_{\rho} e^{tz}R(z)\mathrm{d}z,$$

where $\rho \subset (-\infty, 0) \times \mathrm{i}\mathbb{R}$ is a closed curve, which encircles the zeros of $\det P(z)$. Since Γ is a covariance matrix, $\Gamma(-t) = \Gamma(t)^\top$. It is clear that the autocovariance function of the discrete process $(Y_{n\Delta})_{n\in\mathbb{N}}$ is $\Gamma(\Delta n)$, so by the inversion formula for discrete processes the spectral density can be calculated as

$$\begin{aligned}
f_\Delta(\omega) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-\mathrm{i}k\omega} \Gamma(\Delta k) \\
&= \frac{1}{2\pi} \left[\sum_{k=-\infty}^{0} e^{-\mathrm{i}k\omega} \Gamma(-\Delta k)^\top + \sum_{k=1}^{\infty} e^{-\mathrm{i}k\omega} \Gamma(\Delta k) \right] \\
&= \frac{1}{4\pi^2 \mathrm{i}} \left[\int_\rho \sum_{k=0}^{\infty} e^{k(\Delta z + \mathrm{i}\omega)} R(z)^\top \mathrm{d}z + \int_\rho \sum_{k=1}^{\infty} e^{k(\Delta z - \mathrm{i}\omega)} R(z) \mathrm{d}z \right] \\
&= \frac{1}{4\pi^2 \mathrm{i}} \left[\int_\rho \frac{1}{1 - e^{\Delta z + \mathrm{i}\omega}} R(z)^\top \mathrm{d}z + \int_\rho \frac{e^{\Delta z - \mathrm{i}\omega}}{1 - e^{\Delta z - \mathrm{i}\omega}} R(z) \mathrm{d}z \right], \qquad (22)
\end{aligned}$$

where $\omega \in (-\pi, \pi)$; the change of the sum and integration is justified, since $\Re z < 0$ on the curve ρ.

For any $\omega \in (-\pi, \pi) \setminus \{0\}$, consider the Taylor series expansions

$$\frac{1}{1 - e^{z + \mathrm{i}\omega}} = \sum_{k=0}^{\infty} \widetilde{c}_k(\omega) z^k, \text{ and } \frac{e^{z - \mathrm{i}\omega}}{1 - e^{z - \mathrm{i}\omega}} = \sum_{k=0}^{\infty} \widetilde{d}_k(\omega) z^k,$$

which converge in the open disk $|z| < |\omega|$. Note that both functions have a simple pole at 0 if $\omega = 0$. Adding the two expressions

$$\frac{1}{1 - e^{z + \mathrm{i}\omega}} + \frac{e^{z - \mathrm{i}\omega}}{1 - e^{z - \mathrm{i}\omega}} = -\frac{e^z - e^{-z}}{e^z + e^{-z} - 2\cos\omega} = -\frac{\sinh z}{\cosh z - \cos\omega},$$

which is an odd function of z; therefore, in the series expansion

$$\widetilde{c}_{2k}(\omega) + \widetilde{d}_{2k}(\omega) = 0.$$

On the other hand,

$$\frac{e^{z - \mathrm{i}\omega}}{1 - e^{z - \mathrm{i}\omega}} - \frac{1}{1 - e^{z + \mathrm{i}\omega}} = -\frac{e^z + e^{-z} - 2e^{-\mathrm{i}\omega}}{e^z + e^{-z} - 2\cos\omega} = -1 - \mathrm{i}\frac{\sin\omega}{\cosh z - \cos\omega}$$

is an even function of z; therefore,

$$\widetilde{d}_{2k+1}(\omega) - \widetilde{c}_{2k+1}(\omega) = 0.$$

Summarizing, we obtain

$$\widetilde{d}_k(\omega) = (-1)^{k+1} \widetilde{c}_k(\omega). \qquad (23)$$

For the coefficient $\widetilde{c}_k(\omega)$ we have

$$-\frac{1}{2}\frac{\sinh z}{\cosh z-\cos\omega}=\sum_{k=0}^{\infty}\widetilde{c}_{2k+1}(\omega)z^{2k+1}, \tag{24}$$

and

$$\frac{1}{2}+\frac{\mathrm{i}}{2}\frac{\sin\omega}{\cosh z-\cos\omega}=\sum_{k=0}^{\infty}\widetilde{c}_{2k}(\omega)z^{2k},$$

so for $k\geq 1$ the coefficient $\widetilde{c}_{2k}(\omega)$ is purely imaginary. From (24), using the notation of Brockwell et al. (2012) formula (18) we see that $\widetilde{c}_{2k+1}(\omega)=-c_k(\omega)/2$.

Inserting the series expansion into (22) (here we use that $\omega\neq 0$ is fixed, and Δ is small enough to make $|\Delta z|<|\omega|$ for each $z\in\rho$) and using that $R(z)^\top=R(-z)$, we obtain

$$f_\Delta(\omega)=\frac{1}{4\pi^2\mathrm{i}}\sum_{k=0}^{\infty}\Delta^k\left(\widetilde{c}_k(\omega)\int_\rho z^kR(-z)\mathrm{d}z+\widetilde{d}_k(\omega)\int_\rho z^kR(z)\mathrm{d}z\right). \tag{25}$$

Then, changing the variables and using (23) and (5) we have

$$\begin{aligned}
&\widetilde{c}_k(\omega)\int_\rho z^kR(-z)\mathrm{d}z+\widetilde{d}_k(\omega)\int_\rho z^kR(z)\mathrm{d}z\\
&=(-1)^{k+1}\widetilde{c}_k(\omega)\int_{-\rho}z^kR(z)\mathrm{d}z+(-1)^{k+1}\widetilde{c}_k(\omega)\int_\rho z^kR(z)\mathrm{d}z\\
&=(-1)^{k+1}\widetilde{c}_k(\omega)\int_{\rho\cup-\rho}z^kR(z)\mathrm{d}z\\
&=(-1)^{k+1}\widetilde{c}_k(\omega)2\pi\mathrm{i}\Theta_k.
\end{aligned} \tag{26}$$

Substituting into (25)

$$f_\Delta(\omega)=\frac{-1}{2\pi}\sum_{k=0}^{\infty}(-\Delta)^k\Theta_k\widetilde{c}_k(\omega).$$

Taking into account that $\Theta_k=0$ for $k\leq 2(p-q)-2$, the proof is ready. □

Proof of Proposition 2 We have

$$R(z)=\frac{\operatorname{adj}P(z)Q(z)\Sigma_L(\operatorname{adj}P(-z)Q(-z))^\top}{\det P(z)\det P(-z)}=\frac{S(z)}{\det P(z)\det P(-z)}.$$

Since the degree of the numerator is less than that of the denominator, the partial fraction decomposition (11) holds for some matrices $\alpha(\ell),\beta(\ell)\in M_d$. Note that

$S(-z)^\top = S(z)$ implies $\beta(\ell)^\top = \alpha(\ell)$. By simple properties of the residue, the second summand in (22) is

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_\rho \frac{e^{\Delta z-\mathrm{i}\omega}}{1-e^{\Delta z-\mathrm{i}\omega}}R(z)\mathrm{d}z &= \sum_{\ell=1}^{pd}\frac{1}{2\pi\mathrm{i}}\int_\rho \frac{e^{\Delta z-\mathrm{i}\omega}}{1-e^{\Delta z-\mathrm{i}\omega}}\left(\frac{\alpha(\ell)}{z-\lambda_\ell}+\frac{\beta(\ell)}{-z-\lambda_\ell}\right)\mathrm{d}z\\ &= \sum_{\ell=1}^{pd}\frac{e^{\Delta\lambda_\ell-\mathrm{i}\omega}}{1-e^{\Delta\lambda_\ell-\mathrm{i}\omega}}\alpha(\ell).\end{aligned}$$

In the same way,

$$\frac{1}{2\pi\mathrm{i}}\int_\rho \frac{R(z)^\top}{1-e^{\Delta z+\mathrm{i}\omega}}\mathrm{d}z = \sum_{\ell=1}^{pd}\frac{1}{1-e^{\Delta\lambda_\ell+\mathrm{i}\omega}}\alpha(\ell)^\top.$$

Therefore, by (22)

$$f_\Delta(\omega) = \frac{1}{2\pi}\sum_{\ell=1}^{pd}\left(\frac{e^{\Delta\lambda_\ell-\mathrm{i}\omega}}{1-e^{\Delta\lambda_\ell-\mathrm{i}\omega}}\alpha(\ell)+\frac{1}{1-e^{\Delta\lambda_\ell+\mathrm{i}\omega}}\alpha(\ell)^\top\right),$$

from which simple manipulation gives (10). □

Proof of Proposition 3 In the general case, the partial fraction decomposition of $R(z)$ reads as (13), with some matrices $\alpha(\ell, j), \beta(\ell, j) \in M_d$. Similarly, as in the previous case we obtain

$$\frac{1}{2\pi\mathrm{i}}\int_\rho \frac{e^{\Delta z-\mathrm{i}\omega}}{1-e^{\Delta z-\mathrm{i}\omega}}R(z)\mathrm{d}z = \sum_{\ell=1}^{m}\sum_{j=1}^{\nu_\ell}\left(D^{(j-1)}\frac{e^{\Delta z-\mathrm{i}\omega}}{1-e^{\Delta z-\mathrm{i}\omega}}\right)_{z=\lambda_\ell}\frac{\alpha(\ell, j)}{(j-1)!},$$

where D stands for differentiation. Noting that

$$\frac{e^{\Delta z-\mathrm{i}\omega}}{1-e^{\Delta z-\mathrm{i}\omega}} = -1+\frac{1}{1-e^{\Delta z-\mathrm{i}\omega}},$$

one can show that for $j \geq 2$

$$\frac{1}{(j-1)!}\left(D^{(j-1)}\frac{1}{1-e^{\Delta z-\mathrm{i}\omega}}\right)_{z=\lambda_\ell} = \frac{s_j^\Delta(\omega, \lambda_\ell)}{(\cosh\Delta\lambda_\ell-\cos\omega)^j},$$

with $s_j^\Delta(\omega, \lambda_\ell)$ being a trigonometric polynomial in ω of degree $j-1$, whose coefficients depend on Δ and λ_ℓ. Similarly, for $j \geq 2$

$$\frac{1}{(j-1)!}\left(D^{(j-1)}\frac{1}{1-e^{\Delta z+\mathrm{i}\omega}}\right)_{z=\lambda_\ell} = \frac{s_j^\Delta(-\omega, \lambda_\ell)}{(\cosh\Delta\lambda_\ell-\cos\omega)^j}.$$

Substituting back into (22), we obtain (12). □

Proof of Lemma 1 Recall definition $\tilde{c}_k$ from (9). To ease the notation put $h(z) = 1/(1 - e^{z+\mathrm{i}\omega})$, and $y = e^{z+\mathrm{i}\omega}$. Then, for the first few derivatives (all the derivatives are meant with respect to z)

$$h'(z) = \frac{y}{(1-y)^2}, \ h''(z) = \frac{y^2 + y}{(1-y)^3}, \ h'''(z) = \frac{y^3 + 4y^2 + y}{(1-y)^4}.$$

In general, using induction it is easy to see that

$$h^{(n)}(z) = \frac{yA_n(y)}{(1-y)^{n+1}}, \ n = 1, 2, \ldots,$$

where A_n is a polynomial of degree $n - 1$, for which the recursion

$$A_{n+1}(y) = (y - y^2)A_n'(y) + A_n(y)(ny + 1) \tag{27}$$

holds. These polynomials are the *Eulerian polynomials* (do not confuse with Euler polynomials and Euler numbers), and the coefficients are the Eulerian numbers, i.e., $A_n(y) = A(n, n-1)y^{n-1} + A(n, n-2)y^{n-2} + \cdots + A(n, 0)$. The combinatorial interpretation of the Eulerian numbers is that $A(n, k)$ is the number of permutations of $\{1, 2, \ldots, n\}$ in which exactly k elements are greater than the previous element. From (27), we obtain

$$A(n+1, k) = (k+1)A(n, k) + (n+1-k)A(n, k-1).$$

Induction gives that $A(n, n-1) = A(n, 0) = 1$, and

$$A(n, k) = A(n, n-1-k), \ k = 0, 1, \ldots, n-1, \tag{28}$$

that is A_n is a symmetric polynomial.

Since $(1 - e^{\mathrm{i}\omega})(1 - e^{-\mathrm{i}\omega}) = 2(1 - \cos\omega)$, from (9) we have

$$n!\,\widetilde{c}_n(\omega) = h^{(n)}(0) = \frac{e^{\mathrm{i}\omega}A_n(e^{\mathrm{i}\omega})}{(1 - e^{\mathrm{i}\omega})^{n+1}} = \frac{e^{\mathrm{i}\omega}A_n(e^{\mathrm{i}\omega})(1 - e^{-\mathrm{i}\omega})^{n+1}}{[2(1 - \cos\omega)]^{n+1}}.$$

For odd indices, $n = 2k - 1$, $k = 1, 2, \ldots$, using (28)

$$\begin{aligned} A_{2k-1}(e^{\mathrm{i}\omega}) &= A(2k-1, 0)e^{(2k-2)\mathrm{i}\omega} + A(2k-1, 1)e^{(2k-3)\mathrm{i}\omega} + \cdots \\ &\quad + A(2k-1, 1)e^{\mathrm{i}\omega} + A(2k-1, 0) \\ &= 2e^{(k-1)\mathrm{i}\omega}\big[A(2k-1, 0)\cos(k-1)\omega + A(2k-1, 1)\cos(k-2)\omega \\ &\qquad + \cdots + 2^{-1}A(2k-1, k-1)\big]. \end{aligned}$$

The second factor above is a polynomial of $\cos\omega$ of degree $k-1$, while for the first factor $e^{\mathrm{i}\omega}(1-e^{-\mathrm{i}\omega})^2 = -2(1-\cos\omega)$; therefore,

$$
\begin{aligned}
(2k-1)!\,\widetilde{c}_{2k-1}(\omega) &= \frac{e^{\mathrm{i}\omega} A_{2k-1}(e^{\mathrm{i}\omega})(1-e^{-\mathrm{i}\omega})^{2k}}{[2(1-\cos\omega)]^{2k}} \\
&= \frac{2(-1)^k}{[2(1-\cos\omega)]^k}\big[A(2k-1,0)\cos(k-1)\omega \\
&\quad + A(2k-1,1)\cos(k-2)\omega + \cdots + 2^{-1}A(2k-1,k-1)\big].
\end{aligned}
\tag{29}
$$

For $n = 2k$, $k = 1, 2, \ldots$, (28) implies $A_{2k}(-1) = 0$, so $(1+y)$ divides $A_{2k}(y)$, i.e., $A_{2k}(y) = (1+y)\widetilde{A}_{2k-1}(y)$, where

$$
\begin{aligned}
\widetilde{A}_{2k-1}(y) = {} & \widetilde{A}(2k-1,0)y^{2k-2} + \widetilde{A}(2k-1,1)y^{2k-3} + \cdots + \widetilde{A}(2k-1,1)y \\
& + \widetilde{A}(2k-1,0)
\end{aligned}
$$

is again a symmetric polynomial of degree $2k-2$. Thus, using the calculation above, and that $(1+e^{\mathrm{i}\omega})(1-e^{-\mathrm{i}\omega}) = 2\mathrm{i}\sin\omega$ we obtain

$$
\begin{aligned}
(2k)!\,\widetilde{c}_{2k}(\omega) &= \frac{e^{\mathrm{i}\omega} A_{2k}(e^{\mathrm{i}\omega})(1-e^{-\mathrm{i}\omega})^{2k+1}}{[2(1-\cos\omega)]^{2k+1}} \\
&= \frac{4(-1)^k \mathrm{i}\sin\omega}{[2(1-\cos\omega)]^{k+1}}\big[\widetilde{A}(2k-1,0)\cos(k-1)\omega \\
&\quad + \widetilde{A}(2k-1,1)\cos(k-2)\omega + \ldots + 2^{-1}\widetilde{A}(2k-1,k-1)\big].
\end{aligned}
\tag{30}
$$

Apart from the constant factors, the statement is proved.

Expressing $\cos n\omega$ as a polynomial of $\cos\omega$ serves as a definition of the Chebishev polynomials T_n, i.e.,

$$
\cos n\omega = T_n(\cos\omega) = \frac{n}{2}\sum_{k=0}^{[n/2]}(-1)^k \frac{(n-k-1)!}{k!(n-2k)!}(2\cos\omega)^{n-2k}.
$$

From this, we see that the coefficient of $(\cos\omega)^n$ equals 2^{n-1}. Thus, the coefficient of $(\cos\omega)^{k-1}$ on the right-hand side of (29) is $(-1)^k 2^{k-1}$, from which we obtain the value of the leading coefficient. After noting that $\widetilde{A}(2k-1,0) = A(2k,0) = 1$, the value of the leading coefficients follows similarly in the even case. Finally, from (27) we see that $A_n(1) = n!$, from which the formula for the product of the roots follows.

Thus, we have shown that the polynomials q_{k-1}, r_{k-1} defined via

$$
\begin{aligned}
q_{k-1}(\cos\omega) &= (-1)^k(2k-1)![2(1-\cos\omega)]^k\widetilde{c}_{2k-1}(\omega), \\
\mathrm{i}\sin\omega\, r_{k-1}(\cos\omega) &= (-1)^k(2k)![2(1-\cos\omega)]^{k+1}\widetilde{c}_{2k}(\omega),
\end{aligned}
$$

have the stated properties. From (29), we see that q_{k-1} and r_{k-1} are linear combinations of Chebishev polynomials

$$q_{k-1}(x) = 2\big[A(2k-1,0)T_{k-1}(x) + A(2k-1,1)T_{k-2}(x) + \cdots + 2^{-1}A(2k-1,k-1)\big],$$
$$r_{k-1}(x) = 4\big[\widetilde{A}(2k-1,0)T_{k-1}(x) + \widetilde{A}(2k-1,1)T_{k-2}(x) + \cdots + 2^{-1}\widetilde{A}(2k-1,k-1)\big].$$

Now we turn to the statement about the roots. Indirectly assume that q_{k-1} has a real root in $(-1,1)$, say $\cos\omega_0$. Then from (29), we see that $A_{2k-1}(e^{\mathrm{i}\omega_0}) = 0$. This is a contradiction, since Frobenius showed in 1910 that the roots of the Eulerian polynomials are real [for a recent work on roots of generalized Eulerian polynomials see Savage and Visontai (2015)]. Similar reasoning shows that $r_{k-1}(x)$ has no real root in $(-1,1)$. □

Proof of Theorem 1 The proof relies on analyzing the spectral density $f^{\Delta}_{\mathrm{MA}}(\omega)$ of the process X^{Δ}_n.

As

$$\cosh\Delta\lambda_j - \cos\omega = 1 - \cos\omega + \sum_{\ell=1}^{\infty}\frac{(\Delta\lambda_j)^{2\ell}}{(2\ell)!},$$

using Corollary 1 and (15) the asymptotics of the spectral density of the moving average process $\Phi^{\Delta}(B)Y^{\Delta}_n$ is

$$f^{\Delta}_{\mathrm{MA}}(\omega) = \frac{-1}{2\pi}2^{pd}e^{\Delta\sum_{j=1}^{pd}\lambda_j}\prod_{j=1}^{pd}(\cosh\Delta\lambda_j - \cos\omega)\sum_{k=0}^{\infty}(-\Delta)^k\Theta_k\widetilde{c}_k(\omega)$$
$$\sim \frac{\Delta^{2(p-q)-1}}{2\pi}2^{pd}(1-\cos\omega)^{pd}\widetilde{c}_{2(p-q)-1}(\omega)\Theta_{2(p-q)-1} \tag{31}$$

as $\Delta \downarrow 0$. Write

$$f^{\Delta}_{\mathrm{MA}}(\omega) \sim \frac{\Delta^{2(p-q)-1}}{2\pi}[2(1-\cos\omega)]^{pd-(p-q)} \times$$
$$\times[2(1-\cos\omega)]^{p-q}\widetilde{c}_{2(p-q)-1}(\omega)\Theta_{2(p-q)-1}. \tag{32}$$

In (32), the factor $[2(1-\cos\omega)]^{p(d-1)+q}$ corresponds to $(I_d - I_d B)^{p(d-1)+q}$ in the moving average representation.

For the factorization of the remaining terms, we need that

$$(1-\eta e^{\mathrm{i}\omega})(1-\eta e^{-\mathrm{i}\omega}) = 2\eta\left(1 - \cos\omega + \frac{(1-\eta)^2}{2\eta}\right);$$

thus, solving the equation $-\xi = (1-\eta)^2/(2\eta)$ we have for the solution

$$\eta(\xi) := 1 - \xi \pm \sqrt{\xi^2 - 2\xi},$$

where the sign is chosen so that $|\eta(\xi)| < 1$. This is possible, since the product of the two roots is 1. That is,

$$1 - \cos\omega - \xi = \frac{1}{2\eta(\xi)}(1 - \eta(\xi)e^{\mathrm{i}\omega})(1 - \eta(\xi)e^{-\mathrm{i}\omega}).$$

Therefore, combining the above with Lemma 1 we obtain

$$[2(1-\cos\omega)]^{p-q}\widetilde{c}_{2(p-q)-1}(\omega) = -2^{p-q-1}\frac{\prod_{j=1}^{p-q-1}(1-\cos\omega - \xi_{2(p-q)-1,j})}{(2(p-q)-1)!}$$
$$= -\frac{\prod_{j=1}^{p-q-1}(1-\eta(\xi_{2(p-q)-1,j})e^{\mathrm{i}\omega})(1-\eta(\xi_{2(p-q)-1,j})e^{-\mathrm{i}\omega})}{(2(p-q)-1)!\prod_{j=1}^{p-q-1}\eta(\xi_{2(p-q)-1,j})}.$$

We conjecture that the zeros $\xi_{2k-1,j}$ are all real and greater than 2. This is true for $k = 1, 2, \ldots, 8$; however, we cannot prove it in general. For real zeros, the η's are also real (we did prove that $\xi_{2k-1,j} \notin (0,2)$); thus, in the factorization everything is real. When there is a nonreal root ξ then necessarily its conjugate $\overline{\xi}$ is also a root, and one can check easily that $\eta(\overline{\xi}) = \overline{\eta(\xi)}$; therefore, in the factorization the coefficients are real. □

Acknowledgements I am grateful to Claudia Klüppelberg and to Peter Brockwell for inspiring conversations on the subject and for comments on the manuscript. I also thank Béla Nagy for discussions on the polynomials appearing in Lemma 1.

References

Andresen, A., Benth, F. E., Koekebakker, S., Zakamulin, V. (2014). The CARMA interest rate model. *International Journal of Theoretical and Applied Finance*, *17*(2), 1450008.

Benth, F. E., Klüppelberg, C., Müller, G., Vos, L. (2014). Futures pricing in electricity markets based on stable CARMA spot models. *Energy Economics*, *44*, 392–406.

Bergstrom, A. R. (1988). The history of continuous-time econometric models. *Econometric Theory*, *4*(3), 365–383.

Bertoin, J. (1996). *Lévy processes, Cambridge Tracts in Mathematics* (Vol. 121). Cambridge: Cambridge University Press.

Brockwell, P. J. (2001a). Continuous-time ARMA processes. *Stochastic processes: theory and methods, Handbook of Statistics* (Vol. 19, pp. 249–276). Amsterdam: North-Holland.

Brockwell, P. J. (2001b). Lévy-driven CARMA processes. *Annals of the Institute of Statistical Mathematics*, *53*(1), 113–124.

Brockwell, P. J., Davis, R. A. (1987). *Time series: Theory and methods. Springer series in statistics*. New York: Springer.

Brockwell, P., Lindner, A. (2012). Lévy-driven time series models for financial data. *Time series analysis: Methods and applications, handbook of statistics* (Vol. 30, pp. 543–563). Amsterdam: Elsevier.

Brockwell, P. J., Schlemm, E. (2013). Parametric estimation of the driving Lévy process of multivariate CARMA processes from discrete observations. *Journal of Multivariate Analysis*, *115*, 217–251.

Brockwell, P. J., Ferrazzano, V., Klüppelberg, C. (2012). High-frequency sampling of a continuous-time ARMA process. *Journal of Time Series Analysis*, *33*(1), 152–160.

Brockwell, P. J., Ferrazzano, V., Klüppelberg, C. (2013). High-frequency sampling and kernel estimation for continuous-time moving average processes. *Journal of Time Series Analysis*, *34*(3), 385–404.

Chihara, T. S. (1978). *An introduction to orthogonal polynomials, Mathematics and its applications* (Vol. 13). New York: Gordon and Breach Science Publishers.

Doob, J. L. (1944). The elementary Gaussian processes. *Annals of Mathematical Statistics*, *15*, 229–282.

Fasen, V. (2014). Limit theory for high frequency sampled MCARMA models. *Advances in Applied Probability*, *46*(3), 846–877.

Fasen, V., Kimmig, S. (2017). Information criteria for multivariate CARMA processes. *Bernoulli* (to appear). arXiv:1505.00901.

Ferrazzano, V., Fuchs, F. (2013). Noise recovery for Lévy-driven CARMA processes and high-frequency behaviour of approximating Riemann sums. *Electronic Journal of Statistics*, *7*, 533–561.

Hannan, E. J. (1970). *Multiple time series*. New York: Wiley.

Marquardt, T., Stelzer, R. (2007). Multivariate CARMA processes. *Stochastic Processes and their Applications*, *117*(1), 96–120.

Phillips, A. W. (1959). The estimation of parameters in systems of stochastic differential equations. *Biometrika*, *46*(1/2), 67–76.

Robinson, P. M. (1976). Fourier estimation of continuous time models. In A. R. Bergstrom (Ed.), *Statistical inference in continuous time economic models* (pp. 215–266). Amsterdam: North-Holland.

Robinson, P. M. (1977). The construction and estimation of continuous time models and discrete approximations in econometrics. *Journal of Econometrics*, *6*(2), 173–197.

Robinson, P. M. (1993). Continuous-time models in econometrics: Closed and open systems, stocks and flows. In Phillips, P.C.B. (ed.) Models, methods, and applications of econometrics—Essays in honor of A.R. Bergstrom (pp. 71–90). Cambridge, Mass: Blackwell.

Savage, C. D., Visontai, M. (2015). The s-Eulerian polynomials have only real roots. *Transactions of the American Mathematical Society*, *367*(2), 1441–1466.

Schlemm, E., Stelzer, R. (2012). Multivariate CARMA processes, continuous-time state space models and complete regularity of the innovations of the sampled processes. *Bernoulli*, *18*(1), 46–63.

Todorov, V., Tauchen, G. (2006). Simulation methods for Lévy-driven continuous-time autoregressive moving average (CARMA) stochastic volatility models. *Journal of Business & Economic Statistics*, *24*(4), 455–469.