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Reflected Brownian motions in the KPZ universality class

Thomas Weiß

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Vorsitzende:		Univ.-Prof. Dr. Caroline Lasser
Prüfer der Dissertation:	1.	Univ.-Prof. Dr. Dr. h.c. Herbert Spohn
	2.	Univ.-Prof. Dr. Nina Gantert
	3.	Prof. Dr. Maurice Duits KTH Royal Institute of Technology, Schweden

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Abstract

In this thesis we study a system of totally asymmetrically reflected Brownian motions under various initial conditions. Totally asymmetric reflection means that each particle is reflected from its left neighbor, which is a special case of oblique reflection. This system can be described as a signed determinantal point process giving exact formulas for the fixed time joint distribution. Steepest descent analysis of these formulas allows to obtain the asymptotic behaviour for large times. As expected from the asymmetry in the reflection, we find the characteristics of the KPZ universality class, i.e. the model shows the same large scale properties as the KPZ equation. Among these properties are a magnitude of fluctuations of order $t^{1/3}$, a spatial correlation scale of order $t^{2/3}$ as well as limit objects given by Tracy-Widom distributions and Airy processes.

The KPZ equation has six distinguished initial profiles, each of one showing a different type of limiting process. This property is shared by our model of reflected Brownian motions. This thesis covers all of the distinguished initial conditions, making this model the second one after the TASEP that has been studied in this thoroughness. Although the underlying methods are the same, there are different non-trivial issues to overcome for each case.

The first and simplest case is the so-called step initial configuration, where all particles start at the same point. Under the characteristic KPZ scaling, this process converges to the Airy_2 process. This case also introduces some of the tools needed for the more complicated cases.

Secondly we study the periodic initial configuration, that is, particles are located at the integer lattice at time zero. Interestingly and somewhat unexpectedly, the Lambert W function appears in the formula for the fixed time distribution. In the appropriate large time scaling limit, the fluctuations in the particle positions are described by the Airy_1 process.

As third and most difficult case, we consider initial configurations distributed according to a Poisson point process with constant intensity, which makes the process space-time stationary. We prove convergence to the

$\text{Airy}_{\text{stat}}$ process. As a byproduct we obtain a novel representation of the finite-dimensional distributions of this process. Our method differs from the one used for the TASEP and the KPZ equation by removing the initial step only after the limit $t \to \infty$. This leads to a new universal cross-over process.

The remaining three geometries are mixtures of the first three and their limit processes are correspondingly crossover processes between the fundamental Airy processes. The system with half-periodic initial configuration, where particles occupy $\mathbb{Z}_{>0}$ at time zero, converges to $\text{Airy}_{2\to 1}$, similarly half-Poisson initial configuration, i.e. particles given by a Poisson process on $\mathbb{R}_{>0}$ at time zero, leads to the limit process $\text{Airy}_{2\to\text{BM}}$. Finally, the Poisson-periodic case, where we have particles on $\mathbb{Z}_{>0}$ as well as particles given by a Poisson process on $\mathbb{R}_{<0}$, shows the limit process $\text{Airy}_{\text{BM}\to 1}$. Although some of the results from the pure cases can be reused, the technical details of the analysis become considerably more involved for the mixed geometries.

Contents

Chapter 1

Introduction

> *If you hear a "prominent" economist using the word 'equilibrium', or 'normal distribution', do not argue with him; just ignore him, or try to put a rat down his shirt.*
>
> – Nassim Nicholas Taleb, *The Black Swan* (2007)

One might wonder why a mathematics dissertation starts with an economics quote. In fact, during my time as a PhD student, I spent quite an amount of thoughts on economics and how it relates to the modern financial capitalism we live in. On many occasions I found myself wishing that economists would share not only the very sharpness and thoroughness in their arguments, but also the respectfulness and sobriety I got used to in and loved about the mathematics community. Although it is debatable that rats are the best solution to this problem, I fully agree with Taleb that mainstream economic theory stretches the applicability of equilibrium and the normal distribution way to far. It was a great achievement by the giants of probability theory to study the Gaussian universality until our understanding reached the level it has today. However, it also can seduce the scientist who is not careful enough to assume this behaviour without sufficiently looking at the underlying presumptions. The KPZ universality class provides a perfect example that non-equilibrium systems often behave fundamentally different and cannot simply be treated as a small perturbation of an equilibrium system, as neoclassical economists like to do. Taleb is aware of this, he specifically studies rare events, called *Black swan events*, that have a high impact on the course of history, science, economics or technology. As an example, it has been empirically known for decades, that the distribution of asset prices follows distributions with heavy tails [Man63], i.e. especially heavy downturns of the economy are far more likely than a

Gaussian model would predict. Yet up to today the financial industry still relies on the Black-Scholes model to a great extent.

The educated mathematician of course recognizes that the Gaussian universality does not cover the whole spectrum of probability theory. One of such counterexamples is classical extreme value statistics, where Weibull, Gumbel and Fréchet distributions appear as the limit. Over the past years however there have been found models which are not described by any of these classes, but instead belong to a new one, which is called the KPZ universality class, after the names of their discoverers Kardar, Parisi and Zhang [KPZ86].

Let us focus on a growth model for now: We consider a thin film that can be regarded as a two-dimensional object. It can occur as two phases A and B, and the one-dimensional interface between them is described by a height function $h(x,t)$. Assuming that both phases are equally stable, the rate of A turning into B is equal to the rate of B turning into A, which means that there is no net motion of the interface. $h(x,t)$ can then be described by the Edwards-Wilkinson equation [EW82],

$$\frac{\partial}{\partial t}h = \frac{1}{2}\partial_x^2 h + \mathcal{W}, \tag{1.0.1}$$

where the Laplacian is a smoothing term coming from the surface tension, and $\mathcal{W}(x,t)$ denotes space-time white noise, with Gaussian distribution and the delta distribution as covariance function, modeling a random influence. It is known that for large t, the order of fluctuations of $h(x,t)$ is $t^{1/4}$ and nontrivial correlations occur on a spatial scale of $t^{1/2}$, with the limiting distribution being Gaussian. Models with microscopically different, but structurally similar behaviour are part of the *Edwards-Wilkinson universality class* and show the same asymptotic properties.

These observations change significantly when we are turning towards a non-equilibrium system, i.e. let the phase A be stable, but the phase B be metastable. In this case the interface will show a net motion, reflecting the growth of the phase A that occupies the whole film over time. This can be integrated into (1.0.1) by introducing a slope-dependent growth term $F(\partial_x h)$. In order for the growth speed to be rotationally invariant the choice $F(x) = \sqrt{1+x^2}$ would be straightforward, however, let us keep only the first non-vanishing term in the Taylor expansion $F(x) = x^2/2$ (up to the constant part, that gives just a trivial linear shift). The result is the famous KPZ equation [KPZ86] for the height function $h(x,t)$:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left(\partial_x h\right)^2 + \mathcal{W}. \tag{1.0.2}$$

For large t the fluctuations of the height function are now of the larger order $t^{1/3}$ and also the spatial correlation length is different, as $t^{2/3}$. The limiting distributions are non-Gaussian, and depend on the initial profile now. Meanwhile, the mathematical predictions have been confirmed experimentally, the most spectacular evidence being the interface of turbulent liquid crystals [TS12], where not only the correct scaling exponents but also distributions and correlation functions have been found. Other examples include bacterial colony growth, facet boundaries, paper wetting, paper burning and more [WIMM97, DSC$^+$06, KHO96, MMMT05].

There are numerous probabilistic models, which are viewed as members of the KPZ universality class: It contains particle systems, like the asymmetric simple exclusion process with its generalizations q-TASEP, q-Hahn TASEP, and more, or some systems of interacting Brownian motions like the one studied in this thesis, but also growth models, like the polynuclear growth or ballistic deposition, and directed polymer models with their zero-temperature limit usually called last passage percolation. There is no generally agreed clear-cut definition of this class, but in a sense a model is in this class, if it shows the same scaling behaviour as the KPZ equation. Usually the first hint towards KPZ behaviour comes from discovering the characteristic fluctuation exponent 1/3.

A closer look at the structure of the solutions to (1.0.2) reveal that this equation is in fact a very intricate object. The problem is that its solutions are nowhere differentiable functions, since, omitting the non-linearity, the solutions look locally like Brownian motion. This leaves one clueless about how to define the square of the derivative, i.e. the stochastic PDE is ill-posed. The first successful approach to circumvent this problem has been to introduce the *Cole-Hopf transformation* $Z(x,t) = e^{h(x,t)}$ [BG97], which turns the KPZ equation into the stochastic heat equation with multiplicative noise, eliminating the nonlinearity:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\mathcal{W}. \tag{1.0.3}$$

If $\mathcal{W}$ would be a smooth function, there were no problem with this transformation, but the roughness of the white noise in fact leads to an infinite Itô correction term that is ignored here. Lacking a better definition, this so-called Cole-Hopf solution to the KPZ equation was anyway used as the basis for further analysis, and heuristic arguments as well as experimental results led to a general acceptance of this solution as the 'correct' one. Finally, in the epic work [Hai13], Hairer developed a general solution theory for the KPZ equation which confirmed the sensibility of the Cole-Hopf solution. This implies that a more accurate way of writing the KPZ equation is actually

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 - \infty + \mathcal{W}, \tag{1.0.4}$$

where the term $-\infty$ denotes a proper renormalization of the diverging square of the derivative. In a recent work Hairer and Quastel also showed that the precise form of the nonlinearity is irrelevant on a macroscopic level, i.e. that for the function F being an arbitrary even polynomial the solution to this generalized KPZ equation converges to the ordinary KPZ equation under proper scaling of space, time and the nonlinearity coefficient.

There are six initial profiles of the KPZ equation that are distinguished and these are self-similar under diffusive rescaling. The first three (sometimes called fundamental geometries) are the narrow wedge or delta initial condition $h(x, 0) = -|x|/\varepsilon$, $\varepsilon \to 0$, the flat initial condition $h(x, 0) = 0$ and the Brownian initial condition $h(x, 0) = B(x)$, with $B(x)$ being a two-sided Brownian motion. The other three distinguished profiles are mixtures of one different of the fundamental geometries on each side $\mathbb{R}_{<0}$ and $\mathbb{R}_{\geq 0}$. As $t \to \infty$, each geometry converges to a different limiting object:

- Under the narrow wedge initial profile, the limiting distribution is given by the GUE Tracy-Widom distribution, and the limit process is the Airy_2 process,
- Under the flat initial profile, the one-point distribution converges to GOE Tracy-Widom, and the limit process is the Airy_1 process,
- Under the Brownian initial profile, the one-point distribution converges to the Baik-Reins distribution, and the limit process is the $\mathrm{Airy}_{\mathrm{stat}}$ process.

The mixed initial profiles converge correspondingly to crossover processes $\mathrm{Airy}_{2\to 1}$, $\mathrm{Airy}_{2\to \mathrm{BM}}$ and $\mathrm{Airy}_{\mathrm{BM}\to 1}$.

Many of these asymptotics are, however, still conjectures. Proofs are known only for one-point distributions in the cases of the wedge [ACQ11, SS10], the two-sided Brownian motion [BCFV15] and the mixture of these, the one-sided Brownian motion [CQ13].

The first discovery of these limit processes were made not through studying the KPZ equation itself, but came from other models in the KPZ universality class, with the first being the discovery of the Airy_2 process in the polynuclear growth model [PS02]. One of the most studied models, however, is the totally asymmetric simple exclusion process, the TASEP. It is a particle system in continuous time on the space $\mathbb{Z}$, where each particle has an independent exponential clock with rate 1 and jumps to the right when it rings – but only if the site it wants to jump to is not already occupied.

The TASEP is very well-understood by now. In particular, the asymptotics of the multi-point distributions have been proven in all six distinguished initial conditions, which has been done for no other system up to now. There are also other interesting result, i.e. on shocks [CFP10, FN15], large deviations [Pro15] or the TASEP speed process [AAV11].

The main focus of this thesis is achieving a similar level of completeness for the system of one-sided reflected Brownian motions. It is not surprising that exact solvability should be possible in this case, as in fact this system can be derived as a limit of TASEP, by subtracting the average speed of 1 and then rescaling space and time diffusively. This limit has been rigorously proven for finite particle TASEP [KPS12].

The model studied in this thesis are Brownian particles that are reflected totally asymmetrically whenever they meet, thus retaining their order. Totally asymmetric means that the lower order particle is not influenced by the higher order particle bouncing off it. The asymmetry is crucial: The same system with symmetric collisions instead is part of the Edwards-Wilkinson class, its fluctuations scale as $t^{1/4}$ and converge to a Gaussian distribution [Har65].

Models in the KPZ universality class can be scaled to the KPZ equation if they have some additional parameter for the strength of the noise or the asymmetry. This weak scaling is in fact the way most rigorous results about the KPZ equation have been proven. Unfortunately there is no free parameter in our model, but there are two closely related models containing such:

The first one is a system of Brownian motions that are not subject to hard reflections but instead interacting through an exponential potential. This can be written as a system of coupled stochastic differential equations:

$$\mathrm{d}x_n(t) = e^{-\beta(x_n - x_{n-1})}\mathrm{d}t + \mathrm{d}B_n(t). \tag{1.0.5}$$

This system can be also interpreted as a directed polymer (see Section 2.2.1), which actually has been used to prove the asymptotics of the KPZ equation with Brownian initial profile in [BCFV15].

Alternatively one can change the totally asymmetric reflection to a partially asymmetric one. This is in analogy to the ASEP, which is a version of the TASEP, but where the particles can jump both to the left and to the right. In [SS15] a determinantal formula for this system of partially asymmetric reflected Brownian motions under half-Poisson initial conditions is derived and asymptotically analyzed in a non-rigorous way. Presumably one could extend these results to derive the asymptotic distribution of the KPZ equation under the half-Brownian initial profile.

In almost all works on KPZ-type models the analyzed distributions are for correlations along space-like paths, i.e. along the direction that corresponds to space in the KPZ equation. The correlations along the time direction behave quite differently, for once they decorrelate not on a scale of $t^{2/3}$ but t. Recently a formula for the two-point distribution along this direction has been found [Joh15] for precisely the model studied in this thesis. The definition of the distribution function, however, fills more than one page and does not seem amenable to a numerical evaluation, so the question of a deeper mathematical understanding seems still open.

The outline of the thesis is as follows:

Chapter 2 deals with the definition of the system of one-sided reflected Brownian motions. Especially in the case of infinitely many particles, it is not trivial to prove this is a well-defined mathematical object. Previous results and relations to other models are also discussed therein. Finally a nice stationarity property is proven for the system with Poisson initial conditions.

Chapter 3 introduces the theory of determinantal point processes and Fredholm determinants. Only definitions and theorems necessary for this thesis are given in order to keep the section tight.

Chapters 4 to 7 constitute the core of this work. For each of the distinguished initial conditions determinantal expressions for the multi-point distribution at a fixed time are developed and analyzed asymptotically to show convergence to an Airy-type process. The Poisson case is distinct in the sense that in addition to the usual strategy, a careful analytic continuation is necessary.

In Chapter 8 it is shown that all of the asymptotic theorems hold not only in the strict sense, but allow for relaxation in some of the assumptions. For one, our system shows the phenomenon of *slow decorrelations*, which is common among models in the KPZ universality class, so that the limit holds not only for the fixed time limit process, but also e.g. for a tagged particle. On the other hand, our system satisfies *attractiveness*, which implies that small modifications of the initial condition stay small for all times.

Finally, Chapter 9 contains the definition of the Airy-type processes as well as some historical information and their basic properties.

It should be noted, that the study of the periodic system that is contained in Chapter 5 is the subject of the work [FSW15b], which has been accepted as my Master's Thesis. Also some parts of the proofs in Chapter 2 and Chapter 8 appeared already in a similar form therein. The case of Poisson initial conditions has also been published [FSW15a], covering

Chapter 6 together with its prerequisites from Chapter 4 and Section 2.3.4, as well as the proof of Theorem 9.17.

Chapter 2

One-sided reflected Brownian motions and related models

2.1 Skorokhod construction

Our definition of reflected Brownian motions is the so-called Skorokhod representation [Sko61, AO76], which is a deterministic function of the driving Brownian motions. This representation is the following: the process $x(t)$, driven by the Brownian motion $B(t)$, starting from $x(0) \in \mathbb{R}$ and being reflected (in the positive direction) at some continuous function $f(t)$ with $f(0) \leq x(0)$ is defined as:

$$\begin{aligned} x(t) &= x(0) + B(t) - \min\left\{0, \inf_{0\leq s\leq t}(x(0) + B(s) - f(s))\right\} \\ &= \max\left\{x(0) + B(t), \sup_{0\leq s\leq t}(f(s) + B(t) - B(s))\right\}. \end{aligned} \tag{2.1.1}$$

Let B_n, $n \in \mathbb{Z}$, be independent standard Brownian motions starting at 0. Throughout this thesis, B_n always denotes these Brownian motions, allowing for coupling arguments. The non-positive indices will be used from Section 2.3 on.

Definition 2.1. *The half-infinite system of one-sided reflected Brownian motions* $\{x_n(t), n \geq 1\}$ *with initial condition* $\vec{x}(0) = \vec{\zeta}$, $\zeta_n \leq \zeta_{n+1}$ *is defined recursively by* $x_1(t) = \zeta_1 + B_1(t)$ *and*

$$x_n(t) = \max\left\{\zeta_n + B_n(t), \sup_{0\leq s\leq t}(x_{n-1}(s) + B_n(t) - B_n(s))\right\} \tag{2.1.2}$$

for $n \geq 2$.

Introducing the random variables

$$Y_{k,n}(t) = \sup_{0 \le s_k \le \ldots \le s_{n-1} \le t} \sum_{i=k}^{n} (B_i(s_i) - B_i(s_{i-1})) \tag{2.1.3}$$

for $k \le n$, with the convention $s_{k-1} = 0$ and $s_n = t$, allows for an equivalent explicit expression:

$$x_n(t) = \max_{k \in [1,n]} \{Y_{k,n}(t) + \zeta_k\}. \tag{2.1.4}$$

Although we are constructing an infinite system of particles, well-definedness is clear in this case, as each process $x_n(t)$ is a deterministic function of only finitely many Brownian motions B_k, $1 \le k \le n$.

Adopting a stochastic analysis point of view, the system $\{x_n(t), n \ge 1\}$ satisfies

$$x_n(t) = \zeta_n + B_n(t) + L^n(t), \quad \text{for } n \ge 0, \tag{2.1.5}$$

Here, $L^1(t) = 0$, while L^n, $n \ge 2$, are continuous non-decreasing processes increasing only when $x_n(t) = x_{n-1}(t)$. In fact, L^n is twice the semimartingale local time at zero of $x_n - x_{n-1}$.

2.2 Step initial conditions

A canonical and in fact the most studied initial condition for the system $\{x_n(t), n \ge 1\}$ is the one where all particles start at zero. Following the nomenclature of the discrete analogue, the TASEP, we call this *step initial condition*:

$$\vec{x}(0) = \vec{\zeta}^{\text{step}} = 0. \tag{2.2.1}$$

Using the monotonicity,

$$\begin{aligned} Y_{k-1,n}(t) &= \sup_{0 \le s_{k-1} \le s_k \le \ldots \le s_{n-1} \le t} \sum_{i=k-1}^{n} (B_i(s_i) - B_i(s_{i-1})) \\ &\ge \sup_{0 = s_{k-1} \le s_k \le \ldots \le s_{n-1} \le t} \sum_{i=k-1}^{n} (B_i(s_i) - B_i(s_{i-1})) = Y_{k,n}(t), \end{aligned} \tag{2.2.2}$$

and inserting this initial condition into (2.1.4), leads to:

$$x_n(t) = Y_{1,n}(t) = \sup_{0 \le s_1 \le \ldots \le s_{n-1} \le t} \sum_{i=1}^{n} (B_i(s_i) - B_i(s_{i-1})), \tag{2.2.3}$$

again with the convention $s_0 = 0$ and $s_n = t$.

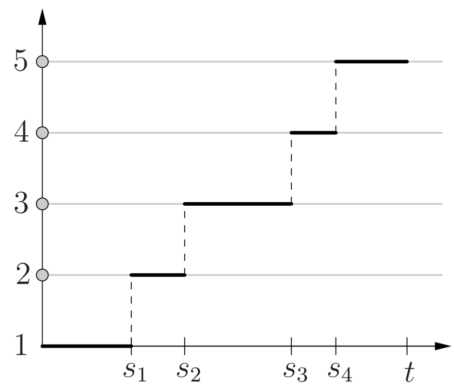

Figure 2.1: A path $\pi \in \Pi(0,1;t,5)$ (thick black) and the random background noise (grey).

2.2.1 Queues, last passage percolation and directed polymers

There are other interpretations for the quantity $x_n(t)$ than the system of reflected Brownian motions focused on in this work, and each interpretation has inspired different results over the last decades. One of these is seeing it as a sequence of *Brownian queues in series*. A famous theorem of queueing theory, *Burke's theorem*, which states that the output of a stable, stationary M/M/1 queue is Poisson, can be adapted to the Brownian setting [OY01]. This will be employed in Section 2.3.4.

Furthermore, $x_n(t)$ can be viewed as a model of directed last-passage percolation through a random medium, or equivalently a zero-temperature directed polymer in a random environment. This model is constructed as follows:

Consider the space $\mathbb{R}_+ \times \mathbb{Z}$ and assign white noise $\mathrm{d}B_n$ as random background weight on each line $\mathbb{R}_+ \times \{n\}$ for $n \geq 1$. An up-right path is characterized by its jumping points s_i and consists of line segments $[s_{n-1}, s_n] \times \{n\}$, see Figure 2.1. The set of up-right paths going from (t_1, n_1) to (t_2, n_2) can then be parameterized by

$$\Pi(t_1, n_1; t_2, n_2) = \left\{ \vec{s} \in \mathbb{R}^{n_2 - n_1 + 2} | t_1 = s_{n_1 - 1} \leq s_{n_1} \leq \dots \leq s_{n_2} = t_2 \right\}. \tag{2.2.4}$$

The *percolation time* or *weight* of a path $\vec{\pi} \in \Pi$ is the integral over the background weights along the path. Explicitly, we have:

$$w(\vec{\pi}) = \sum_{i=n_1}^{n_2} \left(B_i(s_i) - B_i(s_{i-1})\right). \tag{2.2.5}$$

The *last passage percolation time* is given by the supremum over all such paths:

$$L_{(t_1,n_1)\to(t_2,n_2)} := \sup_{\vec{\pi}\in\Pi(t_1,n_1;t_2,n_2)} w(\vec{\pi}). \tag{2.2.6}$$

The supremum is almost surely attained by a unique path $\vec{\pi}^*$, called the maximizer. It exists because the supremum can be rewritten as a composition of a finite maximum and a supremum of a continuous function over a compact set. Uniqueness follows from elementary properties of the Brownian measure. Most importantly, from the definition, we have

$$x_n(t) = L_{(0,1)\to(t,n)}. \tag{2.2.7}$$

This representation will be used repeatedly throughout this work as it nicely visualizes coupling arguments, however, it also gives some connections to different works. Our model can be seen as the semi-continuous limit of a more widely studied discrete last passage percolation model (see for example [Joh00, Joh03]).

This last passage percolation model is also the zero temperature limit of a directed polymer model, which has been studied thoroughly in the recent past [SV10, BCFV15]. In the directed polymer setting we have a parameter β representing the inverse temperature, consider $w(\vec{\pi})$ as an *energy* and assign a Gibbs measure on the set of paths according to the density $e^{\beta w(\vec{\pi})}$, i.e. paths with higher energy have a higher probability. The partition function of the polymer is given by

$$Z_{(t_1,n_1)\to(t_2,n_2)}(\beta) = \int_{\Pi(t_1,n_1;t_2,n_2)} \mathrm{d}\vec{\pi}\, e^{\beta w(\vec{\pi})}, \tag{2.2.8}$$

and satisfies the limit

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log Z_{(t_1,n_1)\to(t_2,n_2)}(\beta) = L_{(t_1,n_1)\to(t_2,n_2)}. \tag{2.2.9}$$

Although apparently more difficult to handle, attention has been turning to these positive temperature models recently, because, among other things, they allow for a scaling limit to the KPZ equation by tuning the parameter β in the right way.

2.2.2 Previous results

The behaviour of $x_n(t)$ under the step initial condition is quite well understood by now. Notice that by Brownian scaling $\{x_n(t), n \geq 1\} \overset{d}{=} \{\sqrt{t}x_n(1), n \geq 1\}$. The first result has been a law of large numbers, i.e. $x_n(1)/\sqrt{n}$ converges to a constant [GW91], they already conjectured it equals 2, which has been proven subsequently in [Sep97]. Much of the results that followed exploited connections to a random matrix model, so let us introduce it in full generality right away:

Let $b_{i,i}(t)$ for $1 \leq i \leq N$ and $b_{i,j}(t)$, $b'_{i,j}(t)$ for $1 \leq i < j \leq N$, be independent Brownian motions. Define a stochastic process $H(t)$, $t \geq 0$, on the space of $N \times N$ Hermitian matrices by

$$\begin{aligned} H_{i,i}(t) &= b_{i,i}(t) && \text{for } 1 \leq i \leq N \\ H_{i,j}(t) &= \sqrt{2}\left(b_{i,j}(t) + \mathrm{i}b'_{i,j}(t)\right) && \text{for } 1 \leq i < j \leq N \\ H_{i,j}(t) &= \sqrt{2}\left(b_{i,j}(t) - \mathrm{i}b'_{i,j}(t)\right) && \text{for } 1 \leq j < i \leq N \end{aligned} \tag{2.2.10}$$

Denote by $\lambda_1^n(t) \leq \lambda_2^n(t) \leq \cdots \leq \lambda_n^n(t)$ the ordered eigenvalues of the $n \times n$ principal minor of the matrix $H(t)$. The process $\{\lambda_k^n(t), 1 \leq k \leq n \leq N\}$ is called the *Dyson Brownian minor process*. It is a classical result, that the eigenvalues of consecutive minors are interlaced, i.e. $\lambda_k^{n+1}(t) \leq \lambda_k^n(t) \leq \lambda_{k+1}^{n+1}(t)$, so this process lives in the Gelfand-Tsetlin cone:

$$\mathrm{GT}_N = \left\{x_k^n \in \mathbb{R}, 1 \leq k \leq n \leq N, x_k^{n+1} \leq x_k^n \leq x_{k+1}^{n+1}\right\}. \tag{2.2.11}$$

Restricted to one layer n, the process $\{\lambda_k^n(t), 1 \leq k \leq n\}$ is called *Dyson's Brownian motion* [Dys62]. It is a Markov process and satisfies the coupled stochastic differential equation

$$\mathrm{d}\lambda_k^n = \mathrm{d}W_k + \sum_{i \neq k} \frac{\mathrm{d}t}{\lambda_k^n - \lambda_i^n}. \tag{2.2.12}$$

Its fixed time distribution is the eigenvalue distribution of the Gaussian unitary ensemble (GUE):

$$\mathbb{P}\left(\lambda_k^n(t) \in \mathrm{d}y_k\right) = \frac{1}{Z_n(t)} \prod_{k=1}^{N} e^{-\frac{y_k^2}{2t}} \prod_{k<i} (y_i - y_k)^2 \mathrm{d}\vec{y}. \tag{2.2.13}$$

Recognizing the second product as the square of the Vandermonde determinant, one is able to describe this eigenvalue distribution as a determinantal point process governed by the Hermite kernel. Asymptotic analysis of the distribution of the largest eigenvalue in the appropriate *edge scaling* gives the GUE Tracy-Widom distribution [TW94].

The first connections between our system of reflecting Brownian motions and the matrix diffusion were found by [GTW01], proving that for every $n \geq 1$, $x_n(1) \overset{d}{=} \lambda_n^n(1)$, and [Bar01] generalizing this to $\{x_n(1), n \geq 1\} \overset{d}{=} \{\lambda_n^n(1), n \geq 1\}$ by a combinatorial procedure originating from group representation theory, called the Robinson-Schensted-Knuth (RSK) correspondence. [Bar01] also showed the remarkable fact that conditioned on the top layer $\{\lambda_k^N(1), 1 \leq k \leq N\}$ the distribution of $\{\lambda_k^n(1), 1 \leq k \leq n < N\}$ is uniform on the compact set given by the Gelfand-Tsetlin interlacing inequality. Restricting the Dyson Brownian minor process to a fixed time gives the *GUE minor process* $\{\lambda_k^n(1), 1 \leq k \leq n \leq N\}$, whose full distribution has been found in [JN06], again in the form of a determinantal point process.

There is a natural extension of the system of reflecting Brownian motions $\{x_n(t), n \geq 1\}$ to a process in the Gelfand-Tsetlin cone that is constructed in the following way: Let $B_1^1(t)$ be a Brownian motion. Let $B_1^2(t)$ and $B_2^2(t)$ be Brownian motions, which are reflected downwards resp. upwards from $B_1^1(t)$. Iteratively construct $B_k^n(t)$ as a Brownian motion reflected downwards from $B_k^{n-1}(t)$ and upwards from $B_{k-1}^{n-1}(t)$, with the peripheral processes $B_k^n(t)$ for $k = 1$ or $k = n$ being reflected from one process only. By construction, we have $x_n(t) = B_n^n(t)$. The process $\{B_k^n(t), 1 \leq k \leq n \leq N\}$ is called *Warren's process*, and has been introduced and studied in [War07]. Restricted to one layer n, it is distributed as a Dyson's Brownian motion. Warren's process shares the fixed time distribution with the GUE minor process, $\{\lambda_k^n(1), 1 \leq k \leq n \leq N\} \overset{d}{=} \{B_k^n(1), 1 \leq k \leq n \leq N\}$. There are also formulas for the transition density of the system along the edge, $\{B_n^n(t), 1 \leq n \leq N\}$ as well as for the system of two consecutive layers $\{B_n^k(t), 1 \leq k \leq n, N-1 \leq n \leq N\}$.

The connection between Warren's process and the Dyson Brownian minor process does not, however, hold in full generality. The common dynamics of any amount of consecutive layers of Warrens process, $\{B_n^k(t), 1 \leq k \leq n, N_1 \leq n \leq N_2\}$, is simply given by Dyson's SDE (2.2.12) for the layer $n = N_1$ and the reflection SDE's for the higher order layers. In the Dyson Brownian minor process, on the other hand, the common dynamics of two consecutive layers is given by a more complicated SDE (see (2.30) in [ANvM14]) and the evolution of three or more consecutive layers is not even a Markov process anymore. Interestingly, both processes still show the same distribution along so-called space-like paths, i.e. sequences of points (n_i, t_i) satisfying $t_i \leq t_{i+1}$ and $n_i \geq n_{i+1}$, in which case also determinantal formulas exist [ANvM10, FF10].

The determinantal formulas coming from the random matrix model are suitable for asymptotic analysis to show multi-point scaling limits, where the Airy_2 process arises. It appears both for correlations of $x_n(t)$ along the n direction and the t direction, as well as along general space-like paths. [Joh03] gives a sketch of the proof for both directions, [AvM05] prove the scaling limit with correlations along t rigorously. A complete proof for correlations along n is given in Chapter 4.

2.3 Infinite particle systems

The main focus of this work is showing determinantal formulas and scaling limits for other initial conditions. Unfortunately the connection to random matrices breaks down in this case. In fact, neither the Airy_1 process nor the $\mathrm{Airy}_{\mathrm{stat}}$ process have ever been found in a scaling limit of a random matrix model. This has been rather surprising, as the one-point distribution of the Airy_1 process, the GOE Tracy-Widom distribution, *does* arise in such a model, namely as the limiting distribution of the largest eigenvalue of a Gaussian real symmetric matrix, the GOE ensemble [BFP08].

2.3.1 Definition

At first sight it might seem trivial to extend the definition of the half-infinite system of one-sided reflected Brownian motions to an infinite number of particles $\{x_n(t), n \in \mathbb{Z}\}$, by letting the index k run over $(\infty, n]$ in (2.1.4). However, it has to be shown that this maximum is finite, which is only the case for initial conditions which are not too closely spaced together. Roughly said, the growth rate of $-\zeta_{-k}$ has to be faster than $\sqrt{k}$ for large k. We call those initial conditions *admissible*.

Definition 2.2. *A random vector $\vec{\zeta} \in \mathbb{R}^{\mathbb{Z}}$ with $\zeta_n \leq \zeta_{n+1}$ for all $n \in \mathbb{Z}$ is called an* admissible initial condition, *if there exists a $\chi > \frac{1}{2}$ such that for any $n \in \mathbb{Z}$ the sum*

$$\sum_{M \geq 0} \mathbb{P}\left(\zeta_n - \zeta_{-M} \leq M^{\chi}\right) \tag{2.3.1}$$

is finite.

Definition 2.3. *Let $\vec{\zeta} \in \mathbb{R}^{\mathbb{Z}}$ be an admissible initial condition. The infinite system of one-sided reflected Brownian motions $\{x_n(t), n \in \mathbb{Z}\}$ with initial* *condition* $\vec{x}(0) = \vec{\zeta}$ *is defined by*

$$x_n(t) = \max_{k \le n}\{Y_{k,n}(t) + \zeta_k\}. \tag{2.3.2}$$

By Proposition 2.4 this maximum exists and is finite. More specifically, we will show that for $\vec{\zeta}$ being any admissible initial condition, the infinite system $\{x_n(t), n \in \mathbb{Z}\}$ is the limit of certain half-infinite systems $\{x_n^{(M)}(t), n \ge -M\}$ as $M \to \infty$, where

$$x_n^{(M)}(t) = \max_{k \in [-M,n]}\{Y_{k,n}(t) + \zeta_k\}, \quad n \ge -M. \tag{2.3.3}$$

Notice that these processes indeed satisfy the Skorokhod equation,

$$x_n^{(M)}(t) = \max\left\{\zeta_n + B_n(t), \sup_{0 \le s \le t}(x_{n-1}^{(M)}(s) + B_n(t) - B_n(s))\right\}, \tag{2.3.4}$$

for $n > -M$, while the leftmost process is simply

$$x_{-M}^{(M)}(t) = \zeta_{-M} + B_{-M}(t). \tag{2.3.5}$$

Thus as desired $x_n^{(M)}(t)$ is a Brownian motion starting from ζ_n and reflected off by $x_{n-1}^{(M)}$ for $n > -M$.

Proposition 2.4. *For any* $t > 0$, $n \in \mathbb{Z}$ *there exists almost surely a* $k \le n$ *maximizing* $Y_{k,n}(t) + \zeta_k$, *i.e. the maximum in* (2.3.2) *exists. Furthermore, for any* $T > 0$,

$$\sup_{t \in [0,T]} |x_n(t)| < \infty, \quad \text{a.s.}, \tag{2.3.6}$$

as well as

$$\lim_{M \to \infty} \sup_{t \in [0,T]} |x_n^{(M)}(t) - x_n(t)| = 0, \quad \text{a.s.}. \tag{2.3.7}$$

The convergence result allows for taking the limit in (2.3.4), implying that the system $\{x_n(t), n \in \mathbb{Z}\}$ satisfies the Skorokhod equation, too.

The initial conditions we are actually interested in are corresponding to the two remaining fundamental geometries in the KPZ universality class. The first one is the flat surface, which translates into periodic initial conditions $\vec{x}(0) = \vec{\zeta}^{\text{flat}}$, defined by

$$\zeta_n^{\text{flat}} = n, \text{ for } n \in \mathbb{Z}, \tag{2.3.8}$$

which is obviously admissible.

Finally we will study the case where the model starts in its random stationary distribution, which is in our case a Poisson point process on the real line. However, as already familiar from other models in the KPZ universality class [BFP10,BCFV15,FS06,IS04], Theorem 6.1 will be proven via a sequence of approximating initial conditions.

Let therefore be $\{\mathrm{Exp}_n, n \in \mathbb{Z}\}$ be i.i.d. random variables with exponential distribution with parameter 1. For parameters $\lambda > 0$ and $\rho > 0$ define the initial condition $\vec{x}(0) = \vec{\zeta}^{\mathrm{stat}}(\lambda, \rho)$ by

$$\begin{aligned} \zeta_0^{\mathrm{stat}} &= 0, \\ \zeta_n^{\mathrm{stat}} - \zeta_{n-1}^{\mathrm{stat}} &= \begin{cases} \lambda^{-1}\,\mathrm{Exp}_n, & \text{for } n > 0, \\ \rho^{-1}\,\mathrm{Exp}_n, & \text{for } n \leq 0. \end{cases} \end{aligned} \tag{2.3.9}$$

Admissibility of this initial condition is also not hard to prove.

To recover the uniform Poisson process on the whole real line, we will set $\lambda = 1$ and carefully take the limit $\rho \to 1$ in the determinantal formulas that hold in the case $\rho < \lambda$. Finally, setting $\zeta_0 = 0$ will induce a difference of order one as compared to the true Poisson process case. By Proposition 8.5 this difference will stay bounded at all times, and consequently be irrelevant in the scaling limit. Thus it is enough to prove Theorem 6.1 for the initial conditions $\vec{x}(0) = \vec{\zeta}^{\mathrm{stat}}(1, 1)$.

2.3.2 Well-definedness

For the proof of Proposition 2.4 we first need the following concentration inequality:

Proposition 2.5. *For each $T > 0$ there exists a constant $C > 0$ such that for all $k < m$, $\delta > 0$,*

$$\mathbb{P}\left(\frac{Y_{k,m}(T)}{\sqrt{(m-k+1)T}} \geq 2 + \delta\right) \leq \mathrm{const} \cdot e^{-(m-k+1)^{2/3}\delta}. \tag{2.3.10}$$

This proposition is proven in Section 4.2. Another necessary lemma, that will be proven an intuitive way in Section 2.3.3, is:

Lemma 2.6. *Consider $0 \leq t_1 \leq t_2$ and m, M_{t_1}, M_{t_2} such that*

$$x_m(t_i) = x_m^{(M_{t_i})}(t_i) = \widetilde{x}_m^{(M_{t_i})}(t_i), \quad \textit{for } i = 1, 2. \tag{2.3.11}$$

Then

$$x_m(t_1) = x_m^{(M_{t_2})}(t_1) = \widetilde{x}_m^{(M_{t_2})}(t_1). \tag{2.3.12}$$

Proof of Proposition 2.4. Let us define an auxiliary system of processes, which we will use later in proving Proposition 2.7, by

$$\widetilde{x}_{-M}^{(M)}(t) = \zeta_{-M} + B_{-M}(t) + \rho t, \tag{2.3.13}$$

and

$$\widetilde{x}_n^{(M)}(t) = \max\big\{\zeta_n + B_n(t), \sup_{0\leq s\leq t}(\widetilde{x}_{n-1}^{(M)}(s) + B_n(t) - B_n(s))\big\} \tag{2.3.14}$$

for $n > -M$. This system differs from $x_n^{(M)}(t)$ just in the drift of the leftmost particle, which of course influences all other particles as well (the choice of the extra drift is because the system with infinite many particles in $\mathbb{R}_-$ generates a drift ρ). This system of particles satisfies

$$\widetilde{x}_n^{(M)}(t) = \max\big\{\widetilde{Y}_{-M,n}(t) + \zeta_{-M}, \max_{k\in[-M+1,n]}\{Y_{k,n}(t) + \zeta_k\}\big\}, \tag{2.3.15}$$

with

$$\widetilde{Y}_{k,n}(t) = \sup_{0\leq s_{k+1}\leq\ldots\leq s_m\leq t}\Big(\rho s_{k+1} + \sum_{i=k}^{n}(B_i(s_{i+1}) - B_i(s_i))\Big). \tag{2.3.16}$$

Also, we have the inequalities

$$Y_{k,n}(t) \leq \widetilde{Y}_{k,n}(t) \leq Y_{k,n}(t) + \rho t. \tag{2.3.17}$$

Consider the event

$$\begin{aligned} A_M := \{Y_{-M,n}(T) \geq 3\sqrt{(M+n+1)T}\} \cup \{\zeta_n - \zeta_{-M} \leq M^\chi\} \\ \cup \{Y_{n,n}(T) \leq \rho T + 3\sqrt{(M+n+1)T} - M^\chi\}. \end{aligned} \tag{2.3.18}$$

It is now straightforward to show $\sum_{M=0}^\infty \mathbb{P}(A_M) < \infty$. In fact, summability of the probabilities of the first set in (2.3.18) is a consequence of Proposition 2.5, applied with $\delta = 1$, while the second set is covered by the definition of an admissible initial condition. For the third set, notice that the left hand side is a Gaussian distribution independent of M, while the right hand side is dominated by the M^χ term for large M. Finiteness of the sum allows applying Borel-Cantelli, i.e. A_M occurs only finitely many times almost surely. This means, that a.s. there exists a M_T, such that for all $M \geq M_T$ the following three inequalities hold:

$$\begin{aligned} Y_{-M,n}(T) &< 3\sqrt{(M+n+1)T} \\ M^\chi &< \zeta_n - \zeta_{-M} \\ -Y_{n,n}(T) &< -\rho T - 3\sqrt{(M+n+1)T} + M^\chi \end{aligned} \tag{2.3.19}$$

Adding up these, $Y_{-M,n}(T)+\zeta_{-M}+\rho T<Y_{n,n}(T)+\zeta_n$ for all $M\geq M_T$ and dropping the term ρT shows us that the maximizing element in (2.3.2) cannot be a $k\leq -M_T$, or

$$x_n(T)=x_n^{(M_T)}(T). \tag{2.3.20}$$

Moreover, applying (2.3.17), gives

$$\widetilde{Y}_{-M_T,n}(t)+\zeta_{-M_T}\leq Y_{-M_T,n}(t)+\zeta_{-M_T}+\rho T<Y_{n,n}(T)+\zeta_n, \tag{2.3.21}$$

resulting in

$$\widetilde{x}_n^{(M_T)}(T)=x_n^{(M_T)}(T). \tag{2.3.22}$$

Repeating the same argument, we see that for every $t\in[0,T]$ there exists M_t such that $x_n(t)=x_n^{(M_t)}(t)=\widetilde{x}_n^{(M_t)}(t)$. Applying Lemma 2.6 then gives $x_n(t)=x_n^{(M_T)}(t)=\widetilde{x}_n^{(M_T)}(t)$ for every $t\in[0,T]$. This settles the convergence and the existence of a finite maximizing k in (2.3.2).

To see (2.3.6), which is equivalent to $\sup_{t\in[0,T]}|x_m^{(M_T)}(t)|<\infty$, we apply the bound

$$|Y_{k,n}(t)|\leq\sum_{i=k}^{n}\Big(\sup_{0\leq s\leq t}B_i(s)-\inf_{0\leq s\leq t}B_i(s)\Big)<\infty. \tag{2.3.23}$$

□

2.3.3 Last passage percolation

It is also possible to extend the last passage percolation interpretation to nontrivial initial conditions. In order to do this, add non-negative Dirac background weights $\zeta_k-\zeta_{k-1}$ on $(0,k)$, $k\in\mathbb{Z}$. The weight of a path is explicitly given by

$$w(\vec{\pi})=\sum_{i=n_1}^{n_2}\left(B_i(s_i)-B_i(s_{i-1})+(\zeta_i-\zeta_{i-1})\mathbb{1}_{s_{i-1}=0}\right), \tag{2.3.24}$$

and the percolation time $L_{(0,n_1)\to(t,n_2)}$ again as the supremum over the weight of all paths. As $t\to 0$ it is clear that any contribution from the Brownian background weight will converge to 0, so the path tries to accumulate as much of the Dirac weights as possible, i.e. we have the initial condition

$$\lim_{t\to 0}L_{(0,n_1)\to(t,n_2)}=\sum_{i=n_1}^{n_2}(\zeta_i-\zeta_{i-1})=\zeta_{n_2}-\zeta_{n_1-1}. \tag{2.3.25}$$

By defining a normalized percolation time,

$$\widehat{L}_{(0,n_1)\to(t,n_2)} = L_{(0,n_1)\to(t,n_2)} + \zeta_{n_1-1}, \tag{2.3.26}$$

we recover the system

$$x_n^{(M)}(t) = \widehat{L}_{(0,-M)\to(t,n)}. \tag{2.3.27}$$

For any $M \leq n$, $M = -\infty$ included, we can define an *exit point* of a path $\pi = (\dots, s_{n_2-1}, s_{n_2}) \in \Pi(0,-M;t,n)$ by

$$\inf\{k \in [n_1,n_2], s_i > 0\}, \tag{2.3.28}$$

which is of course the maximizing index k in (2.3.3).

We also can reproduce the system $\widetilde{x}_n^{(M)}(t)$ by adding a Lebesgue measure of density ρ on the line $\{-M\} \times \mathbb{R}_+$.

Proof of Lemma 2.6. For $t_1 = t_2$ there is nothing to prove, so let $t_1 < t_2$. For each i, the equation $x_m(t_i) = x_m^{(M_{t_i})}(t_i)$ implies that the maximizing paths of the LHS and the RHS are equal on the restriction to $(s_i, i \geq M_i)$, i.e. they have the same exit point e_i that satisfies $e_i \geq M_{t_i}$. Now if $e_1 \geq e_2$, then also $e_1 \geq M_{t_2}$, which means that the path maximizing $x_m(t_1)$ is contained in the set $\Pi(0,-M_{t_2};t_1,m)$, resulting in $x_m(t_1) = x_m^{(M_{t_2})}(t_1)$.

If, however, $e_1 < e_2$, then the maximizing path segments $(0,e_1) \to (t_1,m)$ and $(0,e_2) \to (t_2,m)$ would need to have an intersection point (t^*,m^*). We can then construct a new maximizing path for $x_m(t_1)$ by stringing together the segments $(0,e_1) \to (0,e_2) \to (t^*,m^*) \to (t_1,m)$, where the middle segment is part of the $x_m(t_2)$-maximizing path and the last segment is part of the original $x_m(t_1)$-maximizing path. This contradicts the uniqueness of the maximizing path.

The equality $x_m(t_1) = \widetilde{x}_m^{(M_{t_2})}(t_1)$ is shown in the same way. □

2.3.4 Stationarity

We establish a useful property which will allow us to study our system of interacting Brownian motions through a system with a left-most Brownian particle.

Proposition 2.7. *Under the initial condition $\vec{x}(0) = \vec{\zeta} = \vec{\zeta}^{\,\mathrm{stat}}(\lambda,\rho)$, for each $n \leq 0$ the process*

$$x_n(t) - \zeta_n - \rho t \tag{2.3.29}$$

is a standard Brownian motion.

Remark 2.8. Proposition 2.7 allows us to restrict our attention to the half-infinite system. In fact, conditioned on the path of x_0, the systems of particles $\{x_n(t), n < 0\}$ and $\{x_n(t), n > 0\}$ are independent, as it is clear by the definition of the system. Then (2.3.29) implies that the law of $\{x_n(t), n > 0\}$ is the same as the one obtained replacing the infinitely many particles $\{x_m(t), m \leq 0\}$ with a single Brownian motion $x_0(t)$ which has a drift ρ. This property will be used to derive our starting result, Proposition 6.2.

Remark 2.9. From a stochastic analysis point of view, we find that the system $\{x_n(t), n \geq 0\}$ satisfies

$$\begin{aligned} x_n(t) &= \zeta_n + B_n(t) + L^n(t), \quad \text{for } n \geq 1, \\ x_0(t) &= \widetilde{B}_0(t) + \rho t. \end{aligned} \tag{2.3.30}$$

Here L^n, $n \geq 2$, are continuous non-decreasing processes increasing only when $x_n(t) = x_{n-1}(t)$. In fact, L^n is twice the semimartingale local time at zero of $x_n - x_{n-1}$. Notice that $\widetilde{B}_0(t)$ is a standard Brownian motion independent of $\{\zeta_n, B_n(t), n \geq 1\}$, but not equal to $B_0(t)$.

Proof of Proposition 2.7. First notice that for any M,

$$\widetilde{x}_{-M}^{(M)}(t) - \zeta_{-M} - \rho t, \tag{2.3.31}$$

is a Brownian motion. Now assume $\widetilde{x}_{n-1}^{(M)}(t) - \zeta_{n-1} - \rho t$ is a Brownian motion. By definition,

$$\widetilde{x}_{n}^{(M)}(t) - \zeta_{n-1} = \max\left\{\zeta_n - \zeta_{n-1} + B_n(t), \sup_{0\leq s\leq t}\left(\widetilde{x}_{n-1}^{(M)}(s) - \zeta_{n-1} + B_n(t) - B_n(s)\right)\right\}, \tag{2.3.32}$$

which allows us to apply Proposition 2.10, i.e., we have that

$$\widetilde{x}_{n}^{(M)}(t) - \zeta_{n-1} - (\zeta_n - \zeta_{n-1}) - \rho t = \widetilde{x}_{n}^{(M)}(t) - \zeta_n - \rho t \tag{2.3.33}$$

is a Brownian motion. Since $\widetilde{x}_{n}^{(M_T)}(t) = x_n(t)$ the proof is completed. □

It is clear, that in the case $\lambda = \rho$ the process (2.3.29) is a Brownian motion for $n > 0$, too, i.e., the system is stationary in n. We also have stationarity in t, in the sense that for each $t \geq 0$ the random variables $\{x_n(t) - x_{n-1}(t), n \in \mathbb{Z}\}$ are independent and distributed exponentially with parameter ρ. The following result is a small modification of Theorem 2 in [OY01].

Proposition 2.10 (Burke's theorem for Brownian motions)**.** *Fix $\rho > 0$ and let $B(t)$, $C(t)$ be standard Brownian motions, as well as $\zeta \sim \exp(\rho)$, independent. Define the process*

$$D(t) = \max\big\{\zeta + C(t), \sup_{0\le s\le t}(B(s) + \rho s + C(t) - C(s))\big\}. \tag{2.3.34}$$

Then

$$D(t) - \zeta - \rho t \tag{2.3.35}$$

is distributed as a standard Brownian motion.

Proof. Extend the processes $B(t)$, $C(t)$ to two-sided Brownian motions indexed by $\mathbb{R}$. Defining

$$q(t) = \sup_{-\infty<s\le t}\{B(t) - B(s) + C(t) - C(s) - \rho(t-s)\} \tag{2.3.36}$$

and

$$d(t) = B(t) + q(0) - q(t), \tag{2.3.37}$$

we can apply Theorem 2 [OY01], i.e., $d(t)$ is a Brownian motion. Now,

$$q(0) = \sup_{s\le 0}\{-B(s) - C(s) + \rho s\} \overset{d}{=} \sup_{s\ge 0}\{\sqrt{2}B(s) - \rho s\} \overset{d}{=} \sup_{s\ge 0}\big\{B(s) - \frac{\rho}{2}s\big\}, \tag{2.3.38}$$

so by Lemma 2.11 it has exponential distribution with parameter ρ. As it is independent of the processes $\{B(t), C(t), t \ge 0\}$ we can write $q(0) = \zeta$. Dividing the supremum into $s < 0$ and $s \ge 0$ we arrive at:

$$\begin{aligned} -d(t) &= q(t) - B(t) - q(0) \\ &= \max\Big\{C(t) - \rho t, \sup_{0\le s\le t}\{-B(s) + C(t) - C(s) - \rho(t-s)\} - \zeta\Big\}, \end{aligned} \tag{2.3.39}$$

which is (2.3.35) up to a sign flip of $B(s)$. □

Lemma 2.11. *Fix $\rho > 0$ and let $B(t)$ be a standard Brownian motion. Then*

$$\sup_{s\ge 0}(B(s) - \rho s) \sim \exp(2\rho). \tag{2.3.40}$$

Proof. The random variable

$$\sup_{0\le s\le t}(B(s) - \rho s) \tag{2.3.41}$$

is distributed as a Brownian motion with drift $-\rho$ started at zero and being reflected (upwards) at zero, at time t. As $t \to \infty$, this converges to the stationary distribution of this process, which is the exponential distribution with parameter 2ρ. □

Chapter 3

Determinantal point processes

The main tool for studying the systems of reflected Brownian motions is the theory of determinantal point processes and Fredholm determinants. It leads to formulas for the marginal distributions of measures given by a product of determinants in a form that is suitable for asymptotic analysis. However, not much of the underlying theory is needed for our purposes, so we just introduce the main definitions and the crucial Lemma 3.5. Up to minor modifications, we follow [Joh06], which is a well-written introduction to the topic.

3.1 Definition

Definition 3.1. *Let Λ be a complete separable metric space and let $\mathcal{N}(\Lambda)$ denote the space of all counting measures μ on Λ which are boundedly finite, i.e. $\mu(X) < \infty$ for all bounded $X \subseteq \Lambda$. Define a σ-algebra $\mathcal{F}$ on $\mathcal{N}(\Lambda)$ by taking the smallest σ-algebra for which $X \mapsto \mu(X)$ is measurable for all Borel sets X in Λ.*

A point process *is a probability measure on $\mathcal{N}(\Lambda)$. A* signed point process *is a normalized signed measure on $\mathcal{N}(\Lambda)$.*

Typically, Λ will be either $\mathbb{R}$ or $S \times \mathbb{R}$ for some discrete set S. For any realization μ of a point process Ξ, $\mu(X)$ is interpreted as the number of points in the set X. For each bounded set X, the restriction of the point process to this set is given by a finite sum of Dirac measures:

$$\mu|_X = \sum_{i=1}^{\mu(X)} \delta(x_i). \tag{3.1.1}$$

For a given point process Ξ we can construct a measure M_n over Λ^n, called the factorial moment measure, by

$$M_n = \mathbb{E}\left[\sum_{x_{i_1}\neq\cdots\neq x_{i_n}} \delta(x_{i_1},\dots,x_{i_n})\right], \tag{3.1.2}$$

where we abused the notation for the expectation in the case of a signed point process. The measure M_n is an intensity measure for n-tuples of distinct points in the original process.

Definition 3.2. *If M_n is absolutely continuous with respect to the Lebesgue measure, i.e.*

$$M_n(X_1,\dots,X_n) = \int_{X_1\times\cdots\times X_n} \rho_n(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n \tag{3.1.3}$$

for all Borel sets X_i in Λ, we call $\rho_n(x_1,\dots,x_n)$ the n-th correlation function *or* joint intensity*.*

Often times we will construct a point process from a symmetric measure on $\mathbb{R}^N$ with a density $p_N(x_1,\dots,x_N)$ by employing the canonical map $(x_1,\dots,x_N)\mapsto\sum_{i=1}^N\delta(x_i)$, and speak interchangeably of the measure and its associated point process. The correlation functions are then given by

$$\rho_n(x_1,\dots,x_n) = \frac{N!}{(N-n)!}\int_{\mathbb{R}^{N-n}} p_N(x_1,\dots,x_N)\,\mathrm{d}x_{n+1}\dots\mathrm{d}x_N. \tag{3.1.4}$$

If the density p_N is not symmetric, but instead normalized over the Weyl chamber $W^N=\{\vec{x}\in\mathbb{R}^N|x_1\leq\cdots\leq x_N\}$, it can be transformed into a symmetric one with density

$$\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N} p_N(x_{\sigma(1)},\dots,x_{\sigma(N)}). \tag{3.1.5}$$

We will study point processes whose correlation function are given by determinants:

Definition 3.3. *Consider a (signed) point process on Λ, all of whose correlation functions exist. If there is a function $K\colon\Lambda\times\Lambda\to\mathbb{C}$ such that*

$$\rho_n(x_1,\dots,x_n) = \det_{1\leq i,j\leq n}[K(x_i,x_j)], \tag{3.1.6}$$

for all $x_1,\dots,x_n\in\Lambda$, $n\geq 1$, then we say that it is a (signed) determinantal process*, and call K its* correlation kernel*.*

Correlation functions allow for a convenient calculation of *hole probabilities*, i.e. the probability of finding no particle in some set X:

$$\mathbb{P}\left(\mu(X)=0\right)=\sum_{n\geq 0}\frac{(-1)^n}{n!}\int_{X^n}\rho_n(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{3.1.7}$$

If $\Lambda=\mathbb{R}$, then choosing $X=(s,\infty)$ gives the distribution function of the rightmost particle $\mathbb{P}(x_{\max}\leq s)$, provided the series converges absolutely.

3.2 Fredholm determinants

Let $\mathcal{H}$ be a separable Hilbert space and A be a bounded linear operator acting on $\mathcal{H}$. Let $|A|=\sqrt{A^*A}$ be the unique square root of the operator A^*A, and $\{e_i, i\in I\}$ be an orthonormal basis of $\mathcal{H}$. The *trace norm* of A is given by $||A||_1=\sum_i\langle e_i,|A|e_i\rangle$, and A is called *trace class*, if $||A||_1<\infty$. Similarly, the *Hilbert-Schmidt norm* of A is given by $||A||_2=\left(\sum_i||Ae_i||^2\right)^{1/2}$, and A is called *Hilbert-Schmidt*, if $||A||_2<\infty$. Both norms are independent of the choice of the basis, and, with $||\cdot||_{\mathrm{op}}$ denoting the usual operator norm, satisfy:

$$||A||_{\mathrm{op}}\leq||A||_2\leq||A||_1. \tag{3.2.1}$$

With B being another bounded linear operator acting on $\mathcal{H}$ we also have the inequalities

$$||AB||_1\leq||A||_2||B||_2, \tag{3.2.2}$$

as well as

$$||AB||_1\leq||A||_1||B||_{\mathrm{op}}. \tag{3.2.3}$$

We call A an *integral operator* on the space $L^2(\Lambda)$ if there is a function $A\colon\Lambda\times\Lambda\to\mathbb{R}$, called its *integral kernel*, such that $(Af)(x)=\int_\Lambda A(x,y)f(y)\mathrm{d}y$. We abuse notation by denoting the operator and its integral kernel by the same letter. The Hilbert-Schmidt norm of an integral operator is given by

$$||A||_2^2=\int_{\Lambda^2}|A(x,y)|^2\mathrm{d}x\mathrm{d}y. \tag{3.2.4}$$

Definition 3.4. *Let $A\colon L^2(\Lambda)\to L^2(\Lambda)$ be a trace class operator with integral kernel $A(x,y)$. Then the* Fredholm determinant *of A is given by*

$$\det(\mathbb{1}+A)_{L^2(\Lambda)}=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\mathrm{d}x_1\ldots\mathrm{d}x_N\det_{1\leq i,j\leq n}[K(x_i,x_j)]. \tag{3.2.5}$$

Instead of using the series expansion, one can define the Fredholm determinant in a more abstract way for operators on a general separable Hilbert space $\mathcal{H}$, see [Sim00]. It can be seen as a natural generalization of the ordinary determinant as it satisfies $\det(\mathbb{1}+A) = \prod_n (1+\lambda_n)$, with λ_n being the eigenvalues of A, and has also the following properties:

- Continuity, specifically:
$$|\det(\mathbb{1}+A) - \det(\mathbb{1}+B)| \leq ||A-B|| \exp(||A||_1 + ||B||_1 + 1), \tag{3.2.6}$$
- Multiplicativity:
$$\det(\mathbb{1}+A+B+AB) = \det(\mathbb{1}+A)\det(\mathbb{1}+B), \tag{3.2.7}$$
- Sylvester's determinant theorem:
$$\det(\mathbb{1}+AB) = \det(\mathbb{1}+BA). \tag{3.2.8}$$

By the last identity, the Fredholm determinant is invariant under conjugations $A \mapsto U^{-1}AU$. Definition 3.4 can thus be extended to operators that are not necessarily trace class themselves, but have a conjugate that is trace class.

For a determinantal point process, the formula (3.1.7) for the hole probability can be written as a Fredholm determinant:

$$\begin{aligned}\mathbb{P}\left(\mu(X) = 0\right) &= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{X^n} \det_{1\leq i,j\leq n} [K(x_i,x_j)] \mathrm{d}x_1 \dots \mathrm{d}x_n \\ &= \det(\mathbb{1} - \mathbb{1}_X K \mathbb{1}_X)_{L^2(\Lambda)},\end{aligned} \tag{3.2.9}$$

where $\mathbb{1}_X$ denotes the projection operator on the set X. In the case $\Lambda = \mathbb{R}$ we recover the distribution function of the rightmost particle by choosing $X = (s,\infty)$.

In the case $\Lambda = S \times \mathbb{R}$, with some discrete set S, we have a point process that has particles in each layer $\{r_k\} \times \mathbb{R}$, $r_k \in S$. The choice $X = \bigcup_k \{r_k\} \times (s_k,\infty)$ gives the joint distribution of the rightmost particles in the layers r_k. For this choice of X we use the shorthand $\mathbb{1}_X = \chi_s(r_k,x)$. The integrals over the discrete measure can be written out explicitly as sums, resulting in the formula

$$\begin{aligned}\mathbb{P}\left(\bigcap_{k=1}^m \left\{x_{\max}(r_k) \leq s_k\right\}\right) &= \det(\mathbb{1} - \chi_s K \chi_s)_{L^2(S\times\mathbb{R})} = \\ &= \sum_{n\geq 0} \frac{(-1)^n}{n!} \sum_{i_1,\dots,i_n=1}^m \int_{\mathbb{R}^n} \mathbb{1}_{x_k > s_{i_k}} \det_{1\leq k,l\leq n} \left[K(r_{i_k},x_k;r_{i_l},x_l)\right] \mathrm{d}x_1 \dots \mathrm{d}x_n.\end{aligned} \tag{3.2.10}$$

3.3 Eynard-Mehta Theorem

Whenever a point process comes from a measure that is given by the product of two or more determinants in a certain way, the correlation functions are determinantal and there is an explicit formula for the correlation kernel. This has been discovered in [EM98] and generalized to various similar settings later on [FNH99, FS03, Joh03]. This explicit formula, however, involves the inverse of a quite complicated matrix. Instead of finding this inverse, which is usually not feasible, one relies on the *orthogonal polynomial method*, i.e. chooses functions in the right way, such that the resulting matrix is diagonal. A variant of this method is the basis of our analysis:

Lemma 3.5 (Corollary of Theorem 4.2 [BF08])**.** *Assume we have a signed measure on $\{x_i^n, n=1,\ldots,N, i=1,\ldots,n\}$ given in the form,*

$$\frac{1}{Z_N}\prod_{n=1}^{N}\det[\phi_n(x_i^{n-1},x_j^n)]_{1\le i,j\le n}\det[\Psi^N_{N-i}(x_j^N)]_{1\le i,j\le N}, \tag{3.3.1}$$

where x_{n+1}^n are some "virtual" variables and Z_N is a normalization constant. If $Z_N\neq 0$, then the correlation functions are determinantal.

To write down the kernel we need to introduce some notations. Define

$$\phi^{(n_1,n_2)}(x,y)=\begin{cases}(\phi_{n_1+1}*\cdots*\phi_{n_2})(x,y), & n_1<n_2,\\ 0, & n_1\ge n_2,\end{cases}. \tag{3.3.2}$$

where $(a*b)(x,y)=\int_{\mathbb{R}}\mathrm{d}z\,a(x,z)b(z,y)$, and, for $1\le n<N$,

$$\Psi^n_{n-j}(x):=(\phi^{(n,N)}*\Psi^N_{N-j})(y),\quad j=1,\ldots,N. \tag{3.3.3}$$

Then the functions

$$\{\phi^{(0,n)}(x_1^0,x),\ldots,\phi^{(n-2,n)}(x_{n-1}^{n-2},x),\phi_n(x_n^{n-1},x)\} \tag{3.3.4}$$

are linearly independent and generate the n-dimensional space V_n. Define a set of functions $\{\Phi^n_{n-j}(x), j=1,\ldots,n\}$ spanning V_n defined by the orthogonality relations

$$\int_{\mathbb{R}}\mathrm{d}x\,\Phi^n_{n-i}(x)\Psi^n_{n-j}(x)=\delta_{i,j} \tag{3.3.5}$$

for $1\le i,j\le n$.

Further, if $\phi_n(x_n^{n-1},x)=c_n\Phi_0^n(x)$, for some $c_n\neq 0$, $n=1,\ldots,N$, then the kernel takes the simple form

$$K(n_1,x_1;n_2,x_2)=-\phi^{(n_1,n_2)}(x_1,x_2)+\sum_{k=1}^{n_2}\Psi^{n_1}_{n_1-k}(x_1)\Phi^{n_2}_{n_2-k}(x_2). \tag{3.3.6}$$

3.4 Outline of the proofs

The basic concepts of the asymptotic behaviour of our system of one-sided reflected Brownian motions in the three fundamental initial conditions as well as the three mixed initial ones have quite some overlap.

The starting point is always the formula for the transition density provided by Proposition 4.3. After that one can insert resp. integrate over the initial condition, subject to some simplifications using stationarity. This is trivial for step and periodic initial conditions, but involves some subtle algebraic handling in the Poisson case. After obtaining the fixed time measure in this way, one has to introduce virtual particles to obtain a version of the measure that allows applying Lemma 3.3.2.

The resulting Fredholm determinant expression for the joint distribution of particles at a fixed time can then be analyzed asymptotically using steep descent. In order for the Fredholm determinants to converge, one has to show not only pointwise convergence of the kernel but also some uniform exponential bounds.

The situation is more complex for the Poisson case, as a determinantal structure exists only for a positive difference $\lambda - \rho$ of the Poisson densities on the positive and negative half-axis. The rigorous limit $\rho \to \lambda$ is the topic of Section 6.4.

Chapter 4

Step initial conditions

We start with the simplest case of step initial conditions. Although this case has been studied extensively in the literature, we give a complete proof here, as nearly all of the methods and results will be needed for the more complicated cases anyway. The first result is a determinantal expression for the fixed time distribution:

Proposition 4.1. *Let* $\{x_n(t), n \geq 1\}$ *be the system of one-sided reflected Brownian motions with initial condition* $\vec{x}(0) = \vec{\zeta}^{\text{step}}$*. Then for any finite subset* S *of* $\mathbb{Z}_{>0}$*, it holds*

$$\mathbb{P}\bigg(\bigcap_{n\in S}\{x_n(t) \leq a_n\}\bigg) = \det(\mathbb{1} - \chi_a \mathcal{K}_{\text{step}} \chi_a)_{L^2(S\times\mathbb{R})}, \tag{4.0.1}$$

where $\chi_a(n,\xi) = \mathbb{1}_{\xi > a_n}$*. The kernel* $\mathcal{K}_{\text{step}}$ *is given by*

$$\mathcal{K}_{\text{step}}(n_1,\xi_1;n_2,\xi_2) = -\phi_{n_1,n_2}(\xi_1,\xi_2)\mathbb{1}_{n_2>n_1} + \mathcal{K}_0(n_1,\xi_1;n_2,\xi_2), \tag{4.0.2}$$

with

$$\begin{aligned} \phi_{n_1,n_2}(\xi_1,\xi_2) &= \frac{(\xi_2-\xi_1)^{n_2-n_1-1}}{(n_2-n_1-1)!}\mathbb{1}_{\xi_1\leq\xi_2} \\ \mathcal{K}_0(n_1,\xi_1;n_2,\xi_2) &= \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathrm{i}\mathbb{R}-\varepsilon}\mathrm{d}w \oint_{\Gamma_0}\mathrm{d}z\, \frac{e^{tw^2/2+\xi_1 w}}{e^{tz^2/2+\xi_2 z}}\frac{(-w)^{n_1}}{(-z)^{n_2}}\frac{1}{w-z}. \end{aligned} \tag{4.0.3}$$

This proposition is the basis for proving the asymptotic theorem:

Theorem 4.2. *With* $\{x_n(t), n \geq 1\}$ *being the system of one-sided reflected Brownian motions with initial condition* $\vec{x}(0) = \zeta^{\text{step}}$*, define the rescaled process*

$$r \mapsto X_t^{\text{step}}(r) = t^{-1/3}\big(x_{\lfloor t+2rt^{2/3}\rfloor}(t) - 2t - 2rt^{2/3}\big). \tag{4.0.4}$$

In the sense of finite-dimensional distributions,

$$\lim_{t\to\infty} X_t^{\mathrm{step}}(r) \stackrel{d}{=} \mathcal{A}_2(r) - r^2. \tag{4.0.5}$$

4.1 Determinantal structure

The first step in proving Proposition 4.1 is a formula for the transition density of a finite system of one-sided reflected Brownian motions. It generalizes Proposition 4.1 [FSW15b], which has been first shown in [War07], to the case of non-zero drifts.

Proposition 4.3. *Let $W^N = \{\vec{x} \in \mathbb{R}^N | x_1 \leq \cdots \leq x_N\}$ be the Weyl chamber. The transition probability density of N one-sided reflected Brownian motions with drift $\vec{\mu}$ from $\vec{x}(0) = \vec{\zeta} \in W^N$ to $\vec{x}(t) = \vec{\xi} \in W^N$ at time t has a continuous version, which is given as follows:*

$$\mathbb{P}\left(\vec{x}(t) \in \mathrm{d}\vec{\xi} | \vec{x}(0) = \vec{\zeta}\right) = r_t(\vec{\zeta}, \vec{\xi})\, \mathrm{d}\vec{\xi}, \tag{4.1.1}$$

where

$$r_t(\vec{\zeta}, \vec{\xi}) = \left(\prod_{n=1}^{N} e^{\mu_n(\xi_n - \zeta_n) - t\mu_n^2/2}\right) \det_{1\leq k,l\leq N} [F_{k,l}(\xi_{N+1-l} - \zeta_{N+1-k}, t)], \tag{4.1.2}$$

and

$$F_{k,l}(\xi, t) = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\mu} \mathrm{d}w\, e^{tw^2/2+\xi w} \frac{\prod_{i=1}^{k-1}(w + \mu_{N+1-i})}{\prod_{i=1}^{l-1}(w + \mu_{N+1-i})}, \tag{4.1.3}$$

with $\mu > -\min\{\mu_1, \ldots, \mu_N\}$.

Proof. We follow the proof of Proposition 8 in [War07]. The strategy is to show that the transition density satisfies three equations, the backwards equation, boundary condition and initial condition, the latter one being contained in Lemma 4.4. These equations are then used for Itô's formula to prove that it indeed is the transition density.

We start with the backwards equation and boundary condition:

$$\frac{\partial r_t}{\partial t} = \sum_{n=1}^{N} \left(\frac{1}{2}\frac{\partial^2}{\partial \zeta_n^2} + \mu_n \frac{\partial}{\partial \zeta_n}\right) r_t. \tag{4.1.4}$$

$$\frac{\partial r_t}{\partial \zeta_i} = 0, \qquad \text{whenever } \zeta_i = \zeta_{i-1},\ 2 \leq i \leq N \tag{4.1.5}$$

To see (4.1.5), move the prefactor $e^{-\mu_i \zeta_i}$ inside the integral in the $(N+1-i)$-th row of the determinant and notice that the differential operator transforms $F_{k,l}$ into $-F_{k+1,l}$. Consequently, $\zeta_i = \zeta_{i-1}$ implies the $(N+1-i)$-th being the negative of the $(N+2-i)$-th row. (4.1.4) can be obtained by the computation

$$\frac{\partial r_t}{\partial t} = \frac{1}{2}\sum_{n=1}^{N}\left(-\mu_n^2 + e^{-\mu_n\zeta_n}\frac{\partial^2}{\partial \zeta_n^2}e^{\mu_n\zeta_n}\right) r_t. \tag{4.1.6}$$

Let $f\colon W^N \to \mathbb{R}$ be a C^∞ function, whose support is compact and has a distance of at least some $\varepsilon > 0$ to the boundary of W^N. Define a function $F\colon (0,\infty)\times W^N \to \mathbb{R}$ as

$$F(t,\vec{\zeta}) = \int_{W^N} r_t(\vec{\zeta},\vec{\xi}) f(\vec{\xi})\,\mathrm{d}\vec{\xi}. \tag{4.1.7}$$

The previous identities (4.1.5) and (4.1.4) carry over to the function F in the form of:

$$\frac{\partial F}{\partial \zeta_i} = 0, \qquad \text{for } \zeta_i = \zeta_{i-1},\ 2 \le i \le N \tag{4.1.8}$$

$$\frac{\partial F}{\partial t} = \sum_{n=1}^{N}\left(\frac{1}{2}\frac{\partial^2}{\partial \zeta_n^2} + \mu_n \frac{\partial}{\partial \zeta_n}\right) F. \tag{4.1.9}$$

Our processes satisfy $x_n(t) = \zeta_n + \mu_n t + B_n(t) + L^n(t)$, where B_n are independent Brownian motions, $L^1 \equiv 0$ and L^n, $n \ge 2$, are continuous non-decreasing processes increasing only when $x_n(t) = x_{n-1}(t)$. In fact, L^n is twice the semimartingale local time at zero of $x_n - x_{n-1}$. Now fix some $\varepsilon > 0$, $T > 0$, define a process $F(T+\varepsilon-t, \vec{x}(t))$ for $t \in [0,T]$ and apply Itô's formula:

$$\begin{aligned} F\left(T+\varepsilon-t,\vec{x}(t)\right) = F\left(T+\varepsilon,\vec{x}(0)\right) &+ \int_0^t -\frac{\partial}{\partial s}F\left(T+\varepsilon-s,\vec{x}(s)\right)\mathrm{d}s \\ &+ \sum_{n=1}^{N}\int_0^t \frac{\partial}{\partial \zeta_n}F\left(T+\varepsilon-s,\vec{x}(s)\right)\mathrm{d}x_n(s) \\ &+ \frac{1}{2}\sum_{m,n=1}^{N}\int_0^t \frac{\partial^2}{\partial\zeta_m\partial\zeta_n}F\left(T+\varepsilon-s,\vec{x}(s)\right)\mathrm{d}\left\langle x_m(s), x_n(s)\right\rangle. \end{aligned} \tag{4.1.10}$$

From the definition it follows that $\mathrm{d}x_n(t) = \mu_n \mathrm{d}t + \mathrm{d}B_n(t) + \mathrm{d}L^n(t)$ and

$$\mathrm{d}\left\langle x_m(t), x_n(t)\right\rangle = \mathrm{d}\left\langle B_m(t), B_n(t)\right\rangle = \delta_{m,n}\mathrm{d}t, \tag{4.1.11}$$

because continuous functions of finite variation do not contribute to the quadratic variation. Inserting the differentials, by (4.1.9) the integrals with respect to ds integrals cancel, which results in:

$$
\begin{aligned}
(4.1.10) = F\left(T+\varepsilon, \vec{x}(0)\right) &+ \sum_{n=1}^{N} \int_0^t \frac{\partial}{\partial \zeta_n} F\left(T+\varepsilon-s, \vec{x}(s)\right) \mathrm{d}B_n(s) \\
&+ \sum_{n=1}^{N} \int_0^t \frac{\partial}{\partial \zeta_n} F\left(T+\varepsilon-s, \vec{x}(s)\right) \mathrm{d}L^n(s).
\end{aligned} \tag{4.1.12}
$$

Since the measure $\mathrm{d}L^n(t)$ is supported on $\{x_n(t) = x_{n-1}(t)\}$, where the spatial derivative of F is zero (see (4.1.8)), the last term vanishes, too. So $F\left(T+\varepsilon-t, \vec{x}(t)\right)$ is a local martingale and, being bounded, even a true martingale. In particular its expectation is constant, i.e.:

$$
F(T+\varepsilon, \vec{\zeta}) = \mathbb{E}\left[F\left(T+\varepsilon, \vec{x}(0)\right)\right] = \mathbb{E}\left[F\left(\varepsilon, \vec{x}(T)\right)\right]. \tag{4.1.13}
$$

Applying Lemma 4.4 we can take the limit $\varepsilon \to 0$, leading to

$$
F(T, \vec{\zeta}) = \mathbb{E}\left[f\left(\vec{x}(T)\right)\right]. \tag{4.1.14}
$$

Because of the assumptions we made on f it is still possible that the distribution of $\vec{x}(T)$ has positive measure on the boundary. We thus have to show that $r_t(\vec{\zeta}, \vec{\xi})$ is normalized over the interior of the Weyl chamber:

Start by integrating (4.1.2) over $\xi_N \in [\xi_{N-1}, \infty)$. Pull the prefactor indexed by $n = N$ as well as the integration inside the $l = 1$ column of the determinant. The $(k, 1)$ entry is then given by:

$$
\begin{aligned}
&e^{-\mu_N \zeta_N - t\mu_N^2/2} \int_{\xi_{N-1}}^{\infty} \mathrm{d}\xi_N e^{\mu_N \xi_N} F_{k,1}(\xi_N - \zeta_{N+1-k}, t) \\
&\quad = e^{-\mu_N \zeta_N - t\mu_N^2/2} e^{\mu_N x} F_{k,2}(x - \zeta_{N+1-k}, t) \Big|_{x=\xi_{N-1}}^{x=\infty}.
\end{aligned} \tag{4.1.15}
$$

The contribution from $x = \xi_{N-1}$ is a constant multiple of the second column and thus cancels out. The remaining terms are zero for $k \geq 2$, since all these functions $F_{k,2}$ have Gaussian decay. The only non-vanishing term comes from $k = 1$ and returns exactly 1 by an elementary residue calculation.

The determinant can thus be reduced to the index set $2 \leq k, l \leq N$. Successively carrying out the integrations of the remaining variables in the same way, we arrive at the claimed normalization. This concludes the proof. □

Lemma 4.4. *For fixed $\vec{\zeta} \in W^N$, the transition density $r_t(\vec{\zeta}, \vec{\xi})$ as given by* (4.1.2)*, satisfies*

$$\lim_{t\to 0} \int_{W^N} r_t(\vec{\zeta}, \vec{\xi}) f(\xi)\, \mathrm{d}\vec{\xi} = f(\vec{\zeta}) \tag{4.1.16}$$

for any C^∞ function $f\colon W^N \to \mathbb{R}$, whose support is compact and has a distance of at least some $\varepsilon > 0$ to the boundary of W^N.

Proof. At first consider the contribution to the determinant in (4.1.2) coming from the diagonal. For $k = l$ the products in (4.1.3) cancel out, so we are left with a simple Gaussian density. This contribution is thus given by the multidimensional heat kernel, which is well known to converge to the delta distribution. The remaining task is to prove that for all other permutations the integral vanishes in the limit.

Let σ be such a permutation. Its contribution is

$$\int_{\mathbb{R}^N} \mathrm{d}\vec{\xi}\, f(\vec{\xi}) \prod_{k=1}^{N} F_{k,\sigma(k)}(\xi_{N+1-\sigma(k)} - \zeta_{N+1-k}, t), \tag{4.1.17}$$

where we have extended the domain of f to $\mathbb{R}^N$, being identically zero outside of W_N. We also omitted the prefactor since it is bounded for ξ in the compact domain of f.

There exist $i < j$ with $\sigma(j) \leq i < \sigma(i)$. Let

$$\begin{aligned} \widetilde{W}_1 &= \{\vec{\xi} \in \mathbb{R}^N \colon \xi_{N+1-\sigma(i)} - \zeta_{N+1-i} < -\varepsilon/2\} \\ \widetilde{W}_2 &= \{\vec{\xi} \in \mathbb{R}^N \colon \xi_{N+1-\sigma(j)} - \zeta_{N+1-j} > \varepsilon/2\}. \end{aligned} \tag{4.1.18}$$

It is enough to restrict the area of integration to these two sets, since on the complement of $\widetilde{W}_1 \cup \widetilde{W}_2$, we have

$$\xi_{N+1-\sigma(i)} \geq \zeta_{N+1-i} - \varepsilon/2 \geq \zeta_{N+1-j} - \varepsilon/2 \geq \xi_{N+1-\sigma(j)} - \varepsilon, \tag{4.1.19}$$

so we are not inside the support of f.

We start with the contribution coming from $\widetilde{W}_1$. Notice that by

$$F_{k,l}(\xi, t) = e^{-\xi\mu_{N+1-l}} \frac{\mathrm{d}}{\mathrm{d}\xi}\Big(e^{\xi\mu_{N+1-l}} F_{k,l+1}(\xi, t)\Big), \tag{4.1.20}$$

all functions $F_{k,l}$ with $k > l$ can be written as iterated derivatives of $F_{k,k}$ and some exponential functions. For each $k \neq i$ with $k > \sigma(k)$ we write $F_{k,\sigma(k)}$ in this way and then use partial integration to move the exponential factors and derivatives onto f. The result is

$$\int_{\widetilde{W}_1} \mathrm{d}\vec{\xi}\, \tilde{f}(\vec{\xi}) F_{i,\sigma(i)}(\xi_{N+1-\sigma(i)} - \zeta_{N+1-i}, t) \prod_{k\neq i} F_{k,\max\{k,\sigma(k)\}}(\xi_{N+1-\sigma(k)} - \zeta_{N+1-k}, t) \tag{4.1.21}$$

for a new C^∞ function $\tilde{f}$, which has compact support and is therefore bounded, too. We can bound the contribution by first integrating the variables $\xi_{N+1-\sigma(k)}$ with $k \geq \sigma(k)$, $k \neq i$, where we have a Gaussian factor $F_{k,k}$:

$$|(4.1.21)| \leq \sup_{\vec{x}} \left|\tilde{f}(\vec{x})\right| \int_{\widetilde{W}_1'} \left|F_{i,\sigma(i)}(\xi_{N+1-\sigma(i)} - \zeta_{N+1-i}, t)\right| \mathrm{d}\xi_{N+1-\sigma(i)} \prod_{k<\sigma(k), k\neq i} \left|F_{k,\sigma(k)}(\xi_{N+1-\sigma(k)} - \zeta_{N+1-k}, t)\right| \mathrm{d}\xi_{N+1-\sigma(k)}. \tag{4.1.22}$$

$\widetilde{W}_1'$ consists of the yet to be integrated ξ-components that are contained in the set $\widetilde{W}_1 \cap \operatorname{supp}(\widetilde{f})$. In particular, $\widetilde{W}_1'$ is compact, so the functions $F_{k,\sigma(k)}$, $k \neq i$, are bounded uniformly in t by Lemma 4.5. The remaining integral gives:

$$|(4.1.21)| \leq \mathrm{const} \int_{-\infty}^{-\varepsilon/2} \left|F_{i,\sigma(i)}(x,t)\right| \mathrm{d}x, \tag{4.1.23}$$

which converges to 0 as $t \to 0$ by (4.1.25).

The contribution of $\widetilde{W}_2$ can be bounded analogously with j playing the role of i. The final convergence is then given by (4.1.24). □

Lemma 4.5. *For each $\varepsilon > 0$ we have*

$$\lim_{t\to 0} \int_{\varepsilon}^{\infty} |F_{k,l}(x,t)|\,\mathrm{d}x = 0, \qquad 1 \leq l \leq k \leq N, \tag{4.1.24}$$

$$\lim_{t\to 0} \int_{-\infty}^{-\varepsilon} |F_{k,l}(x,t)|\,\mathrm{d}x = 0, \qquad 1 \leq k, l \leq N. \tag{4.1.25}$$

In addition, for each $1 \leq k < l \leq N$ the function $F_{k,l}(x,t)$ is bounded uniformly in t on compact sets.

Proof. Let $x < -\varepsilon$, and choose a μ which is positive. By a transformation of variable we have

$$\begin{aligned} |F_{k,l}(x,t)| &= \left| \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\mu} \mathrm{d}w\, e^{tw^2/2+xw} \frac{\prod_{i=1}^{k-1}(w+\mu_{N+1-i})}{\prod_{i=1}^{l-1}(w+\mu_{N+1-i})} \right| \\ &= \left| \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\mu} \mathrm{d}v\, \sqrt{t}^{\,l-k-1} e^{v^2/2+xv/\sqrt{t}} \frac{\prod_{i=1}^{k-1}(v+\sqrt{t}\mu_{N+1-i})}{\prod_{i=1}^{l-1}(v+\sqrt{t}\mu_{N+1-i})} \right| \\ &\leq (2\pi)^{-1} \sqrt{t}^{\,l-k-1} e^{x\mu/\sqrt{t}} \int_{\mathrm{i}\mathbb{R}+\mu} |\mathrm{d}v|\, e^{\mathrm{Re}(v^2/2)} g(|v|), \end{aligned} \tag{4.1.26}$$

where $g(|v|)$ denotes a bound on the fraction part of the integrand, which grows at most polynomial in $|v|$. Convergence of the integral is ensured by the exponential term, so integrating and taking the limit $t \to 0$ gives (4.1.25). To see (4.1.24), notice that by $l \leq k$ the integrand has no poles, so we can shift the contour to the right, such that μ is negative, and obtain the convergence analogously.

We are left to prove uniform boundedness of $F_{k,l}$ on compact sets for $k < l$. For $x \leq 0$ we can use (4.1.26) to get

$$|F_{k,l}(x,t)| \leq (2\pi)^{-1} \int_{\mathrm{i}\mathbb{R}+\mu} |\mathrm{d}v|\, e^{\mathrm{Re}(v^2/2)} g(|v|) \tag{4.1.27}$$

for $t \leq 1$. In the case $x > 0$ we shift the contour to negative μ, thus obtaining contributions from residua as well as from the remaining integral. The latter can be bounded as before, while the residua are well-behaved functions, which converge uniformly on compact sets. $\square$

Proof of Proposition 4.1. Applying Proposition 4.3 for $\vec{\mu} = 0$ and $\vec{\zeta} = 0$ gives

$$\mathbb{P}\left(\vec{x}(t) \in \mathrm{d}\vec{\xi} | \vec{x}(0) = 0\right) = \det_{1 \leq k,l \leq N} [F_{k-l}(\xi_{N+1-l}, t)], \tag{4.1.28}$$

where

$$F_k(\xi, t) = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+1} \mathrm{d}w\, e^{tw^2/2+\xi w} w^k. \tag{4.1.29}$$

Using repeatedly the identity

$$F_k(\xi, t) = \int_{-\infty}^{\xi} \mathrm{d}x\, F_{k+1}(x, t), \tag{4.1.30}$$

relabeling $\xi_1^k := \xi_k$, and introducing new variables ξ_l^k for $2 \leq l \leq k \leq N$, we can write

$$\det_{1 \leq k,l \leq N} \left[F_{k-l}(\xi_1^{N+1-l}, t)\right] = \int_{\mathcal{D}'} \det_{1 \leq k,l \leq N} \left[F_{k-1}(\xi_l^N, t)\right] \prod_{2 \leq l \leq k \leq N} \mathrm{d}\xi_l^k, \tag{4.1.31}$$

where $\mathcal{D}' = \{\xi_l^k \in \mathbb{R}, 2 \leq l \leq k \leq N | x_l^k \leq x_{l-1}^{k-1}\}$. Using the antisymmetry of the determinant and encoding the constraint on the integration variables into indicator functions, we obtain that the measure (4.1.28) is a marginal of

$$\begin{aligned} &\mathrm{const} \cdot \prod_{n=2}^{N} \det_{1 \leq i,j \leq n} \left[\mathbb{1}_{\xi_i^{n-1} \leq \xi_j^n}\right] \det_{1 \leq k,l \leq N} \left[F_{k-1}(\xi_l^N, t)\right] \\ &= \mathrm{const} \cdot \prod_{n=1}^{N} \det_{1 \leq i,j \leq n} \left[\phi_n(\xi_i^{n-1}, \xi_j^n)\right] \det_{1 \leq k,l \leq N} \left[F_{k-1}(\xi_l^N, t)\right] \end{aligned} \tag{4.1.32}$$

with

$$\phi_n(x,y) = \mathbb{1}_{x\le y} \tag{4.1.33}$$

and using the convention that $\xi_n^{n-1} \le y$ always holds.

The measure (4.1.32) has the appropriate form for applying Lemma 3.5. The composition of the ϕ functions can be evaluated explicitly as

$$\phi_{m,n}(x,y) = (\phi_{m+1} * \cdots * \phi_n)(x,y) = \frac{(y-x)^{n-m-1}}{(n-m-1)!}\mathbb{1}_{x\le y}, \tag{4.1.34}$$

for $n > m \ge 0$. Define

$$\Psi_{n-k}^n(\xi) := \frac{(-1)^{n-k}}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w\; e^{tw^2/2+\xi w} w^{n-k}, \tag{4.1.35}$$

for some $\varepsilon > 0$. In the case $n \ge k$ the integrand has no poles, which implies $\Psi_{n-k}^n = (-1)^{n-k} F_{n-k}$. The straightforward recursion

$$(\phi_n * \Psi_{n-k}^n)(\xi) = \Psi_{n-1-k}^{n-1}(\xi) \tag{4.1.36}$$

eventually leads to condition (3.3.3) being satisfied.

The space V_n is generated by

$$\{\phi_{0,n}(\xi_1^0, x), \ldots, \phi_{n-2,n}(\xi_{n-1}^{n-2}, x), \phi_{n-1,n}(\xi_n^{n-1}, x)\}, \tag{4.1.37}$$

so a basis for V_n is given by

$$\{x^{n-1}, x^{n-2}, \ldots, x, 1\}. \tag{4.1.38}$$

Choose functions Φ_{n-k}^n as follows

$$\Phi_{n-k}^n(\xi) = \frac{(-1)^{n-k}}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, e^{-tz^2/2-\xi z} z^{-n+k-1}, \tag{4.1.39}$$

which are polynomials of order $n-k$ by elementary residue calculating rules, so these functions indeed generate V_n. To show (3.3.5), we decompose the scalar product as follows:

$$\int_{\mathbb{R}_-} \mathrm{d}\xi\, \Psi_{n-k}^n(\xi)\Phi_{n-\ell}^n(\xi) + \int_{\mathbb{R}_+} \mathrm{d}\xi\, \Psi_{n-k}^n(\xi)\Phi_{n-\ell}^n(\xi). \tag{4.1.40}$$

Since $n-k \ge 0$ we are free to choose the sign of ε as necessary. For the first term, we choose $\varepsilon < 0$ and the path Γ_0 close enough to zero, such that always $\mathrm{Re}(w-z) > 0$. Then, we can take the integral over ξ inside and obtain

$$\int_{\mathbb{R}_-} \mathrm{d}\xi\, \Psi^n_{n-k}(\xi)\Phi^n_{n-\ell}(\xi) = \frac{(-1)^{k-l}}{(2\pi\mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z\, \frac{e^{tw^2/2} w^{n-k}}{e^{tz^2/2} z^{n-\ell+1}(w-z)}. \tag{4.1.41}$$

For the second term, we choose $\varepsilon > 0$ to obtain $\mathrm{Re}(w-z) < 0$. Then again, we can take the integral over ξ inside and arrive at the same expression up to a minus sign. The net result of (4.1.40) is a residue at $w = z$, which is given by

$$\frac{(-1)^{k-l}}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, z^{\ell-k-1} = \delta_{k,\ell}. \tag{4.1.42}$$

Furthermore, both $\phi_n(\xi_n^{n-1}, x)$ and $\Phi_0^n(\xi)$ are constants, so the kernel has a simple form (compare with (3.3.6))

$$\mathcal{K}_{\mathrm{step}}(n_1, \xi_1; n_2, \xi_2) = -\phi_{n_1,n_2}(\xi_1, \xi_2)\mathbb{1}_{n_2>n_1} + \sum_{k=1}^{n_2} \Psi^{n_1}_{n_1-k}(\xi_1)\Phi^{n_2}_{n_2-k}(\xi_2). \tag{4.1.43}$$

Note that we are free to extend the summation over k up to infinity, since the integral expression for $\Phi^n_{n-k}(\xi)$ vanishes for $k > n$ anyway. Taking the sum inside the integrals we can write

$$\sum_{k\geq 1} \Psi^{n_1}_{n_1-k}(\xi_1)\Phi^{n_2}_{n_2-k}(\xi_2) = \frac{1}{(2\pi\mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z\, \frac{e^{tw^2/2+\xi_1 w}(-w)^{n_1}}{e^{tz^2/2+\xi_2 z}(-z)^{n_2}} \sum_{k\geq 1} \frac{z^{k-1}}{w^k}. \tag{4.1.44}$$

By choosing contours such that $|z| < |w|$, we can use the formula for a geometric series, resulting in

$$(4.1.44) = \frac{1}{(2\pi\mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z\, \frac{e^{tw^2/2+\xi_1 w}(-w)^{n_1}}{e^{tz^2/2+\xi_2 z}(-z)^{n_2}(w-z)} = \mathcal{K}_0(n_1, \xi_1; n_2, \xi_2). \tag{4.1.45}$$

□

4.2 Asymptotic analysis

According to (4.0.4) we use the scaled variables

$$\begin{aligned} n_i &= t + 2t^{2/3} r_i \\ \xi_i &= 2t + 2t^{2/3} r_i + t^{1/3} s_i. \end{aligned} \tag{4.2.1}$$

Correspondingly, consider the rescaled (and conjugated) kernel

$$\mathcal{K}^{\mathrm{resc}}_{\mathrm{step}}(r_1,s_1;r_2,s_2)=t^{1/3}e^{\xi_1-\xi_2}\mathcal{K}_{\mathrm{step}}(n_1,\xi_1;n_2,\xi_2), \tag{4.2.2}$$

which naturally decomposes into

$$\mathcal{K}^{\mathrm{resc}}_{\mathrm{step}}(r_1,s_1;r_2,s_2)=-\phi^{\mathrm{resc}}_{r_1,r_2}(s_1,s_2)\mathbb{1}_{r_1<r_2}+\mathcal{K}^{\mathrm{resc}}_0(r_1,s_1;r_2,s_2). \tag{4.2.3}$$

In order to establish the asymptotics of the joint distributions, one needs both a pointwise limit of the kernel, as well as uniform bounds to ensure convergence of the Fredholm determinant expansion. The first time this approach was used is in [GTW01]. These results are contained in the following propositions.

Proposition 4.6. *Consider any r_1,r_2 in a bounded set and fixed L. Then, the kernel converges as*

$$\lim_{t\to\infty}\mathcal{K}^{\mathrm{resc}}_{\mathrm{step}}(r_1,s_1;r_2,s_2)=K_{\mathcal{A}_2}(r_1,s_1;r_2,s_2) \tag{4.2.4}$$

uniformly for $(s_1,s_2)\in[-L,L]^2$.

Corollary 4.7. *Consider r_1,r_2 fixed. For any L there exists t_0 such that for $t>t_0$ the bound*

$$\left|\mathcal{K}^{\mathrm{resc}}_{\mathrm{step}}(r_1,s_1;r_2,s_2)\right|\leq \mathrm{const}_L \tag{4.2.5}$$

holds for all $(s_1,s_2)\in[-L,L]^2$.

Proposition 4.8. *For fixed r_1,r_2,L there exists $t_0>0$ such that the estimate*

$$|\mathcal{K}^{\mathrm{resc}}_0(r_1,s_1;r_2,s_2)|\leq\frac{1}{2}e^{-(s_1+s_2)} \tag{4.2.6}$$

holds for any $t>t_0$ and $s_1,s_2>0$.

Proposition 4.9 (Proposition 5.4 of [FSW15b])**.** *For fixed $r_1<r_2$ there exists $t_0>0$ and $C>0$ such that*

$$\left|\phi^{\mathrm{resc}}_{r_1,r_2}(s_1,s_2)\right|\leq Ce^{-|s_1-s_2|} \tag{4.2.7}$$

holds for any $t>t_0$ and $s_1,s_2\in\mathbb{R}$.

Now we can prove the asymptotic theorem:

Proof of Theorem 4.2. The joint distributions of the rescaled process $X_t(r)$ are given by the Fredholm determinant with series expansion

$$\begin{aligned}&\mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t(r_k)\leq s_k\}\Big)\\&\quad=\sum_{N\geq 0}\frac{(-1)^N}{N!}\sum_{i_1,\ldots,i_N=1}^{m}\int\prod_{k=1}^{N}\mathrm{d}x_k\,\mathbb{1}_{x_k>\xi_{i_k}}\det_{1\leq k,l\leq N}\left[\mathcal{K}_{\mathrm{step}}(n_{i_k},x_k;n_{i_l},x_l)\right],\end{aligned}\tag{4.2.8}$$

where n_i and ξ_i are given in (4.2.1). By employing the change of variables $\sigma_k = t^{-1/3}(x_k - 2t - 2t^{2/3}r_{i_k})$ and a conjugation we obtain

$$\begin{aligned}(4.2.8)=&\sum_{N\geq 0}\frac{(-1)^N}{N!}\sum_{i_1,\ldots,i_N=1}^{m}\int\prod_{k=1}^{N}\mathrm{d}\sigma_k\,\mathbb{1}_{\sigma_k>s_{i_k}}\\&\times\det_{1\leq k,l\leq N}\left[\mathcal{K}^{\mathrm{resc}}_{\mathrm{step}}(r_k,\sigma_k;r_l,\sigma_l)\frac{(1+\sigma_l^2)^{m+1-i_l}}{(1+\sigma_k^2)^{m+1-i_k}}\right],\end{aligned}\tag{4.2.9}$$

where the fraction inside the determinant is the new conjugation, which does not change the value of the determinant. Using Corollary 4.7 and Propositions 4.8, 4.9, we can bound the (k,l)-coefficient inside the determinant by

$$\mathrm{const}_1\left(e^{-|\sigma_k-\sigma_l|}\mathbb{1}_{i_k<i_l}+e^{-(\sigma_k+\sigma_l)}\right)\frac{(1+\sigma_l^2)^{m+1-i_l}}{(1+\sigma_k^2)^{m+1-i_k}},\tag{4.2.10}$$

assuming the r_k are ordered. The bounds

$$\begin{aligned}\frac{(1+x^2)^i}{(1+y^2)^j}e^{-|x-y|}&\leq\mathrm{const}_2\frac{1}{1+y^2},\quad\text{for } i<j,\\\frac{(1+x^2)^i}{(1+y^2)^j}e^{-(x+y)}&\leq\mathrm{const}_3\frac{1}{1+y^2},\quad\text{for } j\geq 1,\end{aligned}\tag{4.2.11}$$

which hold for x, y bounded from below, lead to

$$(4.2.10)\leq\mathrm{const}_4\frac{1}{1+\sigma_k^2}.\tag{4.2.12}$$

Using the Hadamard bound on the determinant, the integrand of (4.2.9) is therefore bounded by

$$\mathrm{const}_4^N N^{N/2}\prod_{k=1}^{N}\mathbb{1}_{\sigma_k>s_{i_k}}\frac{\mathrm{d}\sigma_k}{1+\sigma_k^2},\tag{4.2.13}$$

which is integrable. Furthermore,

$$|(4.2.8)| \leq \sum_{N\geq 0} \frac{\mathrm{const}_5^N N^{N/2}}{N!}, \tag{4.2.14}$$

which is summable, since the factorial grows like $(N/e)^N$, i.e., much faster than the nominator. Dominated convergence thus allows to interchange the limit $t \to \infty$ with the integral and the infinite sum. The pointwise convergence comes from Proposition 4.6, thus

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{X_t(r_k) \leq s_k\}\right) &= \det\left(\mathbb{1} - \chi_s K_{\mathcal{A}_2} \chi_s\right)_{L^2(\{r_1,\dots,r_m\}\times\mathbb{R})} \\ &= \mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_2(r_k) - r_k^2 \leq s_k\}\right).\end{aligned} \tag{4.2.15}$$

□

By using just the exponential bounds on the kernel instead of the pointwise convergence, we can now prove the concentration inequality required for Proposition 2.4 in a similar way:

Proof of Proposition 2.5. First notice that by $Y_{k,m}(t) \stackrel{d}{=} Y_{1,m-k+1}(t)$ we can restrict ourselves to $k = 1$ without loss of generality, and thus have to show

$$\mathbb{P}\left(\frac{x_m(T)}{\sqrt{mT}} \geq 2 + \delta\right) \leq \mathrm{const} \cdot e^{-m^{2/3}\delta}, \tag{4.2.16}$$

with $\vec{x}(t)$ being the system of reflected Brownian motions with step initial conditions.

Applying Proposition 4.1, we get by the Fredholm series expansion

$$\mathbb{P}\Big(x_t(t) \leq \xi\Big) = \sum_{N\geq 0} \frac{(-1)^N}{N!} \int \prod_{k=1}^{N} \mathrm{d}x_k\, \mathbb{1}_{x_k > \xi} \det_{1\leq k,l\leq N} \left[\mathcal{K}_{\mathrm{step}}(t, x_k; t, x_l)\right]. \tag{4.2.17}$$

We recognize that the $N = 0$ term is exactly 1, so the probability of the complementary event is simply the negative of the series started at $N = 1$. Setting $\xi = 2t + t^{1/3}s$ and using the change of variables $\sigma_k = t^{-1/3}(x_k - 2t)$, we recognize the scaling (4.2.1) and obtain

$$\mathbb{P}\left(\frac{x_t(t)}{t} \geq 2 + \frac{s}{t^{2/3}}\right) \leq \sum_{N\geq 1} \frac{1}{N!} \int \prod_{k=1}^{N} \mathrm{d}\sigma_k\, \mathbb{1}_{\sigma_k > s} \left|\det_{1\leq k,l\leq N} \left[\mathcal{K}_{\mathrm{step}}^{\mathrm{resc}}(0, \sigma_k; 0, \sigma_l)\right]\right|. \tag{4.2.18}$$

By Proposition 4.8 the (k,l)-coefficient inside the determinant is bounded by a constant times $e^{-(\sigma_k+\sigma_l)}$. Using the Hadamard bound on the determinant, the integrand is therefore bounded by

$$\mathrm{const}_1^N N^{N/2} \prod_{k=1}^{N} \mathbb{1}_{\sigma_k>s} e^{-\sigma_k} \mathrm{d}\sigma_k, \tag{4.2.19}$$

leading to

$$\mathbb{P}\left(\frac{x_t(t)}{t} \geq 2 + t^{-2/3}s\right) \leq \sum_{N\geq 1} \frac{\mathrm{const}_1^N N^{N/2}}{N!} e^{-Ns}. \tag{4.2.20}$$

For positive s we have $e^{-Ns} \leq e^{-s}$, and the remaining sum over N is finite, since the factorial grows like $(N/e)^N$, i.e., much faster than the nominator. We rename $m := t$ and notice that for any positive T, by Brownian scaling:

$$\frac{x_m(m)}{m} \overset{d}{=} \frac{x_m(T)}{\sqrt{mT}}, \tag{4.2.21}$$

which implies

$$\mathbb{P}\left(\frac{x_m(T)}{\sqrt{mT}} \geq 2 + m^{-2/3}s\right) \leq \mathrm{const} \cdot e^{-s}. \tag{4.2.22}$$

Inserting $s = m^{2/3}\delta$ finishes the proof. ☐

Before showing Propositions 4.6 and 4.8, we introduce some auxiliary functions and establish asymptotic results for them.

Definition 4.10. *Using the scaling*

$$\begin{aligned} n(t,r) &= t + 2t^{2/3}r \\ \xi(t,r,s) &= 2t + 2t^{2/3}r + t^{1/3}s, \end{aligned} \tag{4.2.23}$$

define the functions

$$\begin{aligned} \alpha_t(r,s) &:= \frac{t^{1/3}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \mathrm{d}w\, e^{t(w^2-1)/2+\xi(w+1)} (-w)^n \\ &= \frac{t^{1/3}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \mathrm{d}w\, e^{t(w^2-1)/2+(2t+2t^{2/3}r+t^{1/3}s)(w+1)} (-w)^{t+2t^{2/3}r}, \\ \beta_t(r,s) &:= \frac{t^{1/3}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, e^{-t(z^2-1)/2-\xi(z+1)} (-z)^{-n} \\ &= \frac{t^{1/3}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, e^{-t(z^2-1)/2-(2t+2t^{2/3}r+t^{1/3}s)(z+1)} (-z)^{-t-2t^{2/3}r}. \end{aligned} \tag{4.2.24}$$

Lemma 4.11. *The limits*

$$\begin{aligned}\alpha(r,s) &:= \lim_{t\to\infty}\alpha_t(r,s) = \mathrm{Ai}(r^2+s)e^{-\frac{2}{3}r^3-rs}\\ \beta(r,s) &:= \lim_{t\to\infty}\beta_t(r,s) = -\mathrm{Ai}(r^2+s)e^{\frac{2}{3}r^3+rs}\end{aligned}\tag{4.2.25}$$

hold uniformly for s and r in a compact set.

Proof. We start by analyzing β_t. Defining functions as

$$\begin{aligned}f_3(z) &= -(z^2-1)/2 - 2(z+1) - \ln(-z)\\ f_2(z) &= -2r(z+1+\ln(-z))\\ f_1(z) &= -s(z+1),\end{aligned}\tag{4.2.26}$$

we can write $G(z) = tf_3(z) + t^{2/3}f_2(z) + t^{1/3}f_1(z)$, leading to

$$\beta_t(r,s) = \frac{t^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z\, e^{G(z)}.\tag{4.2.27}$$

These types of limits of contour integrals appear frequently when studying models in the KPZ universality class. As usual, they will be computed via *steep descent analysis*. The idea is that the limit is dominated by the leading order term $tf_3(z)$ at a point where $\mathrm{Re}(z)$ is maximal. One therefore chooses a contour that passes through a *critical point* z_0, satisfying $f'(z_0) = 0$, in such a way that $\mathrm{Re}(z)$ is strictly decreasing when moving away from z_0 along the contour. The vanishing derivative also ensures that the imaginary part is stationary at the maximum of the real part, such that no rapid oscillations occur, which might cause cancellations.

Let $\theta \in (\pi/6, \pi/4)$. We choose $\Gamma = \gamma_1 \cup \overline{\gamma_1(R)} \cup \gamma_2(R)$ as our steep descent contour, where

$$\begin{aligned}\gamma_1(R) &= \{-1+ue^{\mathrm{i}\theta}, u\in[0,R]\},\\ \gamma_2(R) &= \{-1+Re^{\mathrm{i}u}, u\in[-\theta,\theta]\},\end{aligned}\tag{4.2.28}$$

with the direction of integration as in Figure 4.1.

The integrand is dominated by the $\exp(-z^2)$ term for large $|z|$ as $\theta < \pi/4$. Thus the contribution coming from $\gamma_2(R)$ converges to 0 as $R\to\infty$. With $\gamma_1 = \lim_{R\to\infty}\gamma_1(R)$ the remaining contour of integration is now $\gamma_1\cup\overline{\gamma_1}$. Let us show that the real part is indeed decreasing along γ_1:

$$\begin{aligned}\frac{\mathrm{d}\mathrm{Re}f_3(-1+ue^{\mathrm{i}\theta})}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{Re}\left(-\frac{u^2}{2}e^{2\mathrm{i}\theta} - ue^{\mathrm{i}\theta} - \log(1-ue^{\mathrm{i}\theta})\right)\\ &= \mathrm{Re}\frac{u^2e^{3\mathrm{i}\theta}}{1-ue^{\mathrm{i}\theta}} = \frac{u^2}{||1-ue^{\mathrm{i}\theta}||^2}\mathrm{Re}\left(e^{3\mathrm{i}\theta}-ue^{2\mathrm{i}\theta}\right) < 0.\end{aligned}\tag{4.2.29}$$

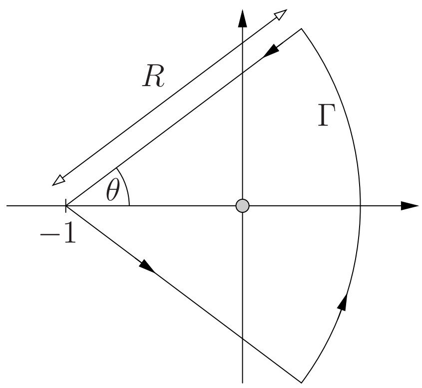

Figure 4.1: The contour $\Gamma = \gamma_1(R) \cup \overline{\gamma_1(R)} \cup \gamma_2(R)$ used in the steep descent analysis.

Replacing θ by $-\theta$ gives the result for $\overline{\gamma_1}$. It is also evident that f_3 decreases quadratically in u, while f_2 and f_1 increase at most linearly in u. Convergence of the integral is therefore clear for arbitrary finite t.

We want to restrict the contour of integration to a small neighbourhood of our critical point $\Gamma_\delta = \{z \in \Gamma | \, |z+1| \leq \delta\}$. Since f_3 is a steep descent contour, the contribution coming from the remaining part will be bounded by a constant times $e^{-\mu t}$ where μ is a constant of order δ^3,

$$\beta_t(r,s) = \mathcal{O}\left(t^{1/3} e^{-\mu t}\right) + \frac{t^{1/3}}{2\pi \mathrm{i}} \int_{\Gamma_\delta} \mathrm{d}z \, e^{G(z)} \tag{4.2.30}$$

For the computation of the integral along Γ_δ we can now use Taylor expansion:

$$\begin{aligned} t f_3(-1+\omega) &= t\left(\omega^3/3 + \mathcal{O}(\omega^4)\right) \\ t^{2/3} f_2(-1+\omega) &= t^{2/3} r \left(\omega^2 + \mathcal{O}(\omega^3)\right) \\ t^{1/3} f_1(-1+\omega) &= -\omega s t^{1/3}. \end{aligned} \tag{4.2.31}$$

All error terms are to be understood uniformly in s, t, r. Note that our critical point $z_0 = -1$ is actually a *doubly critical* point, i.e. the second derivative $f_3''(z_0)$ vanishes, too. This is a core feature of formulas appearing in the KPZ universality class. In a usual Gaussian scaling setting, the leading term of tf_3 would be $t\omega^2$, which required rescaling the integration variable ω by $t^{-1/2}$. This in turn led to the fluctuation scaling $st^{1/2}$ in order to obtain a non-degenerate limit. The limiting function had a second order polynomial in the exponent, resulting in the Fourier transform of a Gaussian function, i.e. again a Gaussian function. So a non-vanishing second derivative is connected to both the scaling exponent $1/2$ and the

Gaussian limiting distribution. The actual term $t\omega^3$ appearing here is the reason for the fluctuation exponent $1/3$ as well as a third order polynomial in the exponent, giving an Airy function.

Let $\tilde{f}_i(-1+\omega)$ be the expression $f_i(-1+\omega)$ but without the error term, and define also $\tilde{G}(-1+\omega)$ correspondingly. We use the inequality $|e^x - 1| \leq |x|e^{|x|}$ to estimate the error we make by integrating over $\exp(\tilde{G})$ instead of $\exp(G)$:

$$
\begin{aligned}
&\Big|\frac{t^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_\delta}\mathrm{d}z\,\left(e^{G(z)}-e^{\tilde{G}(z)}\right)\Big| \\
&\qquad\leq \int_{\Gamma_\delta+1}\mathrm{d}\omega\,\left|e^{\tilde{G}(-1+\omega)}\right| e^{\mathcal{O}(\omega^4 t+\omega^3 t^{2/3}+\omega^2 t^{1/3})}\mathcal{O}\left(\omega^4 t+\omega^3 t^{2/3}+\omega^2 t^{1/3}\right) \\
&\qquad\leq \int_{\Gamma_\delta+1}\mathrm{d}\omega\,\left|e^{t\tilde{f}_3(-1+\omega)(1+\chi_3)+t^{2/3}\tilde{f}_2(-1+\omega)(1+\chi_2)+t^{1/3}\tilde{f}_1(-1+\omega)(1+\chi_1)}\right| \\
&\qquad\quad\times\mathcal{O}\left(\omega^4 t+\omega^3 t^{2/3}+\omega^2 t^{1/3}\right),
\end{aligned}
\tag{4.2.32}
$$

where χ_1, χ_2 and χ_3 are constants, which can be made as small as desired for δ small enough. The leading term in the exponential is

$$
t\widetilde{f}_3(-1+\omega)(1+\chi_3)=\frac{1}{3}\omega^3(1+\chi_3)t,
\tag{4.2.33}
$$

which has negative real part and therefore ensures the integral to stay bounded for $t\to\infty$. By the change of variables $\omega = t^{-1/3}Z$ the prefactor $t^{1/3}$ cancels and the remaining $\mathcal{O}$-terms imply that the overall error is $\mathcal{O}(t^{-1/3})$.

$$
\begin{aligned}
\frac{t^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_\delta}\mathrm{d}z\,e^{\tilde{G}(z)} &= \frac{t^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_\delta+1}\mathrm{d}\omega\,e^{t\omega^3/3+t^{2/3}r\omega^2-t^{1/3}s\omega} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{e^{\theta\mathrm{i}}\delta t^{1/3}}^{-e^{\theta\mathrm{i}}\delta t^{1/3}}\mathrm{d}Z\,e^{Z^3/3+rZ^2-sZ}
\end{aligned}
\tag{4.2.34}
$$

Letting $t\to\infty$ now just extends the integration contour up to infinity. Noticing that we are free to choose an arbitrary angle $\pi/6<\theta<\pi/2$, this is indeed the integral expression for $\beta(r,s)$.

One can carry out an analogous analysis for α_t. Notice that with the same definition of the function G we now have:

$$
\alpha_t(r,s)=\frac{t^{1/3}}{2\pi\mathrm{i}}\oint_{\mathrm{i}\mathbb{R}}\mathrm{d}z\,e^{-G(z)}.
\tag{4.2.35}
$$

We choose the contour $\Gamma' = \{-1+|u|e^{\mathrm{sgn}(u)2\pi\mathrm{i}/3}, u \in \mathbb{R}\}$ and show that it is a steep descent curve:

$$\frac{\mathrm{dRe}\left(-f_3(-1+ue^{2\pi\mathrm{i}/3})\right)}{\mathrm{d}u} = \frac{u^2}{||1-ue^{2\pi\mathrm{i}/3}||^2}\mathrm{Re}\left(-1+ue^{4\pi\mathrm{i}/3}\right) < 0. \tag{4.2.36}$$

Repeating the other steps of the steep descent analysis in the obvious way one arrives at:

$$\lim_{t\to\infty} \alpha_t(r,s) = \frac{1}{2\pi\mathrm{i}} \int_{e^{-2\pi\mathrm{i}/3}\infty}^{e^{2\pi\mathrm{i}/3}\infty} \mathrm{d}W\, e^{-W^3/3-rW^2+sW}. \tag{4.2.37}$$

□

Lemma 4.12. *For fixed r and L, there exist t_0, c_L such that for all $t > t_0$ and $s > -L$ the following bounds hold*

$$\begin{aligned} |\alpha_t(r,s)| &\leq c_L e^{-s} \\ |\beta_t(r,s)| &\leq c_L e^{-s} \end{aligned} \tag{4.2.38}$$

Proof. In the case $s < L$ the result follows from the previous lemma. Let us assume $s \geq L$ from now on. Notice that we can require L to be as large as necessary, since the claim of the lemma is stronger for L large.

Start by analyzing β_t. We have again

$$\beta_t(r,s) = \frac{t^{1/3}}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, e^{G(z)}, \tag{4.2.39}$$

with $G(z)$ as in the proof of Proposition 4.11.

Define a new parameter ω given by

$$\omega = \min\left\{t^{-1/3}\sqrt{s}, \varepsilon\right\}, \tag{4.2.40}$$

for some small, positive ε chosen in the following, and let $\theta \in (\pi/6, \pi/4)$. We change the contour Γ_0 to $\overline{\Gamma} = \gamma_1(R) \cup \overline{\gamma_1(R)} \cup \gamma_2(R) \cup \gamma_3$ as shown in Figure 4.2, with

$$\begin{aligned} \gamma_1(R) &= \{-1+ue^{\mathrm{i}\theta}, u \in [\omega/\cos\theta, R]\} \\ \gamma_2(R) &= \{-1+Re^{\mathrm{i}u}, u \in [-\theta,\theta]\} \\ \gamma_3 &= \{-1+\omega(1+\mathrm{i}u\tan\theta), u \in [-1,1]\}. \end{aligned} \tag{4.2.41}$$

If t and s are fixed, the integrand is dominated by the $\exp(-z^2)$ term for large $|z|$. Thus the contribution coming from $\gamma_2(R)$ converges to 0 as

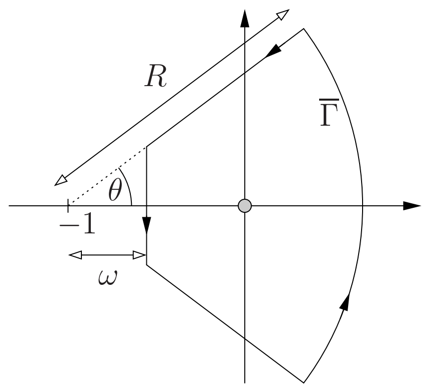

Figure 4.2: The contour $\overline{\Gamma} = \gamma_1(R) \cup \overline{\gamma_1(R)} \cup \gamma_2(R) \cup \gamma_3$ used for obtaining the uniform bounds.

$R \to \infty$. With $\gamma_1 = \lim_{R\to\infty} \gamma_1(R)$ our choice for the contour of integration is now $\gamma_1 \cup \overline{\gamma_1} \cup \gamma_3$.

We start by analyzing the contribution coming from γ_3,

$$\frac{t^{1/3}}{2\pi \mathrm{i}} \int_{\gamma_3} \mathrm{d}z\, e^{G(z)} = e^{G(z_0)} \frac{t^{1/3}}{2\pi} \int_{[-\omega\tan\theta, \omega\tan\theta]} \mathrm{d}u\, e^{G(z_0+\mathrm{i}u)-G(z_0)}, \tag{4.2.42}$$

where $z_0 = -1 + \omega$.

Let us consider the prefactor $e^{G(z_0)}$ at first. Since ω is small we can use Taylor expansion, as well as (4.2.40), to obtain the bounds

$$\begin{aligned} t f_3(z_0) &= t\left(\omega^3/3 + \mathcal{O}(\omega^4)\right) \le \frac{1}{3}\omega s t^{1/3}\left(1 + \mathcal{O}(\varepsilon)\right) \\ t^{2/3} f_2(z_0) &= t^{2/3} r \left(\omega^2 + \mathcal{O}(\omega^3)\right) \le \omega\sqrt{s} t^{1/3} |r| \left(1 + \mathcal{O}(\varepsilon)\right) \\ t^{1/3} f_1(z_0) &= -\omega s t^{1/3}. \end{aligned} \tag{4.2.43}$$

All error terms are to be understood uniformly in s, t, r. The f_1 term dominates both f_2, if L is chosen large enough, and f_3, for ε being small. This results in

$$e^{G(z_0)} \le e^{-\frac{1}{2}\omega t^{1/3} s} \le e^{-s}, \tag{4.2.44}$$

since $\omega t^{1/3}$ can be made as large as desired by increasing t_0 and L while keeping ε fixed.

To show convergence of the integral part of (4.2.42) we first bound the real part of the exponent:

$$
\begin{aligned}
&\mathrm{Re}\left(G(z_0+\mathrm{i}u)-G(z_0)\right)\\
=&\mathrm{Re}\left[t\left(\frac{u^2-2z_0\mathrm{i}u}{2}-2\mathrm{i}u-\ln\frac{z_0+\mathrm{i}u}{z_0}\right)+t^{2/3}\cdot 2r\left(\mathrm{i}u+\ln\frac{z_0+\mathrm{i}u}{z_0}\right)-t^{1/3}s\mathrm{i}u\right]\\
=&t\left(\frac{u^2}{2}-\frac{1}{2}\ln\left(1+\frac{u^2}{z_0^2}\right)\right)+t^{2/3}r\ln\left(1+\frac{u^2}{z_0^2}\right)\\
\leq&t\frac{u^2}{2}\left(1-\frac{1}{z_0^2}+\frac{u^2}{2z_0^4}\right)+t^{2/3}r\frac{u^2}{z_0^2}=:-\eta t^{2/3}u^2.
\end{aligned}
\tag{4.2.45}
$$

η satisfies:

$$
\begin{aligned}
\eta&=\frac{t^{1/3}}{2}\left(\frac{1}{(1-\omega)^2}-1-\frac{u^2}{2(1-\omega)^4}\right)-\frac{r}{(1-\omega)^2}\\
&=t^{1/3}\omega\left(1+\mathcal{O}(\omega)\right)-r\left(1+\mathcal{O}(\omega)\right),
\end{aligned}
\tag{4.2.46}
$$

where we used $|u|<\omega$. Given any ε we can now choose both L and t_0 large, such that the first term dominates. Consequently η will be bounded from below by some positive constant η_0. The integral contribution coming from γ_3 can thus be bounded as

$$
\begin{aligned}
|(4.2.42)|&=e^{G(z_0)}\frac{t^{1/3}}{2\pi}\left|\int_{[-\omega\tan\theta,\omega\tan\theta]}\mathrm{d}u\,e^{G(z_0+iu)-G(z_0)}\right|\\
&\leq e^{-s}\frac{t^{1/3}}{2\pi}\int_{\mathbb{R}}\mathrm{d}u\,e^{-\eta_0 t^{2/3}u^2}=\frac{e^{-s}}{2\pi}\int_{\mathbb{R}}\mathrm{d}u\,e^{-\eta_0 u^2}=\frac{e^{-s}}{2\sqrt{\pi\eta_0}}.
\end{aligned}
\tag{4.2.47}
$$

Finally we need a corresponding bound on the γ_1 contribution to the integral. By symmetry this case covers also the contour $\overline{\gamma_1}$. Write

$$
\frac{t^{1/3}}{2\pi\mathrm{i}}\int_{\gamma_1}\mathrm{d}z\,e^{G(z)}=e^{G(z_1)}\frac{t^{1/3}e^{\mathrm{i}\theta}}{2\pi\mathrm{i}}\int_{\mathbb{R}_+}\mathrm{d}u\,e^{G(z_1+ue^{\mathrm{i}\theta})-G(z_1)},
\tag{4.2.48}
$$

with $z_1=-1+\omega(1+\mathrm{i}\tan\theta)$. From the previous estimates one easily gets

$$
\left|e^{G(z_1)}\right|\leq e^{G(z_0)}\leq e^{-s},
\tag{4.2.49}
$$

so the remaining task is to show boundedness of the integral part of (4.2.48).

At first notice that the real part of the f_1 contribution in the exponent is negative, so we can omit it, avoiding the problem of large s. By elementary calculus, we have for all $u\geq\omega/\cos\theta$,

$$
\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{Re}\left(f_3(-1+ue^{\mathrm{i}\theta})\right)<0,
\tag{4.2.50}
$$

that is, γ_1 is a steep descent curve for f_3. We can therefore restrict the contour to a neighbourhood of the critical point z_1, which we choose of size δ. The error we make is exponentially small in t, so can be bounded by 1 through choosing t_0 large enough:

$$\left|\frac{t^{1/3}}{2\pi}\int_{\mathbb{R}_+} \mathrm{d}u\, e^{G(z_1+ue^{\mathrm{i}\theta})-G(z_1)}\right| \leq 1+\int_0^\delta \mathrm{d}u\,\left|e^{t\hat{f}_3(ue^{\mathrm{i}\theta})+t^{2/3}\hat{f}_2(ue^{\mathrm{i}\theta})}\right| \tag{4.2.51}$$

where $\hat{f}_i(z) = f_i(z_1+z)-f_i(z_1)$. Taylor expanding these functions leads to

$$\begin{aligned}\mathrm{Re}(t\hat{f}_3(ue^{\mathrm{i}\theta})) &= t\mathrm{Re}(e^{3\mathrm{i}\theta})u\frac{\omega^2}{\cos^2\theta}\,(1+\mathcal{O}(\delta))\,(1+\mathcal{O}(\varepsilon))\\ &\leq -\chi_3 t^{1/3}\omega\cdot t^{2/3}u\omega\\ \mathrm{Re}(t^{2/3}\hat{f}_2(ue^{\mathrm{i}\theta})) &= 2\mathrm{Re}(e^{2\mathrm{i}\theta})t^{2/3}ru\frac{\omega}{\cos\theta}\,(1+\mathcal{O}(\delta))\,(1+\mathcal{O}(\varepsilon))\\ &\leq \chi_2|r|\cdot t^{2/3}u\omega,\end{aligned} \tag{4.2.52}$$

for some positive constants χ_2, χ_3, by choosing δ and ε small enough. For large L and t_0, $-\chi_3 t^{1/3}\omega$ dominates over $\chi_2|r|$, so we can further estimate:

$$\int_0^\delta \mathrm{d}u\,\left|e^{t\hat{f}_3(ue^{\mathrm{i}\theta})+t^{2/3}\hat{f}_2(ue^{\mathrm{i}\theta})+\hat{f}_0(ue^{\mathrm{i}\theta})}\right| \leq \int_0^\infty \mathrm{d}u\, e^{-\chi_3 t\omega^2 u/2} \leq \frac{2}{\chi_3 t\omega^2}. \tag{4.2.53}$$

Combining this with (4.2.47) gives us

$$|\beta_t(r,s)| \leq e^{-s}\left(\frac{1}{2\sqrt{\pi\eta_0}}+2\left(1+\frac{2}{\chi_3 t\omega^2}\right)\right) \leq c_L e^{-s} \tag{4.2.54}$$

The bound on α_t can be obtained by the same line of arguments. In this case choose the contour $\Gamma' = \gamma_1' \cup \overline{\gamma_1'} \cup \gamma_3'$, with

$$\begin{aligned}\gamma_1' &= \{-1+ue^{2\pi\mathrm{i}/3}, u\in[2\omega,R]\}\\ \gamma_3' &= \{-1+\omega(-1+\mathrm{i}u\sqrt{3}), u\in[-1,1]\}.\end{aligned} \tag{4.2.55}$$

□

Proof of Proposition 4.6. We start with the first part of the kernel. It has an integral representation:

$$\phi_{n_1,n_2}(\xi_1,\xi_2) = \int_{\mathrm{i}\mathbb{R}-\delta} \mathrm{d}z\,\frac{e^{z(\xi_1-\xi_2)}}{(-z)^{n_2-n_1}}. \tag{4.2.56}$$

Inserting the scaling gives

$$t^{1/3}e^{\xi_1-\xi_2}\phi_{n_1,n_2}(\xi_1,\xi_2) = \frac{t^{1/3}}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\delta} \mathrm{d}z\,\frac{e^{(z+1)(\xi_1-\xi_2)}}{(-z)^{n_2-n_1}}. \tag{4.2.57}$$

Setting $\delta = 1$ and using the change of variables $z = -1 + t^{-1/3}\zeta$ as well as the shorthand $r = r_2 - r_1$ and $s = s_2 - s_1$, we have

$$(4.2.57) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \mathrm{d}\zeta \, \frac{e^{t^{-1/3}\zeta(\xi_1 - \xi_2)}}{(1 - t^{-1/3}\zeta)^{n_2 - n_1}} = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \mathrm{d}\zeta \, e^{-s\zeta} f_t(\zeta, r) \quad (4.2.58)$$

with

$$f_t(\zeta, r) = \frac{e^{-2t^{1/3} r\zeta}}{(1 - t^{-1/3}\zeta)^{2t^{2/3}r}} = e^{-2t^{1/3}r\zeta - 2t^{2/3} r \log(1 - t^{-1/3}\zeta)}. \quad (4.2.59)$$

Since this integral is 0 for $r \leq 0$ we can assume $r > 0$ from now on. The function $f_t(\zeta, r)$ satisfies the pointwise limit $\lim_{t\to\infty} f_t(\zeta, r) = e^{r\zeta^2}$, which is easy to see by Taylor expanding the logarithm in the exponent. Applying Bernoulli's inequality, we also obtain a t-independent integrable bound

$$\begin{aligned} |f_t(\zeta, r)| &= |1 - t^{-1/3}\zeta|^{-2t^{2/3}r} = \left(1 + t^{-2/3}|\zeta|^2\right)^{-t^{2/3}r} \\ &\leq (1 + r|\zeta|^2)^{-1}. \end{aligned} \quad (4.2.60)$$

Thus by dominated convergence

$$\left| \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \mathrm{d}\zeta \, \left(e^{-s\zeta} f_t(\zeta, r) - e^{-s\zeta + r\zeta^2} \right) \right| \leq \frac{1}{2\pi} \int_{\mathrm{i}\mathbb{R}} |\mathrm{d}\zeta| \left| f_t(\zeta, r) - e^{r\zeta^2} \right| \xrightarrow{t\to\infty} 0. \quad (4.2.61)$$

This implies that the convergence of the integral is uniform in s. The limit is easily identified as

$$\lim_{t\to\infty} -\phi^{\mathrm{resc}}_{r_1, r_2}(s_1, s_2) = -\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \mathrm{d}\zeta \, e^{-s\zeta + r\zeta^2} \mathbb{1}_{r>0} = -\frac{1}{\sqrt{4\pi r}} e^{-s^2/4r} \mathbb{1}_{r>0}, \quad (4.2.62)$$

which is the first part of the kernel $K_{\mathcal{A}_2}$.

The remaining kernel can be rewritten as integrals over the previously defined functions α and β. Therefore, choose the contours in such a way that $\mathrm{Re}(z - w) > 0$ is ensured.

$$\begin{aligned} &\mathcal{K}^{\mathrm{resc}}_0(r_1, s_1; r_2, s_2) = t^{1/3} e^{\xi_1 - \xi_2} \mathcal{K}_0(n_1, \xi_1; n_2, \xi_2) \\ &= \frac{t^{1/3}}{(2\pi \mathrm{i})^2} \int_{\mathrm{i}\mathbb{R} - \varepsilon} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z \frac{e^{tw^2/2 + \xi_1(w+1)}}{e^{tz^2/2 + \xi_2(z+1)}} \frac{(-w)^{n_1}}{(-z)^{n_2}} \frac{1}{w - z} \\ &= \frac{-t^{1/3}}{(2\pi \mathrm{i})^2} \int_{\mathrm{i}\mathbb{R} - \varepsilon} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z \frac{e^{t(w^2-1)/2 + \xi_1(w+1)}}{e^{t(z^2-1)/2 + \xi_2(z+1)}} \frac{(-w)^{n_1}}{(-z)^{n_2}} \int_0^\infty \mathrm{d}x \, t^{1/3} e^{-t^{1/3}x(z-w)} \\ &= -\int_0^\infty \mathrm{d}x \, \alpha_t(r_1, s_1 + x) \beta_t(r_2, s_2 + x) \end{aligned} \quad (4.2.63)$$

Using the previous lemmas we can deduce compact convergence of the kernel. Indeed (omitting the r-dependence for greater clarity) we can write:

$$\begin{aligned}\sup_{s_1,s_2\in[-L,L]}&\left|\int_0^\infty \mathrm{d}x\,\alpha_t(s_1+x)\beta_t(s_2+x)-\int_0^\infty \mathrm{d}x\,\alpha(s_1+x)\beta(s_2+x)\right|\\&\leq\int_0^\infty \mathrm{d}x\sup_{s_1,s_2\in[-L,L]}\left|\alpha_t(s_1+x)\beta_t(s_2+x)-\alpha(s_1+x)\beta(s_2+x)\right|.\end{aligned}\tag{4.2.64}$$

By Lemma 4.11 the integrand converges to zero for every $x>0$. Using Lemma 4.12 we can bound it by const $\cdot\, e^{-2x}$, thus ensuring that (4.2.64) goes to zero, i.e., $\mathcal{K}_0^{\mathrm{resc}}$ converges compactly. Applying the limit in (4.2.63) and inserting the expressions for α and β finishes the proof. □

Proof of Proposition 4.8. Inserting the bounds from Lemma 4.12 into (4.2.63) results in

$$|\mathcal{K}_0^{\mathrm{resc}}(r_1,s_1;r_2,s_2)|\leq\int_0^\infty \mathrm{d}x\,e^{-(s_1+x)}e^{-(s_2+x)}=\frac{1}{2}e^{-(s_1+s_2)}.\tag{4.2.65}$$

□

Chapter 5

Periodic initial conditions

The periodic initial conditions have been analyzed in detail in [FSW15b]. We refer the reader to this work for details and give only the main results as well as a sketch of the idea of the arguments. Notice that here the direction of space is reversed as compared to [FSW15b].

5.1 Determinantal structure

The first result is an expression for the joint distribution at fixed time t.

Proposition 5.1. *Let $\{x_n(t), n \in \mathbb{Z}\}$ be the system of one-sided reflected Brownian motions with initial condition $\vec{x}(0) = \vec{\zeta}^{\text{flat}}$. Then, for any finite subset S of $\mathbb{Z}$, it holds*

$$\mathbb{P}\left(\bigcap_{n\in S}\{x_n(t) \le a_k\}\right) = \det(\mathbb{1} - P_a K_t^{\text{flat}} P_a)_{L^2(\mathbb{R}\times S)}, \tag{5.1.1}$$

where $P_a(x,k) = \mathbb{1}_{(a_k,\infty)}(x)$ and the kernel K_t^{flat} is given by

$$\begin{aligned} K_t^{\text{flat}}(x_1,n_1;x_2,n_2) &= -\frac{(x_2-x_1)^{n_2-n_1-1}}{(n_2-n_1-1)!}\mathbb{1}(x_2\ge x_1)\mathbb{1}(n_2>n_1) \\ &+ \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_-} \mathrm{d}z \frac{e^{tz^2/2}e^{zx_1}(-z)^{n_1}}{e^{t\varphi(z)^2/2}e^{\varphi(z)x_2}(-\varphi(z))^{n_2}}. \end{aligned} \tag{5.1.2}$$

Here Γ_- is any path going from $\infty e^{-\theta\mathrm{i}}$ to $\infty e^{\theta\mathrm{i}}$ with $\theta \in [\pi/2, 3\pi/4)$, crossing the real axis to the left of -1, and such that the function

$$\varphi(z) = L_0(ze^z) \tag{5.1.3}$$

is continuous and bounded. Here L_0 is the Lambert-W function, i.e., the principal solution for w in $z = we^w$, see Figure 5.1.

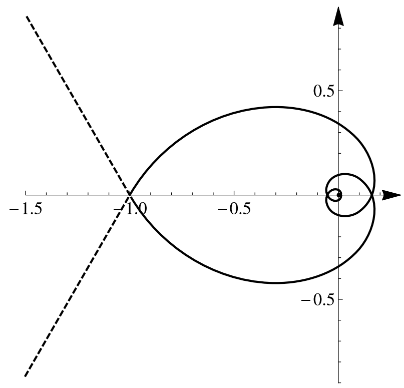

Figure 5.1: A possible choice for the contour Γ_- (dashed line) and its image by φ (solid line).

Interesting and quite unexpected is the appearance of the Lambert function, defined as the multi-valued inverse of the function $z \mapsto ze^z$. It has a branch structure similar to the logarithm, but slightly more complicated. The Lambert function is of use in many different areas like combinatorics, exponential towers, delay-differential equations [CM02] and several problems from physics [BJ00, JK04, CJV00]. This function has been studied in detail, *e.g.* see [BPL+00, CJK97, CGHJ93], with [CGH+96] the standard reference. However, the specific behaviour needed for our asymptotic analysis does not seem to be covered in the literature.

The proof of Proposition 5.1 proceeds in several steps. The first step develops a determinantal for the half-periodic system $\{x_n^{\text{half-flat}}(t), n \geq 1\}$ with initial condition $\vec{x}^{\,\text{half-flat}}(0) = \vec{\zeta}^{\,\text{half-flat}}$ (see Section 7.1), in a similar way as in the proof of Proposition 4.1 but with a slightly more complex choice of the orthogonal polynomials. This system is subsequently scaled to the full periodic case by

$$x_n(t) = \lim_{M\to\infty} \big(x_{n+M}^{\text{half-flat}}(t) - M\big). \tag{5.1.4}$$

The resulting kernel has a main part that is given by a double contour integral,

$$-\frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \int_{\Gamma_-} \mathrm{d}z \frac{e^{tz^2/2} e^{zx_1} (-z)^{n_1}}{e^{tw^2/2} e^{wx_2} (-w)^{n_2}} \frac{(1+w)e^w}{ze^z - we^w} \left(\frac{we^w}{ze^z}\right)^{n_2}, \tag{5.1.5}$$

which can then be simplified to the form of (5.1.2) using the Lambert-W function.

5.2 Asymptotics

The second main result contains a characterization of the law for the positions of the interacting Brownian motions in the large time limit. Due to the asymmetric reflections, the particles have an average velocity 1, so that the macroscopic position of $x_{\lfloor t \rfloor}(t)$ is around $2t$. For large time t the KPZ scaling theory predicts the positional fluctuations relative to the characteristic to be of order $t^{1/3}$. Nontrivial correlations between particles occur if the particle indices are of order $t^{2/3}$ apart from each other. The scaling is thus the same as the one used in the step initial conditions case.

Theorem 5.2. *With $\{x_n(t), n \in \mathbb{Z}\}$ being the system of one-sided reflected Brownian motions with initial condition $\vec{x}(0) = \zeta^{\mathrm{flat}}$, define the rescaled process*

$$r \mapsto X_t^{\mathrm{flat}}(r) = t^{-1/3} \left(x_{\lfloor t + 2t^{2/3} r \rfloor}(t) - 2t - 2t^{2/3} r \right). \tag{5.2.1}$$

In the sense of finite-dimensional distributions,

$$\lim_{t \to \infty} X_t^{\mathrm{flat}}(r) = 2^{1/3} \mathcal{A}_1(2^{-2/3} r). \tag{5.2.2}$$

The proof of this theorem relies on the usual asymptotic analysis of the kernel via steep descent. However, rather unconventional claims about the behaviour of the Lambert-W function $L_k(z)$ are necessary, which do not seem to be covered in the literature. For a special choice of the integration contour, these claims can be derived from the differential identity

$$L_k'(z) = \frac{L_k(z)}{z(1 + L_k(z))} \tag{5.2.3}$$

as well as some information about the branch structure (see also the proof of Proposition 7.3).

Remark 5.3. Due to the translational invariance, other choices of the rescaled process are possible, for which the asymptotic theorem still holds. For example, one could choose

$$r \mapsto \bar{X}_t^{\mathrm{flat}}(r) = t^{-1/3} \left(x_{\lfloor -t + 2t^{2/3} r \rfloor}(t) - 2t^{2/3} r \right), \tag{5.2.4}$$

which is the way it has been proven in [FSW15b] (up to constant factors).

Chapter 6

Poisson initial conditions

The last of the fundamental KPZ initial conditions is the stationary initial condition given by the Poisson process in our case. The final result is again an asymptotic theorem:

Theorem 6.1. *With $\{x_n(t), n \in \mathbb{Z}\}$ being the system of one-sided reflected Brownian motions with initial condition $\vec{x}(0) = \zeta^{\mathrm{stat}}(1,1)$, define the rescaled process*

$$r \mapsto X_t^{\mathrm{stat}}(r) = t^{-1/3}\big(x_{\lfloor t+2rt^{2/3}\rfloor}(t) - 2t - 2rt^{2/3}\big). \tag{6.0.1}$$

In the sense of finite-dimensional distributions,

$$\lim_{t\to\infty} X_t^{\mathrm{stat}}(r) \stackrel{d}{=} \mathcal{A}_{\mathrm{stat}}(r). \tag{6.0.2}$$

A crucial step towards this theorem is a determinantal formula for the fixed time distribution of the process under the initial condition with two different densities on $\mathbb{R}_+$ and $\mathbb{R}_-$:

Proposition 6.2. *Let $\{x_n(t), n \in \mathbb{Z}\}$ be the system of one-sided reflected Brownian motions with initial condition $\vec{x}(0) = \zeta^{\mathrm{stat}}(\lambda,\rho)$ for any $\lambda > \rho > 0$. For any finite subset S of $\mathbb{Z}_{\geq 0}$, it holds*

$$\mathbb{P}\Big(\bigcap_{n\in S}\{x_n(t)\leq a_n\}\Big) = \Big(1 + \frac{1}{\lambda-\rho}\sum_{n\in S}\frac{\mathrm{d}}{\mathrm{d}a_n}\Big)\det(\mathbb{1} - \chi_a\mathcal{K}_{\mathrm{stat}}\chi_a)_{L^2(S\times\mathbb{R})}, \tag{6.0.3}$$

where $\chi_a(n,\xi) = \mathbb{1}_{\xi>a_n}$. The kernel $\mathcal{K}_{\mathrm{stat}}$ is given by

$$\begin{aligned}\mathcal{K}_{\mathrm{stat}}(n_1,\xi_1;n_2,\xi_2) = &-\phi_{n_1,n_2}(\xi_1,\xi_2)\mathbb{1}_{n_2>n_1} + \mathcal{K}_0(n_1,\xi_1;n_2,\xi_2)\\ &+ (\lambda-\rho)f(n_1,\xi_1)g(n_2,\xi_2).\end{aligned} \tag{6.0.4}$$

where

$$\begin{aligned} \phi_{0,n_2}(\xi_1,\xi_2) &= \rho^{-n_2} e^{\rho\xi_2}, && \textit{for } n_2 \geq 0, \\ \phi_{n_1,n_2}(\xi_1,\xi_2) &= \frac{(\xi_2-\xi_1)^{n_2-n_1-1}}{(n_2-n_1-1)!}\mathbb{1}_{\xi_1\leq\xi_2}, && \textit{for } 1 \leq n_1 < n_2, \end{aligned} \tag{6.0.5}$$

and

$$\begin{aligned} \mathcal{K}_0(n_1,\xi_1;n_2,\xi_2) &= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathrm{i}\mathbb{R}-\varepsilon}\mathrm{d}w \oint_{\Gamma_0}\mathrm{d}z\, \frac{e^{tw^2/2+\xi_1 w}}{e^{tz^2/2+\xi_2 z}}\frac{(-w)^{n_1}}{(-z)^{n_2}}\frac{1}{w-z}, \\ f(n_1,\xi_1) &= \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\varepsilon}\mathrm{d}w\, \frac{e^{tw^2/2+\xi_1 w}(-w)^{n_1}}{w+\lambda}, \\ g(n_2,\xi_2) &= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{0,-\rho}}\mathrm{d}z\, \frac{e^{-tz^2/2-\xi_2 z}(-z)^{-n_2}}{z+\rho}, \end{aligned} \tag{6.0.6}$$

for any fixed $0 < \varepsilon < \lambda$.

Notice that this result holds for $\lambda > \rho$ only and not for the most interesting case $\lambda = \rho$. The latter can be accessed through a careful analytic continuation of the formula (6.0.3). One of the novelties of this work is to perform the analytic continuation *after* the scaling limit. This allows us to discover a new process, called *finite-step Airy*$_{\text{stat}}$ *process*, describing the large time limit close to stationarity.

In general, the limits $t \to \infty$ and $\lambda - \rho \downarrow 0$ do not commute. Therefore we have to consider $\lambda - \rho > 0$ (to be able to apply Proposition 6.2), but vanishing with a tuned scaling exponent as $t \to \infty$, a critical scaling. We set $\lambda - \rho = \delta t^{-1/3}$ for $\delta > 0$. These considerations give rise to the following positive-step asymptotic theorem:

Theorem 6.3. *With* $\{x_n^\rho(t), n \in \mathbb{Z}\}$ *being the system of one-sided reflected Brownian motions with initial condition* $\vec{x}(0) = \zeta^{\text{stat}}(1,\rho)$*, define the rescaled process*

$$r \mapsto X_t^{(\delta)}(r) = t^{-1/3}\Big(x_{\lfloor t+2rt^{2/3}\rfloor}^{(1-t^{-1/3}\delta)}(t) - 2t - 2rt^{2/3}\Big). \tag{6.0.7}$$

For every $\delta > 0$*, the rescaled process converges to the* finite-step Airy$_{\text{stat}}$ process

$$\lim_{t\to\infty} X_t^{(\delta)}(r) \stackrel{d}{=} \mathcal{A}_{\text{stat}}^{(\delta)}(r), \tag{6.0.8}$$

in the sense of finite-dimensional distributions.

Recognizing $X_t^{(0)}(r) = X_t^{\text{stat}}(r)$, in order to finally arrive at Theorem 6.1, we have to prove that the limit $t \to \infty$ commutes with $\delta \downarrow 0$ on the left hand side, as well as convergence of the finite-step Airy$_{\text{stat}}$ process to the standard Airy$_{\text{stat}}$ process. This is the content of Section 6.4.

6.1 Determinantal structure

To obtain a representation as a signed determinantal point process we have to introduce a new measure. This measure $\mathbb{P}_+$ coincides with $\mathbb{P}$ on the sigma algebra which is generated by $\zeta_{k+1}^{\rm stat} - \zeta_k^{\rm stat}$, $k \in \mathbb{Z}$, and the driving Brownian motions B_k, $k \in \mathbb{Z}$. But under $\mathbb{P}_+$, $\zeta_0^{\rm stat}$ is a random variable with an exponential distribution instead of being fixed at zero. Formally, $\mathbb{P}_+ = \mathbb{P} \otimes \mathbb{P}_0$, with $\mathbb{P}_0$ giving rise to $\zeta_0^{\rm stat} \sim \exp(\lambda - \rho)$, so that $\mathbb{P}$ is the result of conditioning $\mathbb{P}_+$ on the event $\{\zeta_0^{\rm stat} = 0\}$. This new measure satisfies a determinantal formula for the joint distribution at a fixed time.

Proposition 6.4. *Under the modified initial condition specified by $\mathbb{P}_+$, the joint density of the positions of the asymmetrically reflected Brownian motions $\{x_n(t), 0 \leq n \leq N-1\}$ is given by*

$$\mathbb{P}_+(\vec{x}(t) \in \mathrm{d}\vec{\xi}) = (\lambda - \rho)\lambda^{N-1} e^{-t\rho^2/2+\rho\xi_0} \det_{1\leq k,l\leq N}[\widetilde{F}_{k-l}(\xi_{N-l}, t)]\,\mathrm{d}\vec{\xi} \tag{6.1.1}$$

with

$$\widetilde{F}_k(\xi, t) := \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w\, \frac{e^{tw^2/2+\xi w} w^k}{w+\lambda}. \tag{6.1.2}$$

Proof of Proposition 6.4. The fixed time distribution can be obtained by integrating the transition density (4.1.1) over the initial condition. Denote by $p_+(\vec{\xi})$ the probability density of $\vec{x}(t)$, i.e., $\mathbb{P}_+(\vec{x}(t) \in \mathrm{d}\vec{\xi}) = p_+(\vec{\xi})\mathrm{d}\vec{\xi}$. $p_+(\vec{\xi})$ equals

$$\begin{aligned}
&\int_{W^N\cap\{\zeta_0>0\}} \mathrm{d}\vec{\zeta}\, e^{\rho(\xi_0-\zeta_0)-t\rho^2/2} (\lambda-\rho)\lambda^{N-1} e^{\rho\zeta_0} e^{-\lambda\zeta_{N-1}} \det_{1\leq k,l\leq N}[F_{k,l}(\xi_{N-l} - \zeta_{N-k}, t)] \\
=&(\lambda-\rho)\lambda^{N-1} e^{-t\rho^2/2+\rho\xi_0} \int_{W^N\cap\{\zeta_0>0\}} \mathrm{d}\vec{\zeta}\, e^{\lambda\zeta_{N-1}} \\
&\quad\times \det_{1\leq k,l\leq N}\left[\frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\mu} \mathrm{d}w_k\, e^{tw_k^2/2} e^{\xi_{N-l}w_k} e^{-\zeta_{N-k}w_k} w_k^{k-l}\right] \\
=&(\lambda-\rho)\lambda^{N-1} e^{-t\rho^2/2+\rho\xi_0} \sum_{\sigma\in\mathcal{S}_N} (-1)^{|\sigma|} \prod_{k=1}^N \int_{\mathrm{i}\mathbb{R}+\mu} \frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\, e^{tw_k^2/2} e^{\xi_{N-\sigma(k)}w_k} w_k^{k-\sigma(k)} \\
&\quad\times \int_0^\infty \mathrm{d}\zeta_0 \ldots \int_{\zeta_{N-2}}^\infty \mathrm{d}\zeta_{N-1}\, e^{-\lambda\zeta_{N-1}} e^{-\zeta_{N-1}w_1} e^{-\zeta_{N-2}w_2} \ldots e^{-\zeta_0 w_N} \\
=&(\lambda-\rho)\lambda^{N-1} e^{-t\rho^2/2+\rho\xi_0} \sum_{\sigma\in\mathcal{S}_N} (-1)^{|\sigma|} \prod_{k=1}^N \int_{\mathrm{i}\mathbb{R}+\mu} \frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\, \frac{e^{tw_k^2/2} e^{\xi_{N-\sigma(k)}w_k} w_k^{k-\sigma(k)}}{w_1+\cdots+w_k+\lambda}.
\end{aligned} \tag{6.1.3}$$

Since all w_k are integrated over the same contour, we can replace w_k by $w_{\sigma(k)}$:

$$\begin{aligned}(6.1.3) = (\lambda-\rho)\lambda^{N-1}e^{-t\rho^2/2+\rho\xi_0}& \\ \times \sum_{\sigma\in\mathcal{S}_N}(-1)^{|\sigma|}\prod_{k=1}^{N}\int_{\mathrm{i}\mathbb{R}+\mu}\frac{\mathrm{d}w_k}{2\pi\mathrm{i}}&\,\frac{e^{tw_k^2/2}e^{\xi_{N-\sigma(k)}w_{\sigma(k)}}w_{\sigma(k)}^{k-\sigma(k)}}{w_{\sigma(1)}+\dots+w_{\sigma(k)}+\lambda} \\ = (\lambda-\rho)\lambda^{N-1}e^{-t\rho^2/2+\rho\xi_0}\prod_{k=1}^{N}\int_{\mathrm{i}\mathbb{R}+\mu}&\frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\,e^{tw_k^2/2}e^{\xi_{N-k}w_k}w_k^{-k} \\ \times \sum_{\sigma\in\mathcal{S}_N}(-1)^{|\sigma|}\prod_{k=1}^{N}&\frac{w_{\sigma(k)}^k}{w_{\sigma(1)}+\dots+w_{\sigma(k)}+\lambda}.\end{aligned} \tag{6.1.4}$$

We apply Lemma 6.5 below to the sum and finally obtain

$$\begin{aligned}p_+(\vec{x}) &= (\lambda-\rho)\lambda^{N-1}e^{-t\rho^2/2+\rho\xi_0}\prod_{k=1}^{N}\int_{\mathrm{i}\mathbb{R}+\mu}\frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\,e^{tw_l^2/2}e^{\xi_{N-l}w_l}w_l^{-l}\det_{1\le k,l\le N}\left[\frac{w_l^k}{w_l+\lambda}\right] \\ &= (\lambda-\rho)\lambda^{N-1}e^{-t\rho^2/2+\rho\xi_0}\det_{1\le k,l\le N}\left[\widetilde{F}_{k-l}(\xi_{N-l},t)\right].\end{aligned} \tag{6.1.5}$$

□

Lemma 6.5. *Given $N\in\mathbb{Z}_{>0}$, $\lambda>0$ and $w_1,\dots,w_N\in\mathbb{C}\backslash\mathbb{R}_-$, the following identity holds:*

$$\sum_{\sigma\in\mathcal{S}_N}(-1)^{|\sigma|}\prod_{k=1}^{N}\frac{w_{\sigma(k)}^k}{w_{\sigma(1)}+\dots+w_{\sigma(k)}+\lambda} = \det_{1\le k,l\le N}\left[\frac{w_l^k}{w_l+\lambda}\right]. \tag{6.1.6}$$

Proof. We use induction on N. For $N=1$ the statement is trivial. For arbitrary N, rearrange the left hand side of (6.1.6) as

$$\begin{aligned}&\sum_{\sigma\in\mathcal{S}_N}(-1)^{|\sigma|}\prod_{k=1}^{N}\frac{w_{\sigma(k)}^k}{w_{\sigma(1)}+\dots+w_{\sigma(k)}+\lambda} \\ &\quad=\sum_{l=1}^{N}\frac{w_l^N}{w_1+\dots+w_N+\lambda}\sum_{\sigma\in S_N,\sigma(N)=l}(-1)^{|\sigma|}\prod_{k=1}^{N-1}\frac{w_{\sigma(k)}^k}{w_{\sigma(1)}+\dots+w_{\sigma(k)}+\lambda} \\ &\quad=\sum_{l=1}^{N}\frac{w_l^N}{w_1+\dots+w_N+\lambda}\sum_{\sigma\in S_N,\sigma(N)=l}(-1)^{|\sigma|}\prod_{k=1}^{N-1}\frac{w_{\sigma(k)}^k}{w_{\sigma(k)}+\lambda},\end{aligned} \tag{6.1.7}$$

where we applied the induction hypothesis to the second sum. Further,

$$
\begin{aligned}
(6.1.7) &= \sum_{\sigma\in\mathcal{S}_N}(-1)^{|\sigma|}\frac{w_{\sigma(N)}+\lambda}{w_1+\cdots+w_N+\lambda}\prod_{k=1}^N\frac{w^k_{\sigma(k)}}{w_{\sigma(k)}+\lambda}\\
&= \frac{1}{w_1+\cdots+w_N+\lambda}\prod_{l=1}^N\frac{w_l}{w_l+\lambda}\\
&\quad\times\left(\sum_{\sigma\in\mathcal{S}_N}(-1)^{|\sigma|}w_{\sigma(N)}\prod_{k=1}^N w^{k-1}_{\sigma(k)}+\lambda\sum_{\sigma\in\mathcal{S}_N}(-1)^{|\sigma|}\prod_{k=1}^N w^{k-1}_{\sigma(k)}\right)\\
&= \frac{1}{w_1+\cdots+w_N+\lambda}\prod_{l=1}^N\frac{w_l}{w_l+\lambda}\\
&\quad\times\left(\det_{1\le k,l\le N}\left[w_l^{k-1+\delta_{k,N}}\right]+\lambda\det_{1\le k,l\le N}\left[w_l^{k-1}\right]\right),
\end{aligned}
\tag{6.1.8}
$$

with $\delta_{k,N}$ being the Kronecker delta. Inserting the identity

$$
\det_{1\le k,l\le N}\left[w_l^{k-1+\delta_{k,N}}\right]=(w_1+\cdots+w_N)\det_{1\le k,l\le N}\left[w_l^{k-1}\right],
\tag{6.1.9}
$$

we arrive at

$$
(6.1.7)=\left(\prod_{l=1}^N\frac{w_l}{w_l+\lambda}\right)\det_{1\le k,l\le N}\left[w_l^{k-1}\right]=\det_{1\le k,l\le N}\left[\frac{w_l^k}{w_l+\lambda}\right].
\tag{6.1.10}
$$

To show (6.1.9) we introduce the variable w_{N+1} and consider the factorization

$$
\det_{1\le k,l\le N+1}\left[w_l^{k-1}\right]=\prod_{i=1}^N(w_{N+1}-w_i)\det_{1\le k,l\le N}\left[w_l^{k-1}\right],
\tag{6.1.11}
$$

which follows directly from the explicit formula for a Vandermonde determinant. Expanding the determinant on the left hand side along the $(N+1)$-th column gives an explicit expression in terms of monomials in w_{N+1}. Examining the coefficient of w_{N+1}^{N-1} on the left and right hand side respectively provides (6.1.9). □

We can rewrite the measure in Proposition 6.4 in terms of a conditional L-ensemble (see Lemma 3.4 of [BFPS07] reported here as Lemma 3.5) and obtain a Fredholm determinant expression for the joint distribution of any subsets of particles position. Then it remains to relate the law under $\mathbb{P}_+$ and

$\mathbb{P}$, which is the law of the reflected Brownian motions specified by the initial condition (2.3.9). This is made using a *shift argument*, analogue to the one used for the polynuclear growth model with external sources [BR00, IS04] or in the totally asymmetric simple exclusion process [PS04, FS06, BFP10].

Proof of Proposition 6.2. The proof is divided into two steps. In *Step 1* we determine the distribution under $\mathbb{P}_+$ and in *Step 2* we extend this result via a shift argument to $\mathbb{P}$.

Step 1. We consider the law of the process under $\mathbb{P}_+$ for now. The first part of the proof is identical to the proof of Proposition 3.5 [FSW15b], so it is only sketched here. Using repeatedly the identity

$$\widetilde{F}_k(\xi,t) = \int_{-\infty}^{\xi} \mathrm{d}x\, \widetilde{F}_{k+1}(x,t), \tag{6.1.12}$$

relabeling $\xi_1^k := \xi_{k-1}$, and introducing new variables ξ_l^k for $2 \le l \le k \le N$, we can write

$$\det_{1\le k,l\le N}\left[\widetilde{F}_{k-l}(\xi_1^{N+1-l},t)\right] = \int_{\mathcal{D}'} \det_{1\le k,l\le N}\left[\widetilde{F}_{k-1}(\xi_l^N,t)\right] \prod_{2\le l\le k\le N} \mathrm{d}\xi_l^k, \tag{6.1.13}$$

where $\mathcal{D}' = \{\xi_l^k \in \mathbb{R}, 2 \le l \le k \le N | x_l^k \le x_{l-1}^{k-1}\}$. Using the antisymmetry of the determinant and encoding the constraint on the integration variables into indicator functions, we obtain that the measure (6.1.1) is a marginal of

$$\begin{aligned} &\mathrm{const}\cdot e^{\rho\xi_1^1} \prod_{n=2}^{N} \det_{1\le i,j\le n}\left[\mathbb{1}_{\xi_i^{n-1}\le \xi_j^n}\right] \det_{1\le k,l\le N}\left[\widetilde{F}_{k-1}(\xi_l^N,t)\right] \\ &= \mathrm{const}\cdot \prod_{n=1}^{N} \det_{1\le i,j\le n}\left[\tilde{\phi}_n(\xi_i^{n-1},\xi_j^n)\right] \det_{1\le k,l\le N}\left[\widetilde{F}_{k-1}(\xi_l^N,t)\right] \end{aligned} \tag{6.1.14}$$

with

$$\begin{aligned} \tilde{\phi}_n(x,y) &= \mathbb{1}_{x\le y}, \quad \text{for } n \ge 2 \\ \tilde{\phi}_1(x,y) &= e^{\rho y}, \end{aligned} \tag{6.1.15}$$

and using the convention that $\xi_n^{n-1} \le y$ always holds.

The measure (6.1.14) has the appropriate form for applying Lemma 3.5. The composition of the $\tilde{\phi}$ functions can be evaluated explicitly as

$$\begin{aligned} \tilde{\phi}_{0,n}(x,y) &= (\tilde{\phi}_1 * \cdots * \tilde{\phi}_n)(x,y) = \rho^{1-n} e^{\rho y}, && \text{for } n \ge 1, \\ \tilde{\phi}_{m,n}(x,y) &= (\tilde{\phi}_{m+1} * \cdots * \tilde{\phi}_n)(x,y) = \frac{(y-x)^{n-m-1}}{(n-m-1)!}\mathbb{1}_{x\le y}, && \text{for } n > m \ge 1. \end{aligned} \tag{6.1.16}$$

Define

$$\Psi^n_{n-k}(\xi) := \frac{(-1)^{n-k}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \, \frac{e^{tw^2/2+\xi w} w^{n-k}}{w+\lambda}, \tag{6.1.17}$$

for $n, k \geq 1$ and some $0 < \varepsilon < \lambda$. In the case $n \geq k$ the integrand has no poles in the region $|w| < \lambda$, which implies $\Psi^n_{n-k} = (-1)^{n-k} \widetilde{F}_{n-k}$. The straightforward recursion

$$(\tilde{\phi}_n * \Psi^n_{n-k})(\xi) = \Psi^{n-1}_{n-1-k}(\xi) \tag{6.1.18}$$

eventually leads to condition (3.3.3) being satisfied.

The space V_n is generated by

$$\{\tilde{\phi}_{0,n}(\xi^0_1, x), \dots, \tilde{\phi}_{n-2,n}(\xi^{n-2}_{n-1}, x), \tilde{\phi}_{n-1,n}(\xi^{n-1}_n, x)\}, \tag{6.1.19}$$

so a basis for V_n is given by

$$\{e^{\rho x}, x^{n-2}, x^{n-3}, \dots, x, 1\}. \tag{6.1.20}$$

Choose functions Φ^n_{n-k} as follows

$$\Phi^n_{n-k}(\xi) = \begin{cases} \frac{(-1)^{n-k}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, \frac{z+\lambda}{e^{tz^2/2+\xi z} z^{n-k+1}} & 2 \leq k \leq n, \\ \frac{(-1)^{n-1}}{2\pi \mathrm{i}} \oint_{\Gamma_{0,-\rho}} \mathrm{d}z \, \frac{z+\lambda}{e^{tz^2/2+\xi z} z^{n-1}(z+\rho)} & k = 1. \end{cases} \tag{6.1.21}$$

By residue calculating rules, Φ^n_{n-k} is a polynomial of order $n-k$ for $k \geq 2$ and a linear combination of 1 and $e^{\rho\xi}$ for $k = 1$, so these functions indeed generate V_n. To show (3.3.5) for $\ell \geq 2$, we decompose the scalar product as follows:

$$\int_{\mathbb{R}_-} \mathrm{d}\xi \, \Psi^n_{n-k}(\xi) \Phi^n_{n-\ell}(\xi) + \int_{\mathbb{R}_+} \mathrm{d}\xi \, \Psi^n_{n-k}(\xi) \Phi^n_{n-\ell}(\xi). \tag{6.1.22}$$

Since $n - k \geq 0$ we are free to choose the sign of ε as necessary. For the first term, we choose $\varepsilon < 0$ and the path Γ_0 close enough to zero, such that always $\mathrm{Re}(w - z) > 0$. Then, we can take the integral over ξ inside and obtain

$$\int_{\mathbb{R}_-} \mathrm{d}\xi \, \Psi^n_{n-k}(\xi) \Phi^n_{n-\ell}(\xi) = \frac{(-1)^{k-l}}{(2\pi \mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z \, \frac{e^{tw^2/2} w^{n-k}(z+\lambda)}{e^{tz^2/2} z^{n-\ell+1}(w+\lambda)(w-z)}. \tag{6.1.23}$$

For the second term, we choose $\varepsilon > 0$ to obtain $\mathrm{Re}(w - z) < 0$. Then again, we can take the integral over ξ inside and arrive at the same expression up to a minus sign. The net result of (6.1.22) is a residue at $w = z$, which is given by

$$\frac{(-1)^{k-l}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, z^{\ell-k-1} = \delta_{k,\ell}. \tag{6.1.24}$$

The case $\ell = 1$ uses the same decomposition and requires the choice $\varepsilon > \rho$ resp. $\varepsilon < 0$, finally leading to

$$(6.1.22) = \frac{(-1)^{k-1}}{2\pi \mathrm{i}} \oint_{\Gamma_{0,-\rho}} \mathrm{d}z \, \frac{z^{1-k}}{z+\rho} = \delta_{k,1}. \tag{6.1.25}$$

Furthermore, both $\tilde{\phi}_n(\xi_n^{n-1}, x)$ and $\Phi_0^n(\xi)$ are constants, so the kernel has a simple form (compare with (3.3.6))

$$\tilde{\mathcal{K}}(n_1,\xi_1;n_2,\xi_2) = -\tilde{\phi}_{n_1,n_2}(\xi_1,\xi_2)\mathbb{1}_{n_2>n_1} + \sum_{k=1}^{n_2} \Psi^{n_1}_{n_1-k}(\xi_1)\Phi^{n_2}_{n_2-k}(\xi_2). \tag{6.1.26}$$

However, the relabeling $\xi_1^k := \xi_{k-1}$ included an index shift, so the kernel of our system is actually

$$\begin{aligned} \mathcal{K}_{\mathrm{stat}}(n_1,\xi_1;n_2,\xi_2) &= \tilde{\mathcal{K}}(n_1+1,\xi_1;n_2+1,\xi_2) \\ &= -\phi_{n_1,n_2}(\xi_1,\xi_2)\mathbb{1}_{n_2>n_1} + \sum_{k=1}^{n_2} \Psi^{n_1+1}_{n_1-k+1}(\xi_1)\Phi^{n_2+1}_{n_2-k+1}(\xi_2). \end{aligned} \tag{6.1.27}$$

Note that we are free to extend the summation over k up to infinity, since the integral expression for $\Phi^n_{n-k}(\xi)$ vanishes for $k > n$ anyway. Taking the sum inside the integrals we can write

$$\sum_{k\geq 1} \Psi^{n_1+1}_{n_1-k+1}(\xi_1)\Phi^{n_2+1}_{n_2-k+1}(\xi_2) = \frac{1}{(2\pi\mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \oint_{\Gamma_{0,-\rho}} \mathrm{d}z \, \frac{e^{tw^2/2+\xi_1 w}(-w)^{n_1}}{e^{tz^2/2+\xi_2 z}(-z)^{n_2}} \eta(w,z), \tag{6.1.28}$$

with

$$\eta(w,z) = \frac{z+\lambda}{(w+\lambda)(z+\rho)} + \sum_{k\geq 2} \frac{z^{k-2}(z+\lambda)}{w^{k-1}(w+\lambda)}. \tag{6.1.29}$$

By choosing contours such that $|z| < |w|$, we can use the formula for a geometric series, resulting in

$$\begin{aligned} \eta(w,z) &= \frac{z+\lambda}{(w+\lambda)(z+\rho)} + \frac{z+\lambda}{(w+\lambda)w} \frac{1}{1-z/w} \\ &= \frac{1}{w-z} + \frac{\lambda-\rho}{(w+\lambda)(z+\rho)}. \end{aligned} \tag{6.1.30}$$

Inserting this expression back into (6.1.28) gives the kernel (6.0.4), which governs the multidimensional distributions of $x_n(t)$ under the measure $\mathbb{P}_+$, namely

$$\mathbb{P}_+\Big(\bigcap_{n\in S} \{x_n(t) \leq a_n\}\Big) = \det(\mathbb{1} - \chi_a \mathcal{K}_{\mathrm{stat}} \chi_a)_{L^2(S\times\mathbb{R})}. \tag{6.1.31}$$

Step 2. The distributions under $\mathbb{P}$ and under $\mathbb{P}_+$ can be related via the following *shift argument*. Introducing the shorthand

$$\widetilde{\mathcal{E}}(S,\vec{a}) := \bigcap_{n\in S}\{x_n(t) \le a_n\}, \tag{6.1.32}$$

we have

$$\begin{aligned}
\mathbb{P}_+(\widetilde{\mathcal{E}}(S,\vec{a})) &= \int_{\mathbb{R}_+} \mathrm{d}x\, \mathbb{P}_+(x_0(0)\in \mathrm{d}x)\mathbb{P}_+(\widetilde{\mathcal{E}}(S,\vec{a})|x_0(0)=x) \\
&= \int_{\mathbb{R}_+} \mathrm{d}x\,(\lambda-\rho)e^{-(\lambda-\rho)x}\mathbb{P}(\widetilde{\mathcal{E}}(S,\vec{a}-x)) \\
&= -e^{-(\lambda-\rho)x}\,\mathbb{P}(\widetilde{\mathcal{E}}(S,\vec{a}-x))\Big|_0^\infty + \int_{\mathbb{R}_+} \mathrm{d}x\, e^{-(\lambda-\rho)x}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\widetilde{\mathcal{E}}(S,\vec{a}-x)) \\
&= \mathbb{P}(\widetilde{\mathcal{E}}(S,\vec{a})) - \int_{\mathbb{R}_+} \mathrm{d}x\, e^{-(\lambda-\rho)x}\sum_{n\in S}\frac{\mathrm{d}}{\mathrm{d}a_n}\mathbb{P}(\widetilde{\mathcal{E}}(S,\vec{a})) \\
&= \mathbb{P}(\widetilde{\mathcal{E}}(S,\vec{a})) - \frac{1}{\lambda-\rho}\sum_{n\in S}\frac{\mathrm{d}}{\mathrm{d}a_n}\mathbb{P}_+(\widetilde{\mathcal{E}}(S,\vec{a})).
\end{aligned} \tag{6.1.33}$$

Combining the identity

$$\mathbb{P}(\widetilde{\mathcal{E}}(S,\vec{a})) = \left(1+\frac{1}{\lambda-\rho}\sum_{n\in S}\frac{\mathrm{d}}{\mathrm{d}a_n}\right)\mathbb{P}_+(\widetilde{\mathcal{E}}(S,\vec{a})) \tag{6.1.34}$$

with (6.1.31) finishes the proof. □

6.2 Asymptotic analysis

Noticing that the change in variables

$$x \mapsto \lambda^{-1}x \qquad t\mapsto \lambda^{-2}t \tag{6.2.1}$$

reproduces the same system with new parameters $\widetilde{\lambda}=1$ and $\widetilde{\rho}=\frac{\rho}{\lambda}$, we can restrict our considerations to $\lambda = 1$ without loss of generality. According to (6.0.7) we use the scaled variables

$$\begin{aligned}
n_i &= t + 2t^{2/3}r_i \\
\xi_i &= 2t + 2t^{2/3}r_i + t^{1/3}s_i \\
\rho &= 1 - t^{-1/3}\delta,
\end{aligned} \tag{6.2.2}$$

with $\delta > 0$. Correspondingly, consider the rescaled (and conjugated) kernel

$$\mathcal{K}^{\mathrm{resc}}(r_1, s_1; r_2, s_2) = t^{1/3} e^{\xi_1 - \xi_2} \mathcal{K}(n_1, \xi_1; n_2, \xi_2), \tag{6.2.3}$$

which decomposes into

$$\begin{aligned} \mathcal{K}^{\mathrm{resc}}(r_1, s_1; r_2, s_2) = -\phi^{\mathrm{resc}}_{r_1, r_2}(s_1, s_2) \mathbb{1}_{r_1 < r_2} + \mathcal{K}_0^{\mathrm{resc}}(r_1, s_1; r_2, s_2) \\ + \delta f^{\mathrm{resc}}(r_1, s_1) g^{\mathrm{resc}}(r_2, s_2), \end{aligned} \tag{6.2.4}$$

by

$$\begin{aligned} f^{\mathrm{resc}}(r_1, s_1) &= e^{-t/2 + \xi_1} f(n_1, \xi_1) \\ g^{\mathrm{resc}}(r_2, s_2) &= e^{t/2 - \xi_2} g(n_2, \xi_2). \end{aligned} \tag{6.2.5}$$

As before, the proof of Theorem 6.3 relies both on the pointwise convergence as well as a uniform bound for the rescaled kernel:

Proposition 6.6. *Consider any r_1, r_2 in a bounded set and fixed L. Then, the kernel converges as*

$$\lim_{t \to \infty} \mathcal{K}^{\mathrm{resc}}_{\mathrm{stat}}(r_1, s_1; r_2, s_2) = K^{\delta}(r_1, s_1; r_2, s_2) \tag{6.2.6}$$

uniformly for $(s_1, s_2) \in [-L, L]^2$.

Corollary 6.7. *Consider r_1, r_2 fixed. For any L there exists t_0 such that for $t > t_0$ the bound*

$$|\mathcal{K}^{\mathrm{resc}}_{\mathrm{stat}}(r_1, s_1; r_2, s_2)| \leq \mathrm{const}_L \tag{6.2.7}$$

holds for all $(s_1, s_2) \in [-L, L]^2$.

Proposition 6.8. *For fixed r_1, r_2, L and $\delta > 0$ there exists $t_0 > 0$ such that the estimate*

$$\left| \delta f^{\mathrm{resc}}(r_1, s_1) g^{\mathrm{resc}}(r_2, s_2) \right| \leq \mathrm{const} \cdot e^{-\min\{\delta, 1\} s_2} \tag{6.2.8}$$

holds for any $t > t_0$ and $s_1, s_2 > 0$.

Now we can prove the asymptotic theorem:

Proof of Theorem 6.3. The joint distributions of the rescaled process $X_t^{(\delta)}(r)$ under the measure $\mathbb{P}_+$ are given by the Fredholm determinant with series expansion

$$\begin{aligned} &\mathbb{P}_+\left(\bigcap_{k=1}^{m} \left\{ X_t^{(\delta)}(r_k) \leq s_k \right\} \right) \\ &\quad = \sum_{N \geq 0} \frac{(-1)^N}{N!} \sum_{i_1, \ldots, i_N = 1}^{m} \int \prod_{k=1}^{N} \mathrm{d}x_k \, \mathbb{1}_{x_k > \xi_{i_k}} \det_{1 \leq k, l \leq N} \left[\mathcal{K}_{\mathrm{stat}}(n_{i_k}, \xi_k; n_{i_l}, \xi_l) \right], \end{aligned} \tag{6.2.9}$$

where n_i and ξ_i are understood as in (6.2.2). Using the change of variables $\sigma_k = t^{-1/3}(x_k - 2t - 2t^{2/3} r_{i_k})$, we obtain

$$\begin{aligned}(6.2.9) = \sum_{N\geq 0} \frac{(-1)^N}{N!} \sum_{i_1,\dots,i_N=1}^{m} \int \prod_{k=1}^{N} \mathrm{d}\sigma_k \, \mathbb{1}_{\sigma_k > s_{i_k}} \\ \times \det_{1\leq k,l\leq N} \left[\mathcal{K}_{\mathrm{stat}}^{\mathrm{resc}}(r_k, \sigma_k; r_l, \sigma_l) \frac{(1+\sigma_l^2)^{m+1-i_l}}{(1+\sigma_k^2)^{m+1-i_k}} \right],\end{aligned} \tag{6.2.10}$$

where the fraction inside the determinant is a new conjugation, which does not change the value of the determinant. By Proposition 4.8 we have

$$|\mathcal{K}_0^{\mathrm{resc}}(r_1, s_1; r_2, s_2)| \leq \frac{1}{2} e^{-(s_1+s_2)} \leq \mathrm{const} \cdot e^{-\min\{\delta,1\} s_2}. \tag{6.2.11}$$

Using this estimate together with Corollary 6.7 and Propositions 6.8, 4.9, we can bound the (k, l)-coefficient inside the determinant by

$$\mathrm{const}_1 \left(e^{-|\sigma_k - \sigma_l|} \mathbb{1}_{i_k < i_l} + e^{-\min\{\delta,1\}\sigma_l} \right) \frac{(1+\sigma_l^2)^{m+1-i_l}}{(1+\sigma_k^2)^{m+1-i_k}}, \tag{6.2.12}$$

assuming the r_k are ordered. The bounds

$$\begin{aligned}\frac{(1+x^2)^i}{(1+y^2)^j} e^{-|x-y|} &\leq \mathrm{const}_2 \frac{1}{1+y^2}, \quad \text{for } i < j, \\ \frac{(1+x^2)^i}{(1+y^2)^j} e^{-\min\{\delta,1\}x} &\leq \mathrm{const}_3 \frac{1}{1+y^2}, \quad \text{for } j \geq 1,\end{aligned} \tag{6.2.13}$$

lead to

$$(6.2.12) \leq \mathrm{const}_4 \frac{1}{1+\sigma_k^2}. \tag{6.2.14}$$

Using the Hadamard bound on the determinant, the integrand of (6.2.10) is therefore bounded by

$$\mathrm{const}_4^N N^{N/2} \prod_{k=1}^{N} \mathbb{1}_{\sigma_k > s_{i_k}} \frac{\mathrm{d}\sigma_k}{1+\sigma_k^2}, \tag{6.2.15}$$

which is integrable. Furthermore,

$$|(6.2.9)| \leq \sum_{N\geq 0} \frac{\mathrm{const}_5^N N^{N/2}}{N!}, \tag{6.2.16}$$

which is summable, since the factorial grows like $(N/e)^N$, i.e., much faster than the nominator. Dominated convergence thus allows to interchange the limit $t \to \infty$ with the integral and the infinite sum. The pointwise convergence comes from Proposition 6.6, thus

$$\lim_{t\to\infty} \mathbb{P}_+\left(\bigcap_{k=1}^{m} \left\{X_t^{(\delta)}(r_k) \leq s_k\right\}\right) = \det\left(\mathbb{1} - \chi_s K^\delta \chi_s\right)_{L^2(\{r_1,\dots,r_m\}\times\mathbb{R})}. \tag{6.2.17}$$

It remains to show that the convergence carries over to the measure $\mathbb{P}$. The identity

$$\frac{\mathrm{d}s_i}{\mathrm{d}\xi_i} = t^{-1/3} = \delta^{-1}(1-\rho) \tag{6.2.18}$$

leads to

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \left\{X_t^{(\delta)}(r_k) \leq s_k\right\}\right) = \left(1 + \frac{1}{\delta}\sum_{i=1}^{m}\frac{\mathrm{d}}{\mathrm{d}s_i}\right)\mathbb{P}_+\left(\bigcap_{k=1}^{m} \left\{X_t^{(\delta)}(r_k) \leq s_k\right\}\right). \tag{6.2.19}$$

Notice that in (6.2.10), s_i appears only in the indicator function, so differentiation just results in one of the σ_k not being integrated but instead being set to s_i. Using the same bounds as before we can again show interchangeability of the limit $t \to \infty$ with the remaining integrals and the infinite sum. □

Proof of Proposition 6.6. The kernel $\mathcal{K}_{\mathrm{stat}}^{\mathrm{resc}}$ consists of three parts, where compact convergence for the first two parts of the kernel comes directly from Proposition 4.6.

As we did with $\mathcal{K}_0^{\mathrm{resc}}$, we rewrite the third part, which is the product of f^{resc} and g^{resc}, as integrals over the previously defined functions α and β:

$$\begin{aligned} f^{\mathrm{resc}}(r_1, s_1) &= \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w\, \frac{e^{t(w^2-1)/2+\xi_1(w+1)}(-w)^{n_1}}{w+1} \\ &= 1 + \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\varepsilon-1} \mathrm{d}w\, \frac{e^{t(w^2-1)/2+\xi_1(w+1)}(-w)^{n_1}}{w+1} \\ &= 1 - \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\varepsilon-1} \mathrm{d}w\, e^{t(w^2-1)/2+\xi_1(w+1)}(-w)^{n_1}\int_0^\infty \mathrm{d}x\, t^{1/3}e^{t^{1/3}x(w+1)} \\ &= 1 - \int_0^\infty \mathrm{d}x\, \alpha_t(r_1, s_1+x). \end{aligned} \tag{6.2.20}$$

Now,

$$\begin{aligned} &\sup_{s_1\in[-L,L]}\left|\int_0^\infty \mathrm{d}x\, \alpha_t(r_1,s_1+x) - \int_0^\infty \mathrm{d}x\, \alpha(r_1,s_1+x)\right| \\ &\quad\leq \int_0^\infty \mathrm{d}x \sup_{s_1,s_2\in[-L,L]} \left|\alpha_t(r_1,s_1+x) - \alpha(r_1,s_1+x)\right|. \end{aligned} \tag{6.2.21}$$

By Lemma 4.11 the integrand converges to zero for every $x > 0$. Using Lemma 4.12 we can bound it by const $\cdot e^{-2x}$, thus ensuring that (6.2.21) goes to zero, i.e., $f^{\mathrm{resc}}(r_1, s_1)$ converges to $f_{r_1}(s_1)$ uniformly on compact sets.

Similarly,

$$g^{\mathrm{resc}}(r_2, s_2) = \mathrm{Res}_{g,-\rho} + \int_0^\infty \mathrm{d}x\, \beta_t(r_2, s_2 + x) e^{\delta x}, \tag{6.2.22}$$

with

$$\mathrm{Res}_{g,-\rho} = e^{t^{2/3}\delta - t^{1/3}\delta^2/2 - \xi_2 t^{-1/3}\delta}(1 - t^{-1/3}\delta)^{-n_2}. \tag{6.2.23}$$

The residuum satisfies the limit

$$\lim_{t\to\infty} \mathrm{Res}_{g,-\rho} = e^{\delta^3/3 + r_2\delta^2 - s_2\delta} \tag{6.2.24}$$

uniformly in s_2. By the same argument, uniform convergence holds again.

□

Proof of Proposition 6.8. To bound the product $\delta f^{\mathrm{resc}}(r_1, s_1) g^{\mathrm{resc}}(r_2, s_2)$, we use Lemma 4.12 in the representations (6.2.20) and (6.2.22):

$$\begin{aligned}\left|\delta f^{\mathrm{resc}}(r_1, s_1) g^{\mathrm{resc}}(r_2, s_2)\right| &\le \delta \left(1 + \int_0^\infty \mathrm{d}x\, e^{-(s_1+x)}\right) \\ &\quad \cdot \left(\mathrm{Res}_{g,-\rho} + \int_0^\infty \mathrm{d}x\, e^{-(s_2+x)} e^{\delta x}\right) \\ &= \delta\left(1 + e^{-s_1}\right)\left(\mathrm{Res}_{g,-\rho} + \frac{e^{-s_2}}{1-\delta}\right)\end{aligned} \tag{6.2.25}$$

Since the convergence (6.2.24) is uniform in s_2 we can deduce

$$\left|\mathrm{Res}_{g,-\rho}\right| \le \mathrm{const}_1 \cdot e^{-s_2\delta}, \tag{6.2.26}$$

resulting in

$$|(6.2.25)| \le \mathrm{const} \cdot e^{-\min\{\delta,1\} s_2}. \tag{6.2.27}$$

□

6.3 Path-integral style formula

Using the results from [BCR15] we can transform the formula for the multidimensional probability distribution of the finite-step $\mathrm{Airy}_{\mathrm{stat}}$ process from the current form involving a Fredholm determinant over the space $L^2(\{r_1,\dots,r_m\}\times\mathbb{R})$ into a path-integral style form, where the Fredholm determinant is over the simpler space $L^2(\mathbb{R})$. The result of [BCR15] can not be applied at the stage of finite time as one of the assumption is not satisfied.

Proposition 6.9. *For any parameters $\chi_k\in\mathbb{R}$, $1\le k\le m$, satisfying*

$$0<\chi_m<\dots<\chi_2<\chi_1<\max_{i<j}\{r_j-r_i,\delta\}, \tag{6.3.1}$$

define the multiplication operator $(M_{r_i}f)(x)=m_{r_i}(x)f(x)$, with

$$m_{r_i}(x)=\begin{cases} e^{-\chi_i x} & \text{for } x\ge 0\\ e^{x^2} & \text{for } x<0.\end{cases} \tag{6.3.2}$$

Writing $K^\delta_{r_i}(x,y):=K^\delta(r_i,x;r_i,y)$, the finite-dimensional distributions of the finite-step Airy$_{\rm stat}$ process are given by

$$\mathbb{P}\left(\bigcap_{k=1}^m\{\mathcal{A}^{(\delta)}_{\rm stat}(r_k)\le s_k\}\right)=\left(1+\frac{1}{\delta}\sum_{i=1}^m\frac{\mathrm{d}}{\mathrm{d}s_i}\right)\det\left(\mathbb{1}+M_{r_1}QM_{r_1}^{-1}\right)_{L^2(\mathbb{R})}, \tag{6.3.3}$$

with

$$Q=-K^\delta_{r_1}+\bar P_{s_1}V_{r_1,r_2}\bar P_{s_2}\cdots V_{r_{m-1},r_m}\bar P_{s_m}V_{r_m,r_1}K^\delta_{r_1}, \tag{6.3.4}$$

where $\bar P_s=\mathbb{1}-P_s$ denotes the projection operator on $(-\infty,s)$.

Remark 6.10. The operator V_{r_j,r_i} for $r_i<r_j$ is defined only on the range of $K^\delta_{r_i}$ and acts on it in the following way:

$$V_{r_j,r_i}K_{r_i,r_k}=K_{r_j,r_k},\qquad V_{r_j,r_i}f_{r_i}=f_{r_j}. \tag{6.3.5}$$

In particular, we have also $V_{r_j,r_i}\mathbf{1}=\mathbf{1}$.

Proof. We will denote conjugations by the operator M by a hat in the following way:

$$\begin{aligned}\widehat V_{r_i,r_j}&=M_{r_i}V_{r_i,r_j}M_{r_j}^{-1}, & \widehat f_{r_i}&=M_{r_i}f_{r_i},\\ \widehat K^\delta_{r_i}&=M_{r_i}K^\delta_{r_i}M_{r_i}^{-1}, & \widehat g_{r_i}&=g_{r_i}M_{r_i}^{-1},\\ \widehat K_{r_i,r_j}&=M_{r_i}K_{r_i,r_j}M_{r_j}^{-1}.\end{aligned} \tag{6.3.6}$$

Applying the conjugation also in the determinant in (9.2.10), the identity we have to show is:

$$\begin{aligned}&\det\left(\mathbb{1}-\chi_s\widehat K^\delta\chi_s\right)_{L^2(\{r_1,\dots,r_m\}\times\mathbb{R})}\\ &=\det\left(\mathbb{1}-\widehat K^\delta_{r_1}+\bar P_{s_1}\widehat V_{r_1,r_2}\bar P_{s_2}\cdots\widehat V_{r_{m-1},r_m}\bar P_{s_m}\widehat V_{r_m,r_1}\widehat K^\delta_{r_1}\right)_{L^2(\mathbb{R})}\end{aligned} \tag{6.3.7}$$

This is done by applying Theorem 1.1 [BCR15].

It has three groups of assumptions we have to prove. We merged them into two by choosing the multiplication operators of Assumption 3 to be the identity.

Assumption 1

(i) The operators $P_{s_i}\widehat{V}_{r_i,r_j}$, $P_{s_i}\widehat{K}^{\delta}_{r_i}$, $P_{s_i}\widehat{V}_{r_i,r_j}\widehat{K}^{\delta}_{r_j}$ and $P_{s_j}\widehat{V}_{r_j,r_i}\widehat{K}^{\delta}_{r_i}$ for $r_i < r_j$ preserve $L^2(\mathbb{R})$ and are trace class in $L^2(\mathbb{R})$.

(ii) The operator $\widehat{V}_{r_i,r_1}\widehat{K}^{\delta}_{r_1} - \bar{P}_{s_i}\widehat{V}_{r_i,r_{i+1}}\bar{P}_{s_{i+1}}\cdots\widehat{V}_{r_{m-1},r_m}\bar{P}_{s_m}\widehat{V}_{r_m,r_1}\widehat{K}^{\delta}_{r_1}$ preserves $L^2(\mathbb{R})$ and is trace class in $L^2(\mathbb{R})$.

Assumption 2

(i) Right-invertibility: $\widehat{V}_{r_i,r_j}\widehat{V}_{r_j,r_i}\widehat{K}^{\delta}_{r_i} = \widehat{K}^{\delta}_{r_i}$

(ii) Semigroup property: $\widehat{V}_{r_i,r_j}\widehat{V}_{r_j,r_k} = \widehat{V}_{r_i,r_k}$

(iii) Reversibility relation: $\widehat{V}_{r_i,r_j}\widehat{K}^{\delta}_{r_j} = \widehat{K}^{\delta}_{r_i}\widehat{V}_{r_i,r_j}$

The semigroup property is clear. To see the reversibility relation, start from the contour integral representation (9.1.4) of K_{r_j,r_j} and (9.2.13) of f_{r_j} and use the Gaussian identity:

$$\int_{\mathbb{R}} \mathrm{d}z \frac{1}{\sqrt{4\pi(r_j - r_i)}} e^{-(z-x)^2/4(r_j-r_i)} e^{-r_j W^2 + zW} = e^{-r_i W^2 + xW}. \tag{6.3.8}$$

This results in $\widehat{V}_{r_i,r_j}\widehat{K}^{\delta}_{r_j} = \widehat{K}_{r_i,r_j} + \delta \widehat{f}_{r_i} \otimes \widehat{g}_{r_j}$. On the other hand we have

$$\int_{\mathbb{R}} \mathrm{d}z \frac{1}{\sqrt{4\pi(r_j - r_i)}} e^{-(z-y)^2/4(r_j-r_i)} e^{r_i Z^2 - zZ} = e^{r_j Z^2 - yZ}, \tag{6.3.9}$$

so $\widehat{K}^{\delta}_{r_i}\widehat{V}_{r_i,r_j} = \widehat{K}_{r_i,r_j} + \delta \widehat{f}_{r_i} \otimes \widehat{g}_{r_j}$, which proves Assumption 2 (iii). Noticing Remark 6.10, the right-invertibility follows immediately.

Assumption 1 (ii) can be deduced from Assumption 1 (i) as shown in Remark 3.2, [BCR15]. Using the previous identities we thus are left to show that the three operators $P_{s_i}\widehat{V}_{r_i,r_j}$, for $r_i < r_j$, as well as $P_{s_i}\widehat{K}_{r_i,r_j}$ and $P_{s_i}\widehat{f}_{r_i} \otimes \widehat{g}_{r_j}$, for arbitrary r_i, $r_j \in \mathbb{R}$, are all L^2-bounded and trace class.

First notice that $V_{r_i,r_j}(x,y) = V_{0,r_j-r_i}(-x,-y)$. Using the shorthand $r = r_j - r_i$ and inserting this into the integral representation (9.1.4) of V we have

$$V_{r_i,r_j}(x,y) = e^{\frac{2}{3}r^3}\int_{\mathbb{R}} \mathrm{d}\lambda\, \mathrm{Ai}(-x+\lambda)e^{r(-y+\lambda)}\mathrm{Ai}(r^2-y+\lambda) = \left(V^{(1)}V_r^{(2)}\right)(x,y), \tag{6.3.10}$$

with the new operators

$$\begin{aligned} V^{(1)}(x,y) &= \mathrm{Ai}(-x+y) \\ V_r^{(2)}(x,y) &= e^{\frac{2}{3}r^3}e^{r(x-y)}\mathrm{Ai}(r^2+x-y). \end{aligned} \tag{6.3.11}$$

Introducing yet another operator, $(Nf)(x) = \exp\left(-(\chi_i+\chi_j)x/2\right)f(x)$, we can write

$$P_{s_i}\widehat{V}_{r_i,r_j} = (P_{s_i}M_{r_i}V^{(1)}N^{-1})(NV_r^{(2)}M_{r_j}^{-1}). \tag{6.3.12}$$

The Hilbert-Schmidt norm of the first factor is given by

$$\begin{aligned} &\int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\, \left|(P_{s_i}M_{r_i}V^{(1)}N^{-1})(x,y)\right|^2 \\ &\qquad = \int_{s_1}^{\infty}\mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y\, m_{r_i}^2(x)\mathrm{Ai}^2(-x+y)e^{(\chi_i+\chi_j)y} \\ &\qquad = \int_{s_1}^{\infty}\mathrm{d}x\, m_{r_i}^2(x)e^{(\chi_i+\chi_j)x}\int_{\mathbb{R}}\mathrm{d}z\,\mathrm{Ai}^2(z)e^{(\chi_i+\chi_j)z}. \end{aligned} \tag{6.3.13}$$

The asymptotic behaviour of the Airy function and the inequalities $\chi_i > \chi_j > 0$ imply that both integrals are finite. Similarly,

$$\begin{aligned} &\int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\, \left|(NV_r^{(2)}M_{r_j}^{-1})(x,y)\right|^2 \\ &\qquad = e^{\frac{4}{3}r^3}\int_{\mathbb{R}^2}\mathrm{d}x\,\mathrm{d}y\, e^{-(\chi_i+\chi_j)x}e^{2r(x-y)}\mathrm{Ai}^2(r^2+x-y)m_{r_j}^{-2}(y) \\ &\qquad = e^{\frac{4}{3}r^3}\int_{\mathbb{R}}\mathrm{d}z\, e^{-(\chi_i+\chi_j)z}e^{2rz}\mathrm{Ai}^2(r^2+z)\int_{\mathbb{R}}\mathrm{d}y\, m_{r_j}^{-2}(y)e^{-(\chi_i+\chi_j)y} < \infty, \end{aligned} \tag{6.3.14}$$

where we used $2r > \chi_i + \chi_j$ as well. As a product of two Hilbert-Schmidt operators, $P_{s_i}\widehat{V}_{r_i,r_j}$ is thus L^2-bounded and trace class.

We decompose the operator $\widehat{K}_{r_i,r_j}$ as

$$P_{s_i}\widehat{K}_{r_i,r_j} = (P_{s_i}M_{r_i}K'_{-r_i}P_0)(P_0K'_{r_j}M_{r_j}^{-1}) \tag{6.3.15}$$

where

$$K'_r(x,y) = e^{\frac{2}{3}r^3}e^{r(x+y)}\mathrm{Ai}(r^2+x+y). \tag{6.3.16}$$

Again, we bound the Hilbert-Schmidt norms,

$$\begin{aligned}\int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\,\left|(P_{s_i}M_{r_i}K'_{-r_i}P_0)(x,y)\right|^2 \\ &= e^{-\frac{4}{3}r_i^3}\int_{s_i}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,m_{r_i}^2(x)e^{-2r_i(x+y)}\mathrm{Ai}^2(r_i^2+x+y) \quad (6.3.17)\\ &\le e^{-\frac{4}{3}r_i^3}\int_{s_i}^{\infty}\mathrm{d}x\,m_{r_i}^2(x)\int_{s_i}^{\infty}\mathrm{d}z\,e^{-2r_iz}\mathrm{Ai}^2(r_i^2+z)<\infty,\end{aligned}$$

as well as

$$\begin{aligned}\int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\,\left|(P_0K'_{r_j}M_{r_j}^{-1})(x,y)\right|^2 \\ &= e^{\frac{4}{3}r_j^3}\int_0^{\infty}\mathrm{d}x\int_{\mathbb{R}}\mathrm{d}y\,e^{2r_j(x+y)}\mathrm{Ai}^2(r_j^2+x+y)m_{r_j}^{-2}(y) \quad (6.3.18)\\ &= e^{\frac{4}{3}r_j^3}\int_{\mathbb{R}}\mathrm{d}y\,m_{r_j}^{-2}(y)\int_y^{\infty}\mathrm{d}z\,e^{2r_jz}\mathrm{Ai}^2(r_j^2+z).\end{aligned}$$

The superexponential decay of the Airy function implies that for every $c_1>|r_j|$ we can find c_2 such that $e^{2r_jz}\mathrm{Ai}^2(r_j^2+z)\le c_2e^{-c_1z}$. This proves finiteness of the integrals.

Regarding the last operator, start by decomposing it as

$$P_{s_i}\widehat{f}_{r_i}\otimes\widehat{g}_{r_j}=(P_{s_i}\widehat{f}_{r_i}\otimes\phi)(\phi\otimes\widehat{g}_{r_j}) \quad (6.3.19)$$

for some function ϕ with L^2-norm 1. Next, notice that

$$\int_{\mathbb{R}^2}\mathrm{d}x\,\mathrm{d}y\,\left|(P_{s_i}M_{r_i}f_{r_i}\otimes\phi)(x,y)\right|^2=\int_{s_i}^{\infty}\mathrm{d}x\,m_{r_i}^2(x)f_{r_i}^2(x) \quad (6.3.20)$$

It is easy to see that $\lim_{s\to\infty}f_{r_i}(s)=1$, so f_{r_i} is bounded on the area of integration. But then the $m_{r_i}^2$ term ensures the decay, implying that the integral is finite. Furthermore,

$$\int_{\mathbb{R}^2}\mathrm{d}x\,\mathrm{d}y\,\left|(\phi\otimes g_{r_j}M_{r_j}^{-1})(x,y)\right|^2=\int_{\mathbb{R}}\mathrm{d}y\,m_{r_j}^{-2}(y)g_{r_j}^2(y). \quad (6.3.21)$$

Analyzing the asymptotic behaviour of g_{r_j} we see that for large positive arguments, the first part decays exponentially with rate $-\delta$ and the second part even superexponentially. $\delta>\chi_j$ thus gives convergence on the positive half-line. For negative arguments, it is sufficient to see that g_{r_j} does not grow faster than exponentially. □

6.4 Analytic continuation

We know from Theorem 6.3 and Proposition 6.9 that:

$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \left\{X_t^{(\delta)}(r_k) \leq s_k\right\}\right) = \left(1 + \frac{1}{\delta}\sum_{i=1}^{m} \frac{\mathrm{d}}{\mathrm{d}s_i}\right) \det(\mathbb{1} - \widehat{\mathcal{P}}\widehat{K}_{r_1}^{\delta}). \quad (6.4.1)$$

In this section we prove the main Theorem 6.1 by extending this equation to $\delta = 0$. The right hand side can actually be analytically continued for all $\delta \in \mathbb{R}$ (see Proposition 6.13). Additionally we have to show that the left hand side is continuous at $\delta = 0$. This proof relies mainly on Proposition 6.11, which gives a bound on the exit point of the maximizing path from the lower boundary in the last passage percolation model.

Proof of Theorem 6.1. We adopt the point of view of last passage percolation discussed in the Sections 2.2.1 and 2.3.3. By the stationarity property, we know that $x_n(t) \stackrel{d}{=} \widetilde{x}_n^{(0)}(t)$ for $n \geq 0$. We use the latter interpretation, i.e. consider $x_n(t)$ as being constructed from ζ_n with $n \geq 1$, B_n with $n \geq 1$ and $\widetilde{B}_0$, and base coupling arguments also on these variables being fixed. In this way, we have

$$x_n(t) = L_{(0,0)\to(t,n)}, \quad (6.4.2)$$

with background weights on both boundaries, i.e. Dirac weights $\zeta_k - \zeta_{k-1}$ on $(0, k)$, $k \geq 1$, and a Lebesgue measure of density ρ on the line $\{0\} \times \mathbb{R}_+$ additionally to the white noise $\mathrm{d}\widetilde{B}_0$.

We add superscripts to x, L and w indicating the choice of ρ, while λ is always fixed at 1. It is clear that for any path $\vec{\pi}$ the weight $w^{(\rho)}(\vec{\pi})$ is non-decreasing in ρ. But then the supremum is non-decreasing, too, and:

$$x_n^{(\rho)}(t) \leq x_n^{(1)}(t), \quad (6.4.3)$$

for $\rho < 1$. We know that there exists a unique maximizing path $\vec{\pi}^* \in \Pi(0, 0; t; n)$. We can therefore define $Z_n(t) := s_0^*$, the exit point from the lower boundary specifically with $\rho = 1$. We want to derive the inequality

$$x_n^{(1)}(t) \leq x_n^{(\rho)}(t) + (1 - \rho)Z_n(t). \quad (6.4.4)$$

This can be seen as follows:

$$\begin{aligned} L^{(1)}_{(0,0)\to(t,n)} - (1-\rho)Z_n(t) &= \sup_{\vec{\pi}\in\Pi(0,0;t,n)} w^{(1)}(\vec{\pi}) - (1-\rho)Z_n(t) \\ &= w^{(1)}(\vec{\pi}^*) - (1-\rho)s_0^* = w^{(\rho)}(\vec{\pi}^*). \end{aligned} \quad (6.4.5)$$

Note that $\vec{\pi}^*$ maximizes $w^{(1)}(\vec{\pi})$ and not necessarily $w^{(\rho)}(\vec{\pi})$. In particular we have

$$w^{(\rho)}(\vec{\pi}^*) \leq \sup_{\vec{\pi}\in\Pi(0,0;t,n)} w^{(\rho)}(\vec{\pi}) = L^{(\rho)}_{(0,0)\to(t,n)}. \tag{6.4.6}$$

Combining the last two equations results in (6.4.4).

(6.4.3) and (6.4.4) imply that for the rescaled processes $X_t^{(\delta)}$, see (6.0.7), we have

$$X_t^{(\delta)}(r) \leq X_t^{\mathrm{stat}}(r) \leq X_t^{(\delta)}(r) + \delta t^{-2/3} Z_{t+2t^{2/3}r}(t). \tag{6.4.7}$$

For any $\varepsilon > 0$ it holds

$$\begin{aligned}\mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t^{(\delta)}(r_k) \leq s_k\}\Big) &\geq \mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t^{\mathrm{stat}}(r_k) \leq s_k\}\Big)\\ &\geq \mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t^{(\delta)}(r_k) + \delta t^{-2/3} Z_{t+2t^{2/3}r}(t) \leq s_k\}\Big)\\ &\geq \mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t^{(\delta)}(r_k) \leq s_k - \varepsilon\}\Big) - \sum_{k=1}^{m}\mathbb{P}\left(\delta t^{-2/3} Z_{t+2t^{2/3}r}(t) > \varepsilon\right).\end{aligned} \tag{6.4.8}$$

Then, taking $t \to \infty$, we obtain

$$\begin{aligned}\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_{\mathrm{stat}}^{(\delta)}(r_k) \leq s_k\}\Big) &\geq \limsup_{t\to\infty}\mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t^{\mathrm{stat}}(r_k) \leq s_k\}\Big)\\ &\geq \liminf_{t\to\infty}\mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t^{\mathrm{stat}}(r_k) \leq s_k\}\Big)\\ &\geq \mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_{\mathrm{stat}}^{(\delta)}(r_k) \leq s_k - \varepsilon\}\Big) - \sum_{k=1}^{m}\limsup_{t\to\infty}\mathbb{P}\left(Z_{t+2t^{2/3}r}(t) > t^{2/3}\varepsilon/\delta\right).\end{aligned} \tag{6.4.9}$$

Using Proposition 6.11 on the last term and Proposition 6.13 on the other terms, we can now take the limit $\delta \downarrow 0$, resulting in

$$\begin{aligned}\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_{\mathrm{stat}}(r_k) \leq s_k\}\Big) &\geq \limsup_{t\to\infty}\mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t^{\mathrm{stat}}(r_k) \leq s_k\}\Big)\\ &\geq \liminf_{t\to\infty}\mathbb{P}\Big(\bigcap_{k=1}^{m}\{X_t^{\mathrm{stat}}(r_k) \leq s_k\}\Big) \geq \mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_{\mathrm{stat}}(r_k) \leq s_k - \varepsilon\}\Big).\end{aligned} \tag{6.4.10}$$

Continuity of (9.1.15) in the s_k finishes the proof. □

Proposition 6.11. *For any $r \in \mathbb{R}$,*

$$\lim_{\beta\to\infty} \limsup_{t\to\infty} \mathbb{P}\left(Z_{t+2t^{2/3}r}(t) > \beta t^{2/3}\right) = 0. \tag{6.4.11}$$

Proof. By scaling of t and β, (6.4.11) is equivalent to

$$\lim_{\beta\to\infty} \limsup_{t\to\infty} \mathbb{P}\left(Z_t(t + 2t^{2/3}r) > \beta t^{2/3}\right) = 0, \tag{6.4.12}$$

for any $r \in \mathbb{R}$, which is the limit we are showing. We introduce some new events:

$$\begin{aligned} M_\beta &:= \{Z_t(t + 2t^{2/3}r) > \beta t^{2/3}\} \\ E_\beta &:= \{L_{(0,0)\to(\beta t^{2/3},0)} + L_{(\beta t^{2/3},0)\to(t+2t^{2/3}r,t)} \leq 2t + 2t^{2/3}r + st^{1/3}\} \\ N_\beta &:= \{L_{(0,0)\to(t+2t^{2/3}r,t)} \leq 2t + 2t^{2/3}r + t^{1/3}s\}, \end{aligned} \tag{6.4.13}$$

where L is to be understood as in the proof of Theorem 6.1, with $\rho = 1$. Notice that if M_β occurs, then

$$L_{(0,0)\to(t+2t^{2/3}r,t)} = L_{(0,0)\to(\beta t^{2/3},0)} + L_{(\beta t^{2/3},0)\to(t+2t^{2/3}r,t)}, \tag{6.4.14}$$

resulting in $M_\beta \cap E_\beta \subseteq N_\beta$. We arrive at the inequality:

$$\mathbb{P}(M_\beta) = \mathbb{P}(M_\beta \cap E_\beta) + \mathbb{P}(M_\beta \cap E_\beta^c) \leq \mathbb{P}(N_\beta) + \mathbb{P}(E_\beta^c). \tag{6.4.15}$$

We further define new random variables

$$\begin{aligned} \xi^{(t)}_{\text{spiked}} &= \frac{L_{(\beta t^{2/3},0)\to(t+2t^{2/3}r,t)} - 2t - 2t^{2/3}(r-\beta/2)}{t^{1/3}} + (r-\beta/2)^2, \\ \xi^{(t)}_{\text{GUE}} &= \frac{L^{\text{step}}_{(0,1)\to(t+2t^{2/3}r,t)} - 2t - 2t^{2/3}r}{t^{1/3}} + r^2, \\ \xi^{(t)}_{\text{N}} &= \frac{L_{(0,0)\to(\beta t^{2/3},0)} - \beta t^{2/3}}{\sqrt{\beta} t^{1/3}}, \end{aligned} \tag{6.4.16}$$

where L^{step} is to be understood as the last passage percolation time *without* boundary weights. By Theorem 8.1, for any fixed $r \in \mathbb{R}$,

$$\xi^{(t)}_{\text{GUE}} \xrightarrow{d} \xi_{\text{GUE}}, \tag{6.4.17}$$

where ξ_{GUE} has the GUE Tracy-Widom distribution. $\xi^{(t)}_{\text{spiked}}$ follows the distribution of the largest eigenvalue of a critically spiked GUE matrix, as will be shown in Lemma 6.12. $\xi^{(t)}_{\text{N}}$ has the distribution of a standard normal random variable ξ_{N} for any $\beta > 0$, $t > 0$.

Combining these definitions, we have:

$$\mathbb{P}(E_\beta) = \mathbb{P}\big(\sqrt{\beta}\xi_{\mathrm{N}}^{(t)} + \xi_{\mathrm{spiked}}^{(t)} \le (r-\beta/2)^2 + s\big). \tag{6.4.18}$$

Fix $s = 3r^2 - \beta^2/16$, such that:

$$\left(r - \frac{\beta}{2}\right)^2 + s = 4r^2 - r\beta + \frac{\beta^2}{16} + \frac{\beta^2}{8} \ge \frac{\beta^2}{8}. \tag{6.4.19}$$

Using the independence of $\xi_{\mathrm{N}}^{(t)}$ and $\xi_{\mathrm{spiked}}^{(t)}$, we obtain

$$\begin{aligned}\mathbb{P}(E_\beta) &\ge \mathbb{P}\left(\sqrt{\beta}\xi_{\mathrm{N}}^{(t)} + \xi_{\mathrm{spiked}}^{(t)} \le \frac{\beta^2}{16} + \frac{\beta^2}{16}\right)\\ &\ge \mathbb{P}\left(\xi_{\mathrm{N}}^{(t)} \le \frac{\beta^{3/2}}{16} \text{ and } \xi_{\mathrm{spiked}}^{(t)} \le \frac{\beta^2}{16}\right)\\ &= \mathbb{P}\left(\xi_{\mathrm{N}}^{(t)} \le \frac{\beta^{3/2}}{16}\right)\mathbb{P}\left(\xi_{\mathrm{spiked}}^{(t)} \le \frac{\beta^2}{16}\right)\end{aligned} \tag{6.4.20}$$

Further, the inequality

$$L^{\mathrm{step}}_{(0,1)\to(t+2t^{2/3}r,t)} \le L_{(0,0)\to(t+2t^{2/3}r,t)} \tag{6.4.21}$$

leads to

$$\mathbb{P}(N_\beta) \le \mathbb{P}\big(\xi_{\mathrm{GUE}}^{(t)} \le 4r^2 - \beta^2/16\big). \tag{6.4.22}$$

Inserting (6.4.20) and (6.4.22) into (6.4.15), we arrive at

$$\mathbb{P}(M_\beta) \le \mathbb{P}\left(\xi_{\mathrm{GUE}}^{(t)} \le 4r^2 - \frac{\beta^2}{16}\right) + 1 - \mathbb{P}\left(\xi_{\mathrm{N}}^{(t)} \le \frac{\beta^{3/2}}{16}\right)\mathbb{P}\left(\xi_{\mathrm{spiked}}^{(t)} \le \frac{\beta^2}{16}\right) \tag{6.4.23}$$

By (6.4.17) and Lemma 6.12 we can take limits:

$$\begin{aligned}0 &\le \limsup_{\beta\to\infty}\limsup_{t\to\infty}\mathbb{P}(M_\beta)\\ &\le \lim_{\beta\to\infty}\left[\mathbb{P}\left(\xi_{\mathrm{GUE}} \le 4r^2 - \frac{\beta^2}{16}\right) + 1 - \mathbb{P}\left(\xi_{\mathrm{N}} \le \frac{\beta^{3/2}}{16}\right)\mathbb{P}\left(\xi_{\mathrm{spiked}}(\beta) \le \frac{\beta^2}{16}\right)\right]\\ &= 0.\end{aligned} \tag{6.4.24}$$

□

Lemma 6.12. *Let $r \in \mathbb{R}$ be fixed. For any $\beta > 2(r+1)$, as $t \to \infty$, the random variable*

$$\xi_{\mathrm{spiked}}^{(t)} = \frac{L_{(\beta t^{2/3},0)\to(t+2t^{2/3}r,t)} - 2t - 2t^{2/3}(r-\beta/2)}{t^{1/3}} + (r-\beta/2)^2 \tag{6.4.25}$$

converges in distribution,

$$\xi^{(t)}_{\text{spiked}} \xrightarrow{d} \xi_{\text{spiked}}(\beta). \tag{6.4.26}$$

In addition, $\xi_{\text{spiked}}(\beta)$ satisfies

$$\lim_{\beta\to\infty} \mathbb{P}\left(\xi_{\text{spiked}}(\beta) \leq \beta^2/16\right) = 1. \tag{6.4.27}$$

Proof. The family of processes $L_{(\beta t^{2/3},0)\to(\beta t^{2/3}+t,n)}$ indexed by $n \in \mathbb{Z}_{\geq 0}$ and time parameter $t \geq 0$ is precisely a marginal of Warren's process with drifts, starting at zero, as defined in [FF14]. In our case only the first particle has a drift of 1, and all the others zero. By Theorem 2 [FF14], the fixed time distribution of this process is given by the distribution of the largest eigenvalue of a spiked $n \times n$ GUE matrix, where the spikes are given by the drifts.

Thus we can apply the results on spiked random matrices, more concretely we want to apply Theorem 1.1 [BW11], with the potential given by $V(x) = -x^2/2$. Since

$$L^* := L_{(\beta t^{2/3},0)\to(t+2t^{2/3}r,n)} \tag{6.4.28}$$

represents a $n\times n$ GUE matrix diffusion $M(t)$ at time $t = t+2t^{2/3}(r-\beta/2)$, it is distributed according to the density

$$p_n(M) = \frac{1}{Z_n} \exp\left(-\frac{\operatorname{Tr}(M - t\mathrm{I}_{11})^2}{2t}\right), \tag{6.4.29}$$

where I_{11} is a $n \times n$ matrix with a one at entry $(1, 1)$ and zeros elsewhere. In order to apply the theorem we need the density given in equation (1) [BW11], i.e., consider the scaled quantity $L^*/\sqrt{nt}$. The size of the first-order spike is then:

$$a = t/\sqrt{nt} = \sqrt{1 + 2t^{-1/3}(r - \beta/2)} = 1 + (r - \beta/2)t^{-1/3} + \mathcal{O}(t^{-2/3}). \tag{6.4.30}$$

We are thus in the neighbourhood of the critical value $\mathbf{a}_c = 1$. For $\alpha \geq 0$, let

$$C_\alpha(\xi) = \int_{-\infty}^{0} e^{\alpha x} \operatorname{Ai}(x + \xi)\mathrm{d}x. \tag{6.4.31}$$

With $F_{\text{GUE}}(s)$ being the cumulative distribution function of the GUE Tracy-Widom distribution, and $K_{0,0}(s_1, s_2)$ as in (9.1.2), define:

$$F_1(s;\alpha) = F_{\text{GUE}}(s)\Big(1 - \big\langle(1 - P_s K_{0,0} P_s)^{-1} C_\alpha, P_s \operatorname{Ai}\big\rangle\Big). \tag{6.4.32}$$

Applying (28) [BW11], we have

$$n^{2/3}(L^*/\sqrt{nt}-2) \to \xi_{\text{spiked}}(\beta), \tag{6.4.33}$$

with

$$\mathbb{P}\left(\xi_{\text{spiked}}(\beta) \leq \beta^2/16\right) = F_1(\beta^2/16, \alpha), \tag{6.4.34}$$

where $\alpha = \beta/2 - r$. Since in our case $\alpha > 1$, we can estimate:

$$|C_\alpha(\xi)| \leq \int_{-\infty}^{0} e^{\alpha x} e^{-x-\xi} \mathrm{d}x = e^{-\xi}\frac{1}{\alpha - 1}. \tag{6.4.35}$$

Combining this with the usual bounds on the Airy kernel and the Airy function, we see that as $\beta \to \infty$, the scalar product in (6.4.32) converges to zero and we are left with the limit of F_0 which is one.

On the other hand,

$$n^{2/3}(L^*/\sqrt{nt}-2) \leq s \quad \Leftrightarrow \quad L^* \leq \sqrt{nt}(2+n^{-2/3}s), \tag{6.4.36}$$

and

$$\sqrt{nt}(2+n^{-2/3}s) = 2t+2t^{2/3}(r-\beta/2)+t^{1/3}\left(s-(r-\beta/2)^2\right)+\mathcal{O}(1), \tag{6.4.37}$$

from which the claim follows. ∎

Proposition 6.13. *The function $\delta \mapsto \delta^{-1}\det(\mathbb{1}-\widehat{\mathcal{P}}\widehat{K}^{\delta}_{r_1})$ can be extended analytically in the domain $\delta \in \mathbb{R}$. Its value at $\delta = 0$ is given by*

$$G_m(\vec{r},\vec{s})\det\left(\mathbb{1}-\mathcal{P}K\right)_{L^2(\mathbb{R})}. \tag{6.4.38}$$

Proof. We use the identity $\det(\mathbb{1}+A)\det(\mathbb{1}+B) = \det(\mathbb{1}+A+B+AB)$ and Lemma 9.8 to factorize

$$\begin{aligned}\delta^{-1}\det\left(\mathbb{1}-\widehat{\mathcal{P}}\widehat{K}^{\delta}_{r_1}\right) &= \delta^{-1}\det(\mathbb{1}-\widehat{\mathcal{P}}\widehat{K}^{\delta}_{r_1}) = \delta^{-1}\det\left(\mathbb{1}-\widehat{\mathcal{P}}\widehat{K}-\delta\widehat{\mathcal{P}}\widehat{f}_{r_1}\otimes\widehat{g}_{r_1}\right)\\ &= \delta^{-1}\det\left(\mathbb{1}-\delta(\mathbb{1}-\widehat{\mathcal{P}}\widehat{K})^{-1}\widehat{\mathcal{P}}\widehat{f}_{r_1}\otimes\widehat{g}_{r_1}\right)\cdot\det\left(\mathbb{1}-\widehat{\mathcal{P}}\widehat{K}\right)\\ &= \left(\delta^{-1}-\left\langle(\mathbb{1}-\widehat{\mathcal{P}}\widehat{K})^{-1}\widehat{\mathcal{P}}\widehat{f}_{r_1},\widehat{g}_{r_1}\right\rangle\right)\cdot\det\left(\mathbb{1}-\widehat{\mathcal{P}}\widehat{K}\right)\\ &= \left(\delta^{-1}-\left\langle(\mathbb{1}-\mathcal{P}K)^{-1}\mathcal{P}f_{r_1},g_{r_1}\right\rangle\right)\cdot\det\left(\mathbb{1}-\mathcal{P}K\right).\end{aligned} \tag{6.4.39}$$

Since the second factor is independent of δ, the remaining task is the analytic continuation of the first. Using (9.2.13), decompose f_{r_1} as

$$f_{r_1}(s) = 1 + \frac{1}{2\pi\mathrm{i}}\int_{\langle 0}\mathrm{d}W\frac{e^{-W^3/3-r_1W^2+sW}}{W} = 1 + f^*(s). \tag{6.4.40}$$

Now,

$$\begin{aligned}
\langle P_{s_1}\mathbf{1}, g_{r_1}\rangle &= \int_{s_1}^{\infty} \mathrm{d}s\, \frac{1}{2\pi \mathrm{i}} \int_{0\langle\delta} \mathrm{d}Z \frac{e^{Z^3/3 + r_1 Z^2 - sZ}}{Z-\delta} \\
&= \frac{1}{2\pi \mathrm{i}} \int_{0\langle\delta} \mathrm{d}Z \frac{e^{Z^3/3 + r_1 Z^2 - s_1 Z}}{Z(Z-\delta)} \\
&= \frac{1}{\delta} + \frac{1}{2\pi \mathrm{i}} \int_{\langle 0,\delta} \mathrm{d}Z \frac{e^{Z^3/3 + r_1 Z^2 - s_1 Z}}{Z(Z-\delta)} = \frac{1}{\delta} - \mathcal{R}_\delta .
\end{aligned} \tag{6.4.41}$$

The function $\mathcal{R}_\delta$ is analytic in $\delta \in \mathbb{R}$. Using these two identities as well as $(\mathbb{1} - \mathcal{P}K)^{-1} = \mathbb{1} + (\mathbb{1} - \mathcal{P}K)^{-1}\mathcal{P}K$, we can rearrange the inner product as follows:

$$\begin{aligned}
\frac{1}{\delta} &- \left\langle (\mathbb{1} - \mathcal{P}K)^{-1}\mathcal{P} f_{r_1}, g_{r_1} \right\rangle \\
&= \frac{1}{\delta} - \left\langle (\mathbb{1} - \mathcal{P}K)^{-1}\mathcal{P}\mathbf{1} + (\mathbb{1} - \mathcal{P}K)^{-1}\mathcal{P} f^*, g_{r_1} \right\rangle \\
&= \frac{1}{\delta} - \left\langle \mathcal{P}\mathbf{1} + (\mathbb{1} - \mathcal{P}K)^{-1}(\mathcal{P}K\mathcal{P}\mathbf{1} + \mathcal{P} f^*), g_{r_1} \right\rangle \\
&= \frac{1}{\delta} - \langle P_{s_1}\mathbf{1}, g_{r_1}\rangle - \left\langle (\mathcal{P} - P_{s_1})\mathbf{1} + (\mathbb{1} - \mathcal{P}K)^{-1}(\mathcal{P}K\mathcal{P}\mathbf{1} + \mathcal{P} f^*), g_{r_1} \right\rangle \\
&= \mathcal{R}_\delta - \left\langle (\mathbb{1} - \mathcal{P}K)^{-1}\left(\mathcal{P} f^* + \mathcal{P}K P_{s_1}\mathbf{1} + (\mathcal{P} - P_{s_1})\mathbf{1}\right), g_{r_1} \right\rangle
\end{aligned} \tag{6.4.42}$$

Since g_{r_1} is evidently analytic in $\delta \in \mathbb{R}$, we are left to show convergence of the scalar product.

All involved functions are locally bounded, so to establish convergence it is enough to investigate their asymptotic behaviour. g_{r_1} may grow exponentially at arbitrary high rate, depending on r_1 and δ, for both large positive and large negative arguments. We therefore need superexponential bounds on the function:

$$(\mathbb{1} - \mathcal{P}K)^{-1}\left(\mathcal{P} f^* + \mathcal{P}K P_{s_1}\mathbf{1} + (\mathcal{P} - P_{s_1})\mathbf{1}\right). \tag{6.4.43}$$

For this purpose we first need an expansion of the operator $\mathcal{P}$:

$$\mathcal{P} = \sum_{k=1}^{n} \bar{P}_{s_1} V_{r_1,r_2} \dots \bar{P}_{s_{k-1}} V_{r_{k-1},r_k} P_{s_k} V_{r_k,r_1}. \tag{6.4.44}$$

Notice that all operators P_{s_i}, $\bar{P}_{s_i}$ and V_{r_i,r_j} map superexponentially decaying functions onto superexponentially decaying functions. Moreover P_{s_i} and $\bar{P}_{s_i}$ generate superexponential decay for large negative resp. positive arguments.

The function f^* decays superexponentially for large arguments but may grow exponentially for small ones. Since every part of the sum contains one projection P_{s_k}, $\mathcal{P}f^*$ decays superexponentially on both sides.

Examining $(\mathcal{P} - P_{s_1})\mathbf{1}$, notice that the $k = 1$ contribution in (6.4.44) is equal to P_{s_1}, which is cancelled out here. All other contributions contain both $\bar{P}_{s_1}$ and P_{s_k}, which ensure superexponential decay.

Using the usual asymptotic bound on the Airy function, we see that the operator K maps any function in its domain onto one which is decreasing superexponentially for large arguments. By previous arguments, functions in the image of $\mathcal{P}K$ decay on both sides, in particular $\mathcal{P}KP_{s_1}\mathbf{1}$.

Now, in order to establish the finiteness of the scalar product, decompose the inverse operator as $(\mathbb{1}-\mathcal{P}K)^{-1} = \mathbb{1}+\mathcal{P}K(\mathbb{1}-\mathcal{P}K)^{-1}$. The contribution coming from the identity has just been settled. As inverse of a bounded operator, $(\mathbb{1} - \mathcal{P}K)^{-1}$ is also bounded. Because of the rapid decay, the functions $\mathcal{P}f^*$, $\mathcal{P}KP_{s_1}\mathbf{1}$ and $(\mathcal{P} - P_{s_1})\mathbf{1}$ are certainly in $L^2(\mathbb{R})$ and thus mapped onto $L^2(\mathbb{R})$ by this operator. Finally, the image of an $L^2(\mathbb{R})$-function under the operator $\mathcal{P}K$ is decaying superexponentially on both sides.

The expression (6.4.42) is thus an analytic function in δ in the domain $\mathbb{R}$. Setting $\delta = 0$ returns the value of $G_m(\vec{r}, \vec{s})$. Combining these results with (6.4.39) finishes the proposition. □

Chapter 7

Mixed initial conditions

Besides the fundamental initial geometries analyzed up to now, there are three initial conditions that are mixtures of these fundamental ones. They all satisfy an asymptotic theorem, where the limit process is a crossover Airy process, which interpolates between two of the processes Airy_2, Airy_1 and $\mathrm{Airy}_{\mathrm{stat}}$.

7.1 Half-Periodic

Let $\vec{\zeta}^{\text{half-flat}} \in \mathbb{R}^{\mathbb{Z}_{>0}}$ be the vector defined by $\zeta_n^{\text{half-flat}} = n$. The determinantal structure for this case has already been obtained as a byproduct of the full periodic case. We can thus directly state the kernel, in its alternative form as mentioned in Remark 4.3 [FSW15b]. Notice again that the direction of space is reversed in our convention.

Proposition 7.1 (Proposition 4.2 of [FSW15b])**.** *Let $\{x_n(t), n \geq 1\}$ be the system of one-sided reflected Brownian motions with initial condition $\vec{x}(0) = \vec{\zeta}^{\text{half-flat}}$. Then for any finite subset S of $\mathbb{Z}_{>0}$, it holds*

$$\mathbb{P}\left(\bigcap_{n\in S}\{x_n(t) \leq a_n\}\right) = \det(\mathbb{1} - \chi_a \mathcal{K}_{\text{half-flat}} \chi_a)_{L^2(S\times\mathbb{R})}, \tag{7.1.1}$$

where $\chi_a(n,\xi) = \mathbb{1}_{\xi > a_n}$. The kernel $\mathcal{K}_{\text{half-flat}}$ is given by

$$\mathcal{K}_{\text{half-flat}}(n_1,\xi_1;n_2,\xi_2) = -\phi_{n_1,n_2}(\xi_1,\xi_2)\mathbb{1}_{n_2>n_1} + \mathcal{K}_1(n_1,\xi_1;n_2,\xi_2), \tag{7.1.2}$$

with

$$\phi_{n_1,n_2}(\xi_1,\xi_2) = \frac{(\xi_2-\xi_1)^{n_2-n_1-1}}{(n_2-n_1-1)!}\mathbb{1}_{\xi_1\leq\xi_2}$$
$$\mathcal{K}_1(n_1,\xi_1;n_2,\xi_2) = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathrm{i}\mathbb{R}-1}\mathrm{d}w\oint_{\Gamma_0}\mathrm{d}z\,\frac{e^{tw^2/2+\xi_1 w}}{e^{tz^2/2+\xi_2 z}}\frac{(-w)^{n_1}}{(-z)^{n_2}}\frac{(1+z)e^z}{we^w-ze^z}, \tag{7.1.3}$$

where Γ_0 *is chosen in such a way that* $|ze^z|<|we^w|$ *holds always.*

The law of large numbers now depends on the region we examine:

$$\lim_{t\to\infty}\frac{x_{\alpha^2 t}(t)}{t} = \begin{cases}2\alpha, & \text{for } 0\leq\alpha\leq 1\\ 1+\alpha^2, & \text{for } \alpha>1.\end{cases} \tag{7.1.4}$$

Notice that the first case is the same scaling as for the step initial condition and the second case is the scaling for the full periodic case. This analogy carries over to the behaviour of the fluctuations around the macroscopic position. For $0<\alpha<1$ they are of order $t^{1/3}$ and given by the Airy$_2$ process, while for $\alpha>1$ they are of the same order and given by the Airy$_1$ process. These limits are not proven here, instead we focus on the more interesting transition regions. The transition at $\alpha=0$ is simple, as for any finite n, $x_n(t)$ is a bounded modification of the system with step initial condition. By Lemma 8.5 this bounded modification stays bounded and is therefore irrelevant on the scale $\sqrt{t}$. This implies that $\lim_{t\to\infty}x_n(t)/\sqrt{t}$ behaves as the top line of a n-particle Dyson's Brownian motion.

At the transition point $\alpha=1$ we find a new process that interpolates between $\mathcal{A}_2$ and $\mathcal{A}_1$ and is therefore called the Airy$_{2\to1}$ process, $\mathcal{A}_{2\to1}$:

Theorem 7.2. *With* $\{x_n(t),n\geq1\}$ *being the system of one-sided reflected Brownian motions with initial condition* $\vec{x}(0)=\vec{\zeta}^{\,\text{half-flat}}$*, define the rescaled process*

$$r\mapsto X_t^{\text{half-flat}}(r) = t^{-1/3}\big(x_{\lfloor t+2rt^{2/3}\rfloor}(t)-2t-2rt^{2/3}\big). \tag{7.1.5}$$

In the sense of finite-dimensional distributions,

$$\lim_{t\to\infty}X_t^{\text{half-flat}}(r) \stackrel{d}{=} \mathcal{A}_{2\to1}(r). \tag{7.1.6}$$

Proof. With

$$\begin{aligned} n_i &= t+2t^{2/3}r_i\\ \xi_i &= 2t+2t^{2/3}r_i+t^{1/3}s_i, \end{aligned} \tag{7.1.7}$$

define a rescaled and conjugated kernel by

$$\mathcal{K}^{\text{resc}}_{\text{half-flat}}(r_1, s_1; r_2, s_2) = t^{1/3} e^{\xi_1 - \xi_2} \mathcal{K}_{\text{half-flat}}(n_1, \xi_1; n_2, \xi_2). \tag{7.1.8}$$

Once the Propositions 7.3 and 7.4 are established, the result follows in the same way as in the proof of Theorem 6.3. . □

Proposition 7.3. *Consider any r_1, r_2 in a bounded set and fixed L. Then, the kernel converges as*

$$\lim_{t\to\infty} \mathcal{K}^{\text{resc}}_{\text{half-flat}}(r_1, s_1; r_2, s_2) = K_{\mathcal{A}_{2\to 1}}(r_1, s_1; r_2, s_2) \tag{7.1.9}$$

uniformly for $(s_1, s_2) \in [-L, L]^2$.

Proof. The proof is conceptually similar to the case of step initial conditions. However, some new issues arise, mainly due to the *double* contour integral. The convergence of ϕ is clear from previous cases so we jump directly to the main part of the kernel. Defining functions as

$$\begin{aligned} f_3(z) &= -(z^2 - 1)/2 - 2(z + 1) - \ln(-z) \\ f_2(z, r) &= -2r(z + 1 + \ln(-z)) \\ f_1(z, s) &= -s(z + 1), \end{aligned} \tag{7.1.10}$$

we can write $G(z, r, s) = t f_3(z) + t^{2/3} f_2(z, r) + t^{1/3} f_1(z, s)$, leading to

$$\mathcal{K}^{\text{resc}}_1(r_1, s_1; r_2, s_2) = \frac{t^{1/3}}{(2\pi \mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}-1} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z\, e^{G(z, r_2, s_2)} e^{-G(w, r_1, s_1)} \frac{(1 + z)e^z}{we^w - ze^z}. \tag{7.1.11}$$

The contour of the variable w is already a steep descent curve for the leading order function $-f_3(w)$:

$$\frac{\mathrm{dRe}\,(-f_3(-1 + \mathrm{i}u))}{\mathrm{d}u} = \mathrm{Re}\left(\frac{(\mathrm{i}u)^2}{1 + \mathrm{i}u}\mathrm{i}\right) = \frac{-u^3}{1 + u^2}, \tag{7.1.12}$$

which means that the real part is maximal at $w = -1$ and strictly decreasing when moving away from it, quadratically fast for large $|w|$.

The choice for the contour of z is trickier. Not only do we need a steep descent curve that comes close to the critical point $z = -1$, but we also have to ensure that we are not crossing any poles by deforming it, i.e. respect the inequality $|ze^z| < |we^w|$. Let $0 < \omega < \sqrt{2}$ be a parameter and $L_0(z)$ the principal branch of the *Lambert W function* defined by the inverse of the function $D_L = \{a + \mathrm{i}b \in \mathbb{C}, a + b\cot(b) > 0 \text{ and } -\pi < b < \pi\} \to \mathbb{C}$, $z \mapsto ze^z$. Choose

$$\gamma^\omega(u) = L_0\left(-(1 - \omega^2/2)e^{-1+\mathrm{i}u}\right). \tag{7.1.13}$$

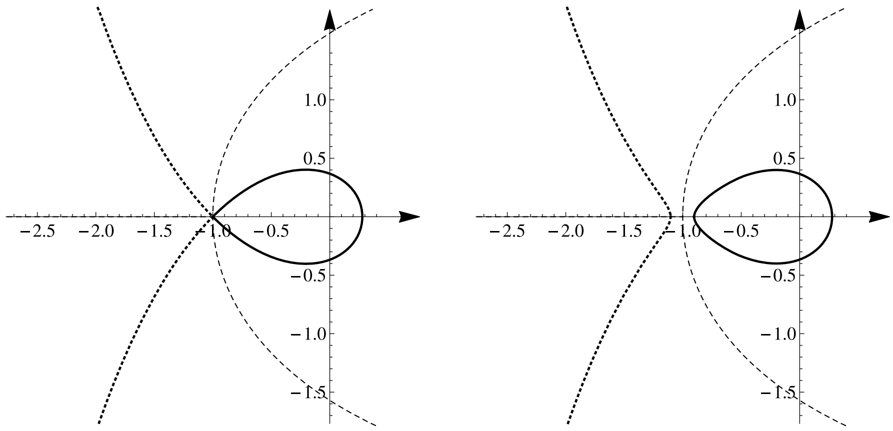

Figure 7.1: The contours γ^ω (solid line) and $\widehat{\gamma}^\omega$ (dotted line) for $\omega = 0$ (left picture) and some small positive ω (right picture). The dashed lines separate the ranges of the principal branch 0 (right) and the branches 1 (top left) and −1 (bottom left) of the Lambert W function.

We have to show that this is actually a simple loop around the origin. For points in the domain D_L of the principal branch of the Lambert function, we have

$$\operatorname{sgn}\operatorname{Im}\big((a+\mathrm{i}b)e^{a+\mathrm{i}b}\big) = \operatorname{sgn}(a\sin b + b\cos b) = \operatorname{sgn} b. \tag{7.1.14}$$

So the function $z \mapsto ze^z$ preserves the sign of the imaginary part, and consequently its inverse does the same. $\gamma^\omega(u)$ is thus in the lower half-plane for $0 < u < \pi$ and in the upper one for $\pi < u < 2\pi$ and meets the real line in the two points $\gamma^\omega(0) < 0$ and $\gamma^\omega(\pi) > 0$. The latter two inequalities follow from the monotonicity of $z \mapsto ze^z$ on $[-1,\infty)$.

It immediately follows that for any $z \in \gamma^\omega$,

$$|ze^z| = |1-\omega^2/2|e^{-1} < \sqrt{1+u^2}e^{-1} = |we^w|. \tag{7.1.15}$$

To see the steep descent property, first notice that L_0 satisfies the following differential identity:

$$L_0'(z) = \frac{L_0(z)}{z(1+L_0(z))}. \tag{7.1.16}$$

Combining this with $f_3'(z) = -(z+1)^2/z$, using the shorthand $z(u) = -(1-\omega^2/2)e^{-1+\mathrm{i}u}$ leads to

$$\begin{aligned}\frac{\mathrm{d}\mathrm{Re}\,(f_3(\gamma^\omega(u)))}{\mathrm{d}u} &= \mathrm{Re}\left(-\frac{(\gamma^\omega(u)+1)^2}{\gamma^\omega(u)}\cdot\frac{\gamma^\omega(u)}{z(u)(1+\gamma^\omega(u))}\cdot z(u)\cdot \mathrm{i}\right)\\ &= \mathrm{Re}\,(-\mathrm{i}(\gamma^\omega(u)+1)) = \mathrm{Im}(\gamma^\omega(u)).\end{aligned} \tag{7.1.17}$$

So the real part is strictly decreasing along $0 < u < \pi$, reaches its minimum at $\gamma^\omega(\pi)$ and then increases along $\pi < u < 2\pi$ back to its maximum at $\gamma^\omega(0) = \gamma^\omega(2\pi)$.

By (4.22) in [CGH+96] the Lambert W function can be expanded around the branching point $-e^{-1}$ as

$$L_0(z) = -1 + p - \frac{1}{3}p^2 + \frac{11}{72}p^3 + \cdots, \tag{7.1.18}$$

with $p(z) = \sqrt{2(ez+1)}$. This means that

$$\gamma^\omega(0) = -1 + \omega + \mathcal{O}(\omega^2) \tag{7.1.19}$$

for small ω.

As both contours are steep descent curves, we can restrict them to a neighbourhood of their critical point while making an error which is exponentially small in t. We do this by simply intersecting both curves with the ball $B_{-1}(2\omega) = \{z \in \mathbb{C}, |z+1| < 2\omega\}$.

Now we can apply Taylor expansion of the function G as in (4.2.31) and estimate the error made by omitting the higher order terms in the same way as in (4.2.32). With $z = -1 + Zt^{-1/3}$ and $w = -1 + Wt^{-1/3}$ the term constant in t converges as

$$t^{-1/3}\frac{(1+z)e^z}{we^w - ze^z} = \frac{t^{-2/3}Ze^{Zt^{-1/3}}}{-1 + W^2t^{-2/3}/2 + 1 - Z^2t^{-2/3}/2 + \mathcal{O}(t^{-1})} \to \frac{2Z}{W^2 - Z^2}. \tag{7.1.20}$$

Applying this transformation of variable to the integral results in

$$\begin{aligned}\mathcal{K}_1^{\text{resc}}(r_1, s_1; r_2, s_2) &+ \mathcal{O}(t^{1/3}e^{-\text{const}t}) + \mathcal{O}(t^{-1/3}) \\ &= \frac{1}{(2\pi \mathrm{i})^2}\int_{-2\omega \mathrm{i}t^{1/3}}^{2\omega \mathrm{i}t^{1/3}} \mathrm{d}W \int_{\widetilde{\gamma}_t^\omega} \mathrm{d}Z\, \frac{e^{Z^3/3 + r_2Z^2 - s_2Z}}{e^{W^3/3 + r_1W^2 - s_1W}} \frac{2Z}{W^2 - Z^2},\end{aligned} \tag{7.1.21}$$

with $\widetilde{\gamma}_t^\omega = t^{1/3}(\gamma^\omega \cap B_{-1}(2\omega) + 1)$. The endpoints of this contour are at $\pm 2\omega e^{i\theta}$, where θ is close to $\pi/4$ for ω being small. The limit $t \to \infty$ now extends both contours up to infinity. While $t \to \infty$ we have to keep deforming $\widetilde{\gamma}_t^\omega$ in order for it to not vanish to infinity. This is allowed, since as long as $|\arg(Z)| < \pi/4$, we have $\operatorname{Re}(W^2 - Z^2) < 0$, i.e. we are not crossing any poles. We can then change the contours to the generic Airy contours with the added restriction given in Definition 9.9 to arrive at the final result:

$$\lim_{t\to\infty} \mathcal{K}_1^{\text{resc}}(r_1, s_1; r_2, s_2) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\gamma_W} \mathrm{d}W \int_{\gamma_Z} \mathrm{d}Z\, \frac{e^{Z^3/3 + r_2Z^2 - s_2Z}}{e^{W^3/3 + r_1W^2 - s_1W}} \frac{2Z}{W^2 - Z^2} \tag{7.1.22}$$

□

Proposition 7.4. *For fixed r_1, r_2, L there exists $t_0 > 0$ such that the estimate*

$$|\mathcal{K}^{\text{resc}}_{\text{half-flat}}(r_1, s_1; r_2, s_2)| \leq \text{const} \cdot e^{-s_1} \tag{7.1.23}$$

holds for any $t > t_0$ and $s_1, s_2 > 0$.

Proof. Presumably we could show exponential decay in s_2, too, but the analysis would be even more involved, and the single-sided decay is enough for proving the convergence of the Fredholm determinant.

We start with another representation of the kernel, which can be found in (4.38), [FSW15b] combined with (4.51) therein. With a new contour

$$\widehat{\gamma}^{\omega} = \{L_{\lfloor u \rfloor}(-(1-\omega^2/2)e^{2\pi \mathrm{i} u - 1}), u \in \mathbb{R} \setminus [0,1)\} \tag{7.1.24}$$

it is given by

$$\mathcal{K}^{\text{resc}}_{1}(r_1, s_1; r_2, s_2) = \mathcal{K}^{\text{resc}}_{1(a)}(r_1, s_1; r_2, s_2) + \mathcal{K}^{\text{resc}}_{1(b)}(r_1, s_1; r_2, s_2), \tag{7.1.25}$$

with

$$\begin{aligned}
\mathcal{K}^{\text{resc}}_{1(a)}(r_1, s_1; r_2, s_2) &= \frac{t^{1/3}}{(2\pi \mathrm{i})^2} \int_{\widehat{\gamma}^{\omega}} \mathrm{d}w \oint_{\gamma^0} \mathrm{d}z\, e^{G(z, r_2, s_2)} e^{-G(w, r_1, s_1)} \frac{(1+z)e^z}{we^w - ze^z} \\
\mathcal{K}^{\text{resc}}_{1(b)}(r_1, s_1; r_2, s_2) &= \frac{t^{1/3}}{2\pi \mathrm{i}} \int_{\widehat{\gamma}^{0}} \mathrm{d}w\, \frac{e^{tw^2/2 + (w+1)\xi_1}(-w)^{n_1}}{e^{t\varphi(w)^2/2 + (\varphi(w)+1)\xi_2}(-\varphi(w))^{n_2}},
\end{aligned} \tag{7.1.26}$$

where $\varphi(w) = L_0(we^w)$. For a visualization of $\widehat{\gamma}^{\omega}$ see Figure 7.1 and proven results about its pathway can be found in Lemma A.1 [FSW15b]. Notice that now we have $|ze^z| = e^{-1} > |we^w|$ for any $\omega > 0$.

By Corollary 5.2 and Proposition 5.3 [FSW15b] $\mathcal{K}^{\text{resc}}_{1(b)}$ is bounded by a constant times $e^{-(s_1+s_2)}$ under the scaling

$$\begin{aligned}
n_i &= -t + 2^{5/3} r_i \\
\xi_i &= 2^{5/3} r_i + (2t)^{1/3}.
\end{aligned} \tag{7.1.27}$$

Noticing $\varphi(w)e^{\varphi(w)} = we^w$ we see that the kernel is invariant under simultaneous shifts $n_i \to n_i + k$, $\xi_i \to \xi_i + k$. Using $k = 2t$ we recover our scaling up to constant factors, which are irrelevant for our purposes.

We are thus left to show boundedness of $\mathcal{K}_{1(a)}$:

$$
\begin{aligned}
&|\mathcal{K}^{\mathrm{resc}}_{1(a)}(r_1,s_1;r_2,s_2)| \\
&\quad\leq \sup_{w\in\widehat{\gamma}^\omega, z\in\gamma^0}\left|\frac{e^z}{we^w-ze^z}\right| \oint_{\gamma^0}\mathrm{d}z\,\left|e^{G(z,r_2,s_2)}(1+z)\right| \frac{t^{1/3}}{(2\pi)^2}\int_{\widehat{\gamma}^\omega}\mathrm{d}w\,\left|e^{-G(w,r_1,s_1)}\right| \\
&\quad\leq e^{-G(w_0,r_1,s_1)} \sup_{w\in\widehat{\gamma}^\omega, z\in\gamma^0}\left|\frac{t^{-2/3}e^z}{we^w-ze^z}\right| \frac{t^{2/3}}{2\pi}\oint_{\gamma^0}\mathrm{d}z\,\left|e^{G(z,r_2,s_2)}(1+z)\right| \\
&\qquad\times\frac{t^{1/3}}{2\pi}\int_{\widehat{\gamma}^\omega}\mathrm{d}w\,\left|e^{-G(w,r_1,s_1)+G(w_0,r_1,s_1)}\right|.
\end{aligned}
\tag{7.1.28}
$$

The point w_0 is defined by

$$
w_0=\widehat{\gamma}^\omega(0)=-1-\omega+\mathcal{O}(\omega^2), \tag{7.1.29}
$$

and ω is chosen specifically as $\omega := \min\{t^{-1/3}\sqrt{s_2},\varepsilon\}$ for some small positive ε chosen in the following.

We can estimate the factor $e^{G(w_0,r_1,s_1)}$ as in the proof of Lemma 4.12, see (4.2.44) and the preceding arguments:

$$
e^{-G(w_0,r_1,s_1)} \leq e^{-\frac{1}{2}\omega t^{1/3}s_1} \leq c_L e^{-s_1}. \tag{7.1.30}
$$

Noticing $|ze^z| = e^{-1}$ and $|we^w| = |1-\omega^2/2|e^{-1}$ the remaining prefactor can be estimated as

$$
\sup_{w\in\widehat{\gamma}^\omega, z\in\gamma^0}\left|\frac{t^{-2/3}e^z}{we^w-ze^z}\right| \leq \mathrm{const}\cdot\frac{1}{\omega^2 t^{2/3}} \tag{7.1.31}
$$

by choosing ε small enough. This can be bounded, as for any fixed ε, $\omega t^{1/3}$ is large by choosing t_0 and L large.

It remains to show that the two integral expressions converge, starting with the z integral. First notice that by $s_2 \geq 0$, and $\mathrm{Re}(z) \geq -1$, we have $|e^{G(z,r_2,s_2)}| \leq |e^{G(z,r_2,0)}|$, avoiding the problem of large s_2. Now we can apply standard steep descent analysis, i.e. restrict the contour to a δ neighbourhood of the critical point -1, Taylor expand the integrand and estimate the error term as in (4.2.32). The analogue of (4.2.34) becomes

$$
\begin{aligned}
\frac{t^{2/3}}{2\pi}\int_{\gamma^0_\delta}|\mathrm{d}z|\,|e^{\tilde{G}(z)}(1+z)| &= \frac{t^{2/3}}{2\pi}\int_{\gamma^0_\delta+1}|\mathrm{d}\omega|\,|e^{t\omega^3/3+t^{2/3}r\omega^2}\omega| \\
&\leq \frac{1}{2\pi}\int_{e^{\pi\mathrm{i}/4}\infty}^{-e^{\pi\mathrm{i}/4}\infty}|\mathrm{d}Z|\,|e^{Z^3/3+rZ^2}Z| < \infty
\end{aligned}
\tag{7.1.32}
$$

Similarly the inequalities $\mathrm{Re}(w-w_0)<0$ and $s_1\geq 0$ allow us to simply set $s_1=0$ in the w integral. The differential identity (7.1.16) holds for any branch of the Lambert function, so (7.1.17) can be derived analogously for $\widehat{\gamma}^\omega(u)=L_{\lfloor u\rfloor}(-(1-\omega^2/2)e^{2\pi\mathrm{i}u-1}), u\in\mathbb{R}\setminus[0,1)$:

$$\frac{\mathrm{dRe}\left(-f_3(\gamma(u))\right)}{\mathrm{d}u}=-\mathrm{Im}(\gamma(u)). \tag{7.1.33}$$

$\widehat{\gamma}^\omega$ is thus a steep descent curve for the leading order term and the rate of decline along the path is even quadratic in $|w|$, allowing us to restrict the contour to a small neighbourhood of w_0. The path $\widehat{\gamma}^\omega$ is close to vertical in a neighbourhood of w_0, thus we can proceed as in the proof of Lemma 4.12 from here on. □

7.2 Half-Poisson

In this section we will study the initial condition given by a Poisson process on the positive half-line and no particles on the negative one. Let therefore be $\{\mathrm{Exp}_n, n\in\mathbb{Z}_{\geq 0}\}$ be i.i.d. random variables with exponential distribution with parameter 1. For a parameter $\lambda>0$ define the initial condition $\vec{x}(0)=\vec{\zeta}^{\,\text{half-stat}}(\lambda)$ by

$$\begin{aligned}\zeta_0^{\text{half-stat}}&=\lambda^{-1}\,\mathrm{Exp}_0,\\ \zeta_n^{\text{half-stat}}-\zeta_{n-1}^{\text{half-stat}}&=\lambda^{-1}\,\mathrm{Exp}_n,\qquad \text{for } n\geq 1.\end{aligned} \tag{7.2.1}$$

As before, a variation of ζ_0 introduces only a bounded modification of the initial condition and is thus irrelevant in the scaling limit as long as it stays bounded almost surely. We chose it in this way to keep the determinantal structure simple.

Notice that if we let $\rho\to 0$ in the initial condition $\vec{\zeta}^{\,\text{stat}}(\lambda,\rho)$ all particles with negative label vanish to $-\infty$ and can consequently be ignored. So $\vec{\zeta}^{\,\text{half-stat}}(\lambda)$ is simply $\vec{\zeta}^{\,\text{stat}}(\lambda,0)$ shifted by $\zeta_0^{\text{half-stat}}$. This means that we can obtain the determinantal structure as a corollary of Proposition 6.2. Strictly speaking, $\vec{\zeta}^{\,\text{half-stat}}(\lambda)$ equals $\vec{\zeta}^{\,\text{stat}}(\lambda,0)$ *under the modified measure* $\mathbb{P}_+$. This means the shift argument, i.e. *Step 2* of the proof of Proposition 6.2, is not necessary and we obtain the determinantal formula by simply specifying (6.1.31) to $\rho=0$:

Proposition 7.5. *Let $\{x_n(t), n\in\mathbb{Z}_{\geq 0}\}$ be the system of one-sided reflected Brownian motions with initial condition $\vec{x}(0)=\zeta^{\text{half-stat}}(\lambda)$ for any $\lambda>0$. For any finite subset S of $\mathbb{Z}_{\geq 0}$, it holds*

$$\mathbb{P}\left(\bigcap_{n\in S}\{x_n(t)\leq a_n\}\right)=\det(\mathbb{1}-\chi_a\mathcal{K}_{\text{half-stat}}\chi_a)_{L^2(S\times\mathbb{R})}, \tag{7.2.2}$$

where $\chi_a(n,\xi) = \mathbb{1}_{\xi > a_n}$. *The kernel* $\mathcal{K}_{\text{half-stat}}$ *is given by*

$$\begin{aligned}\mathcal{K}_{\text{half-stat}}(n_1,\xi_1;n_2,\xi_2) = -\phi_{n_1,n_2}(\xi_1,\xi_2)\mathbb{1}_{n_2>n_1} + \mathcal{K}_0(n_1,\xi_1;n_2,\xi_2) \\ + \lambda f(n_1,\xi_1) g_0(n_2,\xi_2).\end{aligned} \tag{7.2.3}$$

where

$$\begin{aligned}\phi_{n_1,n_2}(\xi_1,\xi_2) &= \frac{(\xi_2-\xi_1)^{n_2-n_1-1}}{(n_2-n_1-1)!}\mathbb{1}_{\xi_1\le\xi_2}, \quad \textit{for } 0 \le n_1 < n_2,\\ \mathcal{K}_0(n_1,\xi_1;n_2,\xi_2) &= \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathrm{i}\mathbb{R}-\varepsilon}\mathrm{d}w\oint_{\Gamma_0}\mathrm{d}z\,\frac{e^{tw^2/2+\xi_1 w}}{e^{tz^2/2+\xi_2 z}}\frac{(-w)^{n_1}}{(-z)^{n_2}}\frac{1}{w-z},\\ f(n_1,\xi_1) &= \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\varepsilon}\mathrm{d}w\,\frac{e^{tw^2/2+\xi_1 w}(-w)^{n_1}}{w+\lambda},\\ g_0(n_2,\xi_2) &= \frac{-1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z\,e^{-tz^2/2-\xi_2 z}(-z)^{-n_2-1},\end{aligned} \tag{7.2.4}$$

for any fixed $0<\varepsilon<\lambda$.

We assume $\lambda = 1$ from now on. It is clear that the initial macroscopic shape in this case is the same as in the half periodic case. We thus have again the law of large numbers,

$$\lim_{t\to\infty}\frac{x_{\alpha^2 t}(t)}{t} = \begin{cases}2\alpha, & \text{for } 0\le\alpha\le 1\\ 1+\alpha^2, & \text{for } \alpha > 1.\end{cases} \tag{7.2.5}$$

The fluctuations in the case $0<\alpha<1$ are of order $t^{1/3}$ and given by the Airy$_2$ process, while for $\alpha > 1$ they are now governed by the Airy$_{\text{stat}}$ process. For finite n, $x_n(t)$ fluctuates on the scale $\sqrt{t}$ according to Dyson's Brownian motion. At the interesting transition point $\alpha = 1$, however, the Airy$_{2\to\text{BM}}$ process will appear:

Theorem 7.6. *With* $\{x_n(t), n\in\mathbb{Z}_{\ge 0}\}$ *being the system of one-sided reflected Brownian motions with initial condition* $\vec{x}(0) = \zeta^{\text{half-stat}}(1)$, *define the rescaled process*

$$r \mapsto X_t^{\text{half-stat}}(r) = t^{-1/3}\big(x_{\lfloor t+2rt^{2/3}\rfloor}(t) - 2t - 2rt^{2/3}\big). \tag{7.2.6}$$

In the sense of finite-dimensional distributions,

$$\lim_{t\to\infty} X_t^{\text{half-stat}}(r) \stackrel{d}{=} \mathcal{A}_{2\to\text{BM}}(r) - r^2. \tag{7.2.7}$$

Proof of Theorem 7.6. With

$$\begin{aligned} n_i &= t + 2t^{2/3} r_i \\ \xi_i &= 2t + 2t^{2/3} r_i + t^{1/3} s_i, \end{aligned} \tag{7.2.8}$$

define the rescaled kernel

$$\mathcal{K}^{\text{resc}}_{\text{half-stat}}(r_1, s_1; r_2, s_2) = t^{1/3} e^{\xi_1 - \xi_2} \mathcal{K}_{\text{half-stat}}(n_1, \xi_1; n_2, \xi_2). \tag{7.2.9}$$

It decomposes into

$$\begin{aligned} \mathcal{K}^{\text{resc}}_{\text{half-stat}}(r_1, s_1; r_2, s_2) = -\phi^{\text{resc}}_{r_1,r_2}(s_1, s_2) \mathbb{1}_{r_1<r_2} + \mathcal{K}^{\text{resc}}_0(r_1, s_1; r_2, s_2) \\ + f^{\text{resc}}(r_1, s_1) \mathscr{g}^{\text{resc}}_0(r_2, s_2), \end{aligned} \tag{7.2.10}$$

by

$$\begin{aligned} f^{\text{resc}}(r_1, s_1) &= e^{-t/2+\xi_1} f(n_1, \xi_1) \\ \mathscr{g}^{\text{resc}}_0(r_2, s_2) &= e^{t/2-\xi_2} t^{1/3} \mathscr{g}_0(n_2, \xi_2). \end{aligned} \tag{7.2.11}$$

We can apply the proof of Theorem 6.3 once we have shown compact convergence as well as uniform boundedness of the kernel. The former means that for any r_1, r_2 in a bounded set and fixed L, the kernel converges as

$$\lim_{t\to\infty} \mathcal{K}^{\text{resc}}_{\text{half-stat}}(r_1, s_1; r_2, s_2) = \frac{e^{\frac{2}{3}r_2^3 + r_2 s_2}}{e^{\frac{2}{3}r_1^3 + r_1 s_1}} K_{\mathcal{A}_{2\to\text{BM}}}(r_1, s_1 + r_1^2, ; r_2, s_2 + r_2^2) \tag{7.2.12}$$

uniformly for $(s_1, s_2) \in [-L, L]^2$.

The first two parts of (7.2.10) are simply $\mathcal{K}^{\text{resc}}_{\text{step}}$, so we can apply Proposition 4.6 to them. In the proof of Proposition 6.6 it is shown that $f^{\text{resc}}(r_1, s_1)$ converges compactly to $f_{r_1}(s_1)$. Finally,

$$\mathscr{g}^{\text{resc}}_0(r_2, s_2) = -\beta_t(r + t^{-2/3}/2, s - t^{-1/3}), \tag{7.2.13}$$

allows applying Lemma 4.11. Putting this together, we have that as $t \to \infty$, the kernel $\mathcal{K}^{\text{resc}}_{\text{half-stat}}$ converges compactly to

$$\mathcal{K}_{\mathcal{A}_2}(r_1, s_1; r_2, s_2) + \left(1 - e^{-\frac{2}{3}r_1^3 - r_1 s_1} \int_0^\infty \mathrm{d}x\, \mathrm{Ai}(r_1^2 + s_1 + x) e^{-r_1 x}\right) \mathrm{Ai}(r_2^2 + s_2) e^{\frac{2}{3}r_2^3 + r_2 s_2}, \tag{7.2.14}$$

which is the right hand side of (7.2.12).

Since we can apply Proposition 4.9 and Proposition 4.8 on the first two parts of the rescaled kernel, the only bound we have to show is the following:

For fixed r_1, r_2, L there exists $t_0 > 0$ such that the estimate

$$\left|f^{\text{resc}}(r_1, s_1) g_0^{\text{resc}}(r_2, s_2)\right| \leq \text{const} \cdot e^{-s_2} \tag{7.2.15}$$

holds for any $t > t_0$ and $s_1, s_2 > 0$.

This can be seen by:

$$\begin{aligned}\left|f^{\text{resc}}(r_1, s_1) g_0^{\text{resc}}(r_2, s_2)\right| &= \left|1 - \int_0^\infty \mathrm{d}x\, \alpha_t(r_1, s_1 + x)\right| \left|\beta_t(r_2 + t^{-2/3}/2, s_2 - t^{-1/3})\right| \\ &\leq \left(1 + \int_0^\infty \mathrm{d}x\, e^{-(s_1+x)}\right) e^{-s_2 + t^{-1/3}} \leq \text{const} \cdot e^{-s_2},\end{aligned} \tag{7.2.16}$$

which finishes the proof. □

7.3 Poisson-Periodic

The remaining mixed initial condition consists of equally spaced particles on the positive half-axis and a Poisson process on the negative one. Let therefore be $\{\text{Exp}_n, n \in \mathbb{Z}_{\leq 1}\}$ be i.i.d. random variables with exponential distribution with parameter 1. For a parameter $\rho > 0$ define the initial condition $\vec{x}(0) = \vec{\zeta}^{\text{stat-flat}}(\rho)$ by

$$\begin{aligned} \zeta_n^{\text{stat-flat}} &= n, && \text{for } n \geq 1 \\ \zeta_n^{\text{stat-flat}} - \zeta_{n-1}^{\text{half-stat}} &= \rho^{-1}\,\text{Exp}_n, && \text{for } n \leq 1. \end{aligned} \tag{7.3.1}$$

By the stationarity property, we know that $x_1(t) = 1 + \widetilde{B}(t) + \rho t$ for a Brownian motion $\widetilde{B}(t)$ that is independent of $B_n(t)$, $n \geq 2$. We can therefore restrict our analysis to a half-infinite system, which is in fact the same system as in the half-periodic setting except for the drift of the first particle $x_1(t)$.

Proposition 7.7. *Let $\{x_n(t), n \in \mathbb{Z}\}$ be the system of one-sided reflected Brownian motions with initial condition $\vec{x}(0) = \vec{\zeta}^{\text{stat-flat}}(\rho)$, $\rho > 0$. Then for any finite subset S of $\mathbb{Z}_{>0}$, it holds*

$$\mathbb{P}\left(\bigcap_{n \in S} \{x_n(t) \leq a_n\}\right) = \det(\mathbb{1} - \chi_a \mathcal{K}_{\text{stat-flat}} \chi_a)_{L^2(S \times \mathbb{R})}, \tag{7.3.2}$$

where $\chi_a(n, \xi) = \mathbb{1}_{\xi > a_n}$. The kernel $\mathcal{K}_{\text{stat-flat}}$ is given by

$$\mathcal{K}_{\text{stat-flat}}(n_1, \xi_1; n_2, \xi_2) = \mathcal{K}_{\text{half-flat}}(n_1, \xi_1; n_2, \xi_2) + \Psi_{n_1-1}^{n_1}(\xi_1)\left(\widehat{\Phi}_{(1)}^{n_2}(\xi_2) + \widehat{\Phi}_{(2)}^{n_2}(\xi_2)\right), \tag{7.3.3}$$

with $\mathcal{K}_{\text{half-flat}}$ *as in* (7.1.2) *and*

$$\begin{aligned}
\Psi^n_{n-1}(\xi) &= \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w\; e^{tw^2/2+w(\xi-1)}(-w)^{n-1} \\
\widehat{\Phi}^n_{(1)}(\xi) &= \frac{(-1)^n}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z\,\frac{e^{-tz^2/2-z(\xi-1)}}{z^n}\frac{\rho(1+z)}{\rho+ze^{\rho+z}}, \\
\widehat{\Phi}^n_{(2)}(\xi) &= \rho^{1-n}e^{-t\rho^2/2+\rho(\xi-1)}.
\end{aligned} \tag{7.3.4}$$

Proof of Proposition 7.7. Applying Proposition 4.3 with parameters $\mu_1 = \rho$, $\mu_k = 0$ for $2 \leq k \leq N$ gives

$$\mathbb{P}\left(\vec{x}(t)\in\mathrm{d}\vec{\xi}\right) = e^{\rho(\xi_1-1)-t\rho^2/2}\det_{1\leq k,l\leq N}[F_{k-l}(\xi_{N+1-l}-\zeta_{N+1-k},t)], \tag{7.3.5}$$

where

$$F_k(\xi,t) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+1}\mathrm{d}w\, e^{tw^2/2+\xi w}w^k. \tag{7.3.6}$$

Using repeatedly the identity

$$F_k(\xi,t) = \int_{-\infty}^{\xi}\mathrm{d}x\, F_{k+1}(x,t), \tag{7.3.7}$$

relabeling $\xi^k_1 := \xi_k$, and introducing new variables ξ^k_l for $2 \leq l \leq k \leq N$, we can write

$$\det_{1\leq k,l\leq N}\left[F_{k-l}(\xi_1^{N+1-l}-\zeta_{N-1+k},t)\right] = \int_{\mathcal{D}'}\det_{1\leq k,l\leq N}\left[F_{k-1}(\xi_l^N-\zeta_{N-1+k},t)\right]\prod_{2\leq l\leq k\leq N}\mathrm{d}\xi_l^k, \tag{7.3.8}$$

where $\mathcal{D}' = \{\xi^k_l \in \mathbb{R}, 2 \leq l \leq k \leq N | x^k_l \leq x^{k-1}_{l-1}\}$. Using the antisymmetry of the determinant and encoding the constraint on the integration variables into indicator functions, we obtain that the measure (7.3.5) is a marginal of

$$\begin{aligned}
&\text{const}\cdot e^{\rho\xi_1^1}\prod_{n=2}^N\det_{1\leq i,j\leq n}\left[\mathbb{1}_{\xi_i^{n-1}\leq\xi_j^n}\right]\det_{1\leq k,l\leq N}\left[F_{k-1}(\xi_l^N-\zeta_{N-1+k},t)\right] \\
&= \text{const}\cdot\prod_{n=1}^N\det_{1\leq i,j\leq n}\left[\phi_n(\xi_i^{n-1},\xi_j^n)\right]\det_{1\leq k,l\leq N}\left[F_{k-1}(\xi_l^N-\zeta_{N-1+k},t)\right]
\end{aligned} \tag{7.3.9}$$

with

$$\begin{aligned}
\tilde{\phi}_n(x,y) &= \mathbb{1}_{x\leq y}, \quad \text{for } n\geq 2 \\
\tilde{\phi}_1(x,y) &= e^{\rho y},
\end{aligned} \tag{7.3.10}$$

and using the convention that $\xi_n^{n-1} \leq y$ always holds.

The measure (7.3.9) has the appropriate form for applying Lemma 3.5. The composition of the $\tilde{\phi}$ functions can be evaluated explicitly as

$$\begin{aligned} \tilde{\phi}_{0,n}(x,y) &= (\tilde{\phi}_1 * \cdots * \tilde{\phi}_n)(x,y) = \rho^{1-n} e^{\rho y}, && \text{for } n \geq 1, \\ \tilde{\phi}_{m,n}(x,y) &= (\tilde{\phi}_{m+1} * \cdots * \tilde{\phi}_n)(x,y) = \frac{(y-x)^{n-m-1}}{(n-m-1)!} \mathbb{1}_{x \leq y}, && \text{for } n > m \geq 1. \end{aligned} \tag{7.3.11}$$

Define

$$\Psi_{n-k}^n(\xi) := \frac{(-1)^{n-k}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \; e^{tw^2/2 + w(\xi - k)} w^{n-k}, \tag{7.3.12}$$

for $n, k \geq 1$ and some $\varepsilon > 0$. In the case $n \geq k$ the integrand has no poles, which implies $\Psi_{n-k}^n(\xi) = (-1)^{n-k} F_{n-k}(\xi - \zeta_k)$, now specifically with $\zeta_k = \zeta_k^{\text{stat-flat}} = k$. Reversing the order of the index k one sees that the second determinant in (7.3.9) is equal to $\det_{k,l}\left[\Psi_{N-k}^N(\xi_l^N)\right]$. The straightforward recursion

$$(\tilde{\phi}_n * \Psi_{n-k}^n)(\xi) = \Psi_{n-1-k}^{n-1}(\xi) \tag{7.3.13}$$

eventually leads to condition (3.3.3) being satisfied. A basis for the space V_n is given by

$$\{e^{\rho x}, x^{n-2}, x^{n-3}, \dots, x, 1\}. \tag{7.3.14}$$

Choose functions Φ_{n-k}^n as follows

$$\Phi_{n-k}^n(\xi) = \begin{cases} \widetilde{\Phi}_{n-k}^n(\xi) & 2 \leq k \leq n, \\ \widetilde{\Phi}_{n-1}^n(\xi) + \widehat{\Phi}_{(1)}^n(\xi) + \widehat{\Phi}_{(2)}^n(\xi) & k = 1, \end{cases} \tag{7.3.15}$$

where

$$\begin{aligned} \widetilde{\Phi}_{n-k}^n(\xi) &= \frac{(-1)^{n-k}}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, \frac{e^{-tz^2/2 - z(\xi-k)}}{z^{n-k+1}} (1+z), \\ \widehat{\Phi}_{(1)}^n(\xi) &= \frac{(-1)^n}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, \frac{e^{-tz^2/2 - z(\xi-1)}}{z^n} \frac{\rho(1+z)}{\rho + z e^{\rho+z}}, \\ \widehat{\Phi}_{(2)}^n(\xi) &= \rho^{1-n} e^{-t\rho^2/2 + \rho(\xi-1)}. \end{aligned} \tag{7.3.16}$$

The functions $\widetilde{\Phi}_{n-k}^n$ are polynomials of order $n-k$ by elementary residue calculating rules. $\widetilde{\Phi}_{n-1}^n + \widehat{\Phi}_{(1)}^n$ is in fact a polynomial of order $n-2$, since the poles of order n in each integrand cancel each other out exactly. So the functions (7.3.15) indeed generate V_n. To show the orthogonality (3.3.5), we decompose the scalar product as follows:

$$\int_{\mathbb{R}_-} \mathrm{d}\xi \; \Psi_{n-k}^n(\xi) \widetilde{\Phi}_{n-\ell}^n(\xi) + \int_{\mathbb{R}_+} \mathrm{d}\xi \; \Psi_{n-k}^n(\xi) \widetilde{\Phi}_{n-\ell}^n(\xi). \tag{7.3.17}$$

Since $n-k \geq 0$ we are free to choose the sign of ε as necessary. For the first term, we choose $\varepsilon < 0$ and the path Γ_0 close enough to zero, such that always $\mathrm{Re}(w-z) > 0$. Then, we can take the integral over ξ inside and obtain

$$\int_{\mathbb{R}_-} \mathrm{d}\xi\, \Psi^n_{n-k}(\xi)\widetilde{\Phi}^n_{n-\ell}(\xi) = \frac{(-1)^{k-l}}{(2\pi\mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z\, \frac{e^{tw^2/2} w^{n-k} e^{-wk}(1+z)}{e^{tz^2/2} z^{n-\ell+1} e^{-z\ell}(w-z)}. \tag{7.3.18}$$

For the second term, we choose $\varepsilon > 0$ to obtain $\mathrm{Re}(w-z) < 0$. Then again, we can take the integral over ξ inside and arrive at the same expression up to a minus sign. The net result of (7.3.17) is a residue at $w = z$, which is given by

$$\frac{(-1)^{k-l}}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, (ze^z)^{\ell-k}\, \frac{1+z}{z} = \frac{(-1)^{k-l}}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}Z\, Z^{\ell-k-1} = \delta_{k,\ell}, \tag{7.3.19}$$

where we made the change of variables $Z = ze^z$. In the same way we get

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}\xi\, \Psi^n_{n-k}(\xi)\widehat{\Phi}^n_{(1)}(\xi) &= \frac{(-1)^k}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, (ze^z)^{1-k}\, \frac{\rho(1+z)}{(\rho + ze^{\rho+z})z} \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}Z\, (-Z)^{-k}\, \frac{1}{(1+Ze^\rho/\rho)} \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}Z\, (-Z)^{-k} \sum_{i\geq 0} \left(-Z\frac{e^\rho}{\rho}\right)^i = -\left(\rho e^{-\rho}\right)^{1-k}.\end{aligned} \tag{7.3.20}$$

Regarding the scalar product with $\widehat{\Phi}^n_{(2)}$, choose $\varepsilon < \rho$ for the integral over $\mathbb{R}_-$ and $\varepsilon > \rho$ for the one over $\mathbb{R}_+$. We are left with the following:

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}\xi\, \Psi^n_{n-k}(\xi)\widehat{\Phi}^n_{(2)}(\xi) &= \frac{(-1)^{n-k}\rho^{1-n}}{2\pi\mathrm{i}} \oint_{\Gamma_{-\rho}} \mathrm{d}w\, e^{t(w^2-\rho^2)/2 - wk - \rho} w^{n-k} \frac{1}{w+\rho} \\ &= \left(\rho e^{-\rho}\right)^{1-k},\end{aligned} \tag{7.3.21}$$

which cancels out (7.3.20), proving the orthogonality $\left\langle \Psi^n_{n-k}(\xi), \Phi^n_{n-l}(\xi)\right\rangle = \delta_{k,l}$.

Furthermore, both $\phi_n(\xi^{n-1}_n, x)$ and $\Phi^n_0(\xi)$ are constants, so the kernel has a simple form (3.3.6):

$$\mathcal{K}_{\text{stat-flat}}(n_1,\xi_1;n_2,\xi_2) = -\phi_{n_1,n_2}(\xi_1,\xi_2)\mathbb{1}_{n_2>n_1} + \sum_{k=1}^{n_2} \Psi^{n_1}_{n_1-k}(\xi_1)\Phi^{n_2}_{n_2-k}(\xi_2). \tag{7.3.22}$$

Note that we are free to extend the summation over k up to infinity, since the integral expression for $\Phi^n_{n-k}(\xi)$ vanishes for $k > n$ anyway. Taking the sum inside the integrals we can write

$$\sum_{k\geq 1} \Psi^{n_1}_{n_1-k}(\xi_1)\widetilde{\Phi}^{n_2}_{n_2-k}(\xi_2) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathrm{i}\mathbb{R}-\varepsilon}\mathrm{d}w \oint_{\Gamma_0}\mathrm{d}z\, \frac{e^{tw^2/2+\xi_1 w}(-w)^{n_1}}{e^{tz^2/2+\xi_2 z}(-z)^{n_2}}\frac{z+1}{z}\sum_{k\geq 1}\frac{(ze^z)^k}{(we^w)^k}. \tag{7.3.23}$$

By choosing contours such that $|z| < |w|$, we can use the formula for a geometric series, resulting in

$$(7.3.23) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathrm{i}\mathbb{R}-\varepsilon}\mathrm{d}w \oint_{\Gamma_0}\mathrm{d}z\, \frac{e^{tw^2/2+\xi_1 w}(-w)^{n_1}}{e^{tz^2/2+\xi_2 z}(-z)^{n_2}}\frac{(z+1)e^z}{we^w - ze^z} = \mathcal{K}_1(n_1,\xi_1;n_2,\xi_2). \tag{7.3.24}$$

Note also that we required $n_1 > 0$, in which case the definition of the function ϕ is equal to the one in (7.1.3), allowing us to combine these equations to (7.3.3). □

The asymptotic behaviour of the Poisson-periodic initial condition depends substantially on the parameter ρ. For $\rho > 1$ we have a *shock* moving in positive direction, where the macroscopic particle density changes discontinuously. In the case $0 < \rho < 1$ there is a region of linearly increasing density between the two plateaus of density ρ and 1, and at the right edge of this region we have again the Airy$_{2\to 1}$ process. We will not focus more on these two cases and refer the reader to [BFS09] where these results are derived for TASEP.

Instead we will study the case $\rho = 1$, that where the last of the crossover Airy processes appears. There is a constant macroscopic density of 1, which also results in the particles having an average speed of 1. The law of large numbers therefore looks like this:

$$\lim_{t\to\infty}\frac{x_{\alpha t}(t)}{t} = 1+\alpha. \tag{7.3.25}$$

The behaviour of the fluctuations is not clear immediately. One thing that follows directly from the Poisson case, is that for $\alpha \leq 0$ we have fluctuations of order $t^{1/3}$ described by $\mathcal{A}_{\mathrm{stat}}$, if we normalize along the characteristic direction, i.e. consider the position relative to $x_{(\alpha-1)t}(0)$ (otherwise it is just Gaussian fluctuations on a $t^{1/2}$ scale). It is also to be expected, that for large α, the periodic initial condition will dominate, leading to $t^{1/3}$ fluctuations given by $\mathcal{A}_1$. As it turns out, there is again a single transition point, and it is $\alpha = 1$. The limit process is $\mathcal{A}_{\mathrm{stat}}$ for $\alpha < 1$, $\mathcal{A}_1$ for $\alpha > 1$, and a transition process $\mathcal{A}_{\mathrm{BM}\to 1}$ for $\alpha = 1$. We will prove the limit only in the transition case.

Theorem 7.8. *With $\{x_n(t), n \in \mathbb{Z}\}$ being the system of one-sided reflected Brownian motions with initial condition $\vec{x}(0) = \vec{\zeta}^{\text{stat-flat}}(1)$, define the rescaled process*

$$r \mapsto X_t^{\text{stat-flat}}(r) = t^{-1/3}\big(x_{\lfloor t+2rt^{2/3}\rfloor}(t) - 2t - 2rt^{2/3}\big). \tag{7.3.26}$$

In the sense of finite-dimensional distributions,

$$\lim_{t\to\infty} X_t^{\text{stat-flat}}(r) \stackrel{d}{=} \mathcal{A}_{\text{BM}\to 1}(r) - r^2. \tag{7.3.27}$$

Proof. With

$$\begin{aligned} n_i &= t + 2t^{2/3} r_i \\ \xi_i &= 2t + 2t^{2/3} r_i + t^{1/3} s_i, \end{aligned} \tag{7.3.28}$$

define the rescaled kernel

$$\mathcal{K}_{\text{stat-flat}}^{\text{resc}}(r_1, s_1; r_2, s_2) = t^{1/3} e^{\xi_1 - \xi_2} \mathcal{K}_{\text{stat-flat}}(n_1, \xi_1; n_2, \xi_2). \tag{7.3.29}$$

Once the Propositions 7.9 and 7.10 are established, the result follows in the same way as in the proof of Theorem 6.3. □

Proposition 7.9. *Consider any r_1, r_2 in a bounded set and fixed L. Then, the kernel converges as*

$$\lim_{t\to\infty} \mathcal{K}_{\text{stat-flat}}^{\text{resc}}(r_1, s_1; r_2, s_2) = K_{\mathcal{A}_{\text{BM}\to 1}}(r_1, s_1; r_2, s_2) \tag{7.3.30}$$

uniformly for $(s_1, s_2) \in [-L, L]^2$.

Proof. We can apply Proposition 7.3 to the $\mathcal{K}_{\text{half-flat}}^{\text{resc}}$ part of the kernel and are left with studying the additional part of (7.3.3) given as a product.

We attach a factor $e^{-t/2-1}$ to Ψ and its inverse to both $\widehat{\Phi}_{(1)}$ and $\widehat{\Phi}_{(2)}$. Now,

$$\begin{aligned} t^{1/3} e^{\xi_1 - t/2 - 1} \Psi_{n_1-1}^{n_1}(\xi_1) &= \frac{t^{1/3}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}-\varepsilon} \mathrm{d}w \; e^{t(w^2-1)/2 + (w+1)(\xi_1 - 1)} (-w)^{n_1 - 1} \\ &= \alpha_t(r_1 - t^{-2/3}/2, s_1) \end{aligned} \tag{7.3.31}$$

which converges to $e^{-\frac{2}{3}r_1^3 - r_1 s_1}\mathrm{Ai}(r_1^2 + s_1)$ uniformly for s_1, r_1 in a compact set.

The function $\widehat{\Phi}_{(2)}$ satisfies

$$e^{-\xi_2 + t/2 + 1} \widehat{\Phi}_{(2)}^{n}(\xi_1) = 1. \tag{7.3.32}$$

We are thus left to prove the limit of $\widehat{\Phi}_{(1)}$. Recognize that

$$e^{-\xi_2+t/2+1}\widehat{\Phi}^n_{(1)}(\xi_1) = \frac{t^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_0} \mathrm{d}z\, e^{-t(z^2-1)/2-(\xi_2-1)(z+1)}(-z)^{-n_2}\frac{(1+z)t^{-1/3}}{1+ze^{1+z}} \tag{7.3.33}$$

is precisely $\beta_t(r_2, s_2 - t^{-1/3})$ up to the fraction appearing in the integrand. We can thus carry over the proof of Lemma 4.11 almost completely. Up to (4.2.34) the generalization is straightforward, and the analogue of this equation becomes:

$$\begin{aligned}\frac{t^{1/3}}{2\pi\mathrm{i}}\int_{\widetilde{\Gamma}_\delta} \mathrm{d}z\, e^{\tilde{G}(z)}\frac{(1+z)t^{-1/3}}{1+ze^{1+z}} &= \frac{t^{1/3}}{2\pi\mathrm{i}}\int_{\widetilde{\Gamma}_{\delta+1}} \mathrm{d}\omega\, e^{t\omega^3/3+t^{2/3}r\omega^2-t^{1/3}s\omega}\frac{2}{\omega t^{1/3}}(1+\mathcal{O}(\omega))\\ &= \frac{1+\mathcal{O}(\delta)}{2\pi\mathrm{i}}\int_{e^{\theta\mathrm{i}}\delta t^{1/3},\ \text{right of }0}^{-e^{\theta\mathrm{i}}\delta t^{1/3}} \mathrm{d}Z\, e^{Z^3/3+rZ^2-sZ}\frac{2}{Z},\end{aligned} \tag{7.3.34}$$

where $\widetilde{\Gamma}$ is a small deformation of Γ such that it passes the real line to the right of -1. Letting $t\to\infty$ and $\delta\to 0$ we express the $1/Z$ term as an integral and recognize the contour integral representation of the Airy function to arrive at the desired limit

$$-2e^{\frac{2}{3}r_2^3+r_2s_2}\int_0^\infty \mathrm{d}x\,\mathrm{Ai}(r_2^2+s_2+x)e^{r_2x}. \tag{7.3.35}$$

□

Proposition 7.10. *For fixed r_1, r_2, L there exists $t_0 > 0$ such that the estimate*

$$|\mathcal{K}^{\mathrm{resc}}_{\text{stat-flat}}(r_1,s_1;r_2,s_2)| \le \mathrm{const}\cdot e^{-s_1} \tag{7.3.36}$$

holds for any $t > t_0$ and $s_1, s_2 > 0$.

Proof. The bound on the first part of the kernel is already established by Proposition 7.4. We decompose the product as in the proof of the pointwise convergence:

$$|t^{1/3}e^{\xi_1-t/2-1}\Psi^{n_1}_{n_1-1}(\xi_1)| = |\alpha_t(r_1-t^{-2/3}/2,s_1)| \le c_Le^{-s_1}, \tag{7.3.37}$$

as shown in Lemma 4.12. Recalling (7.3.32), we are finished if we show that $\widehat{\Phi}_{(1)}$ has a s_2-independent upper bound.

Start by

$$|e^{-\xi_2+t/2+1}\widehat{\Phi}^n_{(1)}(\xi_1)| \le \frac{t^{1/3}}{2\pi}\oint_{\widetilde{\Gamma}}|\mathrm{d}z|\,e^{\mathrm{Re}\,G(z,r_2,s_2)}\left|\frac{(1+z)t^{-1/3}}{1+ze^{1+z}}\right|. \tag{7.3.38}$$

We can now use $e^{\mathrm{Re}(-t^{1/3}s_2(z+1))} \leq 1$ since $s_2 \geq 0$ and z stays to the right of -1. Continuing with the steep descent analysis as in the proof of the convergence we arrive at the absolute value analogue of equation (7.3.34),

$$|e^{-\xi_2+t/2+1}\widehat{\Phi}^n_{(1)}(\xi_1)| \leq \mathrm{const} \int_{e^{\theta \mathrm{i}}\infty,\text{ right of }0}^{-e^{\theta \mathrm{i}}\infty} |\mathrm{d}Z|\, e^{\mathrm{Re}(Z^3/3+rZ^2)} \left|\frac{2}{Z}\right|, \quad (7.3.39)$$

which is finite.

□

Chapter 8

Generalizations of the results

8.1 Asymptotics along space-like paths and slow decorrelations

The rescaled process at fixed time is not the only one in which the Airy limit processes appears. It is also the case for the joint distributions of the positions of a tagged Brownian motion at different times, which means correlations along the t direction. This is the content of Theorem 8.1 below. It is a consequence of a phenomenon shared by many models in the KPZ universality class, which is called *slow decorrelations* [Fer08, CFP12]. This means that in the time-like direction the correlation length is not of order $t^{2/3}$ but t, as proven in Proposition 8.3. Thus instead of evaluating the distribution along points with fixed t only, we can shift the points in the time-like direction up to some t^ν with $\nu < 1$, and still keep the same limit result. These statistics on space-like paths are proven in Theorem 8.2.

Theorem 8.1. *For each one of the initial conditions*

$$\vec{\zeta}^* \in \{\vec{\zeta}^{\text{step}}, \vec{\zeta}^{\text{flat}}, \vec{\zeta}^{\text{stat}}, \vec{\zeta}^{\text{half-flat}}, \vec{\zeta}^{\text{half-stat}}, \vec{\zeta}^{\text{stat-flat}}\}, \tag{8.1.1}$$

define a rescaled process

$$\widetilde{X}_t^* \in \{\widetilde{X}_t^{\text{step}}, \widetilde{X}_t^{\text{flat}}, \widetilde{X}_t^{\text{stat}}, \widetilde{X}_t^{\text{half-flat}}, \widetilde{X}_t^{\text{half-stat}}, \widetilde{X}_t^{\text{stat-flat}}\} \tag{8.1.2}$$

by

$$\tau \mapsto \widetilde{X}_t^*(\tau) := t^{-1/3}\left(x_{\lfloor t\rfloor}(t + 2\tau t^{2/3}) - 2t - 2\tau t^{2/3}\right), \tag{8.1.3}$$

where the process $\vec{x}(t)$ on the right hand side is subject to $\vec{x}(0) = \vec{\zeta}^*$.

In the large time limit,

$$\begin{aligned}
\lim_{t\to\infty} \widetilde{X}_t^{\text{step}}(\tau) &= \mathcal{A}_2(\tau),\\
\lim_{t\to\infty} \widetilde{X}_t^{\text{flat}}(\tau) &= 2^{1/3}\mathcal{A}_1(2^{-2/3}\tau),\\
\lim_{t\to\infty} \widetilde{X}_t^{\text{stat}}(\tau) &= \mathcal{A}_{\text{stat}}(\tau),\\
\lim_{t\to\infty} \widetilde{X}_t^{\text{half-flat}}(\tau) &= \mathcal{A}_{2\to 1}(\tau),\\
\lim_{t\to\infty} \widetilde{X}_t^{\text{half-stat}}(\tau) &= \mathcal{A}_{2\to \text{BM}}(\tau),\\
\lim_{t\to\infty} \widetilde{X}_t^{\text{stat-flat}}(\tau) &= \mathcal{A}_{\text{BM}\to 1}(\tau),
\end{aligned} \tag{8.1.4}$$

hold in the sense of finite-dimensional distributions.

Theorem 8.1 is a corollary of the following result on space-like paths:

Theorem 8.2. *Let $\vec{x}(t)$ be the system of one-sided reflected Brownian motions subject to some initial condition $\vec{x}(0) = \vec{\zeta}$. Let us fix a $\nu \in [0,1)$, choose any $\theta_1,\ldots,\theta_m \in [-t^\nu, t^\nu]$, $r_1,\ldots,r_m \in \mathbb{R}$ and define the rescaled random variables*

$$\widehat{X}_t(r_k,\theta_k) := t^{-1/3}\left(x_{[t+2r_kt^{2/3}+\theta_k]}(t+\theta_k) - 2t - 2\theta_k - 2r_kt^{2/3}\right). \tag{8.1.5}$$

Let $s_1,\ldots,s_m \in \mathbb{R}$ and $\widehat{X}_t$ satisfy the limit

$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^m \left\{\widehat{X}_t(r_k,0) \le s_k\right\}\right) = \mathbb{P}\left(\bigcap_{k=1}^m \left\{\mathcal{A}_*(r_k) \le s_k\right\}\right), \tag{8.1.6}$$

for some process $\mathcal{A}_*(r)$ whose joint distribution function is continuous.
Then it holds

$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^m \left\{\widehat{X}_t(r_k,\theta_k) \le s_k\right\}\right) = \mathbb{P}\left(\bigcap_{k=1}^m \left\{\mathcal{A}_*(r_k) \le s_k\right\}\right). \tag{8.1.7}$$

Notice that specifying (8.1.5) to $\theta_k = 0$ gives exactly the familiar scaling where all our asymptotic results hold. Thus this theorem is applicable to all six of the treated initial conditions.

Proof of Theorem 8.1. This follows by taking $\theta_k = 2\tau_k t^{2/3}$ and $r_k = -\tau_k$ in Theorem 8.2. ☐

For the proof of Theorem 8.2 we need this slow decorrelation property:

Proposition 8.3. *Let* $\widehat{X}_t(r,\theta)$ *be defined as in Theorem 8.2 and also satisfy* (8.1.6)*. For a* $\nu \in [0,1)$*, consider* $\theta \in [-t^\nu, t^\nu]$ *and some* $r \in \mathbb{R}$*. Then, for any* $\varepsilon > 0$*,*

$$\lim_{t\to\infty} \mathbb{P}\left(|\widehat{X}_t(r,\theta) - \widehat{X}_t(r,0)| \geq \varepsilon\right) = 0. \tag{8.1.8}$$

Proof. Without loss of generality we consider $\theta \geq 0$. For $\theta < 0$ one just have to denote $\tilde{t} = t + \theta$ so that $\tilde{t} - \theta = t$ and the proof remains valid with t replaced by $\tilde{t}$. Recall that by definition we have

$$x_m(t) = \max_{k\leq m}\{Y_{k,m}(t) + \zeta_k\}. \tag{8.1.9}$$

First we need an inequality, namely

$$\begin{aligned} x_{n+\theta}(t+\theta) &= \max_{k\leq n+\theta}\{\zeta_k + Y_{k,n+\theta}(t+\theta)\} \geq \max_{k\leq n}\{\zeta_k + Y_{k,n+\theta}(t+\theta)\} \\ &= \max_{k\leq n}\{\zeta_k + \sup_{0\leq s_k\leq\ldots\leq s_{n+\theta}=t+\theta} \sum_{i=k}^{n+\theta}(B_i(s_i) - B_i(s_{i-1}))\} \\ &\geq \max_{k\leq n}\{\zeta_k + \sup_{\substack{0\leq s_k\leq\ldots\leq s_{n+\theta}=t+\theta \\ \text{with } s_n=t}} \sum_{i=k}^{n+\theta}(B_i(s_i) - B_i(s_{i-1}))\} \\ &= x_n(t) + Y_{n+1,n+\theta}(t,t+\theta), \end{aligned} \tag{8.1.10}$$

with

$$Y_{n+1,n+\theta}(t,t+\theta) = \sup_{t\leq s_{n+1}\leq\ldots\leq s_{n+\theta}=t+\theta} \sum_{i=n+1}^{n+\theta}(B_i(s_i) - B_i(s_{i-1})). \tag{8.1.11}$$

Remark that $x_n(t)$ and $Y_{n+1,n+\theta}(t,t+\theta)$ are independent. Specifying this to $n = t + 2rt^{2/3}$, the inequality can be rewritten as

$$\widehat{X}_t(r,\theta) \geq \widehat{X}_t(r,0) + \chi(t), \tag{8.1.12}$$

where

$$\chi(t) = \frac{Y_{n+1,n+\theta}(t,t+\theta) - 2\theta}{t^{1/3}} \stackrel{d}{=} \frac{Y_{1,\theta}(\theta) - 2\theta}{t^{1/3}}. \tag{8.1.13}$$

From Theorem 4.2 we know that $(t/\theta)^{1/3}\chi(t)$ converges to a Tracy-Widom distributed random variable, which means that $\chi(t)$ itself converges to 0 in distribution. By (8.1.6), we have

$$\widehat{X}_t(r,0) \stackrel{D}{\Longrightarrow} \mathcal{A}_*(r), \tag{8.1.14}$$

and also

$$\widehat{X}_t(r,\theta) \overset{D}{\Longrightarrow} \mathcal{A}_*(r), \tag{8.1.15}$$

which follows from $\widehat{X}_t(r,\theta) = \widehat{X}_{t+\theta}(r + \mathcal{O}(t^{\nu-1)}), 0)$.

Since both sides of (8.1.12) converge in distribution to $\mathcal{A}_*(r)$, by Lemma 4.1 of [BC11] (reported below) we have $\widehat{X}_t(r,\theta) - \widehat{X}_t(r,0) - \chi(t) \to 0$ in probability as $t \to \infty$. As $\chi(t) \to 0$ in probability, too, the proof is finished. ☐

Lemma 8.4 (Lemma 4.1 of [BC11]). *Consider two sequences of random variables $\{X_n\}$ and $\{\tilde{X}_n\}$ such that for each n, X_n and $\tilde{X}_n$ are defined on the same probability space Ω_n. If $X_n \geq \tilde{X}_n$ and $X_n \Rightarrow D$ as well as $\tilde{X}_n \Rightarrow D$ then $X_n - \tilde{X}_n$ converges to zero in probability. Conversely if $\tilde{X}_n \Rightarrow D$ and $X_n - \tilde{X}_n$ converges to zero in probability then $X_n \Rightarrow D$ as well.*

Finally we come to the proof of Theorem 8.2.

Proof of Theorem 8.2. Let us define the random variables

$$\Xi_k := \widehat{X}_t(r_k,\theta_k) - \widehat{X}_t(r_k,0). \tag{8.1.16}$$

such that

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{\widehat{X}_t(r_k,\theta_k) \leq s_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{m}\{\widehat{X}_t(r_k,0) + \Xi_k \leq s_k\}\right). \tag{8.1.17}$$

The slow decorrelation result (Proposition 8.3) implies $\Xi_k \to 0$ in probability as $t \to \infty$. Introducing $\varepsilon > 0$ we can use inclusion-exclusion to decompose

$$(8.1.17) = \mathbb{P}\left(\bigcap_{k=1}^{m}\{\widehat{X}_t(r_k,0) + \Xi_k \leq s_k\} \cap \{|\Xi_k| \leq \varepsilon\}\right) + \sum_j \mathbb{P}\left(R_j\right), \tag{8.1.18}$$

where the sum on the right hand side is finite and each R_j satisfies $R_j \subset \{|\Xi_k| > \varepsilon\}$ for at least one k, implying $\lim_{t\to\infty} \mathbb{P}(R_j) = 0$. Using (8.1.6) leads to

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m}\{\widehat{X}_t(r_k,\theta_k) \leq s_k\}\right) &\leq \mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_*(r_k) \leq s_k + \varepsilon\}\right),\\ \liminf_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m}\{\widehat{X}_t(r_k,\theta_k) \leq s_k\}\right) &\geq \mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_*(r_k) \leq s_k - \varepsilon\}\right).\end{aligned} \tag{8.1.19}$$

Finally, the continuity assumption allows us to take the limit $\varepsilon \to 0$ and obtain

$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m}\{\widehat{X}_t(r_k,\theta_k) \le s_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_*(r_k) \le s_k\}\right). \tag{8.1.20}$$

□

8.2 Attractiveness and a more general class of initial data

We can relax the strict assumptions on the initial conditions by recognizing that our model shows *attractiveness*. A stochastic particle system is called attractive, if for two distinct initial configurations evolving under the same noise their order is preserved.

Proposition 8.5. *Let us consider two admissible initial conditions, denoted by $\vec{a} \in \mathbb{R}^{\mathbb{Z}}$, $\vec{b} \in \mathbb{R}^{\mathbb{Z}}$. Under the same noise they evolve to $x_m^a(t)$ and $x_m^b(t)$. If there is $M > 0$ such that $|a_m - b_m| \le M$ for all $m \in \mathbb{Z}$, then also*

$$\left|x_m^a(t) - x_m^b(t)\right| \le M \qquad \forall m \in \mathbb{Z}, t > 0. \tag{8.2.1}$$

The same property holds for the standard coupling of the TASEP, as explained in Section 2.1 of [BFS08].

As an immediate consequence, the limit result of Theorem 5.2 holds for bounded modifications of the initial condition $\vec{x}(0) = \vec{\zeta}^{\,\mathrm{flat}}$, since an error of size M vanishes under the $t^{1/3}$ scaling. For example, one could choose a unit cell of length 1 and take an arbitrary initial condition with the only restriction that there are ℓ particles in each cell. Then the convergence to the Airy_1 process holds.

Furthermore, this Proposition allows us to choose $\zeta_0 = 0$ in Theorem 6.1 instead of considering the true Poisson case, since ζ_0 gives rise to a global shift, that is bounded with probability 1.

Proof of Proposition 8.5. By definition,

$$\begin{aligned} x_m^a(t) &= -\max_{k\le m}\bigl\{Y_{k,m}(t) - a_k\bigr\}, \\ x_m^b(t) &= -\max_{k\le m}\bigl\{Y_{k,m}(t) - b_k\bigr\}. \end{aligned} \tag{8.2.2}$$

Since the inequality

$$Y_{k,m}(t) - a_k \le Y_{k,m}(t) - b_k + M \tag{8.2.3}$$

holds for each k, the maximum can be taken on each side, resulting in

$$\begin{aligned} \max_{k\leq m}\{Y_{k,m}(t)-a_k\} &\leq \max_{k\leq m}\{Y_{k,m}(t)-b_k\}+M, \\ x_m^a(t) &\geq x_m^b(t)-M. \end{aligned} \tag{8.2.4}$$

Correspondingly, one has $x_m^b(t) \geq x_m^a(t) - M$. □

Chapter 9

Airy processes

In this section we give definitions and properties of the Airy processes that arise as limit process of our system of reflected Brownian motions, as well as connections between these processes.

9.1 Elementary Airy processes

9.1.1 Airy$_2$ process

The first appearance of an Airy-type process has been the Airy$_2$ process as the limit of the top layer in the polynuclear growth model [PS02]. In this initial paper, it has already been noted that this process is stationary, has continuous sample paths, its one-point distribution is the GUE Tracy-Widom distribution, as well as that the correlation $\mathbb{E}\left(\mathcal{A}_2(r)\mathcal{A}_2(0)\right)-\mathbb{E}\mathcal{A}_2(r)\mathbb{E}\mathcal{A}_2(0)$ is positive and decays as r^{-2}. Later it was proven that its sample paths are locally absolutely continuous to Brownian motion [CH14], which implies they are Hölder continuous with exponent $\frac{1}{2}-$.

The Airy$_2$ process is defined by its finite-dimensional distribution function:

Definition 9.1 (Airy$_2$ process)**.** *Let*

$$K_{\mathcal{A}_2}(s_1, r_1; s_2, r_2) = -V_{r_1,r_2}(s_1, s_2)\mathbb{1}_{r_1<r_2} + K_{r_1,r_2}(s_1, s_2), \qquad (9.1.1)$$

with

$$
\begin{aligned}
V_{r_1,r_2}(s_1,s_2) &= \frac{e^{-\frac{(s_2-s_1)^2}{4(r_2-r_1)}}}{\sqrt{4\pi(r_2-r_1)}} \\
K_{r_1,r_2}(s_1,s_2) &= \frac{e^{\frac{2}{3}r_2^3+r_2s_2}}{e^{\frac{2}{3}r_1^3+r_1s_1}}\int_0^\infty \mathrm{d}x\, e^{x(r_2-r_1)}\mathrm{Ai}(r_1^2+s_1+x)\mathrm{Ai}(r_2^2+s_2+x).
\end{aligned}
\tag{9.1.2}
$$

The Airy$_2$ process, $\mathcal{A}_2$, *is the process with* m*-point joint distributions at* $r_1 < r_2 < \cdots < r_m$ *given by the Fredholm determinant*

$$
\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_2(r_k)\le s_k+r_k^2\}\right) = \det(\mathbb{1}-\chi_s K_{\mathcal{A}_2}\chi_s)_{L^2(\{r_1,\dots,r_m\}\times\mathbb{R})}, \tag{9.1.3}
$$

where $\chi_s(r_k,x) = \mathbb{1}_{x>s_k}$.

Remark 9.2. The parts of the kernel, (9.1.2), have alternative representations:

$$
\begin{aligned}
K_{r_1,r_2}(s_1,s_2) &= \frac{-1}{(2\pi\mathrm{i})^2}\int_{e^{-2\pi\mathrm{i}/3}\infty}^{e^{2\pi\mathrm{i}/3}\infty}\mathrm{d}W\int_{e^{\pi\mathrm{i}/3}\infty}^{e^{-\pi\mathrm{i}/3}\infty}\mathrm{d}Z\,\frac{e^{Z^3/3+r_2Z^2-s_2Z}}{e^{W^3/3+r_1W^2-s_1W}}\frac{1}{Z-W} \\
V_{r_1,r_2}(s_1,s_2) &= \frac{e^{\frac{2}{3}r_2^3+r_2s_2}}{e^{\frac{2}{3}r_1^3+r_1s_1}}\int_{\mathbb{R}}\mathrm{d}x\, e^{-x(r_1-r_2)}\mathrm{Ai}(r_1^2+s_1+x)\mathrm{Ai}(r_2^2+s_2+x).
\end{aligned}
\tag{9.1.4}
$$

In the integral defining K, the path for W and Z do not have to intersect.

Remark 9.3. Using the integral representation 9.1.4 for V, inserting the parameter shift $s_i \to s_i - r_i^2$ and finally applying a conjugation, leads to another way to describe the Airy$_2$ process. With

$$
K'_{\mathcal{A}_2}(s_1,r_1;s_2,r_2) = \begin{cases} \int_0^\infty \mathrm{d}x\, e^{-x(r_1-r_2)}\mathrm{Ai}(s_1+x)\mathrm{Ai}(s_2+x) & \text{for } r_1\ge r_2 \\ -\int_{-\infty}^0 \mathrm{d}x\, e^{-x(r_1-r_2)}\mathrm{Ai}(s_1+x)\mathrm{Ai}(s_2+x) & \text{for } r_1 < r_2. \end{cases} \tag{9.1.5}
$$

its multi-dimensional distributions are given by

$$
\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_2(r_k)\le s_k\}\right) = \det(\mathbb{1}-\chi_s K'_{\mathcal{A}_2}\chi_s)_{L^2(\{r_1,\dots,r_m\}\times\mathbb{R})}. \tag{9.1.6}
$$

9.1.2 Airy$_1$ process

The Airy$_1$ process has been discovered three years later as the limit of the TASEP with flat initial condition [Sas05]. It is also a stationary with the one-point distribution being the GOE Tracy-Widom distribution, i.e. $\mathbb{P}(\mathcal{A}_1(r) \leq s) = F_{\mathrm{GOE}}(2s)$, which has already been conjectured in the initial paper, and soon been confirmed [FS05]. More recently, it was proven that the sample paths have locally Brownian fluctuations [QR13], and are consequently Hölder continuous with exponent $\frac{1}{2}-$, too.

The Airy$_1$ process is also defined in terms of its finite-dimensional distributions:

Definition 9.4 (Airy$_1$ process)**.** *Let $B_0(x,y) = \mathrm{Ai}(x+y)$, with* Ai *the standard Airy function, Δ the one-dimensional Laplacian and the kernel $K_{\mathcal{A}_1}$ defined by*

$$K_{\mathcal{A}_1}(s_1,r_1;s_2,r_2) = -\big(e^{(r_2-r_1)\Delta}\big)(s_1,s_2)\mathbb{1}_{r_2>r_1} + \big(e^{-r_1\Delta}B_0e^{r_2\Delta}\big)(s_1,s_2). \tag{9.1.7}$$

The Airy$_1$ process*, $\mathcal{A}_1$, is the process with m-point joint distributions at $r_1 < r_2 < \cdots < r_m$ given by the Fredholm determinant*

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\big\{\mathcal{A}_1(r_k) \leq s_k\big\}\right) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_1}\chi_s)_{L^2(\{r_1,\dots,r_m\}\times\mathbb{R})}, \tag{9.1.8}$$

where $\chi_s(r_k,x) = \mathbb{1}_{x>s_k}$.

Remark 9.5. There is a more explicit formula for the kernel, which is also used in the asymptotic analysis in [FSW15b]. It is given by

$$\begin{aligned} K_{\mathcal{A}_1}(s_1,r_1;s_2,r_2) = &- V_{r_1,r_2}(s_1,s_2)\mathbb{1}_{r_1<r_2} \\ &+ \mathrm{Ai}\big(s_1+s_2+(r_2-r_1)^2\big)e^{(r_2-r_1)(s_1+s_2)+\frac{2}{3}(r_2-r_1)^3}. \end{aligned} \tag{9.1.9}$$

The equivalence of these formulas is proven in Appendix A [BFPS07].

9.1.3 Airy$_{\mathrm{stat}}$ process

In spite of what the name might suggest, the Airy$_{\mathrm{stat}}$ process is not stationary. The name is derived from the fact that it arises as limit process of models started in their respective stationary initial condition. The one-point distribution $\mathcal{A}_{\mathrm{stat}}(0)$ has been identified in [BR00] as the limit of the PNG model, it has mean zero and is often called Baik-Rains distribution. The multi-point distribution has been discovered some time later in the TASEP [BFP10].

The increments of the $\mathrm{Airy}_{\mathrm{stat}}$ process are identical to those of a Brownian motion, this is straightforward to see by indirect arguments:

Since $\{x_n(t), n \in \mathbb{Z}\}$ is a Poisson point process for every $t \geq 0$, the process $X_t^{\mathrm{stat}}(r) - X_t^{\mathrm{stat}}(0)$ is a scaled Poisson jump process up to a linear part and

$$\lim_{t\to\infty} \left(X_t^{\mathrm{stat}}(r) - X_t^{\mathrm{stat}}(0)\right) \stackrel{d}{=} B(2r)\,. \tag{9.1.10}$$

By Theorem 6.1 the limit process $\mathcal{A}_{\mathrm{stat}}(r) - \mathcal{A}_{\mathrm{stat}}(0)$ must also have the statistics of two-sided Brownian motion. This property is not so easily inferred from the formulas in Definition 9.6. We will provide a direct proof of this fact in Section 9.4. The structure of the $\mathrm{Airy}_{\mathrm{stat}}$ is nevertheless quite rich, as these Brownian increments are non-trivially correlated with the random height shift $\mathcal{A}_{\mathrm{stat}}(0)$.

Definition 9.6 ($\mathrm{Airy}_{\mathrm{stat}}$ process)**.** *Let P_s be the projection operator on $[s,\infty)$ and $\bar{P}_s = \mathbb{1} - P_s$ the one on $(-\infty, s)$. With $V_{r_1,r_2}(s_1,s_2)$ as in (9.1.2), define*

$$\mathcal{P} = \mathbb{1} - \bar{P}_{s_1} V_{r_1,r_2} \bar{P}_{s_2} \cdots V_{r_{m-1},r_m} \bar{P}_{s_m} V_{r_m,r_1}, \tag{9.1.11}$$

as well as an operator K with integral kernel

$$K(s_1,s_2) = K_{r_1,r_1}(s_1,s_2) = e^{r_1(s_2-s_1)} \int_{r_1^2}^{\infty} \mathrm{d}x\, \mathrm{Ai}(s_1+x)\mathrm{Ai}(s_2+x). \tag{9.1.12}$$

Further, define the functions

$$\begin{aligned} \mathcal{R} &= s_1 + e^{\frac{2}{3}r_1^3} \int_{s_1}^{\infty} \mathrm{d}x \int_x^{\infty} \mathrm{d}y\, \mathrm{Ai}(r_1^2+y) e^{r_1 y}, \\ f^*(s) &= -e^{-\frac{2}{3}r_1^3} \int_s^{\infty} \mathrm{d}x\, \mathrm{Ai}(r_1^2+x) e^{-r_1 x}, \\ g(s) &= 1 - e^{\frac{2}{3}r_1^3} \int_s^{\infty} \mathrm{d}x\, \mathrm{Ai}(r_1^2+x) e^{r_1 x}. \end{aligned} \tag{9.1.13}$$

With these definitions, set

$$G_m(\vec{r},\vec{s}) = \mathcal{R} - \left\langle (\mathbb{1} - \mathcal{P}K)^{-1} \left(\mathcal{P}f^* + \mathcal{P}KP_{s_1}\mathbf{1} + (\mathcal{P} - P_{s_1})\mathbf{1}\right), g \right\rangle, \tag{9.1.14}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2(\mathbb{R})$. Then, the $\mathrm{Airy}_{\mathrm{stat}}$ process, *$\mathcal{A}_{\mathrm{stat}}$, is the process with m-point joint distributions at $r_1 < r_2 < \cdots < r_m$ given by*

$$\mathbb{P}\left(\bigcap_{k=1}^m \{\mathcal{A}_{\mathrm{stat}}(r_k) \leq s_k\}\right) = \sum_{i=1}^m \frac{\mathrm{d}}{\mathrm{d}s_i} \left(G_m(\vec{r},\vec{s}) \det(\mathbb{1} - \mathcal{P}K)_{L^2(\mathbb{R})}\right). \tag{9.1.15}$$

Remark 9.7. In this definition there appears actually yet another representation for the joint distributions of the Airy$_2$ process:

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_2(r_k) \le s_k + r_k^2\}\Big) = \det(\mathbb{1} - \mathcal{P}K)_{L^2(\mathbb{R})}. \tag{9.1.16}$$

This actually the version first obtained in [PS02] for $m = 2$. Equivalence of this formula to our definition is proven in [BCR15].

For well-definedness of the formula (9.1.14) we need the following lemma:

Lemma 9.8. *The operator* $\mathbb{1} - \mathcal{P}K$ *is invertible.*

Proof. We employ the same strategy as in Appendix B [BFP10]. For that purpose we use the following equivalence

$$\det(\mathbb{1} + A) \neq 0 \iff \mathbb{1} + A \text{ is invertible.} \tag{9.1.17}$$

Let $s_{\min} = \min_k s_k$.

$$\begin{aligned}\det(\mathbb{1} - \mathcal{P}K) &= \mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_2(r_k) - r_k^2 \le s_k\}\Big)\\ &\ge \mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_2(r_k) - r_k^2 \le s_{\min}\}\Big) \ge \mathbb{P}\Big(\max_{r\in\mathbb{R}}(\mathcal{A}_2(r) - r^2) \le s_{\min}\Big)\\ &= F_{\mathrm{GOE}}(2^{2/3}s_{\min}) > 0\end{aligned} \tag{9.1.18}$$

for any $s_{\min} > -\infty$, where F_{GOE} is the GOE Tracy-Widom distribution function. For the last equality see Section 9.3. The tails of the GOE Tracy-Widom distribution have been studied in great detail in various publications, see for instance [BBD08]. □

Our Definition 9.6 is actually an alternative formula for the joint distributions of the Airy$_{\mathrm{stat}}$ process compared to the one given in [BFP10] (Definition 1.1 and Theorem 1.2 therein). The main difference is that in [BFP10] the joint distributions are given in terms of a Fredholm determinant on $L^2(\{1,\dots,m\}\times\mathbb{R})$, while here we have a Fredholm determinant on $L^2(\mathbb{R})$. A similar twist was already visible in [PS02] and has been generalized in [BCR15].

9.2 Crossover Processes

In this section we introduce the three mixed Airy processes that are transition processes between two of the elementary Airy processes, as well as a new process with a parameter that interpolates between an elementary and a crossover Airy process.

9.2.1 Airy$_{2\to1}$ process

The Airy$_{2\to1}$ process has been discovered as the limit of the TASEP under half-periodic initial conditions [BFS08]. It is again defined in terms of its finite-dimensional distributions:

Definition 9.9 (Airy$_{2\to1}$ process)**.** *The Airy$_{2\to1}$ process, $\mathcal{A}_{2\to1}$, is the process with m-point joint distributions at $r_1 < r_2 < \cdots < r_m$ given by*

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_{2\to1}(r_k) \le s_k + r_k^2\,\mathbb{1}_{r_k\le 0}\}\right) = \det\left(\mathbb{1} - \chi_s K_{\mathcal{A}_{2\to1}}\chi_s\right)_{L^2(\{r_1,\ldots,r_m\}\times\mathbb{R})}, \tag{9.2.1}$$

where $\chi_s(r_k,x) = \mathbb{1}_{x>s_k}$ and the kernel $K_{\mathcal{A}_{2\to1}}$ is defined by

$$\begin{aligned} K_{\mathcal{A}_{2\to1}}(r_1,s_1;r_2,s_2) &= K_{\mathcal{A}_2}(s_1,r_1;s_2,r_2) \\ &\quad + \frac{e^{\frac{2}{3}r_2^3 + r_2 s_2}}{e^{\frac{2}{3}r_1^3 + r_1 s_1}} \int_0^\infty \mathrm{d}x\, e^{x(r_1+r_2)} \mathrm{Ai}(r_1^2 + s_1 - x)\mathrm{Ai}(r_2^2 + s_2 + x), \end{aligned} \tag{9.2.2}$$

with $K_{\mathcal{A}_2}$ as in (9.1.1).

The Airy$_{2\to1}$ process satisfies the limits:

$$\begin{aligned} \lim_{w\to\infty} \mathcal{A}_{2\to1}(r+w) &= 2^{-1/3}\mathcal{A}_1(2^{2/3}r), \\ \lim_{w\to\infty} \mathcal{A}_{2\to1}(r-w) &= \mathcal{A}_2(r). \end{aligned} \tag{9.2.3}$$

Remark 9.10. There exists a contour integral representation, which will be used in proving Theorem 7.2:

$$\begin{aligned} K_{\mathcal{A}_{2\to1}}(r_1,s_1;r_2,s_2) &= -V_{r_1,r_2}(s_1,s_2)\mathbb{1}_{r_1<r_2} \\ &\quad + \frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_W}\mathrm{d}W\int_{\gamma_Z}\mathrm{d}Z\,\frac{e^{Z^3/3 + r_2 Z^2 - s_2 Z}}{e^{W^3/3 + r_1 W^2 - s_1 W}}\frac{2Z}{W^2 - Z^2}. \end{aligned} \tag{9.2.4}$$

The contours $\gamma_W\colon e^{-2\pi \mathrm{i}/3}\infty \to e^{2\pi \mathrm{i}/3}\infty$ and $\gamma_Z\colon e^{\pi \mathrm{i}/3}\infty \to e^{-\pi \mathrm{i}/3}\infty$ are chosen in such a way that both γ_W and $-\gamma_W$ pass left of γ_Z. (9.2.2) can be derived from (9.2.4) by noticing the identity

$$\frac{2Z}{W^2-Z^2} = \frac{1}{W-Z} - \frac{1}{W+Z} = -\int_0^\infty \mathrm{d}x\, e^{x(W-Z)} - \int_0^\infty \mathrm{d}x\, e^{-x(W+Z)}, \tag{9.2.5}$$

and employing the definition of the Airy function.

9.2.2 Airy$_{2\to\mathrm{BM}}$ process

The Airy$_{2\to\mathrm{BM}}$ process has been discovered in [IS04]. Therein it was already shown that $\mathcal{A}_{2\to\mathrm{BM}}(r)$ converges to the GUE Tracy Widom distribution for $r \to -\infty$ and to a Gaussian distribution as $r \to \infty$. They also identified the distribution at $r = 0$, which is given by $\mathbb{P}(\mathcal{A}_{2\to\mathrm{BM}}(0) \le s) = (F_{\mathrm{GOE}}(s))^2$, i.e. the distribution of the maximum of two independent GOE-distributed random variables.

Definition 9.11 (Airy$_{2\to\mathrm{BM}}$ process)**.** *The Airy$_{2\to\mathrm{BM}}$ process, $\mathcal{A}_{2\to\mathrm{BM}}$, is the process with m-point joint distributions at $r_1 < r_2 < \cdots < r_m$ given by*

$$\mathbb{P}\left(\bigcap_{k=1}^m \{\mathcal{A}_{2\to\mathrm{BM}}(r_k) \le s_k\}\right) = \det\left(\mathbb{1} - \chi_s K_{\mathcal{A}_{2\to\mathrm{BM}}} \chi_s\right)_{L^2(\{r_1,\ldots,r_m\}\times\mathbb{R})}, \tag{9.2.6}$$

where $\chi_s(r_k, x) = \mathbb{1}_{x>s_k}$ and the kernel $K_{\mathcal{A}_{2\to\mathrm{BM}}}$ is defined by

$$\begin{aligned} K_{\mathcal{A}_{2\to\mathrm{BM}}}(r_1,s_1;r_2,s_2) &= K'_{\mathcal{A}_2}(r_1,s_1;r_2,s_2) \\ &\quad + \left(e^{-\frac{1}{3}r_1^3 + r_1 s_1} - \int_0^\infty \mathrm{d}x\, \mathrm{Ai}(s_1+x) e^{-r_1 x}\right) \mathrm{Ai}(s_2). \end{aligned} \tag{9.2.7}$$

Here, $K'_{\mathcal{A}_2}$ is defined as in (9.1.5).

9.2.3 Airy$_{\mathrm{BM}\to 1}$ process

The last one of the mixed Airy processes is the Airy$_{\mathrm{BM}\to 1}$ process, which first appeared in [BFS09]. The limit process that appears therein is actually more general, denoted by $\mathcal{A}_{2\to 1,M,\kappa}$ with parameters $M \in \mathbb{Z}_{\ge 0}$ and $\kappa \in \mathbb{R}$. In our case this more general process would appear through modifying the model asymptotically analyzed in Section 7.3: Instead of applying the drift ρ to the particle $x_1(t)$ only, the first M particles have drift ρ. Furthermore, ρ is not equal to 1, but scaled critically as $\rho = 1 - \kappa t^{1/3}$.

We will not give this general definition, but only the version that appears in this thesis, i.e. $\mathcal{A}_{\mathrm{BM}\to 1}(r) = \mathcal{A}_{2\to 1,1,0}(r)$:

Definition 9.12 (Airy$_{\mathrm{BM}\to 1}$ process)**.** *The Airy*$_{\mathrm{BM}\to 1}$ *process,* $\mathcal{A}_{\mathrm{BM}\to 1}$*, is the process with m-point joint distributions at* $r_1 < r_2 < \cdots < r_m$ *given by*

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_{\mathrm{BM}\to 1}(r_k) \leq s_k\}\right) = \det\left(\mathbb{1} - \chi_s K_{\mathcal{A}_{\mathrm{BM}\to 1}} \chi_s\right)_{L^2(\{r_1,\ldots,r_m\}\times\mathbb{R})}, \tag{9.2.8}$$

where $\chi_s(r_k, x) = \mathbb{1}_{x>s_k}$ *and the kernel* $K_{\mathcal{A}_{\mathrm{BM}\to 1}}$ *is defined by*

$$\begin{aligned} K_{\mathcal{A}_{\mathrm{BM}\to 1}}(r_1, s_1; r_2, s_2) &= K_{\mathcal{A}_{2\to 1}}(r_1, s_1; r_2, s_2) \\ &+ e^{-\frac{2}{3}r_1^3 - r_1 s_1}\mathrm{Ai}(s_1 + r_1^2)\left(1 - 2e^{\frac{2}{3}r_2^3 + r_2 s_2}\int_0^\infty \mathrm{d}x\, \mathrm{Ai}(r_2^2 + s_2 + x)e^{r_2 x}\right). \end{aligned} \tag{9.2.9}$$

Here, $K_{\mathcal{A}_{2\to 1}}$ *is defined as in (9.2.2).*

Remark 9.13. In the original definition of the Airy$_{2\to 1,M,\kappa}$ process, Definition 18 [BFS09], there is an issue with incompletely specified contours, that can be misleading. For the triple contour integral (5.8), with integration variables $u \in \Gamma_\kappa$, $w_1 \in \gamma_1$ and $w_2 \in \gamma_2$, it is only required that γ_1, γ_2 pass on the left of Γ_κ. In fact, for the formula to be correct, both γ_2 and $-\gamma_2$ have to pass to the left of u. As Γ_κ is a loop around κ, this requires some intricate choices especially in the case $\kappa = 0$. A good way to avoid this problem is by calculating one of the residues, so that γ_2 can be chosen in a more usual form, resulting in Definition 9.12 after specifying $M = 1$ and $\kappa = 0$.

9.2.4 Finite-step Airy$_{\mathrm{stat}}$ process

The finite-step Airy$_{\mathrm{stat}}$ process is a new presumably universal limit process that appears in the course of proving Theorem 6.1. It has a parameter δ, which one can interpret as a step size, that gives the transition from the standard Airy$_{\mathrm{stat}}$ process at $\delta = 0$ towards the Airy$_{2\to\mathrm{BM}}$ process as $t \to \infty$. It is defined again by its finite-dimensional distribution:

Definition 9.14 (Finite-step Airy$_{\mathrm{stat}}$ process)**.** *The finite-step Airy*$_{\mathrm{stat}}$ *process with parameter* $\delta > 0$*,* $\mathcal{A}_{\mathrm{stat}}^{(\delta)}$*, is the process with m-point joint distributions at* $r_1 < r_2 < \cdots < r_m$ *given by*

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_{\mathrm{stat}}^{(\delta)}(r_k) \leq s_k\}\right) = \left(1 + \frac{1}{\delta}\sum_{i=1}^{m}\frac{\mathrm{d}}{\mathrm{d}s_i}\right)\det\left(\mathbb{1} - \chi_s K^\delta \chi_s\right)_{L^2(\{r_1,\ldots,r_m\}\times\mathbb{R})}, \tag{9.2.10}$$

where $\chi_s(r_k, x) = \mathbb{1}_{x>s_k}$ *and the kernel* K^δ *is defined by*

$$K^\delta(r_1, s_1; r_2, s_2) = -V_{r_1,r_2}(s_1, s_2)\mathbb{1}_{r_1<r_2} + K_{r_1,r_2}(s_1, s_2) + \delta f_{r_1}(s_1)g_{r_2}(s_2). \tag{9.2.11}$$

Here, V_{r_1,r_2} *and* $K_{r_1,r_2}(s_1, s_2)$ *are defined as in (9.1.2), and*

$$\begin{aligned} f_{r_1}(s_1) &= 1 - e^{-\frac{2}{3}r_1^3 - r_1 s_1} \int_0^\infty \mathrm{d}x\, \mathrm{Ai}(r_1^2 + s_1 + x)e^{-r_1 x} \\ g_{r_2}(s_2) &= e^{\delta^3/3 + r_2\delta^2 - s_2\delta} - e^{\frac{2}{3}r_2^3 + r_2 s_2} \int_0^\infty \mathrm{d}x\, \mathrm{Ai}(r_2^2 + s_2 + x)e^{(\delta + r_2)x}. \end{aligned} \tag{9.2.12}$$

Remark 9.15. Instead of integrals over Airy functions, the functions (9.2.12) can also be written as contour integrals:

$$\begin{aligned} f_{r_1}(s_1) &= \frac{1}{2\pi\mathrm{i}} \int_{e^{-2\pi\mathrm{i}/3}\infty,\text{ right of }0}^{e^{2\pi\mathrm{i}/3}\infty} \mathrm{d}W \frac{e^{-(W^3/3 + r_1 W^2 - s_1 W)}}{W} \\ g_{r_2}(s_2) &= \frac{1}{2\pi\mathrm{i}} \int_{e^{\pi\mathrm{i}/3}\infty,\text{ left of }\delta}^{e^{-\pi\mathrm{i}/3}\infty} \mathrm{d}Z \frac{e^{Z^3/3 + r_2 Z^2 - s_2 Z}}{Z - \delta}. \end{aligned} \tag{9.2.13}$$

Remark 9.16. The identity $\mathcal{A}^{(0)}_{\mathrm{stat}}(r) = \mathcal{A}_{\mathrm{stat}}(r)$ is a consequence of Proposition 6.13. The limit

$$\lim_{\delta\to\infty} \mathcal{A}^{(\delta)}_{\mathrm{stat}}(r) = \mathcal{A}_{2\to\mathrm{BM}}(r) - r^2 \tag{9.2.14}$$

can be seen as follows. Using the identity (D.3) from [FS06] on $g_{r_2}(s_2)$ we obtain that

$$\begin{aligned} \lim_{\delta\to\infty} \delta \cdot g_{r_2}(s_2) &= e^{\frac{2}{3}r_2^3 + r_2 s_2} \lim_{\delta\to\infty} \int_{-\infty}^0 \mathrm{d}x\, \mathrm{Ai}(r_2^2 + s_2 + x)\delta e^{(\delta + r_2)x} \\ &= e^{\frac{2}{3}r_2^3 + r_2 s_2} \lim_{\delta\to\infty} \int_{-\infty}^0 \mathrm{d}y\, \mathrm{Ai}(r_2^2 + s_2 + y/\delta)e^{y(1 + r_2/\delta)} \\ &= e^{\frac{2}{3}r_2^3 + r_2 s_2} \mathrm{Ai}(r_2^2 + s_2). \end{aligned} \tag{9.2.15}$$

This means precisely,

$$\lim_{\delta\to\infty} K^\delta(r_1, s_1; r_2, s_2) = \frac{e^{\frac{2}{3}r_2^3 + r_2 s_2}}{e^{\frac{2}{3}r_1^3 + r_1 s_1}} K_{\mathcal{A}_{2\to\mathrm{BM}}}(r_1, s_1 + r_1^2, ; r_2, s_2 + r_2^2), \tag{9.2.16}$$

Finally, taking $\delta \to \infty$ in (9.2.10) implies that all the terms with the derivatives vanish, giving (9.2.14).

9.3 Variational identities

In the work [Joh03], an interesting variational identity was given that connects the Airy$_2$ and the Airy$_1$ process:

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}}\left\{\mathcal{A}_2(x)-x^2\right\}\le s\right)=F_{\rm GOE}(2^{2/3}s). \tag{9.3.1}$$

Its proof was however very indirect, and it took some time until the first direct proof appeared [CQR13]. It was conjectured that identities of this structure hold for all Airy-type processes [QR14], as the Airy$_2$ process corresponds to delta initial conditions for the KPZ equation, and it should be possible to derive all other initial profiles by integrating this delta initial condition.

The first proof of all of these identities was given recently in [CLW14] through studying TASEP with general initial conditions:

$$\max_{x\in\mathbb{R}}\left(\mathcal{A}_2(x)-(x-r)^2\right)\stackrel{d}{=}2^{1/3}\mathcal{A}_1(2^{-2/3}r) \tag{9.3.2}$$

$$\max_{x\in\mathbb{R}}\left(\mathcal{A}_2(x)-(x-r)^2+\sqrt{2}B(x)\right)\stackrel{d}{=}\mathcal{A}_{\rm stat}(r), \tag{9.3.3}$$

$$\max_{x\geq 0}\left(\mathcal{A}_2(x)-(x-r)^2\right)\stackrel{d}{=}\mathcal{A}_{2\to 1}(r)-r^2\mathbb{1}_{r\le 0}, \tag{9.3.4}$$

$$\max_{x\geq 0}\left(\mathcal{A}_2(x)-(x-r)^2+\sqrt{2}B(x)\right)\stackrel{d}{=}\mathcal{A}_{{\rm BM}\to 2}(-r)-r^2, \tag{9.3.5}$$

$$\max_{x\in\mathbb{R}}\left(\mathcal{A}_2(x)-(x-r)^2+\sqrt{2}\mathbb{1}_{s\geq 0}B(x)\right)\stackrel{d}{=}\mathcal{A}_{2\to 1,1,0}(-r)-r^2\mathbb{1}_{r\geq 0}. \tag{9.3.6}$$

Notice that all equalities hold for the one-point distribution only.

9.4 Gaussian fluctuations of the Airy$_{\rm stat}$ process

In this section we show that the Airy$_{\rm stat}$ process has Brownian increments for nonnegative arguments. As a rigorous proof of this fact is already established by indirect arguments, we do not care for full mathematical precision, but rather give a sketch of a proof:

Theorem 9.17. *Let* $0\le r_1<r_2<\cdots<r_m$*. Then*

$$\mathbb{P}\left(\bigcap_{k=2}^{m}\{\mathcal{A}_{\rm stat}(r_k)-\mathcal{A}_{\rm stat}(r_{k-1})\in {\rm d}\sigma_k\}\right)=\prod_{k=2}^{m}\frac{e^{-\sigma_k^2/4(r_k-r_{k-1})}}{\sqrt{4\pi(r_k-r_{k-1})}}{\rm d}\vec{\sigma}. \tag{9.4.1}$$

Proof. Without loss of generality we can assume that $r_1 = 0$. Denoting the partial derivative with respect to the i-th coordinate by ∂_i, we have

$$\mathbb{P}\bigg(\bigcap_{k=1}^{m}\{\mathcal{A}_{\rm stat}(r_k) \leq s_k\}\bigg) = \sum_{i=1}^{m} \partial_i \Lambda\left(s_1, \dots, s_m\right), \tag{9.4.2}$$

with

$$\Lambda\left(s_1, \dots, s_m\right) = G_m(\vec{r}, \vec{s}) \det\left(\mathbb{1} - \mathcal{P}K\right)_{L^2(\mathbb{R})}. \tag{9.4.3}$$

With a small abuse of notations, in what follows we will write

$$\mathbb{P}\bigg(\bigcap_{k=1}^{m}\{\mathcal{A}_{\rm stat}(r_k) \in {\rm d}s_k\}\bigg) \equiv \mathbb{P}\bigg(\bigcap_{k=1}^{m}\{\mathcal{A}_{\rm stat}(r_k) = s_k\}\bigg){\rm d}s_1 \cdots {\rm d}s_m. \tag{9.4.4}$$

Then,

$$\mathbb{P}\bigg(\bigcap_{k=1}^{m}\{\mathcal{A}_{\rm stat}(r_k) = s_k\}\bigg) = \prod_{i=1}^{m} \partial_i \sum_{j=1}^{m} \partial_j \Lambda\left(s_1, \dots, s_m\right). \tag{9.4.5}$$

The crucial identity is:

$$\begin{aligned}
&\mathbb{P}\bigg(\bigcap_{k=2}^{m}\{\mathcal{A}_{\rm stat}(r_k) - \mathcal{A}_{\rm stat}(r_{k-1}) = \sigma_k\}\bigg)\\
&= \int_{\mathbb{R}} {\rm d}\sigma_1 \mathbb{P}\bigg(\bigcap_{k=1}^{m}\{\mathcal{A}_{\rm stat}(r_k) = \sigma_1 + \cdots + \sigma_k\}\bigg)\\
&= \int_{\mathbb{R}} {\rm d}\sigma_1 \bigg(\prod_{i=1}^{m} \partial_i \sum_{j=1}^{m} \partial_j\bigg) \Lambda\left(\sigma_1, \sigma_1 + \sigma_2, \dots, \sigma_1 + \cdots + \sigma_m\right)\\
&= \int_{\mathbb{R}} {\rm d}\sigma_1 \frac{\rm d}{{\rm d}\sigma_1} \bigg(\prod_{i=1}^{m} \partial_i\bigg) \Lambda\left(\sigma_1, \sigma_1 + \sigma_2, \dots, \sigma_1 + \cdots + \sigma_m\right)\\
&= \bigg(\prod_{i=1}^{m} \partial_i\bigg) \Lambda\left(\sigma_1, \sigma_1 + \sigma_2, \dots, \sigma_1 + \cdots + \sigma_m\right)\bigg|_{\sigma_1=-\infty}^{\sigma_1=\infty}.
\end{aligned} \tag{9.4.6}$$

We therefore have to study the asymptotics of Λ as $\sigma_1 \to \pm\infty$.

First we decompose Λ as

$$\begin{aligned}
&\Lambda = \Lambda_1 + \Lambda_2,\\
&\Lambda_1 := (\mathcal{R} - 1) \det\left(\mathbb{1} - \mathcal{P}K\right)_{L^2(\mathbb{R})},\\
&\Lambda_2 := \det\Big(\mathbb{1} - \mathcal{P}K - (\mathcal{P}f^* + \mathcal{P}KP_{s_1}\mathbf{1} + (\mathcal{P} - P_{s_1})\mathbf{1}) \otimes g\Big)_{L^2(\mathbb{R})}.
\end{aligned} \tag{9.4.7}$$

Since $r_1 = 0$ some functions simplify as

$$\begin{aligned} \mathcal{R} &= s_1 + \int_{s_1}^{\infty} \mathrm{d}x \int_x^{\infty} \mathrm{d}y\, \mathrm{Ai}(y), \\ f^*(s) &= -\int_s^{\infty} \mathrm{d}x\, \mathrm{Ai}(x), \\ g(s) &= 1 - \int_s^{\infty} \mathrm{d}x\, \mathrm{Ai}(x) = \int_{-\infty}^{s} \mathrm{d}x\, \mathrm{Ai}(x), \\ K(s_1, s_2) &= \int_0^{\infty} \mathrm{d}x\, \mathrm{Ai}(s_1 + x)\mathrm{Ai}(s_2 + x), \end{aligned} \tag{9.4.8}$$

where we used the identity (D.3) from [FS06].

Now consider Λ_1.

$$\left(\prod_{i=1}^{m} \partial_i\right)\Lambda_1(\vec{s}) = (\mathcal{R}-1)\left(\prod_{i=1}^{m} \partial_i\right)\det(\mathbb{1}-\mathcal{P}K)+\partial_1\mathcal{R}\left(\prod_{i=2}^{m} \partial_i\right)\det(\mathbb{1}-\mathcal{P}K). \tag{9.4.9}$$

Regarding the first term, notice that the multiple derivative of the Fredholm determinant gives exactly the multipoint density of the Airy$_2$ process, which is known to decay exponentially for both large positive and negative arguments. This exponential decay dominates over the linear growth of $\mathcal{R}$. Similarly, the $(m-1)$-fold derivative is smaller the $(m-1)$-point density of the Airy$_2$ process, so this contribution vanishes in the limit, too.

Continuing to Λ_2, using $f^* = -K\mathbf{1}$, we first simplify the expression

$$\Lambda_2 = \det\Big(\mathbb{1} - \mathcal{P}K + \big(\mathcal{P}K\bar{P}_{s_1}\mathbf{1} - (\mathcal{P} - P_{s_1})\mathbf{1}\big) \otimes g\Big)_{L^2(\mathbb{R})} \tag{9.4.10}$$

We introduce the shift operator S, $(Sf)(x) = f(x + \sigma_1)$, which satisfies $SV_{r_i,r_j}S^{-1} = V_{r_i,r_j}$ and $P_{a+\sigma_1} = S^{-1}P_aS$, and consequently also

$$\mathbb{1} - \bar{P}_{s_1+\sigma_1}V_{r_1,r_2}\bar{P}_{s_2+\sigma_1}\cdots V_{r_{m-1},r_m}\bar{P}_{s_m+\sigma_1}V_{r_m,r_1} = S^{-1}\mathcal{P}S. \tag{9.4.11}$$

Using $\det(\mathbb{1} - AB) = \det(\mathbb{1} - BA)$, we have

$$\Lambda_2(\vec{s}+\sigma_1) = \det\Big(\mathbb{1}-\mathcal{P}SKS^{-1}+\big(\mathcal{P}SKS^{-1}\bar{P}_{s_1}\mathbf{1} - (\mathcal{P} - P_{s_1})\mathbf{1}\big)\otimes Sg\Big)_{L^2(\mathbb{R})}. \tag{9.4.12}$$

Now the dependence on the vector $\vec{s}$ is only in the projection operators, while the dependence on σ_1 is only in these two operators:

$$\begin{aligned} (Sg)(s) &= \int_{-\infty}^{s+\sigma_1} \mathrm{d}x\, \mathrm{Ai}(x), \\ (SKS^{-1})(s_1, s_2) &= \int_{\sigma_1}^{\infty} \mathrm{d}x\, \mathrm{Ai}(s_1 + x)\mathrm{Ai}(s_2 + x). \end{aligned} \tag{9.4.13}$$

For large σ_1, we have $Sg \to \mathbf{1}$ and $SKS^{-1} \to 0$ (both strong types of convergence from the superexponential Airy decay). So

$$\lim_{\sigma_1\to\infty} \Lambda_2(\vec{s}+\sigma_1) = \det\Big(\mathbb{1} - (\mathcal{P} - P_{s_1})\mathbf{1}\otimes\mathbf{1}\Big)_{L^2(\mathbb{R})} = 1 - \langle(\mathcal{P} - P_{s_1})\mathbf{1},\mathbf{1}\rangle_{L^2(\mathbb{R})} \tag{9.4.14}$$

Applying the expansion (6.4.44), we arrive at:

$$\begin{aligned}\left(\prod_{i=1}^m \partial_i\right) \lim_{\sigma_1\to\infty} \Lambda_2(\vec{s}+\sigma_1) &= -\left(\prod_{i=1}^m \partial_i\right)\sum_{k=2}^m \langle \bar{P}_{s_1} V_{r_1,r_2}\ldots \bar{P}_{s_{k-1}} V_{r_{k-1},r_k} P_{s_k}\mathbf{1},\mathbf{1}\rangle \\ &= -\left(\prod_{i=1}^m \partial_i\right)\langle \bar{P}_{s_1} V_{r_1,r_2}\ldots \bar{P}_{s_{m-1}} V_{r_{m-1},r_m} P_{s_m}\mathbf{1},\mathbf{1}\rangle\end{aligned} \tag{9.4.15}$$

Writing out this scalar product and applying the fundamental theorem of calculus leads to:

$$(9.4.15) = V_{r_1,r_2}(s_1,s_2)V_{r_2,r_3}(s_2,s_3)\ldots V_{r_{m-1},r_m}(s_{m-1},s_m), \tag{9.4.16}$$

which is the desired Gaussian density after setting $s_i = \sum_{k=2}^i \sigma_k$ as in (9.4.6).

For large negative σ_1, we have $Sg \to \mathbf{0}$ and $SKS^{-1} \to \mathbb{1}$. The rank one contribution is thus

$$\big(\mathcal{P}\bar{P}_{s_1}\mathbf{1} - (\mathcal{P} - P_{s_1})\mathbf{1}\big)\otimes 0. \tag{9.4.17}$$

We have to be somewhat careful here, as the convergence is weak (only pointwise) and (Sg) is not even L^2-integrable. But the first factor decays superexponentially on both sides for finite σ_1 and also in the limiting case $\mathcal{P}\bar{P}_{s_1}\mathbf{1} - (\mathcal{P} - P_{s_1})\mathbf{1} = (1-\mathcal{P})P_{s_1}\mathbf{1}$, so one should be able to derive nice convergence properties. Neglecting this rank one contribution we are left with

$$\lim_{\sigma_1\to-\infty} \Lambda_2(\vec{s}+\sigma_1) = \det\Big(\mathbb{1} - \mathcal{P}\mathbb{1}\Big)_{L^2(\mathbb{R})} = 0. \tag{9.4.18}$$

□

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