Chair of Automatic Control

Technische Universität München

Time-Varying Parametric Model Order Reduction by Matrix Interpolation

Model Reduction of Parametrized Systems III

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Motivation for Model Order Reduction

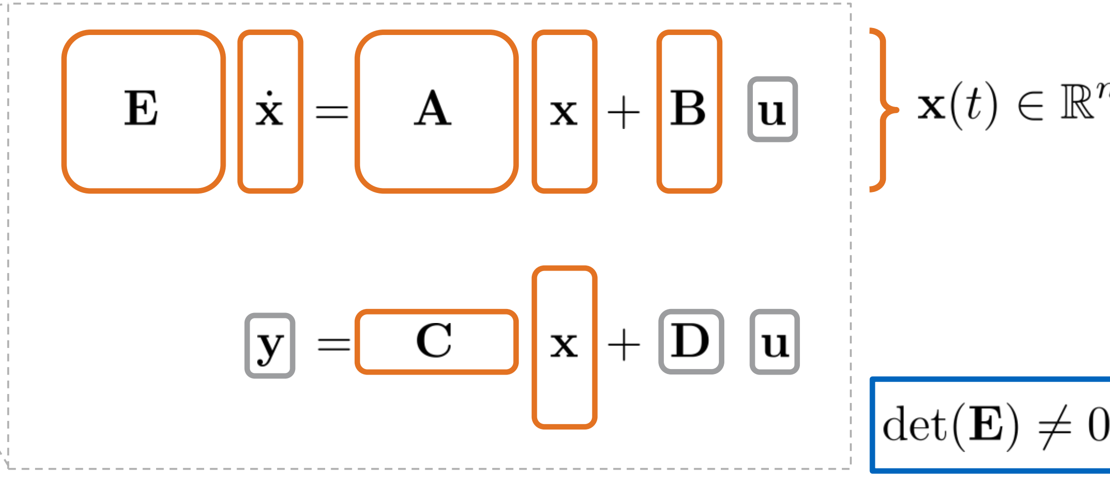
Linear time-invariant system in state-space representation

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t)$$

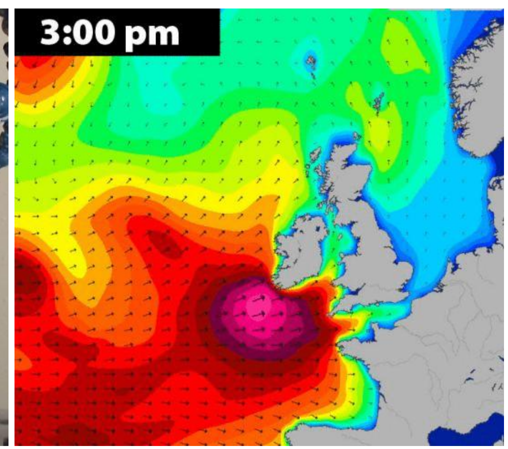
$\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$

$\mathbf{u}(t) \in \mathbb{R}^m, \mathbf{y}(t) \in \mathbb{R}^q$

$m, q \ll n$



$\det(\mathbf{E}) \neq 0$



10^3 — 10^4 — 10^5 — 10^6 → n

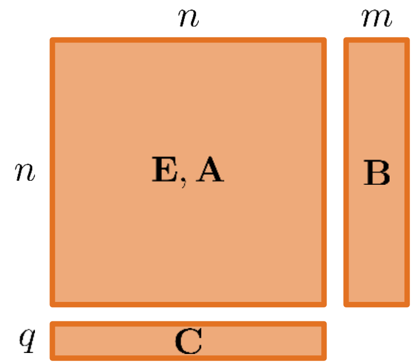
Model Order Reduction (MOR)

Linear time-invariant (LTI) system

$$\mathbf{G}(s): \begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

$\mathbf{E},\ \mathbf{A} \in \mathbb{R}^{n\times n}$

$\mathbf{B} \in \mathbb{R}^{n\times m},\ \mathbf{C} \in \mathbb{R}^{q\times n}$



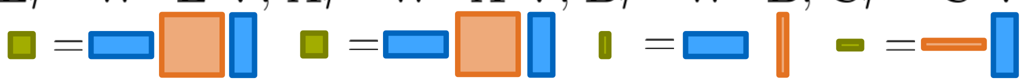
$r \ll n$

MOR

Projection

$\mathbf{V},\ \mathbf{W} \in \mathbb{R}^{n\times r}$

$\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\,\mathbf{V},\ \mathbf{A}_r = \mathbf{W}^T\mathbf{A}\,\mathbf{V},\ \mathbf{B}_r = \mathbf{W}^T\mathbf{B},\ \mathbf{C}_r = \mathbf{C}\,\mathbf{V}$

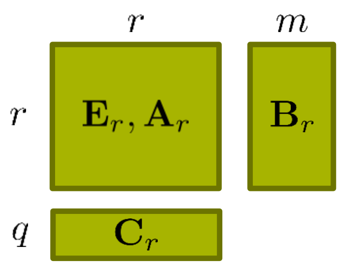


Reduced order model (ROM)

$$\mathbf{G}_r(s): \begin{cases} \mathbf{E}_r\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) = \mathbf{C}_r\mathbf{x}_r(t) \end{cases}$$

$\mathbf{E}_r,\ \mathbf{A}_r \in \mathbb{R}^{r\times r}$

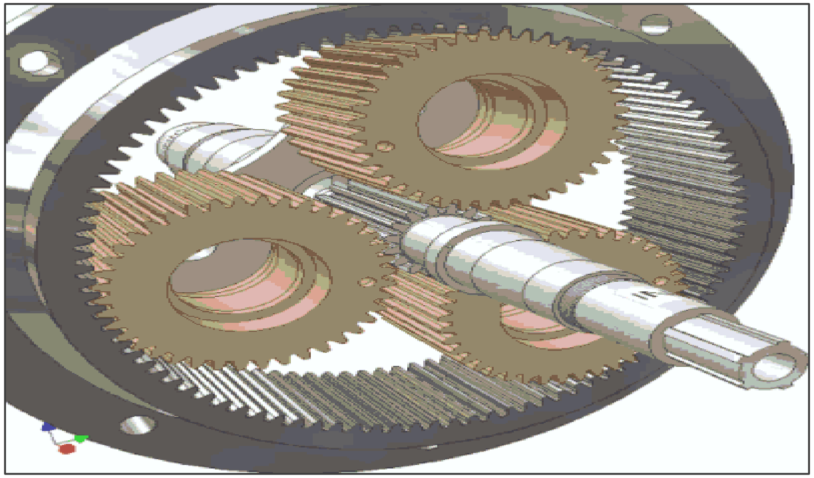
$\mathbf{B_r} \in \mathbb{R}^{r\times m},\ \mathbf{C}_r \in \mathbb{R}^{q\times r}$



Outline

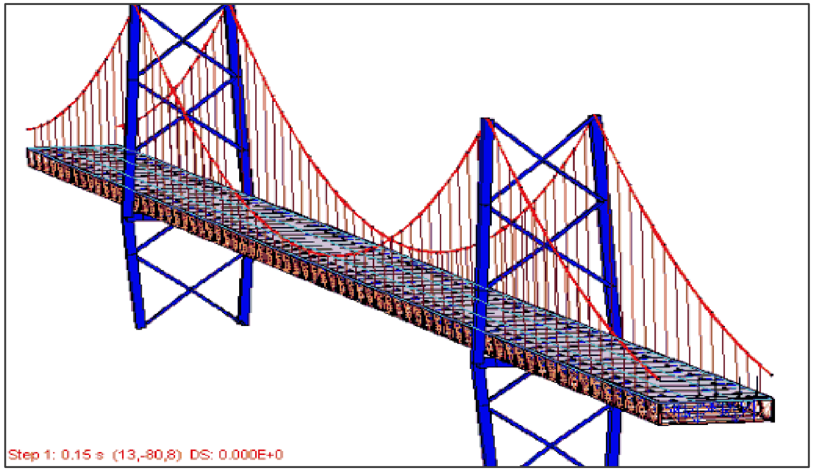
1. **Systems with Moving Loads**
   - Motivation & Examples
   - State-of-the-art: system representation and reduction
2. **Parametric Model Order Reduction (pMOR) by Matrix Interpolation**
   - Main idea
   - Procedure
3. **Time-Varying Parametric Model Order Reduction (p(t)MOR)**
   - Reduction of systems with moving loads: LPV system + Matrix Interpolation
   - Projection-based p(t)MOR
   - p(t)MOR by Matrix Interpolation
   - Numerical example: Timoshenko beam with moving load
4. **Summary and Outlook**
   - Discussion

Systems with Moving Loads



gearing wheels

cable railways



bridge with moving vehicles

circular milling machine

Systems with Moving Loads

- **Applications:** structural dynamics, multibody systems, turning/milling processes
- Position of the load varies over time
- Moving load causes time-varying dynamic behaviour

Moving Loads

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n \times n}$

$\mathbf{B}(t) \in \mathbb{R}^{n \times m}, \ \mathbf{C} \in \mathbb{R}^{q \times n}$

Moving Sensors

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n \times n}$

$\mathbf{B} \in \mathbb{R}^{n \times m}, \ \mathbf{C}(t) \in \mathbb{R}^{q \times n}$

Linear time-varying (LTV) system

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

$\mathbf{E}(t), \mathbf{A}(t) \in \mathbb{R}^{n \times n}$

$\mathbf{B}(t) \in \mathbb{R}^{n \times m}, \ \mathbf{C}(t) \in \mathbb{R}^{q \times n}$

Reduction of Systems with Moving Loads

LTV System

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

Balanced Truncation for LTV systems
[Shokoohi '83, Sandberg '04]

- Solution of two Lyapunov-Differential Equations (LDE)
- high storage and computational effort

Two-step approach
[Stykel/Vasilyev '15]

I) Low-rank approximation of the input matrix
II) Application of LTI-MOR (BT, Krylov)

Switched Linear System

$$\mathbf{E}_\alpha\dot{\mathbf{x}}(t) = \mathbf{A}_\alpha\mathbf{x}(t) + \mathbf{B}_\alpha\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_\alpha\mathbf{x}(t)$$

Switched Linear System + BT
[Lang et al. '14]

- Representation as switched linear system
- Application of BT for each subsystem

LPV System

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t)$$

Parametric LTI system + IRKA
[Lang et al. '14]

- Time-independ. parameter
- Concatenation of the local bases calculated by IRKA

Parametric LTI system + MatrInt
[Fischer '14, Fischer et al. '15]

- Time-independ. parameter
- Application of pMOR by Matrix Interpolation

LPV System + MatrInt
[Cruz/Geuss/Lohmann '15]

- Time-dependent parameter
- Adapted MatrInt with additional time-derivatives

pMOR by Matrix Interpolation

Properties:

- Local pMOR approach
- Analytical expression of the parameter-dependency in general not available
- Model only available at certain parameter sample points

Main idea:

1. Individual reduction of each local model
2. Transformation of the local reduced models
3. Interpolation of the reduced matrices

Advantages
• No analytically expressed parameter-dependency required • Any desired MOR technique applicable for the local reduction • Offline/Online decomposition • Reduced order independent of the number of local models

Drawbacks

• Choice of degrees of freedom
  – Parameter sample points
  – Interpolation method
• Stability preservation
• Error bounds

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure

$\mathbf{V}_1$
$\mathbf{W}_1$

$$\mathbf{E}_1\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}_1\mathbf{x}$$

$$\mathbf{E}_2\dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}_2\mathbf{x}$$

$\mathbf{V}_2$
$\mathbf{W}_2$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t) \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$\mathbf{p}_i, \quad i = 1, \dots, k$
$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$
$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure

$\mathbf{V}_1$
$\mathbf{W}_1$

$$\mathbf{E}_{r,1}\dot{\mathbf{x}}_{r,1} = \mathbf{A}_{r,1}\mathbf{x}_{r,1} + \mathbf{B}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \mathbf{C}_{r,1}\mathbf{x}_{r,1}$$

direct interpolation not meaningful

$$\mathbf{E}_{r,2}\dot{\mathbf{x}}_{r,2} = \mathbf{A}_{r,2}\mathbf{x}_{r,2} + \mathbf{B}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \mathbf{C}_{r,2}\mathbf{x}_{r,2}$$

$\mathbf{V}_2$
$\mathbf{W}_2$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t) \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$\mathbf{p}_i, \quad i = 1, \ldots, k$
$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$
$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure

$$\mathbf{E}_{r,1}\dot{\mathbf{x}}_{r,1} = \mathbf{A}_{r,1}\mathbf{x}_{r,1} + \mathbf{B}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \mathbf{C}_{r,1}\mathbf{x}_{r,1}$$

$$\mathbf{E}_{r,2}\dot{\mathbf{x}}_{r,2} = \mathbf{A}_{r,2}\mathbf{x}_{r,2} + \mathbf{B}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \mathbf{C}_{r,2}\mathbf{x}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t) \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$\mathbf{p}_i, \quad i = 1, \ldots, k$

$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$

$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$

2.) Transformation to generalized coordinates

$\mathbf{M}_i^T \cdot \Big|$

$$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1}$$

$$\mathbf{E}_{r,i}\mathbf{T}_i\dot{\hat{\mathbf{x}}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{T}_i\hat{\mathbf{x}}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{T}_i\hat{\mathbf{x}}_{r,i}(t)$$

$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$

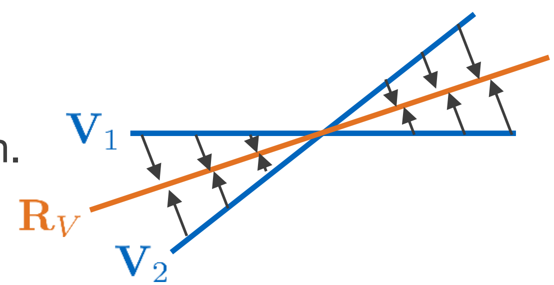
$\mathbf{V}_{all} = [\mathbf{V}_1, \ldots, \mathbf{V}_k]$

$\mathbf{V}_{all} \overset{\text{SVD}}{=} \mathbf{USN}^T$

$\mathbf{R}_V = \mathbf{U}(:, 1:r)$

How do we choose $\mathbf{T}_i$?

Goal: Adjustment of the local bases $\mathbf{V}_i$ to $\hat{\mathbf{V}}_i = \mathbf{V}_i\mathbf{T}_i$, in order to make the gen. coordinates $\hat{\mathbf{x}}_{r,i}$ compatible w.r.t. a reference subspace $\mathbf{R}_V$.



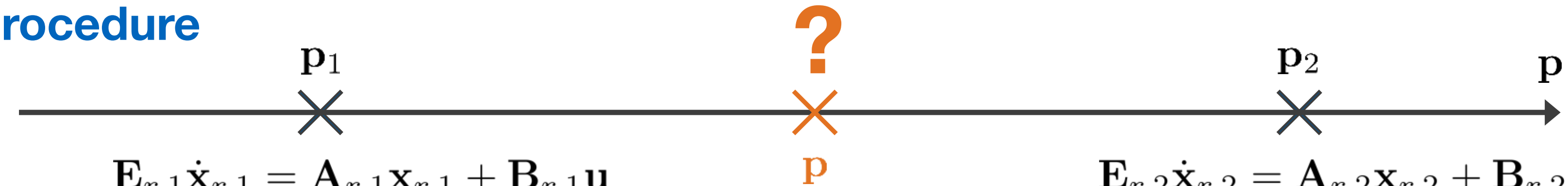
High correlation $\hat{\mathbf{V}}_i \leftrightarrow \mathbf{R}_V$:

$\mathbf{T}_i^T\mathbf{V}_i^T\mathbf{R}_V \overset{!}{=} \mathbf{I}$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure



$$\mathbf{E}_{r,1}\dot{\mathbf{x}}_{r,1} = \mathbf{A}_{r,1}\mathbf{x}_{r,1} + \mathbf{B}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \mathbf{C}_{r,1}\mathbf{x}_{r,1}$$

$$\mathbf{E}_{r,2}\dot{\mathbf{x}}_{r,2} = \mathbf{A}_{r,2}\mathbf{x}_{r,2} + \mathbf{B}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \mathbf{C}_{r,2}\mathbf{x}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t) \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$\mathbf{p}_i, \quad i = 1, \ldots, k$

$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$

$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$

2.) Transformation to generalized coordinates

$\mathbf{M}_i^T \cdot \big|$

$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i}(t) = \underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t) + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t)$$

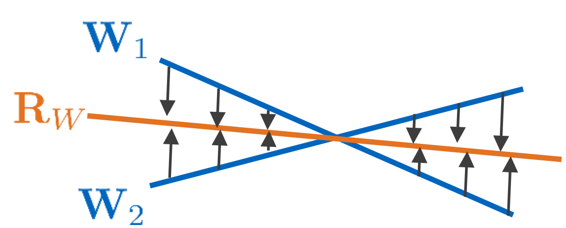
$$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1}$$
$$\mathbf{M}_i = (\mathbf{R}_W^T\mathbf{W}_i)^{-1}$$

$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$

Analogous to $\mathbf{R}_V$ or $\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$

How do we choose $\mathbf{M}_i$?

Goal: Adjustment of the local bases $\mathbf{W}_i$ to $\hat{\mathbf{W}}_i = \mathbf{W}_i\mathbf{M}_i$, in order to describe the local reduced models w.r.t. the same reference basis $\mathbf{R}_W$.



High correlation $\hat{\mathbf{W}}_i \leftrightarrow \mathbf{R}_W$:
$\mathbf{M}_i^T\mathbf{W}_i^T\mathbf{R}_W \overset{!}{=} \mathbf{I}$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure

$\mathbf{p}_1$? $\mathbf{p}_2$ $\mathbf{p}$

$\mathbf{p}$

$\mathbf{T}_1$
$\mathbf{M}_1$

$$\mathbf{E}_{r,1}\dot{\mathbf{x}}_{r,1} = \mathbf{A}_{r,1}\mathbf{x}_{r,1} + \mathbf{B}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \mathbf{C}_{r,1}\mathbf{x}_{r,1}$$

$$\mathbf{E}_{r,2}\dot{\mathbf{x}}_{r,2} = \mathbf{A}_{r,2}\mathbf{x}_{r,2} + \mathbf{B}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \mathbf{C}_{r,2}\mathbf{x}_{r,2}$$

$\mathbf{T}_2$
$\mathbf{M}_2$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t) \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1, \ldots, k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

2.) Transformation to generalized coordinates

$$\mathbf{M}_i^T \cdot \Big|$$

$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i}(t) = \underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t) + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t)$$

$$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1}$$
$$\mathbf{M}_i = (\mathbf{R}_W^T\mathbf{W}_i)^{-1}$$
$$\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$$

$$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$$

$$\mathbf{V}_{all} = [\mathbf{V}_1, \ldots, \mathbf{V}_k]$$
$$\mathbf{V}_{all} \overset{\text{SVD}}{=} \mathbf{USN}^T$$
$$\mathbf{R}_V = \mathbf{U}(:, 1:r)$$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure

$\mathbf{p}_1$? $\mathbf{p}_2$ $\mathbf{p}$

$\mathbf{p}$

$\mathbf{T}_1$, $\mathbf{M}_1$

$$\hat{\mathbf{E}}_{r,1}\dot{\hat{\mathbf{x}}}_{r,1} = \hat{\mathbf{A}}_{r,1}\hat{\mathbf{x}}_{r,1} + \hat{\mathbf{B}}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \hat{\mathbf{C}}_{r,1}\hat{\mathbf{x}}_{r,1}$$

$$\hat{\mathbf{E}}_{r,2}\dot{\hat{\mathbf{x}}}_{r,2} = \hat{\mathbf{A}}_{r,2}\hat{\mathbf{x}}_{r,2} + \hat{\mathbf{B}}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \hat{\mathbf{C}}_{r,2}\hat{\mathbf{x}}_{r,2}$$

$\mathbf{T}_2$, $\mathbf{M}_2$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t)$$

$$\mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1, \dots, k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

2.) Transformation to generalized coordinates

$$\mathbf{M}_i^T \cdot \Big|$$

$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i}(t) = \underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t) + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t)$$

$$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1}$$
$$\mathbf{M}_i = (\mathbf{R}_W^T\mathbf{W}_i)^{-1}$$
$$\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$$

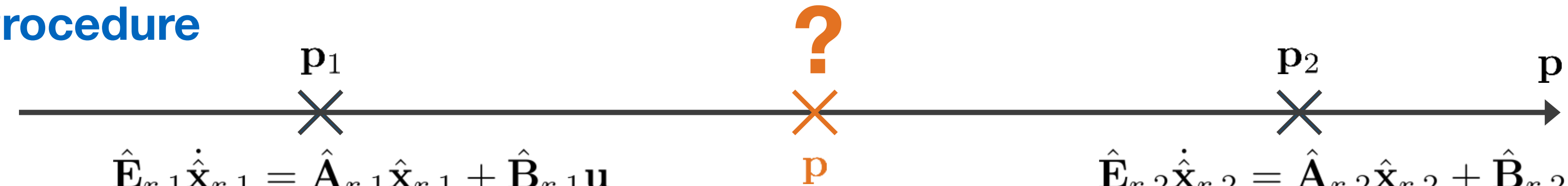
$$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$$

$$\mathbf{V}_{all} = [\mathbf{V}_1, \dots, \mathbf{V}_k]$$
$$\mathbf{V}_{all} \overset{\text{SVD}}{=} \mathbf{U}\mathbf{S}\mathbf{N}^T$$
$$\mathbf{R}_V = \mathbf{U}(:, 1:r)$$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Procedure



$$\hat{\mathbf{E}}_{r,1}\dot{\hat{\mathbf{x}}}_{r,1} = \hat{\mathbf{A}}_{r,1}\hat{\mathbf{x}}_{r,1} + \hat{\mathbf{B}}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \hat{\mathbf{C}}_{r,1}\hat{\mathbf{x}}_{r,1}$$

$$\hat{\mathbf{E}}_{r,2}\dot{\hat{\mathbf{x}}}_{r,2} = \hat{\mathbf{A}}_{r,2}\hat{\mathbf{x}}_{r,2} + \hat{\mathbf{B}}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \hat{\mathbf{C}}_{r,2}\hat{\mathbf{x}}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t)$$

$$\mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1, \dots, k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

2.) Transformation to generalized coordinates

$$\mathbf{M}_i^T \cdot \Big|$$

$$\overbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}^{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i}(t) = \overbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}^{\hat{\mathbf{A}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t) + \overbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}^{\hat{\mathbf{B}}_{r,i}}\mathbf{u}(t)$$
$$\mathbf{y}_{r,i}(t) = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t)$$

$$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1}$$
$$\mathbf{M}_i = (\mathbf{R}_W^T\mathbf{W}_i)^{-1}$$
$$\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$$

$$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$$

$$\mathbf{V}_{all} = [\mathbf{V}_1, \dots, \mathbf{V}_k]$$
$$\mathbf{V}_{all} \overset{\text{SVD}}{=} \mathbf{USN}^T$$
$$\mathbf{R}_V = \mathbf{U}(:, 1:r)$$

3.) Interpolation

$$\hat{\mathbf{E}}_r(\mathbf{p}) = \sum_{i=1}^k \omega_i(\mathbf{p})\hat{\mathbf{E}}_{r,i}, \quad \hat{\mathbf{A}}_r(\mathbf{p}) = \sum_{i=1}^k \omega_i(\mathbf{p})\hat{\mathbf{A}}_{r,i}$$
$$\hat{\mathbf{B}}_r(\mathbf{p}) = \sum_{i=1}^k \omega_i(\mathbf{p})\hat{\mathbf{B}}_{r,i}, \quad \hat{\mathbf{C}}_r(\mathbf{p}) = \sum_{i=1}^k \omega_i(\mathbf{p})\hat{\mathbf{C}}_{r,i}$$

$$\sum_{i=1}^k \omega_i(\mathbf{p}) = 1$$

Reduction of Systems with Moving Loads

LTV System

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

Balanced Truncation for LTV systems
[Shokoohi ’83, Sandberg ’04]

- Solution of two Lyapunov-Differential Equations (LDE)
- high storage and computational effort

Two-step approach
[Stykel/Vasilyev ’15]

I) Low-rank approximation of the input matrix
II) Application of LTI-MOR (BT, Krylov)

Switched Linear System

$$\mathbf{E}_\alpha\dot{\mathbf{x}}(t) = \mathbf{A}_\alpha\mathbf{x}(t) + \mathbf{B}_\alpha\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_\alpha\mathbf{x}(t)$$

Switched Linear System + BT
[Lang et al. ’14]

- Representation as switched linear system
- Application of BT for each subsystem

LPV System

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t)$$

Parametric LTI system + IRKA
[Lang et al. ’14]

- Time-independ. parameter
- Concatenation of the local bases calculated by IRKA

Parametric LTI system + MatrInt
[Fischer ’14, Fischer et al. ’15]

- Time-independ. parameter
- Application of pMOR by Matrix Interpolation

LPV System + MatrInt
[Cruz/Geuss/Lohmann ’15]

- Time-dependent parameter
- Adapted MatrInt with additional time-derivatives

Time-Varying Parametric Model Order Reduction: p(t)MOR

Linear parameter-varying (LPV) system

$$\begin{aligned}\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}} &= \mathbf{A}(\mathbf{p}(t))\mathbf{x} + \mathbf{B}(\mathbf{p}(t))\mathbf{u} \qquad && \mathbf{p}(t) \in \mathcal{D} \subset \mathbb{R}^d \\ \mathbf{y} &= \mathbf{C}(\mathbf{p}(t))\mathbf{x} && \mathbf{x} \in \mathbb{R}^n\end{aligned}$$

$r \ll n$ p(t)MOR

Approximation of the full state vector:

$$\mathbf{x} = \mathbf{V}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{e},$$

$$\dot{\mathbf{x}} = \dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{V}(\mathbf{p}(t))\dot{\mathbf{x}}_r + \dot{\mathbf{e}}$$

Petrov-Galerkin condition: $\mathbf{W}(\mathbf{p}(t)) \perp \boldsymbol{\epsilon}$

$\mathbf{W}(\mathbf{p}(t))^T \cdot \Big|$

$$\begin{aligned}\mathbf{E}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))\dot{\mathbf{x}}_r &= \Big(\mathbf{A}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t)) - \mathbf{E}(\mathbf{p}(t))\dot{\mathbf{V}}(\mathbf{p}(t))\Big)\mathbf{x}_r + \mathbf{B}(\mathbf{p}(t))\mathbf{u} + \boldsymbol{\epsilon} \\ \mathbf{y}_r &= \mathbf{C}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))\mathbf{x}_r\end{aligned}$$

Time-Varying Parametric Model Order Reduction: p(t)MOR

Linear parameter-varying (LPV) system

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}} = \mathbf{A}(\mathbf{p}(t))\mathbf{x} + \mathbf{B}(\mathbf{p}(t))\mathbf{u} \qquad \mathbf{p}(t) \in \mathcal{D} \subset \mathbb{R}^d$$

$$\mathbf{y} = \mathbf{C}(\mathbf{p}(t))\mathbf{x} \qquad \mathbf{x} \in \mathbb{R}^n$$

$r \ll n$ p(t)MOR

Approximation of the full state vector:

$$\mathbf{x} = \mathbf{V}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{e},$$

$$\dot{\mathbf{x}} = \dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{V}(\mathbf{p}(t))\dot{\mathbf{x}}_r + \dot{\mathbf{e}}$$

Petrov-Galerkin condition: $\mathbf{W}(\mathbf{p}(t)) \perp \boldsymbol{\epsilon}$

$$\overbrace{\mathbf{W}(\mathbf{p}(t))^T\mathbf{E}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))}^{\mathbf{E}_r(\mathbf{p}(t))}\dot{\mathbf{x}}_r = \left(\overbrace{\mathbf{W}(\mathbf{p}(t))^T\mathbf{A}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))}^{\mathbf{A}_r(\mathbf{p}(t))} - \mathbf{W}(\mathbf{p}(t))^T\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_r + \overbrace{\mathbf{W}(\mathbf{p}(t))^T\mathbf{B}(\mathbf{p}(t))}^{\mathbf{B}_r(\mathbf{p}(t))}\mathbf{u}$$

$$\mathbf{y}_r = \underbrace{\mathbf{C}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))}_{\mathbf{C}_r(\mathbf{p}(t))}\mathbf{x}_r$$

Time-Varying Parametric Model Order Reduction: p(t)MOR

Linear parameter-varying (LPV) system

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}} = \mathbf{A}(\mathbf{p}(t))\mathbf{x} + \mathbf{B}(\mathbf{p}(t))\mathbf{u} \qquad \mathbf{p}(t) \in \mathcal{D} \subset \mathbb{R}^d$$

$$\mathbf{y} = \mathbf{C}(\mathbf{p}(t))\mathbf{x} \qquad \mathbf{x} \in \mathbb{R}^n$$

$r \ll n$ p(t)MOR

Approximation of the full state vector:

$$\mathbf{x} = \mathbf{V}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{e},$$

$$\dot{\mathbf{x}} = \dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{x}_r + \mathbf{V}(\mathbf{p}(t))\dot{\mathbf{x}}_r + \dot{\mathbf{e}}$$

Petrov-Galerkin condition: $\mathbf{W}(\mathbf{p}(t)) \perp \boldsymbol{\epsilon}$

Parameter-varying reduced order model

$$\mathbf{E}_r(\mathbf{p}(t))\dot{\mathbf{x}}_r = \left(\mathbf{A}_r(\mathbf{p}(t)) - \mathbf{W}(\mathbf{p}(t))^T\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_r + \mathbf{B}_r(\mathbf{p}(t))\mathbf{u}$$

$$\mathbf{y}_r = \mathbf{C}_r(\mathbf{p}(t))\mathbf{x}_r$$

$$\mathbf{E}_r(\mathbf{p}(t)) = \mathbf{W}(\mathbf{p}(t))^T\mathbf{E}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t)), \qquad \mathbf{A}_r(\mathbf{p}(t)) = \mathbf{W}(\mathbf{p}(t))^T\mathbf{A}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))$$

$$\mathbf{B}_r(\mathbf{p}(t)) = \mathbf{W}(\mathbf{p}(t))^T\mathbf{B}(\mathbf{p}(t)), \qquad \mathbf{C}_r(\mathbf{p}(t)) = \mathbf{C}(\mathbf{p}(t))\mathbf{V}(\mathbf{p}(t))$$

p(t)MOR by Matrix Interpolation

Procedure

$\mathbf{V}_1$, $\mathbf{W}_1$

$$\mathbf{E}_{r,1}\dot{\mathbf{x}}_{r,1} = \mathbf{A}_{r,1}\mathbf{x}_{r,1} + \mathbf{B}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \mathbf{C}_{r,1}\mathbf{x}_{r,1}$$

$\mathbf{p}$

direct interpolation not meaningful

$$\mathbf{E}_{r,2}\dot{\mathbf{x}}_{r,2} = \mathbf{A}_{r,2}\mathbf{x}_{r,2} + \mathbf{B}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \mathbf{C}_{r,2}\mathbf{x}_{r,2}$$

$\mathbf{V}_2$, $\mathbf{W}_2$

1.) Individual reduction

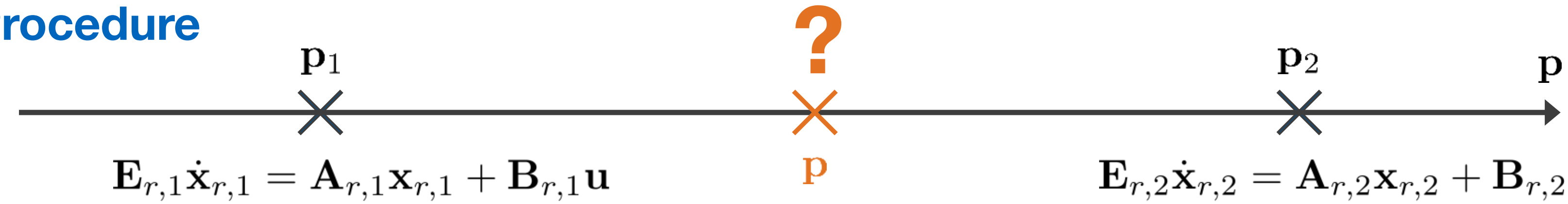
$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i} = \left(\mathbf{A}_{r,i} - \mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_{r,i} + \mathbf{B}_{r,i}\mathbf{u}$$
$$\mathbf{y}_{r,i} = \mathbf{C}_{r,i}\mathbf{x}_{r,i}$$

$$\mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1, \dots, k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

p(t)MOR by Matrix Interpolation

Procedure



$$\mathbf{E}_{r,1}\dot{\mathbf{x}}_{r,1} = \mathbf{A}_{r,1}\mathbf{x}_{r,1} + \mathbf{B}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \mathbf{C}_{r,1}\mathbf{x}_{r,1}$$

$$\mathbf{E}_{r,2}\dot{\mathbf{x}}_{r,2} = \mathbf{A}_{r,2}\mathbf{x}_{r,2} + \mathbf{B}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \mathbf{C}_{r,2}\mathbf{x}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i} = \left(\mathbf{A}_{r,i} - \mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_{r,i} + \mathbf{B}_{r,i}\mathbf{u}$$
$$\mathbf{y}_{r,i} = \mathbf{C}_{r,i}\mathbf{x}_{r,i}$$

$$\mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1, \dots, k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

2.) Transformation to generalized coordinates

$$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$$
$$\dot{\mathbf{x}}_{r,i} = \dot{\mathbf{T}}_i\hat{\mathbf{x}}_{r,i} + \mathbf{T}_i\dot{\hat{\mathbf{x}}}_{r,i}$$

$$\mathbf{M}_i^T \cdot \Big|$$

$$\mathbf{E}_{r,i}\mathbf{T}_i\,\dot{\hat{\mathbf{x}}}_{r,i} = \left(\mathbf{A}_{r,i}\mathbf{T}_i - \mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\,\mathbf{T}_i - \mathbf{E}_{r,i}\dot{\mathbf{T}}_i\right)\hat{\mathbf{x}}_{r,i} + \mathbf{B}_{r,i}\,\mathbf{u}$$
$$\mathbf{y}_{r,i} = \mathbf{C}_{r,i}\mathbf{T}_i\,\hat{\mathbf{x}}_{r,i}$$

p(t)MOR by Matrix Interpolation

Procedure

$\mathbf{p}_1$? $\mathbf{p}$ $\mathbf{p}_2$ $\mathbf{p}$

$\mathbf{T}_1$, $\mathbf{M}_1$

$$\hat{\mathbf{E}}_{r,1}\dot{\hat{\mathbf{x}}}_{r,1} = \hat{\mathbf{A}}_{r,1}\hat{\mathbf{x}}_{r,1} + \hat{\mathbf{B}}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \hat{\mathbf{C}}_{r,1}\hat{\mathbf{x}}_{r,1}$$

$$\hat{\mathbf{E}}_{r,2}\dot{\hat{\mathbf{x}}}_{r,2} = \hat{\mathbf{A}}_{r,2}\hat{\mathbf{x}}_{r,2} + \hat{\mathbf{B}}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \hat{\mathbf{C}}_{r,2}\hat{\mathbf{x}}_{r,2}$$

$\mathbf{T}_2$, $\mathbf{M}_2$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i} = \left(\mathbf{A}_{r,i} - \mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_{r,i} + \mathbf{B}_{r,i}\mathbf{u}$$
$$\mathbf{y}_{r,i} = \mathbf{C}_{r,i}\mathbf{x}_{r,i}$$

$$\mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$\mathbf{p}_i, \quad i = 1, \dots, k$

$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$

$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$

2.) Transformation to generalized coordinates

$\mathbf{M}_i^T \cdot \Big|$

$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i} = \underbrace{\Big(\underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}} - \mathbf{M}_i^T\mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\,\mathbf{T}_i - \mathbf{M}_i^T\mathbf{E}_{r,i}\dot{\mathbf{T}}_i\Big)}_{\hat{\mathbf{A}}_{\text{new }r,i}}\hat{\mathbf{x}}_{r,i} + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}$$
$$\mathbf{y}_{r,i} = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}$$

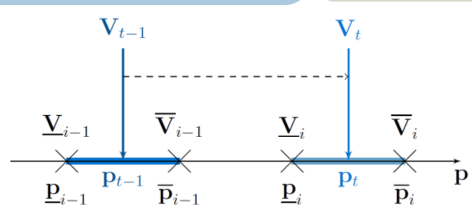
$$\mathbf{T}_i = (\mathbf{R}^T\mathbf{V}_i)^{-1}$$
$$\mathbf{M}_i = (\mathbf{R}^T\mathbf{W}_i)^{-1}$$

$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$

$\dot{\mathbf{x}}_{r,i} = \dot{\mathbf{T}}_i\hat{\mathbf{x}}_{r,i} + \mathbf{T}_i\dot{\hat{\mathbf{x}}}_{r,i}$

$\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$

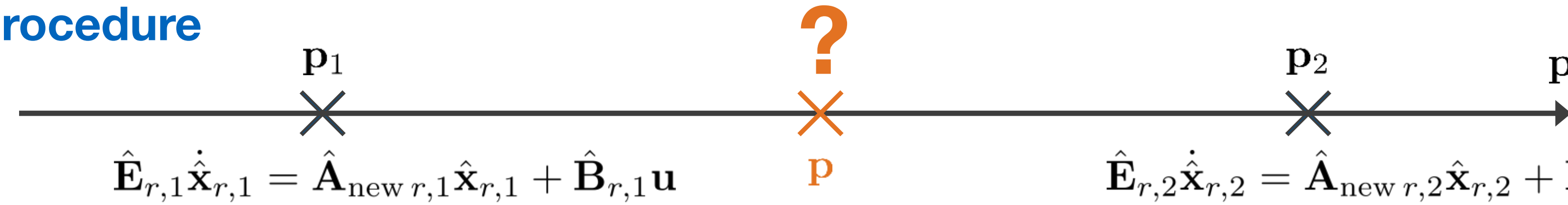
Calculation of $\dot{\mathbf{V}}(\mathbf{p}(t))$: $\dot{\mathbf{V}}(\mathbf{p}(t)) = \dfrac{\partial \mathbf{V}}{\partial \mathbf{p}}\dot{\mathbf{p}} = \dfrac{\overline{\mathbf{V}}_i - \underline{\mathbf{V}}_i}{\overline{\mathbf{p}}_i - \underline{\mathbf{p}}_i}\dot{\mathbf{p}}$



Calculation of $\dot{\mathbf{T}}_i$: $\dot{\mathbf{T}}_i = -(\mathbf{R}^T\mathbf{V}_i)^{-1}\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))(\mathbf{R}^T\mathbf{V}_i)^{-1} = -\mathbf{T}_i\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{T}_i$

p(t)MOR by Matrix Interpolation

Procedure



$$\hat{\mathbf{E}}_{r,1}\dot{\hat{\mathbf{x}}}_{r,1} = \hat{\mathbf{A}}_{\text{new}\,r,1}\hat{\mathbf{x}}_{r,1} + \hat{\mathbf{B}}_{r,1}\mathbf{u} \qquad \hat{\mathbf{E}}_{r,2}\dot{\hat{\mathbf{x}}}_{r,2} = \hat{\mathbf{A}}_{\text{new}\,r,2}\hat{\mathbf{x}}_{r,2} + \hat{\mathbf{B}}_{r,2}\mathbf{u}$$

$$\mathbf{y}_{r,1} = \hat{\mathbf{C}}_{r,1}\hat{\mathbf{x}}_{r,1} \qquad \mathbf{y}_{r,2} = \hat{\mathbf{C}}_{r,2}\hat{\mathbf{x}}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i} = \left(\mathbf{A}_{r,i} - \mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_{r,i} + \mathbf{B}_{r,i}\mathbf{u} \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i,\ \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$

$$\mathbf{y}_{r,i} = \mathbf{C}_{r,i}\mathbf{x}_{r,i} \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1, \ldots, k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

2.) Transformation to generalized coordinates

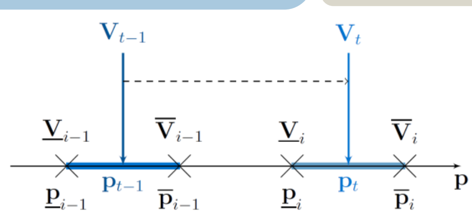
$$\mathbf{M}_i^T \cdot \Big|$$

$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i} = \underbrace{\left(\underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}} - \mathbf{M}_i^T\mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\,\mathbf{T}_i - \mathbf{M}_i^T\mathbf{E}_{r,i}\dot{\mathbf{T}}_i\right)}_{\hat{\mathbf{A}}_{\text{new}\,r,i}}\hat{\mathbf{x}}_{r,i} + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}$$

$$\mathbf{y}_{r,i} = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i} \qquad \mathbf{T}_i = (\mathbf{R}^T\mathbf{V}_i)^{-1}, \quad \mathbf{M}_i = (\mathbf{R}^T\mathbf{W}_i)^{-1}$$

$$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$$
$$\dot{\mathbf{x}}_{r,i} = \dot{\mathbf{T}}_i\hat{\mathbf{x}}_{r,i} + \mathbf{T}_i\dot{\hat{\mathbf{x}}}_{r,i}$$
$$\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$$

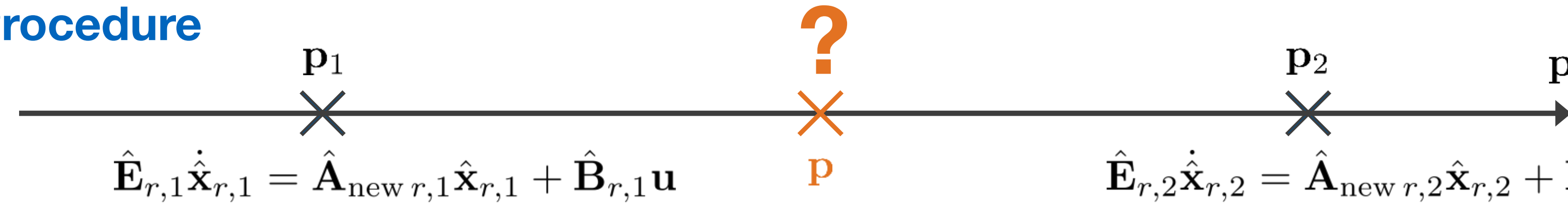
Calculation of $\dot{\mathbf{V}}(\mathbf{p}(t))$: $\dot{\mathbf{V}}(\mathbf{p}(t)) = \frac{\partial \mathbf{V}}{\partial \mathbf{p}}\dot{\mathbf{p}} = \frac{\overline{\mathbf{V}}_i - \underline{\mathbf{V}}_i}{\overline{\mathbf{p}}_i - \underline{\mathbf{p}}_i}\dot{\mathbf{p}}$



Calculation of $\dot{\mathbf{T}}_i$: $\dot{\mathbf{T}}_i = -(\mathbf{R}^T\mathbf{V}_i)^{-1}\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))(\mathbf{R}^T\mathbf{V}_i)^{-1} = -\mathbf{T}_i\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{T}_i$

p(t)MOR by Matrix Interpolation

Procedure



$$\hat{\mathbf{E}}_{r,1}\dot{\hat{\mathbf{x}}}_{r,1} = \hat{\mathbf{A}}_{\text{new }r,1}\hat{\mathbf{x}}_{r,1} + \hat{\mathbf{B}}_{r,1}\mathbf{u}$$
$$\mathbf{y}_{r,1} = \hat{\mathbf{C}}_{r,1}\hat{\mathbf{x}}_{r,1}$$

$$\hat{\mathbf{E}}_{r,2}\dot{\hat{\mathbf{x}}}_{r,2} = \hat{\mathbf{A}}_{\text{new }r,2}\hat{\mathbf{x}}_{r,2} + \hat{\mathbf{B}}_{r,2}\mathbf{u}$$
$$\mathbf{y}_{r,2} = \hat{\mathbf{C}}_{r,2}\hat{\mathbf{x}}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i} = \left(\mathbf{A}_{r,i} - \mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_{r,i} + \mathbf{B}_{r,i}\mathbf{u} \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i} = \mathbf{C}_{r,i}\mathbf{x}_{r,i} \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$$\mathbf{p}_i, \quad i = 1, \ldots, k$$
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$$
$$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$

2.) Transformation to generalized coordinates

$$\mathbf{M}_i^T \cdot \Big|$$

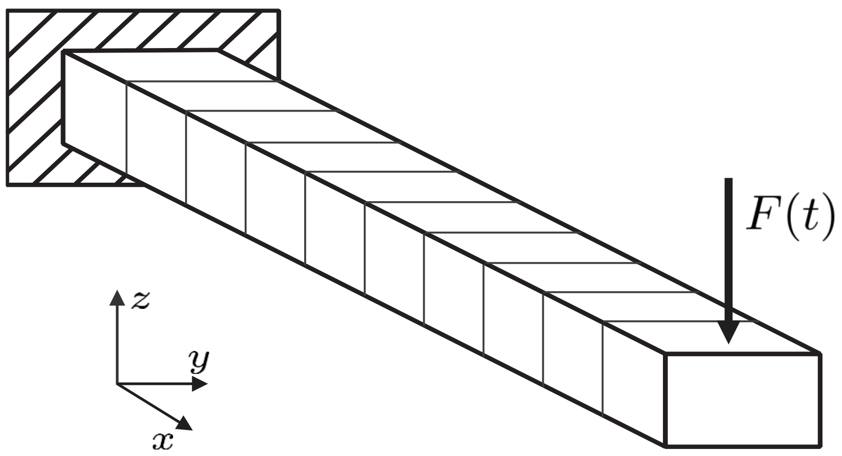
$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i} = \underbrace{\Big(\underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}} - \mathbf{M}_i^T\mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{T}_i - \mathbf{M}_i^T\mathbf{E}_{r,i}\dot{\mathbf{T}}_i\Big)}_{\hat{\mathbf{A}}_{\text{new }r,i}}\hat{\mathbf{x}}_{r,i} + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}$$
$$\mathbf{y}_{r,i} = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}$$
$$\mathbf{T}_i = (\mathbf{R}^T\mathbf{V}_i)^{-1}$$
$$\mathbf{M}_i = (\mathbf{R}^T\mathbf{W}_i)^{-1}$$

$$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$$
$$\dot{\mathbf{x}}_{r,i} = \dot{\mathbf{T}}_i\hat{\mathbf{x}}_{r,i} + \mathbf{T}_i\dot{\hat{\mathbf{x}}}_{r,i}$$
$$\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$$

3.) Interpolation

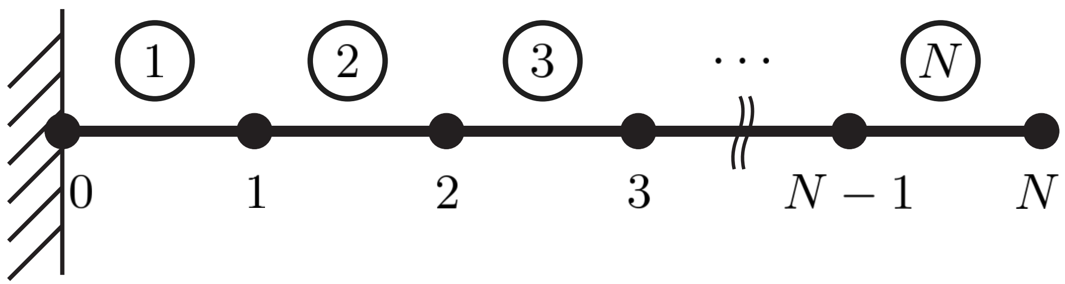
$$\tilde{\mathbf{E}}_r(\mathbf{p}(t)) = \sum_{i=1}^{k}\omega_i(\mathbf{p}(t))\hat{\mathbf{E}}_{r,i}, \quad \tilde{\mathbf{A}}_{\text{new }r}(\mathbf{p}(t)) = \sum_{i=1}^{k}\omega_i(\mathbf{p}(t))\hat{\mathbf{A}}_{\text{new }r,i}$$
$$\tilde{\mathbf{B}}_r(\mathbf{p}(t)) = \sum_{i=1}^{k}\omega_i(\mathbf{p}(t))\hat{\mathbf{B}}_{r,i}, \quad \tilde{\mathbf{C}}_r(\mathbf{p}(t)) = \sum_{i=1}^{k}\omega_i(\mathbf{p}(t))\hat{\mathbf{C}}_{r,i}$$
$$\sum_{i=1}^{k}\omega_i(\mathbf{p}(t)) = 1$$

Numerical example: Timoshenko beam with moving load



Parameters of the beam
Length: L
Height: h
Thickness: t
Density of material: ρ
Mass: m
Young‘s modulus: E
Poisson‘s ratio: ν
Shear modulus: G

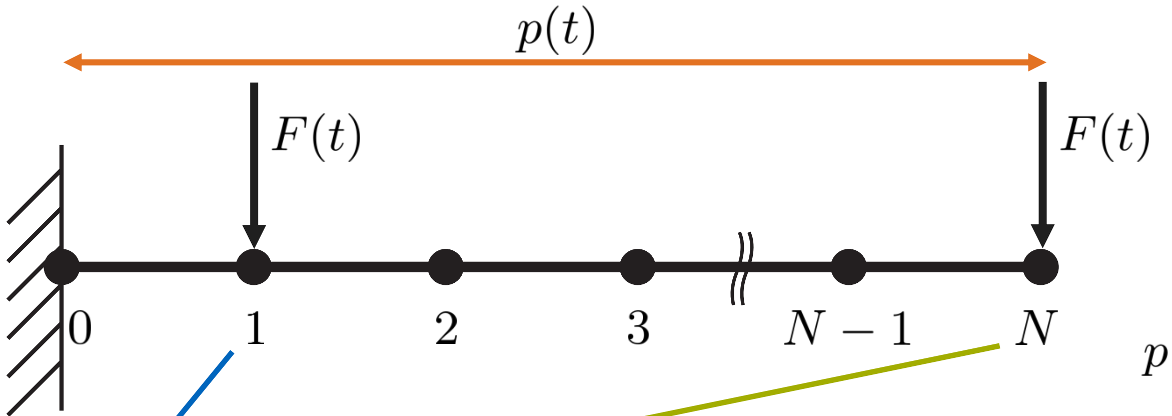
- Load position is considered as time-varying parameter
- Spatial discretization with finite element method (FEM)



N finite elements with length $l = \dfrac{L}{N}$

Numerical example: Timoshenko beam with moving load

[Panzer et al. '09]



N : finite elements

$p(t)$: varying load position

$$-\tilde{\mathbf{b}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots, -\tilde{\mathbf{b}}_N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Interpolation of the input vector:

$$\tilde{\mathbf{b}}(p(t)) = \sum_{i=1}^{N} \omega_i(p(t))\tilde{\mathbf{b}}_i$$

LPV first-order model:

$$\overbrace{\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}^{\mathbf{E}} \overbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \end{bmatrix}}^{\dot{\mathbf{x}}}(t) = \overbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}^{\mathbf{x}}(t) + \overbrace{\begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}(p(t)) \end{bmatrix}}^{\mathbf{b}(p(t))} F(t)$$

$$y(t) = \underbrace{\begin{bmatrix} \tilde{\mathbf{c}}^T & \mathbf{0}^T \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}(t)$$

with:

$\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{6N \times 6N}, \quad \tilde{\mathbf{c}}^T = -\tilde{\mathbf{b}}_N^T \in \mathbb{R}^{1 \times 6N}$

$\mathbf{F} = \mathbf{K}$ chosen ⇨ $\mathbf{A}$ dissipative, $\mathbf{E}$ pos. def.

$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{2 \cdot 6N \times 2 \cdot 6N}, \quad \mathbf{c}^T \in \mathbb{R}^{1 \times 2 \cdot 6N}$

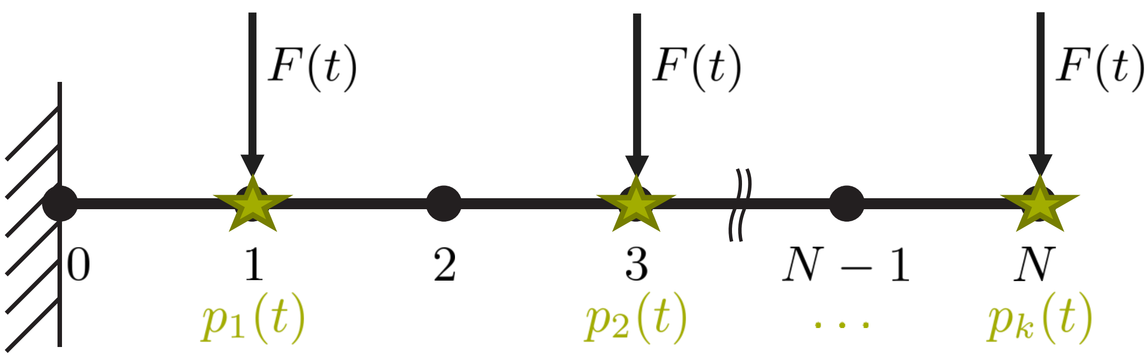
Original order: $2 \cdot 6N$

Reduction of the Timoshenko beam with moving load

Reduction with p(t)MOR by Matrix Interpolation

<u>Offline phase:</u>

1. Choose k appropriate parameter sample points $p_i(t), \; i = 1, \ldots, k \; (k \leq N)$
2. Build local models with respective input vector at the sample points
3. Reduce the local models separately via orthogonal projection ($\mathbf{W} = \mathbf{V}$) with desired MOR technique (e.g. one-sided rational Krylov method)
4. Transform the local reduced models to generalized coordinates



$$\mathbf{b}_1 = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}_1 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}_3 \end{bmatrix} \quad \cdots \quad \mathbf{b}_k = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}_N \end{bmatrix}$$

$$p_i(t), \; i = 1, \ldots, k$$

$$G_i(s) : (\mathbf{A}, \mathbf{b}_i, \mathbf{c}^T, \mathbf{E})$$

$$G_{r,i}(s) : (\mathbf{A}_{r,i}, \mathbf{b}_{r,i}, \mathbf{c}_{r,i}^T, \mathbf{E}_{r,i})$$

$$\hat{G}_{r,i}(s) : (\hat{\mathbf{A}}_{r,i}, \hat{\mathbf{b}}_{r,i}, \hat{\mathbf{c}}_{r,i}^T, \hat{\mathbf{E}}_{r,i})$$

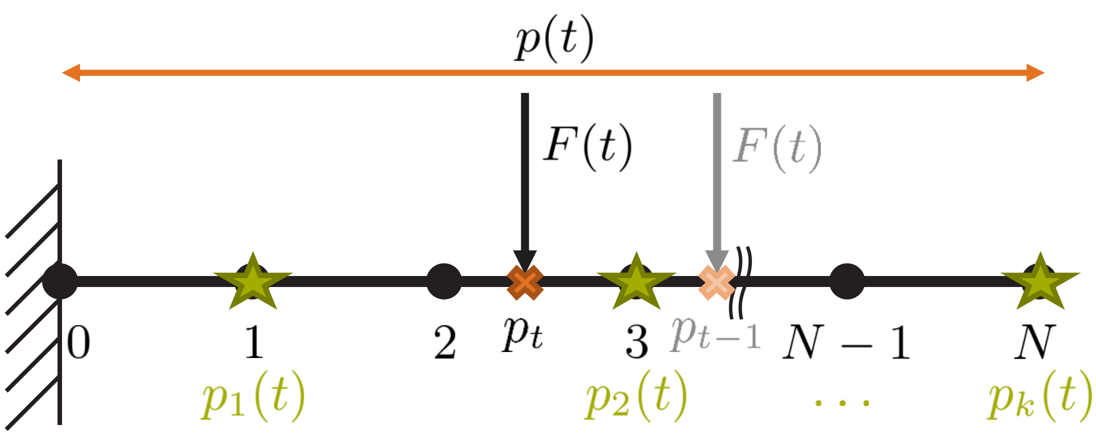
$$\Downarrow$$

Reduction of the Timoshenko beam with moving load

Reduction with p(t)MOR by Matrix Interpolation

Online phase:

1. Determine the actual parameter value $p(t)$ depending on the load position
2. Compute the time-derivatives $\dot{\mathbf{V}}(p(t))$, $\dot{\mathbf{T}}_i$ and calculate $\hat{\mathbf{A}}_{\mathrm{new}\,r,i}$
3. Calculate the weights $\omega_i(p(t)),\ i = 1, \dots, k$ depending on the actual $p(t)$
4. Interpolate between the reduced and adapted matrices
5. Simulate the reduced and interpolated model



$$\hat{G}_{r,i}(s) : (\hat{\mathbf{A}}_{r,i}, \hat{\mathbf{b}}_{r,i}, \hat{\mathbf{c}}_{r,i}^T, \hat{\mathbf{E}}_{r,i})$$

$$p(t)$$

$$\dot{\mathbf{V}}(p(t)) = \frac{\mathbf{V}_2 - \mathbf{V}_1}{p_2 - p_1}\,\dot{p}, \quad \dot{\mathbf{T}}_i$$

$$\Rightarrow \hat{\mathbf{A}}_{\mathrm{new}\,r,i}$$

$$\omega_i(p(t)),\ i = 1, \dots, k$$

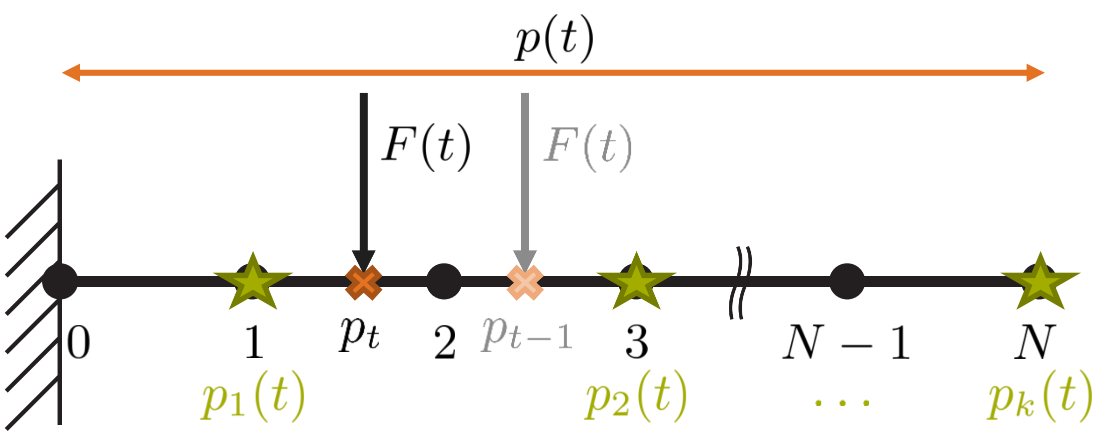
$$\tilde{G}_r^{\mathrm{int}}(s) : (\tilde{\mathbf{A}}_{\mathrm{new}\,r}(p(t)), \tilde{\mathbf{b}}_r(p(t)), \tilde{\mathbf{c}}_r(p(t))^T, \tilde{\mathbf{E}}_r(p(t)))$$

Reduction of the Timoshenko beam with moving load

Reduction with p(t)MOR by Matrix Interpolation

Online phase:

1. Determine the actual parameter value $p(t)$ depending on the load position
2. Compute the time-derivatives $\dot{\mathbf{V}}(p(t))$, $\dot{\mathbf{T}}_i$ and calculate $\hat{\mathbf{A}}_{\mathrm{new}\, r,i}$
3. Calculate the weights $\omega_i(p(t)),\ i = 1, \ldots, k$ depending on the actual $p(t)$
4. Interpolate between the reduced and adapted matrices
5. Simulate the reduced and interpolated model



$\hat{G}_{r,i}(s) : (\hat{\mathbf{A}}_{r,i}, \hat{\mathbf{b}}_{r,i}, \hat{\mathbf{c}}_{r,i}^T, \hat{\mathbf{E}}_{r,i})$

$p(t)$

$$\dot{\mathbf{V}}(p(t)) = \frac{\mathbf{V}_2 - \mathbf{V}_1}{p_2 - p_1}\dot{p},\quad \dot{\mathbf{T}}_i$$

$$\Rightarrow\ \hat{\mathbf{A}}_{\mathrm{new}\, r,i}$$

$\omega_i(p(t)),\ i = 1, \ldots, k$

$$\tilde{G}_r^{\mathrm{int}}(s) : (\tilde{\mathbf{A}}_{\mathrm{new}\, r}(p(t)), \tilde{\mathbf{b}}_r(p(t)), \tilde{\mathbf{c}}_r(p(t))^T, \tilde{\mathbf{E}}_r(p(t)))$$

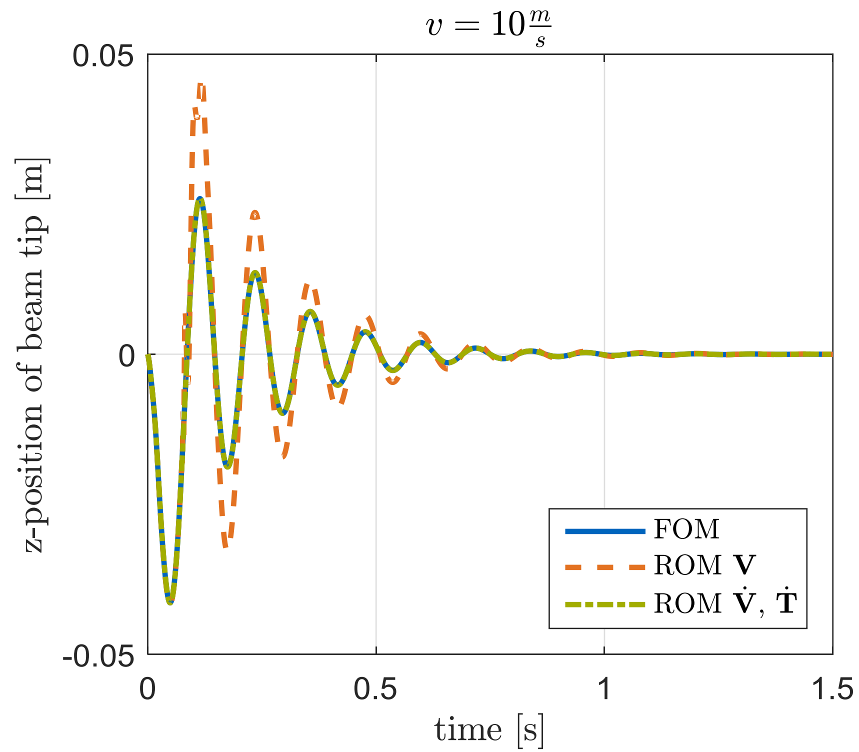
Simulation Results with Input Krylov Subspace

Reduction with p(t)MOR by matrix interpolation and input Krylov subspace

$$\mathbf{V}(p(t)) := \left[\mathbf{A}_{s_0}^{-1}\mathbf{b}(p(t)) \quad \mathbf{A}_{s_0}^{-1}\mathbf{E}\mathbf{A}_{s_0}^{-1}\mathbf{b}(p(t)) \quad \ldots \quad (\mathbf{A}_{s_0}^{-1}\mathbf{E})^{r-1}\mathbf{A}_{s_0}^{-1}\mathbf{b}(p(t))\right]$$

$$\mathbf{W}(p(t)) = \mathbf{V}(p(t))$$ ⇒ Parameter-varying projection matrices

Length of the beam	L = 1 m
Load amplitude	F(t) = 20 N
Velocity of the moving load	v = 1..10 m/s
Number of finite elements	N = 151
Original order	n = 1812
Number of local models	k = 76
Reduced order	r = 10
Expansion points	$s_0 = 0$
Implicit Euler method	dt = 0.001 s



Reduced order model with adapted matrix interpolation (ROM $\dot{\mathbf{V}}$, $\dot{\mathbf{T}}$) yields better results than standard matrix interp.

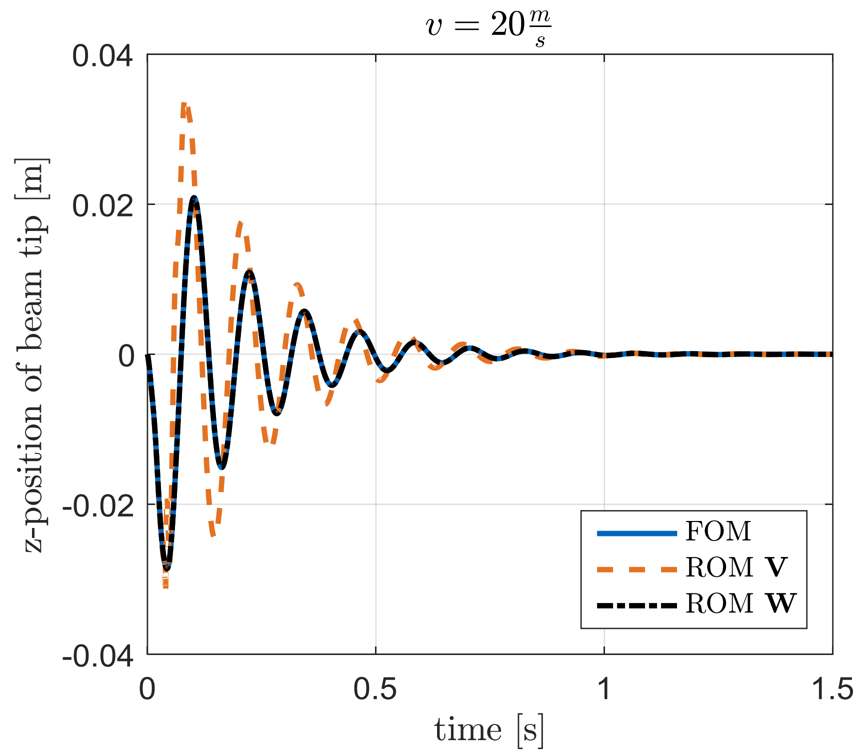
Simulation Results with Output Krylov Subspace

Reduction with p(t)MOR by matrix interpolation and output Krylov subspace

$$\mathbf{W} := \left[\mathbf{A}_{s_0}^{-T}\mathbf{c}^T \quad \mathbf{A}_{s_0}^{-T}\mathbf{E}^T\mathbf{A}_{s_0}^{-T}\mathbf{c}^T \quad \dots \quad (\mathbf{A}_{s_0}^{-T}\mathbf{E}^T)^{r-1}\mathbf{A}_{s_0}^{-T}\mathbf{c}^T\right]$$

$\mathbf{V} = \mathbf{W}$ ⇨ Time-independent projection matrices ⇨ $\dot{\mathbf{V}} = \mathbf{0}$

Length of the beam	L = 1 m
Load amplitude	F(t) = 20 N
Velocity of the moving load	v = 5..20 m/s
Number of finite elements	N = 151
Original order	n = 1812
Number of local models	k = 76
Reduced order	r = 10
Expansion points	$s_0 = 0$
Implicit Euler method	dt = 0.001 s



Reduced order model obtained with output Krylov subspace (ROM **W**) yields in our case the best results

Summary and Outlook

Summary:

- **Goal:** Reduction of high dimensional LPV systems (e.g. systems with moving load) by matrix interpolation
- Projection-based p(t)MOR for the reduction of LPV systems
- Extension of matrix interpolation to the parameter-varying case
- Application of p(t)MOR by matrix interpolation to Timoshenko beam with moving load
  - Consideration of the emerged time-derivative terms during the reduction process yields better results than the standard matrix interpolation
  - Reduction with output Krylov subspace is particularly suitable in our case

Outlook:

- Further development of the matrix interpolation for the reduction of LPV systems and investigation of the influence of the additional time-derivative terms
- Performance analysis and validation of the algorithm through testing on other benchmarks and real-life models

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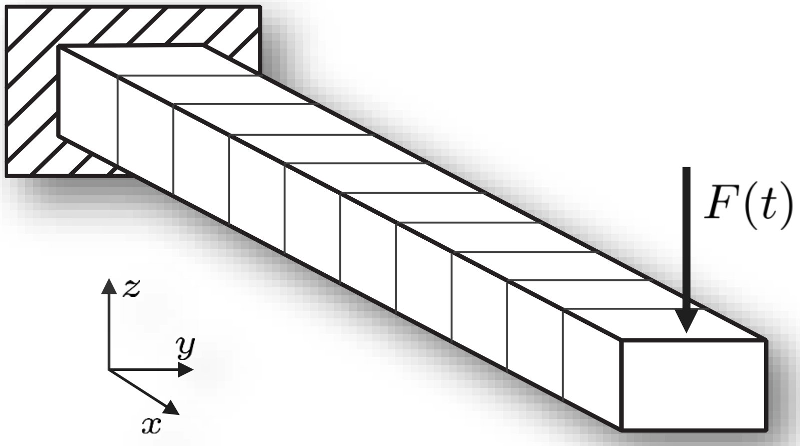
Chair of Automatic Control

Technische Universität München

Time-Varying Parametric Model Order Reduction by Matrix Interpolation



Model **Re**duction of **Pa**rametrized **S**ystems III

Trieste, 13th October 2015

Thank you for your attention

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Backup

Model Order Reduction (MOR)

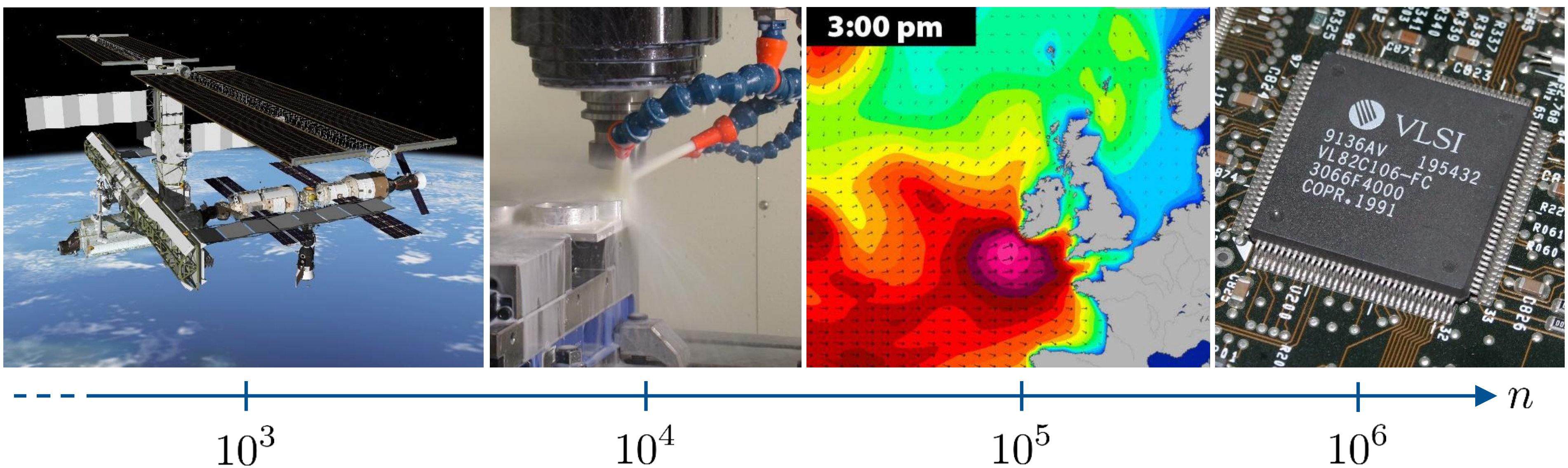
Linear time-invariant system in state-space representation

$$\mathbf{G}(s) \left\{ \begin{aligned} \mathbf{E}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \right.$$

MOR

$$\left. \begin{aligned} \mathbf{E}_r\dot{\mathbf{x}}_r &= \mathbf{A}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{u} \\ \mathbf{y}_r &= \mathbf{C}_r\mathbf{x}_r + \mathbf{D}_r\mathbf{u} \end{aligned} \right\} \mathbf{G}_r(s)$$

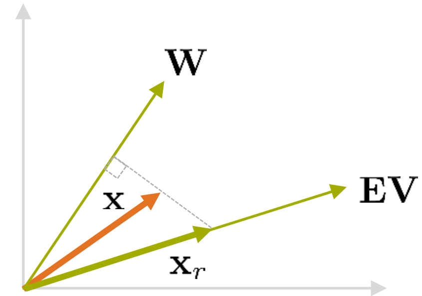
$$\mathbf{x}_r \in \mathbb{R}^r, \; r \ll n$$



Projection-based MOR

Approximation in dem Unterraum $\mathcal{V} = \mathrm{span}(\mathbf{V})$

$$\mathbf{x} = \mathbf{V}\,\mathbf{x}_r + \mathbf{e}, \qquad \mathbf{V} \in \mathbb{R}^{n \times r}$$



Procedure:

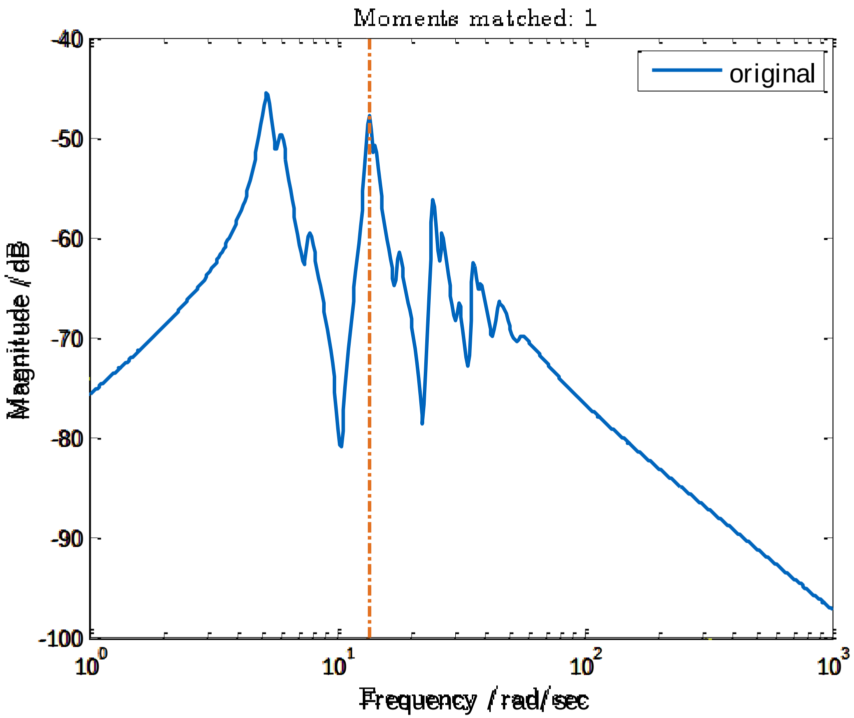
1. Ansatz in die Zustandsgleichung einsetzen
2. Anzahl der Gleichungen reduzieren (via projection with $\mathbf{\Pi} = \mathbf{EV}(\mathbf{W}^T\mathbf{EV})^{-1}\mathbf{W}^T$)
3. Petrov-Galerkin condition

$$\underbrace{\mathbf{W}^T\mathbf{E}\mathbf{V}}_{\mathbf{E}_r}\dot{\mathbf{x}}_r = \underbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}_{\mathbf{A}_r}\mathbf{x}_r + \underbrace{\mathbf{W}^T\mathbf{B}}_{\mathbf{B}_r}\mathbf{u}$$

$$\mathbf{y} \approx \mathbf{y}_r = \underbrace{\mathbf{C}\mathbf{V}}_{\mathbf{C}_r}\mathbf{x}_r + \underbrace{\mathbf{D}}_{\mathbf{D}_r}\mathbf{u}$$

Krylov Subspace Methods

Grundidee: Lokale Approximation der Übertragungsfunktion $\mathbf{G}(s)$ um die Frequenz(en) s_0



Momente einer Übertragungsfunktion

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}$$

$$= \mathbf{G}(\Delta s + s_0) = -\sum_{i=0}^{\infty} \mathbf{M}_i(s_0)(s-s_0)^i$$

s_0 : Entwicklungspunkt (Shift)

$\mathbf{M}_i(s_0)$: i-tes Moment um s_0

Moment Matching mit Krylow-Unterräume

$$\mathbf{A}_{s_0} := \mathbf{A} - s_0\mathbf{E}$$

- Basis für Eingangs-Krylow-Raum:

$$\mathbf{V} = \left[\mathbf{A}_{s_0}^{-1}\mathbf{B},\quad \mathbf{A}_{s_0}^{-1}\mathbf{E}\mathbf{A}_{s_0}^{-1}\mathbf{B},\quad \ldots,\quad (\mathbf{A}_{s_0}^{-1}\mathbf{E})^{r-1}\mathbf{A}_{s_0}^{-1}\mathbf{B}\right]$$

- Basis für Ausgangs-Krylow-Raum:

$$\mathbf{W} = \left[\mathbf{A}_{s_0}^{-T}\mathbf{C}^T,\quad \mathbf{A}_{s_0}^{-T}\mathbf{E}^T\mathbf{A}_{s_0}^{-T}\mathbf{C}^T,\quad \ldots,\quad (\mathbf{A}_{s_0}^{-T}\mathbf{E}^T)^{r-1}\mathbf{A}_{s_0}^{-T}\mathbf{C}^T\right]$$

⇨ $2\,r$ Momente um s_0 von original und reduziertem System stimmen überein

Comparison: BT vs. Krylov Subspace Methods

Balanced Truncation (BT)

+ Stabilitätserhaltung
+ Automatisierbar
+ Fehlerschranke (a priori)

– rechenintensiv
– speicherintensiv
– $n < 5000$

Krylov Subspace Methods

+ numerisch effizient
+ $n < 10^6$
+ H_2 -optimal (IRKA)
+ viele Freiheitsgrade

– viele Freiheitsgrade
– Stabilität i.A. nicht erhalten
– keine Fehlerschranke

Focus of research

- Numerisch effiziente Lösung hochdimensionaler Lyapunow-Gleichungen

⇨ Niedrig-Rang Approximation

  - ADI (Alternating Directions Implicit)
  - RKSM (Rational Krylov Subspace Method)

Focus of research

- Adaptive Wahl der Freiheitsgrade
  - Reduzierte Ordnung
  - Entwicklungspunkte
- Stabilitätserhaltung
- Numerisch effiziente Berechnung rigoroser Fehlerschranken

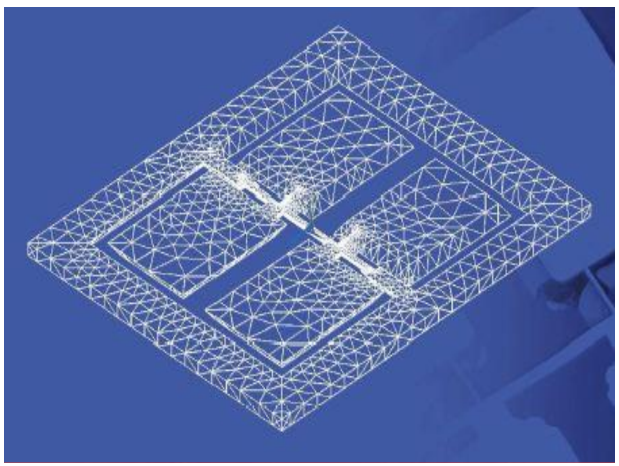
Parametric Model Order Reduction (pMOR)

Motivation:

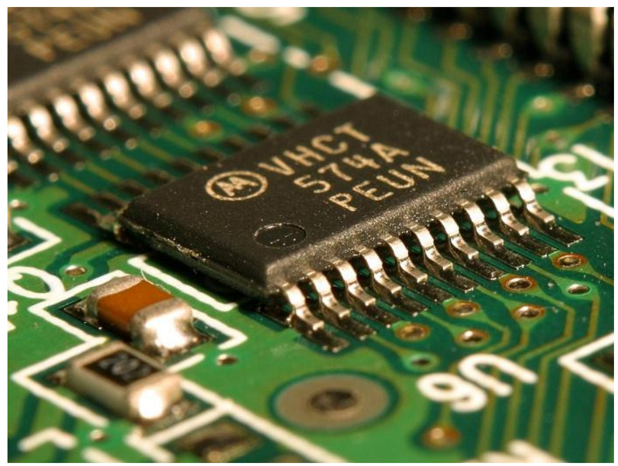
- Numerische Simulation und Designoptimierung komplexer technischer Systeme
- Typische Designparameter (z.B. Materialeigenschaften, Abmessungen) als offene Parameter im Modell

FEM ⇒ Hochdimensionales, parametrisches Modell

Examples:



MEMS Gyroskop



Integrierte Schaltungen

Solarzellen

Goal: Reduktion des Originalmodells und Erhaltung der Parameterabhängigkeit

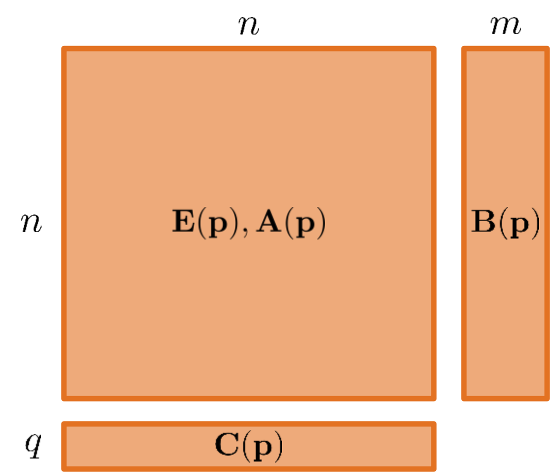
Projection-based pMOR

Parametric LTI system

$$\mathbf{G}(\mathbf{p}) : \begin{cases} \mathbf{E}(\mathbf{p})\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p})\mathbf{x}(t) + \mathbf{B}(\mathbf{p})\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(\mathbf{p})\mathbf{x}(t) \end{cases}$$

$$\mathbf{x}(t) \in \mathbb{R}^n, \mathbf{u}(t) \in \mathbb{R}^m, \mathbf{y}(t) \in \mathbb{R}^q, \mathbf{p} \in \mathcal{D} \subset \mathbb{R}^d$$

Hoher Rechenaufwand und Speicherbedarf für Simulation, Optimierung und Regelung erforderlich



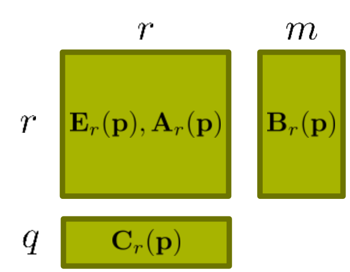
pMOR $r \ll n$

Parametric reduced order model

$$\mathbf{G}_r(\mathbf{p}) : \begin{cases} \mathbf{E}_r(\mathbf{p})\dot{\mathbf{x}}_r(t) = \mathbf{A}_r(\mathbf{p})\mathbf{x}_r(t) + \mathbf{B}_r(\mathbf{p})\mathbf{u}(t) \\ \mathbf{y}_r(t) = \mathbf{C}_r(\mathbf{p})\mathbf{x}_r(t) \end{cases}$$

$$\mathbf{E}_r(\mathbf{p}) = \mathbf{W}(\mathbf{p})^T\mathbf{E}(\mathbf{p})\mathbf{V}(\mathbf{p}), \quad \mathbf{A}_r(\mathbf{p}) = \mathbf{W}(\mathbf{p})^T\mathbf{A}(\mathbf{p})\mathbf{V}(\mathbf{p})$$

$$\mathbf{B}_r(\mathbf{p}) = \mathbf{W}(\mathbf{p})^T\mathbf{B}(\mathbf{p}), \quad \mathbf{C}_r(\mathbf{p}) = \mathbf{C}(\mathbf{p})\mathbf{V}(\mathbf{p})$$



State-of-the-art: pMOR approaches

Globale Verfahren

Gemeinsame Unterräume $\mathbf{V}(\mathbf{p}), \mathbf{W}(\mathbf{p})$ für alle $\mathbf{p} \in \mathcal{D} \subset \mathbb{R}^d$

Multi-Parameter Moment Matching
[Weile '99, Daniel '04]

- ➕ Moment Matching bzgl. s und $\mathbf{p}$
- ➖ Explizite Parameterabhängigkeit nötig
- ➖ Fluch der Dimensionalität

Verkettung von lokalen Basen
[Leung '05, Li '05, Baur et al. '11]

- ➕ Berechnung von $\mathbf{V}_1, \mathbf{W}_1, \ldots, \mathbf{V}_k, \mathbf{W}_k$ mittels RK, BT, IRKA oder POD
- ➕ Verkettung der lokalen Basen
  $\mathbf{V}(\mathbf{p}) = [\mathbf{V}_1, \ldots, \mathbf{V}_k], \mathbf{W}(\mathbf{p}) = [\mathbf{W}_1, \ldots, \mathbf{W}_k]$
- ➖ Reduzierte Ordnung: $r = k \cdot r'$
- ➖ Affine Parameterabhängigkeit nötig

Lokale Verfahren

Individuelle Unterräume $\mathbf{V}(\mathbf{p}_i), \mathbf{W}(\mathbf{p}_i)$ für lokale Systeme $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$

Interpolation von Übertragungsfunkt.
[Baur '09]

- ➕ Lokale Reduktion mittels BT
  ⇨ Fehlerschranken und Stabilität
- ➖ Reduzierte Ordnung: $r = k \cdot r'$

Interpolation von Unterräume
[Amsallem '08]

- ◎ Interpolation der Projektionsmatrizen
- ➕ Reduzierte Ordnung: $r = r'$

Interpolation von reduzierten Matrizen
[Eid '09, Panzer '10, Amsallem '11]

- ➕ Keine explizite oder affine Parameter-abhängigkeit notwendig
- ➕ Reduzierte Ordnung: $r = r'$

pMOR by Matrix Interpolation

[Panzer et al. '10]

Offline/Online decomposition

Offline phase:

1. Choose appropriate sample points $\mathbf{p}_i, \; i = 1, \dots, k$ in the parameter space
2. Build local models at the parameter sample points
3. Reduce the local models separately with desired MOR technique (e.g. rational Krylov method, IRKA, BT, …)
4. Compute $\mathbf{R}$ and all transformation matrices $\mathbf{T}_i, \mathbf{M}_i$ and transform the local reduced models to generalized coordinates

Online phase:

1. Calculate the weights $\omega_i(\mathbf{p}), i = 1, \dots, k$ depending on the actual parameter value $\mathbf{p}$ and the chosen interpolation method (linear, spline, …)
2. Interpolate between the reduced system matrices

pMOR by Matrix Interpolation

Evaluation of the method according to different criteria

Criterion	Evaluation
Structure preservation	
Reduced order	
Storage effort	
Computational cost	
Offline/Online decomposition	
Stability preservation	
Error bounds	

Extensions for Matrix Interpolation

Vereinheitlichendes Framework
[Geuss et al. ’13]

Framework mit folgenden Schritten:
1.) Wahl der Parameterstützstellen
2.) Reduktion der lokalen Modelle
3.) Anpassung der lokalen Basen
4.) Wahl der Interpolationsmannigfaltigkeit
5.) Wahl der Interpolationsmethode

Stabilitätserhaltung
[Geuss et al. ’14a]

- Interpolation (selbst stabiler) reduzierter Modelle garantiert i.A. keine Stabilität
- **Idee:** Stabile reduzierte Modelle auf dissipative Form bringen, damit ein stabiles interpoliertes System resultiert
  → Lösung von Lyapunov-Gleichungen

Interpolation zwischen Modellen verschiedener reduzierter Ordnung
[Geuss et al. ’14b]

- Interpolation zwischen Modellen mit unterschiedlicher reduzierter Ordnung r_i nicht möglich
- **Idee:** Basen $\mathbf{V}_i, \mathbf{W}_i$ auf dieselbe Größe r_0 bringen durch die Berechnung von $\mathbf{T}_i, \mathbf{M}_i$ mittels Pseudoinversen

Black-Box Methode
[Geuss et al. ’15]

- **Ziel:** Automatisierte pMOR-Methode
- **Idee:** Kreuzvalidierungsfehler für die iterative Ermittlung von Stützstellen und die optimale Wahl der Interpolations-mannigfaltigkeit und Interpolations-methode verwenden

Overview of the Research Project

State-of-the-art

Matrix interpolation for parametric linear time-invariant systems

1.) Individual reduction

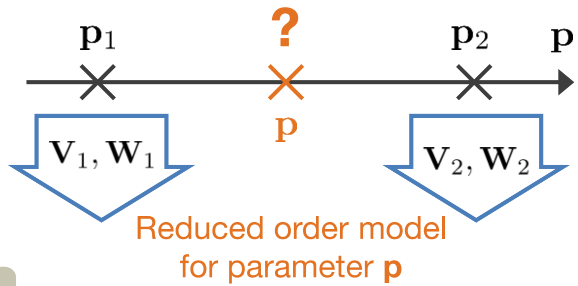
$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t) \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$
$$\mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t) \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

2.) Transformation to generalized coordinates

$$\overbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}^{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i}(t) = \overbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}^{\hat{\mathbf{A}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t) + \overbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}^{\hat{\mathbf{B}}_{r,i}}\mathbf{u}(t) \qquad \mathbf{T}_i = (\mathbf{R}^T\mathbf{V}_i)^{-1}$$
$$\mathbf{y}_{r,i}(t) = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i}(t) \qquad \mathbf{M}_i = (\mathbf{R}^T\mathbf{W}_i)^{-1}$$

3.) Interpolation

$$\hat{\mathbf{E}}_r(\mathbf{p}) = \sum_{i=1}^{k}\omega_i(\mathbf{p})\hat{\mathbf{E}}_{r,i}, \quad \hat{\mathbf{A}}_r(\mathbf{p}) = \sum_{i=1}^{k}\omega_i(\mathbf{p})\hat{\mathbf{A}}_{r,i} \qquad \sum_{i=1}^{k}\omega_i(\mathbf{p}) = 1$$
$$\hat{\mathbf{B}}_r(\mathbf{p}) = \sum_{i=1}^{k}\omega_i(\mathbf{p})\hat{\mathbf{B}}_{r,i}, \quad \hat{\mathbf{C}}_r(\mathbf{p}) = \sum_{i=1}^{k}\omega_i(\mathbf{p})\hat{\mathbf{C}}_{r,i}$$



Further system classes

Linear parameter-varying systems (LPV)

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t)$$

Nonlinear systems

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \qquad \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \qquad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Reduction of Systems with Moving Loads

Balanced Truncation for LTV systems

Linear time-varying system:

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

Solution of two Lyapunov-DE (LDE):

$$\mathbf{A}(t)\mathbf{P}(t)\mathbf{E}(t)^T + \mathbf{E}(t)\mathbf{P}(t)\mathbf{A}(t)^T + \mathbf{B}(t)\mathbf{B}(t)^T = \dot{\mathbf{P}}(t)$$
$$\mathbf{P}(t_0) = \mathbf{0}$$
$$\mathbf{A}(t)^T\mathbf{Q}(t)\mathbf{E}(t) + \mathbf{E}(t)^T\mathbf{Q}(t)\mathbf{A}(t) + \mathbf{C}(t)^T\mathbf{C}(t) = \dot{\mathbf{Q}}(t)$$
$$\mathbf{Q}(t_e) = \mathbf{0}$$

Switched Linear System + BT

Switched linear system:

$$\mathbf{E}_\alpha\dot{\mathbf{x}}(t) = \mathbf{A}_\alpha\mathbf{x}(t) + \mathbf{B}_\alpha\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_\alpha\mathbf{x}(t)$$

BT for each subsystem:

$$\left.\begin{aligned}\mathbf{A}_\alpha\mathbf{P}_\alpha\mathbf{E}_\alpha^T + \mathbf{E}_\alpha\mathbf{P}_\alpha\mathbf{A}_\alpha^T + \mathbf{B}_\alpha\mathbf{B}_\alpha^T = \mathbf{0}\\ \mathbf{A}_\alpha^T\mathbf{Q}_\alpha\mathbf{E}_\alpha + \mathbf{E}_\alpha^T\mathbf{Q}_\alpha\mathbf{A}_\alpha + \mathbf{C}_\alpha^T\mathbf{C}_\alpha = \mathbf{0}\end{aligned}\right. \Rightarrow \mathbf{V}_\alpha, \mathbf{W}_\alpha$$

Model reduction: $\mathbf{E}_{r,\alpha},\ \mathbf{A}_{r,\alpha},\ \mathbf{B}_{r,\alpha},\ \mathbf{C}_{r,\alpha}$

Two-step approach

I) Low-rank approximation: $\mathbf{B}(t) \approx \hat{\mathbf{B}}\mathbf{\Psi}(t)$

$\mathbf{u}(t) \in \mathbb{R}^m$ $\quad \xrightarrow{m \gg \hat{m}} \quad$ $\hat{\mathbf{u}}(t) = \mathbf{\Psi}(t)\mathbf{u}(t) \in \mathbb{R}^{\hat{m}}$

$\mathbf{B}(t) \in \mathbb{R}^{n\times m}$ $\qquad$ $\hat{\mathbf{B}} \in \mathbb{R}^{n\times\hat{m}}$

II) LTI-MOR: Reduction of the resulting LTI system with Rational Krylov, IRKA, BT, …

Parametric LTI system + pMOR

Global IRKA: $\mathbf{E}_i,\ \mathbf{A}_i,\ \mathbf{B}_i,\ \mathbf{C}_i$

$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i),\ \mathbf{W}_i := \mathbf{W}(\mathbf{p}_i) \qquad \mathbf{p}_i,\ i = 1,\dots,k$$
$$\mathbf{V} = [\mathbf{V}_1,\dots,\mathbf{V}_k],\ \mathbf{W} = [\mathbf{W}_1,\dots,\mathbf{W}_k]$$

Matrix Interpolation:

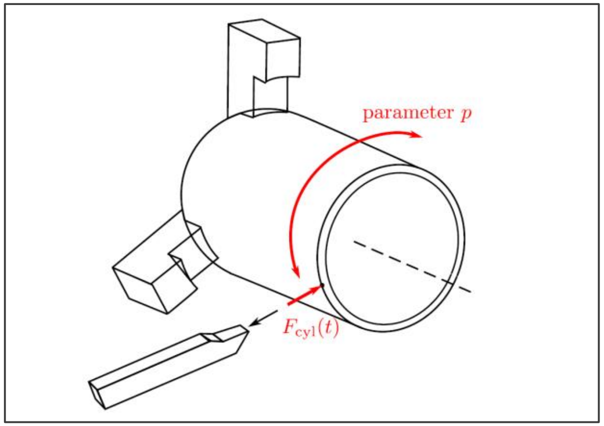
$$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i),\ \mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$$
$$\mathbf{T}_i,\ \mathbf{M}_i$$

Interpolation of reduced system matrices

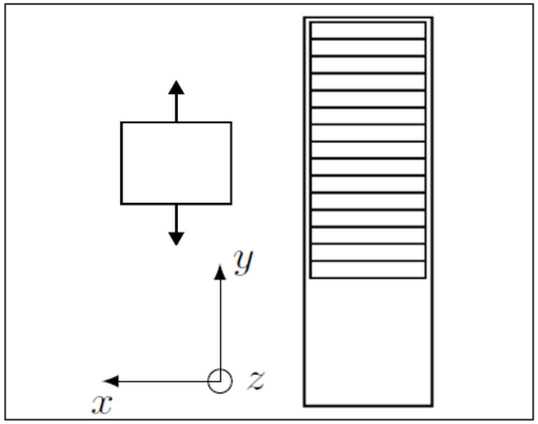
Reduction of Moving Loads by Matrix Interpolation

Systems with Moving Loads:

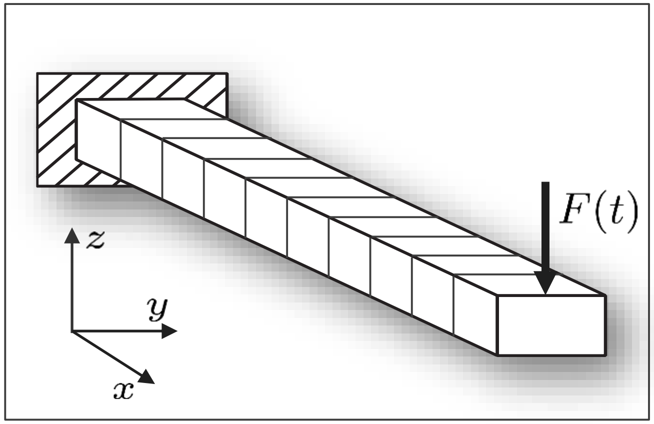
- Location of the load varies with time
- Moving load is considered as time-dependent parameter



thin-walled cylinder



thermo-elastic machine stand



Timoshenko beam

Linear parameter-varying (LPV) system:

$$\mathbf{E}(\mathbf{p}(t))\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t) \qquad \mathbf{p}(t) \in \mathcal{D} \subset \mathbb{R}^d$$
$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) \qquad \mathbf{x}(t) \in \mathbb{R}^n$$

- System matrices explicitly depend on time-varying parameters
- Special class of linear time-varying (LTV) or nonlinear systems

Goal: Reduction of high dimensional LPV systems by matrix interpolation

p(t)MOR by Matrix Interpolation

Procedure

$\mathbf{p}_1$ — $?$ $\mathbf{p}$ — $\mathbf{p}_2$ — $\mathbf{p}$

$$\hat{\mathbf{E}}_{r,1}\dot{\hat{\mathbf{x}}}_{r,1} = \hat{\mathbf{A}}_{\text{new}\,r,1}\hat{\mathbf{x}}_{r,1} + \hat{\mathbf{B}}_{r,1}\mathbf{u} \qquad \hat{\mathbf{E}}_{r,2}\dot{\hat{\mathbf{x}}}_{r,2} = \hat{\mathbf{A}}_{\text{new}\,r,2}\hat{\mathbf{x}}_{r,2} + \hat{\mathbf{B}}_{r,2}\mathbf{u}$$

$$\mathbf{y}_{r,1} = \hat{\mathbf{C}}_{r,1}\hat{\mathbf{x}}_{r,1} \qquad \mathbf{y}_{r,2} = \hat{\mathbf{C}}_{r,2}\hat{\mathbf{x}}_{r,2}$$

1.) Individual reduction

$$\mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i} = \left(\mathbf{A}_{r,i} - \mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\right)\mathbf{x}_{r,i} + \mathbf{B}_{r,i}\mathbf{u} \qquad \mathbf{E}_{r,i} = \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \quad \mathbf{A}_{r,i} = \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i$$

$$\mathbf{y}_{r,i} = \mathbf{C}_{r,i}\mathbf{x}_{r,i} \qquad \mathbf{B}_{r,i} = \mathbf{W}_i^T\mathbf{B}_i, \quad \mathbf{C}_{r,i} = \mathbf{C}_i\mathbf{V}_i$$

$\mathbf{p}_i, \quad i = 1, \dots, k$

$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i)$

$\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i)$

2.) Transformation to generalized coordinates

$\mathbf{M}_i^T \cdot$

$$\underbrace{\mathbf{M}_i^T\mathbf{E}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{E}}_{r,i}}\dot{\hat{\mathbf{x}}}_{r,i} = \underbrace{\Big(\underbrace{\mathbf{M}_i^T\mathbf{A}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{A}}_{r,i}} - \mathbf{M}_i^T\mathbf{W}_i^T\mathbf{E}_i\dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{T}_i - \mathbf{M}_i^T\mathbf{E}_{r,i}\dot{\mathbf{T}}_i\Big)}_{\hat{\mathbf{A}}_{\text{new}\,r,i}}\hat{\mathbf{x}}_{r,i} + \underbrace{\mathbf{M}_i^T\mathbf{B}_{r,i}}_{\hat{\mathbf{B}}_{r,i}}\mathbf{u}$$

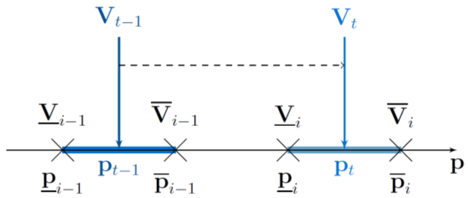
$$\mathbf{y}_{r,i} = \underbrace{\mathbf{C}_{r,i}\mathbf{T}_i}_{\hat{\mathbf{C}}_{r,i}}\hat{\mathbf{x}}_{r,i} \qquad \mathbf{T}_i = (\mathbf{R}^T\mathbf{V}_i)^{-1}, \quad \mathbf{M}_i = (\mathbf{R}^T\mathbf{W}_i)^{-1}$$

$\mathbf{x}_{r,i} = \mathbf{T}_i\hat{\mathbf{x}}_{r,i}$

$\dot{\mathbf{x}}_{r,i} = \dot{\mathbf{T}}_i\hat{\mathbf{x}}_{r,i} + \mathbf{T}_i\dot{\hat{\mathbf{x}}}_{r,i}$

$\mathbf{R}_W = \mathbf{R}_V := \mathbf{R}$

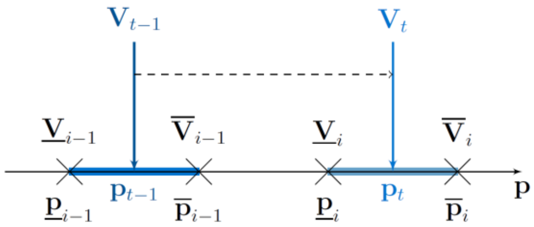
Calculation of $\dot{\mathbf{V}}(\mathbf{p}(t))$: $\dot{\mathbf{V}}(\mathbf{p}(t)) = \dfrac{\partial \mathbf{V}}{\partial \mathbf{p}}\dot{\mathbf{p}} = \dfrac{\overline{\mathbf{V}}_i - \underline{\mathbf{V}}_i}{\overline{\mathbf{p}}_i - \underline{\mathbf{p}}_i}\dot{\mathbf{p}}$



Calculation of $\dot{\mathbf{T}}_i$ **:** $\dot{\mathbf{T}}_i = -(\mathbf{R}^T\mathbf{V}_i)^{-1}\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))(\mathbf{R}^T\mathbf{V}_i)^{-1} = -\mathbf{T}_i\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{T}_i$

p(t)MOR by Matrix Interpolation

Calculation of time-derivatives:



Calculation of $\dot{\mathbf{V}}(\mathbf{p}(t))$**:** $\dot{\mathbf{V}}(\mathbf{p}(t)) = \dfrac{\partial \mathbf{V}}{\partial \mathbf{p}}\dot{\mathbf{p}} = \dfrac{\overline{\mathbf{V}}_i - \underline{\mathbf{V}}_i}{\overline{\mathbf{p}}_i - \underline{\mathbf{p}}_i}\dot{\mathbf{p}}$

Calculation of $\dot{\mathbf{T}}_i$ **:** $\dot{\mathbf{T}}_i = -(\mathbf{R}^T\mathbf{V}_i)^{-1}\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))(\mathbf{R}^T\mathbf{V}_i)^{-1} = -\mathbf{T}_i\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{T}_i$

Time-derivative of inverse matrix

Definition: Is $\mathbf{H}$ a regular matrix, then the time-derivative of the inverse matrix is given by

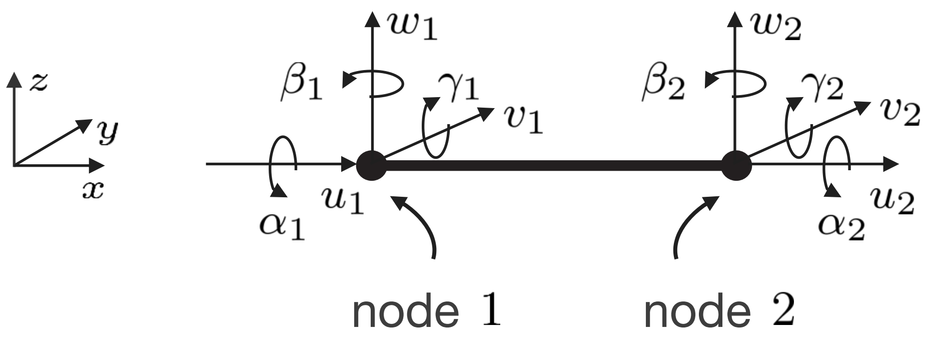
$$\frac{\mathrm{d}\mathbf{H}^{-1}}{\mathrm{d}t} = -\mathbf{H}^{-1}\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}\mathbf{H}^{-1}$$

Thereby, one obtains for $\mathbf{T}_i = (\mathbf{R}^T\mathbf{V}_i)^{-1} := \mathbf{H}^{-1}$:

$$\dot{\mathbf{T}}_i = \frac{\mathrm{d}\mathbf{H}^{-1}}{\mathrm{d}t} = -(\mathbf{R}^T\mathbf{V}_i)^{-1}\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))(\mathbf{R}^T\mathbf{V}_i)^{-1} = -\mathbf{T}_i\mathbf{R}^T\dot{\mathbf{V}}(\mathbf{p}(t))\mathbf{T}_i$$

Numerical example: Timoshenko beam with moving load

[Panzer et al. '09]



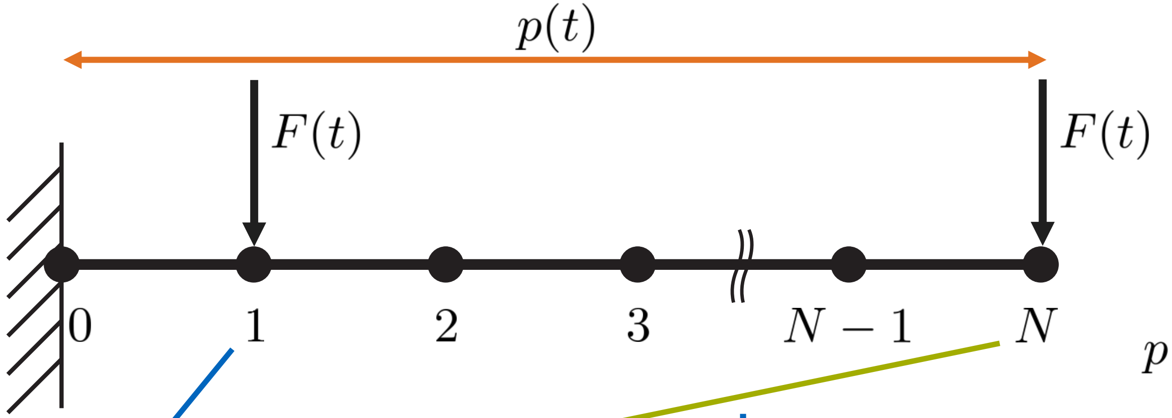
- Every node has 6 degrees of freedom:
  - 3 translational (u, v, w)
  - 3 rotational (α, β, γ)
- Dirichlet boundary condition: degrees of freedom at node 0 are zero
- Remaining degrees of freedom collected in state vector:

$$\mathbf{z} = \left[u^{(1)} v^{(1)} w^{(1)} \alpha^{(1)} \beta^{(1)} \gamma^{(1)} \; u^{(2)} v^{(2)} w^{(2)} \alpha^{(2)} \beta^{(2)} \gamma^{(2)} \; \ldots \; u^{(N)} v^{(N)} w^{(N)} \alpha^{(N)} \beta^{(N)} \gamma^{(N)} \right]^T$$

- Force $F(t)$ acts along the negative z-axis ⇨ affects only the state variable w

Numerical example: Timoshenko beam with moving load

[Panzer et al. '09]



N : finite elements

$p(t)$: varying load position

$$-\tilde{\mathbf{b}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots, -\tilde{\mathbf{b}}_N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Interpolation of the input vector:

$$\tilde{\mathbf{b}}(p(t)) = \sum_{i=1}^{N} \omega_i(p(t))\tilde{\mathbf{b}}_i$$

LPV second-order model:

$$\mathbf{M}\,\ddot{\mathbf{z}}(t) + \mathbf{D}\,\dot{\mathbf{z}}(t) + \mathbf{K}\,\mathbf{z}(t) = \tilde{\mathbf{b}}(p(t))\,F(t)$$
$$y(t) = \tilde{\mathbf{c}}^T\,\mathbf{z}(t)$$

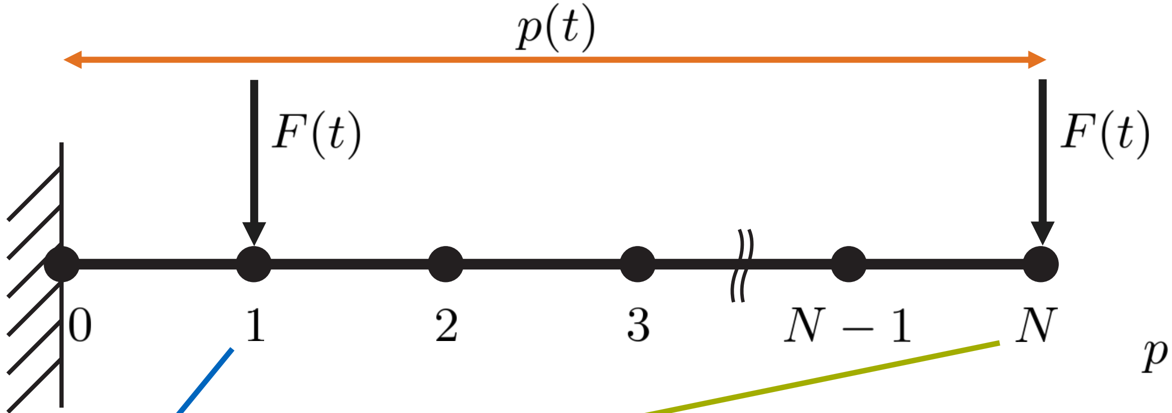
with:

$$\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{6N\times 6N}, \quad \tilde{\mathbf{c}}^T = -\tilde{\mathbf{b}}_N^T \in \mathbb{R}^{1\times 6N}$$

⇓ Reformulation as LPV first-order model

Numerical example: Timoshenko beam with moving load

[Panzer et al. '09]



N : finite elements

$p(t)$: varying load position

$$-\tilde{\mathbf{b}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots, -\tilde{\mathbf{b}}_N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Interpolation of the input vector:

$$\tilde{\mathbf{b}}(p(t)) = \sum_{i=1}^{N} \omega_i(p(t))\tilde{\mathbf{b}}_i$$

LPV first-order model:

$$\overbrace{\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}^{\mathbf{E}} \overbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \end{bmatrix}}^{\dot{\mathbf{x}}}(t) = \overbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}^{\mathbf{x}}(t) + \overbrace{\begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}(p(t)) \end{bmatrix}}^{\mathbf{b}(p(t))} F(t)$$

$$y(t) = \underbrace{\begin{bmatrix} \tilde{\mathbf{c}}^T & \mathbf{0}^T \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}(t)$$

with:

$\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{6N \times 6N}, \quad \tilde{\mathbf{c}}^T = -\tilde{\mathbf{b}}_N^T \in \mathbb{R}^{1 \times 6N}$

$\mathbf{F} = \mathbf{K}$ chosen ⇨ $\mathbf{A}$ dissipative, $\mathbf{E}$ pos. def.

$\mathbf{E}, \mathbf{A} \in \mathbb{R}^{2 \cdot 6N \times 2 \cdot 6N}, \quad \mathbf{c}^T \in \mathbb{R}^{1 \times 2 \cdot 6N}$

Original order: $2 \cdot 6N$