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Event-triggered Scheduling for Stochastic Multi-loop Networked Control Systems with Packet Dropouts

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***Abstract*— In this paper, we study event-triggered data scheduling for stochastic multi-loop control systems communicating over a shared network with communication uncertainties. We introduce a novel dynamic scheduling scheme which allocates the channel access according to an error-dependent policy. The proposed scheduler deterministically excludes subsystems with lower error values from the medium access competition in favor of those with larger errors. Subsequently, the scheduler probabilistically allocates the communication resource to the eligible entities. We model the overall network-induced error as a homogeneous Markov chain and show its boundedness in expectation over a multi time-step horizon. In addition, analytical upper bound for the associated average cost is derived. Furthermore, we show that our proposed policy is robust against packet dropouts. Numerical results demonstrate a significant performance improvement in terms of error level in comparison with periodic and random scheduling policies.**

I. INTRODUCTION

Control over shared communication resources imposes various imperfections, such as capacity limitation, congestion, time delays and packet dropouts, that impair the control performance and can even lead to instability. These network-induced phenomena give rise to the notion of Networked Control Systems (NCSs) [1]. In order to utilize the limited communication and energy resources in NCSs more efficiently, event-triggered control and scheduling strategies have been proposed recently [2]–[7].

Along with [7]–[9], the aforementioned references suggest that it is often more beneficial to sample analog signals and transmit the sampled values upon the occurrence of certain events rather than at predefined (and periodic) time instants. This is even more so when large-scale systems are of interest due to the sheer amount of information that needs to be exchanged. While stability of single-loop NCSs is well addressed (e.g., [10], [11]), stability of multi-loop NCSs requires further investigations (refer to [12]–[14] for some notable exceptions). In [13], the authors show the Lyapunov mean square stability of multi-loop stochastic NCSs under static scheduling laws, and additive increasing/decreasing laws depending on the delay time i.e. not necessarily error-dependent. In addition, events typically trigger by either deterministic or probabilistic scheduling policies [12], [15]–[17]. Lossy communication channels in NCSs are investigated in, among others, [18], [19]. In most of the available results, probabilistic packet dropouts are considered. Hence, a scheduling design for stochastic multi-loop NCSs that combines benefits of both deterministic and probabilistic event-triggered policies, and is robust against the channel imperfections is certainly of interest.

In this paper, we propose an error-dependent, locally implementable and flexible bi-character scheduling protocol for a network of multiple heterogeneous control loops communicating over a shared channel. This policy deterministically blocks transmission requests with local errors not exceeding predefined thresholds. Subsequently, the medium access is granted to the remaining transmission requests in a probabilistic manner. Unlike purely deterministic policies, which require centralized knowledge of all entities, the probabilistic nature of our scheduler facilitates an approximative decentralized implementation. On the other hand, the deterministic feature of our scheduler enhances the performance of NCSs (especially as the number of loops increases). It is worth mentioning that, we adopt an emulation-based approach meaning that we are given stabilizing controllers in the absence of communication network. To investigate performance vs. energy trade-offs of our bi-character scheduler, a cost function is introduced. Afterwards, analytic uniform upper-bound for the associated average cost is derived.

Notation In this paper, the Euclidean norm and conditional expectation are denoted by $\| \cdot \|_2$ and $\mathsf{E}[\cdot|\cdot]$, respectively. The Gaussian distribution with mean μ and covariance matrix X is represented by $\mathcal{N}(\mu, X)$. A state variable with superscript i indicates that it belongs to the subsystem i, and the subscript k denotes the time step. For matrices though, subscript i indicates the corresponding subsystem while superscript n denotes the matrix power. Lastly, we use $(x, y) := [x^\top \quad y^\top]^\top$ in order to simplify notation.

II. PROBLEM STATEMENT AND PRELIMINARIES

We consider an NCS composed of N LTI control loops which are coupled through a shared communication network, as schematically depicted in Fig. 1. Each individual loop consists of a stochastic plant $\mathcal{P}_i$, a controller $\mathcal{C}_i$, and a sensor $\mathcal{S}_i$. An event-based scheduler decides when the state vector $x_k^i \in \mathbb{R}^{n_i}$ is an event to be scheduled for channel access. The plant $\mathcal{P}_i$ is modeled by the stochastic difference equation

$$x_{k+1}^i = A_i x_k^i + B_i u_k^i + w_k^i, \tag{1}$$

where $w_k^i \in \mathbb{R}^{n_i}$ is i.i.d. with $w_k^i \sim \mathcal{N}(0, W_i)$ at each time step k, and $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m_i}$ describe system and input matrices, respectively. Since our results are independent

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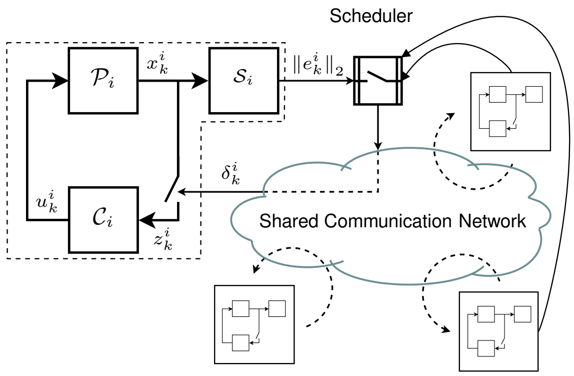

Fig. 1. A multi-loop NCS with a shared communication channel and error-dependent scheduler.

of initial states, x_0^i is allowed to be any random variable with an arbitrary distribution and a bounded second moment.

At each time step k, the scheduler decides which subsystems have the channel access via the variable $\delta_k^i \in \{0,1\}$:

$$\delta_k^i = \begin{cases} 1, & x_k^i \text{ sent through the channel} \\ 0, & x_k^i \text{ blocked.} \end{cases}$$

This implies that the received signal z_k at the controller is

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1 \\ \varnothing, & \delta_k^i = 0. \end{cases}$$

Each subsystem is assumed to be steered by a local state feedback controller updated at every time step k either by the true state values x_k^i or by the estimated states $\hat{x}_k^i$. Essentially, we consider control laws γ^i's given by

$$u_k^i = \gamma_k^i(Z_k^i) = -L_i \mathsf{E}\left[x_k^i | Z_k^i\right], \tag{2}$$

where $Z_k^i = \{z_0^i, \dots, z_k^i\}$ is the update history and L_i is any stabilizing feedback gain. Notice that, in accordance with the emulation-based frameworks, the control law synthesis is not explicitly addressed (i.e., we do not design L_i's). Instead, knowing that the controllers stabilize the plants in the absence of communication network, we focus on the scheduler design. It is also assumed that the i^{th} controller is provided with local information A_i, B_i, W_i, Z_k^i and x_0^i.

In case a transmission request is blocked, the least-square estimate of x_k^i is computed by a Kalman-like estimator

$$\mathsf{E}\left[x_k^i | Z_k^i\right] = (A_i - B_i L_i)\mathsf{E}\left[x_{k-1}^i | Z_{k-1}^i\right], \tag{3}$$

with initial distribution $\mathsf{E}\left[x_0^i | Z_0^i\right] = 0$. The estimate in (3) is well-behaved since a stabilizing L_i ensures that the closed-loop matrix $(A_i - B_i L_i)$ is Hurwitz. The network-induced error is defined as $e_k^i = x_k^i - \mathsf{E}\left[x_k^i | Z_{k-1}^i\right]$ and evolves as

$$e_{k+1}^i = \left(1 - \delta_{k+1}^i\right) A_i e_k^i + w_k^i. \tag{4}$$

According to (1)-(4), the aggregate state (x_k^i, e_k^i) has a triangular dynamics within each local loop implying that the evolution of e_k^i is independent of the system state x_k^i. This implies that the stability of e_k is sufficient to show the overall stability of the NCS of interest.

To measure the performance of the designed scheduler, we define a cost function per time-step for all subsystems as

$$J_{e_k} = \sum_{j=1}^{N} e_k^{j\top} Q_j e_k^j + \eta_j \delta_k^j, \tag{5}$$

where η_j is a non-negative constant and Q_j is a symmetric positive semi-definite weight matrix for the j^{th} loop. We adopt the following notation to avoid lengthy expressions:

$$\|e_k^j\|_{Q_j}^2 := e_k^{j\top} Q_j e_k^j.$$

The cost function in (5) states that at each time k, the total cost is the weighted norm of the error of all loops plus the imposed cost incurred when a loop transmits. Moreover, we evaluate the performance of the scheduler independently of the control law and system state, i.e. there is no penalty on the size of u_k^i and x_k^i signals. However, terms involving u_k^i and x_k^i can easily be added because the stabilizing control law is designed independently of the scheduling law.

Now we introduce the novel scheduling rule which dynamically prioritizes the channel access for a multi-loop NCS with shared communication channel according to an error-dependent policy. Assume that the communication channel has the capacity constraint $c < N$. Our scheduling rule defines the probability of channel access for each loop at each time-step by the following deterministic-probabilistic policy:

$$\mathsf{P}[\delta_{k+1}^i = 1 | e_k^j, \lambda_j] = \begin{cases} 0 & \|e_k^i\|_{Q_i}^2 \le \lambda_i \\ 1 & \|e_k^i\|_{Q_i}^2 > \lambda_i \wedge n_\lambda \le c \\ \frac{\|e_k^i\|_{Q_i}^2}{\sum_{n_\lambda} \|e_k^j\|_{Q_j}^2} & \|e_k^i\|_{Q_i}^2 > \lambda_i \wedge n_\lambda > c \end{cases} \tag{6}$$

where λ_i is the local error threshold for subsystem i, n_λ is the number of subsystems satisfying $\|e_k^j\|_{Q_j}^2 > \lambda_j$, and c is the channel capacity. The probability distribution above is supported on the semi-infinite interval $[0, \infty)$.

Remark 1: Since $c < N$, transmission requests from some subsystems should be blocked. The blocking is performed in two steps. First, according to the first argument in (6), if $\|e_k^i\|_{Q_i}^2 \le \lambda_i$, then a transmission request is not submitted and the corresponding subsystem is excluded from channel access competition. If the number of remaining subsystems is less than or equal to the capacity of the channel, they all transmit. Otherwise, the channel is allocated probabilistically until the capacity is reached while remaining transmission requests are blocked. In this allocation process, the subsystems with higher error have a higher channel access probability.

In the interest of reducing the paper length, we impose the following hard capacity constraint for every $k \ge 0$:

$$\sum\nolimits_{i=1}^{N} \delta_k^i = 1. \tag{7}$$

Namely, only one subsystem is allowed to transmit at each time-step. The presented results readily extend towards $c > 1$.

A. Decentralized Implementation of the Scheduler

The stochastic nature of the policy (6) enables a decentralized implementation within the CSMA protocol. As the details of implementation go beyond the scope of this paper, we focus on sketching the main idea of the protocol.

The CSMA model follows the assumptions: (i) sensing the carrier is instantaneous, (ii) there are no hidden nodes, (iii) the backoff intervals are exponentially distributed with error-dependent exponents, (iv) the mean backoff time is negligible with respect to the sampling interval, (v) data packets are discarded after one retransmission trial.

It should be noted that, due to the assumptions (i) and (ii), no packet collisions may occur. The assumptions (iii)-(v) are tailored to the discrete-time nature of the control process. In particular, assumption (v) reflects the idea of the try-once discard (TOD) protocol introduced in [16]. Moreover, assumption (vi) guarantees that the transmission is accomplished at the end of each sampling interval.

At the beginning of every sampling instance, each eligible subsystem waits to transmit according to its chosen backoff interval. The duration is chosen randomly according to assumption (iii) and depends on the current error norm of the subsystem. The subsystem with the smallest interval is permitted to transmit, while all other subsystems are blocked. Furthermore, the mean backoff interval decreases accordingly with increasing error norm. This naturally leads to a prioritization of the control loops, as subsystems with larger error norms are more likely to transmit.

III. UNIFORM PERFORMANCE BOUNDS

In this section, we investigate the behavior of a multi-loop NCS with a shared communication channel under policy (6). Our goal is to show that the expectation of error state e_k^i remains bounded. First, let us define the aggregate the error $e_k \in \mathbb{R}^n$, where $n = \sum_{i=1}^N n_i$, at each time-step k as

$$e_k := (e_k^1, \ldots, e_k^N). \tag{8}$$

The scheduling policy (6) is a randomized policy and depends on the most recent error values. Hence, the error evolution in (8) is a Markov chain. The Markov chain e_k is homogeneous because the difference equation in (3) is time-invariant and the noise process w_k^i is i.i.d. for $i = \{1, \ldots, N\}$. Since the noise distribution is absolute continuous having a positive density function, it is furthermore concluded that the chain is aperiodic and ψ-irreducible.

To show boundedness of $\mathsf{E}\left[e_k\right]$ in the limit, we restrict our attention to intervals of length N. The intuition behind this choice is as follows. Between two consecutive transmissions of each subsystem, these subsystems operate in open loop. Hence, in general, the respective local errors are expected to grow. After each transmission, the respective local errors reset. Now, say that at some time step k_1 we have arbitrary $\mathsf{E}\left[e_{k_1}\right]$. If one is to obtain boundedness of $\mathsf{E}\left[e_{k_1+N}\right]$, which is typically shown by requiring a negative drift according to the f-Norm Ergodic theorem, [20, Chapter 14], all subsystems need to have a chance of transmitting within $[k_1, k_1+N]$. Due to the capacity restriction (7), one infers that the interval of interest needs to be of length greater or equal to N, (see [21] for a comprehensive discussion). This observation becomes more evident from the exposition provided below. We first show $\mathsf{E}\left[\|e_k^j\|_{Q_j}^2\right]$ is bounded over an interval of length N, and then we invoke the f-Norm-ergodic theorem to prove the ergodicity of (8). *Lemma 1* is the main tool to obtain the upper bound for the average cost.

Lemma 1: [22] Let e_k represents a Markov chain evolving in state space $\mathbb{R}^n$. Introduce $J_{e_k} : \mathbb{R}^n \to \mathbb{R}$ and a measurable function $h : \mathbb{R}^n \to \mathbb{R}$. Define the average cost as

$$J_{\text{ave}} = \lim_{n\to\infty} \sup \frac{1}{n} \sum\nolimits_{k=0}^{n-1} \mathsf{E}\left[J_{e_k}\right].$$

If $h\left(e_k\right) \geq 0$ for all $e_k \in \mathbb{R}^n$, then

$$J_{\text{ave}} \leq \sup_{e_k} \left\{J_{e_k} + \mathsf{E}\left[h\left(e_{k+1}\right)|e_k\right] - h\left(e_k\right)\right\}.$$

As discussed earlier, we are interested in intervals of length N while Lemma 1 refers to one step transitions. However, since e_k is a ψ-irreducible Markov chain evolving in uncountable space $\mathbb{R}^n$, one can generate another Markov chain that samples the states of the original chain at steps $\{0, N, 2N, ...\}$. It is straightforward to show that ψ-irreducibility and aperiodicity are carried over to the generated chain. Moreover, time-homogeneity of the original Markov chain implies homogeneity of the constructed chain. Refer to [20, Chapter 1] for a comprehensive discussion.

Therefore, we rewrite the upper bound for the average cost J_{ave} from Lemma 1 as follows:

$$J_{\text{ave}} \leq \sup_{e_k} \left\{J_{e_k} + \mathsf{E}\left[h\left(e_{k+N}\right)|e_k\right] - h\left(e_k\right)\right\}. \tag{9}$$

Introducing the non-negative quadratic function $h\left(e_k\right) = \sum_{i=1}^N \|e_k^i\|_{Q_i}^2$, and considering the cost function (5), the upper bound for the average cost in (9) is reduced to

$$\begin{aligned} J_{\text{ave}} &\leq \sup_{e_k} \left[J_{e_k} + \mathsf{E}\left[h\left(e_{k+N}\right)|e_k\right] - \sum\nolimits_{i=1}^N \|e_k^i\|_{Q_i}^2\right] \\ &= \sup_{e_k} \left[\mathsf{E}\left[h\left(e_{k+N}\right)|e_k\right] + \sum\nolimits_{i=1}^N \eta_i \delta_k^i\right] \\ &= \sup_{e_k} \sum\nolimits_{i=1}^N \left[\mathsf{E}\left[\|e_{k+N}^i\|_{Q_i}^2|e_k\right] + \eta_i \delta_k^i\right]. \end{aligned} \tag{10}$$

Theorem 1: Consider an NCS with N heterogeneous LTI control loops, with the plants modeled as in (1), which share a communication channel subject to the constraint (7). Given the control and scheduling laws as in (2) and (6), respectively, the average cost J_{ave} is uniformly upper bounded.

Proof: In order to obtain an upper bound on J_{ave} over some interval $[k, k+N]$, we condition the expectation $\mathsf{E}[e_{k+N}]$ over all possible scenarios during the interval $[k, k+N-1]$ under the policy (6). Thus, we define two disjoint time-varying sets $S_{k'}^1$ and $S_{k'}^2$ where $k' \in [k, k+N-1]$, such that for a subsystem i at time step k' we have

$$i \in \begin{cases} S_{k'}^1, & \|e_{k'}^i\|_{Q_i}^2 \leq \lambda_i \\ S_{k'}^2, & \|e_{k'}^i\|_{Q_i}^2 > \lambda_i \end{cases}. \tag{11}$$

Notice that the subsystems in $S_{k'}^1$ are not eligible to transmit at k' while the subsystems belonging to $S_{k'}^2$ are. Clearly, $S_{k'}^1 \cup S_{k'}^2 = N$. According to the policy (6), we identify the following four complementary and mutually exclusive cases regarding the evolution of the error Markov chain e_k as

c_1) Subsystem i has not transmitted during $[k, k+N-1]$ and $i \in S^1_{k+N-1}$,

c_2) Subsystem i has transmitted at least once during $[k, k+N-1]$ and $i \in S^1_{k+N-1}$,

c_3) Subsystem i has transmitted at least once during $[k, k+N-1]$ and $i \in S^2_{k+N-1}$,

c_4) Subsystem i has not transmitted during $[k, k+N-1]$ and $i \in S^2_{k+N-1}$.

We proceed by deriving uniform upper bounds for J_{ave} in each of four cases, which assures boundedness of $\mathsf{E}[e_k]$.

Suppose that some subsystems belong to c_1 at time step $k+N-1$, i.e., $i \in c_1$. Since those subsystems do not transmit until time step $k+N$, the bound in (10) reduces to

$$\begin{aligned} J^{i\in c_1}_{\text{ave}} &\leq \sup_{e_k} \sum_{i\in c_1} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q_i}|e_k\right] \\ &= \sup_{e_k} \sum_{i\in c_1} \mathsf{E}\left[\|A_i e^i_{k+N-1} + w^i_{k+N-1}\|^2_{Q_i}|e_k\right] \\ &\leq \sup_{e_k} \sum_{i\in c_1} \|A_i\|^2_2 \mathsf{E}\left[\|e^i_{k+N-1}\|^2_{Q_i}\big|\, e_k\right] + \mathsf{E}\left[\|w^i_{k+N-1}\|^2_{Q_i}\right] \end{aligned}$$

Since $i \in S^1_{k+N-1}$, it follows that $\|e^i_{k+N-1}\|^2_{Q_i} \leq \lambda_i$. Thus,

$$J^{i\in c_1}_{\text{ave}} \leq \sum\nolimits_{i\in c_1} \|A_i\|^2_2 \lambda_i + \mathsf{E}\left[\|w^i_{k+N-1}\|^2_{Q_i}\right]. \quad (12)$$

For the subsystems belonging to the second case, the upper bound can be similarly obtained, noticing that the transmission penalty must be considered. So, we have

$$J^{i\in c_2}_{\text{ave}} \leq \sum\nolimits_{i\in c_2} \|A_i\|^2_2 \lambda_i + \eta_i + \mathsf{E}\left[\|w^i_{k+N-1}\|^2_{Q_i}\right]. \quad (13)$$

For the third case, assume that the last transmission has occurred at time $k+r'$, where $r' \in [0, N-1]$, i.e. $\delta^i_{k+r'} = 1$. We express e^i_{k+N} as a function of the error at time $k+r'$ as

$$\begin{aligned} e^i_{k+N} &= \prod\nolimits_{j=r'+1}^{N} \left(1-\delta^i_{k+j}\right) A_i^{N-r'} e^i_{k+r'} \\ &+ \sum\nolimits_{r=r'}^{N-1} \left[\prod\nolimits_{j=r+2}^{N} \left(1-\delta^i_{k+j}\right) A_i^{N-r-1} w^i_{k+r}\right], \quad (14) \end{aligned}$$

where for $r' = N-1$, we have $\prod_{N+1}^{N}(1-\delta^i_{k+j}) = 1$. Now, employing (14), (10) is reduced to

$$\begin{aligned} J^{i\in c_3}_{\text{ave}} &\leq \sup_{e_k} \sum\nolimits_{i\in c_3} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q_i}|e_k\right] + \eta_i \\ &\leq \sum\nolimits_{i\in c_3} \sum\nolimits_{r=r'}^{N-1} \mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|^2_{Q_i}\right] + \eta_i. \quad (15) \end{aligned}$$

As for the fourth case, there is no guarantee that a system with a higher priority for channel access wins the competition against a system with a lower priority, due to the probabilistic feature of our scheduling policy. To show boundedness of the error, we consider two complementary and mutually exclusive sub-cases within the fourth case as follows:

$l^{c_4}_1$ Subsystem i has not transmitted during $[k, k+N-1]$, but has been in the set $S^1_{k'}$ at least once,

$l^{c_4}_2$ Subsystem i has not transmitted during $[k, k+N-1]$, but has been in the set $S^2_{k'}$ for all $k' \in [k, k+N-1]$.

For both sub-cases no transmission occurs so we can rewrite (10) as follows

$$J^{i\in c_4}_{\text{ave}} \leq \sup_{e_k} \sum\nolimits_{i\in c_4} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q_i}|e_k\right]. \quad (16)$$

The sub-case $l^{c_4}_1$ is similar to the case c_1. The difference is that the subsystem in c_1 is in S^1_{k+N-1} at time step $k+N-1$, while in $l^{c_4}_1$ is in S^1 in some prior time steps. Supposing that in the last step the system i was in $S^1_{k+r'}$ is $k+r'$, an upper bound for (10) is readily obtained employing (14):

$$J^{i\in l^{c_4}_1}_{\text{ave}} \leq \sum_{i\in l^{c_4}_1} \left[\|A_i^{N-r'}\|^2_2 \lambda_i + \sum_{r=r'}^{N-1} \mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|^2_{Q_i}\right]\right] \quad (17)$$

The sub-case $l^{c_4}_2$ presents the situation in which a certain subsystem has always been a candidate for channel access, i.e., $i \in S^2_{[k,k+N-1]}$, even though it has never transmitted. Hence, $\|e^i_{k'}\|^2_{Q_i} > \lambda_i$ for all steps $k' \in [k, k+N-1]$. With the upper bound for the average cost in (16) still valid and considering (14) with $r' = 0$, we obtain

$$\begin{aligned} J^{i\in l^{c_4}_2}_{\text{ave}} &\leq \sup_{e_k} \sum_{i\in l^{c_4}_2} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q_i}|e_k\right] \\ &\leq \sup_{e_k} \sum_{i\in l^{c_4}_2} \mathsf{E}\left[\|A_i^N e^i_k + \sum_{r=0}^{N-1}\left[A_i^{N-r-1} w^i_{k+r}\right]\|^2_{Q_i}|e_k\right] \\ &\leq \sup_{e_k} \sum_{i\in l^{c_4}_2} \mathsf{E}\left[\|\sum_{r=0}^{N-1}\left[A_i^{N-r-1} w^i_{k+r}\right]\|^2_{Q_i}\right] + \|A_i^N\|^2_2 \|e^i_k\|^2_{Q_i}. \end{aligned}$$

According to the scheduling policy (6), we calculate the probability for the sub-case $l^{c_4}_2$ to occur. Recall that the length of the interval of interest is N. Thus, if one system, say j, does not transmit during the entire interval, there exists another subsystem, say i, who transmits more than once. Let $k+\bar{r}$ denote the most recent step in which system i transmitted. The probability that $\delta^i_{k+N} = 1$ is computed as

$$\begin{aligned} &\mathsf{P}[\delta^i_{k+N} = 1|\delta^i_{k+\bar{r}} = 1, e^j_k, \|e^j_{\bar{k}}\|^2_{Q_j} > \lambda_j] \\ &= \mathsf{E}\Big[\mathsf{P}[\delta^i_{k+N} = 1|e^j_k, \lambda_j]|\delta^i_{k+\bar{r}} = 1, \|e^j_{\bar{k}}\|^2_{Q_j} > \lambda_j\Big] \\ &= \mathsf{E}\left[\frac{\|e^i_{k+N-1}\|^2_{Q_i}}{\sum_{j\in S^2}\|e^j_{k+N-1}\|^2_{Q_j}}|e^j_k, \delta^i_{k+\bar{r}} = 1, \|e^j_{\bar{k}}\|^2_{Q_j} > \lambda_j\right]. \quad (18) \end{aligned}$$

Since we are interested in the worst case scenario (i.e. upper bounds), we take $\|e^{j\in l^{c_4}_2}_{k'}\|^2_{Q_j} \leq \|e^{j\in l^{c_4}_2}_{k'+1}\|^2_{Q_j}$. Then we have

$$\begin{aligned} &\mathsf{P}[\delta^i_{k+N} = 1|\delta^i_{k+\bar{r}} = 1, e^j_k, \|e^j_{\bar{k}}\|^2_{Q_j} > \lambda_j] \\ &\leq \mathsf{E}\left[\frac{\|\sum_{r=\bar{r}}^{N-2} A_i^{N-r-1} w^i_{k+r}\|^2_{Q_i}}{\sum_{j\in l^{c_4}_1}\|e^j_{k+N-1}\|^2_{Q_j} + \sum_{j\in l^{c_4}_2}\|e^j_{k+N-1}\|^2_{Q_j}}\Big|z_{i,j}\right] \\ &\leq \mathsf{E}\left[\frac{\|\sum_{r=\bar{r}}^{N-2} A_i^{N-r-1} w^i_{k+r}\|^2_{Q_i}}{\sum_{j\in l^{c_4}_1}\lambda_j + \sum_{j\in l^{c_4}_2}\|e^j_k\|^2_{Q_j}}\Big|z_{i,j}\right] \\ &= \frac{\sum_{r=\bar{r}}^{N-2} \mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|^2_{Q_i}\right]}{\sum_{j\in l^{c_4}_1}\lambda_j + \sum_{j\in l^{c_4}_2}\|e^j_k\|^2_{Q_j}} = \mathsf{P}_{l^{c_4}_2}, \quad (19) \end{aligned}$$

where $z_{i,j}$ abbreviates the conditions of the expectation. From (19) one infers that the probability of a subsequent transmission for a certain subsystem, in the presence of large errors, can be made arbitrarily close to zero by selecting the appropriate values for λ_j's and Q_j's. Now, the average cost for the sub-case $l_2^{c_4}$ becomes

$$J_{\text{ave}}^{j\in l_2^{c_4}} \leq \mathsf{P}_{l_2^{c_4}}\left[\|A_j^N\|_2^2\|e_k^j\|_{Q_j}^2 + \sum_{j\in l_2^{c_4}}\mathsf{E}\left[\|\sum_{r=0}^{N-1}A_j^p w_{k+r}^j\|_{Q_j}^2\right]\right]$$

$$\leq \sum\nolimits_{j\in l_2^{c_4}}\|A_j\|_2^2\sum\nolimits_{r=\bar{r}}^{N-2}\mathsf{E}\left[\|A_i^p w_{k+r}^i\|_{Q_i}^2|z_{i,j}\right] \quad (20)$$

$$+\frac{\sum_{j\in l_2^{c_4}}\sum_{r=0}^{N-1}\mathsf{E}\Big[\|A_j^p w_{k+r}^j\|_{Q_j}^2\Big]\Big[\sum_{r=\bar{r}}^{N-2}\mathsf{E}\left[\|A_i^p w_{k+r}^i\|_{Q_i}^2\right]\Big]}{\sum_{j\in l_1^{c_4}}\lambda_j},$$

where $p = N - r - 1$. The upper bound (20) is uniform and independent of the initial state e_k^i, and shows that the bound can be made small by increasing λ_j's and decreasing Q_j's, but not arbitrarily small due to its first term. It confirms that, despite having unstable plants and sparsity of the capacity, which might cause a subsystem with large error waiting for channel access, the aggregate error remains bounded.

In order to calculate the upper bound for $l_2^{c_4}$, we considered the worst case possible error evolution by assuming $\|e_{k'}^{j\in l_2^{c_4}}\|_{Q_j}^2 \leq \|e_{k'+1}^{j\in l_2^{c_4}}\|_{Q_j}^2$. This assumption places a condition on the noise values $w_{k'}^{j\in l_2^{c_4}}$ for all $k' \in [k, k+N]$. Basically, the distribution of the vector $A_j^{N-1}w_k^j + \ldots + w_{k+N-1}^j = W\left(A_j, w^j\right)$ is therefore restricted to distributions which enlarge the error-dependent term $A_j^N e_k^j$. The worst case occurs when the noise-dependent and error-dependent terms have the same signs element wise (either positive or negative). Due to the symmetry of the noise-dependent distribution $W\left(A_j, w^j\right)$, both positive and negative parts of the corresponding distribution turn out to have the same values. Since the noise variables are independent and $W\left(A_j, w^j\right)$ has a zero-mean multi-dimensional Gaussian distribution with covariance matrix $\Sigma = \left(A_j^{N-1} + \ldots + A_j + I\right)C_j$, the following probability density function is obtained

$$f\left(w\right) = \frac{1}{\sqrt{(2\pi)^{n_j}|\Sigma|}}\text{exp}\left(\frac{-w^{\mathrm{T}}\Sigma^{-1}w}{2}\right), \quad (21)$$

where w is the n_j-dimensional noise-dependent random vector and $|\Sigma|$ is the determinant of the covariance matrix. Employing the law of *unconscious statistician* yields

$$\mathsf{E}\left[\|W\left(A_j, w^j\right)\|_{Q_j}^2|W \geq 0\right] = \frac{\|Q_j\|_2^2}{\sqrt{(2\pi)^{n_j}|\Sigma|}}\int_0^\infty\cdots\int_0^\infty\|w\|_2^2\text{exp}\left(\frac{-w^T\Sigma^{-1}w}{2}\right)dw.$$

Finally, we derived the uniform upper bounds (12),(13), (15), (17) and (20) for all four cases. Having an upperbound for (10) implies the boundedness of $\|e_k^i\|_{Q_i}^2$, and consequently boundedness of the error variance over an interval of length N. Therefore, the average cost, which is obtained as the summation of all four cases, remains bounded. Hence, the Markov chain e_k given by (8) has a bounded quadratic size over the interval $[k, k+N]$. ■

Having finite values for the $\mathsf{E}\left[\|e_k^i\|_{Q_i}^2\right]$, for all time-steps k, we can always find appropriate quadratic functions f such that the conditions of the *f-Norm Ergodic Theorem* hold. This implies that the Markov chain e_k is ergodic meaning that there exists an invariant finite transition function $\pi(e)$ over the entire evolution [20, Chapter 14].

IV. SCHEDULING OF COMMUNICATION RESOURCE SUBJECT TO PACKET DROPOUTS

In this section, we investigate the robustness of the proposed scheduler (6) with respect to data packet dropouts. Although, the centralized structure of our scheduler implies that the communication channel is collision-free, the consideration of dropouts makes it feasible to handle the collisions while implementing the scheduler in decentralized fashion. So far in this paper, we assumed that every data packet which is awarded the channel access, will be successfully received by the controller. In this section, we relax this assumption and consider the possibility of packet dropouts in the channel. Assume that at every time step the scheduler is aware whether the transmission has been successful or not. This is achieved via the binary variable $\theta_k \in \{0, 1\}$, i.e.,

$$\theta_k = \begin{cases} 1 & \text{packet is successfully sent} \\ 0 & \text{packet is dropped out} \end{cases}$$

Hence, taking the possibility of dropouts into account, the error evolution (4) also depends on θ_k as follows

$$e_{k+1}^i = \left(1 - \delta_{k+1}^i\theta_{k+1}\right)A_i e_k^i + w_k^i. \quad (22)$$

In what follows, we show that the quadratic measure of the aggregate error (5) remains bounded in expectation even if the scheduled data packets are dropped in the communication channel. The way we model the dropouts is as follows – whenever a packet is dropped, we assume that a virtual loop has successfully transmitted instead of the subsystem with dropped data packet. Basically, when a dropout occurs, N real and one virtual subsystems share the communication channel, and the channel is awarded to the virtual system. The virtual loops have the same discrete LTI dynamics as in (4) with appropriately chosen system parameters. Over the interval $[k, k+N]$, we assume having as many virtual loops connected to the NCS as the dropped packets. This assumption affects the scheduling process and has no influence on the average cost for which we seek the upper bound.

Let the channel experience $m < N - 1$ dropouts over the interval $[k, k + N - 1]$. At time step $k + N$ we have N real and m virtual subsystems (all virtual ones have transmitted). Consider again the cases $c_1, \ldots, c_3$ as well as the sub-case $l_1^{c_4}$. It is clear from the equations (12), (13), (15) and (17) that the aforementioned cases are uniformly upper bounded regardless of the access probability. The intuition is that the scheduler has a priori knowledge about the Markov chain at a specific prior time knowing that the error is below the threshold for those cases. Since the error evolves as a homogeneous Markov chain, driven by the variance-bounded standard Gaussian noise, it suffices to know that the Markov chain has been bounded at a prior time step in order to show boundedness of the chain in the future over a finite horizon.

Proposition 1: Let an NCS be composed of a finite number of LTI loops with dynamics (1). Assume that the constraint (7) holds, and the channel experiences m packet dropouts during the interval $[k, k+N]$. Then, employing the scheduling policy in (6), the error evolves according to a bounded probability distribution, if $m < N-1$.

Proof: The critical case is the sub-case $l_2^{c_4}$ in which the error has always been above the threshold but the respective subsystem has not transmitted. Thus, the scheduler has no bounded knowledge about the error. Considering the worst case over all possibilities of error evolution, we aim to find the upper bound for the average cost. With this in mind, we calculate again the probability that the sub-case $l_2^{c_4}$ occurs:

$$\begin{aligned}
&\mathsf{P}[\delta^i_{k+N+m} = 1 | \delta^i_{k+\bar{r}} = 1, e^j_k, d_j, \|e^j_k\|^2_{Q_j} > \lambda_j] \\
&= \mathsf{E}\left[\frac{\|e^i_{k+k_{d_m}}\|^2_{Q_i}}{\sum_{j\in S^2}\|e^j_{k+k_{d_m}}\|^2_{Q_j}} | e^j_k, m, \delta^i_{k+\bar{r}} = 1, \|e^j_k\|^2_{Q_j} > \lambda_j\right] \\
&\leq \mathsf{E}\left[\frac{\|\sum_{r=\bar{r}}^{N+m-2} A_i^{N-r-1} w^i_{k+r}\|^2_{Q_i}}{\sum_{j\in l_1^{c_4}} \|e^j_{k+k_{d_m}}\|^2_{Q_j} + \sum_{j\in l_2^{c_4}} \|e^j_{k+k_{d_m}}\|^2_{Q_j}} | z_{i,j}\right] \\
&\leq \frac{\sum_{r=\bar{r}}^{N+m-2} \mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|^2_{Q_i}\right]}{\sum_{j\in l_1^{c_4}} \lambda_j + \sum_{j\in l_2^{c_2}} \|e^j_k\|^2_{Q_j}}. \qquad (23)
\end{aligned}$$

where $k_{d_m} = N+m-1$. The last inequality holds considering $\|e^{l_2^{c_4}}_{k'}\|^2_{Q_j} \leq \|e^{l_2^{c_4}}_{k'+1}\|^2_{Q_j}$ for all $k' \in [k, k+N+m-1]$. In addition, we take $\sum_{l_1^{c_4}} \|e^j_{k+N+m-1}\|^2_{Q_j} > \sum_{l_1^{c_4}} \lambda_j$ resulting from being in the subset $l_1^{c_4}$. Considering packet dropouts, we calculate the upper bound for the average cost as:

$$J_{\text{ave}}^{j\in l_2^{c_4}} \leq \sum_{j\in l_2^{c_4}} \|A_j\|^2_2 \sum_{r=\bar{r}}^{N+m-2} \mathsf{E}\left[\|A_i^p w^i_{k+r}\|^2_{Q_i} | z_{i,j}\right] + \frac{\sum_{j\in l_2^{c_4}} \sum_{r=0}^{N-1} \mathsf{E}\left[\|A_j^p w^j_{k+r}\|^2_{Q_j}\right]\left[\sum_{r=\bar{r}}^{N+m-2} \mathsf{E}\left[\|A_i^p w^i_{k+r}\|^2_{Q_i}\right]\right]}{\sum_{j\in l_1^{c_4}} \lambda_j}$$

■

Comparing the last inequality with (20), one infers that λ_j needs to be increased and Q_j decreased in order to achieve the same upper bound (the value in numerator gets larger). Basically, to compensate the effect of dropouts, λ_j's are increased and Q_j's are decreased, which in turn increases the chance of transmission for the subsystems in $l_2^{c_4}$.

V. NUMERICAL RESULTS

In this section, the performance of our event-triggered scheduler is compared with TDMA and idealized CSMA policies. We also demonstrate that the deterministic feature of our bi-character scheduler yields performance improvements in comparison with the pure stochastic scheduler from [12].

Consider an NCS comprised of two classes of subsystems. The first class includes control loops with unstable plants while the second class contains control loops with stable processes. The system parameters are $A_1 = 1.25$, $B_1 = 1$ and $A_2 = 0.75$, $B_2 = 1$, respectively. In both classes, the initial condition is $x_0^1 = x_0^2 = 0$ and the random disturbance is given by $w_k^i \sim \mathcal{N}(0,1)$. We consider N loops with equal number of loops belonging to each of two classes. In order

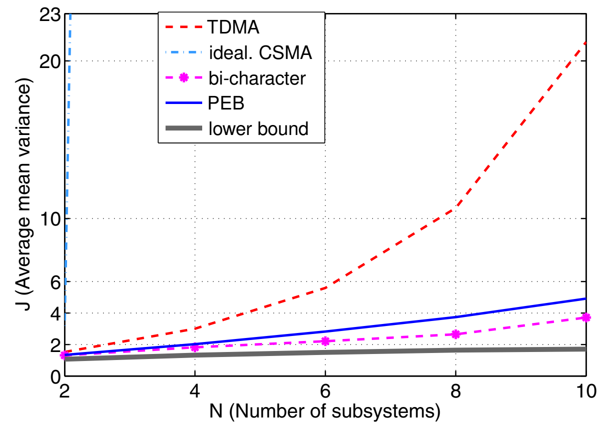

Fig. 2. Comparison of the average mean of error variance vs. the number of control loops for different scheduling policies.

to stabilize the subsystems, we choose a deadbeat control law $L_i = A_i$ for $i \in \{1,2\}$ and a model-based observer given by (3). Next we select $Q_i = I$, for each class $i \in \{1,2\}$.

Figure 2 provides the obtained numerical results for our and related scheduling protocols with different $N \in \{2,4,6,8,10\}$. Note that for $N > 2$, we have more unstable systems than the available transmission slots per time-step (which is one). The averages are calculated via Monte Carlo simulations over a horizon of 10^5. The lower bound is determined by relaxing the constraint in (7), (refer to [7] for more details). For the results to be comparable, we disregard the communication penalty, meaning that the channel usage is costless. We calculate the mean variance by considering equal λ's for NCSs with different N, according to Table I.

The results indicate that the mean error variance is smaller than the purely stochastic PEB protocol [12]. The increase of λ as the number of loops increases follows from the fixed channel capacity. Hence, the subsystems need to tolerate greater error values. The superiority of our approach is also evident in comparison with TDMA and idealized CSMA protocols. TDMA is a time-triggered access scheme, where subsystems update their controllers periodically. The idealized CSMA operates such that the probability of updating the controller is $\frac{1}{N}$ for each subsystem at each time step. As the number of subsystems increases, the performance efficiency of our scheduler becomes more evident. As it can be seen from Fig. 2, the static idealized CSMA protocol results in an acceptable performance only for $N = 2$. This is expected as this protocol allocates the channel in a static manner resulting in a probable non-transmission for an unstable system when the number of subsystems grows, and one free transmission slot is available. For $N \geq 6$, the variance of e_k takes values of magnitude 10^{15} which suggests an unbounded variance. This is in accordance with [7, Theorem 2], where the stability condition is violated for $N \geq 6$ for the considered system parameters. In Fig. 3, the mean variances are calculated with the same thresholds as in Table I, considering one packet dropout for an unstable loop. It can be seen that the variance increases for the same error thresholds.

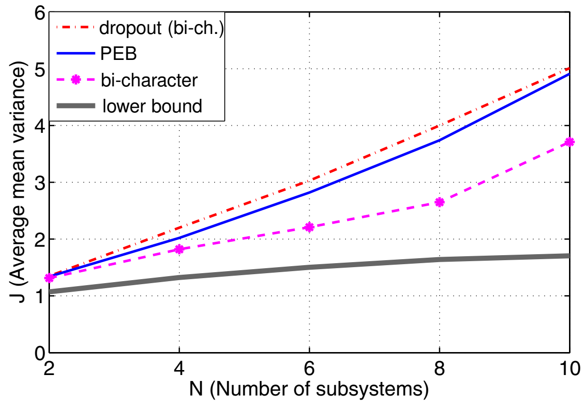

Fig. 3. Effect of packet dropout on error variance.

Number of plants (N)	2	4	6	8	10
Error threshold (λ)	0.2	6	10	15	18
Upper bounds on J_{ave}	3.12	3.99	6.14	12.28	25.60

TABLE I

THEORETICALLY PREDICTED UPPER BOUNDS FOR THE AVERAGE COST WITH COMMUNICATION PENALTY.

Next, we provide uniform upperbounds for the average cost in (10). We employ the upper bounds in (12), (13), (15), (17) and (20) for NCSs with $\{2, 4, 6, 8, 10\}$ subsystems. In Table I the analytical upper bounds are derived using the same error thresholds for the simulation. For the sake of simplicity, we assumed $\eta_i = \lambda_i$. Note that the average cost in Table I considers the communication penalty, as discussed in (5), while it is not considered in simulations. The absence of the communication penalty in the average cost (10) can decrease the overall cost as shown in Table II.

Number of plants (N)	2	4	6	8	10
Error threshold (λ)	0.2	6	10	15	18
Numeric mean variance	1.31	1.87	2.05	2.34	3.05
Upper bounds on J_{ave}	1.65	3.25	5.30	11.35	24.70
Upper bounds with dropout	2.45	3.76	5.70	11.68	25.00

TABLE II

NUMERICAL MEAN VARIANCE VS. THEORETICAL UPPER BOUNDS FOR COSTLESS COMMUNICATION.

VI. CONCLUSIONS

In this paper, we examine stability and performance of resource-constrained NCSs under a novel error-dependent scheduling scheme which combines deterministic and probabilistic scheduling. Given the stabilizing feedback controllers, we show boundedness of the overall NCS error. In addition, we derive analytical uniform performance bounds for the error variance under the proposed scheduling policy. We also study the possibility of packet dropouts and show robustness of our approach against this network-induced phenomenon. Numerical simulations validate boundedness of the error variance and display a major performance improvement in comparison with other randomized protocols, especially as the number of subsystems increases.

REFERENCES

[1] R. Murray, K. Astrom, S. Boyd, R. Brockett, and G. Stein, "Future directions in control in an information-rich world," *Control Systems, IEEE*, vol. 23, pp. 20–33, Apr 2003.

[2] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *Automatic Control, IEEE Transactions on*, vol. 52, no. 9, pp. 1680 –1685, 2007.

[3] D. Dimarogonas and K. Johansson, "Event-triggered control for multi-agent systems," in *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pp. 7131–7136, Dec 2009.

[4] D. Tolić and R. Fierro, "Decentralized output synchronization of heterogeneous linear systems with fixed and switching topology via self-triggered communication," in *American Control Conference*, pp. 4655–4660, June 2013.

[5] T. Henningsson, E. Johannesson, and A. Cervin, "Sporadic event-based control of first-order linear stochastic systems," *Automatica*, vol. 44, no. 11, pp. 2890 – 2895, 2008.

[6] A. Molin and S. Hirche, "On the optimality of certainty equivalence for event-triggered control systems," *Automatic Control, IEEE Transactions on*, vol. 58, no. 2, pp. 470 –474, 2013.

[7] A. Molin and S. Hirche, "A bi-level approach for the design of event-triggered control systems over a shared network," *Discrete Event Dynamic Systems*, vol. 24, no. 2, pp. 153–171, 2014.

[8] A. Cervin and T. Henningsson, "Scheduling of event-triggered controllers on a shared network," in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pp. 3601–3606, Dec 2008.

[9] W. P. M. H. Heemels, J. H. Sandee, and P. P. J. Van Den Bosch, "Analysis of event-driven controllers for linear systems," *International Journal of Control*, vol. 81, no. 4, pp. 571–590, 2008.

[10] M. Donkers, W. Heemels, D. Bernardini, A. Bemporad, and V. Shneer, "Stability analysis of stochastic networked control systems," *Automatica*, vol. 48, no. 5, pp. 917–925, 2012.

[11] M. Tabbara and D. Nesic, "Input–output stability of networked control systems with stochastic protocols and channels," *Automatic Control, IEEE Transactions on*, vol. 53, no. 5, pp. 1160–1175, 2008.

[12] M. H. Mamduhi, A. Molin, and S. Hirche, "On the stability of prioritized error-based scheduling for resource-constrained networked control systems," in *Distributed Estimation and Control in Networked Systems (NecSys), 4th IFAC Workshop on*, pp. 356–362, 2013.

[13] C. Ramesh, H. Sandberg, and K. Johansson, "Stability analysis of multiple state-based schedulers with CSMA," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pp. 7205–7211, 2012.

[14] A. Cervin and T. Henningsson, "A simple model for the interference between event-based control loops using a shared medium," in *Proc. 49th IEEE Conference on Decision and Control*, (Atlanta, USA), 2010.

[15] D. Nesic and A. Teel, "Input-output stability properties of networked control systems," *Automatic Control, IEEE Transactions on*, vol. 49, no. 10, pp. 1650–1667, 2004.

[16] G. C. Walsh, H. Ye, and L. G. Bushnell, "Stability analysis of networked control systems," *Control Systems Technology, IEEE Transactions on*, vol. 10, no. 3, pp. 438–446, 2002.

[17] Y. Xu and J. P. Hespanha, "Communication logic design and analysis for networked control systems," in *Current trends in nonlinear systems and control* (L. Menini, L. Zaccarian, and C. T. Abdallah, eds.), Boston: Birkhäuser, 2006.

[18] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, pp. 163–187, Jan 2007.

[19] C. Hadjicostis and R. Touri, "Feedback control utilizing packet dropping network links," in *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, vol. 2, pp. 1205–1210 vol.2, Dec 2002.

[20] S. Meyn and R. Tweedie, *Markov chains and stochastic stability*. Springer London, 1996.

[21] S. P. Meyn and R. Tweedie, "State-dependent criteria for convergence of markov chains," *The Annals of Applied Probability*, pp. 149–168, 1994.

[22] R. Cogill and S. Lall, "Suboptimality bounds in stochastic control: A queueing example," in *American Control Conference, 2006*, pp. 1642–1647, 2006.