Coding Schemes for Discrete Memoryless Broadcast Channels with Rate-Limited Feedback

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Objective

Investigate *whether* and *how* rate-limited feedback increases the nofeedback capacity of some discrete memoryless broadcast channels (DMBC)

Channel Model

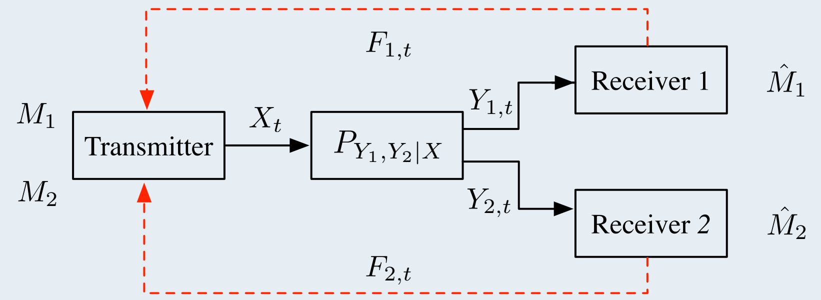

Figure 1 : DMBC with rate-limited feedback

- Messages: $M_i \in [1 : 2^{nR_i}]$, for $i \in \{1,2\}$
- Channel law: $P_{Y_1Y_2|X}$
- Feedback: $F_{i,t} \in \mathcal{F}_i$ satisfy

$$|\mathcal{F}_{i,1} \times \cdots \times \mathcal{F}_{i,n}| \le 2^{nR_{\mathrm{Fb},i}}, \text{ for } i \in \{1,2\}$$

- Encoder:

$$X_t = f_t^{(n)}(M_1, M_2, F_1^{t-1}, F_2^{t-1}), \text{ for } t \in [1:n]$$

- Decoder: $\hat{M}_i = \Phi_i^{(n)}(Y_i^n)$
- Error probability:

$$P_e^{(n)} \triangleq \Pr\{\hat{M}_1 \ne M \text{ or } \hat{M}_2 \ne M\}$$

Definitions

Definition 1: A DMBC is called *strictly* less-noisy if

$$I(U;Y_2) > I(U;Y_1)$$

holds for all auxiliary $U - X - (Y_1,Y_2)$ with $I(U;Y_1) > 0$

Definition 2: A DMBC is called *strictly* essentially less-noisy ($Y_2 \succ Y_1$) if given any $P_X \in \mathcal{P}_\mathcal{X}$

$$I(U;Y_2) > I(U;Y_1)$$

holds for all auxiliary $U - X - (Y_1,Y_2)$ with $I(U;Y_1) > 0$, where $\mathcal{P}_\mathcal{X}$ is a sufficient class: Given a BC, for any joint pmf P_{UVX} there exists a joint pmf P'_{UVX} that satisfies

$$\begin{aligned}
&\textstyle\sum_{u,v} P'_{UVX} \quad (u,v,x) \in \mathcal{P}_\mathcal{X}\\
I_P(U;Y_1) &\le I_{P'}(U;Y_1)\\
I_P(V;Y_2) &\le I_{P'}(V;Y_2)\\
I_P(U;Y_1) + I_P(X;Y_2|U) &\le I_{P'}(U;Y_1) + I_{P'}(X;Y_2|U)\\
I_P(V;Y_2) + I_P(X;Y_1|V) &\le I_{P'}(V;Y_2) + I_{P'}(X;Y_1|V)
\end{aligned}$$

Remark 1: Superposition coding is optimal for strictly (essentially) less-noisy DMBC: for some $P_{UX}P_{Y_1Y_2|X}$

$$\begin{aligned}
R_1 &\le I(U;Y_1)\\
R_2 &\le I(X;Y_2|U)
\end{aligned}$$

New Achievable Regions

Theorem 1:
For DMBCs with feedback, the capacity region $\mathcal{C}_{\mathrm{Fb}}$ includes the region $\mathcal{R}_{\mathrm{in},1}$:

$$\begin{aligned}
R_1 &\le I(U;Y_1|Q)\\
R_1 &\le I(U;Y_2|Q) - I(\tilde{Y}_1;Y_1|U,Y_2,Q)\\
R_2 &\le I(X;\tilde{Y}_1,Y_2|U,Q)
\end{aligned}$$

for some pmf $P_Q P_{U|Q} P_{X|UQ} P_{Y_1Y_2|X} P_{\tilde{Y}_1|UY_1Q}$ satisfying

$$I(\tilde{Y}_1;Y_1|U,Y_2,Q) \le R_{\mathrm{Fb},1}$$

Theorem 2:
For DMBCs with feedback, $\mathcal{C}_{\mathrm{Fb}}$ includes the region $\mathcal{R}_{\mathrm{in},2}$:

$$\begin{aligned}
R_1 &\le I(U_0,U_1;Y_1,\tilde{Y}_2|Q) - I(\tilde{Y}_2;Y_2|Y_1,Q)\\
R_2 &\le I(U_0,U_2;Y_2,\tilde{Y}_1|Q) - I(\tilde{Y}_1;Y_1|Y_2,Q)\\
R_1+R_2 &\le I(U_0,U_1;Y_1,\tilde{Y}_2|Q) - I(\tilde{Y}_2;Y_2|Y_1,Q) + I(U_2;Y_2,\tilde{Y}_1|U_0,Q) - I(U_1;U_2|U_0,Q)\\
R_1+R_2 &\le I(U_0,U_2;Y_2,\tilde{Y}_1|Q) - I(\tilde{Y}_1;Y_1|Y_2,Q) + I(U_1;Y_1,\tilde{Y}_2|U_0,Q) - I(U_1;U_2|U_0,Q)\\
R_1+R_2 &\le I(U_0,U_1;Y_1,\tilde{Y}_2|Q) - I(\tilde{Y}_2;Y_2|Y_1,Q) + I(U_0,U_2;Y_2,\tilde{Y}_1|Q) - I(\tilde{Y}_1;Y_1|Y_2,Q) - I(U_1;U_2|U_0,Q)
\end{aligned}$$

for some pmf $P_Q P_{U_0U_1U_2|Q} P_{X|U_0U_1U_2Q} P_{\tilde{Y}_1|Y_1Q} P_{\tilde{Y}_2|Y_2Q}$ satisfying

$$I(\tilde{Y}_1;Y_1|Y_2,Q) \le R_{\mathrm{Fb},1} \quad \text{and} \quad I(\tilde{Y}_2;Y_2|Y_1,Q) \le R_{\mathrm{Fb},2}$$

Remark 2: When setting $\tilde{Y}_1 = \text{const.}$, $\mathcal{R}_{\mathrm{in},1}$ reduces to superposition coding region

Remark 3: $\mathcal{R}_{\mathrm{in},2}$ strictly contains $\mathcal{C}_{\mathrm{NoFb}}$ for some more general DMBCs: some essentially less-noisy and some more capable BCs. This is the case e.g., for BS/BE-BCs

Usefulness of Feedback

Theorem 3:
Assume $R_{\mathrm{Fb},1} > 0$. For strictly essentially less-noisy DMBCs:

1. If $(R_1 > 0, R_2 > 0) \in (\mathrm{bd}(\mathcal{C}_{\mathrm{NoFb}}) \cap \mathrm{int}(\mathcal{C}_{\mathrm{Enh}})$, then $(R_1,R_2) \in \mathrm{int}(\mathcal{C}_{\mathrm{Fb}})$
2. If $\mathcal{C}_{\mathrm{NoFb}} \ne \mathcal{C}_{\mathrm{Enh}}$, then $\mathcal{C}_{\mathrm{NoFb}} \subset \mathcal{C}_{\mathrm{Fb}}$, *i.e. feedback strictly increases the nofeedback capacity region*

Examples

$Y_i = X \oplus Z_i$ with inde. $Z_i \sim$Bern(p_i) and $0 < p_2 < p_1 < 1/2$

- let $U_1 = \tilde{Y}_2 = const.$, $U_0 \sim$Bern(q), $U_2 = U_0 \oplus W_1$ and $\tilde{Y}_1 = Y_1 \oplus W_2$ with $W_1 \sim$Bern(r), $W_2 \sim$Bern(s)

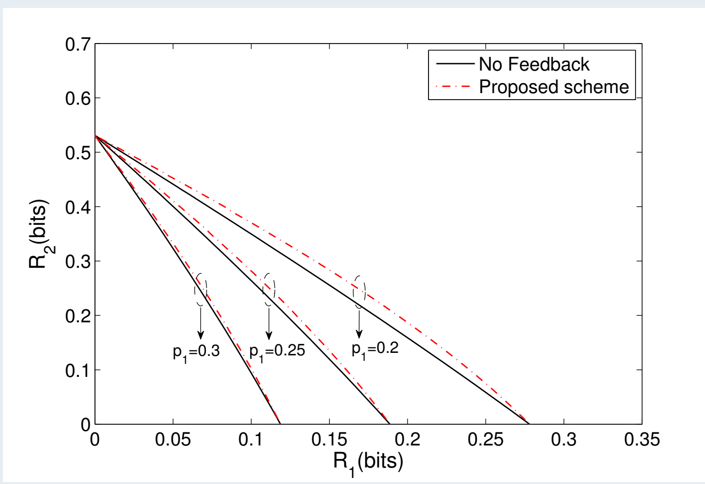

Figure 2 : $p_1 \in \{0.2, 0.25, 0.3\}$, $p_2 = 0.1$ and $R_{\mathrm{Fb},1} = 0.8$

$X \to Y_1$: BSC(p), $X \to Y_2$: BEC(e) with $p, e \in (0,1)$

- $0 < e < H(p)$: More capable but NOT $Y_2 \succ Y_1$
- $H(p) < e < 1$: $Y_2 \succ Y_1$

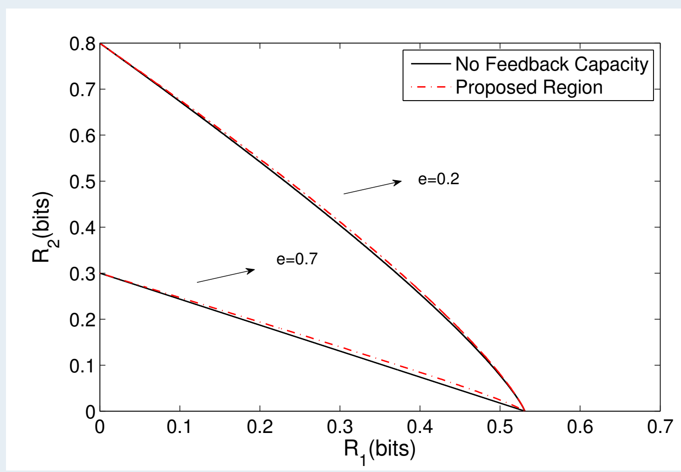

Figure 3 : $p = 0.1$, $e \in \{0.2, 0.7\}$ and $R_{\mathrm{Fb},1}, R_{\mathrm{Fb},2} = 0.8$

Coding Schemes

Scheme achieving $\mathcal{R}_{\mathrm{in},1}$

- *Codebook:* In block $b \in [1:B+1]$, generate

$$u_b^n(m_{1,b}, l_{b-1}) \sim \textstyle\prod_{t=1}^n P_U(u_{b,t}) \text{ for } m_{1,b} \in [1:2^{nR_1}]$$

$$x_b^n(m_{2,b}|m_{1,b}, l_{b-1}) \sim \textstyle\prod_{t=1}^n P_{X|U}(x_{b,t}|u_{b,t}) \text{ for } m_{2,b} \in [1:2^{nR_2}]$$

$$\tilde{y}_b^n(k_b, l_b|m_{1,b}, l_{b-1}) \sim \textstyle\prod_{t=1}^n P_{\tilde{Y}|U}(\tilde{y}_{b,t}|u_{b,t})$$

for $k_b \in [1:2^{nR'}]$, $l_b \in [1:2^{nR_{\mathrm{Fb},1}}]$

- *Transmitter:* In block $b \in [1:B+1]$, sends

$$x_b^n(m_{2,b}|m_{1,b}, l_{b-1})$$

- Sliding-window decoding: In block $b \in [1:B+1]$,
Receiver 1: looks for an index $\hat{m}_{1,b}^{(1)}$ s.t.

$$(u_b^n(\hat{m}_{1,b}^{(1)}, l_{b-1}), y_{1,b}^n) \in \mathcal{T}_\epsilon^n(P_{UY_1}).$$

Next, looks for a pair (k_b, l_b) s.t.

$$(u_b^n(\hat{m}_{1,b}^{(1)}, l_{b-1}), \tilde{y}_b^n(k_b, l_b|\hat{m}_{1,b}^{(1)}, l_{b-1}), y_{1,b}^n) \in \mathcal{T}_\epsilon^n(P_{U\tilde{Y}Y_1}),$$

and sends l_b back to the transmitter

Receiver 2: looks for a pair $(\hat{m}_{1,b}^{(2)}, \hat{l}_{b-1})$ s.t.

$$(u_b^n(\hat{m}_{1,b}^{(2)}, \hat{l}_{b-1}), y_{2,b}^n) \in \mathcal{T}_\epsilon^n(P_{UY_2}).$$

Then, it looks for a pair $(\hat{k}_{b-1}, \hat{l}_{b-1})$ s.t.

$$\begin{aligned}
(u_{b-1}^n \quad &(\hat{m}_{1,b-1}^{(2)}, \hat{l}_{b-2}), y_{2,b-1}^n,\\
&\tilde{y}_{b-1}^n(\hat{k}_{b-1}, \hat{l}_{b-1}|\hat{m}_{1,b-1}^{(2)}, \hat{l}_{b-2})) \in \mathcal{T}_\epsilon^n(P_{U\tilde{Y}Y_2}),
\end{aligned}$$

and finally searches an index $\hat{m}_{2,b-1}$ s.t.

$$\begin{aligned}
(u_{b-1}^n(\hat{m}_{1,b-1}^{(2)}, \hat{l}_{b-2}), &x_{b-1}^n(\hat{m}_{2,b-1}|\hat{m}_{1,b-1}^{(2)}, \hat{l}_{b-2}),\\
\tilde{y}_{b-1}^n(\hat{k}_{b-1}, \hat{l}_{b-1}|\hat{m}_{1,b-1}^{(2)}, \hat{l}_{b-2}), &y_{2,b-1}^n) \in \mathcal{T}_\epsilon^n(P_{UX\tilde{Y}Y_2})
\end{aligned}$$

Note: $\mathcal{R}_{\mathrm{in},2}$ is achieved by applying Marton's coding, two-sided feedback and backward decoding to the scheme for $\mathcal{R}_{\mathrm{in},1}$

Summary

- Feedback can increase the capacity of strictly essentially less-noisy DMBC
- Feedback can even increase the capacity for some more capable but not strictly essentially less-noisy DMBC
- Recover all previously known capacity and degrees of freedom results for memoryless BCs with feedback
- Results hold up under noisy feedback links if the receivers can code over them

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