Capacity Bounds for Diamond Networks with an Orthogonal Broadcast Channel

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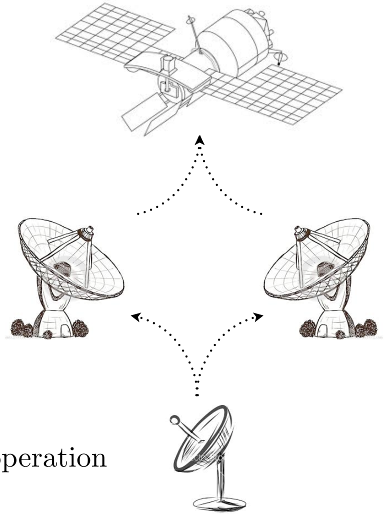
Motivation

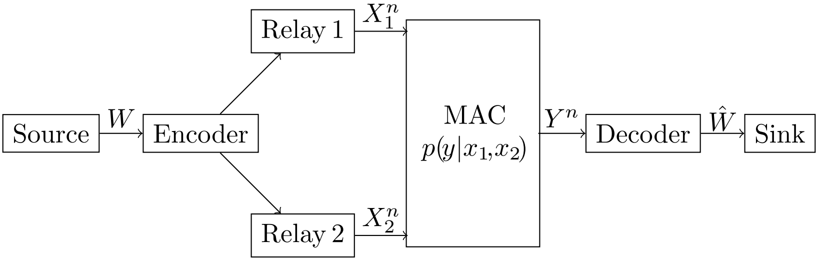


- Challenge: diversity vs. cooperation

Outline

- The Problem Setup
- A Lower Bound
- An Upper-Bound
- Examples
    - The Gaussian MAC
    - The binary adder MAC

The Problem Setup



- W message of rate R

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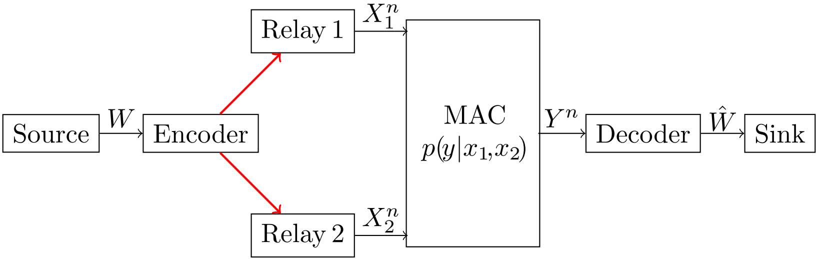
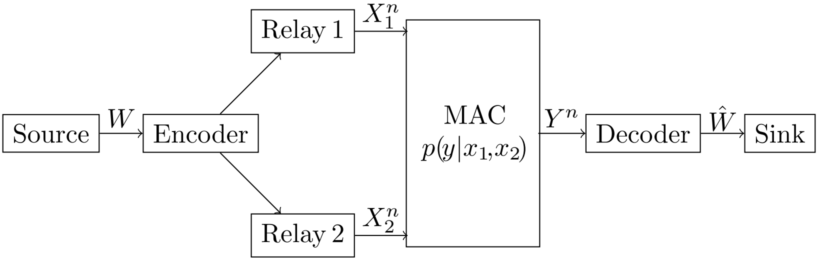


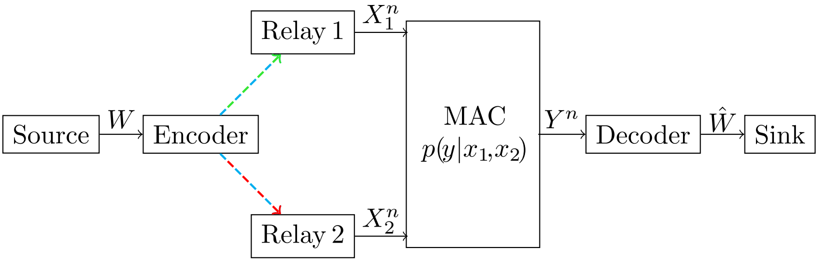
- W message of rate R
- Bit-pipes of capacities C_1, C_2

The Problem Setup



- W message of rate R
- Bit-pipes of capacities C_1, C_2
- Goal: What is the highest rate R such that $\mathbb{P}\mathrm{r}(W \neq \hat{W}) \to 0$?

A Lower Bound



- Rate splitting: $W = (W_{12}, W_1, W_2)$
- Superposition Coding:
  W_{12} encoded in V^n.
  X_1^n, X_2^n superposed on V^n.
- Marton's Coding

A Lower Bound (Cont.)

Theorem (Lower Bound)

The rate R is achievable if it satisfies the following condition for some pmf $p(v, x_1, x_2, y) = p(v, x_1, x_2)p(y|x_1, x_2)$:

$$R \leq \min \left\{ \begin{array}{l} C_1 + C_2 - I(X_1; X_2|V) \\ C_2 + I(X_1; Y|X_2 V) \\ C_1 + I(X_2; Y|X_1 V) \\ \frac{1}{2}(C_1 + C_2 + I(X_1 X_2; Y|V) - I(X_1; X_2|V)) \\ I(X_1 X_2; Y) \end{array} \right\}$$

$V \in \mathcal{V}, \ |\mathcal{V}| \leq \min\{|\mathcal{X}_1\|\mathcal{X}_2| + 2, |\mathcal{Y}| + 4\}$

The Cut-Set Bound

Cut-Set bound: R is achievable only if it satisfies the following bounds for some $p(x_1, x_2)$:

$$
\begin{aligned}
R &\leq C_1 + C_2 \\
R &\leq C_1 + I(X_2; Y | X_1) \\
R &\leq C_2 + I(X_1; Y | X_2) \\
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\end{aligned}
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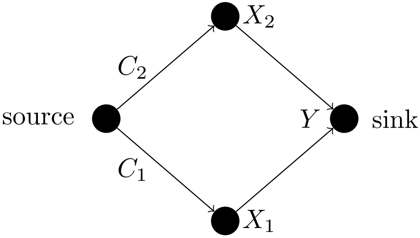
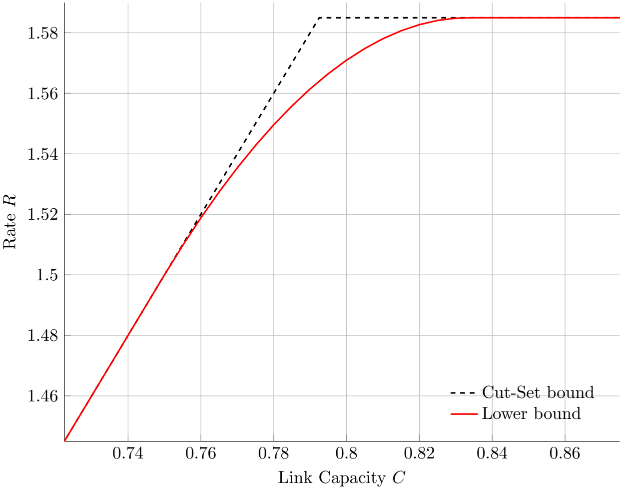


Example I: binary adder MAC

- $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}, \quad \mathcal{Y} = \{0, 1, 2\}$
- $Y = X_1 + X_2$



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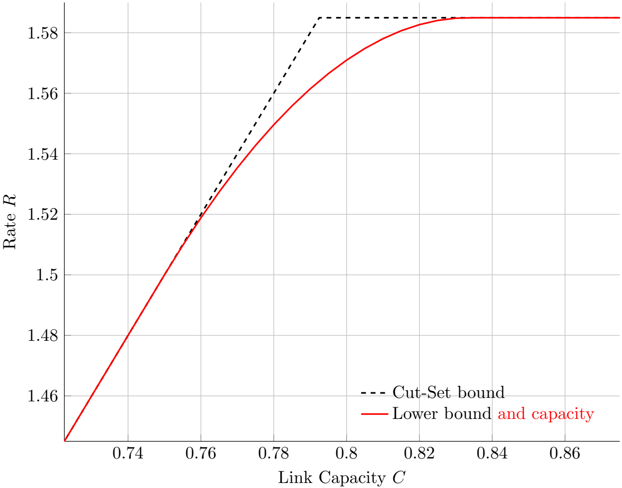
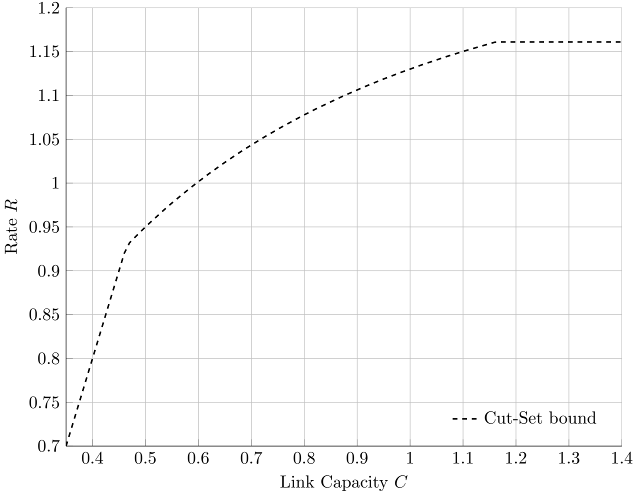


Example II: Gaussian MAC

- $Y = X_1 + X_2 + Z, \quad Z \sim \mathcal{N}(0,1)$
- $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \le P_1, \quad \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{2,i}^2) \le P_2, \quad P_1 = P_2 = 1$



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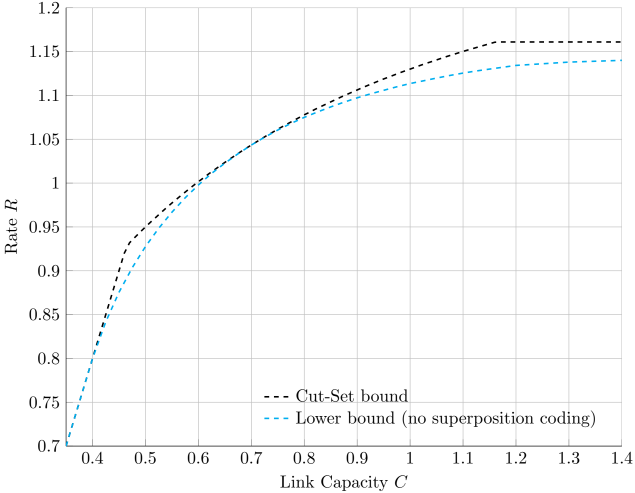

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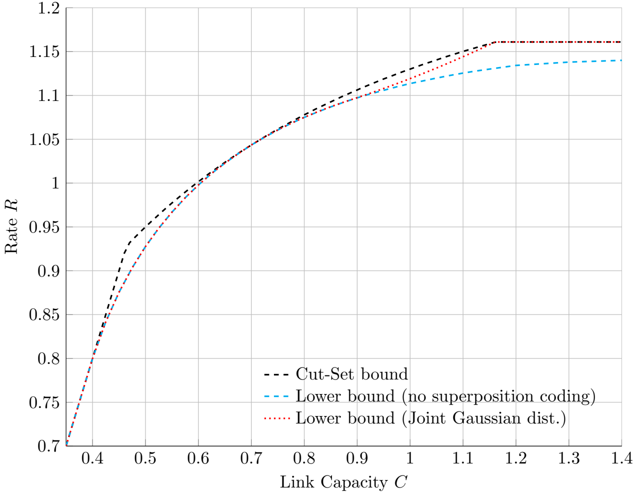

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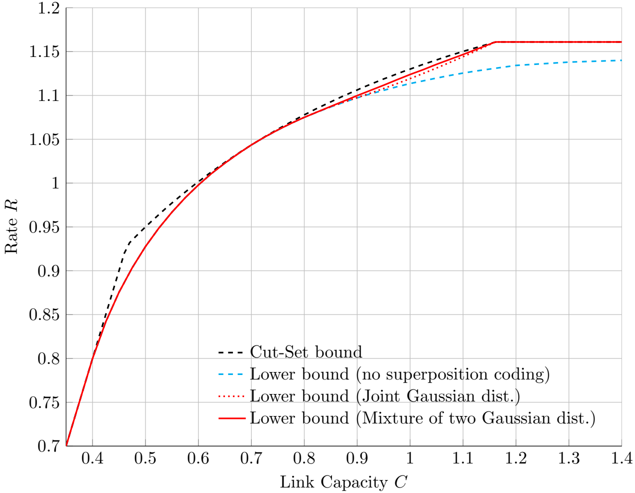

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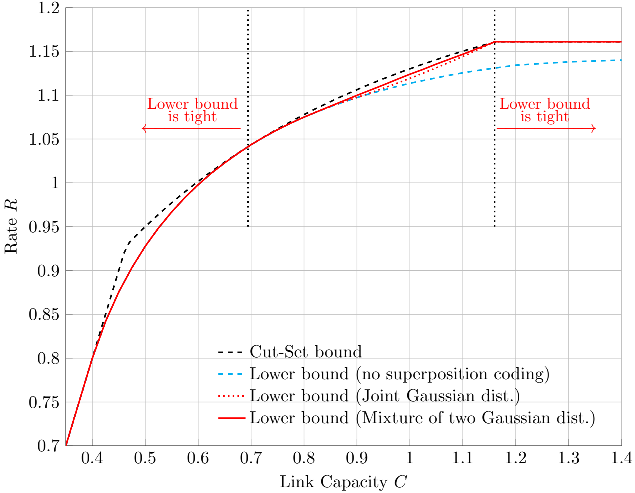
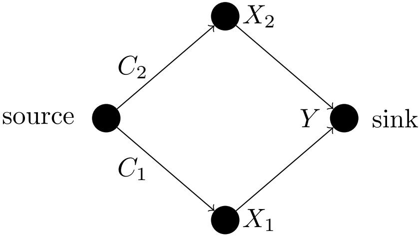


Is the Cut-Set Bound Tight?

Cut-Set bound:

$$\begin{aligned} R &\le C_1 + C_2 \\ R &\le C_1 + I(X_2;Y|X_1) \\ R &\le C_2 + I(X_1;Y|X_2) \\ R &\le I(X_1X_2;Y). \end{aligned}$$

Maximize over $p(x_1, x_2)$.



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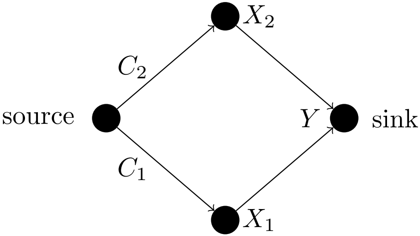

It turns out that the cut-set bound is not tight.

Refining the Cut-Set Bound

- Motivated by [Ozarow'80, KangLiu'11]

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$$nR \leq nC_1 + nC_2 - I(X_1^n; X_2^n)$$

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$$nR \leq nC_1 + nC_2 - I(X_1^n; X_2^n)$$

- For any U^n:

$$\begin{aligned} I(X_1^n; X_2^n) = & \; I(X_1^n X_2^n; U^n) - I(X_1^n; U^n | X_2^n) - I(X_2^n; U^n | X_1^n) \\ & + I(X_1^n; X_2^n | U^n) \end{aligned}$$

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- For any U^n:

$$I(X_1^n; X_2^n) \geq I(X_1^n X_2^n; U^n) - I(X_1^n; U^n | X_2^n) - I(X_2^n; U^n | X_1^n)$$

Refining the Cut-Set Bound (Cont.)

$$nR \leq nC_1 + nC_2 - I(X_1^n X_2^n; U^n) + I(X_1^n; U^n | X_2^n) + I(X_2^n; U^n | X_1^n)$$

Refining the Cut-Set Bound (Cont.)

$$nR \leq nC_1 + nC_2 - I(X_1^n X_2^n; U^n) + I(X_1^n; U^n | X_2^n) + I(X_2^n; U^n | X_1^n)$$

choose U_i as follows:

$$Y_i \longrightarrow \boxed{p_{U|Y}} \longrightarrow U_i$$

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$$2nR \le nC_1 + nC_2 + I(X_1^n X_2^n; Y^n | U^n) + I(X_1^n; U^n | X_2^n) + I(X_2^n; U^n | X_1^n)$$

Refining the Cut-Set Bound (Cont.)

$$nR \le nC_1 + nC_2 - I(X_1^n X_2^n; U^n) + I(X_1^n; U^n | X_2^n) + I(X_2^n; U^n | X_1^n)$$

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$$\ldots \le n\left(C_1 + C_2 + I(X_1 X_2; Y | U) + I(X_1; U | X_2) + I(X_2; U | X_1)\right)$$

New Upper-Bounds (1)

Theorem (Upper Bound I)

The rate R is achievable only if there exists a joint distribution $p(x_1, x_2)$ for which the following inequalities hold for every auxiliary channel $p(u|x_1, x_2, y) = p(u|y)$

$$
\begin{aligned}
R &\leq C_1 + C_2 \\
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- max-min problem

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- max-min problem
- $2R \leq C_1 + C_2 + I(X_1 X_2; Y) - I(X_1; X_2) + I(X_1; X_2|U)$

New Upper-Bounds (2)

Theorem (Upper Bound II)

The capacity is bounded from above by

$$\max_{p(x_1,x_2)} \min_{\substack{p(u|x_1,x_2,y)\\ =p(u|y)}} \max_{\substack{p(q|x_1,x_2,y,u)\\ =p(q|x_1,x_2)}} \min \begin{Bmatrix} C_1+C_2, \\ C_1+I(X_2;Y|X_1Q), \\ C_2+I(X_1;Y|X_2Q), \\ I(X_1X_2;Y|Q), \\ C_1+C_2-I(X_1;X_2|Q)+I(X_1;X_2|UQ) \end{Bmatrix}$$

- $|\mathcal{Q}| \leq |\mathcal{X}_1||\mathcal{X}_2| + 3.$

New Upper-Bounds (2)

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- $|\mathcal{Q}| \leq |\mathcal{X}_1||\mathcal{X}_2| + 3$.
- last term is related to the Hekstra-Willems dependence balance bound and can be written as

$$R \leq C_1+C_2-I(X_1X_2;U|Q)+I(X_2;U|X_1Q)+I(X_1;U|X_2Q)$$

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The Gaussian MAC

$$Y = X_1 + X_2 + Z \qquad \begin{array}{l} Z \sim \mathcal{N}(0,1), \\ \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \le P, \\ \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{2,i}^2) \le P \end{array}$$

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$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{2,i}^2) \leq P$

Max-Min-Max problem

$$R \leq 2C$$
$$R \leq C + I(X_1; Y | X_2 Q)$$
$$R \leq C + I(X_2; Y | X_1 Q)$$
$$R \leq I(X_1 X_2; Y | Q)$$
$$R \leq \quad C_1 + C_2 - I(X_1 X_2; U | Q) + I(X_1; U | X_2 Q) + I(X_2; U | X_1 Q)$$

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\end{aligned}
$$

Choose $U = Y + Z_N$

$Z_N \sim \mathcal{N}(0, N)$

N to be optimized.

The Gaussian MAC

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$Z \sim \mathcal{N}(0,1)$,
$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \leq P$,
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Choose $U = Y + Z_N$
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N to be optimized.

$$R \leq 2C$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq C + I(X_2; Y|X_1 Q)$$
$$R \leq I(X_1 X_2; Y|Q)$$
$$R \leq C_1 + C_2 - I(X_1 X_2; U|Q) + I(X_1; U|X_2 Q) + I(X_2; U|X_1 Q)$$

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$Z_N \sim \mathcal{N}(0, N)$
N to be optimized.

The Gaussian MAC

$$Y = X_1 + X_2 + Z$$

$$Z \sim \mathcal{N}(0,1),$$
$$\tfrac{1}{n}\textstyle\sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \leq P,$$
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Choose $U = Y + Z_N$

$Z_N \sim \mathcal{N}(0, N)$

N to be optimized.

$$R \leq 2C$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq \log\left(1 + 2P(1+\rho)\right)/2$$
$$R \leq C_1 + C_2 - I(X_1X_2;U|Q) + I(X_1;U|X_2Q) + I(X_2;U|X_1Q)$$

The Gaussian MAC

$$Y = X_1 + X_2 + Z$$

$$Z \sim \mathcal{N}(0,1),$$
$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \le P,$$
$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{2,i}^2) \le P$$

$$R \le 2C$$
$$R \le C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \le C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \le \log\left(1 + 2P(1+\rho)\right)/2$$

Choose $U = Y + Z_N$
$Z_N \sim \mathcal{N}(0, N)$
N to be optimized.

$$R \le \quad C_1 + C_2 - I(X_1X_2;U|Q) + \log\left(\frac{1 + N + P\left(1-\rho^2\right)}{1+N}\right)$$

The Gaussian MAC (Cont.)

- $U = Y + Z_N$, $Z_N \sim \mathcal{N}(0, N)$

$$\boxed{I(X_1X_2;U|Q)} = h(U|Q) - h(U|X_1X_2)$$
$$\overset{\text{EPI}}{\geq} \frac{1}{2}\log\left(2\pi eN + 2^{2h(Y|Q)}\right) - \frac{1}{2}\log\left(2\pi e(1+N)\right)$$

$$\boxed{I(X_1X_2;Y|Q)} = h(Y|Q) - \frac{1}{2}\log\left(2\pi e\right) \geq R$$

The Gaussian MAC (Cont.)

- $U = Y + Z_N$, $Z_N \sim \mathcal{N}(0, N)$

$$\begin{aligned} I(X_1X_2;U|Q) &= h(U|Q) - h(U|X_1X_2) \\ &\overset{\text{EPI}}{\geq} \frac{1}{2}\log\left(2\pi eN + 2^{2h(Y|Q)}\right) - \frac{1}{2}\log\left(2\pi e(1+N)\right) \end{aligned}$$

$$I(X_1X_2;Y|Q) = h(Y|Q) - \frac{1}{2}\log\left(2\pi e\right) \geq R$$

$$\begin{aligned} R \leq C_1 + C_2 - \frac{1}{2}\log\left(N + 2^{2R}\right) - \frac{1}{2}\log\left(1+N\right) \\ + \log\left(1 + N + P\left(1-\rho^2\right)\right) \end{aligned}$$

The Gaussian MAC (Cont.)

- $U = Y + Z_N$, $Z_N \sim \mathcal{N}(0, N)$

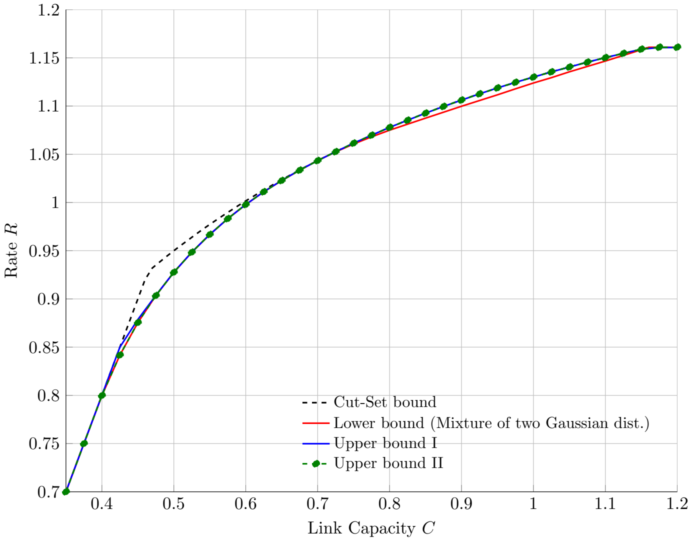
$$\boxed{I(X_1X_2;U|Q)} = h(U|Q) - h(U|X_1X_2)$$
$$\overset{\text{EPI}}{\geq} \frac{1}{2}\log\left(2\pi eN + 2^{2h(Y|Q)}\right) - \frac{1}{2}\log\left(2\pi e(1+N)\right)$$

$$\boxed{I(X_1X_2;Y|Q)} = h(Y|Q) - \frac{1}{2}\log\left(2\pi e\right) \geq R$$

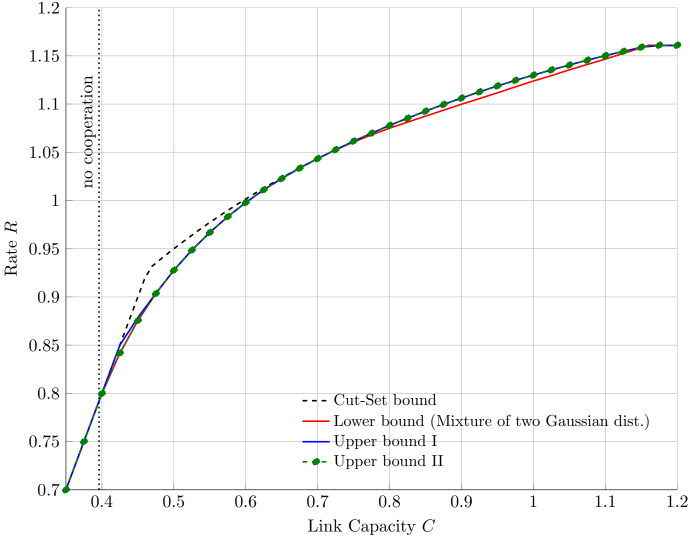
$$R \leq C_1 + C_2 - \frac{1}{2}\log\left(N + 2^{2R}\right) - \frac{1}{2}\log\left(1+N\right)$$
$$+ \log\left(1 + N + P\left(1-\rho^2\right)\right)$$

- Strictly tighter than [KangLiu'11]

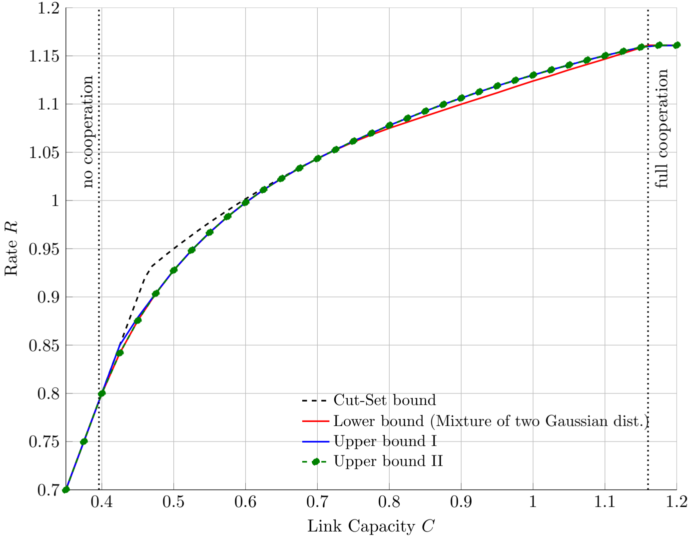
The Gaussian MAC (Cont.)



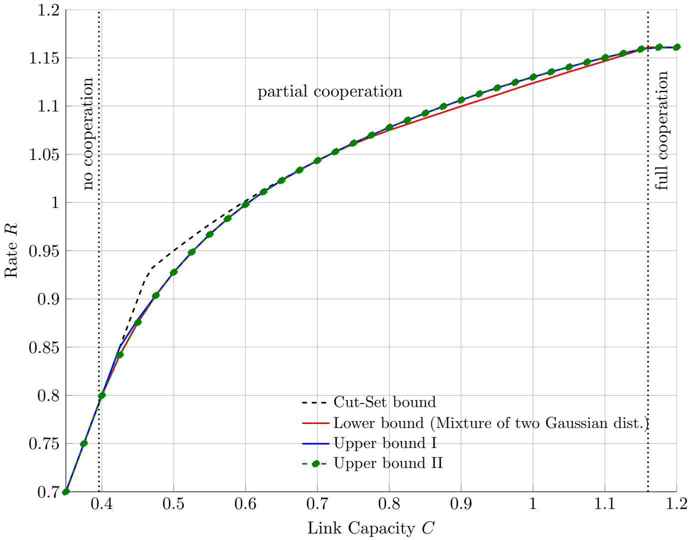
The Gaussian MAC (Cont.)



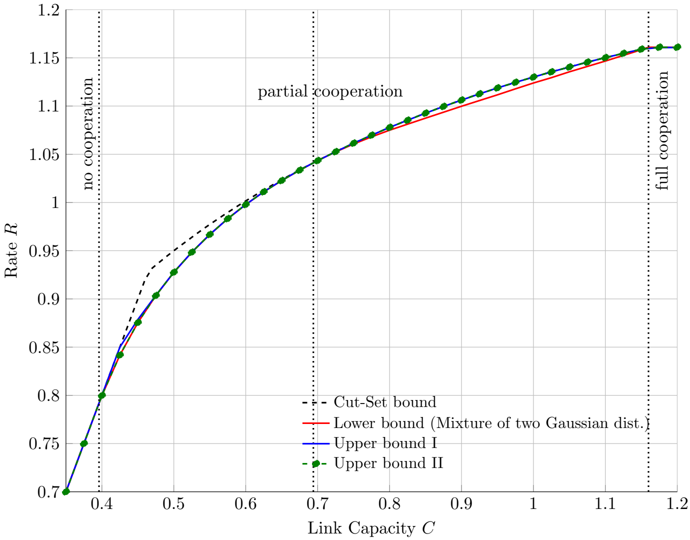
The Gaussian MAC (Cont.)



The Gaussian MAC (Cont.)



The Gaussian MAC (Cont.)



On The Capacity of The Gaussian MAC

Theorem

For a symmetric Gaussian diamond network, the upper bound meets the lower bound for all C such that $C \geq \frac{1}{2}\log(1+4P)$, or

$$C \leq \frac{1}{4}\log\frac{1+2P(1+\rho^{(2)})}{1-\left(\rho^{(2)}\right)^2}$$

where

$$\rho^{(2)} = \sqrt{1+\frac{1}{4P^2}} - \frac{1}{2P}$$

The Optimal Choice of N

- $U = Y + Z_N$ (motivated by [Ozarow'80, KangLiu'11])
- (X_1, X_2) an optimal jointly Gaussian input for the lower bound

$$\begin{bmatrix} P & \lambda^\star P \\ \lambda^\star P & P \end{bmatrix}.$$

- $N = \left(P\left(\frac{1}{\lambda^\star} - \lambda^\star\right) - 1\right)^+$
- $P\left(\frac{1}{\lambda^\star} - \lambda^\star\right) - 1 \geq 0$: $X_1 - U - X_2$ forms a Markov chain– new upper-bound
- $P\left(\frac{1}{\lambda^\star} - \lambda^\star\right) - 1 \leq 0$: the cut-set bound

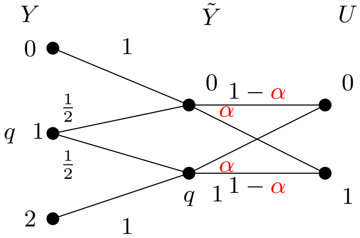
The Binary Adder MAC

$$Y = X_1 + X_2, \quad \mathcal{X}_1 = \mathcal{X} = \{0, 1\}, \quad \mathcal{Y} = \{0, 1, 2\}$$

$$\begin{aligned}
R &\leq C_1 + C_2 \\
R &\leq C_2 + I(X_1; Y | X_2 Q) \\
R &\leq C_1 + I(X_2; Y | X_1 Q) \\
R &\leq I(X_1 X_2; Y | Q) \\
R &\leq C_1 + C_2 - I(X_1 X_2; U | Q) + I(X_1; U | X_2 Q) + I(X_2; U | X_1 Q)
\end{aligned}$$

The Binary Adder MAC

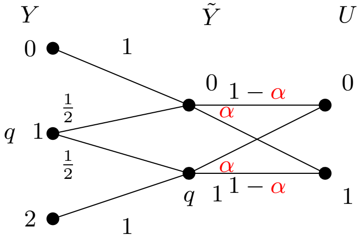
$$Y = X_1 + X_2, \quad \mathcal{X}_1 = \mathcal{X} = \{0,1\}, \quad \mathcal{Y} = \{0,1,2\}$$



$$\begin{aligned}
R &\le C_1 + C_2 \\
R &\le C_2 + I(X_1; Y | X_2 Q) \\
R &\le C_1 + I(X_2; Y | X_1 Q) \\
R &\le I(X_1 X_2; Y | Q) \\
R &\le C_1 + C_2 - I(X_1 X_2; U | Q) + I(X_1; U | X_2 Q) + I(X_2; U | X_1 Q)
\end{aligned}$$

The Binary Adder MAC

$$Y = X_1 + X_2, \quad \mathcal{X}_1 = \mathcal{X} = \{0,1\}, \quad \mathcal{Y} = \{0,1,2\}$$



$$\begin{aligned}
R &\leq C_1 + C_2 \\
R &\leq C_2 + h_2(q) \\
R &\leq C_1 + h_2(q) \\
R &\leq 1 + h_2(q) - q \\
R &\leq C_1 + C_2 - I(X_1X_2; U|Q) + 2h_2(\frac{q}{2} \star \alpha) - 2(1-q)h_2(\alpha) - 2q
\end{aligned}$$

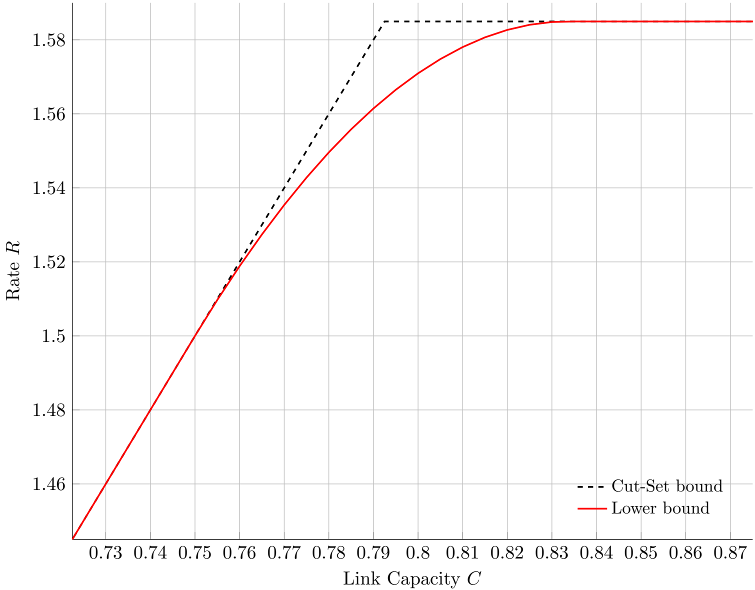
The Interplay in the upper bound

$$\boxed{I(X_1X_2;U|Q)} = H(U|Q) - H(U|X_1X_2)$$

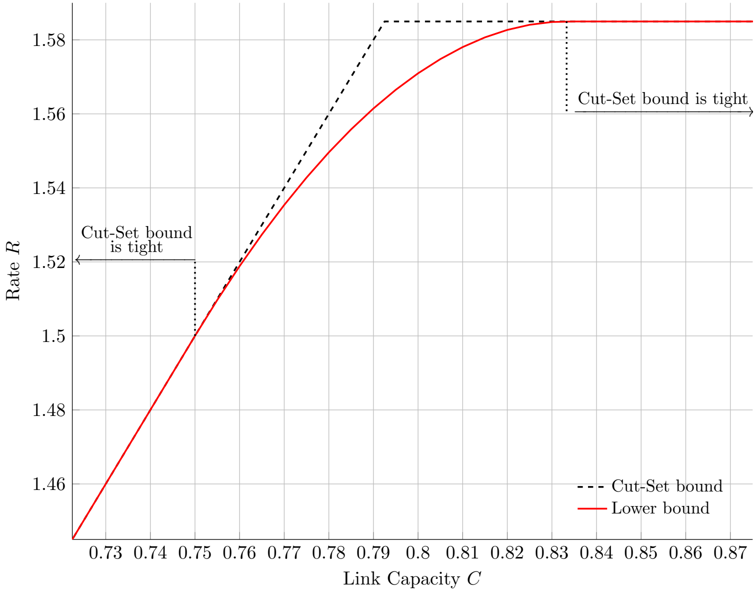
$$\stackrel{\text{MGL}}{\geq} h_2\left(\alpha \star h_2^{-1}\left(H(\tilde{Y}|Q)\right)\right) - (1-q)h_2(\alpha) - q$$

$$\boxed{I(X_1X_2;Y|Q)} = H(\tilde{Y}|Q) + h_2(q) - q \geq R$$

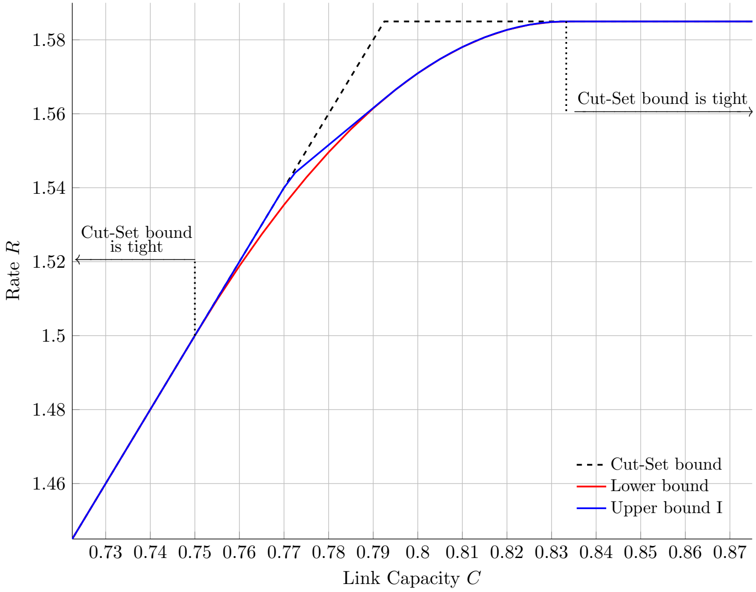
The Binary Adder MAC (Cont.)



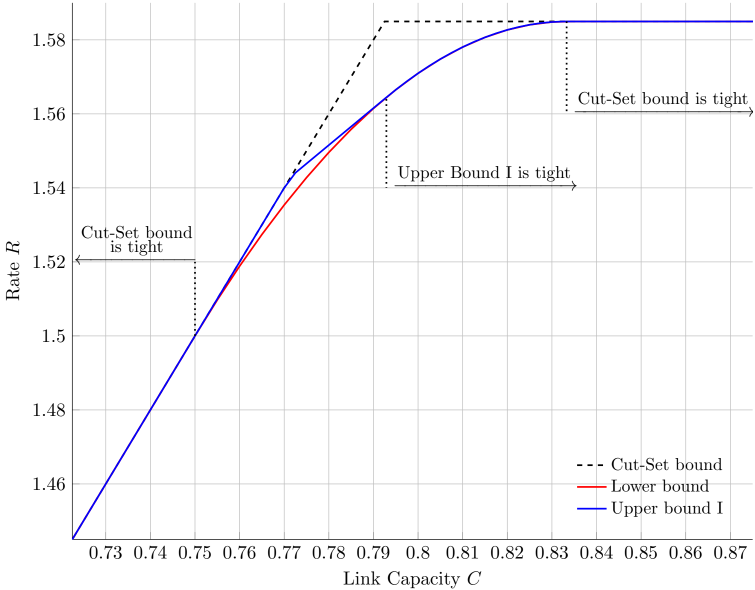
The Binary Adder MAC (Cont.)



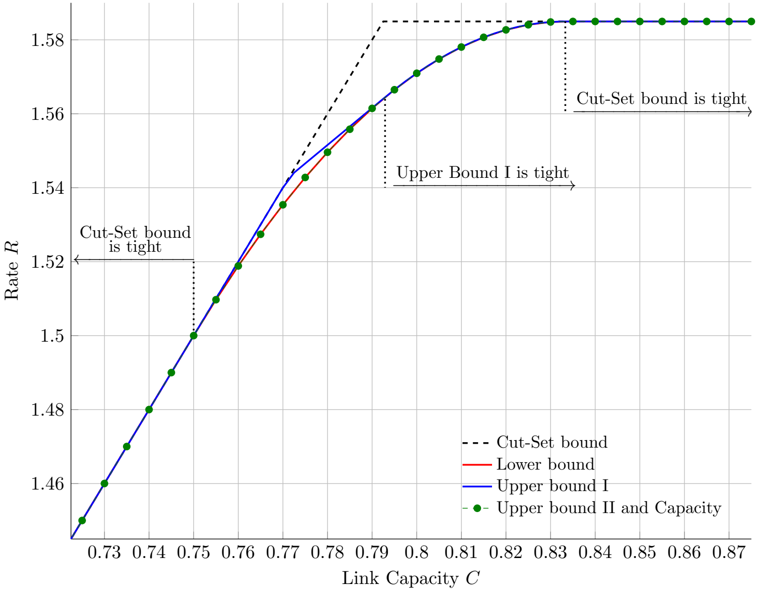
The Binary Adder MAC (Cont.)



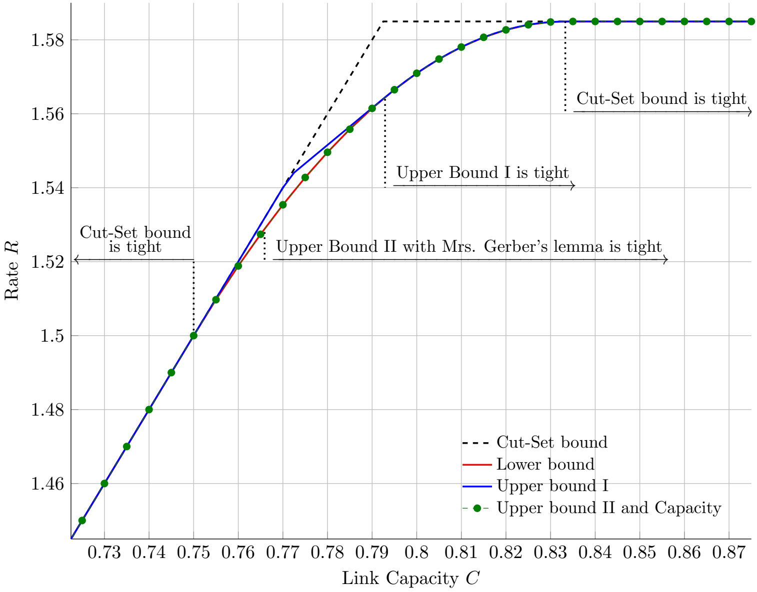
The Binary Adder MAC (Cont.)



The Binary Adder MAC (Cont.)



The Binary Adder MAC (Cont.)



The interplay in the upper bounds

$$R \leq I(X_1 X_2; Y|Q)$$

$$R \leq C_1 + C_2 - I(X_1 X_2; U|Q) + I(X_2; U|X_1 Q) + I(X_1; U|X_2 Q)$$

The interplay in the upper bounds

$$
\begin{aligned}
R \leq & I(X_1 X_2; Y|Q) \\
\leq & H(Y|Q) - H(Y|X_1 X_2) \\
R \leq & C_1 + C_2 - I(X_1 X_2; U|Q) + I(X_2; U|X_1 Q) + I(X_1; U|X_2 Q)
\end{aligned}
$$

The interplay in the upper bounds

$$
\begin{aligned}
R \leq & I(X_1 X_2; Y|Q) \\
\leq & H(Y|Q) - H(Y|X_1 X_2) \\
R \leq & C_1 + C_2 - I(X_1 X_2; U|Q) + I(X_2; U|X_1 Q) + I(X_1; U|X_2 Q) \\
\leq & C_1 + C_2 - H(U|Q) - H(U|X_1 X_2) + H(U|X_1 Q) + H(U|X_2 Q)
\end{aligned}
$$

The interplay in the upper bounds

$$
\begin{aligned}
R \leq & I(X_1X_2;Y|Q) \\
\leq & \boxed{H(Y|Q)} - H(Y|X_1X_2) \\
R \leq & C_1 + C_2 - I(X_1X_2;U|Q) + I(X_2;U|X_1Q) + I(X_1;U|X_2Q) \\
\leq & C_1 + C_2 - \boxed{H(U|Q)} - H(U|X_1X_2) + H(U|X_1Q) + H(U|X_2Q)
\end{aligned}
$$

- Up to now: Entropy Power Inequality, Mrs. Gerber's Lemma
    1. $\min\{H(U)|H(Y)=t\} \geq f(t)$
    2. $f(t)$ is convex in t

The interplay in the upper bounds

$$
\begin{aligned}
R \leq& I(X_1X_2;Y|Q) \\
\leq& \boxed{H(Y|Q)} - H(Y|X_1X_2) \\
R \leq& C_1 + C_2 - I(X_1X_2;U|Q) + I(X_2;U|X_1Q) + I(X_1;U|X_2Q) \\
\leq& C_1 + C_2 - \boxed{H(U|Q)} - H(U|X_1X_2) + \boxed{H(U|X_1Q) + H(U|X_2Q)}
\end{aligned}
$$

- Up to now: Entropy Power Inequality, Mrs. Gerber's Lemma
  1. $\min\{H(U)|H(Y)=t\} \geq f(t)$
  2. $f(t)$ is convex in t

The interplay in the upper bounds

$$
\begin{aligned}
R \leq & I(X_1X_2;Y|Q) \\
\leq & \boxed{H(Y|Q)} - H(Y|X_1X_2) \\
R \leq & C_1 + C_2 - I(X_1X_2;U|Q) + I(X_2;U|X_1Q) + I(X_1;U|X_2Q) \\
\leq & C_1 + C_2 - \boxed{H(U|Q)} - H(U|X_1X_2) + \boxed{H(U|X_1Q) + H(U|X_2Q)}
\end{aligned}
$$

- Up to now: Entropy Power Inequality, Mrs. Gerber's Lemma
  1. $\min\{H(U)|H(Y)=t\} \geq f(t)$
  2. $f(t)$ is convex in t
- What we want to do:
  1. $\min\{H(U) - H(U|X_1) - H(U|X_2)|H(Y)=t\} \geq f(t)$
  2. $f(t)$ is convex in t

The Binary Adder MAC: Upper Bound

$$R \le 2C$$
$$R \le C + h_2(q)$$
$$R \le 1 + h_2(q) - q$$
$$\begin{aligned} R \le\ & 2C - h_2\left(\alpha \star \left(\frac{q}{2} + (1-q)h_2^{-1}\left(\min\left(1, \frac{(R - h_2(q))^+}{1-q}\right)\right)\right)\right) \\ & - (1-q)h_2(\alpha) - q + 2h_2\left(\alpha \star \frac{q}{2}\right) \end{aligned}$$

Capacity of The Binary Adder MAC

Theorem

The capacity of diamond networks with binary adder MACs is

$$\max_{0 \leq p \leq \frac{1}{2}} \min \begin{cases} C_1 + C_2 - 1 + h_2(p) \\ C_1 + h_2(p) \\ C_2 + h_2(p) \\ h_2(p) + 1 - p. \end{cases}$$

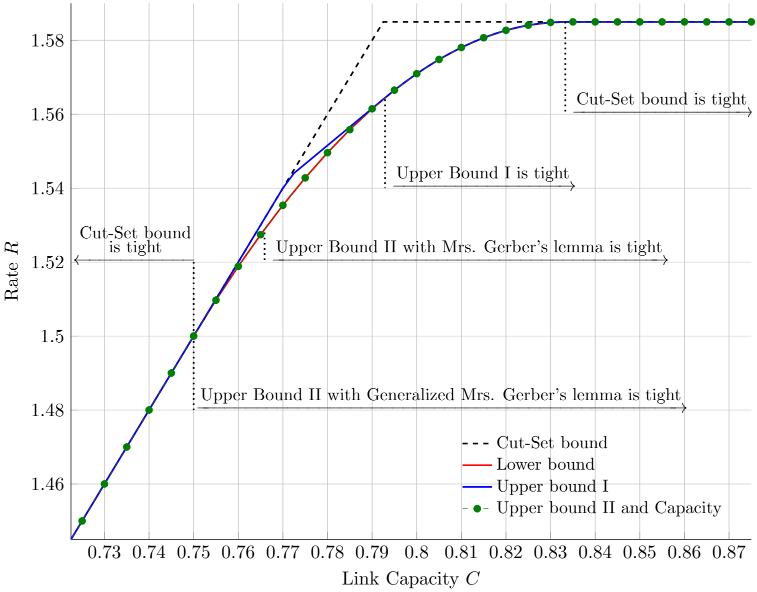
The optimal Choice of α

- Let (X_1, X_2) be an optimizing doubly symmetric binary pmf with parameter $p^\star$ for the lower bound
- α is such that

$$\alpha(1-\alpha) = \left(\frac{p^\star}{2(1-p^\star)}\right)^2$$

and it makes the following Markov chain $X_1 - U - X_2$.

Capacity of The Binary Adder MAC



Summary and Work in Progress

- Lower and Upper bounds on the capacity of a class of diamond networks
- A new upper bound which is in the form of a max-min problem
- Gaussian MACs:
  - improved previous lower and upper bounds
  - characterized the capacity for interesting ranges of bit-pipe capacities.
- Binary adder MAC: fully characterized the capacity
- Work in progress: the general class of 2-relay diamond networks, n-relay diamond networks with orthogonal BC components