THE HAMILTONIAN STRUCTURE OF THE EQUATIONS OF MOTION OF A LIQUID DROP TRAPPED BETWEEN TWO PLATES

HANS-PETER KRUSE

Abstract

The equations of motion of an ideal incompressible liquid drop trapped between two parallel plates under the influence of surface tension and adhesion forces are studied. A main result of this paper is the proof that the equations of motion can be written in Hamiltonian form

$$\dot{F} = \{F, H\} \quad \text{for all } F \in \mathscr{D}.$$

Here $\mathscr{D}$ denotes a class of real-valued functions on the phase space $\mathscr{N}$ of the system and the Hamiltonian $H \in \mathscr{D}$ is the energy function of the system. This allows the derivation of an equation for the (dynamic) contact angle, in which the free fluid surface meets the plates. The behaviour of the dynamic contact angle is a point of great controversy in the capillarity literature and the derivation confirms one of the existing models. In the second part of the paper, which can be read independently, existence and stability questions for rigidly rotating drops are dealt with. The existence of solutions to the equations of motion that describe rotationally symmetric drops which rotate rigidly between the plates with constant angular velocity is proved. These solutions can be regarded as relative equilibria of a mechanical system with symmetry. Using ideas of the energy-momentum method of Lewis, Marsden and Simo, a stability criterion for this kind of motion is provided. To derive this criterion, the second derivative of the so-called augmented energy functional at the relative equilibrium in directions which are transversal to the group orbit of this equilibrium is studied. The stability criterion is applied to rigidly rotating drops of cylindrical shape. These represent solutions to the equations of motion in the case that no adhesion forces act along the plates. The result extends previous work of Vogel and Lewis.

1. *Introduction*

This paper is about the application of Hamiltonian mechanics to capillarity theory. We study the dynamics of an ideal incompressible fluid drop trapped between two parallel flat plates under the influence of surface tension and adhesion forces. The influence of gravity is neglected. Surface tension acts on the free boundary of the drop, and the adhesion forces act at the contact surfaces between the drop and the plates.

The free boundary of the drop meets each plate in a certain angle, called the *contact angle*. In the static case the value of the contact angle has been derived by C. F. Gauss using variational principles (compare Finn [**6**]). The value of the contact angle of a drop in motion, the so-called *dynamic contact angle*, is still a point of controversy in capillarity theory (compare Hocking [**8**]). The Hamiltonian model for the drop dynamics which we introduce in this paper allows us to deduce an equation for the dynamic contact angle which turns out to be the equation satisfied by the static contact angle. In Hocking [**8**] one finds further arguments that this is the correct equation for modelling the dynamic contact angle.

Received 20 December 1994.

1991 *Mathematics Subject Classification* 70H05.

The author was supported by a Feodor Lynen Fellowship of the Alexander von Humboldt Foundation.

J. London Math. Soc. (2) 59 (1999) 742–768

In our Hamiltonian model we describe the state of the drop by giving its free boundary Σ and a spatial velocity field $v: D_\Sigma \longrightarrow \mathbf{R}^3$, where D_Σ is the region bounded by Σ and the plates, that is, the region occupied by the drop. The collection of pairs (Σ, v) is denoted by $\mathscr{N}$. We show that the equations of motion of the drop have a Hamiltonian structure, that is, that they can be written in Poisson bracket form

$$\dot{F} = \{F, H\} \quad \text{for all } F \in \mathscr{D}.$$

Here $\mathscr{D}$ denotes a certain class of functions $\mathscr{N} \longrightarrow R$, called the class of *admissible functions*. The *Poisson bracket* $\{\cdot, \cdot\}$, which will be developed later, associates to two functions $F_1, F_2 \in \mathscr{D}$ another function $\{F_1, F_2\}: \mathscr{N} \longrightarrow \mathbf{R}$. The function $H \in \mathscr{D}$ is the Hamiltonian of our problem. As a Hamiltonian we choose the *total energy* function

$$H(\Sigma, v) = \frac{1}{2}\int_{D_\Sigma} \|v\|^2 dV + \tau \int_\Sigma dA - \sum_{i=0}^{1} \sigma_i \int_{\Sigma_i} dA.$$

The first term on the right-hand side is the kinetic energy of the drop (here we assume that the drop has constant density $\rho \equiv 1$). The second term is the surface energy, where τ is the constant of surface tension. The third term corresponds to the adhesion forces, where Σ_i is the contact surface between the drop and the ith plate, and σ_i/τ is the relative adhesion coefficient at this plate. For an introduction to the physics of surfaces compare Landau and Lifshitz [**13**, §60].

Hamiltonian formulations of fluid dynamical problems play an important role in hydrodynamic stability and bifurcation theory (compare Holm, Marsden, Ratiu and Weinstein [**9**]). Arnold [**2**] showed that the equations of motion of an ideal incompressible fluid filling a fixed domain can be put into Hamiltonian form. Lewis, Marsden, Montgomery and Ratiu [**15**] analysed the case of a free fluid drop moving under the influence of self-gravitation and surface tension. They show that the equations of motion are Hamiltonian and derive (nonlinear) stability results using techniques of Hamiltonian mechanics.

Another important feature of Hamiltonian mechanics is that it allows one to derive equations of motion for physical systems, which have properties corresponding to observations of real-world phenomena. A particularly striking example is the derivation of a certain shallow water equation which has 'peaked' soliton solutions by Camassa and Holm [**4**]. As mentioned above, in the present paper the Hamiltonian formulation allows us to derive an equation for the dynamic contact angle of a moving drop.

Kröner [**11**] studies the flow of a viscous fluid with a free surface along a contact surface and derives an existence result for the corresponding system of partial differential equations. The results cited in Kröner's paper suggest that the usual no-slip boundary condition for viscous fluid is not the correct one for this problem.

In the present paper we are not concerned with the general existence theory for the partial differential equations describing the flow of our Hamiltonian system. Because of this we do not investigate the question of which function spaces one should choose in the Hamiltonian formulation to guarantee the existence of a solution to the equations of motion.

To derive our Poisson bracket, we use the idea of Lewis *et al.* [**15**] to pass from a modified canonical bracket in body representation to a noncanonical bracket in spatial representation via a Marsden–Weinstein reduction. Because we have to deal not only with free boundaries but also with contact surfaces, further modifications of the canonical bracket are necessary. This is important to assure that the natural candidate for the Hamiltonian, the total energy function, is contained in the class of functions $\mathscr{D}$, for which the Poisson bracket is defined. The choice of this class of functions and the definition of functional derivatives is very subtle and there is no simple recipe describing how to choose these objects in our infinite-dimensional system with its free boundaries and contact surfaces. The ideas developed in this paper should be useful in the Hamiltonian analysis of other systems of continuum mechanics with free boundaries and contact surfaces.

Hamiltonian systems with free boundaries are also subjects of research from a more theoretical point of view. Soloviev [**21**] derives criteria under which the Poisson bracket in systems with free boundaries satisfies Jacobi's identity.

In the second part of the paper, which can be read independently, we show the existence of rotationally symmetric solutions to our equations of motion and study their stability. The symmetry is with respect to rotations about an axis perpendicular to the plates. We prove that for every positive constant V and sufficiently small ω there exist solutions of the equations of motion representing drops with volume $< V$ that rotate rigidly with constant angular velocity ω between the two plates.

We investigate the stability of these rotating drops using the energy-momentum method of Lewis, Marsden and Simo [**16**]. In the last section we explain some of the key ideas of this method in our context. We make use of the fact that rigidly rotating drops are critical points of an augmented energy functional V_ω, which is defined on the configuration space of our system. To determine stability, we look at the second derivative of the augmented energy functional at these critical points. (Observe that one can only expect definiteness of the second variation transversal to group orbits of the rotation group, which acts on configuration space as described above and in a canonical way also on phase space.)

As an application, we study the stability of a cylindrical drop with base radius d and height h which rotates rigidly with constant angular velocity ω between the plates. This is a solution to our equations in the absence of adhesion forces, that is, in the case $\sigma_0, \sigma_1 = 0$. We show that this motion is stable if both

$$\frac{h^2}{\pi^2 d^2} + \frac{\omega^2 h^2 d}{\pi^2 \tau} < 1$$

and

$$\frac{3\tau}{d^3} > \omega^2$$

hold. Here τ is, as explained above, the constant of surface tension. This extends a stability result of Vogel [**22**], who considers the static case of a trapped drop at rest between two plates. Notice that, unlike the static situation, one cannot restrict oneself to the study of symmetric perturbations of the drop shape in the case of a rotating drop. Our stability criterion also extends that of Lewis [**14**], who studies 2-dimensional circular drops.

This paper is structured as follows. In Section 2 we describe the geometry of the problem. The Eulerian configuration and phase space are introduced. We introduce also the Lagrangian phase space with its canonical Poisson structure. In Section 3, we discuss the equations of motion of the drop. In Section 4, Poisson structures are defined on Eulerian and Lagrangian phase space. The Hamiltonian structure of the equations of motion is then exhibited in Section 5. The existence theorem stated above is proved in Section 6. Also in Section 6 we discuss the stability of rigidly rotating drops.

2. *The geometry of the problem*

We consider a drop of an ideal incompressible fluid trapped between two parallel flat plates denoted P_0 and P_1 (compare Figure 1). Let h denote the distance between the plates.

Choose in $\mathbf{R}^3$ a Cartesian coordinate system such that plate P_1 lies in the (x, y)-plane and P_0 lies in the plane $z = h$.

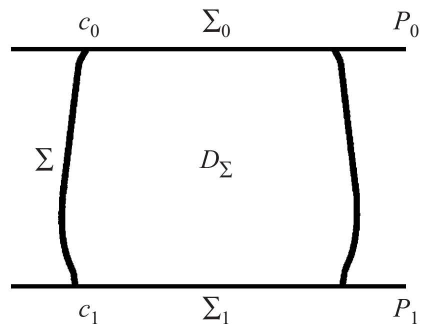

FIGURE 1. *The drop profile.*

There are two ways to describe the instantaneous position of the drop, the *Lagrangian* description and the *Eulerian* description. We first introduce the Lagrangian or *material* description.

We choose as a reference configuration the cylinder

$$D := \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 < 1,\ 0 < z < h\}.$$

The Lagrangian configuration space $\mathscr{C}$ is the set of all volume-preserving embeddings η of the reference configuration D into the region between the two plates, such that the contact surface of D with the plate P_i, that is, $\partial(D) \cap P_i$, is mapped into P_i, $i = 0, 1$. We do not model motions in which the drop loses contact with the plates. It is not clear to us at the moment how to define a Hamiltonian structure in that situation. The assumption that the drop does stay in contact with the plates seems to be physically reasonable if the main forces acting on the drop are surface tension and adhesion forces and the effect of gravity is small compared with them.

Let Σ denote the closure of that part of $\partial(\eta(D))$ which does not touch the plates. We call Σ the *free boundary* of $\eta(D)$. Instead of $\eta(D)$ we also write D_Σ. Let c_i, $i = 0, 1$, denote the contact curve of Σ and the plate P_i and let Σ_i denote the contact surface of D_Σ and P_i. Thus, for $i = 0, 1$ the curve c_i is the boundary of Σ_i in P_i.

Although we have not introduced a manifold structure on $\mathscr{C}$, there is still a notion of tangent bundle of $\mathscr{C}$ using tangents to curves in $\mathscr{C}$. For $\eta \in \mathscr{C}$ we introduce a formal tangent space

$$T_\eta \mathscr{C} = \{\mu : D \longrightarrow \mathbf{R}^3 \mid \operatorname{div}(\mu \circ \eta^{-1}) = 0 \text{ and } \langle \eta \circ \mu(x), (0, 0, 1)^T \rangle = 0 \text{ for } x \in \Sigma_i, i = 0, 1\}.$$

Here $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbf{R}^3$. One arrives at this definition in the following way.

Let $I \longrightarrow \mathscr{C}$, $\lambda \longmapsto c_\lambda$, $c_0 = \eta$, be a motion of the drop, that is, a one-parameter family of functions embedding the reference configuration D into $\mathbf{R}^3$. We assume that the map $(x, \lambda) \longmapsto c_\lambda(x)$ is sufficiently smooth, so that our calculations make sense. One gets the tangent vector $\mu: D \longrightarrow \mathbf{R}^3$ represented by the curve $c: I \longrightarrow \mathscr{C}$ by taking the derivative with respect to λ at $\lambda = 0$:

$$\mu(d) := \frac{d}{d\lambda}\bigg|_{\lambda=0} c_\lambda(d).$$

If $x = \eta(d)$, then $\mu(d)$ is the fluid velocity at the spatial point x at 'time' $\lambda = 0$. Thus, putting

$$\mu_\lambda := \frac{d}{d\sigma}\bigg|_{\sigma=\lambda} c_\sigma$$

$$(x, \lambda) \longmapsto \mu_\lambda \circ c_\lambda^{-1}(x) =: v(x, \lambda)$$

can be interpreted as the spatial velocity field of the motion $\lambda \longmapsto c_\lambda$. Because the maps c_λ are volume preserving, we have

$$\operatorname{div} \mu \circ \eta^{-1} = 0.$$

From the definition of $\mathscr{C}$ and the fact that

$$\partial(c_\lambda(D)) = c_\lambda(\partial(D))$$

it follows that $\langle \mu \circ \eta^{-1}(x), (0,0,1)^T \rangle = 0$ for $x \in \Sigma_i$, $i = 0, 1$.

The elements of the *Eulerian configuration space* $\mathscr{M}$ are those two-dimensional submanifolds of $\mathbf{R}^3$ which arise as free boundaries for some $\eta(D)$, $\eta \in \mathscr{C}$. We might describe an element of $\mathscr{M}$ intrinsically as a two-dimensional submanifold Σ of $\mathbf{R}^3$ with boundary where the boundary consists of two components c_0, c_1 each one isomorphic to S^1. The intersection of the two-dimensional manifold Σ with the plate P_i should be the component c_i of the boundary, $i = 0, 1$. Furthermore, the volume enclosed by Σ and the two plates has to be equal to the volume of the reference configuration D. Note that the conditions on the boundary of Σ reflect the fact that we do not study motions in which the drop loses contact with the plates.

For $\Sigma \in \mathscr{M}$ we introduce the formal tangent space

$$T_\Sigma \mathscr{M} = \{\kappa: \Sigma \longrightarrow \mathbf{R}^3 | \int_\Sigma \langle \kappa, n \rangle \, \mathrm{ds} = 0 \text{ and}$$
$$\langle \kappa, (0,0,1)^T \rangle = 0 \text{ on } c_i, i = 0, 1\}.$$

Here $n: \Sigma \longrightarrow \mathbf{R}^3$ denotes a vector field of outer unit normal vectors to the surface Σ. To motivate this definition, choose a curve $t \longmapsto \eta_t$ in $\mathscr{C}$, such that Σ is the free boundary of $\eta_0 =: \eta$. Then

$$\kappa := \frac{d}{dt}\bigg|_{t=0} \eta_t \circ \eta^{-1}$$

satisfies $\int_\Sigma \langle \kappa, n \rangle \, \mathrm{ds} = 0$ and $\langle \kappa, (0,0,1)^T \rangle = 0$ on c_i for $i = 0, 1$, because elements of $\mathscr{C}$ are volume preserving and preserve contact surfaces with the plates.

Let $\mathscr{G}$ denote the group of volume-preserving automorphisms of the reference configuration D. The group $\mathscr{G}$ acts on $\mathscr{C}$ on the right. By definition of the Eulerian configuration space $\mathscr{M}$ and the Lagrangian configuration space $\mathscr{C}$, one has a natural bijection

$$\mathscr{M} \cong \mathscr{C}/\mathscr{G}.$$

The phase space in the Lagrangian description is the tangent bundle

$$T\mathscr{C} = \{(\eta, \mu) \,|\, \eta \in \mathscr{C}, \mu \in T_\eta \mathscr{C}\}.$$

For $\Sigma \in \mathscr{M}$ let V_Σ be the vector space of all functions $v: D_\Sigma \longrightarrow \mathbf{R}^3$ such that

$$\operatorname{div} v = 0 \tag{2.1}$$

and

$$\langle v, (0,0,1)^T \rangle = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1. \tag{2.2}$$

The phase space in the Eulerian description is

$$\mathscr{N} = \{(\Sigma, v) \mid \Sigma \in \mathscr{M} \text{ and } v \in V_\Sigma\}.$$

We may interpret $v \in V_\Sigma$ as a spatial velocity field on D_Σ. The connection between $T\mathscr{C}$ and $\mathscr{N}$ is given by the map

$$\begin{aligned} \pi_{\mathscr{N}}: T\mathscr{C} &\longrightarrow \mathscr{N} \\ (\eta, \mu) &\longmapsto (\Sigma_\eta, \mu \circ \eta^{-1}) \end{aligned}$$

where Σ_η denotes the free boundary of $\eta(D)$. The action of $\mathscr{G}$ on $\mathscr{C}$ induces an action of $\mathscr{G}$ on $T\mathscr{C}$ via

$$\begin{aligned} \Phi: \mathscr{G} \times T\mathscr{C} &\longrightarrow T\mathscr{C} \\ (g, (\eta, \mu)) &\longmapsto (\eta \circ g, \mu \circ g). \end{aligned}$$

It is easy to check that there is a canonical bijective map between $T\mathscr{C}/\mathscr{G}$ and $\mathscr{N}$:

$$\mathscr{N} \cong T\mathscr{C}/\mathscr{G}.$$

3. *The equations of motion*

The equations of motion for the drop are

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v = -\nabla p \tag{3.1}$$

$$\operatorname{div} v = 0 \tag{3.2}$$

$$\langle v, (0,0,1)^T \rangle = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1 \tag{3.3}$$

$$\langle \dot{\Sigma}, n \rangle = \langle v, n \rangle \tag{3.4}$$

$$p = \tau\kappa \quad \text{on } \mathring{\Sigma} \tag{3.5}$$

$$\cos\gamma_i = \frac{\sigma_i}{\tau} \quad \text{on } c_i,\ i = 0, 1 \tag{3.6}$$

where τ is the constant of surface tension and κ is the mean curvature function of the free surface Σ of the drop.

Physical reasoning shows that τ has to be positive (compare [Landau and Lifshitz [**12**, §139]).

Equations (3.1) and (3.2) are the well-known Euler equations for the spatial velocity field of an ideal incompressible fluid. We assume for simplicity that the density of the drop is equal to 1. Equation (3.3) is the usual tangential (slip) boundary condition for the Euler equations. We study only motions in which the drop does not separate from the plates. Equation (3.4) is a kinematic boundary condition describing the fact that the free boundary of the drop is moving with the fluid. The dot denotes differentiation with respect to the time parameter t.

In equation (3.6), c_i denotes, as explained above, the boundary of Σ_i in P_i; $n_i(x)$ is the outer unit normal of Σ_i at $x \in c_i$ and $w_i(x)$ is the outer conormal on Σ at $x \in c_i$. This means that $w_i(x) \in T_x\Sigma$, $\langle w_i(x), t \rangle = 0$ for $t \in T_x c_i \subseteq \mathbf{R}^3$. Furthermore, $w_i(x)$ has

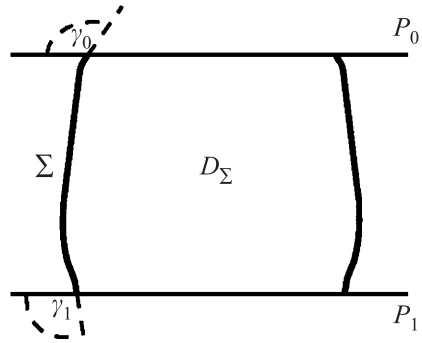

FIGURE 2. *The contact angles between drop and plates.*

length 1 and points outwards of Σ (compare Simon [**20**] for the definition of conormal vectors). Also $\gamma_i(x)$ denotes the angle between $n_i(x)$ and $w_i(x)$ (compare Figure 2); σ_1/τ is the relative adhesion coefficient (compare Finn [**6**]).

Note that (3.6) is just the classic contact angle condition for a static drop, which has been derived by Gauss with the help of variational principles (compare Finn [**6**]). To our knowledge there is no generally accepted model for the *dynamic* contact angle yet. We regard it as a rather striking result that the Hamiltonian formulation of the problem, given in Section 5 below, allows us to derive equation (3.6) for the dynamic contact angles. Hocking [**8**] arrives by a different kind of reasoning at this result. In his paper and in the references therein one finds a thorough discussion of rival contact angle models.

4. *Poisson structures and reduction*

We now show that equations (3.1)–(3.6) are Hamiltonian.

Let $\mathscr{F}_{\mathscr{N}}$ denote the set of real-valued functions on $\mathscr{N}$. We will find

(1) $\mathscr{D} \subseteq \mathscr{F}_{\mathscr{N}}$;

(2) $\{\cdot,\cdot\}_{\mathscr{N}}: \mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{F}_{\mathscr{N}}$;

(3) $\mathrm{H} \in \mathscr{D}$;

such that equations (3.1)–(3.6) can be written in the form

$$\dot{F} = \{F, H\}_{\mathscr{N}} \quad \text{for all } F \in \mathscr{D}.$$

We call $\mathscr{D}$ the set of *admissible functions*; $\{\cdot,\cdot\}_{\mathscr{N}}$ is the *Poisson bracket* and H is the *Hamiltonian* for our system. Similarly to Lewis *et al*. [**15**], we get $\{\cdot,\cdot\}_{\mathscr{N}}$ by applying a Marsden–Weinstein reduction to an appropriate modification of the canonical Poisson bracket on $T\mathscr{C}$. Our situation differs from that considered by Lewis *et al*. because the free surface of our drop is not a closed surface. This affects the definition of our functional derivatives. Unlike the situation for a free drop we have to deal with functions which are integrals over the contact surfaces between the drop and the plates. The modified Poisson bracket will be defined for functions $F, G: \mathscr{N} \longrightarrow \mathbf{R}$ possessing functional derivatives defined as follows.

Let $F: T\mathscr{C} \longrightarrow \mathbf{R}$, $c: I \longrightarrow \mathscr{C}$ be a curve with $c_0 =: \eta$ and

$$\left.\frac{d}{dt}\right|_{t=0} c_t =: \delta\eta.$$

Let $D_\Sigma := \eta(D)$.

DEFINITION 4.1. F has functional derivatives with respect to η in $(\eta, \mu) \in T\mathscr{C}$ if there exist functions

$$\frac{\delta F}{\delta \eta}(\eta, \mu): D_\Sigma \longrightarrow \mathbf{R}^3$$

$$\frac{\delta' F}{\delta \eta}(\eta, \mu): \Sigma \longrightarrow \mathbf{R}^3$$

$$\frac{\delta_i F}{\delta \eta}(\eta, \mu): c_i \longrightarrow \mathbf{R}^3 \quad i = 0, 1$$

such that

$$\left.\frac{d}{dt}\right|_{t=0} F(c_t, \mu) = \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta \eta}(\eta, \mu), \delta\eta \circ \eta^{-1} \right\rangle dV + \int_\Sigma \left\langle \frac{\delta' F}{\delta \eta}(\eta, \mu), \delta\eta \circ \eta^{-1} \right\rangle dA + \sum_{i=0}^{1} \int_{c_i} \left\langle \frac{\delta_i F}{\delta \eta}(\eta, \mu), \delta\eta \circ \eta^{-1} \right\rangle ds.$$

DEFINITION 4.2. F has a functional derivative with respect to μ in $(\eta, \mu) \in T\mathscr{C}$ if there exists a function

$$\frac{\delta F}{\delta \mu}(\eta, \mu): D_\Sigma \longrightarrow \mathbf{R}^3$$

with

$$\operatorname{div}\left(\frac{\delta F}{\delta \mu}(\eta, \mu)\right) = 0 \tag{4.1}$$

and

$$\left\langle \frac{\delta F}{\delta \mu}(\eta, \mu)(x), (0, 0, 1)^T \right\rangle = 0 \quad \text{for } x \in \Sigma_i,\ i \in 0, 1 \tag{4.2}$$

such that for every curve $d: I \longrightarrow T_\eta \mathscr{C}$ with $d(0) = \mu$ and

$$\left.\frac{d}{dt}\right|_{t=0} d_t =: \delta\mu$$

we have

$$\left.\frac{d}{dt}\right|_{t=0} F(\eta, d_t) = \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta \mu}(\eta, \mu), \delta\mu \circ \eta^{-1} \right\rangle dV.$$

Let $\mathscr{F}$ denote the set of functions $F: T\mathscr{C} \longrightarrow \mathbf{R}$ which possess functional derivatives with respect to η and μ at every point $(\eta, \mu) \in T\mathscr{C}$.

We now define a Poisson bracket on $T\mathscr{C}$, a map

$$\{\cdot, \cdot\}_{T\mathscr{C}}: \mathscr{F} \times \mathscr{F} \longrightarrow \{f: T\mathscr{C} \longrightarrow \mathbf{R}\}$$

which maps a pair (F, H) to $\{F, H\}_{T\mathscr{C}}$. We also write just $\{F, H\}$ instead of $\{F, H\}_{T\mathscr{C}}$.

DEFINITION 4.3. We define the Poisson bracket $\{F, H\}: T\mathscr{C} \longrightarrow \mathbf{R}$ of two functions $F, H \in \mathscr{F}$ by

$$\{F, H\}(\eta, \mu) := \int_{D_\Sigma} \left[\left\langle \frac{\delta F}{\delta \eta}, \frac{\delta H}{\delta \mu} \right\rangle - \left\langle \frac{\delta H}{\delta \eta}, \frac{\delta F}{\delta \mu} \right\rangle \right] dV + \int_\Sigma \left[\left\langle \frac{\delta' F}{\delta \eta}, \frac{\delta H}{\delta \mu} \right\rangle - \left\langle \frac{\delta' H}{\delta \eta}, \frac{\delta F}{\delta \mu} \right\rangle \right] dA + \sum_{i=0}^{1} \int_{c_i} \left[\left\langle \frac{\delta_i F}{\delta \eta}, \frac{\delta H}{\delta \mu} \right\rangle - \left\langle \frac{\delta_i H}{\delta \eta}, \frac{\delta F}{\delta \mu} \right\rangle \right] ds.$$

Here functional derivatives are taken at the point $(\eta, \mu) \in T\mathscr{C}$.

REMARK 4.1. The functional derivative with respect to μ is uniquely determined: let $a, b\colon D_\Sigma \longrightarrow \mathbf{R}^3$ be functional derivatives with respect to μ of $F\colon T\mathscr{C} \longrightarrow \mathbf{R}$ at $(\eta, \mu) \in T\mathscr{C}$. Consider the curve $d\colon t \longrightarrow \mu + t(a-b) \circ \eta$. By definition we have

$$\begin{aligned} 0 &= \int_{D_\Sigma} \left\langle a, \frac{d}{dt}\bigg|_{t=0} d_t \circ \eta^{-1} \right\rangle dV - \int_{D_\Sigma} \left\langle b, \frac{d}{dt}\bigg|_{t=0} d_t \circ \eta^{-1} \right\rangle dV \\ &= \int_{D_\Sigma} \langle a-b, a-b \rangle \, dV \end{aligned}$$

and so $a = b$.

REMARK 4.2. Let $F, H \in \mathscr{F}$. By definition $(\delta F/\delta\mu)(\eta, \mu) \circ \eta \in T_\eta \mathscr{C}$. If there are curves $t \longmapsto c_t$, $c_0 = \eta$ and $t \longmapsto e_t$, $e_0 = \eta$, such that

$$\frac{d}{dt}\bigg|_{t=0} c_t = \frac{\delta F}{\delta \mu}(\eta, \mu) \circ \eta \quad \frac{d}{dt}\bigg|_{t=0} e_t = \frac{\delta H}{\delta \mu}(\eta, \mu) \circ \eta,$$

then we have

$$\{F, H\}_{(\eta,\mu)} = \frac{d}{dt}\bigg|_{t=0} F(e_t, \mu) - \frac{d}{dt}\bigg|_{t=0} H(c_t, \mu). \tag{4.3}$$

Thus, if we can find representing curves for every element of $T_\eta \mathscr{C}$, the Poisson bracket is well defined by (4.3) and Remark 4.1.

Formula (4.3) can be used to define the Poisson bracket of functions F, H: $T\mathscr{C} \longrightarrow \mathbf{R}$, which possess functional derivatives with respect to μ, but not necessarily with respect to η.

REMARK 4.3. A computation shows that if $F, H \in \mathscr{F}$ and $g \in \mathscr{G}$, then $F \circ \Phi_g$, $H \circ \Phi_g \in \mathscr{F}$ and

$$\{F, H\} \circ \Phi_g = \{F \circ \Phi_g, H \circ \Phi_g\}.$$

Thus, each of the maps Φ_g is Poisson.

We now use a Poisson reduction to get a Poisson bracket on $\mathscr{N} = T\mathscr{C}/\mathscr{G}$. Let $F, H\colon \mathscr{N} \longrightarrow \mathbf{R}$ be two functions such that their pullbacks to $T\mathscr{C}$, $\bar{F} := F \circ \pi_{\mathscr{N}}$, $\bar{H} := H \circ \pi_{\mathscr{N}}$, possess functional derivatives with respect to η and μ.

DEFINITION 4.4. We define the Poisson bracket $\{F, H\}_{\mathscr{N}}\colon \mathscr{N} \longrightarrow \mathbf{R}$ of F and H by

$$\{F, H\}_{\mathscr{N}}\, \pi_{\mathscr{N}}(\eta, \mu) := \{\bar{F}, \bar{H}\}_{T\mathscr{C}}(\eta, \mu)$$

for $\eta, \mu \in T\mathscr{C}$.

By Remark 4.3, $\{F, H\}_{\mathscr{N}}$ is well defined. We also write just $\{F, H\}$ instead of $\{F, H\}_{\mathscr{N}}$.

We want to specify a class $\mathscr{D}$ of functions $F\colon \mathscr{N} \longrightarrow \mathbf{R}$ such that the pullbacks $F \circ \pi_{\mathscr{N}}\colon T\mathscr{C} \longrightarrow \mathbf{R}$ are in $\mathscr{F}$. These will be functions which possess functional derivatives with respect to Σ and v. Let $\Sigma \in \mathscr{M}$. Let $t \longrightarrow c_t$ be a curve in $\mathscr{M}$ with $c_0 = \Sigma$ and

$$\delta\Sigma := \frac{d}{dt}\bigg|_{t=0} c_t.$$

We consider $\delta\Sigma$ as a map $\Sigma \longrightarrow \mathbf{R}^3$.

DEFINITION 4.5. $F: \mathcal{N} \longrightarrow \mathbf{R}$ has a functional derivative with respect to Σ at $(\Sigma, v) \in \mathcal{N}$, if there exist functions

$$\frac{\delta F}{\delta \Sigma}(\Sigma, v): \Sigma \longrightarrow \mathbf{R}$$

$$\frac{\delta_i F}{\delta \Sigma}(\Sigma, v): c_i \longrightarrow \mathbf{R}, \quad i = 0, 1$$

such that

$$\left.\frac{d}{dt}\right|_{t=0} F(\Sigma_t, v) = \int_\Sigma \frac{\delta F}{\delta \Sigma}(\Sigma, v) \langle \delta\Sigma, n \rangle \, dA + \sum_{i=0}^{1} \int_{c_i} \frac{\delta_i F}{\delta \Sigma}(\Sigma, v) \langle \delta\Sigma, n_i \rangle \, ds.$$

As above, n denotes an outer unit normal vector field on Σ and n_i, $i = 0, 1$, is an outer unit normal vector field to c_i regarded as a submanifold of P_i. Let V_Σ denote the space of vector fields $v: D_\Sigma \longrightarrow \mathbf{R}^3$ satisfying

$$\operatorname{div} v = 0 \tag{4.4}$$

and

$$\langle v, (0, 0, 1)^T \rangle = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1. \tag{4.5}$$

Let $t \longrightarrow v_t$ be a curve in V_Σ and

$$\delta v := \left.\frac{d}{dt}\right|_{t=0} v_t.$$

DEFINITION 4.6. $F: \mathcal{N} \longrightarrow \mathbf{R}$ has a functional derivative with respect to v at $(\Sigma, v) \in \mathcal{N}$ if there exists a vector field

$$\frac{\delta F}{\delta v}(\Sigma, v) \in V_\Sigma$$

such that

$$\left.\frac{d}{dt}\right|_{t=0} F(\Sigma, v_t) = \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}(\Sigma, v), \delta v \right\rangle dV.$$

DEFINITION 4.7. Let $\mathscr{D}$ denote the set of functions $F: \mathcal{N} \longrightarrow \mathbf{R}$ that possess functional derivatives with respect to Σ and to v at every $(\Sigma, v) \in \mathcal{N}$.

CLAIM 4.1. *If $F \in \mathscr{D}$, then $\bar{F} := F \circ \pi_{\mathcal{N}} \in \mathscr{F}$, where $\pi_{\mathcal{N}}: T\mathscr{C} \longrightarrow \mathcal{N}$ denotes the canonical projection.*

Proof. Let $(\eta, \mu) \in T\mathscr{C}$ and $(\Sigma, v) = \pi_{\mathcal{N}}(\eta, \mu)$. Let $t \longmapsto c_t$, $c_0 = \eta$ and

$$\delta\eta = \left.\frac{d}{dt}\right|_{t=0} c_t.$$

An application of the chain rule gives

$$D_\eta \bar{F}(\eta, \mu) = D_\Sigma F(\Sigma, v) \, (D\Sigma(\eta, \mu) \, \delta\eta) + D_v F(\Sigma, v) \, (Dv(\eta, \mu) \, \delta\eta).$$

We have

$$D\Sigma(\eta,\mu)\,\delta\eta = \frac{d}{dt}\bigg|_{t=0} c_t\circ\eta^{-1}$$
$$= \delta\eta\circ\eta^{-1}. \tag{4.6}$$

Because of $v(c_t,\mu)(x) = \mu\circ c_t^{-1}(x)$ we have

$$(Dv(\eta,\mu)\,\delta\eta)(x) = -(\delta\eta\circ\eta^{-1}\cdot\nabla)\,v(x). \tag{4.7}$$

Thus,

$$D_\eta\bar{F}\delta\eta = D_\Sigma F(\Sigma,v)\,\delta\eta\circ\eta^{-1} - D_v F(\Sigma,v)\,(\nabla v(\delta\eta\circ\eta^{-1})). \tag{4.8}$$

Now $F\in\mathscr{D}$, so F possesses functional derivatives with respect to Σ and to v. Then (4.8) becomes

$$\begin{aligned} D_\eta\bar{F}\delta\eta = &\int_\Sigma \frac{\delta F}{\delta\Sigma}(\Sigma,v)\,\langle n,\delta\eta\circ\eta^{-1}\rangle\,dA \\ &+\sum_{i=0}^{1}\int_{c_i}\frac{\delta_i F}{\delta\Sigma}(\Sigma,v)\,\langle n_i,\delta\eta\circ\eta^{-1}\rangle\,ds \\ &-\int_{D_\Sigma}\left\langle\frac{\delta F}{\delta v}(\Sigma,v),\ (\nabla v(\delta\eta\circ\eta^{-1}))\right\rangle dV. \end{aligned} \tag{4.9}$$

Equation (4.9) shows that $\bar{F}$ has a functional derivative with respect to η:

$$\frac{\delta\bar{F}}{\delta\eta}(\eta,\mu) = -(\nabla v)^T\frac{\delta F}{\delta v}$$
$$\frac{\delta'\bar{F}}{\delta\eta}(\eta,\mu) = \frac{\delta F}{\delta\Sigma}(\Sigma,v)\,n$$
$$\frac{\delta_i\bar{F}}{\delta\eta}(\eta,\mu) = \frac{\delta_i F}{\delta\Sigma}(\Sigma,v)\,n_i.$$

One easily shows that $\bar{F}$ also has a functional derivative with respect to μ, given by

$$\frac{\delta\bar{F}}{\delta\mu}(\eta,\mu) = \frac{\delta F}{\delta v}(\Sigma,v).$$

Thus, $\bar{F}\in\mathscr{F}$. □

For functions $F,H\in\mathscr{D}$, the Poisson bracket $\{F,H\}$ can be written explicitly. We have

$$\begin{aligned} \{F,H\}(\Sigma,v) = &\int_{D_\Sigma} -\left(\left\langle\frac{\delta F}{\delta v},\nabla v\frac{\delta H}{\delta v}\right\rangle - \left\langle\frac{\delta H}{\delta v},\nabla v\frac{\delta F}{\delta v}\right\rangle\right)dV \\ &+\int_\Sigma\left(\frac{\delta F}{\delta\Sigma}\left\langle\frac{\delta H}{dv},n\right\rangle - \frac{\delta H}{\delta\Sigma}\left\langle\frac{\delta F}{\delta v},n\right\rangle\right)dA \\ &+\sum_{i=0}^{1}\int_{c_i}\left(\frac{\delta_i F}{\delta\Sigma}\left\langle\frac{\delta H}{\delta v},n_i\right\rangle - \frac{\delta_i H}{\delta\Sigma}\left\langle\frac{\delta F}{\delta v},n_i\right\rangle\right)ds. \end{aligned} \tag{4.10}$$

Here the functional derivatives on the right-hand side of (4.10) are evaluated at (Σ, v).

Putting $\omega := \nabla \times v$, a simple calculation shows that

$$\begin{aligned}\{F, H\} &= \int_{D_\Sigma} \left\langle \omega, \frac{\delta F}{\delta v} \times \frac{\delta H}{\delta v} \right\rangle dV \\ &+ \int_\Sigma \left(\frac{\delta F}{\delta \Sigma} \left\langle \frac{\delta H}{\delta v}, n \right\rangle - \frac{\delta H}{\delta \Sigma} \left\langle \frac{\delta F}{\delta v}, n \right\rangle \right) dA \\ &+ \sum_{i=0}^{1} \int_{c_i} \left(\frac{\delta_i F}{\delta \Sigma} \left\langle \frac{\delta H}{\delta v}, n_i \right\rangle - \frac{\delta_i H}{\delta \Sigma} \left\langle \frac{\delta F}{\delta v}, n_i \right\rangle \right) ds. \end{aligned} \tag{4.11}$$

5. *Equivalence of the classical and the Hamiltonian formulation of the equations of motion*

We will show that a curve $t \longmapsto (\Sigma_t, v_t) \in \mathcal{N}$ satisfies the equations (3.1)–(3.6) if and only if

$$\frac{d}{dt} F(\Sigma_t, v_t) = \{F, H\}(\Sigma_t, v_t) \tag{5.1}$$

for all $F \in \mathscr{D}$. We abbreviate (5.1) as

$$\dot{F} = \{F, H\}. \tag{5.2}$$

Here

$$H = \frac{1}{2} \int_{D_\Sigma} \|v\|^2 dV + \tau \int_\Sigma dA - \sum_{i=0}^{1} \sigma_i \int_{\Sigma_i} dA_i \tag{5.3}$$

is the total energy function of the system. It is easy to show that $H \in \mathscr{D}$ and

$$\frac{\delta H}{\delta \Sigma}(\Sigma, v) = (\tfrac{1}{2}\|v\|^2 + \tau\kappa) \tag{5.4}$$

$$\frac{\delta_i H}{\delta \Sigma}(\Sigma, v) = (\tau \cos\gamma_i - \sigma_i), \quad i = 0, 1 \tag{5.5}$$

$$\frac{\delta H}{\delta v}(\Sigma, v) = v. \tag{5.6}$$

To derive (5.5), we use the transport theorem for shells of [**19**, Chapter 2] and the divergence theorem on the manifold Σ with boundary $c_0 \cup c_1$. Remember that κ denotes the mean curvature function of Σ. Applying this in equation (4.11), we get

$$\begin{aligned}\{F, H\} &= \int_{D_\Sigma} \left\langle \omega, \frac{\delta F}{\delta v} \times v \right\rangle dV + \int_\Sigma \frac{\delta F}{\delta \Sigma}(\Sigma, v) \langle v, n \rangle \, dA \\ &- \int_\Sigma [\tfrac{1}{2}\|v\|^2 + \tau\kappa] \left\langle \frac{\delta F}{\delta v}(\Sigma, v), n \right\rangle dA + \sum_{i=0}^{1} \int_{c_i} \frac{\delta_i F}{\delta \Sigma}(\Sigma, v) \langle v, n_i \rangle \, ds \\ &- \sum_{i=0}^{1} \int_{c_i} \{\tau \cos\gamma_i - \sigma_i] \left\langle \frac{\delta F}{\delta v}(\Sigma, v), n_i \right\rangle ds. \end{aligned} \tag{5.7}$$

First we show that a solution $t \longmapsto (\Sigma_t, v_t) \in \mathcal{N}$ of equations (3.1)–(3.6) also solves Hamilton's equations (5.1). With the help of the divergence theorem and equations (3.5) and (3.6) one gets

$$\{F, H\} = \int_{D_\Sigma} \left\langle v_t \times \omega_t - \tfrac{1}{2}\nabla \|v\|^2 - \nabla p, \frac{\delta F}{\delta v}(\Sigma_t, v_t) \right\rangle dV + \int_\Sigma \frac{\delta F}{\delta \Sigma}(\Sigma_t, v_t) \langle v_t, n_t \rangle \, dA + \sum_{i=0}^{1} \int_{c_i} \frac{\delta_i F}{\delta \Sigma}(\Sigma_t, v_t) \langle v_t, n_{it} \rangle \, ds.$$

Making use of equation (3.4) and the identity

$$-(w \cdot \nabla) w = w \times (\nabla \times w) - \tfrac{1}{2}\nabla \|w\|^2 \tag{5.8}$$

for a vector field $w: D_\Sigma \longrightarrow \mathbf{R}^3$, one has

$$\{F, H\} = \int_{D_\Sigma} \left\langle -(v_t \cdot \nabla) v_t - \nabla p, \frac{\delta F}{\delta v}(\Sigma_t, v_t) \right\rangle dV + \int_\Sigma \frac{\delta F}{\delta \Sigma}(\Sigma_t, v_t) \langle \dot{\Sigma}, n \rangle \, dA + \sum_{i=0}^{1} \int_{c_i} \frac{\delta_i F}{\delta \Sigma}(\Sigma_t, v_t) \langle \dot{\Sigma}, n_i \rangle \, ds.$$

Equation (3.1) then yields

$$\begin{aligned} \{F, H\} &= \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}(\Sigma, v), \dot{v} \right\rangle dV + \int_\Sigma \frac{\delta F}{\delta \Sigma} \langle \dot{\Sigma}, n \rangle \, dA + \sum_{i=0}^{1} \int_{c_i} \frac{\delta F}{\delta v} \langle \dot{\Sigma}, n_i \rangle \, ds \\ &= D_v F(\Sigma, v)\, \dot{v} + D_\Sigma F(\Sigma, v) \dot{\Sigma} \\ &= \dot{F}(\Sigma, v). \end{aligned}$$

Thus (5.1) holds for every $F \in \mathcal{D}$ along solution curves of equations (3.1)–(3.6).

Now we want to show that a solution curve of Hamilton's equations (5.1) also solves equations (3.1)–(3.6). To do so, we first prove two lemmas which will be needed later on.

LEMMA 5.1. *There exist $d \in \mathbf{R}^+$ and a family $(p_r)_{r \in (0,1)}$ of functions $p_r: \mathbf{R} \longrightarrow \mathbf{R}$ such that*

$$\operatorname{supp}(p_r) \subseteq [1-r, 1+r] \tag{5.9}$$

$$|p_r| \leqslant 1 \tag{5.10}$$

$$|p_r'| \leqslant d \tag{5.11}$$

$$p_r(1) = 0 \tag{5.12}$$

$$p_r'(1) = 1. \tag{5.13}$$

Proof. Define a bump function $g \in C_c^\infty(R)$ by

$$g(t) = \begin{cases} \exp\left(-\dfrac{1}{1-t^2}\right) & \text{for } |t| < 1 \\ 0 & \text{for } |t| \geqslant 1. \end{cases}$$

Let

$$h(t) := \frac{t g(t)}{g(0)}.$$

It is easy to check that $p_r(t) := rh((t-1)/r)$ fulfils the requirements. □

We need Lemma 5.1 in the proof of Lemma 5.2.

LEMMA 5.2. *Let* $\|\cdot\|$ *denote the Euclidean norm on* $\mathbf{R}^3$, *with notations as in Section* 3. *Let* $i \in \{0,1\}$ *and* $p \in c_i$. *Then in every neighbourhood* $U \subseteq P_i$ *of* p *there exists a neighbourhood* $V \subseteq U$ *of* p *and a family of vector fields*

$$w_r : \mathbf{R}^3 \longrightarrow \mathbf{R}^3, \quad r \in (0,1) \tag{5.14}$$

and a $K \in \mathbf{R}^+$, *such that*

$$\operatorname{supp}(w_r) \subseteq V \times [1-i-r, 1-i+r] \tag{5.15}$$

$$\max_{q \in \mathbf{R}^3} \|w_r(q)\| \leqslant K \tag{5.16}$$

$$\operatorname{div}(w_r) = 0 \tag{5.17}$$

$$\langle w_r, (0,0,1)^T \rangle = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1 \tag{5.18}$$

$$\langle w_r(p), n_i(p) \rangle > 0 \tag{5.19}$$

$$\langle w_r, n_i \rangle \geqslant 0 \quad \text{on } c_i. \tag{5.20}$$

Proof. Without restriction of generality, we assume that the distance between the two plates is equal to 1. We only prove the case $i = 0$; the case $i = 1$ is treated in the same way. Let

$$n_0(p) = (n_{01}(p), n_{02}(p), 0).$$

We assume that

$$n_{01}(p) > 0.$$

The other cases $n_{01} < 0$, $n_{02} > 0$ and $n_{02} < 0$ are treated similarly. As n_0 is continuous, there exists $l > 0$ and an open ball $B_l(p) := \{q \in P_0 \mid \|q-p\| < 1\}$ around p such that $B_l(p) \subseteq U$ and $n_{01}(q) > 0$ for $q \in c_0 \cap B_l(p)$. Let $V := B_l(p)$. Let $k \in \mathbf{R}^+$ such that $0 < k < l$. With the help of [**1**, Lemma 2.2.7] we choose a bump function $h : \mathbf{R}^2 \longrightarrow \mathbf{R}^+$ with $h(q) = 1$ for $q \in B_k(p)$ and $h(q) = 0$ for $q \notin B_l(p)$. The proof of that lemma shows that we may assume that $h \geqslant 0$. Define $f : \mathbf{R}^3 \longrightarrow \mathbf{R}^+$ by

$$f(x,y,z) = h(x,y)$$

so that $\operatorname{supp}(f) \subseteq V \times \mathbf{R}$. We define $w_r : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ *by*

$$w_r(x,y,z) := \left(f(x,y,z) \cdot p_r'(z), 0, -\frac{\partial f}{\partial x}(x,y,z) \cdot p_r(z) \right)$$

where p_r is as in Lemma 5.1 and the prime denotes the derivative with respect to z. Equation (5.17) is obviously satisfied by w_r. Because p_r satisfies (5.9) and (5.12), equation (5.18) holds. By (5.12) and (5.13) we have for $q \in V \cap c_0$,

$$w_r(q) = (f(q), 0, 0)^T \tag{5.21}$$

and so $\langle w_r(q), n_0(q) \rangle = f(q) \cdot n_{01}(q) \geqslant 0$ for $q \in V \cap c_0$ and $\langle w_r(p), n_0(p) \rangle > 0$ by the choice of V and f. This shows that (5.19) and (5.20) hold. Equation (5.16) is satisfied because of (5.10) and (5.11) and the fact that h has compact support. □

We have seen above that every 'classical' solution of equations (3.1)–(3.6) satisfies Hamilton's equations (5.1). We now want to show that the converse is also true. To do so, let

$$I \longrightarrow \mathcal{N}, \quad t \longmapsto (\Sigma_t, v_t)$$

be a solution to the equations (5.1). Let $p : \{(x,t) \in \mathbf{R}^4 \mid x \in D_{\Sigma_t}, t \in I\} \longrightarrow \mathbf{R}$ solve (for fixed t)

$$\Delta p = -\operatorname{div}((v \cdot \nabla) v)$$

with

$$p = \tau\kappa \quad \text{on } \Sigma$$

and

$$\langle \nabla p, (0,0,1)^T \rangle = -\langle (v \cdot \nabla) v, (0,0,1)^T \rangle \quad \text{on } \Sigma_0 \cup \Sigma_1. \tag{5.22}$$

Then

$$\int_\Sigma \tau\kappa \left\langle \frac{\delta F}{\delta v}, n \right\rangle dA = \int_{D_\Sigma} \operatorname{div}\left(p \frac{\delta F}{\delta v} \right) dV$$
$$= \int_{D_\Sigma} \left\langle \nabla p, \frac{\delta F}{\delta v} \right\rangle dV$$

for $F \in \mathscr{D}$. We apply this in (5.7) to get

$$\{F, H\} = \int_{D_\Sigma} \left\langle v \times \omega - \nabla p, \frac{\delta F}{\delta v} \right\rangle dV$$
$$+ \int_\Sigma \frac{\delta F}{\delta \Sigma} \langle v, n \rangle \, dA - \int_\Sigma \frac{1}{2} \|v\|^2 \left\langle \frac{\delta F}{\delta v}, n \right\rangle dA$$
$$+ \sum_{i=0}^{1} \int_{c_i} \frac{\delta_i F}{\delta \Sigma} \langle v, n_i \rangle \, ds - \sum_{i=0}^{1} \int_{c_i} [\tau \cos\gamma_i - \sigma_i] \left\langle \frac{\delta F}{\delta v}, n_i \right\rangle ds.$$

A simple calculation yields

$$\{F, H\} = \int_{D_\Sigma} \left\langle -(v \cdot \nabla) v - \nabla p, \frac{\delta F}{\delta v} \right\rangle dV$$
$$+ \int_\Sigma \frac{\delta F}{\delta \Sigma} \langle v, n \rangle \, dA + \sum_{i=0}^{1} \int_{c_i} \frac{\delta_i F}{\delta \Sigma} \langle v, n_i \rangle \, ds$$
$$- \sum_{i=0}^{1} \int_{c_i} [\tau \cos\gamma_i - \sigma_i] \left\langle \frac{\delta F}{\delta v}, n_i \right\rangle ds \tag{5.23}$$

where we used the identity (5.8). Because $F \in \mathscr{D}$ we have

$$\dot{F} = \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}, \dot{v} \right\rangle dV + \int_\Sigma \frac{\delta F}{\delta \Sigma} \left\langle \frac{\partial \Sigma}{\partial t}, n \right\rangle dA + \sum_{i=0}^{1} \int_{c_i} \frac{\delta_i F}{\delta \Sigma} \left\langle \frac{\partial \Sigma}{\partial t}, n_i \right\rangle ds. \tag{5.24}$$

Equations (5.23) and (5.24) show that $\dot{F} = \{F, H\}$ is equivalent to

$$0 = \int_{D_\Sigma} \left\langle \dot{v} + (v \cdot \nabla) v + \nabla p, \frac{\delta F}{\delta v} \right\rangle dV + \int_\Sigma \left\langle \frac{\partial \Sigma}{\partial t} - v, n \right\rangle \frac{\delta F}{\delta \Sigma} ds$$
$$+ \sum_{i=0}^{1} \int_{c_i} \left\langle \frac{\partial \Sigma}{\partial t} - v, n_i \right\rangle \frac{\delta_i F}{\delta \Sigma} ds$$
$$+ \sum_{i=0}^{1} \int_{c_i} [\tau \cos\gamma_i - \sigma_i] \left\langle \frac{\delta F}{\delta v}, n_i \right\rangle ds. \tag{5.25}$$

Choosing appropriate 'test functions' $F \in \mathscr{D}$ we show that (3.1)–(3.6) is fulfilled if (5.25) holds for all $F \in \mathscr{D}$. We first consider test functions $F: \mathscr{N} \longrightarrow \mathbf{R}$ of the form

$$F(\Sigma, v) = \int_{D_\Sigma} f(x) \, dV.$$

One checks that $F \in \mathscr{D}$ and that

$$\frac{\delta F}{\delta \Sigma} = f$$

$$\frac{\delta_i F}{\delta \Sigma} = 0, \quad i = 0, 1$$

$$\frac{\delta F}{\delta v} = 0.$$

Choosing $F = \int_{D_\Sigma} f \, dV$ in (5.25) leads to

$$\int_\Sigma \left[\left\langle \frac{\partial \Sigma}{\partial t}, n \right\rangle - \langle v, n \rangle \right] f \, dA = 0.$$

Since f is arbitrary, it follows that

$$\left\langle \frac{\partial \Sigma}{\partial t}, n \right\rangle = \langle v, n \rangle \tag{5.26}$$

on Σ. Thus, $\dot{F} = \{F, H\}$ for all $F \in \mathscr{D}$ is equivalent to

$$\begin{aligned} 0 = & \int_{D_\Sigma} \left\langle \dot{v} + (v \cdot \nabla) v + \nabla p, \frac{\delta F}{\delta v} \right\rangle dV \\ & + \sum_{i=0}^{1} \int_{c_i} \left\langle \frac{\partial \Sigma}{\partial t} - v, n_i \right\rangle \frac{\delta_i F}{\delta \Sigma} ds \\ & + \sum_{i=0}^{1} \int_{c_i} [\tau \cos \gamma_i - \sigma_i] \left\langle \frac{\delta F}{\delta v}, n_i \right\rangle ds \end{aligned} \tag{5.27}$$

for all $F \in \mathscr{D}$.

Choosing test functions of the form $F = \int_{\Sigma_i} f \, dA$ it is easy to see that $\langle (\partial \Sigma / \partial t) - v, n_i \rangle = 0$ for $i = 0, 1$. Thus $\dot{F} = \{F, H\}$ for all $F \in \mathscr{D}$ is equivalent to

$$\begin{aligned} 0 = & \int_{D_\Sigma} \left\langle \dot{v} + (v \cdot \nabla) v + \nabla p, \frac{\delta F}{\delta v} \right\rangle dV \\ & + \sum_{i=0}^{1} \int_{c_i} [\tau \cos \gamma_i - \sigma_i] \left\langle \frac{\delta F}{\delta v}, n_i \right\rangle ds \end{aligned} \tag{5.28}$$

for all $F \in \mathscr{D}$. Now we want to show that (5.28) implies equation (3.6):

$$\tau \cos \gamma_i(p) - \sigma_i = 0 \quad \text{for } p \in c_i, \ i = 0, 1. \tag{5.29}$$

We consider only the case $i = 0$, the case $i = 1$ being similar. Without restriction of generality, we assume that the distance between the two plates is $h = 1$. Suppose there is a $p \in c_0$ such that $\tau \cos \gamma_0(p) - \sigma_0 \neq 0$. Because γ_0 is continuous, we then have $\tau \cos \gamma_0 - \sigma_0 \neq 0$ in a neighbourhood U of p in c_0. We choose vector fields w_r, $r \in (0, 1)$ as in Lemma 5.2. Equation (5.20) shows that $\langle w_r, n_0 \rangle$ does not change sign on c_0. Equation (5.19) says that $\langle w_r(p), n_0(p) \rangle > 0$. Thus, taking $-w_r$ instead of w_r if necessary, we get

$$[\tau \cos \gamma_0 - \sigma_0] \langle w_r, n_i \rangle \geqslant 0 \quad \text{on } c_0$$

and

$$[\tau \cos \gamma_0(p) - \sigma_0] \langle w_r(p), n_i(p) \rangle > 0.$$

By equation (5.21), the left sides of the last two equations are independent of the index r. Thus, C, defined by

$$C := \int_{c_0} [\tau \cos\gamma_0 - \sigma_0] \langle w_r, n_0 \rangle \, ds$$

is positive and independent of the index r. The Cauchy–Schwarz inequality and equations (5.15) and (5.16) yield

$$\lim_{r \longrightarrow 0} \left(\int_{D_\Sigma} \langle \dot{v} + (v \cdot \nabla) v + \nabla p, w_r \rangle \, dV \right) = 0$$

and so

$$\lim_{r \to 0} \left(\int_{D_\Sigma} \langle \dot{v} + (v \cdot \nabla) v + \nabla p, w_r \rangle \, dV + \sum_{i=0}^{1} \int_{c_i} [\tau \cos\gamma_0 - \sigma_0] \langle w_r, n_i \rangle \, ds \right) = C > 0. \quad (5.30)$$

Define test functions $F_r : \mathcal{N} \longrightarrow \mathbf{R}$ by

$$F_r(\Sigma, v) = \int_{D_\Sigma} \langle w_r, v \rangle \, dV.$$

It is easy to show that $F_r \in \mathcal{D}$. In fact, the functional derivatives are given by

$$\frac{\delta F_r}{\delta \Sigma} = \langle w_r, v \rangle$$

$$\frac{\delta_i F_r}{\delta \Sigma} = 0$$

$$\frac{\delta F_r}{\delta v} = w_r.$$

By equation (5.30) there exists an $r \in (0, 1)$ such that

$$\left(\int_{D_\Sigma} \left\langle \dot{v} + (v \cdot \nabla) v + \nabla p, \frac{\delta F_r}{\delta v} \right\rangle dV + \sum_{i=0}^{1} \int_{c_i} [\tau \cos\gamma_0 - \sigma_0] \left\langle \frac{\delta F_r}{\delta v}, n_i \right\rangle ds \right) > 0$$

contradicting (5.28). Hence our assumption was wrong and $\tau \cos\gamma_0 - \sigma_0 = 0$ holds along c_0. Equations (5.28) and (5.29) imply that

$$\int_{D_\Sigma} \left\langle \dot{v}(x, t) + (v \cdot \nabla) v(x, t) + \nabla p(x, t), \frac{\delta F}{\delta v}(x) \right\rangle dV = 0 \quad (5.31)$$

for all t.

We now want to show that equation (5.31) implies equation (3.1); that is,

$$\dot{v}(x, t) + (v \cdot \nabla) v(x, t) + \nabla p(x, t) = 0$$

for all $x, t, x \in D_{\Sigma_t}$. Suppose there is a t such that

$$u : D_\Sigma \longrightarrow \mathbf{R}^3$$
$$x \longmapsto \dot{v}(x, t) + (v \cdot \nabla) v(x, t) + \nabla p(x, t)$$

is not the zero function. Because $\operatorname{div} v = 0$ and because of equation (5.22),

$$\operatorname{div} u = 0.$$

Also, because of (2.2) and (5.22) we have $\langle u, (0, 0, 1)^T \rangle = 0$ on $\Sigma_0 \cup \Sigma_1$. Now define $F : \mathcal{N} \longrightarrow \mathbf{R}$ by

$$F(\Sigma, v) = \int_{D_\Sigma} \langle u, v \rangle \, dV.$$

Then $F \in \mathscr{D}$ and

$$\frac{\delta F}{\delta \Sigma} = \langle u, v \rangle$$

$$\frac{\delta_i F}{\delta \Sigma} = 0$$

$$\frac{\delta F}{\delta v} = u.$$

By assumption, u is not the zero function and so

$$\int_{D_\Sigma} \left\langle \dot{v} + (v \cdot \nabla) v + \nabla p, \frac{\delta F}{\delta v} \right\rangle dV = \int_{D_\Sigma} \langle u, u \rangle \, dV > 0.$$

This contradicts (5.31), so our assumption was wrong and equation (3.1) is also satisfied by any solution of Hamilton's equations (5.1). We have shown the following.

THEOREM 5.1. *A curve* $t \longmapsto (\Sigma_t, v_t) \in \mathscr{N}$ *satisfies equations* (3.1)–(3.6) *if and only if it is a solution to Hamilton's equations* (5.1).

6. *Existence and stability of solutions to the equations of motion*

6.1. *An existence result*

In this section we prove an existence result for solutions to equations (3.1)–(3.6) representing drops that rotate rigidly about the z-axis with constant angular velocity ω. Thus, the velocity field is of the form

$$v(x) = (0, 0, \omega)^T \times x. \tag{6.1}$$

We assume in this section that the free boundary of the drop can be described as the graph of a real-valued function over the reference configuration D which was introduced in Section 2. In the following we speak of the 'drop f' for the drop with profile f. The volume of the drop f is given by

$$\mathrm{Vol}(f) = \frac{1}{2} \int_0^{2\pi} \int_0^h f^2 dz \, d\phi.$$

The free boundary Σ of the drop is the image of the map

$$\begin{aligned} [0, 2\pi] \times [0, h] &\longrightarrow \mathbf{R}^3 \\ (\phi, z) &\longmapsto (f(\phi, z) \cos\phi, f(\phi, z) \sin\phi, \, z)^T. \end{aligned} \tag{6.2}$$

Thus, the state of the drop is fully determined by the function f and the number ω. One easily checks that (Σ, v), where Σ is given by (6.2) and v is given by (6.1), is a steady state solution of equations (3.1)–(3.6) if and only if there is a $c \in \mathbf{R}$ such that $f: [0, 2\pi] \times [0, h] \longrightarrow \mathbf{R}^+$ solves the following boundary value problem:

$$\tau \kappa_f - \tfrac{1}{2} \omega^2 f^2 = c \tag{6.3}$$

$$\cos \gamma_i = \frac{\sigma_i}{\tau} \quad \text{on } c_i. \tag{6.4}$$

Here $\kappa_f: \Sigma \longrightarrow \mathbf{R}$ denotes the mean curvature function of the surface parametrised by the map (6.2). In fact, $\kappa_f(p)$ is the trace of the differential of the Gauss map at $p \in \Sigma$.

We will prove an existence result for rigidly rotating drops that are symmetric with respect to rotations about the z-axis. In this case the function f does not depend on the variable ϕ and we can consider it as a function on the interval $[0, h]$. The mean curvature of the free surface represented by f is then given by

$$\kappa_f(f(z)\cos\phi, f(z)\sin\phi, z) = \frac{1}{f(z)(1+f'^2(z))^{1/2}} - \frac{f''}{(1+f'^2(z))^{3/2}} \tag{6.5}$$

(compare do Carmo [**5**]). The condition $\cos\gamma_0 = \sigma_0/\tau$ can be written in the form

$$f'(h) = \frac{\sigma_0}{\sqrt{\tau^2-\sigma_0^2}} =: \rho_0 \tag{6.6}$$

and $\cos\gamma_1 = \sigma_1/\tau$ is equivalent to

$$f'(0) = -\frac{\sigma_1}{\sqrt{\tau^2-\sigma_1^2}} =: \rho_1. \tag{6.7}$$

In the proof of our existence result we make use of the following property of the mean curvature function.

DEFINITION 6.1. For $f: \mathbf{R} \longrightarrow \mathbf{R}$ and $r > 0$ we define $f_r: \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f_r(z) := rf\left(\frac{z}{r}\right).$$

REMARK 6.1. Let $f: [0, h] \longrightarrow \mathbf{R}^+$ and Σ_f denote the surface that is generated by rotating the graph of f about the z-axis. Let κ_f denote the mean curvature function of this surface. Then for $p \in \Sigma_f$ and $r > 0$, equation (6.5) shows that

$$\kappa_{f_r}(rp) = \frac{1}{r}\kappa_f(p).$$

We prove the following existence result:

THEOREM 6.1. *For every* $V > 0$ *and for sufficiently small* ω *there exists a solution* $f: [0, h] \longrightarrow \mathbf{R}$ *to equations* (6.3) *and* (6.4) *such that* $\operatorname{Vol}(f) < V$.

Proof. Consider first the case $\omega = 0$. For a function f not depending on the variable ϕ, equation (6.3) implies that the surface generated by rotating the graph of f about the z-axis has constant mean curvature. By a theorem of Delaunay (compare Loria [**17**, §212]) one can produce rotationally symmetric constant mean curvature surfaces in the following way.

One rolls an ellipse without slipping along the z-axis. The trace of a focus of the ellipse is described by a function $z \longmapsto f(z)$. The theorem of Delaunay states that one gets a constant mean curvature surface by rotating the graph of f about the z-axis.

Let a and b denote the major and the minor axis of the ellipse, respectively. Let e denote its eccentricity and let s be the arc length parameter. By Loria [**17**, §212] one has

$$f^2 = a^2\left[1 - 2e\cos\frac{s}{a} + e^2\right]. \tag{6.8}$$

By differentiating both sides of (6.8) with respect to z, one gets the estimate

$$\max_{z\in\mathbf{R}} |f'(z)| \geqslant \frac{e}{2}$$

(since f' is periodic, the maximum is defined). Choosing the eccentricity e of the ellipse sufficiently large and by translating the ellipse along the z-axis, we may arrange things so that

$$f'(z_1) < \rho_0 < f'(z_0)$$

for a pair of numbers $z_0 > z_1 > h$ and

$$f'(0) = \rho_1.$$

(The numbers ρ_0 and ρ_1 have been defined in (6.6) and (6.7).) Thus, with the help of Delaunay's construction, we can find a function $f: \mathbf{R} \longrightarrow \mathbf{R}$, satisfying all of the following conditions:

$$\tau\kappa_f = c \tag{6.9}$$

$$f'(0) = \rho_1 \tag{6.10}$$

$$f'(z_0) > \rho_0 \tag{6.11}$$

$$f'(z_1) < \rho_0. \tag{6.12}$$

For the functions f_r, $r > 0$, defined with the help of f in Definition 6.1, the following equations hold:

$$\tau\kappa_{f_r} = \frac{c}{r}$$

$$f_r'(0) = \rho_1$$

$$f_r'(rz_0) > \rho_0 \tag{6.13}$$

$$f_r'(rz_1) < \rho_0. \tag{6.14}$$

For $r \in \mathbf{R}$, $r > 0$, sufficiently small, we have

$$\mathrm{Vol}\,(f_r|_{[0,h]}) < V. \tag{6.15}$$

Thus, by considering an appropriate f_r instead of f we may assume that $\mathrm{Vol}\,(f|_{[0,h]}) < V$. Suppose that $f: [0, h] \longrightarrow \mathbf{R}$ satisfies equation (6.3) for the parameter values $\omega := \omega_0$, $c := c_0$. Then it is easy to check that the functions

$$\begin{aligned} f_1, f_2, \omega, c: [0,h] &\longrightarrow \mathbf{R} \\ f_1(z) &:= f(z) \\ f_2(z) &:= f'(z) \\ \omega(z) &:= \omega_0 \\ c(z) &:= c_0 \end{aligned}$$

satisfy the following first-order system of differential equations without parameters:

$$\begin{aligned} f_1' &= f_2 \\ f_2' &= \left(-\frac{c}{\tau} - \frac{\omega^2}{2\tau} f_1^2 + \frac{1}{f_1\sqrt{1+f_2^2}}\right)(1+f_2^2)^{3/2} \\ \omega' &= 0 \\ c'(0) &= 0. \end{aligned} \tag{6.16}$$

If, on the other hand, $z \longmapsto (f_1(z), f_2(z), \omega(z), c(z))$ is a solution of (6.16), then $f(z) := f_1(z)$ solves equation (6.3) for the parameters $\omega := \omega(0)$, $c := c(0)$. For $p = (z_0, f_{10}, f_{20}, \omega_0, c_0) \in \mathbf{R}^5$ let $I_p \in \mathbf{R}$ denote the maximal interval in which a solution $z \longmapsto (f_1(z), f_2(z), \omega(z), c(z))$ to equations (6.16) exists with initial values $f_1(z_0) = f_{10}$, $f_2(z_0) = f_{20}$, $\omega(z_0) = \omega_0$, $c(z_0) = c_0$. Let

$$B := \{(z, p), z \in I_p\}$$

and let

$$\Phi: B \longrightarrow \mathbf{R}^4, \qquad \Phi(z, z_0, f_{10}, f_{20}, \omega_0, c_0) = (f_1(z), f_2(z), \omega(z), c(z))$$

denote the general solution of the system (6.16). Let $\Phi_i: B \longrightarrow \mathbf{R}$ denote the ith component function, $i = 1, \ldots, 4$.

It is well known that $B \subseteq \mathbf{R}^6$ is an open set and that $\Phi: B \longrightarrow \mathbf{R}^4$ is continuous. We switch from the second-order equation with parameters (6.3) to the first-order system (6.16) just to make use of these facts. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ satisfy (6.9)–(6.12) and let $A := f(0)$. We can rewrite (6.11) and (6.12) as

$$\Phi_2(z_0, 0, A, \rho_1, 0, c) > \rho_0$$
$$\Phi_2(z_1, 0, A, \rho_1, 0, c) < \rho_0.$$

The function $f_r: \mathbf{R} \longrightarrow \mathbf{R}$ defined with the help of f in Definition 6.1 is given by

$$f_r(z) = \Phi\left(z, 0, rA, \rho_1, 0, \frac{c}{r}\right).$$

We consider $r \in [h/z_0, h/z_1]$. Equations (6.13) and (6.14) can be written as

$$\Phi_2\left(h, 0, \frac{h}{z_0} A, \rho_1, 0, \frac{z_0}{h} c\right) > \rho_0 \tag{6.17}$$

$$\Phi_2\left(h, 0, \frac{h}{z_1} A, \rho_1, 0, \frac{z_1}{h} c\right) < \rho_0. \tag{6.18}$$

Since f_r is defined on the whole of $\mathbf{R}$, we have

$$\left\{\left(h, 0, \frac{h}{z} A, \rho_1, 0, \frac{z}{h} c\right), z \in [z_1, z_0]\right\} \subseteq B.$$

Now, B is open, so there is an $\epsilon > 0$, such that for $|\omega| < \epsilon$

$$\left\{\left(h, 0, \frac{h}{2} A, \rho_1, \omega, \frac{z}{h} c\right), z \in [z_1, z_0]\right\} \subseteq B$$

holds. Since $\Phi: B \longrightarrow \mathbf{R}^4$ is continuous and because of (6.15), (6.17) and (6.18) we can choose ϵ such that for $\omega < \epsilon$ we have

$$\Phi_2\left(h, 0, \frac{h}{z_0} A, \rho_1, \omega, \frac{z_0}{h} c\right) > \rho_0$$

$$\Phi_2\left(h, 0, \frac{h}{z_1} A, \rho_1, \omega, \frac{z_1}{h} c\right) < \rho_0$$

and

$$\pi \int_0^h \left[\Phi_1\left(z, 0, rA, \rho_1, \omega, \frac{c}{r}\right)\right]^2 dz < V \quad \text{for } r \in \left[\frac{h}{z_0}, \frac{h}{z_1}\right]. \tag{6.19}$$

By the mean value theorem, there is a $z_\omega \in [z_1, z_0]$ for every $\omega < \epsilon$ such that

$$\Phi_2\left(h, 0, \frac{h}{z_\omega} A, \rho_1, \omega, \frac{z_\omega}{h} c\right) = \rho_0.$$

Thus, $f_\omega : [0, h] \longrightarrow \mathbf{R}$, defined by

$$f_\omega(z) := \Phi_1\left(z, 0, \frac{h}{z_\omega} A, \rho_1, \omega, \frac{z_\omega}{h} c\right)$$

is a solution of the boundary value problem (6.3), (6.4). Because of (6.19), we also have $\mathrm{Vol}\,(f_\omega) < V$. □

6.2. *Stability of rotating drops*

The group S^1 acts from the right on the Lagrangian configuration space $\mathscr{C}$ by rotation about the z-axis. This action can be lifted to Lagrangian phase space $T\mathscr{C}$ and carries over to the Eulerian configuration space and phase space.

A solution (f, ω) of the equations (6.3), (6.4) represents a relative equilibrium of our system in the sense that its dynamic orbit is contained in its group orbit.

We now want to analyse the (orbital) stability of these relative equilibria with the help of the energy-momentum method (compare Marsden [**18**]). In this stability analysis we restrict ourselves to variations of the drop shape which can be described as graphs of real-valued functions on the free boundary of a drop whose stability we investigate. We neither require this drop to be rotationally symmetric nor do we restrict ourselves to rotationally symmetric variations of the drop shape.

We now discuss the energy-momentum method in the context of our system. Let

$$I(\Sigma) = \int_{D_\Sigma} (x^2 + y^2)\, dV$$

denote the moment of inertia about the z-axis of the drop $\Sigma \in \mathscr{M}$. (Remember that the drop has constant density 1.) Define the augmented potential $V_\omega : \mathscr{M} \longrightarrow \mathbf{R}$ by

$$V_\omega(\Sigma) = V - \tfrac{1}{2} I(\Sigma)\, \omega^2$$

where V denotes the potential energy of the drop as given in the introduction to this paper, that is, $V = \tau \int_\Sigma dA - \sum_{i=0}^{1} \sigma_i \int_{\Sigma_i} dA$. If $f : [0, 2\pi] \times [0, h] \longrightarrow \mathbf{R}^+$ describes the free boundary Σ as in (6.2), we also write $V(f)$ instead of $V(\Sigma)$. A direct calculation yields the following.

REMARK 6.2. Let $f : [0, 2\pi] \times [0, h] \longrightarrow \mathbf{R}$ be a solution to the equations (6.3), (6.4). Let $\{f_t : [0, 2\pi] \times [0, h] \longrightarrow \mathbf{R}, t \in I\}$ be a family of functions with $f_0 = f$; then

$$\left.\frac{d}{dt}\right|_{t=0} V_\omega(f_t) - c\, \mathrm{Vol}\,(f_t) = 0.$$

Thus, the drop f is a critical point of the functional $V_\omega - c\,\mathrm{Vol}$ with respect to arbitrary variations of the drop shape that can be represented as functions on the free boundary of the reference cylinder. Let

$$F = \{g : [0, 2\pi] \times [0, h] \longrightarrow \mathbf{R} \mid \mathrm{Vol}\,(g) = \mathrm{Vol}\,(f)\}$$

denote the set of drops which have the same volume as the drop f. Since the function Vol is constant on F, Remark 6.2 shows that

$$\left.\frac{d}{dt}\right|_{t=0} V_\omega(f_t) = 0$$

for every curve $t \longmapsto f_t$, $f_0 = f$, in F. So f is a critical point of V_ω with respect to volume-preserving variations. We restrict ourselves in the following to this kind of variation (note that these variations lie in a transversal subspace at f to the group orbit through f).

We analyse the stability of (f, ω) by checking the second derivative of V_ω at f for positive definiteness. To get an idea about what positive definiteness tells us about stability, observe that if $\Sigma \in \mathscr{M}$ is a critical point of V_ω, then $(f, v) \in \mathscr{N}$ with $v(x) = (0,0,\omega)^T \times x$ is a critical point of the function

$$K_\omega(\Sigma, w) + V_\omega(\Sigma)$$

defined on phase space, where

$$K_\omega(\Sigma, w) = \frac{1}{2}\int_{D_\Sigma} \|w - (0,0,\omega)^T \times x\|^2 dV.$$

Now, K_ω can be interpreted as the kinetic energy of the drop measured in a coordinate system rotating with angular velocity ω about the z-axis. Also, $K_\omega + V_\omega$ can be written as

$$K_\omega + V_\omega = H - \omega J$$

where

$$J(\Sigma, w) = \int_{D_\Sigma} \langle x \times w, (0,0,1)^T \rangle \, dV$$

denotes angular momentum about the z-axis (cf. Marsden [**18**]). Both H and J are constants of motion for our system. If the second variation of V_ω is positive definite at the critical point Σ, then the second variation of the constant of motion $H - \omega J$ is positive definite at its critical point (Σ, v), $v(x) = (0,0,\omega)^T \times x$. By definition, this implies that (Σ, v) is a *formally* stable equilibrium of our system (cf. Lewis [**14**]). Note that by a theorem of Dirichlet (cf. Marsden [**18**]) formally stable equilibria in finite-dimensional systems are nonlinearly stable in the sense of Liapunov. Note also that H and J are constant along group orbits under the S^1-action.

To show that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} V_\omega(f_t) > 0$$

for every curve $t \longmapsto f_t$ in F, $f_0 = f$ it suffices to prove that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} V_\omega(g_t) - c \cdot \mathrm{Vol}\,(g_t) > 0$$

for every curve $t \longmapsto g_t$, $g_t : [0, 2\pi] \times [0, h] \longrightarrow \mathbf{R}$, $g_0 = f$, satisfying the linearised volume constraint

$$\int_0^{2\pi}\int_0^h f \left.\frac{d}{dt}\right|_{t=0} g_t \, dz\, d\phi = 0.$$

This suffices because the volume function Vol restricted to F is constant.

A long but completely straightforward calculation shows that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} (V_\omega - c \cdot \mathrm{Vol})\,(g_t) = \int_0^{2\pi}\int_0^h Pk_z^2 + Qk^2 + Rk_\phi^2 + Sk_z k_\phi \, dz\, d\phi$$
$$+ \int_0^{2\pi} (Tk^2)_{z=h}\, d\phi - \int_0^{2\pi} (Tk^2)_{z=0}\, d\phi$$

where

$$k=\frac{d}{dt}\bigg|_{t=0} g_t$$

the indices ϕ and z denote differentiation with respect to ϕ and z, and where

$$P := \frac{\tau f^2}{\sqrt{f_\phi^2+(1+f_z^2)f^2}}\left(1-\frac{f^2 f_z^2}{f_\phi^2+(1+f_z^2)f^2}\right) \tag{6.20}$$

$$\begin{aligned} Q := \tau\Bigg[&\frac{(f_\phi^2-f^2)(f_z^2+1)}{(f_\phi^2+(1+f_z^2)f^2)^{3/2}} - f\left(\frac{f_z f_\phi^2}{(f_\phi^2+(1+f_z^2)f^2)^{3/2}}\right)_z + \frac{1}{f}\left(\frac{f_\phi}{\sqrt{f_\phi^2+1+(f_z^2)f^2}}\right)_\phi \\ &+ f\left(\frac{f_\phi(f_z^2+1)}{(f_\phi^2+(1+f_z^2)f^2)^{3/2}}\right)_\phi\Bigg] - \omega^2 f^2 \end{aligned} \tag{6.21}$$

$$R := \frac{\tau}{\sqrt{f_\phi^2+(1+f_z^2)f^2}}\left(1-\frac{f_\phi^2}{f_\phi^2+(1+f_z^2)f^2}\right) \tag{6.22}$$

$$S := \frac{\tau f^2 f_z f_\phi}{(f_\phi^2+(1+f_z^2)f^2)^{3/2}} \tag{6.23}$$

$$T := \frac{\tau f f_z f_\phi^2}{(f_\phi^2+(1+f_z^2)f^2)^{3/2}}. \tag{6.24}$$

We have proved the following theorem.

THEOREM 6.2. *A solution* (f, ω) *to* (6.3), (6.4) *is a formally stable relative equilibrium if the quadratic form*

$$\begin{aligned}\beta(k) := \int_0^{2\pi}\int_0^h P k_z^2 + Q k^2 + R k_\phi^2 + S k_z k_\phi \, dz\, d\phi \\ + \int_0^{2\pi} (Tk^2)_{z=h}\, d\phi - \int_0^{2\pi} (Tk^2)_{z=0}\, d\phi\end{aligned}$$

is positive on

$$f^\perp = \left\{k : [0,2\pi]\times[0,h] \longrightarrow \mathbf{R},\ \int_0^{2\pi}\int_0^h fk\, dz\, d\phi = 0,\ k \neq 0\right\}.$$

In the special case of a rotationally symmetric equilibrium, one has $f_\phi = 0$ and the theorem specialises to the following.

THEOREM 6.3. *A solution* (f, ω) *to* (6.3), (6.4) *where* $f_\phi = 0$ *is a formally stable relative equilibrium if the quadratic form*

$$\beta(k) := \int_0^{2\pi}\int_0^h \bar{P}(k_z)^2 + \bar{Q}k^2 + \bar{R}k_\phi^2 \, dz\, d\phi$$

is positive on

$$f^\perp = \left\{k : [0,2\pi]\times[0,h] \longrightarrow \mathbf{R},\ \int_0^{2\pi}\int_0^h fk\, dz\, d\phi = 0,\ k \neq 0\right\}.$$

Here,

$$\bar{P} := \frac{\tau f}{(1+f_z^2)^{3/2}}$$

$$\bar{Q} := -\frac{\tau}{f\sqrt{1+f_z^2}} - \omega^2 f^2$$

$$\bar{R} := \frac{\tau}{f\sqrt{1+f_z^2}}.$$

It is instructive to consider the case $\sigma_0 = \sigma_1 = 0$, that is, no adhesion forces act at the plates. Then every constant function $f: [0, h] \longrightarrow \mathbf{R}$, $f(x) = d$, is a solution to the equations (6.3), (6.4) for arbitrary ω. These solutions represent cylinders rotating about the z-axis with constant angular velocity ω. Denote the free boundary of our cylinder by Z. The Hilbert space $L_2(Z)$ is the closure of the space generated by functions of the form

$$h_{kl}(\phi, z) := \cos(k\phi - \psi)\cos\left(\frac{\pi l}{h} z\right), \quad \psi \in \mathbf{R},\ k, l \in \mathbf{Z}.$$

(We suppress the dependence on ψ and γ in this notation.) For these functions $h_{kl}: Z \longmapsto \mathbf{R}$, integration by parts yields

$$\beta(h_{kl}) = \int_0^{2\pi} \int_0^h L(h_{kl})\, h_{kl}\, dz\, d\phi \tag{6.25}$$

where

$$L(h_{kl}) = \left(\tau d \left(\frac{\pi l}{h}\right)^2 + (k^2 - 1)\frac{\tau}{d} - \omega^2 d^2\right) h_{kl}. \tag{6.26}$$

Putting $k = 0$, we see that β is positive definite on the subspace of $f^\perp$ of functions which do not depend on the ϕ variable if

$$\frac{h^2}{\pi^2 d^2} + \frac{\omega^2 h^2 d}{\pi^2 \tau} < 1. \tag{6.27}$$

In the case $\omega = 0$ this is just the criterion of Vogel [**22**] for stability with respect to axisymmetric disturbances of cylindrical drops at rest.

To study positive-definiteness of β with respect to variations in $f^\perp$, which do not depend on the z variable, we have to evaluate L on the functions h_{k0}. We choose k different from zero, because we only consider volume-preserving variations. We also exclude the case $k = 1$ which corresponds to translations of the drop (compare Lewis [**14**]). The quadratic form β is positive definite with respect to variations of this type if

$$\frac{3\tau}{d^3} > \omega^2. \tag{6.28}$$

This is just the stability criterion for a 2-dimensional circular drop of Lewis [**14**]. From (6.26) one reads off that if (6.27) and (6.28) are satisfied, then β is also positive definite with respect to nonsymmetric variations of the type h_{kl} with $l \neq 0$.

We conclude with the following theorem.

THEOREM 6.4. *In the absence of adhesion forces, a cylindrical drop with radius d rotating between two plates with constant angular velocity ω is stable if both*

$$\frac{h^2}{\pi^2 d^2}+\frac{\omega^2 h^2 d}{\pi^2 \tau}<1 \tag{6.29}$$

and

$$\frac{3\tau}{d^3}>\omega^2 \tag{6.30}$$

hold.

Acknowledgements. This work is an outgrowth of the author's thesis written at the University of Hamburg under the supervision of Professor Jürgen Scheurle. I thank Professor Jerrold Marsden and Prof Jürgen Scheurle for very helpful discussions. I would further like to thank the Alexander von Humboldt Foundation for its support and the Fields Institute for Research in Mathematical Sciences and the University of California at Berkeley for their hospitality.

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Zentrum Mathematik
Technische Universität München
Arcisstrasse 21
D-80290 München
Germany