ON THE ALGORITHMIC COMPLEXITY OF MINKOWSKI'S RECONSTRUCTION THEOREM

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Dedicated to Professor Victor Klee

Abstract

In 1903 Minkowski showed that, given pairwise different unit vectors $u_1, \dots, u_m$ in Euclidean n-space $\mathbb{R}^n$ which span $\mathbb{R}^n$, and positive reals $\mu_1, \dots, \mu_m$ such that $\sum_{i=1}^{m} \mu_i u_i = 0$, there exists a polytope P in $\mathbb{R}^n$, unique up to translation, with outer unit facet normals $u_1, \dots, u_m$ and corresponding facet volumes $\mu_1, \dots, \mu_m$. This paper deals with the computational complexity of the underlying reconstruction problem, to determine a presentation of P as the intersection of its facet halfspaces. After a natural reformulation that reflects the fact that the binary Turing-machine model of computation is employed, it is shown that this reconstruction problem can be solved in polynomial time when the dimension is fixed but is $\#\mathbb{P}$-hard when the dimension is part of the input.

The problem of 'Minkowski reconstruction' has various applications in image processing, and the underlying data structure is relevant for other algorithmic questions in computational convexity.

0. *Introduction*

0.1. *The Minkowski reconstruction problem*

This paper deals with the following problem. Given an $m \times n$ matrix A and positive reals $\mu_1, \dots, \mu_m$, does there exist a polytope $P = \{x : Ax \leqslant b\}$ with m facets $F_1, \dots, F_m$ (indexed so that the ith facet corresponds to the ith row of A) such that $\mathrm{Vol}_{n-1}(F_i) = \mu_i$ for $i = 1, \dots, m$? If such a polytope exists, compute a corresponding right-hand side b of $\mathbb{R}^m$ efficiently!

This algorithmic question is motivated by applications to problems in computer vision [**11**, **12**, **15**, **16**, **28**] and other algorithmic tasks of computational convexity (see [**7**]), and in view of the following classical theorem of Minkowski [**18**, Section 4; **19**, Section 9] it seems fair to say that this problem is a basic question in computational convexity in that it essentially asks for the conversion of one data structure (facet normals and facet volumes) of polytopes into another (intersection of finitely many closed halfspaces).

Proposition 1 (Minkowski 1897/1903). *Let $u_1, \dots, u_m$ be pairwise different unit vectors of Euclidean n-space $\mathbb{R}^n$ which span $\mathbb{R}^n$, and let $\mu_1, \dots, \mu_m$ be positive reals such that $\sum_{i=1}^{m} \mu_i u_i = 0$. Then there exists a polytope P with outer facet normals $u_1, \dots, u_m$ and corresponding facet volumes $\mu_1, \dots, \mu_m$. Further, P is unique up to translation.*

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Note that the assumptions imply that $\mathbb{R}^n$ is actually positively spanned by $u_1, \dots, u_m$, that is,

$$\operatorname{pos}\{u_1, \dots, u_m\} = \mathbb{R}^n.$$

Further, the condition $\Sigma_{i=1}^m \mu_i u_i = 0$ is equivalent to

$$\sum_{i=1}^m \mu_i \langle u_i, y\rangle = 0 \quad \text{for all } y \in \mathbb{R}^n \backslash \{0\},$$

which simply reflects the fact that the orthogonal projections of a polytope in directions $\pm y$ coincide. For extensions of Proposition 1 see [**21–23**].

Proposition 1 solves the theoretical part of our problem completely. Moreover, Minkowski's original proof is constructive. However, the question of efficiency is by no means clear: can a right-hand side b be computed in polynomial time? We will show that (a natural reformulation of) this *Minkowski reconstruction problem* can indeed be solved in polynomial time, if the dimension is fixed, but is $\#P$-hard, if n is part of the input.

0.2. *Problems and main results*

We employ the *binary Turing-machine model* of computation in which the input data is encoded in binary form, and the performance of an algorithm on a given input is measured in terms of the number of operations of a Turing machine. This model leads to the condition that all input and all output be *rational*; hence $A = (a_1, \dots, a_m)^{\mathrm{T}}$ is a rational matrix and $b = (\beta_1, \dots, \beta_m)^{\mathrm{T}}$ is a rational vector of $\mathbb{R}^m$, whence $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is a *rational polytope*. Clearly, for rational polytopes P neither the facet volumes μ_i nor the outer facet *unit* normals need to be rational. Since the vertices of P are rational vectors it follows that for each facet

$$F_i = \{x \in P : \langle a_i, x\rangle = \beta_i\}$$

of P, and each rational point $p \notin \operatorname{aff}(F_i)$,

$$\mu_i = \operatorname{Vol}_{n-1}(F_i) = \frac{n\|a_i\|}{|\beta_i - \langle a_i, p\rangle|} \operatorname{Vol}_n(\operatorname{conv}(F_i \cup \{p\})),$$

whence the product $v_i = \|a_i\|^{-1} \operatorname{Vol}_{n-1}(F_i)$ is rational. (As a notational convenience, $\|\ \ \|$ always denotes the Euclidean norm, while other ℓ_p-norms will be denoted by $\|\ \ \|_p$. Further, $\mathbb{B}^n$ is the Euclidean unit ball in $\mathbb{R}^n$, and Vol is used as an abbreviation for Vol_n.)

Therefore we replace the facet volumes $\mu_1, \dots, \mu_m$ in the input of our problem by the vector

$$v = (v_1, \dots, v_m)^{\mathrm{T}}, \quad \text{where } v_i = \|a_i\|^{-1} \mu_i \text{ for } i = 1, \dots, m.$$

Then for each rationally presented polytope $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ the corresponding *Minkowski data* is rational. This takes care of the rationality of the input.

The rationality requirement for the output b is a different problem since the converse of the above argument fails: even if A and v are rational, the Minkowski reconstruction problem may not admit a rational solution unless $n \leqslant 2$. For instance, let $n \geqslant 3$, let I_n be the $n \times n$ identity matrix, let

$$A = \begin{pmatrix} I_n \\ -I_n \end{pmatrix} \quad \text{and} \quad v = (2^n, \dots, 2^n)^{\mathrm{T}} \in \mathbb{R}^{2n}.$$

One can easily check (using Proposition 1) that the corresponding polytopes are the translates of the cube $P = 2^{1/(n-1)}[-1,1]^n$. Clearly, P is not rational, and neither is any translate of P. Hence, for our algorithmic purposes we need to resort to the following problem, MinkApp.

Instance: $m, n \in \mathbb{N}$; vectors $a_1, \ldots, a_m \in \mathbb{Q}^n \backslash \{0\}$, no two positively dependent, which span $\mathbb{R}^n$; positive rationals $v_1, \ldots, v_m$ such that $\sum_{i=1}^m v_i a_i = 0$; a positive rational error bound ϵ.

Task: Determine a rational vector $\tilde{b} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_m)^{\mathrm{T}}$ such that, for $i = 1, \ldots, m$, the volume $\tilde{\mu}_i$ of the ith facet $\tilde{F}_i = \{x \in \tilde{P} : \langle a_i, x \rangle = \tilde{\beta}_i\}$ of the polytope

$$\tilde{P} = \{x \in \mathbb{R}^n : Ax \leqslant \tilde{b}\}$$

satisfies $|\tilde{\mu}_i - v_i \|a_i\|| \leqslant \epsilon$.

We will often 'collect' the input data of an instance of MinkApp in a string $(n, m; A, v; \epsilon)$, where A is the matrix with rows $a_1^{\mathrm{T}}, \ldots, a_m^{\mathrm{T}}$, and $v = (v_1, \ldots, v_m)^{\mathrm{T}}$. If the dimension n is not part of the input, but a constant that has been fixed in advance, we call the problem n-MinkApp, while the 'exact variant' of MinkApp (where ϵ is 0) will in the following be denoted by MinkRecon.

The original proof of Minkowski models MinkRecon as the task to minimise a linear functional over a closed convex region $C \subset \mathbb{R}^m$; see Section 1.4. Little [**16**] (see also [**11**, **15**]) used standard methods from convex minimisation to develop an algorithm for MinkApp in dimension 3, which was then applied to a practical problem of computer vision. However, even though Little's work [**16**] is restricted to polytopes in $\mathbb{R}^3$, there is no polynomial bound on the running time of his algorithm.

Here we settle the related complexity question completely by showing the following theorems.

THEOREM 1. *For each fixed $n \in \mathbb{N}$, n-MinkApp can be solved in polynomial time.*

THEOREM 2. *MinkApp is #$\mathbb{P}$-equivalent, that is, #$\mathbb{P}$-hard and #$\mathbb{P}$-easy.*

For an introduction to the complexity classes #$\mathbb{P}$, #$\mathbb{P}$-hard, #$\mathbb{P}$-complete and #$\mathbb{P}$-equivalent see for example [**6**, **13**, **25**, **26**].

The proofs of (slightly stronger versions of) our main Theorems 1 and 2 are given in Sections 2 and 3, while Section 1 contains basic prerequisites and tools for these proofs. In particular, we give a brief summary of the 'oracle-based' algorithmic theory of convex bodies of [**10**], collect basic results on the complexity of volume computation, state relevant results of Brunn–Minkowski theory, and outline Minkowski's reduction of MinkRecon to a convex program.

We close this section by posing two open problems. While Theorem 2 shows that MinkApp is #$\mathbb{P}$-hard, it does not rule out the possibility that there is a polynomial-time *randomised* algorithm for MinkApp. The existence of such an algorithm remains, however, open.

If, for a given instance of n-MinkRecon, it is known beforehand that there is a rational solution of binary size bounded by a polynomial in the size of the input, then one can produce an exact solution from one of n-MinkApp (for sufficiently small ϵ)

by standard rounding procedures. (Such a case of special interest is that of 0-1-polytopes.) Then, of course, the full combinatorial structure of the corresponding polytopes is available. One might be tempted to believe that following the approach of [**17**] for dealing with algebraic numbers by encoding the coefficients of their minimal monic polynomial the above approach can easily be extended to the general case. This is, however, not true, so the problem remains open in general as to whether, given an instance of n-MinkRecon, the combinatorial structure of its (up to translation) unique solution can be determined efficiently.

1. *Preliminaries*

1.1. *Presentations of polytopes and general convex bodies*

From an algorithmic point of view, polytopes are dealt with much more easily than general convex bodies, because polytopes can be presented in a finite manner, namely in terms of their vertices ('$\mathscr{V}$-presentation') or in terms of their facet halfspaces ('$\mathscr{H}$-presentation'). For this paper, polytopes are naturally presented in the latter way. Since the underlying model of computation is the binary Turing-machine model (which – in the case of convex bodies – will be augmented by certain oracles (see [**6**, **10**])) we will have to restrict our polytopes to those which can be presented by rational data. Clearly, from an algorithmic point of view it is not the geometric object that is relevant but its presentation. Hence we use the following notation; see for example [**7**].

A string $(n, m; A, b)$, where $n, m \in \mathbb{N}$, A is a rational $m \times n$ matrix and $b \in \mathbb{Q}^m$ such that $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is a polytope is called an *$\mathscr{H}$-polytope* in $\mathbb{R}^n$. Of course, we will often identify $(n, m; A, b)$ with the geometric object P.

The *binary size* (or simply *size*) $\langle P \rangle$ of an $\mathscr{H}$-polytope P is the number of binary digits needed to encode the data of the presentation. Specifically, the size of an integer τ is

$$\langle \tau \rangle = 1 + \lceil \log_2 (|\tau| + 1) \rceil,$$

the size of a rational number ρ written in coprime presentation $\rho = \rho_1/\rho_2$, $\rho_1 \in \mathbb{Z}$, $\rho_2 \in \mathbb{N}$ is $\langle \rho_1 \rangle + \langle \rho_2 \rangle$ and the size of a rational $(m \times n)$-matrix $A = (\alpha_{i,j})$ is $\Sigma_{i=1}^m \Sigma_{j=1}^n \langle \alpha_{ij} \rangle$. Consequently, when P is the $\mathscr{H}$-polytope $(n, m; A, b)$, we have $\langle P \rangle = \langle A \rangle + \langle b \rangle$.

Unlike for polytopes, there is in general no finite manner to present arbitrary convex sets. To deal with convex bodies algorithmically, Grötschel, Lovász and Schrijver [**10**] augment the Turing-machine model by so-called 'oracles' which provide information about the convex body in question. For instance, we say that a convex body K of $\mathbb{R}^n$ is given by a *membership oracle* if there is an algorithm that provides for any given input point y the information '$y \in K$' or '$y \notin K$'. Similarly an *optimisation oracle* accepts as input a linear functional $x \mapsto \langle c, x \rangle$ and outputs a solution of the linear program $\min \{\langle c, x \rangle : x \in K\}$. Note that these oracles function as 'black boxes'; there is no assumption as to how they obtain the output information. Further, the number of calls to an oracle enters the complexity of an algorithm for a convex body. The underlying model of computation is only able to deal with rational numbers; computations are performed only to finite precision. Hence, generally, an *exact* solution to – say – an optimisation problem cannot be stored or processed. For this reason we have to resort to *weak* oracles that allow for rounding errors.

Now, let us make these remarks precise. Let $\mathscr{C}^n$ denote the family of all closed convex subsets of $\mathbb{R}^n$ with non-empty interior, and let $\mathscr{K}^n$ denote the set of all convex bodies, that is, of all compact members of $\mathscr{C}^n$. Further, define for $C \in \mathscr{C}^n$ and $\epsilon \geqslant 0$

$$C(\epsilon) = C + \epsilon\mathbb{B}^n = \{x + \epsilon y : x \in C \wedge y \in \mathbb{B}^n\}, \quad C(-\epsilon) = C \backslash ((\mathbb{R}^n \backslash C) + \epsilon\mathbb{B}^n),$$

the *outer* and the *inner parallel sets* of C, respectively. Then the oracles for sets $C \in \mathscr{C}^n$ that are most relevant for our purposes solve one of the following problems.

WEAK MEMBERSHIP PROBLEM. Given a vector $y \in \mathbb{Q}^n$ and a rational number $\epsilon > 0$,
(i) assert that $y \in C(\epsilon)$; or
(ii) assert that $y \notin C(-\epsilon)$.

WEAK VALIDITY PROBLEM. Given $c \in \mathbb{Q}^n \backslash \{0\}$, $\gamma \in \mathbb{Q}$ and a rational $\epsilon > 0$,
(i) assert that $\langle c, x \rangle \leqslant \gamma + \epsilon$ for every $x \in C(-\epsilon)$; or
(ii) assert that there exists a vector $y \in C(\epsilon)$ such that $\langle c, y \rangle \geqslant \gamma - \epsilon$.

WEAK OPTIMISATION PROBLEM. Given a vector $c \in \mathbb{Q}^n \backslash \{0\}$ and a rational number $\epsilon > 0$,
(i) compute a vector $y \in \mathbb{Q}^n \cap C(\epsilon)$ such that $\langle c, y \rangle \leqslant \langle c, x \rangle + \epsilon$ for every $x \in C(-\epsilon)$; or
(ii) assert that $C(-\epsilon)$ is empty; or
(iii) assert that $x \mapsto \langle c, x \rangle$ is not bounded below on C.

If a set $C \in \mathscr{C}^n$ is given by an algorithm $\mathcal{O}$ that solves the weak membership problem, the weak validity problem or the weak optimisation problem, we say that C is given by a *weak membership oracle*, a *weak validity oracle* or a *weak optimisation oracle* $\mathcal{O}$, respectively. C is called *well-guaranteed* if rational numbers r, R are given in advance such that C contains a ball of radius r and $C \subset R\mathbb{B}^n$. If a vector $a \in \mathbb{Q}^n$ and a positive rational r are given beforehand, such that $a + r\mathbb{B}^n \subset C$ we call C *centred*. Note that a well-guaranteed set $C \in \mathscr{C}^n$ is compact, whence contained in $\mathscr{K}^n$.

The *size* $\langle K \rangle$ of a centred well-guaranteed body $K \in \mathscr{K}^n$ with parameters r, R and a that is given by any of the above oracles is defined as

$$\langle K \rangle = n + \langle r \rangle + \langle R \rangle + \langle a \rangle.$$

Furthermore, the input sizes of the above three problems are, respectively,

$$\langle y \rangle + \langle \epsilon \rangle, \quad \langle c \rangle + \langle \gamma \rangle + \langle \epsilon \rangle, \quad \langle c \rangle + \langle \epsilon \rangle.$$

Using the *ellipsoid algorithm*, Grötschel *et al.* [**10**] show, in particular, that when the underlying class of convex bodies is restricted to those which are centred and well-guaranteed the above three oracles do not differ in their 'algorithmic strength'.

PROPOSITION 2. *The problem whose instance is a centred well-guaranteed body $K \in \mathscr{K}^n$ given by an oracle for one of the problems weak membership, weak validity, or weak optimisation, and whose task it is to solve the other two problems, admits an oracle-polynomial-time algorithm.*

Grötschel *et al.* [**10**] give a slightly sharper version of Proposition 2, in fact the assumptions 'centred' and 'well-guaranteed' are not needed in every single case of the above assertion. (For simplicity, we refrained from stating all six cases separately.) Note that these assumptions can be satisfied for $\mathscr{H}$-polytopes in the following sense; see for example [**10**].

PROPOSITION 3. *The following problems can be solved in polynomial time. Given an $\mathscr{H}$-polytope P,*

(i) *decide whether $P = \emptyset$; or*

(ii) *find an (irredundant) rational $\mathscr{H}$-presentation of $\operatorname{aff}(P)$, and compute a point $a \in P$ and rational numbers r, R such that $(a + r\mathbb{B}^n) \cap \operatorname{aff}(P) \subset P \subset R\mathbb{B}^n$.*

Clearly, $\langle a \rangle$, $\langle r \rangle$ and $\langle R \rangle$ are bounded by a polynomial in $\langle P \rangle$, specifically, $r \geqslant 2^{-4\langle P \rangle}$ and $R \leqslant 2^{4\langle P \rangle}$.

1.2. *Polytope volume computation*

Here we state some results concerning the complexity of computing the volume of $\mathscr{H}$-polytopes. More precisely, we deal with the following problem VolApp (and its variant, VolComp, for $\varepsilon = 0$). Given an $\mathscr{H}$-polytope $P = (n, m; A, b)$, and a positive rational ϵ, compute a rational number $\hat{V}$ such that $|\hat{V} - \operatorname{Vol}(P)| \leqslant \epsilon$.

Let us point out that Lawrence [**14**] has shown that $\langle \operatorname{Vol}(P) \rangle$ cannot be bounded from above by a polynomial in $\langle P \rangle$ if n is part of the input. This means that, in general, $\operatorname{Vol}(P)$ cannot be computed exactly. The non-trivial part of the following proposition is due to Dyer and Frieze [**4**].

PROPOSITION 4. *If the dimension n is fixed, VolComp can be solved in polynomial time; otherwise, VolApp is $\#\mathbb{P}$-equivalent.*

For a survey on volume computation and its application see [**7**]. Let us close this subsection by mentioning that VolApp is $\#\mathbb{P}$-hard in the strong sense: the $\#\mathbb{P}$-hardness persists if the input data in (A, b) is restricted to $\{-1, 0, 1\}$ [**1**].

1.3. *Some results of Brunn–Minkowski theory*

It is one of Minkowski's great achievements to have connected operations on $\mathscr{K}^n$ such as scalar multiplication and (Minkowski)-addition

$$\begin{aligned} \lambda K &= \{\lambda x : x \in K\} & \lambda \in \mathbb{R}; K \in \mathscr{K}^n \\ K_1 + K_2 &= \{x + y : x \in K_1, y \in K_2\} & K_1, K_2 \in \mathscr{K}^n \end{aligned}$$

with the theory of valuations. In particular, he proved the following fundamental theorem [**20**].

PROPOSITION 5. *Let $K_1, K_2 \in \mathscr{K}^n$. Then the function $\xi \mapsto \operatorname{Vol}(K_1 + \xi K_2)$ is a polynomial on $[0, \infty[$ of degree at most n.*

If the coefficient of ξ^i is expressed as

$$\binom{n}{i} V(\overbrace{K_1, \ldots, K_1}^{n-i}, \overbrace{K_2, \ldots, K_2}^{i}),$$

the numbers

$$V_i(K_1, K_2) = V(\overbrace{K_1, \ldots, K_1}^{n-i}, \overbrace{K_2, \ldots, K_2}^{i}) \quad (i = 0, \ldots, n)$$

are called the *mixed volumes* of K_1 and K_2. Mixed volumes are non-negative, monotone, multilinear, and continuous valuations; see [**23**] for an excellent treatment of the Brunn–Minkowski theory, and see [**5**] for a study of algorithmic problems related to mixed volumes and their applications to a number of diverse algorithmic questions including problems in mixture management, combinatorics and algebraic geometry; see also [**7**] for a survey.

Two important results that are relevant to our paper are the following *Brunn–Minkowski theorem* and the *Minkowski inequality*; see [**23**, Chapter 6].

PROPOSITION 6. *Given $K_1, K_2 \in \mathscr{K}^n$, the function*

$$\xi \mapsto g(\xi) = (\mathrm{Vol}\,((1-\xi)\,K_1 + \xi K_2))^{1/n}$$

is concave on $[0, 1]$, *and it is linear on* $[0, 1]$, *if and only if the bodies are homothetic.*

PROPOSITION 7. *Let $K_1, K_2 \in \mathscr{K}^n$, then*

$$V_1^n(K_1, K_2) \geqslant \mathrm{Vol}^{n-1}(K_1)\,\mathrm{Vol}\,(K_2).$$

Equality occurs if and only if the bodies K_1 and K_2 are homothetic.

1.4. *A convex program related to MinkRecon*

Following Minkowski's proof of Proposition 1, we will apply Proposition 7 to polytopes which are 'candidates' for a solution of MinkRecon.

Let A be a rational $m \times n$ matrix with rows $a_1^{\mathrm{T}}, \ldots, a_m^{\mathrm{T}}$, no two positively dependent, such that $\mathrm{pos}\{a_1, \ldots, a_m\} = \mathbb{R}^n$, and for $b = (\beta_1, \ldots, \beta_m)^{\mathrm{T}} \in \mathbb{R}^m$ let

$$P_A(b) = \{x : Ax \leqslant b\}.$$

For simplicity, we will sometimes suppress the subscript A when there is no risk of confusion. For $i = 1, \ldots, m$ let

$$H_i(b) = \{x \in \mathbb{R}^n : \langle a_i, x\rangle = \beta_i\} \quad \text{and} \quad F_i(b) = P_A(b) \cap H_i(b).$$

Note that the input vectors $v = (v_1, \ldots, v_m)^{\mathrm{T}}$ of MinkRecon are contained in the kernel N_A of the linear mapping $y \mapsto y^{\mathrm{T}}\mathrm{A}$. Further, observe that for $t \in \mathbb{R}^n$

$$t + P_A(b) = P_A(b + At).$$

This implies that for each admissible vector $v = (v_1, \ldots, v_m)^{\mathrm{T}} \in N_A$ the solutions of MinkRecon correspond to an equivalence class $b + N_A^{\perp}$, where $N_A^{\perp} = \{At : t \in \mathbb{R}^n\}$. Hence each instance of MinkRecon determines a point in the $(m-n)$-dimensional quotient space $X_A = \mathbb{R}^m / N_A^{\perp}$.

For $i = 1, \ldots, m$ let $\mu_i(b) = \mathrm{Vol}_{n-1}(F_i(b))$, and let $\mu: X_A \to \mathbb{R}^m$ be the mapping

$$\mathbb{R}^m \ni b \mapsto \mu(b) = (\mu_1(b), \mu_2(b), \ldots, \mu_m(b))^{\mathrm{T}}.$$

Then μ is injective, and its image is $Y(N_A)$, where $Y = \mathrm{diag}(\|a_1\|, \ldots, \|a_m\|)$. The aim is to compute its inverse.

The hyperplanes $H_i(b)$ are at distances $\|a_i\|^{-1}\beta_i$ from the origin; hence

$$\mathrm{Vol}(P_A(b)) = \frac{1}{n}\sum_{i=1}^{m} \mu_i(b) \frac{\beta_i}{\|a_i\|}.$$

For $z = (\zeta_1, \ldots, \zeta_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $i = 1, \ldots, m$, let $h_i(z)$ denote the value of the support function of $P_A(z)$ in direction $\mathrm{pos}\{a_i\}$. (In standard terminology, $h_i(z) = h(P_A(z), a_i/\|a_i\|)$.) Then

$$h_i(z) = \max\left\{\left\langle \frac{a_i}{\|a_i\|}, x\right\rangle : x \in P_A(z)\right\} \leqslant \frac{\zeta_i}{\|a_i\|},$$

with equality if and only if $H_i(z)$ supports $P_A(z)$, whence, particularly, if $\mu_i(z) > 0$.

Now let $b, z \in \mathbb{R}^m$ such that $P_A(b)$ has the desired facet-volumes $\mu_i(b) = v_i\|a_i\|$, and that $\mathrm{Vol}(P_A(z)) = 1$. Since

$$V_1(P_A(b), P_A(z)) = \frac{1}{n}\lim_{\xi \to 0} \frac{\mathrm{Vol}(P_A(b) + \xi P_A(z)) - \mathrm{Vol}(P_A(b))}{\xi} = \frac{1}{n}\sum_{i=1}^{m} \mu_i(b)\, h_i(z),$$

Proposition 7 (applied to $K_1 = P_A(b)$ and $K_2 = P_A(z)$) yields

$$\left(\frac{1}{n}\sum_{i=1}^{m} \mu_i(b)\, h_i(z)\right)^n \geqslant \left(\frac{1}{n}\sum_{i=1}^{m} \mu_i(b)\, h_i(b)\right)^{n-1} \mathrm{Vol}(P_A(z)).$$

Since $h_i(z) \leqslant \zeta_i/\|a_i\|$, we have

$$\sum_{i=1}^{m} \mu_i(b)\, h_i(z) \leqslant \langle v, z\rangle, \tag{1}$$

and hence with $\mathrm{Vol}(P_A(z)) = 1$,

$$\langle v, z\rangle^n \geqslant n\langle v, b\rangle^{n-1};$$

equality holds if and only if $P_A(b)$ and $P_A(z)$ are homothetic. Now, with

$$\rho = \left(\frac{n}{\langle v, b\rangle}\right)^{1/n} = \mathrm{Vol}(P_A(b))^{-1/n} \quad \text{and} \quad b^{\diamond} = \rho b,$$

we have $\mathrm{Vol}(P_A(b^{\diamond})) = 1$, and $\langle v, b^{\diamond}\rangle^n = \rho^n\langle v, b\rangle^n = n\langle v, b\rangle^{n-1}$, whence

$$\langle v, z\rangle^n \geqslant \langle v, b^{\diamond}\rangle^n.$$

It follows that $b^{\diamond}$ minimises the function $z \mapsto \langle v, z\rangle$ on the set $\{z \in \mathbb{R}^m : \mathrm{Vol}(P_A(z)) = 1\}$, if and only if $P_A(b^{\diamond})$ is homothetic to a solution of MinkRecon; specifically

$$b = \frac{1}{\rho} b^{\diamond} = \left(\frac{1}{n}\langle v, b\rangle\right)^{1/n} b^{\diamond} = \left(\frac{1}{n}\langle v, b^{\diamond}\rangle\right)^{1/(n-1)} b^{\diamond}.$$

This proves the following proposition.

Proposition 8. *Let* $(n, m; A, v)$ *denote an instance of MinkRecon, let* b *be a solution* (*that is,* $P_A(b)$ *has facet volumes* $v_i\|a_i\|$). *Further, let* $b^{\diamond}$ *be an optimiser of*

$$\min\{\langle v, z\rangle : \mathrm{Vol}(P_A(z)) = 1\}.$$

Then $P_A(b^\diamond)$ is homothetic to $P_A(b)$; specifically, $\lambda b^\diamond$ with $\lambda = (\langle v, b^\diamond\rangle/n)^{1/(n-1)}$ solves MinkRecon.

Clearly, the optimum does not change when we replace the feasible region by

$$C = \{z \in \mathbb{R}^m : \mathrm{Vol}(P_A(z)) \geqslant 1\}.$$

Note that C is closed (since the volume is continuous) and by the Brunn–Minkowski theorem, Proposition 6, C is convex. Hence MinkRecon is equivalent to the following convex program, MinkConvMin.

MinkConvMin. Given an instance $(n, m; A, v)$ of MinkRecon, minimise the linear functional $z \mapsto \langle v, z\rangle$ on $C = \{z \in \mathbb{R}^m : \mathrm{Vol}(P_A(z)) \geqslant 1\}$.

Note that $C + A\mathbb{R}^n \subset C$; in fact, $A\mathbb{R}^n$ is the lineality space of C. This shows again that the condition $v^{\mathrm{T}} A = 0$ is necessary for the solvability of MinkConvMin.

Let us finally point out that though it is constructive, the above characterisation is not fully adequate for our algorithmic purposes. In particular, the convex set C is unbounded, and its lineality space is n-dimensional. This causes problems for applying results from the algorithmic theory of convex bodies. The main difficulty, however, is to handle the fact that only approximative solutions will be available: the oracles are weak. This means, we need strong *stability estimates* for MinkRecon relating the error term ϵ in the optimisation problem to an appropriate distance measure for the corresponding polytopes.

2. *Minkowski reconstruction from a volume oracle*

We will now show that MinkApp can be reduced to VolApp in polynomial time. Theorem 1 and the easiness part of Theorem 2 will then follow from Proposition 4.

Subsection 2.1 derives some bounds that will allow us to replace the *unbounded* feasible regions C of MinkConvMin by convex bodies K. Since it is easy to see that an oracle for approximating the volume of $\mathscr{H}$-polytopes provides a weak membership oracle for K, Proposition 2 allows us then to solve the weak optimisation problem over K in polynomial time. Hence we can construct in polynomial time a polytope which has 'nearly' volume 1 and is 'nearly' homothetic to an exact solution of MinkRecon on the given input. Subsection 2.2 contains some stability estimates which are applied in Subsection 2.3 to show that the 'weak solution' can be used to approximate an exact solution of MinkRecon with respect to the *translative Hausdorff metric*

$$\delta(K_1, K_2) = \min\{d(t + K_1, K_2) : t \in \mathbb{R}^n\} \quad (K_1, K_2 \in \mathscr{K}^n),$$

the translation-invariant version of the *Hausdorff metric*

$$d(K_1, K_2) = \min\{\tau \geqslant 0 : K_1 \subset K_2 + \tau\mathbb{B}^n \wedge K_2 \subset K_1 + \tau\mathbb{B}^n\} \quad (K_1, K_2 \in \mathscr{K}^n).$$

Subsection 2.4 extends these results to give a full solution of MinkApp.

2.1. *Bounds for solutions of MinkConvMin*

In order to solve MinkConvMin by way of Proposition 2, we have to replace the unbounded set C by a suitably centred well-guaranteed convex body $K \subset C$ without changing the optimal value of the objective function. Let L denote the size of the input $(n, m; A, v)$, and let again $C = \{z : \mathrm{Vol}(P_A(z)) \geqslant 1\}$.

First, note that it suffices to consider only $C \cap]0, \infty[^m$ since under our general assumptions, there is a proper polytope $P_A(b)$ that solves MinkRecon, and the translation invariance of the problems allows us to assume that $0 \in \operatorname{int}(P_A(b))$, that is, $b > 0$. Now let

$$\mathbb{1} = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^m, \quad \omega_n = \operatorname{Vol}(\mathbb{B}^n), \quad \kappa_1 = \tfrac{1}{2}\lceil \sqrt{n} \rceil \cdot \max_{i=1,\ldots,m} \lceil \|a_i\| \rceil,$$
$$\kappa_2 = (\min\{v_1, \ldots, v_m\})^{-1} \langle v, \mathbb{1} \rangle \quad \text{and} \quad p = \kappa_1 \mathbb{1}.$$

Then $\frac{1}{2}\sqrt{n}\mathbb{B}^n \subset P_A(p)$, whence $1 \leqslant (\frac{1}{2}\sqrt{n})^n \omega_n \leqslant \operatorname{Vol}(P_A(p))$, and thus $p \in C$. Since then also $p + [0, \infty[^m \subset C$, we have $2p + \kappa_1 \mathbb{B}^m \subset C$. Further, we have for any positive solution $b^\diamond = (\beta_1^\diamond, \ldots, \beta_m^\diamond)^{\mathrm{T}}$ of MinkConvMin, and $i = 1, \ldots, m$,

$$\beta_i^\diamond \leqslant \frac{\langle v, b^\diamond \rangle}{v_i} \leqslant \frac{\langle v, p \rangle}{v_i} \leqslant \kappa_1 \kappa_2.$$

Therefore, the optimum of MinkConvMin remains unchanged if we replace C by the convex body

$$D = \{z \in C : 0 \leqslant z \leqslant 3\kappa_1 \kappa_2 \mathbb{1}\}.$$

Note that $D \subset 2^{2L}[0,1]^m$, and $2p + \kappa_1 \mathbb{B}^m \subset D$; hence D is centred well-guaranteed. Further, D is stable in the following sense. If

$$\tilde{v} \in S(v) = \{\tilde{v} \in N_A : \|\tilde{v} - v\|_\infty \leqslant \tfrac{1}{2}\min\{v_1, \ldots, v_m\}\},$$

then D still contains an optimiser of the modified instance $(n, m; A, \tilde{v})$ of MinkConvMin. This follows from the fact that the hyperplane $\tilde{H} = \{z : \langle \tilde{v}, z \rangle = \langle \tilde{v}, p \rangle\}$ meets the coordinate axes in points that are at most at distance $3\kappa_1\kappa_2$ from the origin.

Further, note that for each $z \in D$ the *circumradius* $R(P_A(z))$ of $P_A(z)$ is bounded above by 2^{12L}.

For the proof of our tractability results we will also need a lower bound on the distance of suitable optimisers from the coordinate hyperplanes. For $z \in D$ we have

$$1 \leqslant \operatorname{Vol}(P_A(z)) \leqslant (2R(P_A(z)))^{n-1} \cdot w(P_A(z)),$$

where $w(P_A(z)) = \min_{\|u\|=1} \max_{x,y \in P_A(z)} \langle u, x - y \rangle$ is the *width* of $P_A(z)$. By Steinhagen's inequality [**24**] we have for the *inradius* $r(P_A(z))$ of $P_A(z)$,

$$r(P_A(z)) \geqslant \frac{1}{2\sqrt{n+1}} w(P_A(z)) \geqslant \frac{1}{2\sqrt{n+1}} (2R(P_A(z)))^{1-n},$$

so we may use the translation invariance of MinkRecon again to assume that

$$2^{-n} R(P_A(z))^{1-n} (n+1)^{-1/2} \mathbb{B}^n \subset P_A(z).$$

Since $2^{-n}(n+1)^{-1/2} R(P_A(z))^{1-n} \|a_i\| \geqslant 2^{-14nL}$, at least one optimiser of MinkConvMin is contained in

$$K = D \cap \bigcap_{i=1}^m \{x : \langle x, e_i \rangle \geqslant 2^{-14nL}\},$$

where e_i is the ith standard unit vector of $\mathbb{R}^m$. Clearly, since the preceding argument only involved A, the body K contains optimisers also for all instances $(n, m;\, A, \tilde{v})$ of MinkConvMin with $\tilde{v} \in S(v)$.

The essence of the above conclusions is summarised in the following lemma. (Note that some of the given bounds are not tight – Lemma 1 is stated as it is to make the technical computations in the following subsections as simple as possible.)

LEMMA 1. *Let $(n, m; A, v)$ be an instance of MinkConvMin of size L, and let $b^\diamond$ denote a positive solution. Further, let $\vartheta = 2^{-16nL}$, $\Theta = \vartheta^{-1}$ and $I = [\vartheta, \Theta]$, and set*

$$K = \{z : 2^{-15nL}\mathbb{1} \leqslant z \leqslant 2^{2L}\mathbb{1} \wedge \mathrm{Vol}\,(P_A(z)) \geqslant 1\}.$$

Then the following statements hold.

(i) *K is a centred, well-guaranteed convex body of $\mathbb{R}^m$, $K \subset I^n$ and $\langle K \rangle$ is bounded by a polynomial in L.*

(ii) $b^\diamond \in \frac{1}{3}2^{2L}[0,1]^m$.

(iii) *Whenever $\tilde{v} \in S(v)$, the body K contains an optimiser $\tilde{b}^\diamond$ for the instance $(n, m; A, \tilde{v})$ of MinkConvMin.*

(iv) *$R(P_A(z)) \leqslant \Theta$ whenever $z \in K$.*

(v) *$\langle \tilde{v}, z \rangle \in nI$ for every $\tilde{v} \in S(v)$ and $z \in K$.*

(vi) *$\tilde{\lambda} = (\langle \tilde{v}, \tilde{b}^\diamond \rangle / n)^{1/(n-1)} \in I$, for every $\tilde{v} \in S(v)$ (λ is the scaling factor defined in the previous subsection).*

(vii) *$R(P_A(\tilde{b})) \leqslant \Theta^2$, where $\tilde{b}$ is any solution of MinkRecon on the input $(n, m; A, \tilde{v})$ with $\tilde{v} \in S(v)$.*

2.2. *Some stability estimates*

We begin with a result of Groemer [**8**]; see also [**23**, p. 318]. (For earlier results see [**2**, **3**, **27**], and see [**9**] for a survey on the stability of geometric inequalities.)

PROPOSITION 9. *Let $K_1, K_2 \in \mathscr{K}^n$ with $\mathrm{Vol}\,(K_1) = \mathrm{Vol}\,(K_2) = 1$. Further, define $R = \max\{R(K_1), R(K_2)\}$, and*

$$\Delta(K_1, K_2) = V_1^n(K_1, K_2) - \mathrm{Vol}^{n-1}(K_1)\,\mathrm{Vol}\,(K_2) = V_1^n(K_1, K_2) - 1.$$

Then

$$\delta(K_1, K_2) \leqslant \left(\frac{1}{2n}\right)^{1/(n+1)} (4 \cdot 6.000\,25 \cdot n) \cdot 2R \cdot \Delta(K_1, K_2)^{1/(n+1)} \leqslant 50Rn\Delta(K_1, K_2)^{1/(n+1)}.$$

Note that $\Delta(K_1, K_2) = 0$ if and only if the bodies K_1 and K_2 are homothetic.

While the proof of Proposition 9 is quite involved, the following lemma is rather easy to show; it will be needed for estimating the error that is induced by rescaling a convex body.

LEMMA 2. *Let $K_1, K_2 \in \mathscr{K}^n$, set $R = \max\{R(K_1), R(K_2)\}$, and let λ_1, λ_2 be non-negative reals. Then*

$$\delta(\lambda_1 K_1, \lambda_2 K_2) \leqslant |\lambda_1 - \lambda_2|\, R + \delta(K_1, K_2) \max\{\lambda_1, \lambda_2\}.$$

Proof. We may assume that $K_2 \subset R\mathbb{B}^n$. Now, suppose that for some points $t_1, t_2 \in \mathbb{R}^n$

$$t_1 + K_1 \subset K_2 + \delta(K_1, K_2)\,\mathbb{B}^n \quad \text{and} \quad t_2 + K_2 \subset K_1 + \delta(K_1, K_2)\,\mathbb{B}^n.$$

Then we have

$$\begin{aligned}\lambda_1 t_1 + \lambda_1 K_1 &\subset \lambda_1 K_2 + \delta(K_1, K_2)\,\lambda_1 \mathbb{B}^n = (\lambda_2 + (\lambda_1 - \lambda_2))\,K_2 + \delta(K_1, K_2)\,\lambda_1 \mathbb{B}^n \\ &\subset \lambda_2 K_2 + |\lambda_1 - \lambda_2|\, R\mathbb{B}^n + \delta(K_1, K_2)\,\lambda_1 \mathbb{B}^n = \lambda_2 K_2 + (|\lambda_1 - \lambda_2|\, R + \delta(K_1, K_2)\,\lambda_1)\,\mathbb{B}^n.\end{aligned}$$

Interchanging the roles of K_1 and K_2 yields a second inclusion which, together with the first, gives the assertion. □

2.3. *Minkowski reconstruction for the translative Hausdorff distance*

We now give proofs of the tractability results for the variant of MinkApp that is obtained by replacing its distance measure $\max_{i=1,\dots,m}\{|\tilde{\mu}_i - \mu_i|\}$ by the translative Hausdorff distance δ. This is in fact the main step in the proofs of our 'positive' results.

Let $(n, m; A, v; \epsilon)$ be an instance of MinkApp, let L denote its size, and suppose that $\epsilon < 1$. We want to construct a vector $\tilde{b} \in \mathbb{Q}^m$ such that

$$\delta(P_A(b), P_A(\tilde{b})) \leqslant \epsilon,$$

where b denotes again an (exact) solution of MinkRecon on the instance $(n, m; A, v)$. Without further notice, we use the notation of Lemma 1 (and we will also make use of the results of Lemma 1 without always referring to it). The construction falls into two steps: first we derive a weak optimisation oracle for the set K and compute a rational polytope $P = P_A(\tilde{\sigma}\tilde{b}^{\diamond})$ with $\delta(P, P_A(b^{\diamond})) \leqslant \epsilon_1$, where $b^{\diamond}$ is an exact solution to MinkConvMin on the input $(n, m; A, v)$, and $\epsilon_1 = \frac{1}{4}\vartheta\epsilon$. In the second step we compute $\tilde{b}$ by means of a suitable scaling.

By Proposition 4, VolApp is #$\mathbb{P}$-easy and, moreover, is in $\mathbb{P}$ when the dimension is fixed. Suppose we have an algorithm $\mathscr{A}$ for VolApp. Then $\mathscr{A}$ can be used to solve the weak membership problem for K in oracle-polynomial time. In fact, checking the coordinate bounds in the definition of K is easy, and – using Proposition 3 and some simple calculations involving mixed volumes – it is also standard fare to show that the approximation measure in VolApp and the weakness notion in the weak membership problem are sufficiently compatible. (Note that when the dimension is fixed even the strong membership problem for K can be solved in polynomial time.) By Proposition 2 we can then solve the weak optimisation problem for K in oracle-polynomial time. Let us call such a weak optimisation oracle with input v and error bound

$$\eta = \frac{1}{(200n)^{n+1} \cdot 8n}\,\vartheta^{n+2}\epsilon_1^{n+1}.$$

Since $K(-\eta) \neq \varnothing$, the oracle outputs a vector $\tilde{b}^{\diamond} = (\tilde{\beta}_1^{\diamond}, \dots, \tilde{\beta}_m^{\diamond})^{\mathrm{T}} \in \mathbb{Q}^m$ such that

$$\tilde{b}^{\diamond} \in K(\eta) \quad \text{and} \quad \langle v, \tilde{b}^{\diamond}\rangle \leqslant \langle v, z\rangle + \eta \quad \text{for all } z \in K(-\eta).$$

With the aid of Lemma 1 it is easy to see that $\tilde{b}^{\diamond} + \eta\mathbb{1} \in C$, and $b^{\diamond} + \eta\mathbb{1} \in K(-\eta)$, whence

$$\mathrm{Vol}\,(P_A(\tilde{b}^{\diamond} + \eta\mathbb{1})) \geqslant 1 \quad \text{and} \quad \langle v, \tilde{b}^{\diamond}\rangle \leqslant \langle v, b^{\diamond}\rangle + \eta\langle v, \mathbb{1}\rangle \leqslant \langle v, b^{\diamond}\rangle + \eta 2^L. \tag{2}$$

Since for $i = 1, \dots, m$,

$$\tilde{\beta}_i^{\diamond} - (1 - 2\eta\Theta)\,(\tilde{\beta}_i^{\diamond} + \eta) \geqslant \eta(2\Theta\tilde{\beta}_i^{\diamond} - 1) \geqslant 0,$$

we have

$$P_A(\tilde{b}^{\diamond}) \supset P_A((1 - 2\eta\Theta)\,(\tilde{b}^{\diamond} + \eta\mathbb{1})),$$

whence

$$\mathrm{Vol}\,(P_A(\tilde{b}^{\diamond})) \geqslant (1 - 2\eta\Theta)^n.$$

Now let $\sigma = \mathrm{Vol}(P_A(\tilde{b}^\diamond))^{-1/n}$. Then $\mathrm{Vol}(P_A(b^\diamond)) = \mathrm{Vol}(P_A(\sigma\tilde{b}^\diamond)) = 1$, and it follows that $\sigma \leqslant (1-2\eta\Theta)^{-1} < \frac{4}{3}$. Using the homogenity of mixed volumes, it follows from Proposition 8 that

$$V_1(P_A(b), P_A(\sigma\tilde{b}^\diamond)) = \frac{1}{n}\langle v, b^\diamond\rangle \cdot V_1(P_A(b^\diamond), P_A(\sigma\tilde{b}^\diamond));$$

and since by (1)

$$V_1(P_A(b), P_A(\sigma\tilde{b}^\diamond)) \leqslant \frac{\sigma}{n}\langle v, \tilde{b}^\diamond\rangle \quad \text{and} \quad \eta\langle v, \mathbb{1}\rangle \leqslant \eta\Theta \leqslant \frac{1}{8n},$$

we have (with the aid of (2) and Lemma 1)

$$\begin{aligned}
\Delta(P_A(b^\diamond), P_A(\sigma\tilde{b}^\diamond)) &= V_1(P_A(b^\diamond), P_A(\sigma\tilde{b}^\diamond))^n - 1 \\
&= \left(\frac{n}{\langle v, b^\diamond\rangle} V_1(P_A(b), P_A(\sigma\tilde{b}^\diamond))\right)^n - 1 \leqslant \sigma^n \left(\frac{\langle v, \tilde{b}^\diamond\rangle}{\langle v, b^\diamond\rangle}\right)^n - 1 \\
&\leqslant \sigma^n\left(1 + \frac{\eta 2^L}{\langle v, b^\diamond\rangle}\right)^n - 1 \leqslant \left(\frac{1+\eta\Theta}{1-2\eta\Theta}\right)^n - 1 \\
&\leqslant (1+4\eta\Theta)^n - 1 \leqslant 4n\eta\Theta + \sum_{j=2}^{n}(4n\eta\Theta)^j \leqslant 8n\eta\Theta.
\end{aligned}$$

Since $\tilde{b}^\diamond \in K(\eta)$ it follows from Lemma 1(iv) that $R(P_A(\tilde{b}^\diamond - \eta\mathbb{1})) \leqslant \Theta$. Further, since $\tilde{\beta}_i^\diamond \leqslant \frac{3}{2}(\tilde{\beta}_i^\diamond - \eta)$, for $i = 1, \ldots, m$, and $\sigma \leqslant \frac{4}{3}$,

$$\max\{R(P_A(b^\diamond)), R(P_A(\sigma\tilde{b}^\diamond))\} \leqslant 2\Theta.$$

Using Proposition 9, we obtain

$$\delta(P_A(\sigma\tilde{b}^\diamond), P_A(b^\diamond)) \leqslant 50n \cdot 2\Theta \cdot (8n\eta\Theta)^{1/(n+1)} \leqslant \tfrac{1}{2}\epsilon_1.$$

Now we use $\mathscr{A}$ to approximate $\mathrm{Vol}(P_A(\tilde{b}^\diamond))$ to absolute error $\vartheta^2\epsilon_1$. Using the fact that $\sigma \leqslant \frac{4}{3}$, and applying the mean value theorem of calculus to the function $x \mapsto x^{-1/n}$, we obtain an estimate $\tilde{\sigma}$ of σ such that

$$|\sigma - \tilde{\sigma}| \leqslant \tfrac{1}{4}\vartheta\epsilon_1.$$

By Lemma 2,

$$\delta(P_A(\tilde{\sigma}\tilde{b}^\diamond), P_A(\sigma\tilde{b}^\diamond)) \leqslant |\tilde{\sigma} - \sigma| \cdot 2\Theta \leqslant \tfrac{1}{2}\epsilon_1,$$

whence

$$\delta(P_A(\tilde{\sigma}\tilde{b}^\diamond), P_A(b^\diamond)) \leqslant \delta(P_A(\tilde{\sigma}\tilde{b}^\diamond), P_A(\sigma\tilde{b}^\diamond)) + \delta(P_A(\sigma\tilde{b}^\diamond), P_A(b^\diamond)) \leqslant \epsilon_1.$$

This concludes the first step of the construction.

To obtain the desired vector $\tilde{b} \in \mathbb{Q}^m$ with $\delta(P_A(b), P_A(\tilde{b})) \leqslant \epsilon$, we will now apply a suitable scaling to $\tilde{b}^\diamond$. Let

$$\lambda = \left(\frac{1}{n}\langle v, b^\diamond\rangle\right)^{1/(n-1)} \quad \text{and} \quad \tilde{\lambda} = \left(\frac{1}{n}\langle v, \tilde{b}^\diamond\rangle\right)^{1/(n-1)};$$

see Proposition 8. Using Lemma 1, we obtain

$$\frac{1}{n}\langle v, \tilde{b}^\diamond\rangle = \frac{1}{n}\langle v, \tilde{b}^\diamond + \eta\mathbb{1}\rangle - \frac{1}{n}\eta\langle v, \mathbb{1}\rangle \geqslant \vartheta - \eta\Theta \geqslant \frac{3}{4}\vartheta \quad \text{and} \quad \frac{1}{n}\langle v, b^\diamond\rangle \geqslant \frac{3}{4}\vartheta.$$

Further,

$$\langle v, b^{\diamond}\rangle \leqslant \sigma\langle v, \tilde{b}^{\diamond}\rangle \leqslant (1+4\eta\Theta)\langle v, \tilde{b}^{\diamond}\rangle \leqslant \langle v, \tilde{b}^{\diamond}\rangle + 4\eta\Theta^2.$$

With the aid of (2) and the mean value theorem applied to the function $x \mapsto x^{1/(n-1)}$, it follows that

$$|\tilde{\lambda} - \lambda| \leqslant \frac{1}{n-1}\left(\frac{3\vartheta}{4}\right)^{(1/(n-1))-1} 4\eta\Theta^2 < 6\eta\Theta^3.$$

Now, we approximate $\tilde{\lambda}$ to absolute error $2\eta\Theta^3$; let $\hat{\lambda}$ denote the corresponding estimate. Then, using $8\eta \leqslant \vartheta^2$ and $\lambda \leqslant \Theta$, we have

$$|\hat{\lambda} - \lambda| \leqslant 8\eta\Theta^3 \quad \text{and} \quad \max\{\lambda, \hat{\lambda}\} \leqslant 2\Theta.$$

Setting $\tilde{b} = \hat{\lambda}\tilde{\sigma}\tilde{b}^{\diamond}$, and using $\eta \leqslant \frac{1}{8}\vartheta^3\epsilon_1$, Lemma 2 yields

$$\delta(P_A(b), P_A(\tilde{b})) = \delta(\lambda P_A(b^{\diamond}), \tilde{\lambda} P_A(\tilde{\sigma}\tilde{b}^{\diamond})) \leqslant 8\eta\Theta^3 \cdot 2\Theta + 2\Theta \cdot \epsilon_1 \leqslant \epsilon,$$

and this is the desired estimate.

2.4. *From Hausdorff to Minkowski*

The following lemma gives a relation between the Hausdorff distances of two polytopes $P_A(z_1)$, $P_A(z_2)$ and the distances of corresponding facets.

LEMMA 3. *Let $a_1, \ldots, a_m \in \mathbb{R}^n \backslash \{0\}$, mutually non-collinear, with* $\operatorname{pos}\{a_1, \ldots, a_m\} = \mathbb{R}^n$*, let A denote the matrix with rows $a_1^{\mathrm{T}}, \ldots, a_m^{\mathrm{T}}$, let $\epsilon > 0$ and let $z_1, z_2 \in \mathbb{R}^m$ such that*

$$d(P_A(z_1), P_A(z_2)) \leqslant \epsilon.$$

For $i = 1, \ldots, m$, let $F_i(z_1)$ and $F_i(z_2)$ denote the facets of $P_A(z_1)$ and $P_A(z_2)$, respectively, that correspond to a_i, and let $t_i \in \epsilon\mathbb{B}^n$ such that $F_i(z_1)$ and $t_i + F_i(z_2)$ lie in a common hyperplane H_i perpendicular to a_i. Then we have for the Hausdorff distance d_H relative to H_i,

$$d_H(F_i(z_1), t_i + F_i(z_2)) \leqslant \frac{2\epsilon}{1-\alpha^2} \leqslant 2\epsilon\Theta,$$

where

$$\alpha = \max\left\{\left\langle \frac{a_k}{\|a_k\|}, \frac{a_l}{\|a_l\|}\right\rangle : 1 \leqslant k < l \leqslant m\right\}.$$

Proof. Let $i \in \{1, \ldots, m\}$. Since $t_i \in \epsilon\mathbb{B}^n$, we have $d(P_A(z_1), t_i + P_A(z_2)) \leqslant 2\epsilon$, so we may prove the assertion under the assumption that $t_i = 0$ and $d(P_A(z_1), P_A(z_2)) \leqslant 2\epsilon$. Let $x \in F_i(z_1)\backslash F_i(z_2)$, and let $y \in F_i(z_2)$ be closest to x. Clearly, y is contained in some other facet $F_j(z_2)$ of $P_A(z_2)$ whose supporting halfspace misses x. Hence the ray $x - [0, \infty[a_j$ intersects the hyperplane $\operatorname{aff}(F_j(z_2))$ in some point x_0 of the form $x - \lambda_0 a_j/\|a_j\|$ with $\lambda_0 > 0$. Let y_0 be the point in $P_A(z_2)$ closest to x. Then $\|x - y_0\| \leqslant 2\epsilon$, and the segment $\operatorname{conv}\{x, y_0\}$ intersects $\operatorname{aff}(F_j(z_2))$. This implies that $\lambda_0 \leqslant 2\epsilon$. Now, consider the (possibly degenerate) triangle $\operatorname{conv}\{x, y, x_0\}$, and let ϕ denote its angle at y. Then

$$\|x - y\| = \frac{\lambda}{\sin\phi} = \frac{\lambda_0}{\sqrt{1-\cos^2\phi}} \leqslant \frac{\lambda_0}{\sqrt{1-\alpha^2}} \leqslant \frac{2\epsilon}{1-\alpha^2},$$

and (since z_1 and z_2 play symmetric roles) this implies the first part of the assertion.

For the second inequality just note that for $1 \leqslant i < j \leqslant m$ the vectors a_i and a_j are linearly independent, whence $\tau_{ij} = \langle a_i, a_i \rangle \langle a_j, a_j \rangle - \langle a_i, a_j \rangle^2$ is a positive rational bounded below by 2^{-4nL}. This implies that

$$1 - \alpha^2 = \min_{1 \leqslant i < j \leqslant m} \frac{\langle a_i, a_i \rangle \langle a_j, a_j \rangle - \langle a_i, a_j \rangle^2}{\|a_i\|^2 \cdot \|a_j\|^2} = \min_{1 \leqslant i < j \leqslant m} \frac{\tau_{ij}}{\|a_i\|^2 \cdot \|a_j\|^2} \geqslant \vartheta,$$

and concludes the proof of Lemma 3. □

LEMMA 4. *Let $K_1, K_2 \in \mathscr{K}^n$, both contained in $R\mathbb{B}^n$. Then, with $\omega_n = \operatorname{Vol}(\mathbb{B}^n)$,*

$$|\operatorname{Vol}(K_1) - \operatorname{Vol}(K_2)| \leqslant \omega_n R^n \left(\left(1 + \frac{d(K_1, K_2)}{R} \right)^n - 1 \right).$$

Proof. Let $d = d(K_1, K_2)$. Then $K_1 \subset K_2 + d\mathbb{B}^n$, and this implies that

$$\operatorname{Vol}(K_1) \leqslant \operatorname{Vol}(\mathrm{K}_2 + \mathrm{d}\mathbb{B}^n) = \operatorname{Vol}(K_2) + \sum_{i=1}^{n} \binom{n}{i} V(\overbrace{K_2, \ldots, K_2}^{n-i}, \overbrace{\mathbb{B}^n, \ldots, \mathbb{B}^n}^{i})\, d^i$$

$$\leqslant \operatorname{Vol}(K_2) + \sum_{i=1}^{n} \binom{n}{i} V(\overbrace{R\mathbb{B}^n, \ldots, R\mathbb{B}^n}^{n-i}, \overbrace{\mathbb{B}^n, \ldots, \mathbb{B}^n}^{i})\, d^i$$

$$= \operatorname{Vol}(K_2) + \omega_n R^n \sum_{i=1}^{n} \binom{n}{i} \left(\frac{d}{R} \right)^i = \operatorname{Vol}(K_2) + \omega_n R^n \left(\left(1 + \frac{d}{R} \right)^n - 1 \right);$$

since the roles of K_1 and K_2 can be reversed, this implies the assertion. □

We are now able to finish the proof of our tractability results. Let $(n, m; A, b; \epsilon)$ be an instance of MinkApp, let L denote its size, and assume that $\epsilon < 1$. Calling the algorithm of Subsection 2.3 with error bound

$$\epsilon_2 = \frac{1}{4n} \vartheta^{3n+1} \frac{1}{\lceil \omega_{n-1} \rceil} \epsilon,$$

we obtain a polytope $P_A(\tilde{b})$ satisfying $\delta(P_A(b), P_A(\tilde{b})) \leqslant \epsilon_2$, where b is a solution of MinkRecon on the input $(n, m; A, b)$. Now let $i \in \{1, \ldots, m\}$. If necessary, we apply a translation t_i to move the faces $F_i(b)$ of $P_A(b)$ and $F_i(\tilde{b})$ of $P_A(\tilde{b})$ into the same hyperplane $H_i = \operatorname{aff}(F_i(b))$. Then by Lemma 3,

$$d = d(F_i(\tilde{b}), t_i + F_i(b)) \leqslant \frac{\vartheta^{3n} \epsilon}{2n\omega_{n-1}}.$$

Now $R(P_A(b)), R(P_A(\tilde{b})) \leqslant \Theta^3$, and hence, clearly, $R(F_i(b)), R(F_i(\tilde{b})) \leqslant \Theta^3$. We can apply Lemma 4 to conclude that

$$|v_i \|a_i\| - \mu_i(\tilde{b})| \leqslant \omega_{n-1} R^{n-1} \left(\left(1 + \frac{d}{R} \right)^{n-1} - 1 \right) \leqslant \omega_{n-1} \sum_{j=1}^{n-1} (dn)^j R^{n-j-1}$$

$$\leqslant \omega_{n-1} \Theta^{3n} \frac{dn}{1 - dn} \leqslant 2\omega_{n-1} \Theta^{3n} dn \leqslant \epsilon.$$

This concludes the proof of the $\#\mathbb{P}$-easiness of MinkApp and the proof of Theorem 1. Note that we have actually shown slightly more: whenever VolApp can be solved in polynomial time for a class of $\mathscr{H}$-polytopes, MinkApp can also be solved in polynomial time. This includes the class of all 'near-simplicial' polytopes in varying dimensions; see [**7**].

3. *Hardness of Minkowski reconstruction*

In the last section we showed that any oracle for computing the volume of an $\mathscr{H}$-polytope can be used to devise an oracle-polynomial-time algorithm for the weak optimisation problem MinkConvMin, whence for MinkApp. In the present section we reverse this argument and show that any oracle for MinkApp gives an oracle-polynomial-time algorithm for computing the volume of $\mathscr{H}$-polytopes. Since the latter problem is #$\mathbb{P}$-hard if the dimension is part of the input, so is the former.

The proof of the #$\mathbb{P}$-hardness of MinkApp falls into three parts. Subsection 3.1 contains the reduction of VolApp to the weak membership problem for the feasible regions C of MinkConvMin. In Subsection 3.2 we use techniques from computational convexity to extend Proposition 2 and show that this weak membership problem (for the unbounded sets C) can be reduced to the weak optimisation problem for the same sets. The weak optimisation problem is then reduced to MinkApp in Subsection 3.3.

3.1. *Reduction of VolApp to a weak membership problem*

Let $(n, m; A, b; \epsilon)$ be an instance of VolApp, let L denote its size, and let P be the corresponding $\mathscr{H}$-polytope. Using Proposition 3, we can decide in polynomial time whether P has empty interior (or, equivalently, $\mathrm{Vol}(P) = 0$), or otherwise find a point p interior to P. Suppose that $\mathrm{Vol}(P) > 0$, and (applying a translation about $-p$, if necessary) that $b = (\beta_1, \ldots, \beta_m)^{\mathrm{T}} > 0$. Let $\beta_{\min} = \min\{\beta_1, \ldots, \beta_m\}$ and $\beta_{\max} = \max\{\beta_1, \ldots, \beta_m\}$.

Again with the aid of Proposition 3, we may rescale P and compute an integer ζ of size bounded by a polynomial in L such that $1 < \mathrm{Vol}(P) < \zeta$. Note that for $\xi > 0$,

$$\xi b \in C \Leftrightarrow \xi^n \mathrm{Vol}(P_A(b)) \geqslant 1 \Leftrightarrow \mathrm{Vol}(P_A(b)) \geqslant \xi^{-n}.$$

Now suppose a weak membership oracle is available for C. On the input $(\xi b, \eta)$ (with $\xi, \eta \in \mathbb{Q}$, $\xi, \eta > 0$), the oracle may supply the answers '$\xi b \in C(\eta)$' or '$\xi b \notin C(-\eta)$'. In the first case, $\xi b + \eta \mathbb{1} \in C$. Using the fact that $\xi b + \eta \mathbb{1} \leqslant (\xi + \eta_1) b$, where $\eta_1 = \eta \beta_{\min}^{-1}$, we see that $(\xi + \eta_1) b \in C$, whence

$$\mathrm{Vol}\,(P_A(b)) \geqslant (\xi + \eta_1)^{-n}.$$

In the second case, that is, when the oracle answers '$\xi b \notin C(-\eta)$', we have $\xi b - \eta \mathbb{1} \notin C$. Using $\xi b - \eta \mathbb{1} \geqslant (\xi - \eta_1) b$, it follows that $(\xi - \eta_1) b \notin C$, whence

$$\mathrm{Vol}\,(P_A(b)) < (\xi - \eta_1)^{-n}.$$

Now choose η specifically to be a positive rational of size bounded by a polynomial in L such that

$$\eta \leqslant \min\left\{\frac{\beta_{\min}}{4^{n+2} n \zeta^2 \epsilon}, \frac{\beta_{\min}}{4 \zeta^{1/n}}\right\},$$

and set, as before, $\eta_1 = \eta / \beta_{\min}$. Since $\eta_1 \leqslant \frac{1}{4} \zeta^{-1/n}$, we can compute some rational number

$$\xi_{\min} \in [\tfrac{1}{2} \zeta^{-1/n}, \zeta^{-1/n} - \eta_1].$$

Then $(\xi_{\min}+\eta_1)^{-n} \geqslant \zeta$, whence the oracle answers '$\xi_{\min} b \notin C(-\eta)$'. Similarly, on $\xi_{\max} = 2 \geqslant 1+\eta_1$ the oracle produces the answer '$\xi_{\max} b \in C(\eta)$' since $\mathrm{Vol}(P_A(b)) > 1$. Standard application of binary search enables us now to compute rational numbers ξ_1, ξ_2 with $0 < \xi_2 - \xi_1 \leqslant 2\eta_1$ on which the oracle answers '$\xi_2 b \in C(\eta)$' and '$\xi_1 b \notin C(-\eta)$'. Note that $\langle \eta_1 \rangle$ is bounded by a polynomial in L; hence the binary search terminates in polynomial time. We obtain

$$(\xi_2+\eta_1)^{-n} \leqslant \mathrm{Vol}(P_A(b)) \leqslant (\xi_1-\eta_1)^{-n},$$

and with the aid of the mean value theorem applied to the function $x \mapsto x^{-n}$, we see that

$$\begin{aligned}(\xi_1-\eta_1)^{-n}-(\xi_2+\eta_1)^{-n} &\leqslant n(\tfrac{1}{4}\zeta^{-1/n})^{-n-1}(2\eta_1+2\eta_1) \leqslant n4^{n+1}\zeta^2 \cdot 4\eta_1 \\ &= \eta 4^{n+2} n \zeta^2 \beta_{\min}^{-1} \leqslant \epsilon.\end{aligned}$$

Hence $\hat{V} = (\xi_2+\eta_1)^{-n}$ is a desired approximation of $\mathrm{Vol}(P)$. This completes our reduction and proves, in particular, that the weak membership problem for the feasible regions C of MinkConvMin is #$\mathbb{P}$-hard.

3.2. *Reduction of weak membership for sets C to weak optimisation*

Now we show how a weak optimisation oracle for MinkConvMin can be used to devise an oracle-polynomial-time algorithm for the weak membership problem for the underlying convex sets C. Note that Proposition 2 is restricted to convex bodies, and hence does not directly apply to the unbounded sets C.

Let $(n, m; A)$ be given as before, let $C = \{z \in \mathbb{R}^m : \mathrm{Vol}(P_A(z)) \geqslant 1\}$, and suppose that a weak optimisation oracle for C is available for positive rational inputs (v, ϵ) with $v \in N_A$. Suppose, now, that (b, ϵ) is a given instance of the corresponding weak membership problem, that is, $b \in \mathbb{Q}^m$ is the query point, and ϵ is the positive rational error bound. Since $C + A\mathbb{R}^n \subset C$, we may assume that $b > 0$.

Note that we are not really interested in points of C whose components are much greater than those of b. This fact can be used to show that for solving the weak membership problem on the input (b, ϵ) we need only calls to the optimisation oracle whose set of optimisers contains points not much greater than b. To make this observation precise, we will now cut off points of C by considering the polar of some translate of C.

Let $p \in C \cap \mathbb{Q}^m$ with $b < p$, let $\rho \in \mathbb{Q}$, $\rho > 0$ such that $2p + \rho\mathbb{B}^m \subset C$, and suppose that $\langle p \rangle$ and $\langle \rho \rangle$ are bounded by a polynomial in the input size. Now, let $\hat{C} = -2p + C$ and let $\hat{b} = b - 2p$; note that $\hat{b} < 0$. Clearly, solving the weak membership problem for C on the input (b, ϵ) is equivalent to solving the weak membership problem for $\hat{C}$ on the input $(\hat{b}, \epsilon)$. To cut off all points of $\hat{C}$ that are not lying in the negative orthant, we use polarity:

$$\hat{C}^\circ = \{y \in \mathbb{R}^m : \langle z, y \rangle \leqslant 1 + 2\langle p, y \rangle \text{ for all } z \in C\}.$$

Since C contains the ball $2p + \rho\mathbb{B}^m$, the set $\hat{C}^\circ$ is bounded. Further, $\hat{C} + [0, \infty[^m + N_A^\perp \subset \hat{C}$, whence $\hat{C}^\circ \subset]-\infty, 0]^m \cap N_A$. Moreover, $\hat{C} \subset -2p + [0, \infty[^m + N_A^\perp$; hence $T \cap N_A \subset \hat{C}^\circ$, where T is the simplex that is cut out of the negative orthant by the halfspace $\{y : \langle p, y \rangle \geqslant -\frac{1}{2}\}$. Since the columns of A^{T} positively span $\mathbb{R}^n$, the linear space $N_A = \ker A^{\mathrm{T}}$ contains a point of $]-\infty, 0[^m$, whence $\hat{C}^\circ$ has non-empty interior relative to N_A.

Clearly, the weak optimisation oracle for C leads to a weak validity oracle for $\hat{C}$. Further, it is not hard to see that one can hence derive an oracle-polynomial-time algorithm that solves the weak membership problem for $\hat{C}^\circ$ relative to N_A; a detailed proof of this fact is given in [**10**, Lemma 4.4.1, pp. 114–115] (it is formulated there only for convex bodies but extends word by word to our situation). By Proposition 2 (applied to $\hat{C}^\circ$ relative to N_A) we obtain an oracle-polynomial-time weak optimisation algorithm for $\hat{C}^\circ$. Now note that $]0,\infty[^m \cap N_A \neq \varnothing$ (since the vector of scaled facet volumes of the polytope $P_A(b)$ is contained in this set). We use linear programming to compute in polynomial time a point $\hat{q}$ of $]0,\infty[^m \cap N_A$, and set $S = \operatorname{conv}(\hat{C}^\circ \cup \{\hat{q}\})$. Clearly, S is again a centred well-guaranteed convex body in N_A, and $0 \in \operatorname{relint} S$. Further, the weak optimisation algorithm for $\hat{C}^\circ$ can easily be extended to S, whence, again by Proposition 2, the weak validity problem for S (relative to N_A) can be solved in oracle-polynomial time. This means that we can solve the weak membership problem in oracle-polynomial time for the polar $\bar{C}$ of S relative to N_A. Note that

$$\bar{C} = \hat{C} \cap N_A \cap \{z \in \mathbb{R}^m : \langle \hat{q}, z\rangle \leqslant 1\}.$$

Further, $\langle \hat{q}, \hat{b}\rangle \leqslant 0 < 1$; thus $\hat{b} \in \hat{C}(\pm\epsilon)$ if and only if the orthogonal projection $\hat{b}'$ of $\hat{b}$ onto N_A is contained in $\bar{C}(\pm\epsilon)$. This concludes our reduction.

3.3. *From Minkowski to weak optimisation*

This subsection completes the #$\mathbb{P}$-hardness proof by showing that an oracle $\mathscr{A}$ for solving MinkApp can be used to produce weak minima of linear functionals $z \mapsto \langle v, z\rangle$ over the sets C, where $v \in]0,\infty[^m \cap N_A$. We use the notation of Subsections 1.4 and 2.1.

Let $(n, m; A; v)$ be an instance of MinkConvMin, let L denote its size, and let $\epsilon \in \mathbb{Q}$ with $0 < \epsilon \leqslant 1$. We have to compute a weak minimum of the linear functional $z \mapsto \langle v, z\rangle$ over $C = \{z : \operatorname{Vol}(P_A(z)) \geqslant 1\}$ or over the restricted body K of Lemma 1. We call the oracle $\mathscr{A}$ on the input $(n, m; A; v; \eta)$, where $\eta = \vartheta^{10}\epsilon$ to produce a rational vector $\tilde{b} \in \mathbb{R}^m$ such that, for the facet volumes $\tilde{\mu}_i$ of $P_A(\tilde{b})$,

$$|\tilde{\mu}_i - v_i\|a_i\|| < \eta \quad \text{for } i = 1, 2, \ldots, m.$$

Let $\tilde{v} = (\tilde{\mu}_1\|a_1\|^{-1}, \ldots, \tilde{\mu}_m\|a_m\|^{-1})^T$; then

$$\|v - \tilde{v}\| \leqslant \frac{\sqrt{m}\eta}{\min\{\|a_1\|, \ldots, \|a_m\|\}} \leqslant \eta 2^L,$$

whence, in particular, $\tilde{v} \in S(v)$. Now, let $b^\diamond$ and $\tilde{b}^\diamond$ be solutions of MinkConvMin in K on the inputs $(n, m; A, v)$ and $(n, m; A, \tilde{v})$, respectively. Then

$$b = \rho^{-1} b^\diamond \quad \text{and} \quad \hat{b} = \tilde{\rho}^{-1}\tilde{b}^\diamond$$

are solutions of MinkRecon on the inputs $(n, m; A; v)$ and $(n, m; A; \tilde{v})$, respectively, where

$$\rho = \operatorname{Vol}(P_A(b))^{-1/n} = \left(\frac{n}{\langle v, b\rangle}\right)^{1/n} \quad \text{and} \quad \tilde{\rho} = \operatorname{Vol}(P_A(\hat{b}))^{-1/n} = \left(\frac{n}{\langle \tilde{v}, \tilde{b}\rangle}\right)^{1/n}.$$

Since $\tilde{v} \in S(v)$, Lemma 1 shows that all entries of $\tilde{b}^\diamond$ and $b^\diamond$ are bounded above by Θ. Further,

$$\begin{aligned} |\langle v, \tilde{b}^\diamond\rangle - \langle v, b^\diamond\rangle| &\leqslant |\langle v, \tilde{b}^\diamond\rangle - \langle \tilde{v}, \tilde{b}^\diamond\rangle| + |\langle \tilde{v}, \tilde{b}^\diamond\rangle - \langle v, b^\diamond\rangle| \\ &\leqslant |\langle v - \tilde{v}, \tilde{b}^\diamond\rangle| + \max\{\langle \tilde{v} - v, b^\diamond\rangle, \langle v - \tilde{v}, \tilde{b}^\diamond\rangle\} \\ &\leqslant \|v - \tilde{v}\|\,(\|\tilde{b}^\diamond\| + \max\{\|\tilde{b}^\diamond\|, \|b^\diamond\|\}) \leqslant \eta 2^L \cdot 2\sqrt{m}\Theta \leqslant \eta\Theta^2. \end{aligned}$$

Note that $\hat{\rho} = ((1/n)\langle v, \tilde{b}\rangle)^{-1/n}$ is an estimate of $\tilde{\rho} = ((1/n)\langle \tilde{v}, \tilde{b}\rangle)^{-1/n}$ (the latter number cannot be computed directly since $\tilde{v}$ is not available), and we may use $\hat{\rho}\tilde{b}$ as an estimate for $\tilde{b}^{\diamond}$.

By Lemma 1, $\vartheta \leqslant \tilde{\rho} \leqslant \Theta$, whence

$$\frac{1}{n}\langle v, \tilde{b}\rangle = \frac{1}{n}\tilde{\rho}^{-1}\langle v, \tilde{b}^{\diamond}\rangle \geqslant \frac{1}{n}\tilde{\rho}^{-1}\langle v, b^{\diamond}\rangle \geqslant \vartheta^2,$$

and

$$\frac{1}{n}|\langle v, \tilde{b}\rangle - \langle \tilde{v}, \tilde{b}\rangle| \leqslant \frac{1}{n}\|v-\tilde{v}\| \cdot \tilde{\rho}^{-1}\|\tilde{b}^{\diamond}\| \leqslant \frac{\sqrt{m}}{n}\eta 2^L \Theta^2.$$

Using the mean value theorem for $x \mapsto x^{-1/n}$ we obtain

$$|\tilde{\rho}-\hat{\rho}| = \left|\left(\frac{1}{n}\langle v, \tilde{b}\rangle\right)^{-1/n} - \left(\frac{1}{n}\langle \tilde{v}, \tilde{b}\rangle\right)^{-1/n}\right| \leqslant \frac{1}{n}\vartheta^{-2(n+1)/n}\frac{\sqrt{m}}{n}\eta 2^L\Theta^2 \leqslant \frac{1}{n}\eta\Theta^7.$$

Since $\hat{\rho}$ can be approximated to any absolute accuracy in polynomial time we can compute a rational number ρ' with $|\rho' - \tilde{\rho}| \leqslant \eta\Theta^7$. Setting $\tilde{b}' = \rho'\tilde{b}$ we have

$$\begin{aligned}|\langle v, b^{\diamond}\rangle - \langle v, b'\rangle| &\leqslant |\langle v, b^{\diamond}\rangle - \langle v, \tilde{b}^{\diamond}\rangle| + |\langle v, \tilde{b}^{\diamond}\rangle - \langle v, b'\rangle| \\ &\leqslant \eta\Theta^2 + |\tilde{\rho}-\rho'|\,\|v\|\,\|\tilde{b}\| \leqslant \eta\Theta^2 + \eta\Theta^7 m 2^L\Theta^2 \leqslant \eta\Theta^{10} = \epsilon.\end{aligned}$$

All that remains to be shown is that $b' \in C(\epsilon)$, but this follows from the fact that $\tilde{b}^{\diamond} \in C$, in conjunction with the inequality

$$\|\tilde{b}^{\diamond} - b'\| \leqslant |\tilde{\rho}-\rho'|\,\|\tilde{b}\| \leqslant \eta\Theta^7\sqrt{m}\Theta^2 \leqslant \epsilon.$$

This completes the final reduction in proof of the hardness of MinkApp.

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