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Inner mappings of Bruck loops

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Abstract

K-loops have their origin in the theory of sharply 2-transitive groups. In this paper a proof is given that K-loops and Bruck loops are the same. For the proof it is necessary to show that in a (left) Bruck loop the left inner mappings $L(b)\,L(a)\,L(ab)^{-1}$ are automorphisms. This paper generalizes results of Glauberman [**3**], Kist [**8**] and Kreuzer [**9**].

1. *Introduction*

In order to describe sharply 2-transitive groups, H. Karzel introduced in [**4**] the notion of a neardomain $(F, \oplus, \cdot)$ (cf. [**16**]). The crucial difficulty of a neardomain is the additive structure $(F, \oplus)$, which need not be associative and no example of a proper neardomain is known (cf. [**6**, **16**]). To obtain partial results, W. Kerby and H. Wefelscheid considered separately the additive structure $(F, \oplus)$ and called such loops K-loops (see definition in Section 2). Since 1988 the interest in K-loops has been revived because A. A. Ungar has found a famous physical example.

A. A. Ungar investigated the relativistic addition $\oplus$ of the velocities $\mathbb{R}^3_c \coloneqq \{v \in \mathbb{R}^3 : |v| < c\}$. He showed that $(\mathbb{R}^3_c, \oplus)$ is a non-associative and non-commutative loop with characteristic automorphisms, which he calls a gyrogroup. Ungar proved that for any two velocities $a, b \in \mathbb{R}^3_c$ there is an automorphism $\delta_{a,b}$ of $(\mathbb{R}^3_c, \oplus)$, the so-called Thomas rotation, satisfying $a \oplus (b \oplus x) = (a \oplus b) \oplus x\delta_{a,b}$ (cf. [**14**, **15**]), i.e. $\delta_{a,b}$ is a left inner mapping of the loop. H. Wefelscheid recognized then that $(\mathbb{R}^3_c, \oplus)$ is a K-loop.

At first it was discovered by G. Kist that there is a connection between K-loops and Bruck loops [**8**, p. 27]. G. Kist remarks, that already from results of G. Glauberman [**3**] one can deduce that every finite Bruck loop of odd order is a K-loop. As a generalization it is proved in [**9**, theorem 1] that every Bruck loop with no element of order 2 is a K-loop.

In this note we prove that K-loops and Bruck loops are the same. For that mainly we have to show that the left inner mappings of a (left) Bruck loop are automorphisms of the loop, denoted as axiom (I). (In general the right inner mappings of a left Bruck loop are not automorphisms, hence Bruck loops are clearly not A-loops in the sense of Bruck and Paige [**2**], but left A-loops by definition 1·1·4 of Nagy and Strambach [**11**], and in particular homogenous loops.) In Sections 1 and 2, we give the definitions and some easy results, partly known, which we need in Section 3. The main results are Theorems 3·1 and 3·3.

In this paper, unlike other papers on K-loops [**5**, **9**, **10**] we use '$\cdot$' instead of '$+$' for the binary operation, as is customary for loops.

2. *Left inner mappings*

Let $(K, \cdot)$ be a loop with the identity element 1, and for $x \in K$ let $x^\lambda, x^\rho \in K$ be the unique elements with $x^\lambda x = x x^\rho = 1$. If $x^\lambda = x^\rho$, then $x^{-1} = x^\lambda = x^\rho$ is the inverse of x. Let $N_\mu = \{b \in K : a \cdot bc = ab \cdot c \text{ for all } a, c \in K\}$ denote the middle nucleus. For any fixed element $a \in K$, the map

$$L(a): K \to K; \quad x \to xL(a) := a \cdot x \tag{2·1}$$

is called *left translation*. The group $\mathsf{M}_\lambda := \langle L(x) : x \in K \rangle$ of all permutations of K which is generated by all left translations (and their inverses) is called the *left multiplication group* of $(K, \cdot)$.

Let $\mathsf{K} := \{L(x) : x \in K\}$ be the subset of all left translations of M_λ.

We recall that the middle nucleus N_μ of a loop is a subgroup (cf. [**12**, theorem (I·3·4)]). Clearly, $b \in N_\mu$ if and only if $ab \cdot c = cL(ab) = a \cdot bc = cL(b)L(a)$, i.e. if and only if $L(ab) = L(b)L(a)$ for every $a \in K$. Assume $L(b)L(a) = L(x) \in \mathsf{K}$, then $1L(b)L(a) = ab = 1L(x) = x$, i.e. $x = ab$. Hence

$$b \in N_\mu \quad \text{if and only if} \quad L(b)L(a) \in \mathsf{K} \quad \text{for every} \quad a \in K. \tag{2·2}$$

We call the permutations of $\mathsf{A} := \{\alpha \in \mathsf{M}_\lambda : 1\alpha = 1\}$ the *left inner mappings* of $(K, \cdot)$.

LEMMA 2·1. *$\mathsf{M}_\lambda = \mathsf{AK}$ and $\mathsf{M}_\lambda = \mathsf{KA}$ are exact decompositions, i.e. for every $\mu \in \mathsf{M}_\lambda$ there are unique elements $L(a), L(b) \in \mathsf{K}, \alpha, \beta \in \mathsf{A}$ with $\mu = \alpha L(a) = L(b)\beta$* and we have $a = b^\rho \mu^2$.

Proof. For $\mu \in \mathsf{M}_\lambda$ let $a = 1\mu, s = 1\mu^{-1} \in K$, i.e. $s\mu = 1$. Set $b = s^\lambda$, then $\mu = \mu L(a)^{-1} L(a) = L(b)L(b)^{-1}\mu$ with $\alpha = \mu L(a)^{-1}, \beta = L(b)^{-1}\mu \in \mathsf{A}$, since $1\mu L(a)^{-1} = aL(a)^{-1} = 1$ and $sL(s^\lambda) = 1$, hence $1L(b)^{-1}\mu = 1L(s^\lambda)^{-1}\mu = s\mu = 1$. Clearly $b^\rho\mu^2 = s\mu\mu = 1\mu^{-1}\mu\mu = 1\mu = a$.

Assume $\mu = \alpha L(a) = \alpha' L(a')$, then $\alpha'^{-1}\alpha = L(a')L(a)^{-1}$ and $1 = 1L(a')L(a)^{-1}$, i.e. $1L(a) = a = a' = 1L(a')$ and $\alpha' = \alpha$. Hence $a \in L, \alpha \in \mathsf{A}$ and also $b \in L, \beta \in \mathsf{A}$ are uniquely determined.

For fixed elements $a, b \in K$ let

$$L(a, b) := L(a)L(b)L(ba)^{-1}. \tag{2·3}$$

In papers on K-loops the notation $\delta_{b,a}$ is used instead of $L(a, b)$ due to the origin of K-loops as the additive structure of neardomains. In this paper we prefer to write $L(a, b)$ rather than $\delta_{b,a}$ to match up papers on Bol and Bruck loops.

Let $\mathsf{A}' := \langle L(x, y) : x, y \in K \rangle$ be the subgroup of M_λ which is generated by all permutations $L(x, y)$. By [**7**, proposition 1] we get (cf. also [**1**, **IV**, lemma 1·2] and [**12**, I·5·2]):

LEMMA 2·2. $\mathsf{A} = \langle L(x, y) : x, y \in K \rangle$.

Clearly definition (2·3) implies for $a, b, x \in K$:

$$a \cdot bx = ab \cdot xL(b, a), \tag{2·4}$$

$$L(1, a) = L(a, 1) = id. \tag{2·5}$$

Lemma 2·3. *In a loop* $(K, \cdot)$ *the following are equivalent:*
(i) $L(a, a^\lambda) = id$,
(ii) $L(a^\lambda) = L(a)^{-1}$ (*left inverse property*).

Proof. Obviously $L(a, a^\lambda) = L(a)\,L(a^\lambda)\,L(1)^{-1} = id$ if and only if $L(a^\lambda) = L(a)^{-1}$.

We recall that the left inverse property implies $a^\lambda = a^\rho = a^{-1}$.

A loop $(K, \cdot)$ is called a left A-loop if (I), a left K-loop if (I), (II) and (III), a left Bol loop if (B), and a left Bruck loop if (B) and (III) are satisfied:

(I) For all $x, y \in K, L(x, y)$ is an automorphism of $(K, \cdot)$.
(II) $L(x, y) = L(xy, y)$ for all $x, y \in K$.
(III) (Automorphic inverse property) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in K$.
(B) (left Bol identity) $a(b \cdot ac) = (a \cdot ba)\,c$ for all $a, b, c \in K$.

In the following we omit the word 'left' and refer by the phrase Bol (Bruck, K-) loop always to left Bol (Bruck, K-) loops.

By (II) and (2·5), $L(a^\lambda, a) = L(a^\lambda a, a) = L(1, a) = id$, hence by (2·4) $a = a \cdot a^\lambda a = aa^\lambda \cdot aL(a^\lambda, a) = aa^\lambda \cdot a$. We obtain (cf. [**10**, (2·10)]) $aa^\lambda = 1$, $L(a, a^\lambda) = L(\alpha\alpha^\lambda, a^\lambda) = L(1, a^\lambda) = \mathrm{id}$ and $\mathrm{id} = L(1, a) = L(a, a)$, i.e. by Lemma 2·3, $a^\lambda \cdot ax = x$ and $a \cdot ax = a^2 \cdot x$, properties which are well known for Bol loops (cf. [**1**, **12**, **13**]). Hence:

Lemma 2·4. *In loops with* (II), *in K-loops and Bol loops the left inverse property* $a^\lambda \cdot ac = c$, *and the left alternative law* $a \cdot ac = a^2c$ *is satisfied.*

Lemma 2·5. *Let* $(K, \cdot)$ *be a loop. Then the following are equivalent:*
(i) (B),
(ii) $L(ba, a) = L(a, b)^{-1}$ *for all* $a, b \in K$,
(iii) $L(a)\,\mathsf{K}L(a) \subset \mathsf{K}$ *for all* $a \in K$.

Proof. By (2·4) $a(b \cdot ac) = a(ba \cdot cL(a, b)) = (a \cdot ba)\,cL(a, b)\,L(ba, a)$, hence $a(b \cdot ac) = (a \cdot ba)\,c$ for every $c \in K$, if and only if $L(a, b)\,L(ba, a) = \mathrm{id}$. Since $a(b \cdot ac) = cL(a)\,L(b)\,(La)$ and $(a \cdot ba)\,c = cL(a \cdot ba)$, (B) is equivalent to $L(a)\,L(b)\,L(a) = L(a \cdot ba) \in \mathsf{K}$ for every $a, b \in K$. Since $1L(a)\,L(b)\,L(a) = a \cdot ba, L(a)\,L(b)\,L(a) \in \mathsf{K}$ implies $L(a)\,L(b)\,L(a) = L(a \cdot ba)$.

By [**9**, (1·2)], [**10**, (2·12)]:

Lemma 2·6. *Every K-loop satisfies the Bol identity and is a Bruck loop.*

3. *Left inner automorphisms*

Now we describe properties of the loop $(K, \cdot)$ in the left multiplication group $\mathsf{M}_\lambda = \mathsf{KA}$.

Theorem 3·1. *An inner mapping* $\alpha \in \mathsf{A}$ *is an automorphism of* $(K, \cdot)$ *if and only if* $\alpha^{-1}\mathsf{K}\alpha \subset \mathsf{K}$.

Proof. Let $x, y \in K$ and $\alpha \in \mathsf{A}$. Then $(xy)\,\alpha = x\alpha \cdot y\alpha$ is equivalent to $xy = (x\alpha \cdot y\alpha)\,\alpha^{-1}$, hence

$$L(x) = \alpha L(x\alpha)\,\alpha^{-1}, \quad \text{i.e.} \quad \alpha^{-1}L(x)\,\alpha = L(x\alpha) \in \mathsf{K}, \tag{3·1}$$

if and only if α is an isomorphism. Assume $\alpha^{-1}L(x)\,\alpha = L(x')\in\mathsf{K}$ for some $x'\in K$, then $1 = 1\alpha^{-1}$ and $1\alpha^{-1}L(x)\,\alpha = x\alpha = 1L(x') = x'$ and (3·1) is satisfied, i.e. α is an automorphism.

THEOREM 3·2. *Let* $(K,\cdot)$ *be a Bol loop and let* $a, b\in K$. *Then the inner mapping* $L(b,a)$ *is an automorphism of* $(K,\cdot)$ *if and only if*

$$ab\cdot(a^{-1}b^{-1})\in N_\mu, \tag{3·2}$$

where N_μ *denotes the middle nucleus.*

Proof. For $L(x)\in\mathsf{K}$ let $\gamma := L(b,a)\,L(x)\,L(b,a)^{-1} = L(b)\,L(a)\,L(ab)^{-1}\,L(x)\,L(ab)\,L(a)^{-1}$ $L(b)^{-1}\in\mathsf{M}_\lambda$. By Theorem 3·1, $L(b,a)$ is an automorphism if and only if $\gamma\in\mathsf{K}$, and by Lemma 2·5 $\gamma\in\mathsf{K}$ if and only if $L(ab)^{-1}L(a)\,L(b)\,\gamma L(b)\,L(a)\,L(ab)^{-1}\in\mathsf{K}$ or

$$L(ab)^{-1}\,L(a)\,L(b)^2\,L(a)\,L(ab)^{-1}\,L(x)\in\mathsf{K}. \tag{3·3}$$

For $z\in K$, the Bol identity implies

$$\begin{aligned} zL(ab)^{-1}\,L(a)\,L(b)^2\,L(a)\,L(ab)^{-1} &= (ab)^{-1}\cdot(a\{b^2[a\cdot(ab)^{-1}\,z]\}) \\ &\overset{(B)}{=} (ab)^{-1}\cdot[(a\cdot b^2a)\cdot(ab)^{-1}\,z] \overset{(B)}{=} [(ab)^{-1}\cdot(a\cdot b^2a)\,(ab)^{-1}]\,z \\ &= zL((ab)^{-1}\cdot(a\cdot b^2a)\,(ab)^{-1}) \end{aligned}$$

and by (2·2) it follows that (3·3) is valid if and only if:

$$s := (ab)^{-1}\cdot(a\cdot b^2a)\,(ab)^{-1}\in N_\mu. \tag{3·4}$$

With (B) and Lemma 2·4, $(a\cdot b^2a)\cdot a^{-1}b^{-1} = a\cdot b^2(a\cdot a^{-1}b^{-1}) = ab$. Hence it follows $1 = (ab)^{-1}\cdot\{(a\cdot b^2a)\cdot[(ab)^{-1}\cdot(ab)\,(a^{-1}b^{-1})]\} = [(ab)^{-1}\cdot(a\cdot b^2a)\,(ab)^{-1}]\cdot(ab)\,(a^{-1}b^{-1})$, i.e. $s^{-1} = ab\cdot(a^{-1}b^{-1})$. Because N_μ is a subgroup of K, $s\in N_\mu$ if and only if $s^{-1}\in N_\mu$. We summarize that $L(b,a)$ is an automorphism if and only if $ab\cdot(a^{-1}b^{-1})\in N_\mu$.

Since $\mathsf{A} = \langle L(a,b): a, b\in K\rangle$, Theorem 3·2 implies:

COROLLARY 3·3. *In every Bruck loop* $(K,\cdot)$, A *is a group of automorphisms of* $(K,\cdot)$, *i.e. the axiom* (I) *is satisfied and* $(K,\cdot)$ *is a left A-loop.*

THEOREM 3·4. *Bruck loop and K-loops are the same.*

Proof. By Lemma 2·6 every K-loop is a Bruck loop. By [**10**, (2·12)] in a loop with (I), (III) and the (left) inverse property, (II) and (B) are equivalent, hence in a loop with (I), (III) and (B), (II) is satisfied, i.e. by Theorem 3·2, every Bruck loop is a K-loop.

The question whether the axioms (II) and (III) also imply (I) is answered to the negative by the following:

Example 3·5. Let $(R,+,\cdot)$ be an associative and commutative ring with zero element O, with $x\cdot x = \mathsf{O} = x+x$ for every $x\in R$ and with four elements p, q, r, s satisfying $pqrs \neq \mathsf{O}$. (For instance for $n\in\mathbb{N}$ with $n\geqslant 4$ let $R := \mathbb{Z}_2^{2^n-1}$ be the vector space over $\mathbb{Z}_2$ with dimension 2^n-1. We write the vectors of a basis B in the following way:

$$B = \{[k_1, k_2, \dots, k_n]: k_i\in\{0,1\} \quad \text{for} \quad i\in\{1,\dots,n\} \quad \text{and} \quad [k_1,\dots,k_n]\neq[0,\dots,0]\}.$$

Let O be the zero vector. We define by $b \cdot O = O \cdot b$ for every $b \in B$ and

$$[k_1, k_2, \ldots, k_n] \cdot [l_1, l_2, \ldots, l_n] := \begin{cases} O & \text{if} \quad k_i + l_i = 2 \quad \text{for some} \quad i \in \{1, \ldots, n\} \\ [k_1 + l_1, k_2 + l_2, \ldots, k_n + l_n] & \text{else} \end{cases}$$

an associative and commutative multiplication on B and extend this multiplication to a distributive multiplication of R. Then obviously $x \cdot x = O$ and

$$[1, 0, 0, 0, \ldots] \cdot [0, 1, 0, 0, \ldots] \cdot [0, 0, 1, 0, \ldots] \cdot [0, 0, 0, 1, \ldots] = [1, 1, 1, 1, \ldots] \neq O).$$

Now we define on $K := R \times R$ the following operation:

$$\oplus : K \times K \to K, (a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_1 + a_1 a_2 b_1 b_2, a_2 + b_2). \tag{3·6}$$

Then for $a = (a_1, a_2), b = (b_1, b_2) \in K, (x_1, x_2) = (a_1 + b_1 + a_1 b_1 a_2 b_2, a_2 + b_2)$ is the unique solution of the equation $(a_1, a_2) \oplus (x_1, x_2) = (b_1, b_2)$ and (O, O) is the zero element, i.e. $(K, \oplus)$ is a commutative loop. Every element of $K \backslash \{(O, O)\}$ has order 2, hence $(K, \oplus)$ satisfies (III). We compute that

$$(x_1, x_2) L(b, a) = (x_1 + a_1 a_2 (b_1 x_2 + b_2 x_1) + (a_1 b_2 + a_2 b_1) x_1 x_2, x_2) \tag{3·7}$$

and $L(b, a) = L(b \oplus a, a)$, i.e. (II) is satisfied. But for the elements $p, q, r, s \in R$ with $pqrs \neq O$ we have: $(p, O) \oplus \{(q, r) \oplus [(p, O) \oplus (O, s)]\} = (q + pqrs, r + s) \neq (q, r + s) = \{(p, O) \oplus [(q, r) \oplus (p, O)]\} \oplus (O, s)$, i.e. the Bol identity (B) is not satisfied and by Lemma 2·6 neither is (I).

Added in proof. The result of Corollary 3·3 can also be found with different proofs in [**17**, corollary 3·12·1] and [**18**, corollary 5·2].

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