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On the topology of the dual of a nilpotent Lie group

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Introduction

In this paper we investigate separation properties in the dual $\widehat{G}$ of a connected, simply connected, nilpotent Lie group G. Following [**4**, **19**], we are particularly interested in the question of when the group G is quasi-standard, in which case the group C^*-algebra $C^*(G)$ may be represented as a continuous bundle of C^*-algebras over a locally compact, Hausdorff, space such that the fibres are primitive throughout a dense subset. The same question for other classes of locally compact groups has been considered previously in [**1**, **5**, **18**]. Fundamental to the study of quasi-standardness is the relation of inseparability in $\widehat{G}$: $\pi \sim \sigma$ in $\widehat{G}$ if π and σ cannot be separated by disjoint open subsets of $\widehat{G}$. Thus we have been led naturally to consider also the set $\operatorname{sep}(\widehat{G})$ of separated points in $\widehat{G}$ (a point in a topological space is *separated* if it can be separated by disjoint open subsets from each point that is not in its closure).

It was shown in [**4**, theorem] that the C^*-algebra of a nilpotent Lie group G with centre Z is quasi-standard if the maximal orbit dimension in $\mathfrak{g}^*$ is equal to the dimension of $\mathfrak{z}^\perp$, and that the converse holds in the two-step case. This is applied in Section 2 to show that the universal, simply connected, two-step nilpotent Lie group W_n is quasi-standard if and only if n is even (Theorem 2·4). It is also shown that in the two-step case (or more generally in cases where every generic Kirillov orbit is flat) $\operatorname{sep}(\widehat{G})$ consists precisely of those irreducible representations for which the associated orbit in $\mathfrak{g}^*$ has maximal dimension (Theorem 2·2). This leads to a description of $\operatorname{sep}(\widehat{W_n})$ in terms of the rank of a skew-symmetric matrix associated with an element of $\mathfrak{w}_n^*$ (Corollary 2·5). The relation $\sim$ is considered in further detail for the universal groups W_n in Theorem 2·7 and Corollary 2·8. In particular, $\sim$ is an equivalence relation on $\widehat{W_n}$ for all $n \geqslant 2$.

In Section 3 we extend from [**4**, **7**] the study of the 'threadlike' nilpotent Lie groups G_N ($N \geqslant 3$). The group G_N is $(N-1)$-step nilpotent and G_3 is the classical Heisenberg group. In fact, we are able to work in the more general context of nilpotent Lie groups

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of the form $G = \mathbf{R} \ltimes \mathbf{R}^d$ (see [**6**]). We show in Theorem 3·1 that such groups fail to be quasi-standard except in the special case of a direct product of G_3 with an abelian group. Our analysis shows in particular that, for $N \geqslant 5$, $C^*(G_N)$ contains a Glimm ideal that is not primal, thereby confirming a view taken in [**4**] (which dealt with the case $N = 5$). Theorem 3·1 is applied to six-dimensional groups in Section 4.

The cortex of a locally compact group G is the closed subset $\mathrm{cor}\,(G)$ of $\widehat{G}$ consisting of those elements which cannot be separated by disjoint open sets from the trivial representation 1_G. It is related to the cohomology of G in unitary representation spaces [**25**]. For some groups, for example those with Kazhdan's property (T), $\mathrm{cor}\,(G)$ is just $\{1_G\}$. Moreover, for connected Lie groups, Sund [**24**] has recently given a characterization of those for which $\mathrm{cor}\,(G) = \{1_G\}$. On the other hand, for groups outside this class the structure of the cortex may be rather complicated [**6**, **7**]. In particular, for the groups G_N the cortex is quite large and has been determined in terms of $\mathfrak{g}_N^*$ by Boidol, Ludwig and Müller [**7**]. Further detailed analysis of the convergence of orbits, extending methods of [**7**], leads to a description of $\mathrm{sep}\,(\widehat{G})$ for the semi-direct products described above (Theorem 3·6). Corollary 3·7 shows, in particular, that $\widehat{G_N} = \mathrm{cor}\,(G_N) \cup \mathrm{sep}\,(\widehat{G_N})$ if and only if N is odd. The relation $\sim$ on $\widehat{G}_N$ is completely determined in terms of elements of $\mathfrak{g}_N^*$ in Theorem 3·9. In contrast to the situation for the groups W_n, it follows that $\sim$ fails to be an equivalence relation on $\widehat{G}_N$ precisely when N is a multiple of 4 exceeding 4 (Corollary 3·10).

Simply connected, nilpotent Lie groups of dimension no greater than six have been classified (see [**20**] and the references therein). The Heisenberg group G_3 is quasi-standard [**4**, **19**] but G_4 is not [**4**]. Of the six five-dimensional groups in [**20**], three are quasi-standard and three are not [**4**]. Turning to dimension six, a straightforward check of the data in [**20**] shows that the maximal orbit dimension is equal to the dimension of $\mathfrak{z}^\perp$ in precisely the following cases:

$$G_{6,16}, \quad G_{6,17}, \quad G_{6,19}, \quad G_{6,20}, \quad G_{6,21}, \quad G_{6,22}, \quad G_{6,23}, \quad G_{6,24}.$$

By the theorem from [**4**] described above, all of these groups are quasi-standard. Surprisingly, we have found that there are six more quasi-standard groups amongst the twenty four in [**20**], thereby refuting the conjecture in [**4**] that the dimension condition is necessary for G to be quasi-standard. We list these six groups in Section 4 and give full details of the proof in the case of $G_{6,4}$, which is isomorphic to the (three-step nilpotent) group of real 4×4 upper triangular matrices. The group $G_{6,4}$ is the unique counterexample to the conjecture which is minimal with respect to both the step of nilpotency and the dimension of the group. The remaining ten groups are not quasi-standard, but in only three cases does $C^*(G)$ contain a non-primal Glimm ideal: $G_{6,10}\,(= G_6)$, $G_{6,15}\,(= W_3)$ and $G_{6,18}$. Like G_6, $G_{6,18}$ has the form $\mathbf{R} \ltimes \mathbf{R}^5$ and so is covered by the results of Section 3.

1. *Preliminaries*

Let A be a C^*-algebra and let $\mathrm{Id}\,(A)$ be the set of all closed (two-sided) ideals of A. An ideal I in $\mathrm{Id}\,(A)$ is called *primal* [**3**] if whenever $J_1, J_2, \ldots, J_n \in \mathrm{Id}\,(A)$, $n \in \mathbb{N}$, are such that $J_1 J_2 \ldots J_n = \{0\}$ then $J_k \subseteq I$ for at least one k. We shall require the following basic result, which shows that primality is closely related to the possible failure of the Hausdorff property in $\mathrm{Prim}\,(A)$.

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LEMMA 1·1 (**3**, Proposition 3·2). *Let A be a C^*-algebra and I a proper closed ideal of A. The following conditions are equivalent.*

(i) *The ideal I is primal.*

(ii) *Whenever $n \geqslant 1$ and $U_1, U_2, \ldots, U_n$ are open subsets of* $\mathrm{Prim}\,(A)$ *such that $U_i \cap \mathrm{Prim}\,(A/I)$ is non-empty $(1 \leqslant i \leqslant n)$ then $\bigcap_{i=1}^n U_i$ is non-empty.*

(iii) *There is a net in* $\mathrm{Prim}\,(A)$ *which converges to every point of* $\mathrm{Prim}\,(A/I)$.

As observed in [**3**], $\mathrm{Prim}\,(A)$ and $\mathrm{Prim}\,(A/I)$ may be replaced by $\widehat{A}$ and $\widehat{A/I}$ in Lemma 1·1. Also, as noted in [**2**, 4·6], the net in (iii) above may be chosen to lie in any prescribed dense subset of $\mathrm{Prim}\,(A)$. Furthermore, the next result shows that the net may be taken to be a sequence if A is a separable C^*-algebra.

LEMMA 1·2. *Let A be a separable C^*-algebra, let S be a dense subset of* $\mathrm{Prim}\,(A)$ *and let J be a proper primal ideal of A. Then there is a sequence in S which converges to every member of* $\mathrm{Prim}\,(A/J)$.

Proof. Since A/J is a separable C^*-algebra, $\mathrm{Prim}\,(A/J)$ has a countable dense subset $\{Q_n \colon n \geqslant 1\}$. For $n \geqslant 1$, let $(U_{(k,n)})_{k \geqslant 1}$ be a decreasing base of open neighbourhoods of Q_n in $\mathrm{Prim}\,(A)$. Since J is primal, for each $k \geqslant 1$ the open set

$$V_k := U_{(k,1)} \cap U_{(k,2)} \cap \ldots \cap U_{(k,k)}$$

is non-empty by Lemma 1·1. Hence there exists $P_k \in V_k \cap S$ $(k \geqslant 1)$. Then, for each $n \geqslant 1$, $P_k \to Q_n$ as $k \to \infty$. Since the set of limits is closed, the sequence (P_k) converges to every member of $\mathrm{Prim}\,(A/J)$. $\square$

If the variable integer $n \in \mathbb{N}$ in the definition of a primal ideal is replaced by a fixed integer $n \geqslant 2$, we obtain the notion of an n-primal ideal. This turns out to be relevant for the study of the universal 2-step nilpotent Lie groups W_n in Section 2. Also, 2-primal and 3-primal ideals arise naturally in [**17**, **22**, **23**]. For each $n \geqslant 2$, there is an example of a C^*-algebra A_n with $I_n \in \mathrm{Id}\,(A_n)$ such that I_n is n-primal but not $(n+1)$-primal [**3**, p. 59]. The following result, which will be used in Section 2, shows that a proper closed ideal I of a C^*-algebra A is n-primal if and only if A has the property (I, n) of [**17**].

LEMMA 1·3. *Suppose that I is a proper closed ideal of a C^*-algebra A and that $n \geqslant 2$. The following conditions are equivalent.*

(i) *The ideal I is n-primal.*

(ii) *The ideal $\bigcap_{i=1}^n P_i$ is primal in A whenever $P_1, P_2, \ldots, P_n$ are primitive ideals of A containing I.*

Proof. (i)⇒(ii). Suppose that $P_1, P_2, \ldots, P_n$ are primitive ideals of A containing I and that $Q := \bigcap_{i=1}^n P_i$ is not primal. Since $S := \{P_1, P_2, \ldots, P_n\}$ is dense in $\mathrm{Prim}\,(A/Q)$, it follows from Lemma 1·1 that there does not exist a net in $\mathrm{Prim}\,(A)$ that is convergent to every member of S. Hence there exist open neighbourhoods U_i of P_i $(1 \leqslant i \leqslant n)$ with empty intersection. Let $J_i \in \mathrm{Id}\,(A)$ be the ideal such that $\mathrm{Prim}\,(J_i) = U_i$. Then $J_1 J_2 \ldots J_n = \{0\}$ and each J_i is not contained in I, and so I is not n-primal.

(ii)⇒(i). Suppose that $J_1 J_2 \ldots J_n = \{0\}$ in $\mathrm{Id}\,(A)$ and that each J_i is not contained in I. Then, for $1 \leqslant i \leqslant n$, there exists $P_i \in \mathrm{Prim}\,(A)$ such that $I \subseteq P_i$ and $J_i \nsubseteq P_i$. Hence $J_i \nsubseteq \bigcap_{j=1}^n P_j$ $(1 \leqslant i \leqslant n)$ and so $\bigcap_{j=1}^n P_j$ is not primal. $\square$

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For elements x and y in a topological space X, we write $x \sim y$ if x and y cannot be separated by disjoint open subsets of X, and we write $x \approx y$ if $f(x) = f(y)$ for all bounded, continuous functions f on X. Then $\sim$ is reflexive and symmetric but not necessarily transitive, whereas the weaker relation $\approx$ is always an equivalence relation. A C^*-algebra A is said to be *quasi-standard* [**5**] if $\sim$ is an open equivalence relation on Prim (A). This condition is a natural substitute for the stronger condition that Prim (A) should be Hausdorff. Since the topology on the spectrum $\widehat{A}$ is induced from that on Prim (A) by the kernel map, routine arguments of general topology show that A is quasi-standard if and only if $\sim$ is an open equivalence relation on $\widehat{A}$. This is particularly transparent in the case of a C^*-algebra of type I (such as the C^*-algebra of a nilpotent Lie group) since $\widehat{A}$ and Prim (A) are homeomorphic in this case.

The Glimm ideal space Glimm (A) of A arises from the complete regularization [**11**] of the primitive ideal space Prim (A). Denoting by $[P]$ the $\approx$-class of P in Prim (A), there is a bijection between the quotient space Prim $(A)/\approx$ and the space of so-called Glimm ideals, given by $[P] \to k([P]) = \bigcap\{Q: \; Q \in [P]\}$. The canonical mapping Φ_A: Prim $(A) \to$ Glimm (A), given by $\Phi_A(P) = k([P])$, defines the *quotient topology* τ_q on Glimm (A). The C^*-algebra A is quasi-standard if and only if every Glimm ideal is minimal primal and Φ_A is an open map [**5**, theorem 3·3].

Let G be a simply connected (and connected) nilpotent Lie group with Lie algebra $\mathfrak{g}$. Kirillov's theory gives a bijection between $\widehat{G}$ and $\mathfrak{g}^*/\mathrm{Ad}^*$, the orbit space of the coadjoint representation of G on the dual vector space $\mathfrak{g}^*$. Indeed, each $f \in \mathfrak{g}^*$ gives rise to an irreducible representation π_f of G, and π_f is unitarily equivalent to π_g $(g \in \mathfrak{g}^*)$ if and only if $g \in \mathrm{Ad}^*(G)f$ (see, for example, [**10**]). The Kirillov correspondence is a homeomorphism provided that $\mathfrak{g}^*/\mathrm{Ad}^*$ carries the quotient topology [**8**]. If $f, f_n \in \mathfrak{g}^*$ $(n \geqslant 1)$ we shall say that the sequence (f_n) is *orbit-convergent* to f $(f_n \xrightarrow{\text{orb}} f)$ if there exists a sequence (x_n) in G such that $\mathrm{Ad}^*(x_n)f_n \to f$. Singleton subsets of $\widehat{G}$ are closed, since coadjoint orbits in $\mathfrak{g}^*$ are closed, so $\pi \in \widehat{G}$ is a separated point if and only if it can be separated by disjoint open sets from every other point of $\widehat{G}$.

It is shown in [**14**, proposition 3] that sep $(\widehat{G})$ contains a dense open subset of $\widehat{G}$. Indeed, generic functionals in $\mathfrak{g}^*$ give rise to such a subset (see Proposition 1·4). The proof of Proposition 1·4 uses $\mathrm{Ad}^*(G)$-invariant polynomials and so one might imagine that they could be used to determine sep $(\widehat{G})$ completely. However, these polynomials are difficult to find (see, for example [**9**, **13**]). Furthermore, Theorem 3·9(i) together with the data from [**9**] shows that $\widehat{G_7}$ contains distinct separated points which cannot be distinguished by $\mathrm{Ad}^*(G_7)$-invariant polynomials on $\mathfrak{g}_7^*$.

Proposition 1·4. *Let G be a connected, simply connected, nilpotent Lie group, and let $\{X_1, \ldots, X_n\}$ be a strong Malcev basis for $\mathfrak{g}$. Suppose that $f \in \mathfrak{g}^*$ is generic for the dual basis. Then π_f is a separated point of $\widehat{G}$ and* ker π_f *is a Glimm ideal.*

Proof. Suppose that $g \in \mathfrak{g}^*$ and that $\pi_g \neq \pi_f$ in $\widehat{G}$. We have to show that π_f and π_g can be separated by a continuous function on $\widehat{G}$. If g is generic for the dual basis then the proof of [**10**, 4·6·4] shows that f and g can be separated by an $\mathrm{Ad}^*(G)$-invariant polynomial on $\mathfrak{g}^*$, whereas if g is not generic then f and g are separated by the $\mathrm{Ad}^*(G)$-invariant Pfaffian associated with the basis $\{X_1, \ldots, X_n\}$ (see [**21**, p. 275]). In either case, the $\mathrm{Ad}^*(G)$-invariant function induces a suitable continuous function on $\widehat{G}$. □

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In the 2-step case, if $f \in \mathfrak{g}^*$ has maximal orbit dimension then it can easily be shown that there exists a strong Malcev basis for $\mathfrak{g}$ such that f is generic for the dual basis. Thus, by Proposition 1·4, π_f is a separated point of $\widehat{G}$. We shall show in Theorem 2·2 that in fact, in this case, the separated points of $\widehat{G}$ correspond exactly to orbits of maximal dimension.

If the step of nilpotency is greater than two then not all orbits of maximal dimension need give separated points of $\widehat{G}$. This is illustrated by the groups G_N, $(N \geqslant 4)$, see Section 3. Inspection of the nilpotent Lie groups of dimension six or less suggests, however, that separated points of $\widehat{G}$ might necessarily have orbits in $\mathfrak{g}^*$ of maximal dimension.

2. *The 2-step nilpotent case*

Before specializing to the 2-step nilpotent case, we shall begin in a more general setting.

PROPOSITION 2·1. *Let G be a connected, simply connected, nilpotent Lie group and suppose that there exists a strong Malcev basis of $\mathfrak{g}$ such that every generic point in $\mathfrak{g}^*$ (with respect to the dual basis) has a flat orbit of dimension d. Let J be any minimal primal ideal of $C^*(G)$. Let $S = \{f \in \mathfrak{g}^*\colon \ker \pi_f \supseteq J\}$, and let $f \in S$. Then there exists a subspace V of $\mathfrak{z}^\perp$ of dimension d such that*

$$S = f + V.$$

Proof. Let $q\colon \mathfrak{g}^* \to \widehat{G}$ be the open, continuous map $f \to \pi_f$, $f \in \mathfrak{g}^*$, and let U be the set of generic points in $\mathfrak{g}^*$. By Lemma 1·2, there exists a sequence (π_k) in $q(U)$ such that

$$\pi_k \to \pi \quad \text{for all } \pi \in (C^*(G)/J)^\wedge.$$

Let $f \in S$. Then $\pi_k \to \pi_f$. Since q is open, by replacing (π_k) by a subsequence, we may obtain a sequence (f_k) in U such that $f_k \to f$ as $k \to \infty$ and $\pi_{f_k} = \pi_k$.

For each $k \geqslant 1$ there exists a subspace V_k of dimension d such that $V_k \subseteq \mathfrak{z}^\perp$ and

$$\mathrm{Ad}^*(G) f_k = f_k + V_k.$$

Suppose that $N = \dim \mathfrak{g}^*$ and identify $\mathfrak{g}^*$ with $\mathbf{R}^N$ by taking coordinates with respect to the given dual basis in $\mathfrak{g}^*$. Using the dot product on $\mathbf{R}^N$, we obtain a real inner product on $\mathfrak{g}^*$. For each $k \geqslant 1$ let $\{\xi_{1,k}, \ldots, \xi_{d,k}\}$ be an orthonormal basis for V_k. By passing to successive subsequences we may suppose that

$$\xi_{i,k} \to \xi_i \quad (1 \leqslant i \leqslant d)$$

as $k \to \infty$. Then $\{\xi_1, \ldots, \xi_d\}$ is an orthonormal set in $\mathfrak{g}^*$, so $V \coloneqq \operatorname{span}\{\xi_1, \ldots, \xi_d\}$ has dimension d. Clearly $V \subseteq \mathfrak{z}^\perp$.

Let

$$g = f + \sum_{i=1}^{d} \lambda_i \xi_i \in f + V.$$

Then $f_k + \sum_{i=1}^{d} \lambda_i \xi_{i,k} \to g$ as $k \to \infty$. Thus $\pi_k = \pi_{f_k} \to \pi_g$. But $\pi_k \to \pi$ for all $\pi \in (C^*(G)/J)^\wedge$ and J is minimal primal, so $g \in S$. Thus $f + V \subseteq S$.

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Conversely, suppose that $f_0 \in S$. Then $\pi_k \to \pi_{f_0}$. Passing to a further subsequence, we may assume that there exists $v_k \in V_k$ such that $f_k + v_k \to f_0$. Hence $v_k \to f - f_0$. Let $v_k = \sum_{i=1}^{d} \mu_{i,k}\xi_{i,k}$. Since (v_k) is convergent in $\mathfrak{g}^*$, $(\|v_k\|)$ is bounded (where $\|\cdot\|$ denotes the norm arising from the inner product), so there exists K such that $|\mu_{i,k}| \leqslant K$ for all i, k. Thus there is a subsequence (v_{k_r}) and $v \in V$ such that $v_{k_r} \to v$ as $r \to \infty$. Hence $f - f_0 = v$, and $f_0 \in f + V$. Thus $S = f + V$ as required. $\square$

Note that V is actually independent of f, for if $S = f' + V'$, where $f' \in S$ and V' is a subspace of $\mathfrak{g}^*$, then $V = V'$.

Theorem 2·2. *Let G be a connected, simply connected, nilpotent Lie group and suppose that there is a strong Malcev basis of $\mathfrak{g}$ such that every generic point in $\mathfrak{g}^*$ has a flat orbit. Then every orbit of maximal dimension is flat, and $\pi \in \mathrm{sep}\,(\widehat{G})$ if and only if the orbit corresponding to π has maximal dimension. In particular, if G is a 2-step nilpotent Lie group then $\pi \in \mathrm{sep}\,(\widehat{G})$ if and only if the orbit corresponding to π has maximal dimension.*

Proof. Let U be the set of generic points and let d be the dimension of the orbits of generic points. Let $f \in \mathfrak{g}^*$ and suppose that $\dim \mathrm{Ad}^*(G)f = d$. Let J be a minimal primal ideal contained in $\ker \pi_f$, and let $S = \{g \in \mathfrak{g}^*\colon \ker \pi_g \supseteq J\}$. By Proposition 2·1 there exists a subspace V of $\mathfrak{g}^*$ of dimension d such that $S = f + V$. Hence $\mathrm{Ad}^*(G)f \subseteq f + V$. Since $\mathrm{Ad}^*(G)f$ is a closed manifold of dimension d, and $f+V$ is a d-dimensional affine space, $\mathrm{Ad}^*(G)f = f + V$. Since $\mathrm{Ad}^*(G)f = S$, $\ker \pi_f$ is equal to the minimal primal ideal J, so $\ker \pi_f$ is a separated point of $\mathrm{Prim}\,(C^*(G))$ [**2**, 4·5]. Thus $\pi_f \in \mathrm{sep}\,(\widehat{G})$.

Conversely, suppose that $\pi \in \mathrm{sep}\,(\widehat{G})$. Let $f \in q^{-1}(\pi)$. Then $\ker \pi_f$ is a minimal primal ideal of $C^*(G)$. Hence there exists a d-dimensional subspace W of $\mathfrak{g}^*$ such that

$$f + W = \{f' \in \mathfrak{g}^*\colon \ker \pi_{f'} \supseteq \ker \pi_f\} = \mathrm{Ad}^*(G)f.$$

The final statement of the theorem follows from the fact that if G is 2-step nilpotent then all orbits in $\mathfrak{g}^*$ are flat.

Remark. Suppose that G satisfies the hypotheses of Theorem 2·2. The set M of points in $\mathfrak{g}^*$ whose orbits have maximal dimension is an open subset, so $q(M)$ is an open subset of $\widehat{G}$ consisting of closed, separated points. The argument of [**12**, proposition 7] shows that each $\pi \in q(M)$ can be separated by continuous functions from every other point in $\widehat{G}$. Thus if $f \in M$, $\ker \pi_f$ is a Glimm ideal.

For $n \geqslant 2$, we now consider the simply connected, two-step nilpotent Lie group W_n, with Lie algebra $\mathfrak{w}_n$ which has the basis

$$\{X_1, \ldots, X_n\} \cup \{Y_{i,j}\colon\ 1 \leqslant i < j \leqslant n\}$$

with non-zero products $[X_i, X_j] = Y_{i,j}$. These algebras are universal in the sense that any 2-step nilpotent Lie algebra is the Lie-homomorphic image of some $\mathfrak{w}_n$. In Nielsen's notation [**20**], $W_2 = G_3$ (the 3-dimensional Heisenberg group) and $W_3 = G_{6,15}$. The centre $\mathfrak{z}_n$ of $\mathfrak{w}_n$ is given by

$$\mathfrak{z}_n = \mathrm{span}\,\{Y_{i,j}\colon\ 1 \leqslant i < j \leqslant n\},$$

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and

$$\mathfrak{z}_n^{\perp} = \mathrm{span}\,\{X_1^*, \ldots, X_n^*\}$$

with dimension n. Since the maximal orbit dimension d_n in $\mathfrak{w}_n^*$ is even, we see that $d_n \leqslant n-1$ when n is odd and $d_n \leqslant n$ when n is even.

We recall from [**4**, theorem], that the C^*-algebra of a two-step nilpotent Lie group G with centre Z is quasi-standard if and only if the maximal orbit dimension in $\mathfrak{g}^*$ is equal to the dimension of $\mathfrak{z}^{\perp}$. It follows at once that $C^*(W_n)$ is not quasi-standard if n is odd. We shall show that equality holds in the two inequalities above for d_n, and hence that $C^*(W_n)$ is quasi-standard if n is even.

LEMMA 2·3. *With the notation above, let*

$$f = \sum_{i=1}^{n} \alpha_i X_i^* + \sum_{1 \leqslant r < s \leqslant n} \beta_{rs} Y_{rs}^* \in \mathfrak{w}_n^*$$

and let B be the skew-symmetric $n \times n$ matrix with entries $b_{rs} = -\beta_{rs}$ for $1 \leqslant r < s \leqslant n$. Then $\mathrm{Ad}^*(W_n)f = f + T_B(\mathbf{R}^n)$ where T_B is the linear mapping from $\mathbf{R}^n$ to $\mathfrak{z}_n^{\perp}$ with matrix B relative to the standard basis in $\mathbf{R}^n$ and the basis $\{X_1^*, \ldots, X_n^*\}$ in $\mathfrak{z}_n^{\perp}$.

Proof. The coadjoint orbit of f is given by

$$\begin{aligned}\mathrm{Ad}^*(W_n)f &= \{f \circ \exp(\mathrm{ad}\,(-X))\colon\ X \in \mathfrak{w}\}\\ &= f + V_f,\end{aligned}$$

where

$$\begin{aligned}V_f &= \{f \circ \mathrm{ad}\, X\colon\ X \in \mathfrak{w}\}\\ &= \left\{\sum_{i=1}^{n} t_i f \circ \mathrm{ad}\,(X_i)\colon\ t_1, \ldots, t_n \in \mathbf{R}\right\}.\end{aligned}$$

Since

$$Y_{rs}^* \circ \mathrm{ad}\,(X_i) = \delta_{ir} X_s^* - \delta_{is} X_r^*,$$

we have that

$$\sum_{i=1}^{n} t_i f \circ \mathrm{ad}\,(X_i) = \sum_{1 \leqslant r < s \leqslant n} \beta_{rs}(t_r X_s^* - t_s X_r^*).$$

Let $T\colon \mathbf{R}^n \to \mathfrak{z}^{\perp}$ be the linear mapping defined by

$$T(t_1, \ldots, t_n) = \sum_{1 \leqslant r < s \leqslant n} \beta_{rs}(t_r X_s^* - t_s X_r^*)$$

Then T has matrix B and so $T_B(\mathbf{R}^n) = T(\mathbf{R}^n) = V_f$, as required.

THEOREM 2·4. *With the notation above, $d_n = n$ if n is even, and $d_n = n-1$ if n is odd. In particular, $C^*(W_n)$ is quasi-standard if and only if n is even.*

Proof. Let

$$f = \sum_{1 \leqslant r < s \leqslant n} Y_{rs}^*,$$

and denote by B_n the associated matrix B of Lemma 2·3. Then B_n is skew-symmetric

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and has (r,s)-entry equal to -1 for $1 \leqslant r < s \leqslant n$. Let $D_n = \det(B_n)$. Since B_n is skew-symmetric, $D_n = 0$ when n is odd. For n even $(n \geqslant 4)$, we may subtract the second column from the first and expand by the new first column to obtain that $D_n = D_{n-2}$ because $D_{n-1} = 0$. Thus $D_n = D_2 = 1$ and $\operatorname{rank}(B_n) = n$ when n is even. When n is odd, B_n has B_{n-1} as a leading submatrix and so $\operatorname{rank}(B_n) = n-1$. Since

$$\dim(\mathrm{Ad}^*(W_n)f) = \dim(V_f) = \operatorname{rank}(B_n),$$

we have $n \leqslant d_n \leqslant n$ when n is even, and $n-1 \leqslant d_n \leqslant n-1$ when n is odd. As discussed above, it now follows from [**4**, theorem] that $C^*(W_n)$ is quasi-standard if and only if n is even.

COROLLARY 2·5. *Let $n \geqslant 2$ and let $f \in \mathfrak{w}_n^*$ have the associated skew symmetric matrix B as in Lemma* 2·3. *If n is even then $\pi_f \in \operatorname{sep}(\widehat{W_n})$ if and only if* $\operatorname{rank}(B) = n$, *and if n is odd then $\pi_f \in \operatorname{sep}(\widehat{W_n})$ if and only if* $\operatorname{rank}(B) = n-1$.

Proof. By Theorem 2·2 and Lemma 2·3, $\pi_f \in \operatorname{sep}(\widehat{W_n})$ if and only if

$$d_n = \dim(\mathrm{Ad}^*(W_n)f) = \operatorname{rank}(B).$$

The values of d_n are given by Theorem 2·4. □

When n is even, it follows from the quasi-standardness of $C^*(W_n)$ that every Glimm ideal is primal. On the other hand, when n is odd the only Glimm ideals which are primal are those which are primitive. This is an immediate consequence of Theorem 2·7(iii) below.

In the next lemma, we shall use the notation of Lemma 2·3 and its proof.

LEMMA 2·6. *Let n be an odd integer $(\geqslant 3)$, and let $f \in \mathfrak{w}_n^*$ have associated matrix B. Suppose that* $\operatorname{rank}(B) < n-1$ *and that S is any $(n-1)$-dimensional subspace of $\mathfrak{z}_n^\perp$ containing the subspace $V_f = T_B(\mathbf{R}^n)$. Then there is a sequence (f_k) in $\mathfrak{w}_n^*$ such that $\pi_{f_k} \to \pi_{(f+g)}$ for all $g \in S$.*

Proof. We define a real inner product on $\mathfrak{z}_n^\perp$ by

$$\left\langle \sum_{i=1}^n a_i X_i^*, \sum_{i=1}^n b_i X_i^* \right\rangle = \sum_{i=1}^n a_i b_i.$$

Let S_0 be the orthogonal complement of V_f in S. Since $\dim(S_0)$ is even, there exists a skew-symmetric $n \times n$ matrix $C = (c_{rs})$ such that $T_C(\mathbf{R}^n) = S_0$. For $k \geqslant 1$, let

$$f_k = f - \frac{1}{k} \sum_{1 \leqslant r < s \leqslant n} c_{rs} Y_{rs}^* \in \mathfrak{w}_n^*.$$

Then, by Lemma 2·3,

$$\begin{aligned} \mathrm{Ad}^*(W_n)f_k &= f_k + T_{(B+\frac{1}{k}C)}(\mathbf{R}^n) \\ &= f_k + T_B(\mathbf{R}^n) + T_C(\mathbf{R}^n) \\ &= f_k + S, \end{aligned}$$

where the second equality holds because B and $\frac{1}{k}C$ are skew-symmetric and V_f is orthogonal to S_0.

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For $g \in S$, we have $f_k + g \to f + g$ and hence

$$\pi_{f_k} = \pi_{(f_k+g)} \longrightarrow \pi_{(f+g)}.$$

THEOREM 2·7. *Let n be an odd integer ($\geqslant 3$), let $f \in \mathfrak{w}_n^*$ and suppose that*

$$d_f := \dim(\mathrm{Ad}^*(W_n)f) < n - 1.$$

(i) *Let $g \in \mathfrak{w}_n^*$. Then*

$$g \in f + \mathfrak{z}_n^\perp \Longleftrightarrow \pi_g \sim \pi_f \Longleftrightarrow \pi_g \approx \pi_f.$$

(ii) *There is a bijection Ψ_f, from the set of $(n-1)$-dimensional subspaces of $\mathfrak{z}_n^\perp$ that contain V_f to the set of minimal primal ideals of $C^*(W_n)$ that are contained in* $\ker \pi_f$, *given by*

$$\Psi_f(S) = \bigcap\{\ker \pi_{(f+g)} \colon\ g \in S\}.$$

(iii) *The unique Glimm ideal J_f of $C^*(W_n)$ that is contained in* $\ker \pi_f$ *is $(n - d_f)$-primal but not $(n+1-d_f)$-primal. In particular, the Glimm ideal contained in* $\ker 1_{W_n}$ *is n-primal but not $(n+1)$-primal.*

Proof. (i) Suppose that $g \in f+\mathfrak{z}_n^\perp$. Then there exists an $(n-1)$-dimensional subspace S of $\mathfrak{z}_n^\perp$ containing both V_f and $g - f$. By Lemma 2·6, $\pi_f \sim \pi_g$.

Suppose next that $g \notin f + \mathfrak{z}_n^\perp$. Then $r(\pi_f) \neq r(\pi_g)$, where r is the (continuous) map from $\widehat{W_n}$ to $\widehat{Z_n}$ obtained via restriction. These distinct elements of $\widehat{Z_n}$ may be separated by a bounded, continuous function g on $\widehat{Z_n}$. Then $g \circ r$ separates π_f and π_g. Since $\sim$ is a stronger relation than $\approx$, the proof of (i) is now complete.

(ii) Let S be an $(n-1)$-dimensional subspace of $\mathfrak{z}_n^\perp$ containing V_f. By Lemma 2·6 and [**3**; p. 60], $\Psi_f(S)$ is a primal ideal of $C^*(W_n)$. It then follows from the general form of minimal primal ideals in the C^*-algebra of a two-step nilpotent Lie group (Proposition 2·1), that $\Psi_f(S)$ is minimal primal and that Ψ is bijective.

(iii) Let $k = n - d_f$ and let $P_1, \ldots, P_k$ be primitive ideals of $C^*(W_n)$ containing J_f. For $1 \leqslant i \leqslant k$, there exist $g_i \in \mathfrak{w}_n^*$ such that $P_i = \ker \pi_{g_i}$. Since $\pi_{g_i} \approx \pi_f$, it follows from (i) that $g_i - f \in \mathfrak{z}_n^\perp$ for $1 \leqslant i \leqslant k$. Hence $V_{g_1} = V_f$ and there is an $(n-1)$-dimensional subspace S of $\mathfrak{z}_n^\perp$ containing V_{g_1} and also $\{g_i - g_1 \colon\ 2 \leqslant i \leqslant k\}$. By (ii), $\Psi_{g_1}(S)$ is a (minimal) primal ideal of $C^*(W_n)$ and is contained in each P_i $(1 \leqslant i \leqslant k)$. Hence $\bigcap_{i=1}^k P_i$ is primal, so J_f is k-primal by Lemma 1·3.

Suppose that J_f is $(n+1-d_f)$-primal and let $\{f_1, \ldots, f_k\}$ be a basis for the orthogonal complement of V_f in $\mathfrak{z}_n^\perp$. For $1 \leqslant i \leqslant k$, let $Q_i = \ker \pi_{(f_i+f)}$. By (i), each $Q_i \supseteq J_f$ and so $(\bigcap_{i=1}^k Q_i) \cap \ker \pi_f$ is primal by Lemma 1·3 and therefore contains a minimal primal ideal I. By Proposition 2·1, there is an $(n-1)$-dimensional subspace T of $\mathfrak{z}_n^\perp$ such that

$$f + T = \{g \in \mathfrak{w}_n^* \colon\ \ker \pi_g \supseteq I\}.$$

Hence T contains V_f and $\{f_1, \ldots, f_k\}$, a contradiction.

COROLLARY 2·8. *Let $n \geqslant 2$ and suppose that $f, g \in \mathfrak{w}_n^*$.*

(i) $\pi_f \sim \pi_g \Longleftrightarrow \pi_f \approx \pi_g$.

(ii) $g \in \mathfrak{z}_n^\perp \Longleftrightarrow \pi_g \in \mathrm{cor}\,(W_n) \Longleftrightarrow \pi_g \approx 1_{W_n}$.

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Proof. (i) Suppose first that n is odd. If $d_f = n-1$ then $\ker \pi_f$ is a Glimm ideal, by the remark following Theorem 2·2, and so

$$\pi_f \sim \pi_g \Longleftrightarrow \pi_f = \pi_g \Longleftrightarrow \pi_f \approx \pi_g.$$

On the other hand, if $d_f < n-1$ then we may apply Theorem 2·7(i).

If n is even then $C^*(W_n)$ is quasi-standard (Theorem 2·4) and so $\sim$ coincides with $\approx$ [**5**, proposition 3·2].

(ii) If n is odd, the result follows from Theorem 2·7(i) by putting $f = 0$.

Now suppose that n is even. Then $d_n = n = \dim(\mathfrak{z}_n^\perp)$ (Theorem 2·4) and so the following general arguments apply. There is a dense (open) subset U of $\mathfrak{w}_n^*$ such that $\mathrm{Ad}^*(W_n)f = f + \mathfrak{z}_n^\perp$ for all $f \in U$. Suppose that $g \in \mathfrak{z}_n^\perp$ and let (f_k) be a sequence in U such that $f_k \to 0$ and hence $\pi_{f_k} \to 1_{W_n}$. Then $\pi_{f_k} = \pi_{(f_k+g)} \to \pi_g$ and so $\pi_g \in \mathrm{cor}\,(W_n)$.

Conversely, suppose that $\pi_g \in \mathrm{cor}\,(W_n)$. Then there exist a sequence (f_k) in U and a sequence (g_k) in $\mathfrak{z}_n^\perp$ such that $f_k \to 0$ and $f_k + g_k \to g$. Hence

$$g = \lim g_k \in \mathfrak{z}_n^\perp.$$

Since $\sim$ coincides with $\approx$, as observed in the proof of (i), the proof of (ii) is now complete.

Remarks. It follows from Corollary 2·8(i) that $\sim$ is an equivalence relation on $\widehat{W_n}$. It is not the case, however, that $\sim$ is an equivalence relation for all 2-step nilpotent Lie groups: counterexamples in dimensions 7 and 8 are given in [**16**] and [**6**] respectively.

The arguments used in the proof of Corollary 2·8(ii) for the case of even n can be easily extended to show that, for $f, g \in \mathfrak{w}_n^*$, $\pi_f \sim \pi_g$ if and only if $f - g \in \mathfrak{z}_n^\perp$, and hence that $\sim$ is an open equivalence relation. This method applies to any nilpotent Lie group G with centre Z for which the maximal orbit dimension in $\mathfrak{g}^*$ is equal to $\dim(\mathfrak{z}^\perp)$, and so provides an alternative proof of [**4**, theorem (i)].

3. *Nilpotent Lie groups of the form* $\mathbf{R} \ltimes \mathbf{R}^d$

In this section we investigate those nilpotent Lie groups G which are semi-direct products of $\mathbf{R}$ with $\mathbf{R}^d$, $G = \mathbf{R} \ltimes \mathbf{R}^d$. We determine the separated points of $\widehat{G}$, and answer the question of when every Glimm ideal of $C^*(G)$ is primal.

These groups are precisely the groups $G = \exp \mathfrak{g}$, where $\mathfrak{g}$ is a nilpotent Lie algebra which can be written as a semi-direct product of some one-dimensional subalgebra and an abelian ideal. This class of Lie algebras contains the so-called threadlike or filiform Lie algebras, defined as follows.

For $N \geqslant 3$, let $\mathfrak{g}_N$ be the N-dimensional nilpotent Lie algebra with basis $X_1, \ldots, X_N$ and non-trivial Lie brackets

$$[X_N, X_{N-1}] = X_{N-2}, \ldots, [X_N, X_2] = X_1;$$

then $\mathfrak{g}_N$ is $(N-1)$-step nilpotent and a semi-direct product of $\mathbf{R}X_N$ with the abelian ideal $\sum_{j=1}^{N-1} \mathbf{R}X_j$. For $f = \sum_{j=1}^{N-1} \xi_j X_j^* \in \mathfrak{g}_N^*$, the coadjoint action is given by

$$\mathrm{Ad}^*(\exp(-tX_N))f = \sum_{j=1}^{N-1} p_j(\xi, t) X_j^*,$$

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where $p_j(\xi, t)$ is a polynomial in t given by

$$p_j(\xi, t) = \sum_{k=0}^{j-1} \frac{t^k}{k!} \xi_{j-k}.$$

Moreover, if $\xi_j \neq 0$ for at least one $j \leqslant N-2$, then $\mathrm{Ad}^*(G)f$ is of dimension two and $\mathrm{Ad}^*(G)f = \mathrm{Ad}^*(G)f + \mathbf{R}X_N^*$.

Let $G_N = \exp \mathfrak{g}_N$. These groups G_N are the building blocks for general nilpotent groups of the form $G = \mathbf{R} \ltimes \mathbf{R}^d$, in the following way. There is a real $d \times d$ matrix A such that the action of $\mathbf{R}$ on $\mathbf{R}^d$ is given by $(t, x) \to \exp(tA)x$, $x \in \mathbf{R}^d$, $t \in \mathbf{R}$. By changing the basis of $\mathbf{R}^d$ if necessary we can assume that $\exp(tA)$ is in real Jordan canonical form. Then, since G is nilpotent, $\exp(tA)$ is block diagonal, with each block B an upper triangular matrix, with entries $b_{i,j} = t^{(j-i)}/(j-i)!$ for $1 \leqslant i \leqslant j \leqslant k$, where k is the size of the block B. Thus $\mathbf{R}^d$ decomposes into a direct product of vector subgroups $V_1, \ldots, V_r$, which are normal in G, such that either V_j is one-dimensional or $\mathbf{R} \ltimes V_j$ is isomorphic to G_{N_j} for some $N_j \geqslant 3$. Equivalently, the Lie algebra $\mathfrak{g}$ of G is a semi-direct product $\mathfrak{g} = \mathbf{R}X \ltimes \mathfrak{a}$, where the abelian ideal decomposes into a direct sum of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ of $\mathfrak{g}$ such that either $\dim \mathfrak{a}_j = 1$ or $\mathbf{R}X \ltimes \mathfrak{a}_j$ is isomorphic to $\mathfrak{g}_{N_j}$.

The cortex of G_N has been described in [**7**], and it was noticed in [**6**] that this description generalizes to groups of the form $\mathbf{R} \ltimes \mathbf{R}^d$. Thus, let $\mathfrak{g} = \mathbf{R}X \ltimes (\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r)$ as above. We can then assume that $\mathbf{R}X \ltimes \mathfrak{a}_j = \mathfrak{g}_{N_j}$ for $1 \leqslant j \leqslant s$ and $\dim \mathfrak{a}_j = 1$ for $s < j \leqslant r$. Then

$$\mathfrak{g} = (\mathbf{R}X \ltimes (\mathfrak{a}_1 + \cdots + \mathfrak{a}_s)) \oplus \mathfrak{b},$$

where $\mathfrak{b}$ is abelian. Let $\{X_{j,1}, \ldots, X_{j,N_j-1}\}$ be a basis of $\mathfrak{a}_j$ such that $[X, X_{j,k}] = X_{j,k-1}$ for $N_j - 1 \geqslant k \geqslant 2$ and $1 \leqslant j \leqslant s$. Let

$$f = \alpha X^* + \sum_{j=1}^{s} \left(\sum_{k=1}^{N_j-1} \xi_{j,k} X_{j,k}^* \right) + g \quad (g \in \mathfrak{b}^*)$$

be an arbitrary element of $\mathfrak{g}^*$. Then π_f belongs to $\mathrm{cor}(G)$ if and only if $g = 0$ and $\xi_{j,k} = 0$ for all $k \leqslant [N_j/2]$, $1 \leqslant j \leqslant s$.

Theorem 3·1. *Let G be a non-abelian nilpotent group of the form $\mathbf{R} \ltimes \mathbf{R}^d$. Then the following conditions are equivalent*:

(i) *every Glimm ideal of $C^*(G)$ is primal*;

(ii) *either $G = G_4 \times A$ or $G = G_3 \times A$, where A is abelian.*

In addition, $C^*(G)$ is quasi-standard if and only if $G = G_3 \times A$.

Proof. Notice first that if $G = G_3 \times A$, where A is abelian, then $C^*(G)$ is quasi-standard. Indeed, $C^*(G_3)$ is quasi-standard and hence so is $C^*(G_3 \times A) = C^*(G_3) \otimes C^*(A)$ [**17**, corollary 2·5]. Next, by [**4**, lemma 4], every Glimm ideal in $C^*(G_4)$ is primal, and hence so is every Glimm ideal in $C^*(G_4 \times A)$, by [**17**, theorems 2·3 and 1·1]. Towards a contradiction, suppose that $C^*(G_4 \times A)$ is quasi-standard. Then $\sim$ is an open equivalence relation on $\mathrm{Prim}(C^*(G_4 \times A))$. Since $\mathrm{Prim}(C^*(G_4 \times A))$ is homeomorphic to the product space $\widehat{A} \times \mathrm{Prim}(C^*(G_4))$, and $\widehat{A}$ is a locally compact, Hausdorff space, $\sim$ must be an open equivalence relation on $\mathrm{Prim}(C^*(G_4))$. Thus $C^*(G_4)$ is quasi-standard, contradicting [**4**, lemma 4].

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Thus it remains to show that (i) implies (ii). For this, let I be the closed two-sided ideal of $C^*(G)$ given by

$$I = \bigcap\{\ker \pi_f \colon \pi_f \in \operatorname{cor}(G)\}.$$

Then I contains the unique Glimm ideal J contained in $\ker 1_G$. By hypothesis J is primal, and hence so is I. Retaining the previous notation, let

$$\mathfrak{g} = (\mathbf{R}X \ltimes (\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_s)) \oplus \mathfrak{b},$$

where $\mathfrak{b}$ is abelian and $\mathbf{R}X \ltimes \mathfrak{a}_j = \mathfrak{g}_{N_j}$, $N_j \geqslant 3$. We have to show that $s = 1$, and $N_1 = 3$ or $N_1 = 4$. Of course, we can assume that $N := N_1 \geqslant N_2 \geqslant \ldots \geqslant N_s$. Towards a contradiction, suppose that either $N_1 \geqslant 5$ or $s \geqslant 2$. The strategy for the proof consists of obtaining a lower estimate for the number of intersections with a grid in $\mathbf{R}^2$ of a projected coadjoint orbit from $\mathfrak{g}^*$. A version of the pigeonhole principle then leads to a polynomial having more zeros than its degree permits.

Writing elements f of $\mathfrak{g}^*$ as

$$f = \alpha X^* + \sum_{j=1}^{s}\left(\sum_{k=1}^{N_j-1} \xi_{j,k} X_{j,k}^*\right) + g,$$

where $g \in \mathfrak{b}^*$, we now define, for $1 \leqslant i, l \leqslant 2N-4$, an open subset $V_{i,l}$ of $\mathfrak{g}^*$ as follows. If $N \geqslant 5$, let

$$V_{i,l} = \{f \in \mathfrak{g}^* \colon \xi_{1,N-1} \in (2i-2, 2i) \text{ and } \xi_{1,N-2} \in (2l-2, 2l)\},$$

and if $4 \geqslant N$, $N_2 \geqslant 3$, put

$$V_{i,l} = \{f \in \mathfrak{g}^* \colon \xi_{1,N-1} \in (2i-2, 2i) \text{ and } \xi_{2,N_2-1} \in (2l-2, 2l)\}.$$

Let $q \colon \mathfrak{g}^* \to \widehat{G}$ be the canonical map given by $q(f) = \pi_f$. Since q is open, $q(V_{i,l})$ is an open subset of $\widehat{G}$. Moreover, by the description of $\operatorname{cor}(G)$,

$$q(V_{i,l}) \cap (C^*(G)/I)^\wedge \supseteq q(V_{i,l}) \cap \operatorname{cor}(G) \neq \varnothing$$

for each (i,l). Since I is primal, it follows from Lemma 1·1 that

$$V := \bigcap_{i,l=1}^{2N-4} q(V_{i,l})$$

is a non-empty subset of $\widehat{G}$.

Let $U = \{f \in \mathfrak{g}^* \colon \xi_{j,1} \neq 0 \text{ for } 1 \leqslant j \leqslant s\}$. Since U is dense in $\mathfrak{g}^*$ and q is a continuous surjection, there exists $f \in U$ such that $\pi_f \in V$. Thus, for $1 \leqslant i, l \leqslant 2N-4$, there exist $v_{i,l} \in V_{i,l}$ such that $v_{i,l} \in \mathrm{Ad}^*(G)f$.

Now define a projection $P \colon \mathfrak{g}^* \to \mathbf{R}^2$ by

$$P(f) = (\xi_{1,N-1}, \xi_{1,N-2}) \quad \text{if } N \geqslant 5$$

and

$$P(f) = (\xi_{1,N-1}, \xi_{2,N_2-1}) \quad \text{if } 4 \geqslant N \geqslant N_2 \geqslant 3.$$

Set $u_{i,l} = P(v_{i,l})$ and

$$U_{i,l} = (2i-2, 2i) \times (2l-2, 2l) \subseteq \mathbf{R}^2, \quad (1 \leqslant i, l \leqslant 2N-4).$$

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From the form of the coadjoint orbits in $\mathfrak{g}^*$, it follows that

$$P(\mathrm{Ad}^*(G)f) = \{(Q_1(t), Q_2(t) \colon t \in \mathbf{R}\},$$

where Q_1 and Q_2 are non-constant polynomials of degree not greater than $N-2$. So for $1 \leqslant i, l \leqslant 2N-4$, there exist $t_{i,l} \in \mathbf{R}$ such that

$$u_{i,l} = (Q_1(t_{i,l}), Q_2(t_{i,l})) \in U_{i,l}.$$

For $(i,l) \neq (i',l')$ we have $u_{i,l} \neq u_{i',l'}$ (since $U_{i,l}$ and $U_{i',l'}$ are disjoint) and hence $t_{i,l} \neq t_{i',l'}$. Suppose that the $t_{i,l}$ are relabelled in increasing order:

$$t_1 < t_2 < \ldots < t_M$$

where $M = (2N-4)^2$.

Now, in $\mathbf{R}^2$, with coordinates (x,y), let L_m be the line $x = 2m$ and M_m be the line $y = 2m$, $(1 \leqslant m \leqslant 2N-5)$. Then for $1 \leqslant n \leqslant M-1$, the points $(Q_1(t_n), Q_2(t_n))$ and $(Q_1(t_{n+1}), Q_2(t_{n+1}))$ are separated by at least one of the $4N-10$ grid lines and so there exists $s_n \in (t_n, t_{n+1})$ such that

$$P_n \coloneqq (Q_1(s_n), Q_2(s_n)) \in \left(\bigcup_{m=1}^{2N-5} L_m\right) \cup \left(\bigcup_{m=1}^{2N-5} M_m\right).$$

We claim that there exists $m \in \{1, \ldots, 2N-5\}$ such that either $\mathrm{card}\,\{n \colon P_n \in L_m\} \geqslant N-1$ or $\mathrm{card}\,\{n \colon P_n \in M_m\} \geqslant N-1$. For otherwise, bearing in mind that a point P_n might lie on more than one of the grid lines, we would have

$$\begin{aligned} M-1 &\leqslant \sum_{m=1}^{2N-5} \mathrm{card}\,\{n \colon P_n \in L_m\} + \sum_{m=1}^{2N-5} \mathrm{card}\,\{n \colon P_n \in M_m\} \\ &\leqslant 2(2N-5)(N-2) \end{aligned}$$

and hence $2N-3 \leqslant 2(N-2)$. From the claim, it now follows that either Q_1 or Q_2 takes the value $2m$ for at least $N-1$ distinct values of its argument, contradicting the fact that both Q_1 and Q_2 are non-constant and of degree not greater than $N-2$. $\square$

For the next two lemmas let $\mathfrak{g}$ be of the form $\mathfrak{g} = \mathbf{R}X \ltimes (\mathfrak{a} \oplus \mathfrak{b})$, where $\mathfrak{a}$ and $\mathfrak{b}$ are abelian ideals of $\mathfrak{g}$ such that $\mathbf{R}X \ltimes \mathfrak{a} = \mathfrak{g}_N$ for some $N \geqslant 3$. Let $\{X_1, \ldots, X_{N-1}\}$ be a basis of $\mathfrak{a}$ such that $[X, X_j] = X_{j-1}$ for $2 \leqslant j \leqslant N-1$. For convenience we shall write X_N in place of X.

LEMMA 3·2. *Let $f = \sum_{j=1}^N \xi_j X_j^* + f'$ and $g = \sum_{j=1}^N \eta_j X_j^* + g'$ be elements of $\mathfrak{g}^*$, where $f', g' \in \mathfrak{b}^*$. Suppose that for some $k \leqslant [N/2]$ we have $\xi_j = \eta_j = 0$ for $j < k$, and $\xi_k^2 \neq \eta_k^2$. Then π_f and π_g can be separated by a continuous function on $\widehat{G}$.*

Proof. Let $P \colon \mathfrak{g}^* \to \mathfrak{g}_N^*$ denote the projection with kernel equal to $\mathfrak{b}^* = \mathfrak{g}_N^{\perp}$. Then for $h \in \mathfrak{g}_N^*$, $h' \in \mathfrak{b}^*$, and $t \in \mathbf{R}$,

$$\begin{aligned} P(\mathrm{Ad}^*(\exp tX_N)(h+h')) &= P(\mathrm{Ad}^*(\exp tX_N)h + \mathrm{Ad}^*(\exp tX_N)h') \\ &= \mathrm{Ad}^*(\exp tX_N)h \end{aligned}$$

since $\mathfrak{g}_N$ is an ideal of $\mathfrak{g}$. Let ϕ be an $\mathrm{Ad}^*(G_N)$-invariant, continuous function on $\mathfrak{g}_N^*$. Since $\mathrm{Ad}^*(G_N)h = \mathrm{Ad}^*(G_N)h + \mathbf{R}X_N^*$ for all h in a dense subset of $\mathfrak{g}_N^*$, it follows that

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$\phi(h+sX_N^*) = \phi(h)$ for all $h \in \mathfrak{g}_N^*$ and all $s \in \mathbf{R}$. Hence, for all $h \in \mathfrak{g}_N^*$, $h' \in \mathfrak{b}^*$ and $s, t \in \mathbf{R}$, we have

$$\begin{aligned}(\phi \circ P)(\mathrm{Ad}^*(\exp tX_N)(h+h') + sX_N^*) &= \phi(\mathrm{Ad}^*(\exp tX_N)(h) + sX_N^*)\\ &= \phi(h)\\ &= (\phi \circ P)(h+h').\end{aligned}$$

Since $\mathrm{Ad}^*(G)(h+h') \subseteq \mathrm{Ad}^*(\exp \mathbf{R}X_N)(h+h') + \mathbf{R}X_N^*$, $\phi \circ P$ is an $\mathrm{Ad}^*(G)$-invariant, continuous function on $\mathfrak{g}^*$.

Now take

$$\phi(t_1, t_2, \ldots, t_N) = t_k^2 + 2\sum_{j=1}^{k-1}(-1)^j t_{k-j}t_{k+j},$$

which is an $\mathrm{Ad}^*(G_N)$-invariant polynomial on $\mathbf{R}^N = \mathfrak{g}_N^*$ [**7**, lemma 1]. By hypothesis, $\phi \circ P(f) \neq \phi \circ P(g)$ and hence the associated continuous function on $\widehat{G}$ separates π_f and π_g. □

We now introduce, for non-negative integers k and l with $k \leqslant l$, the $(l-k+1) \times (l-k+1)$ matrix $A(k,l)$ whose (i,j)th entry is $1/(i+j+k-2)!$. Let $\Delta(k,l) = \det A(k,l)$. For $l > k$, $\Delta(k,l)$ satisfies the recurrence formula

$$\Delta(k,l) = (-1)^{l-k}\frac{(l-k)!}{l!}\Delta(k+2, l+1).$$

This can easily be verified by successively subtracting the appropriate multiple of row $j-1$ from row j, for $l-k+1 \geqslant j \geqslant 2$, thus clearing positions $(l-k+1, 1), \ldots, (2,1)$ in $A(k,l)$. This formula shows, in particular, that $\Delta(k,l) \neq 0$.

Lemma 3·3. *Let $f = \sum_{j=1}^N \xi_j X_j^* + f' \in \mathfrak{g}^*$, with $f' \in \mathfrak{b}^*$, and suppose that $\xi_k \neq 0$ for some k with $2k \leqslant N-1$. Then π_f is a separated point of $\widehat{G}$.*

Proof. Let k be minimal such that $\xi_k \neq 0$, and let $g = \sum_{j=1}^N \eta_j X_j^* + g' \in \mathfrak{g}^*$, with $g' \in \mathfrak{b}^*$. By Lemma 3·2, π_f and π_g can be separated by a continuous function on $\widehat{G}$ whenever $\eta_j \neq 0$ for some $j < k$. Suppose, therefore, that $\eta_j = 0$ for all $j < k$, and that π_f and π_g cannot be separated by open subsets of $\widehat{G}$. Then there exist sequences $(\xi_{j,n})_n \subseteq \mathbf{R}$, $(1 \leqslant j \leqslant N)$, $(t_n) \subseteq \mathbf{R}$, and $(f'_n)_n \subseteq \mathfrak{b}^*$ such that

$$f_n := \sum_{j=1}^N \xi_{j,n}X_j^* + f'_n \to f$$

and $\mathrm{Ad}^*(\exp(-t_nX_N))f_n \to g$. Thus $\xi_{j,n} \to \xi_j$, $(1 \leqslant j \leqslant N)$, and $f'_n \to f'$ as $n \to \infty$. Using the first displayed equation in the proof of Lemma 3·2, we get that

$$\mathrm{Ad}^*(\exp(-t_nX_N))\left(\sum_{j=1}^N \xi_{j,n}X_j^*\right) \to \sum_{j=1}^N \eta_j X_j^* \qquad (*)$$

and hence that $\mathrm{Ad}^*(\exp(-t_nX_n))f'_n \to g'$. We now assume first that $|t_n| \to \infty$ and show that this leads to a contradiction. Using $(*)$ for η_j $(k+1 \leqslant j \leqslant 2k)$ and recalling

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that $\xi_{j,n} \to \xi_j$ $(k+1 \leqslant j \leqslant 2k)$, we obtain

$$\frac{t_n^k}{k!}\xi_{1,n} + \ldots + t_n \xi_{k,n} \to \eta_{k+1} - \xi_{k+1} \tag{A_1}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{t_n^{2k-1}}{(2k-1)!}\xi_{1,n} + \cdots + \frac{t_n^k}{k!}\xi_{k,n} + \cdots + t_n \xi_{2k-1,n} \to \eta_{2k} - \xi_{2k}. \tag{A_k}$$

Let $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$ be arbitrary. Then, after multiplying (A_j) by $\alpha_j t_n^{1-j}$ for $1 \leqslant j \leqslant k$ and summing up, we obtain (since $|t_n| \to \infty$)

$$\sum_{j=1}^k \alpha_j \left(\sum_{l=1}^k \frac{t_n^l}{(j+l-1)!}\xi_{k+1-l,n} \right) \to \alpha_1 \eta_{k+1} - \left(\sum_{j=1}^k \frac{\alpha_j}{(j-1)!} \right) \xi_{k+1}.$$

Thus the sequence (s_n), with

$$s_n = \sum_{l=1}^k t_n^l \xi_{k+1-l,n} \left(\sum_{j=1}^k \alpha_j \frac{1}{(j+l-1)!} \right),$$

converges. Now, since $\Delta(1,k) \neq 0$, there are unique $\alpha_1, \ldots, \alpha_k$ such that

$$\sum_{j=1}^k \alpha_j \frac{1}{(j+l-1)!} = \begin{cases} 1 & \text{for } l = 1 \\ 0 & \text{for } l = 2, \ldots, k. \end{cases}$$

With this choice of $\alpha_1, \ldots, \alpha_k$, it follows that $s_n = t_n \xi_{k,n}$. Since $|t_n| \to \infty$, we have $\xi_k = \lim_{n\to\infty} \xi_{k,n} = 0$, a contradiction.

Thus, passing to a subsequence if necessary, we can assume that $t_n \to t$ for some $t \in \mathbf{R}$. Then, from $\xi_{l,n} \to \xi_l$ for $1 \leqslant l \leqslant N$, we conclude that

$$\eta_j = \lim_{n\to\infty} p_j \left(\sum_{l=1}^N \xi_{l,n} X_l^*, t_n \right) = p_j \left(\sum_{l=1}^N \xi_l X_l^*, t \right)$$

for $1 \leqslant j \leqslant N-1$. Moreover, since $f_n' \to f'$,

$$g' = \lim_{n\to\infty} \mathrm{Ad}^*(\exp(-t_n X_N)) f_n' = \mathrm{Ad}^*(\exp(-tX_N)) f'.$$

Combining these facts gives

$$g - \eta_N X_N^* = \sum_{j=1}^{N-1} \eta_j X_j^* + g'$$

$$= \mathrm{Ad}^*(\exp(-tX_N)) f - \xi_N X_N^*.$$

On the other hand, since $\xi_k \neq 0$ and $k \leqslant N-2$,

$$\mathrm{Ad}^*(G) f = \mathrm{Ad}^*(G) f + \mathbf{R} X_N^*.$$

This proves that $g \in \mathrm{Ad}^*(G) f$, and hence that $\pi_f = \pi_g$, as required.

Lemma 3·4. *Let $N = 2m$ and let $f = \sum_{j=m}^{N-1} \xi_j X_j^*$ and $g = \sum_{j=m}^{N-1} \eta_j X_j^*$ be elements of $\mathfrak{g}_N^*$ satisfying $\xi_m = (-1)^{m-1}\eta_m$. Then there exist sequences $(\xi_{j,n})_{n\in\mathbf{N}}$ in $\mathbf{R}$,*

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$(1 \leqslant j \leqslant N-1)$, *such that with*

$$f_n = \sum_{j=1}^{N-1} \xi_{j,n} X_j^* \in \mathfrak{g}_N^*, \quad (n \in \mathbf{N}),$$

we have in $\mathfrak{g}_N^*$

$$f_n \to f \text{ and } \mathrm{Ad}^*(\exp(-nX_N)) f_n \to g.$$

Proof. For $n \in \mathbf{N}$ and $j = m+1, \ldots, N-1$, put $\xi_{j,n} = \xi_j$. Now, for fixed $n \in \mathbf{N}$ consider the following $m \times m$ system of linear equations in variables $x_1, \ldots, x_m$:

$$\frac{1}{0!} x_m + \frac{n}{1!} x_{m-1} + \cdots + \frac{n^{m-1}}{(m-1)!} x_1 = \eta_m \tag{A_0}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\frac{1}{(m-1)!} x_m + \frac{n}{m!} x_{m-1} + \cdots + \frac{n^{m-1}}{(2m-2)!} x_1 = \frac{1}{n^{m-1}} \left(\eta_{N-1} - \sum_{k=0}^{m-2} \frac{n^k}{k!} \xi_{N-1-k} \right). \tag{A_{m-1}}$$

Since $\Delta(0, m-1) \neq 0$, there is a unique solution for $x_m, nx_{m-1}, \ldots, n^{m-1}x_1$, and hence for $x_m, \ldots, x_1$, which we denote $\xi_{m,n}, \ldots, \xi_{1,n}$. Let

$$f_n = \sum_{j=1}^{N-1} \xi_{j,n} X_j^* \in \mathfrak{g}_N^*, \quad (n \in \mathbf{N}).$$

We are going to show that $f_n \to f$ and $\mathrm{Ad}^*(\exp(-nX_N)) f_n \to g$ in $\mathfrak{g}_N^*$. To this end, we first notice that by the choice of $\xi_{k,n}$, $1 \leqslant k \leqslant N-1$, for each $0 \leqslant j \leqslant m-1$, we can rewrite equation (A_j) as

$$\eta_{m+j} = \frac{n^{m-1+j}}{(m-1+j)!} \xi_{1,n} + \cdots + \frac{n^j}{j!} \xi_{m,n} + \cdots + \xi_{m+j,n}. \tag{B_j}$$

That is $\eta_{m+j} = p_{m+j}(f_n, n)$, the coefficient of X_{m+j}^* in $\mathrm{Ad}^*(\exp(-nX_N)) f_n$.

We next prove that $\xi_{m,n} \to \xi_m$ as $n \to \infty$. Obviously, in equations (A_1) to (A_{m-1}), the righthand side converges to zero as $n \to \infty$. Therefore, for any choice of $\alpha_0, \ldots, \alpha_{m-1} \in \mathbf{R}$,

$$\sum_{k=0}^{m-1} n^k \xi_{m-k,n} \left(\sum_{j=0}^{m-1} \alpha_j \frac{1}{(j+k)!} \right) = \sum_{j=0}^{m-1} \alpha_j \left(\sum_{k=0}^{m-1} \frac{n^k}{(j+k)!} \xi_{m-k,n} \right) \to \alpha_0 \eta_m.$$

Now, take $\alpha_0, \ldots, \alpha_{m-1}$ to be the solutions of the system

$$\sum_{j=0}^{m-1} x_j \frac{1}{(j+k)!} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{for } k = 1, \ldots, m-1. \end{cases}$$

Then, from Cramer's Rule and since $\Delta(0, m-1) = (-1)^{m-1} \Delta(2, m)$, it follows that $\alpha_0 = (-1)^{m-1}$. Thus

$$\xi_{m,n} \to (-1)^{m-1} \eta_m = \xi_m.$$

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It remains to show that $\xi_{j,n} \to 0$ and

$$p_j(f_n, n) = \sum_{k=0}^{j-1} \frac{n^k}{k!} \xi_{j-k,n} \to 0$$

for every $1 \leqslant j \leqslant m-1$. Of course, it suffices to verify that $n^{k-1}\xi_{m-k,n} \to 0$ for $k = 1, \ldots, m-1$. Since $(\xi_{m,n})_n$ converges, dividing equation (B_j) by n^{j+1}, for $1 \leqslant j \leqslant m-1$, yields

$$\frac{1}{2!}\xi_{m-1,n} + \cdots + \frac{n^{m-2}}{m!}\xi_{1,n} \to 0$$

$$\vdots$$

$$\frac{1}{m!}\xi_{m-1,n} + \cdots + \frac{n^{m-2}}{(2m-2)!}\xi_{1,n} \to 0$$

(as $n \to \infty$). Let $A \in M(m-1, \mathbf{R})$ denote the matrix whose (i,j)th entry is $1/(i+j)!$ and let $y_n = (\xi_{m-1,n}, \ldots, n^{m-2}\xi_{1,n}) \in \mathbf{R}^{m-1}$. Then $Ay_n^t \to 0$ and hence, since $A = A(2,m)$ is invertible, $y_n^t = A^{-1}(Ay_n^t) \to 0$. This shows that $n^{k-1}\xi_{m-k,n} \to 0$ for $k = 1, \ldots, m-1$, as required.

Lemma 3·5. *Let $N = 2m+1$ $(m \geqslant 2)$ and let $f = \sum_{j=m+1}^{2m} \xi_j X_j^*$ and $g = \sum_{j=m+1}^{2m} \eta_j X_j^*$ be elements of $\mathfrak{g}_N^*$. Then there exist sequences $(\xi_{j,n})_n$ in $\mathbf{R}$ $(1 \leqslant j \leqslant 2m)$ such that in $\mathfrak{g}_N^*$*

$$f_n = \sum_{j=1}^{2m} \xi_{j,n} X_j^* \to f \quad \textit{and} \quad \mathrm{Ad}^*(\exp(-nX_N))f_n \to g.$$

Proof. The proof is somewhat similar to that of Lemma 3·4. For $n \in \mathbf{N}$ and $j = m+1, \ldots, 2m$, put $\xi_{j,n} = \xi_j$. For fixed $n \in \mathbf{N}$ consider the following $m \times m$ system of linear equations in variables $y_1, \ldots, y_m$:

$$\frac{n}{1!}y_m + \cdots + \frac{n^m}{m!}y_1 = \eta_{m+1} - \xi_{m+1} \qquad (A_1)$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{n}{m!}y_m + \cdots + \frac{n^m}{(2m-1)!}y_1 = \frac{1}{n^{m-1}}\left(\eta_{2m} - \sum_{k=0}^{m-1}\frac{n^k}{k!}\xi_{2m-k}\right). \qquad (A_m)$$

Since $\Delta(1,m) \neq 0$ there is a unique solution for $y_m, \ldots, y_1$ which we shall denote by $\xi_{m,n}, \ldots, \xi_{1,n}$. Hence, by construction, η_j is the jth component of $\mathrm{Ad}^*(\exp(-nX_N))f_n$ for $m+1 \leqslant j \leqslant 2m$.

It remains to show that $\xi_{j,n} \to 0$ and

$$p_j(f_n, n) = \sum_{k=0}^{j-1} \frac{n^k}{k!} \xi_{j-k,n} \to 0$$

as $n \to \infty$ for $j = 1, \ldots, m$. For this, it suffices to show that $n^k\xi_{m-k,n} \to 0$ as $n \to \infty$ for $k = 0, \ldots, m-1$. This is done by multiplying each of the equations $(A_1), \ldots, (A_m)$ by $1/n$, and then arguing as in the last part of the proof of Lemma 3·4, using $A(1,m)$ instead of $A(2,m)$. $\square$

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Let G be a locally compact group which is a direct product $G = A \times H$ of closed subgroups A and H, with A abelian. Then the mapping $(\alpha, \pi) \to \alpha\pi$, where $\alpha\pi(a, x) = \alpha(a)\pi(x)$ for $a \in A$ and $x \in H$, is a homeomorphism between $\widehat{A} \times \widehat{H}$ and $\widehat{G}$. In particular, $\alpha\pi$ is a separated point of $\widehat{G}$ if and only if π is a separated point of $\widehat{H}$. For simplicity we can therefore reduce to the case where G has no abelian direct factor.

Theorem 3·6. *Let G be a simply connected, nilpotent Lie group having no abelian direct factor, and of the form $\mathbf{R} \ltimes \mathbf{R}^d$. Write its Lie algebra $\mathfrak{g}$ as*

$$\mathfrak{g} = \mathbf{R}X \ltimes (\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r),$$

where, for $k = 1, \ldots, r$, $\mathfrak{a}_k$ is an abelian ideal of $\mathfrak{g}$ of dimension d_k, and $\mathbf{R}X \ltimes \mathfrak{a}_k = \mathfrak{g}_{d_k+1}$. Let $\{X, X_{d_k,k}, \ldots, X_{1,k}\}$ denote the usual basis of $\mathfrak{g}_{d_k+1}$, and let

$$f = \alpha X^* + \sum_{k=1}^{r} \left(\sum_{j=1}^{d_k} \xi_{j,k} X^*_{j,k} \right)$$

be an arbitrary element of $\mathfrak{g}^*$. Then π_f is a separated point of $\widehat{G}$ if and only if $\xi_{j,k} \neq 0$ for some pair (j, k) such that $j \leqslant [d_k/2]$.

Proof. Put $N_k = d_k + 1$, $1 \leqslant k \leqslant r$, and suppose first that $\xi_{j,k} \neq 0$ for some pair (j, k) such that $j \leqslant [d_k/2]$. Then $\mathfrak{g} = \mathbf{R}X \ltimes (\mathfrak{a}_k + \mathfrak{b})$, where $\mathfrak{b}$ is the direct sum of all $\mathfrak{a}_l$, $l \neq k$, and

$$f = \alpha X^* + \sum_{i=1}^{d_k} \xi_{i,k} X^*_{i,k} + f',$$

where $f' \in \mathfrak{b}^*$. Since $2j \leqslant 2[d_k/2] \leqslant N_k - 1$, Lemma 3·3 now shows that π_f is a separated point of $\widehat{G}$.

Conversely, suppose that $\xi_{j,k} = 0$ whenever $j \leqslant [d_k/2]$. We have to produce some $g \in \mathfrak{g}^*$ such that $g \notin \mathrm{Ad}^*(G)f$ and π_f and π_g cannot be separated by open subsets of $\widehat{G}$. Of course, we can assume that π_f does not belong to the cortex of G. By the description of $\mathrm{cor}\,(G)$, this implies that there exists k such that N_k is even, $N_k = 2m_k$ say, and $\xi_{m_k,k} \neq 0$. After renumbering, if necessary, we can assume that N_1 is even, $N_1 = 2m_1$, and $\xi_{m_1,1} \neq 0$. We can also assume that $\alpha = 0$. Indeed, since $N_1 - 2 \geqslant m_1$ and $\xi_{m_1,1} \neq 0$, it follows that $\mathrm{Ad}^*(G)f = \mathrm{Ad}^*(G)f + \mathbf{R}X^*$.

For $k = 1, \ldots, r$, let $N_k = 2m_k$ if N_k is even and $N_k = 2m_k + 1$ if N_k is odd; put $f_k = \sum_{j=m_k}^{d_k} \xi_{j,k} X^*_{j,k}$. We are going to define $g_k \in (\mathbf{R}X \ltimes \mathfrak{a}_k)^*$ such that f_k and g_k satisfy the hypotheses of Lemma 3·4 (if N_k is even) or Lemma 3·5 (if N_k is odd) and such that $g_1 \notin \mathrm{Ad}^*(G)f_1$.

We first define g_1. Put $\eta_{m_1,1} = (-1)^{m_1-1}\xi_{m_1,1}$. If m_1 is even, let $\eta_{j,1} = 0$ for all $j \neq m_1$. If m_1 is odd (so that $\eta_{m_1,1} = \xi_{m_1,1}$), there exist $\eta_{m_1+1,1}, \ldots, \eta_{d_1,1}$ such that

$$g_1 = \xi_{m_1,1} X^*_{m_1,1} + \sum_{j=m_1+1}^{d_1} \eta_{j,1} X^*_{j,1}$$

does not belong to the $\mathrm{Ad}^*(G)$-orbit of f_1. In fact, since m_1 is odd, $m_1 \geqslant 3$ and hence $d_1 - m_1 \geqslant 2$. On the other hand, $\mathrm{Ad}^*(G)f_1 = \mathrm{Ad}^*(G)f_1 + \mathbf{R}X^*$ and the orbit dimension is only two.

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For $2 \leqslant k \leqslant r$ with N_k even, we let

$$g_k = (-1)^{m_k-1}\xi_{m_k,k}X^*_{m_k,k}.$$

Finally, if N_k is odd, recall that $\xi_{j,k} = 0$ for all $1 \leqslant j \leqslant m_k$ and put

$$g_k = (-1)^{m_k-1}\xi_{m_k+1,k}X^*_{m_k+1,k}.$$

It is now clear that Lemmas 3·4 and 3·5 apply, whenever N_k is even or odd, respectively. Thus there exist sequences $(\xi_{j,k,n})_n$ in $\mathbf{R}$, $(1 \leqslant k \leqslant r,\ 1 \leqslant j \leqslant d_j)$, such that with

$$f_{k,n} = \sum_{j=1}^{d_j} \xi_{j,k,n}X^*_{j,k} \in (\mathbf{R}X \ltimes \mathfrak{a}_k)^* = \mathfrak{g}^*_{N_k}$$

we have $f_{k,n} \to f_k$ and $\mathrm{Ad}^*(\exp(-nX))f_{k,n} \to g_k$ in $\mathfrak{g}^*_{N_k}$. Let $f_n = \sum_{k=1}^r f_{k,n} \in \mathfrak{g}^*$. Then $f_n \to f$ and $\mathrm{Ad}^*(\exp(-nX))f_n \to g$ in $\mathfrak{g}^*$, where $g = \sum_{k=1}^r g_k$. Since $g \notin \mathrm{Ad}^*(G)f$, this finishes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3·6 and the description of the cortex.

Corollary 3·7. *Let G be as in Theorem* 3·6. *Then the following conditions are equivalent*:

(i) $\widehat{G} = \mathrm{cor}\,(G) \cup \mathrm{sep}\,(\widehat{G})$;

(ii) d_k *is even for every* $k = 1, \ldots, r$.

In particular, $\widehat{G}_N = \mathrm{cor}\,(G_N) \cup \mathrm{sep}\,(\widehat{G}_N)$ *if and only if N is odd.*

Lemma 3·8. *Suppose that N is even and let $m = N/2$. Let $f = \sum_{j=m}^N \xi_j X^*_j$ and $g = \sum_{j=m}^N \eta_j X^*_j$ be elements of $\mathfrak{g}^*_N$. Then $\pi_f \sim \pi_g$ in $\widehat{G}_N$ if and only if either $\pi_f = \pi_g$ or $\xi_m = (-1)^{m-1}\eta_m$.*

Proof. Suppose that $\pi_f \sim \pi_g$. Then there are real sequences $(\xi_{j,n})_n$ $(1 \leqslant j \leqslant N-1)$ and $(t_n)_n$ such that $\xi_{j,n} \to \xi_j$ and

$$\sum_{k=0}^{j-1} \frac{1}{k!}\, t_n^k\, \xi_{j-k,n} \to \eta_j \tag{A_j}$$

as $n \to \infty$. Suppose first of all that $|t_n| \nrightarrow \infty$. By passing to a subsequence we can assume that $t_n \to t$ for some $t \in \mathbf{R}$. Then it follows from (A_j) that

$$\eta_j = \sum_{k=0}^{j-1} \frac{1}{k!} t^k \xi_{j-k} \quad (1 \leqslant j \leqslant N-1);$$

whence $g \in \mathrm{Ad}^*(G_N)f + \mathbf{R}X^*_N$. Either

$$\mathrm{Ad}^*(G_N)f = \mathrm{Ad}^*(G_N)f + \mathbf{R}X^*_N,$$

in which case $\pi_g = \pi_f$, or else $\xi_m = \eta_m = 0$.

Suppose, now, that $|t_n| \to \infty$. Multiplying (A_j) by t_n^{m-j}, $1 \leqslant j \leqslant N-1$, we obtain that

$$\frac{1}{0!}\xi_{m,n} + \frac{t_n}{1!}\xi_{m-1,n} + \cdots + \frac{t_n^{m-1}}{(m-1)!}\xi_{1,n} \to \eta_m$$

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$$\frac{1}{1!}\xi_{m,n} + \frac{t_n}{2!}\xi_{m-1,n} + \cdots + \frac{t_n^{m-1}}{m!}\xi_{1,n} \to 0$$

$$\vdots$$

$$\frac{1}{(m-1)!}\xi_{m,n} + \frac{t_n}{m!}\xi_{m-1,n} + \cdots + \frac{t_n^{m-1}}{(2m-2)!}\xi_{1,n} \to 0.$$

As in the proof of Lemma 3·4, let $\alpha_0 = (-1)^{m-1}, \alpha_1, \ldots, \alpha_{m-1}$ be the real numbers such that

$$\sum_{i=0}^{m-1} \alpha_i \frac{1}{(i+j)!} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{for } j = 1, \ldots, m-1. \end{cases}$$

Then

$$\begin{aligned}\sum_{i=0}^{m-1} \alpha_i \left(\sum_{j=0}^{m-1} \frac{t_n^j}{(i+j)!}\xi_{m-j,n} \right) &= \sum_{j=0}^{m-1} t_n^j \xi_{m-j,n} \left(\sum_{i=0}^{m-1} \alpha_i \frac{1}{(i+j)!} \right) \\ &= \xi_{m,n}\end{aligned}$$

and so, taking limits, $\xi_m = \alpha_0 \eta_m = (-1)^{m-1}\eta_m$.

Conversely, suppose that $\xi_m = (-1)^{m-1}\eta_m$. By Lemma 3·4 there is a sequence (f_n) in $\mathfrak{g}_N^*$ such that

$$f_n \to f - \xi_N X_N^* \text{ and } \mathrm{Ad}^*(\exp(-nX_N))f_n \to g - \eta_N X_N^*.$$

Then $f_n + \xi_N X_N^* \to f$ and

$$\mathrm{Ad}^*(\exp(-nX_N))(f_n + \xi_N X_N^*) \to g + (\xi_N - \eta_N)X_N^*.$$

Let $V = \{X_1, X_2, \ldots, X_{N-2}\}^\perp \subseteq \mathfrak{g}_N^*$. Suppose that $f_n \notin V$ frequently. Then, by passing to a subsequence, we may assume that

$$\mathrm{Ad}^*(G_N)(f_n + \xi_N X_N^*) = \mathrm{Ad}^*(G_N)(f_n + \xi_N X_N^*) + \mathbf{R}X_N^*.$$

Hence $f_n + \xi_N X_N^* \xrightarrow{\text{orb}} g$ and so $\pi_f \sim \pi_g$.

Suppose, on the other hand, that $f_n \in V$ eventually. Then $\mathrm{Ad}^*(G_N)f_n = \{f_n\}$ eventually and so

$$f - \xi_N X_N^* = g - \eta_N X_N^* \in V.$$

Thus $\pi_f \sim \pi_g$ since π_f and π_g cannot be separated in a closed subset of $\widehat{G}_N$ homeomorphic to $\widehat{G}_3$. $\square$

By combining the results of this section, we can now give a complete description of the relation $\sim$ on $\widehat{G}_N$ ($N \geqslant 3$) in terms of functionals in $\mathfrak{g}_N^*$. We note that the cases (i), (ii), and (iii) below are exhaustive but that (ii) and (iii) are not mutually exclusive.

Theorem 3·9. *Let $N \geqslant 3$ and let $f = \sum_{j=1}^N \xi_j X_j^* \in \mathfrak{g}_N^*$.*

(i) *π_f is a separated point of $\widehat{G}_N$ if and only if $\xi_j \neq 0$ for some $j \leqslant [(N-1)/2]$.*

(ii) *Suppose that N is even, $m = N/2$ and $\xi_1 = \ldots = \xi_{m-1} = 0$. Then*

$$\{\pi \in \widehat{G}_N : \pi \sim \pi_f\} = \left\{ \pi_g : g = \sum_{j=m}^N \eta_j X_j^* \textit{ and } \eta_m = (-1)^{m-1}\xi_m \right\}.$$

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(iii) *Suppose that* $\xi_1 = \cdots = \xi_{[N/2]} = 0$, *i.e.* $\pi_f \in \operatorname{cor}(G_N)$. *Then*

$$\{\pi \in \widehat{G}_N\colon \pi \sim \pi_f\} = \operatorname{cor}(G_N).$$

Proof. (i) This is a special case of Theorem 3·6.

(ii) This follows from (i) and Lemma 3·8.

(iii) In view of (i) and Lemma 3·8, we can reduce to the case where N is odd. Let $m = (N-1)/2$ and let $g = \sum_{j=m+1}^{N} \eta_j X_j^*$. We have to show that $\pi_f \sim \pi_g$.

By Lemma 3·5 there is a sequence $(f_n)_n$ in $\mathfrak{g}_N^*$ such that $f_n \to f - \xi_N X_N^*$ and $\mathrm{Ad}^*(\exp(-nX_N))f_n \to g - \eta_N X_N^*$. An argument similar to that in the proof of Lemma 3·8 shows that $\pi_f \sim \pi_g$, as required.

Corollary 3·10. *Let* $N \geqslant 3$. *The relation* $\sim$ *fails to be an equivalence relation on* $\widehat{G}_N$ *if and only if* $N > 4$ *and* $N \equiv 0 \pmod 4$.

Proof. If $N > 4$ and $N \equiv 0 \pmod 4$ then it follows from Theorem 3·9 (ii) that $\sim$ is intransitive on $\widehat{G}_N$. For the other values of N, it follows from Theorem 3·9 that $\sim$ is transitive. □

Theorem 3·9 (i) shows that

$$\operatorname{sep}(\widehat{G}_7) = \left\{\pi_f\colon f = \sum_{j=1}^{7} \xi_j X_j^*,\ \xi_j \neq 0 \quad \text{for some } j \in \{1,2,3\}\right\},$$

a dense open subset of $\widehat{G}_7$. Thus the argument of [**12**, proposition 7] shows that each $\pi \in \operatorname{sep}(\widehat{G}_7)$ can be separated by continuous functions from every other point in $\widehat{G}_7$. Let $a \neq 0$, $f = aX_3^*$, and $g = -f$. Then, although we have just seen that π_f and π_g can be separated by a continuous function, the data from [**9**] shows that no $\mathrm{Ad}^*(G_7)$-invariant polynomial can distinguish f and g.

4. *Six-dimensional nilpotent Lie groups*

In [**4**] it was remarked that the theorem of [**4**] and the behaviour of all the five-dimensional cases (see [**4**, section 4]) lead to the conjecture that for G a simply connected, nilpotent Lie group, $C^*(G)$ is quasi-standard (if and) only if the maximal coadjoint orbit dimension in $\mathfrak{g}^*$ equals the dimension of $\mathfrak{z}^\perp$.

Now, up to topological isomorphism, there are exactly 24 simply connected, nilpotent Lie groups of dimension six (excluding those that are direct products of lower dimensional groups). Surprisingly, it turns out that 14 of them have quasi-standard C*-algebras (see below), and 6 of these, namely $G_{6,4}$, $G_{6,7}$, $G_{6,8}$, $G_{6,12}$, $G_{6,13}$, and $G_{6,14}$, are quasi-standard even though the maximal orbit dimension is strictly less than $\dim \mathfrak{z}^\perp$. We present here the proof for $G_{6,4}$, firstly because $G_{6,4}$ is the only 3-step nilpotent counterexample of dimension six to the above conjecture, and secondly because $G_{6,4}$ is isomorphic to the familiar group of real 4×4 upper triangular matrices.

Proposition 4·1. $C^*(G_{6,4})$ *is quasi-standard.*

Proof. Let $G = G_{6,4}$. For the basis of $\mathfrak{g}$ given in [**20**], the set of generic points in $\mathfrak{g}^*$ is $U = \{\xi \in \mathfrak{g}^*\colon \xi_1 \neq 0\}$ (where we identify $\mathfrak{g}^*$ with $\mathbf{R}^6$ by taking coordinates with respect to the dual basis). The following polynomials on $\mathfrak{g}^*$ are $\mathrm{Ad}^*(G)$-invariant:

$$Q_1(\xi) = \xi_1, \quad Q_2(\xi) = \xi_1\xi_6 + \xi_2\xi_3.$$

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We show first of all that $\{\ker q(\xi): \xi \in U\}$ is a set of Glimm ideals in $\mathrm{Prim}\,(C^*(G))$. We could use Proposition 1·4, but the following short argument is more elementary. Let $\xi \in U$ and $\eta \in \mathfrak{g}^*$ and suppose that $\ker q(\xi) \approx \ker q(\eta)$. Then $\eta_1 = Q_1(\eta) = Q_1(\xi) = \xi_1 \neq 0$ and so, without changing $q(\xi)$ and $q(\eta)$, we may assume that $\xi = (\xi_1, 0, 0, 0, 0, \xi_6)$ and $\eta = (\eta_1, 0, 0, 0, 0, \eta_6)$. Since $Q_2(\xi) = Q_2(\eta)$ we obtain that $\xi_6 = \eta_6$ and hence that $\ker q(\xi) = \ker q(\eta)$.

For $r \in \mathbf{R}$ let $S_r = \{\xi \in \mathfrak{g}^* : \xi_1 = 0,\ \xi_2\xi_3 = r\}$. Let $\xi \in S_r$ and suppose that $((\xi_{1,n}, 0, 0, 0, 0, \xi_{6,n}))_{n\geqslant 1}$ is a sequence in U which is orbit-convergent to ξ (since U is dense in $\mathfrak{g}^*$ and 2–5 are jump indices, such a sequence always exists). By [**20**, p. 32] there exist real sequences $(s_{i,n})_{n\geqslant 1}$ $(1 \leqslant i \leqslant 4)$ such that $s_{i,n} \to \xi_{i+1}$ $(1 \leqslant i \leqslant 4)$ and

$$-\frac{1}{\xi_{1,n}} s_{1,n}s_{2,n} + \xi_{6,n} \to \xi_6.$$

Since $\xi_{1,n} \to 0$, we get that $\xi_{1,n}\xi_{6,n} \to \lim s_{1,n}s_{2,n} = \xi_2\xi_3 = r$. Let $\eta \in S_r$ be arbitrary. We shall show that

$$(\xi_{1,n}, 0, 0, 0, 0, \xi_{6,n}) \xrightarrow{\text{orb}} \eta. \tag{1}$$

For this, we seek real sequences $(t_{i,n})_{n\geqslant 1}$ $(1 \leqslant i \leqslant 4)$ such that $t_{i,n} \to \eta_{i+1}$ $(1 \leqslant i \leqslant 4)$ and

$$-\frac{1}{\xi_{1,n}} t_{1,n}t_{2,n} + \xi_{6,n} \to \eta_6.$$

Put $t_{3,n} = \eta_4$ and $t_{4,n} = \eta_5$ for all $n \geqslant 1$.

Case $\eta_2 \neq 0$.

Put $t_{1,n} = \eta_2$ and $t_{2,n} = (\xi_{1,n}/\eta_2)(\xi_{6,n} - \eta_6)$. Then $t_{2,n} \to r/\eta_2 = \eta_3$ as required.

Case $\eta_2 = 0$, $\eta_3 \neq 0$.

Put $t_{2,n} = \eta_3$ and $t_{1,n} = (\xi_{1,n}/\eta_3)(\xi_{6,n} - \eta_6)$. Then $t_{1,n} \to r/\eta_3 = 0$ as required.

Case $\eta_2 = \eta_3 = 0$.

Let $\lambda_n = \xi_{1,n}(\xi_{6,n} - \eta_6)$. Then $\lambda_n \to 0$ as $n \to \infty$. Put $t_{1,n} = |\lambda_n|^{\frac{1}{2}}$ and

$$t_{2,n} = \begin{cases} 0 & \text{if } \lambda_n = 0 \\ \lambda_n/t_{1,n} & \text{if } \lambda_n \neq 0. \end{cases}$$

Then $t_{1,n}t_{2,n} = \lambda_n$, $t_{1,n} \to 0 = \eta_2$ and $t_{2,n} \to 0 = \eta_3$. This establishes (1). It follows that, for each $r \in \mathbf{R}$,

$$J_r = \bigcap \{\ker q(\xi): \xi \in S_r\}$$

is a primal ideal of $C^*(G)$. On the other hand, if $\eta \in \mathfrak{g}^*$ and $\ker q(\eta) \approx \ker q(\xi)$ for some (and hence all) $\xi \in S_r$ then $\eta \in S_r$ by the $\mathrm{Ad}^*(G)$-invariance of Q_1 and Q_2. Thus J_r is a minimal primal Glimm ideal (and in particular $q(S_0) = \mathrm{cor}\,(G)$).

So far we have established that

$$\mathrm{Glimm}\,(C^*(G)) = \{\ker q(\xi): \xi \in U\} \cup \{J_r: r \in \mathbf{R}\},$$

and that each Glimm ideal is primal. We now show that the quotient map

$$\Phi: \widehat{G} \to \mathrm{Glimm}\,(C^*(G))$$

is open. Let V be an open subset of $\widehat{G}$ and suppose that $\Phi^{-1}(\Phi(V))$ is not open. Then

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there exists $\pi \in \Phi^{-1}(\Phi(V))$ and a sequence (π_n) in $\widehat{G} \setminus \Phi^{-1}(\Phi(V))$ such that $\pi_n \to \pi$. Observe that π cannot be generic, so there exists $r \in \mathbf{R}$ and $\xi, \eta \in S_r$ such that $\pi = q(\xi)$, $q(\eta) \in V$ and $\Phi(\pi) = \Phi(q(\eta))$. By passing to a subsequence we may assume that one or other of the following Cases 1 and 2 holds.

Case 1. $\pi_n \in q(U)$ for all n.

Since q is open, we may assume by passing to a further subsequence if necessary that there exist $\xi^{(n)} = (\xi_{1,n}, 0, 0, 0, 0, \xi_{6,n}) \in U$ $(n \geqslant 1)$ such that $\xi^{(n)} \xrightarrow{\text{orb}} \xi$ and $\pi_n = q(\xi^{(n)})$. Then, by (1), $\xi^{(n)} \xrightarrow{\text{orb}} \eta$ and so $\pi_n \to q(\eta) \in V$. Thus $\pi_n \in V$ eventually, which is a contradiction.

Case 2. For each n, there exists $r_n \in \mathbf{R} \setminus \{r\}$ such that $\pi_n \in q(S_{r_n})$.

Whatever the value of r, we may assume by passing to a subsequence if necessary that $r_n \neq 0$ for all n. Denoting by $\tilde{Q}_2$ the continuous function on $\widehat{G}$ induced by Q_2, we have

$$r_n = \tilde{Q}_2(\pi_n) \to \tilde{Q}_2(\pi) = r.$$

We seek $\eta^{(n)} = (0, \eta_{2,n}, \eta_{3,n}, 0, \eta_{5,n}, 0) \in S_{r_n}$ such that

$$\eta^{(n)} \xrightarrow{\text{orb}} \eta. \tag{2}$$

We therefore require $\eta^{(n)}$ and real sequences $(s_{1,n})_{n \geqslant 1}$, $(s_{2,n})_{n \geqslant 1}$ such that

$$\eta_{2,n}\eta_{3,n} = r_n, \quad \eta_{2,n} \to \eta_2, \quad \eta_{3,n} \to \eta_3, \tag{3}$$

$$s_{1,n} \to \eta_4, \quad s_{2,n} \to \eta_6 \tag{4}$$

and

$$\frac{\eta_{3,n}}{\eta_{2,n}} s_{1,n} + \eta_{5,n} \to \eta_5. \tag{5}$$

Since $r_n \to r = \eta_2\eta_3$, we may satisfy (3) as follows.

Case $\eta_2 \neq 0$.

Put $\eta_{2,n} = \eta_2$ and $\eta_{3,n} = r_n/\eta_2$.

Case $\eta_2 = 0$, $\eta_3 \neq 0$.

Put $\eta_{3,n} = \eta_3$ and $\eta_{2,n} = r_n/\eta_3$.

Case $\eta_2 = \eta_3 = 0$.

Put $\eta_{2,n} = |r_n|^{\frac{1}{2}}$ and $\eta_{3,n} = r_n/\eta_{2,n}$.

For (4) put $s_{1,n} = \eta_4$ and $s_{2,n} = \eta_6$ for all n, and for (5) put

$$\eta_{5,n} = \eta_5 - \frac{\eta_{3,n}}{\eta_{2,n}}\eta_4 \quad (n \geqslant 1).$$

This establishes (2).

Hence $q(\eta^{(n)}) \to q(\eta) \in V$, so $q(\eta^{(n)}) \in V$ eventually. But

$$\Phi(q(\eta^{(n)})) = J_{r_n} = \Phi(\pi_n),$$

so $\pi_n \in \Phi^{-1}(\Phi(V))$ eventually. This is a contradiction.

Thus Φ is open, so $C^*(G_{6,4})$ is quasi-standard. $\square$

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The arguments for $G_{6,8}$ and $G_{6,12}$ are broadly similar, whilst those for $G = G_{6,7}$, $G_{6,13}$, and $G_{6,14}$ utilize the presence of a family of non-generic linear functionals with flat orbits of maximal dimension in order to show that $\widehat{G} = \mathrm{sep}\,(\widehat{G}) \cup \mathrm{cor}\,(G)$ before showing that the canonical mapping from $\widehat{G}$ to Glimm $(C^*(G))$ is open. For $G = G_{6,k}$ ($k = 1, 2, 3, 5, 6, 9, 11$), arguments somewhat similar to those for Proposition 4·1, using invariant polynomials and carefully chosen sequences from the generic subset of $\mathfrak{g}^*$, show that every Glimm ideal of $C^*(G)$ is minimal primal. However, in each case the canonical mapping from Prim $(C^*(G))$ to Glimm $(C^*(G))$ fails to be open (as happens for G_4 [**4**; Lemma 4]) and so G is not quasi-standard. This may be shown by constructing a null sequence in the generic subset of $\mathfrak{g}^*$ that is not orbit-convergent to some η such that $\pi_\eta \in \mathrm{cor}\,(G)$.

The remaining three cases are $G_{6,10}(= G_6)$, $G_{6,15}(= W_3)$ and $G_{6,18}$. By Theorem 3·1, $C^*(G_6)$ contains a Glimm ideal that is not primal and by Theorem 2·7 $C^*(W_3)$ contains a Glimm ideal that is not 4-primal and hence not primal. We show next that $C^*(G_{6,18})$ also contains a Glimm ideal that is not primal.

Let $G = G_{6,18}$ and let $\{X_1, \ldots, X_6\}$ be the basis of $\mathfrak{g}$ given in [**20**]. Then $\mathfrak{g} = \mathbf{R}X \ltimes (\mathfrak{a}_1 + \mathfrak{a}_2)$, where $X = X_6$, $\mathfrak{a}_1 = \mathrm{span}\,\{X_2, X_4, X_5\}$, $\mathfrak{a}_2 = \mathrm{span}\,\{X_1, X_3\}$, $\mathbf{R}X \ltimes \mathfrak{a}_1 = \mathfrak{g}_4$ and $\mathbf{R}X \ltimes \mathfrak{a}_2 = \mathfrak{g}_3$. By Theorem 3·1, $C^*(G)$ contains a non-primal Glimm ideal. In fact it is possible to use arguments specific to this case to show that the Glimm ideal contained in $\ker 1_G$ is 3-primal but not 4-primal. The fact that the corresponding Glimm ideal in $C^*(W_3)$ is also 3-primal but not 4-primal appears to be entirely coincidental.

Summary

Up to topological isomorphism there are 24 simply connected nilpotent Lie groups of dimension six (excluding those which are direct products of lower dimensional groups).

(i) $C^*(G_{6,k})$ is quasi-standard in 14 cases,

$$k = 4, 7, 8, 12, 13, 14, 16, 17, 19, 20, 21, 22, 23, 24.$$

(ii) $C^*(G_{6,k})$ is not quasi-standard, but every Glimm ideal is primal, in 7 cases,

$$k = 1, 2, 3, 5, 6, 9, 11.$$

(iii) $C^*(G_{6,k})$ has a Glimm ideal which is not primal if $k = 10, 15, 18$.

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